Being lucky in the category of commutative ring spectra

j/w Lindenstrauss and Richter

Bjørn Ian Dundas

Nordic topology conference, November 28, 2014[1]

[1] "When in doubt, p is odd"

Prologue: homology and commutativity

- M – abelian group
- X – space

The *homology* $H_*(X; M)$ is represented by

$$X \otimes M = \text{“}\bigoplus_{x \in X} M\text{”}$$

Example:

$$S^n \otimes M = K(M, n).$$

Commutativity of M is vital for functoriality.
From a cellular structure of X one can build $X \otimes M$ by means of $S^d \otimes M$'s.

Prologue: homology and commutativity

Cartan/Serre: let $C^n = H_*(S^n \otimes \mathbf{Z}; \mathbf{F}_p)$. Then

$$C^n = \mathrm{Tor}_*^{C^{n-1}}(\mathbf{F}_p, \mathbf{F}_p).$$

We *really* understand the space $S^1 \otimes \mathbf{Z} \simeq S^1$, and $C^1 = E(z)$, $|z| = 1$, giving a full calculation[2].

[2]Similarly for $H_*(S^n \otimes \mathbf{F}_p; \mathbf{F}_p)$: we understand $S^1 \otimes \mathbf{F}_p$ and $H_*(S^1 \otimes \mathbf{F}_p; \mathbf{F}_p) = E(u) \otimes P(t)$, $|u| = 1$, $|t| = 2$.

Setup

- A – commutative ring spectrum
- X – space

"Higher topological Hochschild homology" is represented by the commutative ring spectrum

$$X \otimes A \quad = \quad \text{“} \bigwedge_{x \in X} A\text{”}.^3$$

Examples:

From a cellular structure of X one can build $X \otimes A$ by means of $S^d \otimes A$'s, e.g.,

$$\Sigma X \otimes A \simeq A \wedge_{X \otimes A} A.$$

[3] McClure, Schwänzl, Vogt. Fancy equivariant versions irrelevant for this talk, but things may be derived without notice.

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- $S^1 \otimes A = \mathrm{THH}(A)$,
- $T^n \otimes A = \mathrm{THH}(\dots \mathrm{THH}(A)\dots)$ $(T^n = S^1 \times \cdots \times S^1)$.

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Some calculations

Theorem

A – discrete $\mathbf{Q}$-algebra.

- $\pi_n(S^1 \otimes A) = \bigoplus_{j=0}^n H_n^{(j)}(A)$ *(Hodge decomp.)*
- $H_n^{(1)}(A) = \lim_{\overrightarrow{k}} \pi_{n+k}(S^k \otimes A) = \mathit{Harrison} = AQ$
- $\pi_n(S^d \otimes A) = \bigoplus_{i+dj=n} H_{i+j}^{(j)}(A)$ *(d odd* [a]*)*

[a] Pirashvili

Some other calculations, e.g., for Thom spectra (Cohen, Blumberg, Schlichtkrull)

Theorem (Schlichtkrull)

$S^n \otimes MU \simeq MU \wedge B^{n+1}U_+$.

Chromatic importance

The "negative multi-cyclic homology"

$$(T^n \otimes A)^{hT^n}$$

detects chromatic behavior of

$$K^{(n)}(A) = K(\dots K(A)\dots):$$

Theorem ($n \leq 2$: Browder, Snaith, Bökstedt, Madsen; $n = 3$: Rognes; $n \leq p$: Veen)

The Rognes classes detect the periodic classes: $v_{n-1} \neq 0$ in

$$k(n\text{-}1)_* K^{(n)}(\mathbf{F}_p)$$

and in

$$k(n\text{-}1)_*(T^n \otimes H\mathbf{F}_p)^{hT^n}.$$

Little comfort in being "close to algebra"

(connective) spectra[4]	$\not\sim$	(simplicial) sets[5]
$H\mathbf{Z}$-modules	$\sim$	abelian groups
$H\mathbf{Z}$-algebras	$\sim$	rings
commutative $H\mathbf{Z}$-algebras	$\not\sim$	commutative rings

$$\begin{array}{ccccc} H\mathbf{Z}\text{-alg.} & \xrightarrow{\mathrm{THH}} & H\mathbf{Z}\text{-mod.} & & \\ \uparrow & & \uparrow & & \\ \text{com. } H\mathbf{Z}\text{-alg.} & \xrightarrow{\mathrm{THH}} & \text{com. } H\mathbf{Z}\text{-alg.} & \xrightarrow{\mathrm{THH}} & \dots \end{array}$$

In order to iterate THH *one needs to understand the* **commutative** *structure...*

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In order to iterate THH one needs to understand the **commutative** *structure...*

BUT occasionally one is lucky!

