Bond Mathematics & Valuation

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Bond Mathematics & Valuation

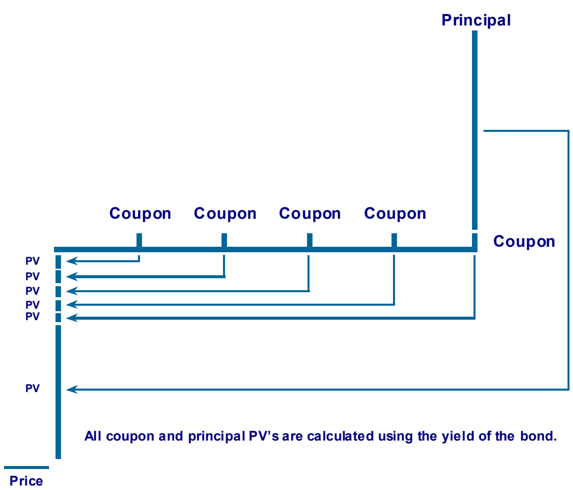
Price Yield Relationship

Yield as a Discount Rate

The price of a bond is the present value of the bond's cash flows. The bond's cash flows consist of coupons paid periodically and principal repaid at maturity.

The present value of each cash flow is calculated using the ***yield to maturity*** **(YTM)** of the bond. Yield to maturity is an ***internal rate of return (IRR)***. That is, yield to maturity is an interest rate that, when used to calculate the present value of each cash flow in the bond, returns the price of the bond as the sum of the present values of the bond's cash flows.

We can picture the price yield relationship as follows:

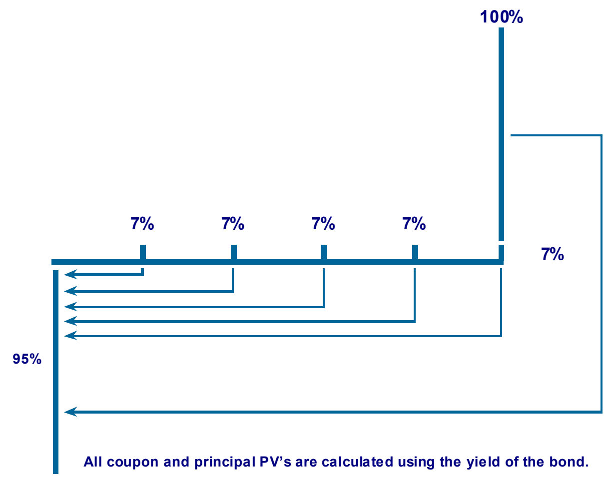


Pricing the Cash Flows of the Bond

Suppose the bond above has annual coupons of 7% and a final principal redemption of 100%. The principal is sometimes referred to as the ***face value*** of the bond.

The market price of the bond—the PV of the five coupons and the face value—is 95% (95% of Par, but in practice no one will include the '%' when quoting a price). This is a given. Market prices are the starting point.

We can picture the bond's cash flows as follows:



The coupons are cash flows—not interest rates. They are stated as 7% of the principal amount. The % only means a cash flow of 7 per 100 of principal. The same is true of the price, which is stated as a per cent of the principal.

We do not yet know the yield to maturity of this bond. Remember that we defined yield to maturity as the IRR of the bond. We have to calculate the yield to maturity as if we were calculating the bond's IRR.

IRR stipulates the following relationship between price and yield. The yield to maturity is the interest rate of the bond. There is only one interest rate (I%) which returns 95% as the sum of the PV's of all the cash flows.

$$95\% = \frac{7\%}{(1+I\%)^1} + \frac{7\%}{(1+I\%)^2} + \frac{7\%}{(1+I\%)^3} + \frac{7\%}{(1+I\%)^4} + \frac{7\%}{(1+I\%)^5} + \frac{100\%}{(1+I\%)^5}$$

Notice how we calculate the PV of each coupon one by one. It is as if we are investing cash for longer and longer periods and earning the yield (the IRR) on each investment.

The future value of our investment each period is calculated by adding the yield to 1 and then compounding it to the number of periods.

For Year 1 our imaginary investment looks like this:

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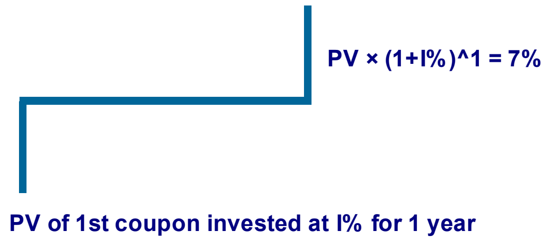

This is the same as saying that we can invest an amount of money today earning a rate of I% for one year. When we get back our invested cash and the interest it has earned for the year, the total will be worth 7%.

For Year 2 our imaginary investment looks like this:

Again we assume we can invest an amount of money today earning a rate of I% for two years. When we get back our invested cash and the interest it has earned after two years, the total will again be worth 7%.

