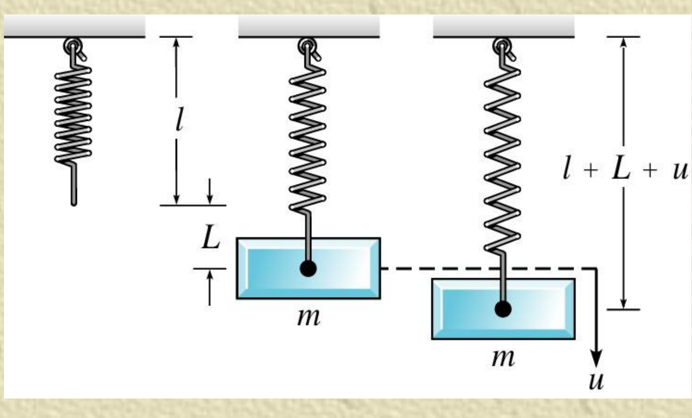
Boyce/DiPrima 9th ed, Ch 3.8: Forced Vibrations

Elementary Differential Equations and Boundary Value Problems, 9th edition, by William E. Boyce and Richard C. DiPrima, ©2009 by John Wiley & Sons, Inc.

- We continue the discussion of the last section, and now consider the presence of a periodic external force:

$$m u''(t) + \gamma u'(t) + k u(t) = F_0 \cos \omega t$$



Forced Vibrations with Damping

- Consider the equation below for damped motion and external forcing funcion $F_0 \cos \omega t$.

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

- The general solution of this equation has the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + A\cos(\omega t) + B\sin(\omega t) = u_C(t) + U(t)$$

where the general solution of the homogeneous equation is

$$u_C(t) = c_1 u_1(t) + c_2 u_2(t)$$

and the particular solution of the nonhomogeneous equation is

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

Homogeneous Solution

- The homogeneous solutions u_1 and u_2 depend on the roots r_1 and r_2 of the characteristic equation:

$$mr^2 + \gamma\, r + kr = 0 \quad \Rightarrow \quad r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- Since m, γ, and k are are all positive constants, it follows that r_1 and r_2 are either real and negative, or complex conjugates with negative real part. In the first case,

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0,$$

while in the second case

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t\right) = 0.$$

- Thus in either case,

$$\lim_{t\to\infty} u_C(t) = 0$$

Transient and Steady-State Solutions

- Thus for the following equation and its general solution,

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_C(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)},$$

we have

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 u_1(t) + c_2 u_2(t)\right) = 0$$

- Thus $u_C(t)$ is called the **transient solution**. Note however that

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

is a steady oscillation with same frequency as forcing function.

- For this reason, $U(t)$ is called the **steady-state solution**, or **forced response**.

Transient Solution and Initial Conditions

- For the following equation and its general solution,

$$mu''(t) + \gamma\, u'(t) + ku(t) = F_0 \cos \omega t$$

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_C(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)}$$

  the transient solution $u_C(t)$ enables us to satisfy whatever initial conditions might be imposed.
- With increasing time, the energy put into system by initial displacement and velocity is dissipated through damping force. The motion then becomes the response $U(t)$ of the system to the external force $F_0 \cos \omega t$.
- Without damping, the effect of the initial conditions would persist for all time.

Example 1 (1 of 2)

- Consider a spring-mass system satisfying the differential equation and initial condition

$$u'' + u' + 1.25u = 0, \quad u(0) = 2, \ u'(0) = 3$$

- Begin by finding the solution to the homogeneous equation
- The methods of Chapter 3.3 yield the solution

$$u_C(t) = c_1 e^{-t/2} \cos t + c_2 e^{-t/2} \sin t$$

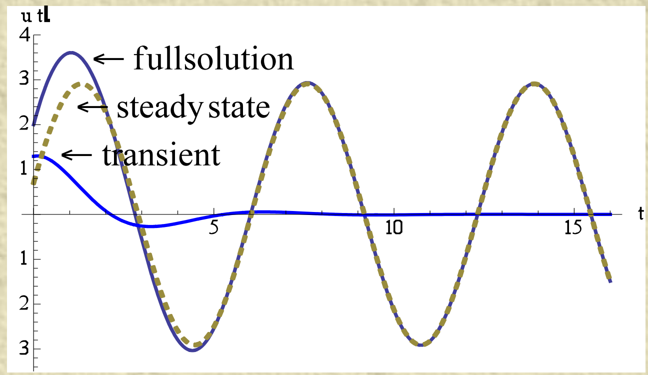
- A particular solution to the nonhomogeneous equation will have the form $U(t)$ = A *cos* t + B *sin* t and substitution gives A = 12/17 and B = 48/17. So

$$U(t) = 12/17 \cos t + 48/17 \sin t$$

$$u'' + u' + 1.25u = 0$$
$$u(0) = 2,\ u'(0) = 3$$

Example 1 (2 of 2)

