2.14.4. *Real Jordan normal form.* If a real matrix has multiple complex eigenvalues and is defective, then its Jordan form can be replaced with an upper block diagonal matrix in a way similar to the diagonal case illustrated in §2.13.2, by replacing the generalized eigenvectors with their real and imaginary parts.

For example, a real matrix which can be brought to the complex Jordan normal form

$$\begin{bmatrix} \alpha + i\beta & 1 & 0 & 0 \\ 0 & \alpha + i\beta & 0 & 0 \\ 0 & 0 & \alpha - i\beta & 1 \\ 0 & 0 & 0 & \alpha - i\beta \end{bmatrix}$$

can be conjugated (by a real matrix) to the real matrix

$$\begin{bmatrix} \alpha & \beta & 1 & 0 \\ -\beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix}$$

2.15. **Block matrices.**

2.15.1. *Multiplication of block matrices.* It is sometimes convenient to work with matrices split in blocks. We have already used this when we wrote

$$M[\mathbf{v}_1, \ldots, \mathbf{v}_n] = [M\mathbf{v}_1, \ldots, M\mathbf{v}_n]$$

More generally, if we have two matrices M, P with dimensions that allow for multiplication (i.e. the number of columns of M equals the number of rows of P) and they are split into blocks:

$$M = \begin{bmatrix} M_{11} & | & M_{12} \\ --- & - & --- \\ M_{21} & | & M_{22} \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & | & P_{12} \\ --- & - & --- \\ P_{21} & | & P_{22} \end{bmatrix}$$

then

$$MP = \begin{bmatrix} M_{11}P_{11} + M_{12}P_{21} & | & M_{11}P_{12} + M_{12}P_{22} \\ ------ & - & ------ \\ M_{21}P_{11} + M_{22}P_{21} & | & M_{21}P_{12} + M_{22}P_{22} \end{bmatrix}$$

if the number of columns of M_{11} equals the number of rows of P_{11}.

Exercise. Prove that the block multiplication formula is correct.

More generally, one may split the matrices M and P into many blocks, so that the number of block-columns of M equal the number of block-rows of P and so that all products $M_{jk}P_{kl}$ make sense. Then MP can be calculated using blocks by a formula similar to that using matrix elements.

In particular, if M, P are block diagonal matrices, having the blocks M_{jj}, P_{jj} on the diagonal, then MP is a block diagonal matrix, having the blocks $M_{jj}P_{jj}$ along the diagonal.

For example, if M is a matrix in Jordan normal form, then it is block diagonal, with Jordan blocks M_{jj} along the diagonal. Then the matrix M^2 is block diagonal, having M_{jj}^2 along the diagonal, and all powers M^k are block diagonal, having M_{jj}^k along the diagonal. Furthermore, any linear combination of these powers of M, say $c_1M+c_2M^2$ is block diagonal, having the corresponding $c_1M_{jj}+c_2M_{jj}^2$ along the diagonal.

2.15.2. *Determinant of block matrices.*

Proposition 17. *Let M be a square matrix, having a triangular block form:*

$$M=\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \quad \textit{or } M=\begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$$

where A and D are square matrices, say A is $k\times k$ and D is $l\times l$.

Then $\det M=\det A\,\det D$.

Moreover, if $a_1,\dots,a_k$ are the eigenvalues of A, and $d_1,\dots,d_l$ are the eigenvalues of D, then the eigenvalues of M are $a_1,\dots,a_k,d_1,\dots,d_l$.

The proof is left to the reader as an exercise.[2]

For a more general 2×2 block matrix, with D invertible[3]

$$M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

the identity

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}=\begin{bmatrix} A-BD^{-1}C & B \\ 0 & D \end{bmatrix}$$

together with Proposition 17 implies that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix}=\det(A-BD^{-1}C)\,\det D$$

which, of course, equals $\det(AD-BD^{-1}CD)$.

For larger number of blocks, there are more complicated formulas.

[2] *Hint:* bring A, D to Jordan normal form, then M to an upper triangular form.