Simple algebra gives us the formula for PV given a future cash flow and the number of periods:

$$\text{Coupon PV}_{\text{Year 1}} = \frac{7\%}{(1+I\%)^1}$$

and

$$\text{Coupon PV}_{\text{Year 2}} = \frac{7\%}{(1+I\%)^2}$$

Extending this logic to the rest of the cash flows gives us the price yield formula we saw above.

$$95\% = \frac{7\%}{(1+I\%)^1} + \frac{7\%}{(1+I\%)^2} + \frac{7\%}{(1+I\%)^3} + \frac{7\%}{(1+I\%)^4} + \frac{7\%}{(1+I\%)^5} + \frac{100\%}{(1+I\%)^5}$$

In this case I% turns out to be 8.2609%. This is the interest rate which prices all the cash flows back to 95%:

$$95\% = \frac{7\%}{(1+8.2609\%)^1} + \frac{7\%}{(1+8.2609\%)^2} + \frac{7\%}{(1+8.2609\%)^3} + \frac{7\%}{(1+8.2609\%)^4} + \frac{107\%}{(1+8.2609\%)^5}$$

Calculators cannot solve for IRR directly. They find it by trying values over and over until the calculated present value equals the given price. This method of calculating is called *iterative*. IRR is an iterative result.

Using a financial calculator to calculate yield is easy. In this case we use a standard Hewlett-Packard business calculator:

Value	Key	Display
5	[N]	5.0000
95	[CHS][PV]	-95.0000
7	[PMT]	7.0000
100	[FV]	100.0000
	[I%]	8.2609%

The IRR or yield to maturity of the above bond is 8.2609%.

Discount Factors Based on Yield to Maturity

Dividing 1 by 1 plus the yield raised to the power of the number of periods is how we calculated the annual discount factors above. These are discount factors based on the bond's yield.

$$DF_{\text{Year 1}} = \frac{1}{(1+0.082609)^1} = 0.923695$$

$$DF_{\text{Year 2}} = \frac{1}{(1+0.082609)^2} = 0.853212$$

$$DF_{\text{Year 3}} = \frac{1}{(1+0.082609)^3} = 0.788107$$

$$DF_{\text{Year 4}} = \frac{1}{(1+0.082609)^4} = 0.727970$$

$$DF_{\text{Year 5}} = \frac{1}{(1+0.082609)^5} = 0.672422$$

There is no real life explanation for this. It is simply how IRR works. There is no promise that we can earn a rate of interest in the market for one year or two years or three years, etc., equal to the yield. In fact, it is entirely implausible—even impossible—that we could earn the yield on cash placed in the market.

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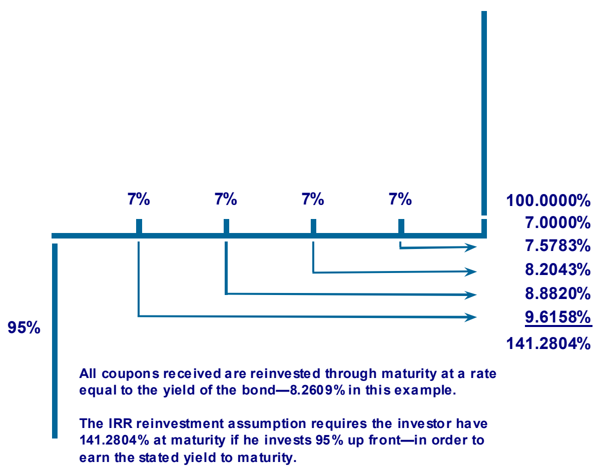
Despite this problem, we still use IRR to calculate bond yields. The key is to always start with a market price and use it to calculate the yield. Never go from yield to price—unless you are absolutely certain that you are using the correct yield for ***that very bond***.

Reinvestment Risk

In fact, the IRR problem is even more interesting. In order to earn the stated yield on the bond, IRR assumes that the bond owner can ***reinvest the coupons through maturity*** at a rate equal to the yield. This is never possible. As a result, no investor has ever actually earned the stated yield on a bond paying him coupons.

The so-called ***reinvestment assumption*** says that we must be able to reinvest all coupons received through the final maturity of the bond at a rate equal to the yield:



If we can reinvest at the yield, the return for the entire five years is 8.2609%:

$$\left(\frac{141.2804\%}{95\%}\right)^{\left(\frac{1}{5}\right)} - 1 = 8.2609\%$$

If we cannot reinvest at the yield, the return over the period does not equal the stated yield. This is the risk of reinvestment.

It is possible to calculate the yield of a bond (its IRR) using a different reinvestment rate—if it makes sense to claim that we know what the actual reinvestment rate will be. Since we do not know what the future will bring with any certainty, this is a mostly fruitless calculation.

Only one kind of bond carries no reinvestment risk. This is a bond that does not pay any coupons, a so-called ***zero-coupon bond***.