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Higher THH of $H\mathbf{F}_p$

Bökstedt/Breen (80 ± 5) $\quad p$ prime

$$\pi_*(S^1 \otimes H\mathbf{F}_p) = \mathbf{F}_p[x] = P(x), \qquad |x| = 2$$

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Basterra/Mandell (∞)

partial results by Veen ($n \leq 2p$ arXiv13),

Bobkova/Lindenstrauss/Poirier/Richter/Zakharevich ($n \leq 2p+2$ arXiv13)

$$\pi_*(S^n \otimes H\mathbf{F}_p) \cong B^n(x),$$

$|x| = 2$, *where*

$$B^1 = P(x), \quad B^n(x) = \mathrm{Tor}^{B^{n-1}(x)}(\mathbf{F}_p, \mathbf{F}_p).$$

$\pi_*(S^2 \otimes H\mathbf{F}_p) \cong B^2(x) = E(\sigma x)$, $|\sigma x| = 3$

a suggestion for sketch of a proof:

$$\begin{aligned} S^2 \otimes H\mathbf{F}_p &= (D^2 \coprod_{S^1} D^2) \otimes H\mathbf{F}_p \\ &\cong (D^2 \otimes H\mathbf{F}_p) \wedge_{S^1 \otimes H\mathbf{F}_p} (D^2 \otimes H\mathbf{F}_p) \\ &\simeq H\mathbf{F}_p \wedge^L_{S^1 \otimes H\mathbf{F}_p} H\mathbf{F}_p, \end{aligned}$$

Tor SS:

$$\mathrm{Tor}_*^{\pi_*(S^1 \otimes H\mathbf{F}_p)}(\mathbf{F}_p, \mathbf{F}_p) = \mathrm{Tor}_*^{P(x)}(\mathbf{F}_p, \mathbf{F}_p) \Rightarrow \pi_*(S^2 \otimes H\mathbf{F}_p),$$

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Unique ht. type of commutative $H\mathbf{F}_p$-algebra,[6] so

$$S^2 \otimes H\mathbf{F}_p \simeq H\mathbf{F}_p \vee \Sigma^3 H\mathbf{F}_p \simeq H(\mathbf{F}_p[S^3])$$

(square zero) as commutative $H\mathbf{F}_p$-algebras(!). . .[7]

[6] Kriz? Learned from Mandell. C.f. also Lazarev's (04) writeup of Basterra/Mandell's identification of $\mathrm{TAQ}(H\mathbf{F}_p)$

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[7] Veen and BLPRZ do it algebraically. I have not seen Basterra/Mandell's proof, but I will return to how one might deal with some of the problems that have to be addressed

Higher THH of integers

D, Lindenstrauss, Richter (∞)

$\pi_*(S^n \otimes H\mathbf{Z})$ is torsion in positive dimensions, and

$$V(0)_*(S^n \otimes H\mathbf{Z}) \cong B^n(x) \otimes B^{n+1}(y), \quad ^a$$

$|x| = 2p$, $|y| = 2p - 2$.

[a] $V(0) = \mathbf{S}/p$ is the mod p Moore spectrum.

$B^1 = P(x), \quad B^n(x) = \mathrm{Tor}^{B^{n-1}(x)}(\mathbf{F}_p, \mathbf{F}_p)$

Higher THH of integers [10]

Relative constructions: $A \to B$, $* \to X$;

$$X \otimes (A, B) = (X \otimes A) \wedge_A B \quad ^8$$

commutative ring spectrum.
Examples:

[8] $* \otimes A = A$

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- $\ell_p \to H\mathbf{F}_p$, $S^n \otimes (\ell_p, H\mathbf{F}_p) \simeq V(1) \wedge (S^n \otimes \ell_p)$ [9]

[8] $* \otimes A = A$

[9] ℓ_p Adams summand, $V(1) = \mathbf{S}/(p, v_1)$

[10] $V(0)_*(S^n \otimes H\mathbf{Z}) \cong B^n(x) \otimes B^{n+1}(y)$

Higher THH of integers [13]