[3] *References:* J.R. Silvester, Determinants of block matrices, Math. Gaz., 84(501) (2000), pp. 460-467, and P.D. Powell, Calculating Determinants of Block Matrices, http://arxiv.org/pdf/1112.4379v1.pdf

3. Solutions of linear differential equations with constant coefficients

In §1.2 we saw an example which motivated the notions of eigenvalues and eigenvectors. General linear first order systems of differential equations with constant coefficients can be solved in a quite similar way. Consider

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u} \tag{32}$$

where M is an $m \times m$ *constant* matrix and $\mathbf{u}$ in an m-dimensional vector.

As in §1.2, it is easy to check that $\mathbf{u}(t) = e^{\lambda t}\mathbf{v}$ is a solution of (32) if λ is an eigenvalue of M, and $\mathbf{v}$ is a corresponding eigenvector. The goal is to find the solution to any initial value problem: find the solution of (32) satisfying

$$\mathbf{u}(0) = \mathbf{u}_0 \tag{33}$$

for any given vector $\mathbf{u}_0$.

3.1. The case when M is diagonalizable.

Assume that M has m independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_m$. Then (32) has the solutions $\mathbf{u}_j(t) = e^{\lambda_j t}\mathbf{v}_j$ for each $j = 1, \ldots, m$.

These solutions are linearly independent. Indeed, assume that for some constants $c_1, \ldots, c_m$ we have $c_1\mathbf{u}_1(t) + \ldots + c_m\mathbf{u}_m(t) = \mathbf{0}$ for all t. Then, in particular, for $t = 0$ it follows that $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$ which implies that all c_j are zero (since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ were assumed independent).

3.1.1. *Fundamental matrix solution.*

Since equation (32) is linear, then any linear combination of solutions is again a solution:

$$\begin{aligned} \mathbf{u}(t) &= a_1\mathbf{u}_1(t) + \ldots + a_m\mathbf{u}_m(t) \\ &= a_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + a_m e^{\lambda_m t}\mathbf{v}_m, \quad a_j \text{ arbitrary constants} \end{aligned} \tag{34}$$

The matrix

$$U(t) = [\mathbf{u}_1(t), \ldots, \mathbf{u}_m(t)] \tag{35}$$

is called a **fundamental matrix solution**. Formula (34) can be written more compactly as

$$\mathbf{u}(t) = U(t)\mathbf{a}, \quad \text{where } \mathbf{a} = (a_1, \ldots, a_m)^T \tag{36}$$

The initial condition (33) determines the constants $\mathbf{a}$, since (33) implies $U(0)\mathbf{a} = \mathbf{u}_0$. Noting that $U(0) = [\mathbf{v}_1, \ldots, \mathbf{v}_m] = S$ therefore $\mathbf{a} = S^{-1}\mathbf{u}_0$ and the initial value problem (32), (33) has the solution

$$\mathbf{u}(t) = U(t)S^{-1}\mathbf{u}_0$$

General results in the theory of differential equations (on existence and uniqueness of solutions to initial value problems) show that this is only one solution.

In conclusion:

Proposition 18. *If the $m \times m$ constant matrix M has has m independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_m$, then equation (32) has m linearly independent solutions $\mathbf{u}_j(t) = e^{\lambda_j t}\mathbf{v}_j$, $j = 1, \ldots, m$ and any solution of (32) is a linear combination of them.*

Example. Solve the initial value problem

$$\begin{aligned} \tfrac{dx}{dt} &= x - 2y, \qquad & x(0) = \alpha \\ \tfrac{dy}{dt} &= -2x + y, \qquad & y(0) = \beta \end{aligned} \tag{37}$$

Denoting $\mathbf{u} = (x, y)^T$, problem (37) is

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u}, \quad \text{where } M = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \quad \text{with } \mathbf{u}(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{38}$$

Calculating the eigenvalues of M, we obtain $\lambda_1 = -1$, $\lambda_2 = 3$, and corresponding eigenvectors $\mathbf{v}_1 = (1, 1)^T$, $\mathbf{v}_2 = (-1, 1)^T$. There are two independent solutions of the differential system:

$$\mathbf{u}_1(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2(t) = e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and a fundamental matrix solution is

$$U(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t)] = \begin{bmatrix} e^{-t} & -e^{3t} \\ e^{-t} & e^{3t} \end{bmatrix} \tag{39}$$