If you hold a zero-coupon bond through final maturity, you will earn the stated yield without any risk. The only cash flow you will receive from the bond is the final repayment of principal on the maturity date. Nothing to reinvest means no reinvestment risk:

The return on this zero-coupon bond is 8.2609%:

$$\text{Yield} = \left(\frac{100\%}{67.2422\%}\right)^{\left(\frac{1}{5}\right)} - 1 = 8.2609\%$$

Real World Bond Prices

When we move into the real world of the market we encounter baggage and distortions to the above calculations in the form of accrual conventions, weekends and holidays. Incorporating these real world issues into the price and yield of a bond is our next task.

Accrual Conventions

Accrual of interest is the first topic when we talk about bonds. In fact, this is a question of how we count time more than how we accrue interest.

Interest accrues over periods of time, and there are a lot of different ways to count time in use in financial markets. Counting time with government bonds became simpler in 1999, as all of Europe's government bonds adopted an approach similar to that already in use in France and the United States.

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The other major issue is the number of coupons payable each year. In the UK, the U.S. and in Italy, government bonds pay semi-annual coupons. In most other countries, coupons are paid annually. A summary of the accrual conventions and coupon payments for a selection of government bond markets follows.

Country	Accrual	Coupon Frequency
Austria	A/A	Annual
Belgium	A/A	Annual
Denmark	A/A	Annual
Finland	A/A	Annual
France	A/A	Annual
Germany	A/A	Annual
Ireland	A/A	Annual
Italy	A/A	Semi-Annual
Luxembourg	A/A	Annual
Netherlands	A/A	Annual
Norway	A/365	Annual or Semi-Annual
Portugal	A/A	Annual
Spain	A/A	Annual
Sweden	30E/360	Annual
Switzerland	30E/360	Annual
United Kingdom	A/A	Semi-Annual

Using Excel's Bond Functions

Day Count Functions

Excel offers several functions for calculating the number of days between any two dates according to different day count conventions. **YEARFRAC** returns a fraction of a year. **COUPDAYBS** returns the number of days from the beginning of the current coupon period to the settlement date. **COUPDAYS** returns the number of days in the current coupon period. **COUPDAYSNC** returns the number of days between the settlement date and the next coupon date. **COUPNCD** returns the next coupon date. **COUPPCD** returns the previous coupon date before the settlement date. All of these functions require similar inputs as explained for the YEARFRAC function immediately following.

YEARFRAC returns the year fraction representing the number of whole days between start_date and end_date. Use YEARFRAC to identify the proportion of a whole year's benefits or obligations to assign to a specific term.

If this function is not available, run the Setup program to install the Analysis ToolPak. After you install the Analysis ToolPak, you must select and enable it in the Add-In Manager.

Syntax	YEARFRAC(start_date, end_date, basis)
Start_date	is a serial date number that represents the start date.
End_date	is a serial date number that represents the end date.
Basis	is the type of day count basis to use.
Basis	Day count basis
0 or omitted	US (NASD) 30/360
1	Actual/actual
2	Actual/360
3	Actual/365
4	European 30/360

If any argument is non-numeric, YEARFRAC returns the #VALUE! error value.

If start_date or end_date are not valid serial date numbers, YEARFRAC returns the #NUM! error value.

If basis < 0 or if basis > 4, YEARFRAC returns the #NUM! error value.

All arguments are truncated to integers.

Examples

YEARFRAC(DATEVALUE("01/01/2006"),DATEVALUE("06/30/2006"),2) = 0.5

YEARFRAC(DATEVALUE("01/01/2006"),DATEVALUE("07/01/2006"),3) = 0.49589

Adjusting for Weekends and Holidays

Coupons cannot be paid on weekends or holidays. Bonds normally do not adjust the size of the coupon paid to reflect this, and thus the investor simply receives the stated coupon one or two—or even three—days late. Contrast this to swaps, where the amount of coupon paid is usually adjusted to reflect waiting days.

Bond yield calculations also normally ignore weekends and holidays, although it is perfectly easy to calculate the yield considering the exact days each

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coupon will be paid. Such calculations are sometimes called ***true yields***.

Bond Price Calculations

Price and Yield

We can check the math of bonds using the following U.S. Treasury bond:

Issuer:	U.S. Treasury
Settlement:	09-Jan-06
Coupon:	4.5%
Issue Date:	15-Nov-05
1st Interest:	15-May-06
Maturity:	15-Nov-15
Mkt. Price:	101 1/64%
Mkt. YTM:	4.37133%
Accrued Int.:	0.6837%
"Dirty" Price:	101.6993%

You can check all of these calculations using a typical; HP business calculator.

What is the calculator actually doing? It is calculating the price of each of the bond's cash flows using the YTM as a discount rate.

The market convention uses the yield to maturity as the discount rate, and discounts each cash flow back over the number of periods as calculated using the accrued interest day-count convention. In the case of Treasuries, this is the A/A s.a. convention, which treats each year as composed of 2 equal periods. Days to the end of the current 6-month period are counted in terms of how many days there actually are. This number of days is divided by the number of actual days in the full 6-month period.

