ECE 101 - Fall 2019

Linear Systems Fundamentals

Final Exam Review Topics

Book Recommendation: Signals & Systems by Oppenheim and Wilsky (Prentice Hall)

1 Signals - Chapter 1

Complex numbers and trigonometry

- Complex arithmetic, magnitude, phase, triangle inequality
- Euler's formula: $e^{j\theta} = \cos\theta + j\sin\theta$
- Evaluation of complex exponential $e^{j\theta}$ at standard angles (e.g., π, $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$)
- Basic trigonometric identities:

$$\cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$
$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

- Infinite geometric series:

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \text{ for } |z| < 1.$$

$$\sum_{n=0}^{\infty} nz^n = \frac{z}{(1-z)^2}, \text{ for } |z| < 1.$$

- Finite geometric series:

$$\sum_{n=0}^{N-1} z^n = \begin{cases} N & z = 1 \\ \frac{1-z^N}{1-z} & \text{for any complex } z \neq 1. \end{cases}$$

CT and DT Signals

- Signal energy and power
- Transformations of independent variable
  - Time shifting, time reversal, time scaling
  - Interpretations of $x(at-b)$

- Periodic signals
  - Periodicity conditions
  - Fundamental period and frequency
  - Finding fund. period/frequency of the sum of periodic signals
  - Periodicity and scaling
- Even and Odd signals
  - Definitions of even and odd signals,
  - Even-odd decomposition theorem: $x(t) = \mathcal{E}v\{x(t)\} + \mathcal{O}d\{x(t)\}$ where $\mathcal{E}v\{x(t)\} = \frac{x(t)+x(-t)}{2}$ and $\mathcal{O}d\{x(t)\} = \frac{x(t)-x(-t)}{2}$
- CT and DT mpulse and unit step signals
  - Relationships
    $\sum_{n=-\infty}^{\infty} \delta[n] = 1$
    $\delta[n] = u[n] - u[n-1]$
    $\delta[n] = u[n] - u[n-1]$
    $u[n] = \sum_{k=0}^{\infty} \delta[n-k] = \sum_{k=-\infty}^{n} \delta[k]$
    $\int_{-\infty}^{\infty} \delta(t)dt = 1$
    $\delta(t) = \frac{d}{dt}u(t)$
    $u(t) = \int_{\infty}^{\infty} \delta(t-\tau)d\tau = \int_{-\infty}^{t} \delta(\tau)d\tau$
  - Sampling and sifting properties
    $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$
    $\int_{\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$
  - Representation property
    $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
    $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$
- Complex exponential signals

  CT: $x(t) = ce^{at}, c, a \in \mathbb{C}$

  DT: $x[n] = c\alpha^n, c, \alpha \in \mathbb{C}$

  $x(t) = e^{j\omega_0 t}$ periodic, fund. frequency ω_0, fund. period $T = 2\pi/\omega_0$

  $x[n] = e^{j\Omega_0 n}$ periodic in n if and only if $\Omega_0 = 2\pi/N$, for $m, N \in \mathbb{Z}, N > 0$

  If $\gcd(m, N) = 1$, fund. period N , fund. frequency $2\pi/N$

  $x[n] = e^{j\Omega n}$ periodic in Ω, period 2π.

2 Systems - Chapters 1 and 2

- **Basic system properties**

  Memoryless: output at time n does not depend on inputs before or after time n

  Invertible: distinct input signals produce distinct output signals

  Causal: output at time n does not depend on inputs after time n

  Stable: bounded input signals produces bounded output signals

  Time-invariant: $x(t)$ produces $y(t) \Rightarrow x(t-t_0)$ produces $y(t-t_0)$

  Linear: additive and scalable

- **System impulse response and step response**

  Input $\delta(t)$ produces impulse response $h(t)$

  Input $u(t)$ produces step response $s(t)$

- **LTI systems**

- **Relationship between impulse response and step response**

  $s(t) = \int_{-\infty}^{t} h(\tau)d\tau \quad h(t) = \frac{ds(t)}{dt}$

  $s[n] = \sum_{k=-\infty}^{n} h[k] \quad h[n] = s[n] - s[n-1]$

- **Convolution formulas**

  Convolution sum formula: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$

  Convolution integral formula $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$

- **Properties of convolution**

  Commutativity, associativity, distributivity over addition

  Convolution with shifted impulse: $x(t) * \delta(t-t_0) = x(t-t_0)$

  Integration: $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)d\tau$

- **Impulse response of serial and parallel concatenations of LTI systems.**

  (shown here for CT systems)