Sketch proof. Bökstedt: ok for $n = 1$:

$$V(0)_*(S^1 \otimes H\mathbf{Z}) \cong B^1(x) \otimes B^2(y) \cong P(x) \otimes E(\sigma y),$$

$|x| = 2p$, $|y| = 2p - 2$, $|\sigma y| = 2p - 1$.
Mapping to the first Postnikov section we get

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[11] unique ht. type ☺ ☺ ☺! – the factorization is a small calculation
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\end{array}$$

So,

$$\begin{aligned}
S^2 \otimes (H\mathbf{Z}, H\mathbf{F}_p) &\simeq H\mathbf{F}_p[S^{2p+1}] \wedge_{H\mathbf{F}_p} \Sigma H\mathbf{F}_p[S^{2p-1}] \\
&\simeq H\left(\mathbf{F}_p[S^{2p+1}] \otimes_{\mathbf{F}_p} \left(\mathbf{F}_p \otimes_{\mathbf{F}_p[S^{2p-1}]} \mathbf{F}_p\right)\right).
\end{aligned}$$

[14] $V(0)_*(S^n \otimes H\mathbf{Z}) \cong \pi_*(S^n \otimes (H\mathbf{Z}, H\mathbf{F}_p)) \cong B^n(x) \otimes B^{n+1}(y)$

Higher THH of integers [15]

$$S^2 \otimes (H\mathbf{Z}, H\mathbf{F}_p) \simeq H\left(\mathbf{F}_p[S^{2p+1}] \otimes_{\mathbf{F}_p} \left(\mathbf{F}_p \otimes_{\mathbf{F}_p[S^{2p-1}]} \mathbf{F}_p\right)\right).$$

Tor SS now gives the result for $n = 2$:

$$\begin{aligned} V(0)_*(S^2 \otimes H\mathbf{Z}) &\cong \pi_*(S^2 \otimes (H\mathbf{Z}, H\mathbf{F}_p)) \\ &\cong B^2(x) \otimes B^3(y) \cong E(\sigma x) \otimes \Gamma(\sigma^2 y). \end{aligned}$$

$|\sigma x| = 2p + 1$, $|\sigma^2 y| = 2p$.

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Higher THH of integers [16]

In higher dimension: do all directions *simultaneously*, remembering that

$$S^2 \otimes (H\mathbf{Z}, H\mathbf{F}_p) \simeq H\left(\mathbf{F}_p[S^{2p+1}] \otimes_{\mathbf{F}_p} \left(\mathbf{F}_p \otimes_{\mathbf{F}_p[S^{2p-1}]} \mathbf{F}_p\right)\right),$$

and that

$$\begin{array}{ccc} S^n \otimes (H\mathbf{Z}, H\mathbf{F}_p) & \longrightarrow & H\mathbf{F}_p \\ \downarrow & & \downarrow \\ H\mathbf{F}_p & \longrightarrow & S^{n+1} \otimes (H\mathbf{Z}, H\mathbf{F}_p) \end{array}$$

is a homotopy pushout of commutative $H\mathbf{F}_p$-algebras; and using a multisimplicial Bar-resolution.

[16] $V(0)_*(S^n \otimes H\mathbf{Z}) \cong \pi_*(S^n \otimes (H\mathbf{Z}, H\mathbf{F}_p)) \cong B^n(x) \otimes B^{n+1}(y)$

Higher THH of rings of integers A

A prime at a time:

$$(S^n \otimes A)_p \simeq \prod_{p \in \mathfrak{m} \in \mathrm{Spec}(A)} (S^n \otimes A\widehat{_\mathfrak{m}})_p$$

D, Lindenstrauss, Richter (∞) A a number ring, $k = A/\mathfrak{m}$

$$\pi_*(S^n \otimes (A\widehat{_\mathfrak{m}}, k)) \cong B^n_k(x_\mathfrak{m}) \otimes B^{n+1}_k(y_\mathfrak{m})$$ [a]

$|x_\mathfrak{m}| = 2p^{r_\mathfrak{m}}$, $|y_\mathfrak{m}| = 2p^{r_\mathfrak{m}} - 2$, *where*

1. $r_\mathfrak{m} = 1$ *if* $A\widehat{_\mathfrak{m}}$ *unramified over* $\mathbf{Z}_p$
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Higher THH of rings of integers

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Proof:

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- Ramified: use Lindenstrauss and Madsen's calculation when $n = 1$ and a slight Postnikov twist of the proof of the $H\mathbf{Z}$-case.

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4. Thank the audience