The number of days to the first coupon, for example, is 126:

09 Jan 06 – 15 May 06:	126 days
15 Nov 05 – 15 May 06:	181 days

Expressing this in periods:

$$\frac{126}{181} = 0.696133$$

The price of the first coupon (its ***present value***) can be calculated in the following way:

N	0.696133
I%YR	4.37133 ÷ 2 = 2.1857
PMT	0
FV	4.5 ÷ 2 = 2.25
PV	⇨ – 2.2164

All the other cash flow present values are calculated in the same manner. Adding them up gives us the price of the bond:

Dates	A/A/ Days	Periods	Cash Flow	CF PV
15-Nov-05				
9-Jan-06	55			101.6993%
15-May-06	126	0.696133	2.2500%	2.2164%
15-Nov-06		1.696133	2.2500%	2.1690%
15-May-07		2.696133	2.2500%	2.1226%
15-Nov-07		3.696133	2.2500%	2.0772%
15-May-08		4.696133	2.2500%	2.0328%
15-Nov-08		5.696133	2.2500%	1.9893%
15-May-09		6.696133	2.2500%	1.9467%
15-Nov-09		7.696133	2.2500%	1.9051%
15-May-10		8.696133	2.2500%	1.8643%
15-Nov-10		9.696133	2.2500%	1.8245%
15-May-11		10.696133	2.2500%	1.7854%
15-Nov-11		11.696133	2.2500%	1.7473%
15-May-12		12.696133	2.2500%	1.7099%
15-Nov-12		13.696133	2.2500%	1.6733%
15-May-13		14.696133	2.2500%	1.6375%
15-Nov-13		15.696133	2.2500%	1.6025%
15-May-14		16.696133	2.2500%	1.5682%
15-Nov-14		17.696133	2.2500%	1.5347%
15-May-15		18.696133	2.2500%	1.5018%
15-Nov-15		19.696133	102.2500%	66.7909%

Dirty Price and Clean Price

Notice that the price of the bond is 101.6993%, not 101.0156%. The so-called "dirty price," i.e. the price of the bond including accrued interest, is the "true" price of the bond.

The dirty price is the sum of the present values of the cash flows in the bond.

The price quoted in the market, the so-called "clean" price, is in fact not the present value of anything. It is only an accounting convention. The market price is the true present value less accrued interest according to the market convention.

The accrued interest from 15 November 2005 to 09 January 2006, is the fractional period remaining

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through the next coupon date subtracted from 1 full period, times the coupon amount:

$$(1-0.696133)\times 4.5\% \div 2 = 0.6837\%$$

This is the same accrued interest figure we calculated above.

Subtracting the accrued interest from the true present value gives us the price as quoted in the market:

$$\begin{array}{r} 101.6993\% \\ -0.6837\% \\ \hline 101.0156\% \end{array}$$

This is the market price we saw above.

Bond Yields and the Influence of the Coupon Size

Imagine the following yield curve made up of bonds with liquid market prices:

Date	Coupon	Price	Yield
19-Sep-06			
19-Sep-07	5.75%	99.75%	6.0150%
19-Sep-08	6.00%	99.00%	6.5496%
19-Sep-09	6.50%	99.00%	6.8802%
19-Sep-10	7.00%	98.00%	7.5985%
19-Sep-11	7.50%	98.50%	7.8744%

We can strip out the discount factors from this market using the bootstrap methodology (outlined in detail in *Product Analysis: Interest Rate Product Structures*) and calculate the par coupon yields for this curve.

The discount factors are calculated using the following relationship: (PVf_t ; discount factor, Cpn: coupon payment, P: present price)

$$PVf_n = \frac{PV}{FV} = \frac{P - Cpn_n \times \sum_{t=1}^{n-1} PVf_t}{1 + Cpn_n}$$

Par coupon yields are calculated using the following relationship:

$$Par\ Cpn_n = \frac{1 - PVf_n}{\sum_{t=1}^{n} PVf_t}$$

Discount factors and par coupon yields are as follows:

Date	PVf	Par Coupons
19-Sep-06		
19-Sep-07	0.943262	6.0150%
19-Sep-08	0.880570	6.5483%
19-Sep-09	0.818264	6.8785%
19-Sep-10	0.743040	7.5908%
19-Sep-11	0.680107	7.8690%

Notice that the par coupon yields are not equal to the yields on the market bonds. In this yield curve all bonds have prices less than par and all bonds also have yields higher than the respective par coupon yield. We observe that bonds with prices nearer to par have coupons closer to the par coupons.

Into this market we introduce a bond with a 10% coupon. Its price will have to be well above par:

Date	PVf	Cash Flows	CF PVs
19-Sep-06			108.6631%
19-Sep-07	0.943262	10.00%	9.4326%
19-Sep-08	0.880570	10.00%	8.8057%
19-Sep-09	0.818264	10.00%	8.1826%
19-Sep-10	0.743040	10.00%	7.4304%
19-Sep-11	0.680107	110.00%	74.8117%
		Yield	7.8394%

The yield on the 10% coupon bond is 7.8394%, some. 0.0296% lower than the par coupon yield, and 0.0350% lower than the market bond yield of 7.8744%.