  Serial: $h(t) = h_1(t) * h_2(t)$

  Parallel: $h(t) = h_1(t) + h_2(t)$

- **Impulse response and properties of LTI systems (shown here for CT systems)**

  Memoryless: $h(t) = a\delta(t)$

  Invertible: There exists $g(t)$ such that $h(t) * g(t) = \delta(t)$

  (necessary condition, also sufficient for inputs $x(t)$ with $x(t) * h(t)$ and $x(t) * g(t)$ both well defined and finite)

Causal: $h(t) = 0$ for $t < 0$

Stable: $h(t)$ is absolutely integrable

- **Differentiation property of CT LTI systems**

  $x(t) \rightarrow y(t) \Rightarrow \frac{dx(t)}{dt} \rightarrow \frac{dy(t)}{dt}$

- **LTI systems defined by differential/difference equations**

  $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$ $\quad\quad$ $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

- **Recursive and non-recursive filters**

  Non-recursive filters: no feedback of the output

  DT blur filters with rectangular impulse response

  $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} \delta[n-k] = \frac{1}{N}(u[n] - u[n-N])$

  Recursive filters: feedback of output

  DT first order filters: $y[n] - ay[n-1] = x[n]$, $\quad$ a real, $|a| < 1$

  a positive: low-pass filter; $\quad h[n] = a^n u[n]$ decaying exponential

  a negative: high-pass filter; $\quad h[n] = a^n u[n]$ alternating-polarity decaying exponential

3 CT and DT Fourier Series - Chapter 3

- **Key equations**

  CTFS equations for periodic signal, fund. period T, fund. frequency $\omega_0 = 2\pi/T$

  Synthesis: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

  Analysis: $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$

  DTFS equations for periodic signal, fund. period N, fund. frequency $\omega_0 = 2\pi/N$

  Synthesis: $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$

  Analysis: $a_k = \frac{1}{N}\sum_{n=<N} x[n] e^{-jk\omega_0 n}$

- **Response of LTI system to complex exponential**

  System Functions $H(s)$ and $H(z)$

  e^{st} (resp. z^n) is an *eigenfunction*

  $H(s)$ (resp. $H(z)$) is the corresponding *eigenvalue*. It is called the *system function*

  $H(j\omega)$ (resp. $H(e^{j\omega})$) is called the *system frequency response*

  $e^{st} \rightarrow H(s)e^{st}$ $\quad\quad$ $H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$

  $z^n \rightarrow H(z) z^n$ $\quad\quad$ $H(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k}$

- **How to determine if system may be LTI by action on complex exponential signals**

  Check if eigenfunction property is satisfied. If violated, system is not LTI.

- **Filtering of periodic signal through an LTI system**

  Assume frequency response $H(j\omega)$ (resp. $H(e^{j\omega})$)

  $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$

  $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} a_k H(e^{jk\omega_0}) e^{jk\omega_0 n}$

- **Key examples**

  Periodic complex exponentials, sinusoidal signals, rectangular waves, impulse train

- **Properties of CTFS/DTFS (Tables 3.1 and 3.2)**

  Periodicity of DTFS $\mathbf{a_k}$: $x[n]$ fund. period $N \Rightarrow a_k$ period N

  Linearity Time shifting, frequency shifting

  Time reversal, time scaling

  Periodic convolution, multiplication

  Differentiation, integration (CT) / First difference, running sum (DT)

  Parseval's relation

  FS and signal properties: real (conjugate symmetry), real & even, imaginary & odd

- **Finding system function and frequency response of causal LTI system from differential/difference equations**

  $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$ $\quad H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$ $\quad H(j\omega)$ - plug in $s = j\omega$

  $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$ $\quad H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$ $\quad H(e^{j\omega})$ - plug in $z = e^{j\omega}$

- **Effect of LTI system with *real* impulse response on sinusoids**

  $\cos(\omega_0 t) \rightarrow |H(j\omega_0)| \cdot \cos(\omega_0 t + \angle H(j\omega_0))$

  $\cos(\omega_0 n) \rightarrow |H(e^{j\omega_0})| \cdot \cos(\omega_0 n + \angle H(e^{j\omega_0}))$

4 CT Fourier Transform - Chapter 4

- **Key equations**

  Synthesis: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{jk\omega t} d\omega$

  Using synthesis equation to evaluate $x(t)$ at specific values of t, such as $t = 0$:

  $x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega$

  Analysis: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

Using analysis equation to evaluate $X(j\omega)$ at specific values of ω, such as $\omega = 0$:

$X(j0) = \int_{-\infty}^{\infty} x(t)dt$

- **Properties of CTFT (Table 4.1)**

  Linearity, Time shifting, frequency shifting

  Conjugation, Time reversal, time scaling

  Convolution, multiplication

  Differentiation (in time/frequency), integration

  Parseval's relation $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$

- FT and signal properties:

  $x(t)$ real $\Rightarrow X(j\omega$ conjugate symmetric

  $x(t)$ real and even $\Rightarrow X(j\omega)$ real and even

  $x(t)$ real and odd $\Rightarrow X(j\omega)$ purely imaginary and odd

  Even-odd decomposition for real signals:

  $\mathcal{E}v\{x(t)\} \longleftrightarrow \mathcal{R}e\{X(j\omega)\}$

  $\mathcal{O}d\{x(t)\} \longleftrightarrow j\mathcal{I}m\{X(j\omega)\}$

- **Basic CT Fourier Transform Pairs (Table 4.2)**

  Periodic signals: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

  $x(t) = 1 \leftrightarrow 2\pi\delta(\omega)$

  Periodic square wave

  Periodic impulse train $\leftrightarrow$ periodic impulse train (picket fence $\leftrightarrow$ picket fence)

  Rectangular pulse $\leftrightarrow$ sinc function

  Sinc function $\leftrightarrow$ rectangular pulse

  Transforms of unit impulse, shifted impulse, and unit step

  $x(t) = e^{-at}u(t), Re\{a\} > 0 \leftrightarrow X(j\omega) = \frac{1}{a+j\omega}$

- **Systems characterized by linear constant-coefficient differential equations**

  $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

  $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$

  Partial Fraction Expansion and tables to find impulse response, calculate inputs/outputs

- **Filtering through LTI systems**

  $y(t) = x(t) * h(t) \leftrightarrow Y(j\omega) = X(j\omega)H(j\omega)$

- **Inverse LTI systems**

  $h(t) * g(t) = \delta(t) \leftrightarrow H(j\omega)G(j\omega) = 1$

5 DT Fourier Transform - Chapter 5

- **Key equations**

  Synthesis: $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

  Analysis: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

- **Periodicity** of $X(e^{j\omega})$ in frequency, period 2π

- **Properties of DTFT (Table 5.1)**

  Similar to CTFT

- **Basic DT Fourier Transform Pairs (Table 5.2)**

  Periodic signals: fund. period N, fund. frequency $\omega_0 = 2\pi/N$

  $x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \leftrightarrow X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$

  Similar to CTFT

- **Systems characterized by linear constant-coefficient difference equations**

  $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

  $H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k (e^{-j\omega})^k}{\sum_{k=0}^{N} a_k (e^{-j\omega})^k}$

  Partial Fraction Expansion and tables to find impulse response, calculate inputs/outputs

- **Filtering through LTI systems**

  $y(t) = x(t) * h(t) \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

- **First-order recursive filters (lowpass, highpass), non-recursive filters (blur)**

  Derivation of frequency response of length-N blur filter:

  $$H(e^{j\omega}) = \frac{1}{N} e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

  Derivation of frequency response of recursive filter: $y[n] - ay[n-1] = x[n],\ |a| < 1$.

  $$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

6 Amplitude Modulation - Chapter 8

- **Amplitude modulation**

  Signal $x(t)$, bandlimited to $[-W, W]$, carrier $c(t) = \cos(\omega_c t)$ ($\omega_c > W$).

  Modulation:

  $y(t) = x(t)c(t) \leftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * C(j\omega) = \frac{1}{2}\big(X(j(\omega - \omega_c)) + X(j(\omega + \omega_c))\big)$

Demodulation:

$z(t) = y(t)c(t) \leftrightarrow$

$Z(j\omega) = \frac{1}{2\pi} Y(j\omega) * C(j\omega) = \frac{1}{4} X(j(\omega - 2\omega_c)) + \frac{1}{2} X(j\omega) + \frac{1}{4} X(j(\omega + 2\omega_c))$

Signal recovery (if $\omega_c > W$):

Pass $z(t)$ through low-pass filter with cutoff frequency ω_c, gain 2

- **Effect of demodulating with different sinusoid:** $\sin(\omega_c t)$, $\cos(\omega_1 t)$ where $\omega_1 \neq \omega_c$.

7 Sampling Theory - Chapter 7

- **Impulse train sampling**

  CT signal, band-limited to $[-W, W]$:

  $x(t) \leftrightarrow X(j\omega)$, $X(j\omega) = 0, \text{ for } |\omega| > W$

  Impulse train, sampling period T, sampling frequency $\omega_s = 2\pi/T$:

  $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \leftrightarrow P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

  Sampled signal:

  $x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \leftrightarrow X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$

- **Sampling Theorem:**

  Let $x(t)$ be band-limited to $[-W, W]$. Then $x(t)$ is uniquely determined by its samples $x(nT),\ n \in \mathbb{Z}$ if $\omega_s = 2\pi/T > 2W$.