The general solution is a linear combination of the two independent solutions

$$\mathbf{u}(t) = a_1 e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = U(t)\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

This solution satisfies the initial condition if

$$a_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

which is solved for a_1, a_2: from

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

it follows that

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{\alpha+\beta}{2} \\ \frac{-\alpha+\beta}{2} \end{bmatrix}$$

therefore

$$\mathbf{u}(t) = \tfrac{\alpha+\beta}{2}e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \tfrac{-\alpha+\beta}{2}e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{40}$$

so

$$\begin{aligned} x(t) &= \tfrac{\alpha+\beta}{2}e^{-t} - \tfrac{-\alpha+\beta}{2}e^{3t} \\ y(t) &= \tfrac{\alpha+\beta}{2}e^{-t} + \tfrac{-\alpha+\beta}{2}e^{3t} \end{aligned}$$

3.1.2. *The matrix e^{Mt}*. It is often preferable to work with a matrix of independent solutions $U(t)$ rather than with a set of independent solutions. Note that the $m \times m$ matrix $U(t)$ satisfies

$$\frac{d}{dt}U(t) = M\,U(t) \tag{41}$$

In dimension one this equation reads $\frac{du}{dt} = \lambda u$ having its general solution $u(t) = Ce^{\lambda t}$. Let us check this fact based on the fact that the exponential is the sum of its Taylor series:

$$e^x = 1 + \frac{1}{1!}\,x + \frac{1}{2!}\,x^2 + \ldots + \frac{1}{n!}\,x^n + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n$$

where the series converges for all $x \in \mathbb{C}$. Then

$$e^{\lambda t} = 1 + \frac{1}{1!}\,\lambda t + \frac{1}{2!}\,\lambda^2 t^2 + \ldots + \frac{1}{n!}\,\lambda^n x^n + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\,\lambda^n t^n$$

and the series can be differentiated term-by-term, giving

$$\frac{d}{dt}e^{\lambda t} = \frac{d}{dt}\sum_{n=0}^{\infty} \frac{1}{n!}\,\lambda^n t^n = \sum_{n=0}^{\infty} \frac{1}{n!}\,\lambda^n \frac{d}{dt}t^n = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\,\lambda^n t^{n-1} = \lambda e^{\lambda t}$$

Perhaps one can define, similarly, the exponential of a matrix and obtain solutions to (41)?

For any square matrix M, one can define polynomials, as in (10), and it is natural to define

$$e^{tM} = 1 + \frac{1}{1!}\,tM + \frac{1}{2!}\,t^2 M^2 + \ldots + \frac{1}{n!}\,t^n M^n + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!}\,t^n M^n \tag{42}$$

provided that the series converges. If, furthermore, the series can differentiated term by term, then this matrix is a solution of (41) since

$$\frac{d}{dt}e^{tM} = \frac{d}{dt}\sum_{n=0}^{\infty} \frac{1}{n!}\,t^n M^n = \sum_{n=0}^{\infty} \frac{1}{n!}\,\frac{d}{dt}t^n M^n = \sum_{n=1}^{\infty} \frac{n}{n!}\,t^{n-1} M^n = Me^{tM} \tag{43}$$

Convergence and term-by-term differentiation can be justified by diagonalizing M.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be independent eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_m$ of M, let $S = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$. Then $M = S\Lambda S^{-1}$ with Λ the diagonal matrix with entries $\lambda_1, \ldots, \lambda_m$.