In the bond market, this effect will often be masked by the strong aversion most investors have to paying a price above par. In Germany, this aversion is economic, as the tax laws do not allow individual investors to reduce their current income from receiving above-market coupons by amortizing the premium part of the bond's price against interest income, as is normal in most other countries. But in all countries, investors do not like to invest more principal today than they will receive at maturity. The prices of premium-priced bonds therefore sag a bit in the market.

If prices sag, yields rise slightly. In fact, in an upward-sloping yield curve yields should be falling. The bonds are therefore cheap in the market, and will be

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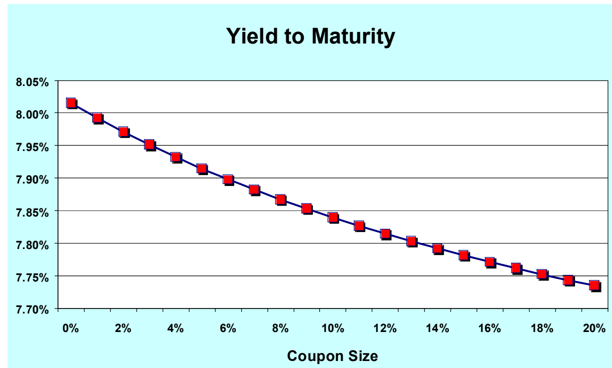
purchased by asset swap traders who have no emotional relationship to cash flows.

This effect is most evident when the yield curve is steeply upward sloping (and would be reversed in an downward-sloping yield curve). In effect, the bond's cash flows are being priced individually by the zero-coupon rates in the yield curves. Higher coupons mean that the cash flow weight of the bond is shifted forward ***down the curve***. This results in premium coupon bonds showing lower yields than par coupon bonds in upward-sloping curves.

Observe the following relationship between coupon size and 5-year yield to maturity in the upward-sloping yield curve of our example bond market.

Smaller coupons mean higher yield to maturity, with the limit at a coupon of 0%. As the coupon grows, the weighted average of the cash flows in the bond moves closer to today and the yield falls—because the curve is upward sloping.



Price Sensitivities

Measuring Price Sensitivity

In this section we examine the sensitivity of the mark-to-market value of a bond to changes in market rates. We approach changing market rates in three ways: 1) Parallel shifts of the entire yield curve; 2) One basis point shifts of each market rate used in establishing the market yield curve; and 3) Non-parallel yield curve shifts of varying amounts.

Parallel Shift Sensitivity

Value of a basis point, also called VBP, BPV, PVBP, PV01, and dollar duration, refers to the average amount by which the mark-to-market value (MMV) of any instrument changes when the entire yield curve is shifted up and down by 0.01%. It is an ***absolute measure*** of pure price change.

Modified duration is a ***relative measure*** of change in the mark-to-market value of an instrument or structure. It measures the average amount of MMV change when the yield curve is shifted up and down by 0.01%. Modified duration is the PV01 divided by the initial MMV. For a par bond or swap, the modified duration is equal to the PV01.

The 0.01% shifts used for PV01 and modified duration are assumed to take place across the entire yield curve. They are thus known as ***parallel yield curve shifts***. PV01 does not capture the risk of change in the shape of the yield curve. This is an important point, as shape changes can have a strong impact on interest rate products.

Individual Market Rate Sensitivities

Factor sensitivity, also called key rate duration, is a measure of the PV01 of any instrument associated to a 0.01% change in ***each market rate*** used to establish the pricing curve. While there is only one measure of PV01 for a given instrument, it has as many ***factor sensitivities*** as there are market-input rates.

In each case, we will shift a single market rate up and down by 0.01%, calculate the average change in the instrument's MMV, put the market rate back to its initial value, and move to the next market input rate.

Non-Parallel Yield Curve Shifts

There is no standard market convention for measuring an instrument's price sensitivity to non-parallel shifts of the yield curve. In the examples outlined in this paper we will move the curve in three ways: steeper, higher and flatter, and lower and flatter. The amounts by which a given curve is adjusted are selected depending on the currency and the level of its curve.

Parallel Shift Sensitivity

We measure the sensitivity to parallel shifts by shifting the entire yield curve by 0.01% up and down and taking the average change.

We can do this with the bond market we are looking at by changing the market yields each by 0.01% up

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and down, calculating the new price of each bond, and then stripping out the discount factors for the now changed curve. Using the new discount factors we can value the coupons and principal of any market bond.