  The signal $x(t)$ can be reconstructed from the signal $x_p(t)$ using a **low-pass** filter with cutoff frequency $\omega_0 = \omega_s/2 = \pi/T$ and gain T.

  If $\omega_s < 2W$, then **aliasing** occurs, and $x(t)$ cannot be reconstructed from $x_p(t)$.

- **Using CTFT properties and transform pairs** to determine the maximum frequency of a signal $y(t)$ obtained from $x(t)$ by various operations, e.g., conjugation, time-shifting, combining with another signal by linear operation, conjugation, multiplication, etc.

- **DT processing of CT signals**

  Define $x(t)$, $p(t)$, $x_p(t)$ as above.

  Define $x_d[n] = x(nT)$, $n \in \mathbb{Z}$

  Using formula for $x_p(t)$, rewrite $X_p(j\omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$

  Using DTFT, $X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega n}$

  Thus, $X_d(e^{j\Omega}) = X_p(j\Omega/T)$ or $X_d(e^{j\omega T}) = X_p(j\omega)$

  The plots of $X_d(e^{j\Omega})$ is obtained from the plot of $X_p(j\omega)$ by multiplying the ω-axis labels of $X_p(j\omega)$by T. So, $\omega_s = 2\pi/T$ becomes $\Omega = 2\pi$.

- **Example of aliasing when undersampling a sinusoidal signal**

  Change of frequency and possible phase shift (reversal)

- **Bandpass sampling techniques** for bandpass signals.

8 Laplace Transform - Chapter 9

- **Laplace Transform:** $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$

  Region of Convergence: ROC $= \{s \in \mathbb{C} |\ X(s)\ \text{exists}\}$

- **Relation to Fourier Transform**

  $X(s)$ exists at $s = \sigma + j\omega \Leftrightarrow$ CTFT of $x(t)e^{-\sigma t}$ exists.

- **Basic LT examples**

  Right-sided exponential, ROC right half-plane

  $x(t) = e^{-at}u(t),\ a \in \mathbb{C} \leftrightarrow X(s) = \frac{1}{s+a}$, ROC $= \{s | Re\{s\} > -Re\{a\}\}$

  Left-sided exponential, ROC left half-plane

  $x(t) = -e^{-at}u(-t),\ a \in \mathbb{C} \leftrightarrow X(s) = \frac{1}{s+a}$, ROC $= \{s | Re\{s\} < -Re\{a\}\}$

- **Rational X(s), poles, zeros, and pole-zero plots**

- **The 8-fold way of the ROC**

  Right-sided signal: ROC contains a right half-plane

  Left-sided signal: ROC contains a left half-plane

  Two-sided signal: ROC is a finite vertical strip

  Absolutely integrable and finite duration signal: ROC is entire s-plane

  Rational $X(s)$: ROC contains no poles; ROC is bounded by poles or extends to infinity

- **ROC and LTI system properties**

  S causal $\Rightarrow$ ROC contains a right half-plane.

  Assume S rational: S causal $\Leftrightarrow$ ROC equals right half-plane to right of rightmost pole.

  S stable $\Leftrightarrow$ ROC contains the $j\omega$-axis.

  Assume S rational and causal: S stable $\Leftrightarrow$ all poles have negative real part.

- **Inverse LT**

  Via integration along line in ROC.

  Via **partial fraction expansion** and tables of LT transform pairs.

- **Geometric evaluation of Fourier Transform using pole-zero plot.**

  Magnitude $|H(j\omega)|$ using lengths of vectors from poles and zeros to $j\omega$, i.e., $(j\omega - \rho)$ and $(j\omega - \zeta)$.

  Phase $\angle H(j\omega)$ using angles of vectors from poles and zeros to $j\omega$, i.e., $(j\omega - \rho)$ and $(j\omega - \zeta)$.

- **Properties of LT (Table 9.1)**

  Linearity, time-shifting, convolution, differentiation/integration in time domain, and effect on ROC.

- **Laplace Transforms of elementary functions (Table 9.2)**

- **Analysis of LTI systems defined by differential equations**

  System function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b^k s^k}{\sum_{k=0}^{N} a^k s^k}$$

  Convert to rational $H(s)$, use PFE, invert using table of transforms

  ROC determined by properties of system (e.g., causal, stable, etc.)

- **System function algebra**

  Serial and parallel concatentation, simple feedback systems.

  Serial: $H(s) = H_1(s)H_2(s)$

  Parallel: $H(s) = H_1(s) + H_2(s)$

  Simple feedback (with positive feedback):

$$H(s) = \frac{H_{ff}(s)}{1 - H_{ff}(s)H_{fb}(s)}$$

  where $H_{ff}(s)$ is system function of feed-forward filter and $H_{fb}(s)$ is system function of feedback filter.