Note that

$$M^2 = \left(S\Lambda S^{-1}\right)^2 = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

then

$$M^3 = M^2 M = \left(S\Lambda^2 S^{-1}\right)\,\left(S\Lambda S^{-1}\right) = S\Lambda^3 S^{-1}$$

and so on; for any power

$$M^n = S\Lambda^n S^{-1}$$

Then the series (42) is

$$e^{tM} = \sum_{n=0}^{\infty} \frac{1}{n!}\, t^n M^n = \sum_{n=0}^{\infty} \frac{1}{n!}\, t^n S\Lambda^n S^{-1} = S\left(\sum_{n=0}^{\infty} \frac{1}{n!}\, t^n \Lambda^n\right) S^{-1} \tag{44}$$

For

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_m \end{bmatrix} \tag{45}$$

it is easy to see that

$$\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 & \dots & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_m^n \end{bmatrix} \text{ for } n = 1, 2, 3 \dots \tag{46}$$

therefore

$$\sum_{n=1}^{\infty} \frac{1}{n!}\, t^n \Lambda^n = \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{n!}\, t^n \lambda_1^n & 0 & \dots & 0 \\ 0 & \sum_{n=1}^{\infty} \frac{1}{n!}\, t^n \lambda_2^n & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sum_{n=1}^{\infty} \frac{1}{n!}\, t^n \lambda_m^n \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & e^{t\lambda_m} \end{bmatrix} = e^{t\Lambda} \tag{48}$$

and (44) becomes

$$e^{tM} = S e^{t\Lambda} S^{-1} \tag{49}$$

which shows that the series defining the matrix e^{tM} converges and can be differentiated term-by-term (since these are true for each of the series in (47)). Therefore e^{tM} is a solution of the differential equation (41).

Multiplying by an arbitrary constant vector $\mathbf{b}$ we obtain vector solutions of (32) as

$$\mathbf{u}(t) = e^{tM}\mathbf{b}, \quad \text{with } \mathbf{b} \text{ an arbitrary constant vector} \tag{50}$$

Noting that $\mathbf{u}(0) = \mathbf{b}$ it follows that *the solution of the initial value problem (32), (33) is*

$$\mathbf{u}(t) = e^{tM}\mathbf{u}_0$$

Note: the fundamental matrix $U(t)$ in (35) is linked to the fundamental matrix e^{tM} by

$$U(t) = S e^{t\Lambda} = e^{tM} S \tag{51}$$

Example. For the example (38) we have

$$S = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \quad e^{t\Lambda} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

and

$$\mathbf{u}_1(t) = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{u}_2(t) = e^{3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The fundamental matrix $U(t)$ is given by (39).

Using (49)

$$e^{tM} = Se^{t\Lambda}S^{-1} = \begin{bmatrix} \frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t} & \frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t} \\ \frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t} & \frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t} \end{bmatrix}$$

and the solution to the initial value problem is

$$e^{tM} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\alpha + \left(\frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\beta \\ \left(\frac{1}{2}\,\mathrm{e}^{-t} - \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\alpha + \left(\frac{1}{2}\,\mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{3\,t}\right)\beta \end{bmatrix}$$

which, of course, is the same as (40).

3.2. **Non-diagonalizable matrix.** The exponential e^{tM} is defined similarly, only a Jordan normal form must be used instead of a diagonal form: writing $S^{-1}MS = J$ where S is a matrix formed of generalized eigenvectors of M, and J is a Jordan normal form, then

$$e^{tM} = Se^{tJ}S^{-1} \tag{52}$$

It only remains to check that the series defining the exponential of a Jordan form converges, and that it can be differentiated term by term.

Also to de determined are m linearly independent solutions, since if M is not diagonalizable, then there are fewer than m independent eigenvectors, hence fewer than m independent solutions of pure exponential type. This can be done using the analogue of (51), namely by considering the matrix

$$U(t) = Se^{tJ} = e^{tM}S \tag{53}$$

The columns of the matrix (53) are linearly independent solutions, and we will see that among them are the purely exponential ones coming from the eigenvectors of M.

Since J is block diagonal (with Jordan blocks along its diagonal), then its exponential will be block diagonal as well, with exponentials of each Jordan block (see §2.15.1 for multiplication of block matrices).