For the sake of this example, we will use the 5-year bond paying a coupon of 7.50% with a market price of 98.50%. First we price the bond with yields unchanged and up and down by 0.01%:

Curve Shift	Coupons	Principal	Bond
MMV			
Curve Δ 0.01%	30.4813%	67.9791%	98.4603%
Initial MMV	30.4893%	68.0107%	98.5000%
Curve Δ -0.01%	30.4974%	68.0423%	98.5397%

By comparing the initial price with the new price for 0.01% shifts up and down, we can measure the **absolute price change** of this bond, also known as the PV01, PVBP, BPV and dollar duration:

Curve Shift	Coupons	Principal	Bond
MMV Δ			
Curve Δ 0.01%	-0.0081%	-0.0316%	-0.0397%
Initial MMV	0.0000%	0.0000%	0.0000%
Curve Δ -0.01%	0.0081%	0.0316%	0.0397%

Relative price change compares the PV01 to the initial market value. This is known as ***modified duration***:

$$\frac{0.0397\%}{98.50\%} \times 10{,}000 = 4.0267$$

The modified duration of this bond is 4.0267 basis points. This means that the relative price change in the bond for a 0.01% shift in market yields is 0.040267%.

We multiply the relative price change by 10,000 because that is the value of 1 basis point:

$$\frac{1}{10{,}000} = 0.01\%$$

You will often see modified duration expressed in years. This is misleading and difficult to interpret. We can have a bond with 7 years remaining with modified duration of 35 or 50. Thinking in years will sometimes result in brain meltdown. Modified duration of 50 simply means that a 0.01% change in market yields will move the bond's price by 0.50% of relative change. This is straightforward—although a bond with modified duration of 50 would be considered very risky…

Calculating Modified Duration

We can also use a formula for modified duration to calculate the modified duration for this bond. The formula represents the first derivative of the bond's price with respect to yield and is based on the formula for the price of a bond using yield to maturity, coupons and principal.

$$\text{Modified Duration} = \frac{\left[\dfrac{\sum_{t=1}^{n}\left(PV_{CF_t} \times t\right)}{\sum_{t=1}^{n} PV_{CF_t}}\right]}{(1 + YTM\%)}$$

where the discount factors are calculated from the bond's yield to maturity as follows:

$$PV\,CF_t = \frac{1}{(1 + YTM\%_n)^t}$$

We can use the formula above to calculate the modified duration of the 7.50% 5-year bond.

Cash Flow	Periods	CF PV	PV × t	PV×t÷TPV
		98.5000%		
7.50%	1	6.9525%	0.069525	0.070584
7.50%	2	6.4450%	0.128900	0.130863
7.50%	3	5.9746%	0.179237	0.181966
7.50%	4	5.5384%	0.221537	0.224911
107.50%	5	73.5895%	3.679473	3.735506
			Duration	4.3438
			Modified Duration	4.0267

The duration of the bond is 4.3438 years. To calculate modified duration, we divide the duration by 1+ YTM:

$$\text{ModifiedDuration} = \frac{4.3438}{1.078744} = 4.0267$$

Using Modified Duration

The formula tying changes in price to modified duration and changes in yield as presented in the Duration module is as follows:

$$\frac{\Delta PV}{PV} \approx -MD \times \Delta YTM$$

This formula says that we can more or less predict the percentage change in the price of a bond for a given

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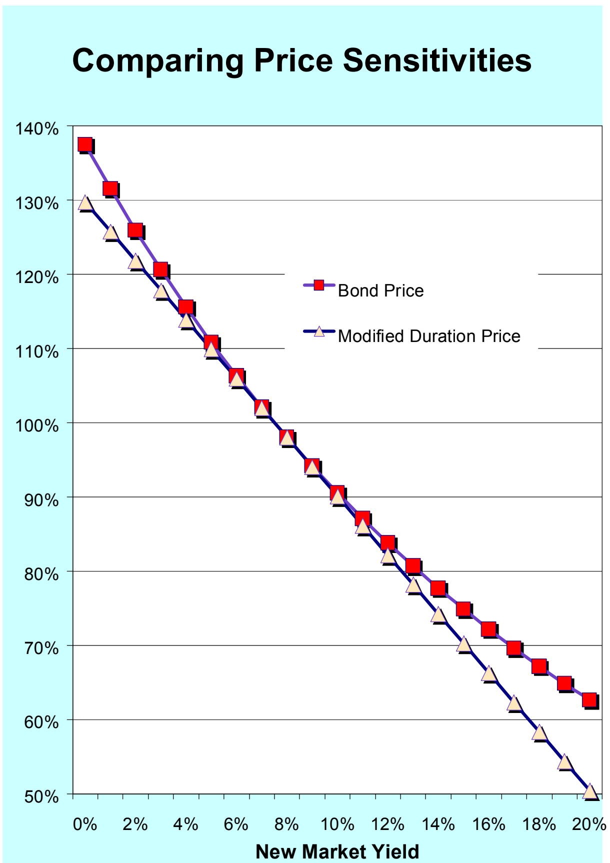
change in its yield to maturity by multiplying its modified duration times the change in YTM.

We can modify the above equation slightly to show the new price of a bond as predicted by modified duration for a given change in the YTM:

$$PV + \Delta PV \approx PV \times \left[1 - \left(MD \times \Delta YTM\right)\right]$$

This formula says that the new price of a bond after a given change in its YTM is more or less equal to the old price times 1 minus the product of its modified duration times the change in YTM.

Using this formula to calculate the price of the 7.50% bond at various possible levels of YTM, and comparing it to the **actual** price of a bond at the same levels of YTM shows something interesting:

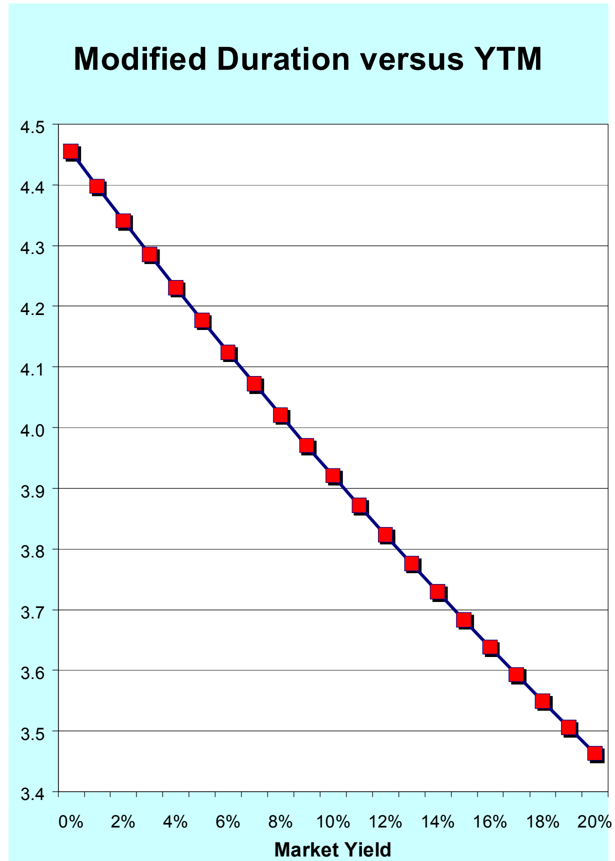


For very small changes in the yield, modified duration predicts the new price fairly well. For larger changes in the yield, it does not do its job very well.

Calculated as the first derivative of the bond price as a function of a change in the yield, modified duration is a linear function. The price of the bond, however, is not a linear function of its yield.

Modified duration changes as yields change.

The following chart shows the modified duration of this bond at various levels of yield:



Modified duration changes fairly sharply as market yields change. This is due to two factors: 1) The price of the bond is not a linear function of its yield; and 2) The measurement of relative change against a static yield change of 0.01%, which represents a larger and larger degree of relative change as yields move lower.

To compensate for the fact that modified duration changes as yields change, we have to add another term to our formulas above, a term which will pick up the changing duration factor. This term is known as ***convexity***—for the obvious reason.

Convexity is a positive quality for owners of bonds:

- As yields fall, prices of bonds gain value faster and faster. The more convex the bond, the faster its price rises as yields fall.

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- As yields rise, prices of bonds fall increasingly slowly. The more convex the bond, the slower its price falls as yields rise.

Investors would like to add convexity to their portfolios for these reasons.

Calculating Convexity

The formula for calculating convexity looks very much like the formula for modified duration. This is because both duration and convexity are ***derivatives*** of the change in the price of a bond as a function of changes in its yield.

Duration is the ***first derivative***. Convexity, depending on how it is measured, is either the ***second derivative*** or the sum of all the other derivatives of the change in price with respect to changes in yield.

Measuring convexity as the second derivative of change in the price of a bond for changes in its yield to maturity, we can use the following formula to calculate a bond's convexity:

$$\text{Convexity} = \frac{\left[\dfrac{\sum_{t=1}^{n}\left[\left(PV_{CF_t} \times t\right) + \left(PV_{CF_t} \times t^2\right)\right]}{\sum_{t=1}^{n} PV_{CF_t}}\right]}{(1 + YTM\%)^2}$$

Using it to calculate the convexity of the 7.50% bond, we obtain the following results:

Cash Flow	Periods	CF PV	PV×t	PV × t^2	Sum ÷TPV
		98.50%			
7.50%	1	6.95%	0.0695	0.0695	0.1411
7.50%	2	6.45%	0.1289	0.2578	0.3926
7.50%	3	5.97%	0.1792	0.5377	0.7278
7.50%	4	5.54%	0.2215	0.8861	1.1246
107.50%	5	73.59%	3.6795	18.3975	22.4132
					24.80
				Convexity	21.31

The bond's convexity is equal to the sum of the columns labeled PV × t and PV × t^2, divided by the price, and then divided by 1 + YTM squared.

The units of convexity are a bit tricky to identify. With duration, we can agree on basis points as the unit, but convexity presents us with something slightly different. It is probably best to not worry too much what the units of convexity are.