3.2.1. *Example:* 2×2 *blocks:* for

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \tag{54}$$

direct calculations give

$$J^2 = \begin{bmatrix} \lambda^2 & 2\,\lambda \\ 0 & \lambda^2 \end{bmatrix},\ J^3 = \begin{bmatrix} \lambda^3 & 3\,\lambda^2 \\ 0 & \lambda^3 \end{bmatrix}, \ldots, J^k = \begin{bmatrix} \lambda^k & k\,\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}, \ldots \tag{55}$$

and then

$$e^{tJ} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k J^k = \begin{bmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{bmatrix} \tag{56}$$

For the equation (32) with the matrix M is similar to a 2×2 Jordan block: $S^{-1}MS = J$ with J as in (54), and $S = [\mathbf{x}_1, \mathbf{x}_2]$ a fundamental matrix solution is $U(t) = Se^{tJ} = [e^{t\lambda}\mathbf{x}_1, e^{t\lambda}(t\mathbf{x}_1 + \mathbf{x}_2)]$ whose columns are two linearly independent solutions

$$\mathbf{u}_1(t) = e^{t\lambda}\mathbf{x}_1, \quad \mathbf{u}_2(t) = e^{t\lambda}(t\mathbf{x}_1 + \mathbf{x}_2) \tag{57}$$

and any linear combination is a solution:

$$\mathbf{u}(t) = a_1 e^{t\lambda}\mathbf{x}_1 + a_2 e^{t\lambda}(t\mathbf{x}_1 + \mathbf{x}_2) \tag{58}$$

Example. Solve the initial value problem

$$\begin{aligned} \tfrac{dx}{dt} &= (1+a)x - y, \qquad && x(0) = \alpha \\ \tfrac{dy}{dt} &= x + (a-1)y, \qquad && y(0) = \beta \end{aligned} \tag{59}$$

Denoting $\mathbf{u} = (x, y)^T$, the differential system (37) is $\frac{d\mathbf{u}}{dt} = M\mathbf{u}$ with M given by (28), matrix for which we found that it has a double eigenvalue a and only one independent eigenvector $\mathbf{x}_1 = (1, 1)^T$.

Solution 1. For this matrix we already found an independent generalized eigenvector $\mathbf{x}_2 = (1, 0)^T$, so we can use formula (58) to write down the general solution of (59).

Solution 2. We know one independent solution to the differential system, namely $\mathbf{u}_1(t) = e^{at}\mathbf{x}_1$. We look for a second independent solution as the same exponential multiplying a polynomial in t, of degree 1: substituting $\mathbf{u}(t) = e^{at}(t\mathbf{b} + \mathbf{c})$ in $\frac{d\mathbf{u}}{dt} = M\mathbf{u}$ we obtain that $a(t\mathbf{b} + \mathbf{c}) + \mathbf{b} = M(t\mathbf{b} + \mathbf{c})$ holds for all t, therefore $M\mathbf{b} = a\mathbf{b}$ and $(M - aI)\mathbf{c} = \mathbf{b}$ which means that $\mathbf{b}$ is an eigenvector of M (or $\mathbf{b} = \mathbf{0}$), and $\mathbf{c}$ is a generalized eigenvector. We have re-obtained the formula (57).

By either method it is found that a fundamental matrix solution is

$$U(t) = [\mathbf{u}_1(t), \mathbf{u}_2(t)] = e^{at} \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix}$$

and the general solution has the form $\mathbf{u}(t) = U(t)\mathbf{c}$ for an arbitrary constant vector $\mathbf{c}$. We now determine $\mathbf{c}$ so that $\mathbf{u}(0) = (\alpha, \beta)^T$, so we solve

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{c} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

which gives

$$\mathbf{c} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha - \beta \end{bmatrix}$$

and the solution to the initial value problem is

$$\mathbf{u}(t) = e^{at} \begin{bmatrix} 1 & t+1 \\ 1 & t \end{bmatrix} \begin{bmatrix} \beta \\ \alpha - \beta \end{bmatrix} = e^{at} \begin{bmatrix} t\,(\alpha - \beta) + \alpha \\ \beta + t\,(\alpha - \beta) \end{bmatrix}$$

or

$$x(t) = e^{at}\left(t\,(\alpha - \beta) + \alpha\right), \ y(t) = e^{at}\left(t\,(\alpha - \beta) + \beta\right)$$