Using Convexity

We can add convexity to the modified duration used above in order to calculate the bond's new price for a given change in yield using the following formula:

$$\frac{\Delta PV}{PV} \approx (-MD \times \Delta YTM) + \left(\frac{C \times \Delta YTM^2}{2}\right)$$

This formula says that we can more or less predict the percentage change in the price of a bond for a given change in its yield to maturity by multiplying its modified duration times the change in YTM and adding the convexity-based "correction factor."

We can modify the above equation slightly to show the new price of a bond as predicted by modified duration and adjusted by convexity for a given change in the YTM:

$$PV + \Delta PV \approx PV \times \left[1 - (MD \times \Delta YTM) + \left(\frac{C \times YTM^2}{2}\right)\right]$$

Using the above formula to calculate the price of the 7.50% bond at various possible levels of YTM produces a much better model of the bond's actual price behavior:

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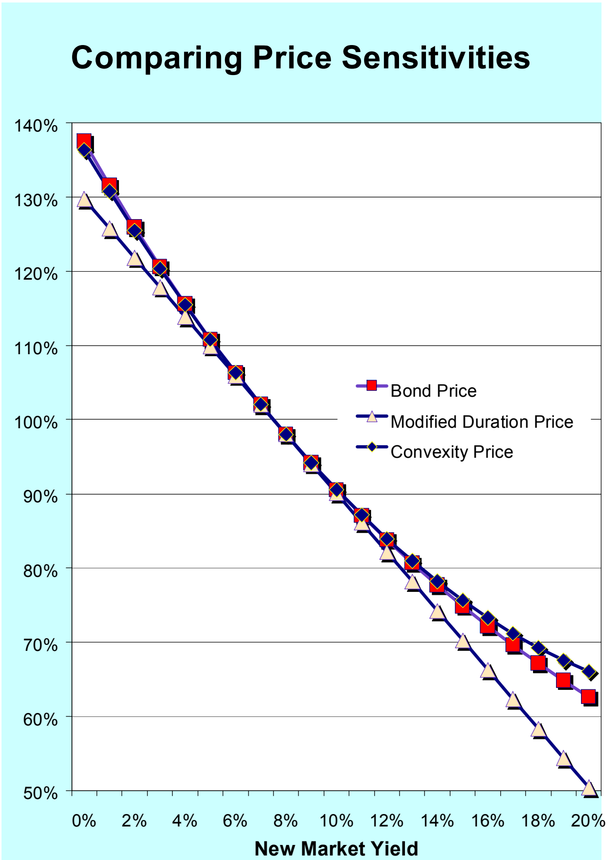

Individual Market Rate Sensitivities

Factor sensitivity, also called key rate duration, is a measure of the PV01 of any instrument associated to a 0.01% change in ***each market rate*** used to establish the pricing curve. While there is only one measure of PV01 for a given instrument, it has as many ***factor sensitivities*** as there are market-input rates.

In our example yield curve, there are five market input rates, each based on the yield to maturity and price of a bond traded in the market. If we shift each of these market input rates (i.e. the yields on each of the five market bonds), recalculate the discount factors and reprice the bond, we can measure its sensitivity to each of the market input rates.

In this case, we will discover that the 5-year bond with the 7.50% coupon only carries exposure to the 5-year market input rate:

Year	Yield	PV01
1	6.0150%	0.0000%
2	6.5496%	0.0000%
3	6.8802%	0.0000%
4	7.5985%	0.0000%
5	7.8744%	-0.0395%

All of its PV01 is located at 5 years. 0.0395% is the PV01 (or dollar duration) we measured above.

At first this might appear counter-intuitive, since it pays coupons every year. If we examine the PV01 on each of the cash flows that results from a change in only one market rate, we see that the effects of a change in a single market rate reverberate through the yield curve.

In the table below, the 1-Year, 2-Year, etc. refer to the market rates that we move by 0.01% one at a time. In each column are the changes in the PV of each cash flow in the far left column when the one market rate for the respective column is shifted.

Cash Flow	1-Year	2-year	3-year	4-year	5-year
7.50%	-0.009660%	0.000000%	0.000000%	0.000000%	0.000000%
7.50%	0.000547%	-0.001274%	0.000000%	0.000000%	0.000000%
7.50%	0.000556%	0.000078%	-0.001838%	0.000000%	0.000000%
7.50%	0.000560%	0.000078%	0.000120%	-0.002320%	0.000000%
107.50%	0.007997%	0.001118%	0.001718%	0.002320%	-0.039500%
Sum	0.000000%	0.000000%	0.000000%	0.000000%	-0.039500%

A 0.01% change in the 1-year bond's market yield forces changes in all the other discount factors across the curve—**because** the other yields do not change and the prices of the other bonds have to stay the same. Because these changes are all driven off each other, the net effect is that the change in the 1-year market yield does not change the price of the 5-year market bond. The effects exactly offset each other.

The same is true for every market rate except the 5-year rate. All of the bond's parallel shift PV01 is in the 5-year market rate.

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