3.2.2. *Example: 3×3 blocks:* for

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \tag{60}$$

direct calculations give

$$J^2 = \begin{bmatrix} \lambda^2 & 2\,\lambda & 1 \\ 0 & \lambda^2 & 2\,\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix}, \ J^3 = \begin{bmatrix} \lambda^3 & 3\,\lambda^2 & 3\,\lambda \\ 0 & \lambda^3 & 3\,\lambda^2 \\ 0 & 0 & \lambda^3 \end{bmatrix}, \ J^4 = \begin{bmatrix} \lambda^4 & 4\,\lambda^3 & 6\,\lambda^2 \\ 0 & \lambda^4 & 4\,\lambda^3 \\ 0 & 0 & \lambda^4 \end{bmatrix}$$

Higher powers can be calculated by induction; it is clear that

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} \tag{61}$$

Then

$$e^{tJ} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k J^k = \begin{bmatrix} e^{t\lambda} & te^{t\lambda} & \frac{1}{2}t^2 e^{t\lambda} \\ 0 & e^{t\lambda} & te^{t\lambda} \\ 0 & 0 & e^{t\lambda} \end{bmatrix} \tag{62}$$

For $M = SJS^{-1}$ with J as in (60) and $S = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]$, a fundamental matrix solution for (32) is

$$Se^{tJ} = [\mathbf{x}_1 e^{\lambda t}, (t\mathbf{x}_1 + \mathbf{x}_2)e^{\lambda t}, (\frac{1}{2}t^2\mathbf{x}_1 + t\mathbf{x}_2 + \mathbf{x}_3)e^{\lambda t}]$$

3.2.3. *In general,* if an eigenvalue λ has multiplicity r, but there are only $k < r$ independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ then, besides the k independent solutions $e^{\lambda t}\mathbf{v}_1, \ldots, e^{\lambda t}\mathbf{v}_k$ there are other $r-k$ independent solutions in the form $e^{\lambda t}\mathbf{p}(t)$ with $\mathbf{p}(t)$ polynomials in t of degree at most $r-k$, with vector coefficients (which turn out to be generalized eigenvectors of M).

Then *the solution of the initial value problem (32), (33) is*

$$\mathbf{u}(t) = e^{tM}\mathbf{u}_0$$

Combined with the results of uniqueness of the solution of the initial value problem (known from the general theory of ordinary differential equations) it follows that:

Theorem 19. *Any linear differential equation $\mathbf{u}' = M\mathbf{u}$ where M is an $m \times m$ constant matrix, and $\mathbf{u}$ is an m-dimensional vector valued function has m linearly independent solutions, and any solution is a linear combination of these. In other words, the solutions of the equation forms a linear space of dimension m.*

3.3. Fundamental facts on linear differential systems.

Theorem 20. *Let M be an $n \times n$ matrix (diagonalizable or not).*

(i) The matrix differential problem

$$\frac{d}{dt}U(t) = M\,U(t), \quad U(0) = U_0 \tag{63}$$

has a unique solution, namely $U(t) = e^{Mt}U_0$.

(ii) Let $W(t) = \det U(t)$. Then

$$W'(t) \;=\; \mathrm{Tr}M\;W(t) \tag{64}$$

therefore

$$W(t) = W(0)\,e^{t\,\mathrm{Tr}M} \tag{65}$$

(iii) If U_0 is an invertible matrix, then the matrix $U(t)$ is invertible for all t, and the columns of $U(t)$ form an independent set of solutions of the system

$$\frac{d\mathbf{u}}{dt} = M\mathbf{u} \tag{66}$$

(iv) Let $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ be solutions of the system (66). If the vectors $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ are linearly independent at some t then they are linearly independent at any t.

Proof.

(i) Clearly $U(t) = e^{Mt}U_0$ is a solution, and it is unique by the general theory of differential equations: (63) is a linear system of n^2 differential equation in n^2 unknowns.

(ii) Using (52) it follows that

$$W(t) = \det U(t) = \det(Se^{tJ}S^{-1}U_0) = \det e^{tJ}\det U_0 = e^{t\sum_{j=1}^n \lambda_j}\det U_0$$

$$= e^{t\mathrm{Tr}M} \det U_0 = e^{t\mathrm{Tr}M} W(0)$$

which is (65), implying (64).

(iii), (iv) are immediate consequences of (65). □