- The general solution for the nonhomogeneous equation is
$$u(t) = c_1 e^{-t/2}\cos t + c_2 e^{-t/2}\sin t + 12/17\cos t + 48/17\sin t$$
- Applying the initial conditions yields
$$\left.\begin{aligned} u(0) &= c_1 + 12/17 = 2 \\ u'(0) &= -1/2\ c_1 + c_2 + 48/17 = 3 \end{aligned}\right\} \Rightarrow c_1 = 22/17,\ c_2 = 14/17$$
- Therefore, the solution to the IVP is
$$u(t) = 22/17\, e^{-t/2}\cos t + 14/17 e^{-t/2}\sin t + 12/17\cos t + 48/17\sin t$$
- The graph breaks the solution into its steady state ($U(t)$) and transient ($u_C(t)$) components



Rewriting Forced Response

- Using trigonometric identities, it can be shown that

$$U(t) = A\cos(\omega t) + B\sin(\omega t)$$

can be rewritten as

$$U(t) = R\cos(\omega t - \delta)$$

- It can also be shown that

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \quad \sin\delta = \frac{\gamma\,\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

where

$$\omega_0^2 = k/m$$

Amplitude Analysis of Forced Response

- The amplitude R of the steady state solution

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}},$$

depends on the driving frequency ω. For low-frequency excitation we have

$$\lim_{\omega \to 0} R = \lim_{\omega \to 0} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

where we recall $(\omega_0)^2 = k/m$. Note that F_0/k is the static displacement of the spring produced by force F_0.

- For high frequency excitation,

$$\lim_{\omega \to \infty} R = \lim_{\omega \to \infty} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = 0$$

Maximum Amplitude of Forced Response

- Thus
$$\lim_{\omega \to 0} R = F_0/k, \quad \lim_{\omega \to \infty} R = 0$$
- At an intermediate value of ω, the amplitude R may have a maximum value. To find this frequency ω, differentiate R and set the result equal to zero. Solving for ω_{max}, we obtain
$$\omega_{max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right)$$
where $(\omega_0)^2 = k/m$. Note $\omega_{max} < \omega_0$, and ω_{max} is close to ω_0 for small γ. The maximum value of R is
$$R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1 - (\gamma^2/4mk)}}$$

Maximum Amplitude for Imaginary ω_{max}

- We have

$$\omega_{max}^2 = \omega_0^2\left(1 - \frac{\gamma^2}{2mk}\right)$$

and

$$R_{max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1 + \frac{\gamma^2}{8mk}\right)$$

where the last expression is an approximation for small γ. If $\gamma^2/(mk) > 2$, then ω_{max} is imaginary. In this case, $R_{max} = F_0/k$, which occurs at $\omega = 0$, and R is a monotone decreasing function of ω. Recall from Section 3.8 that critical damping occurs when $\gamma^2/(mk) = 4$.

Resonance

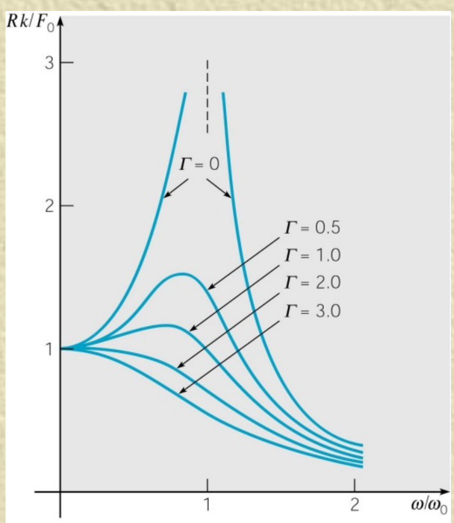
- From the expression

$$R_{\max} = \frac{F_0}{\gamma\omega_0\sqrt{1-(\gamma^2/4mk)}} \cong \frac{F_0}{\gamma\omega_0}\left(1+\frac{\gamma^2}{8mk}\right)$$

  we see that $R_{\max} \cong F_0/(\gamma\,\omega_0)$ for small γ.
- Thus for lightly damped systems, the amplitude R of the forced response is large for ω near ω_0, since $\omega_{\max} \cong \omega_0$ for small γ.
- This is true even for relatively small external forces, and the smaller the γ the greater the effect.
- This phenomena is known as **resonance**. Resonance can be either good or bad, depending on circumstances; for example, when building bridges or designing seismographs.

Graphical Analysis of Quantities

- To get a better understanding of the quantities we have been examining, we graph the ratios $R/(F_0/k)$ vs. ω/ω_0 for several values of $\Gamma = \gamma^2/(mk)$, as shown below.
- Note that the peaks tend to get higher as damping decreases.
- As damping decreases to zero, the values of $R/(F_0/k)$ become asymptotic to $\omega = \omega_0$. Also, if $\gamma^2/(mk) > 2$, then $R_{max} = F_0/k$, which occurs at $\omega = 0$.



Analysis of Phase Angle

- Recall that the phase angle δ given in the forced response

$$U(t) = R\cos(\omega t - \delta)$$

is characterized by the equations

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \ \sin\delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

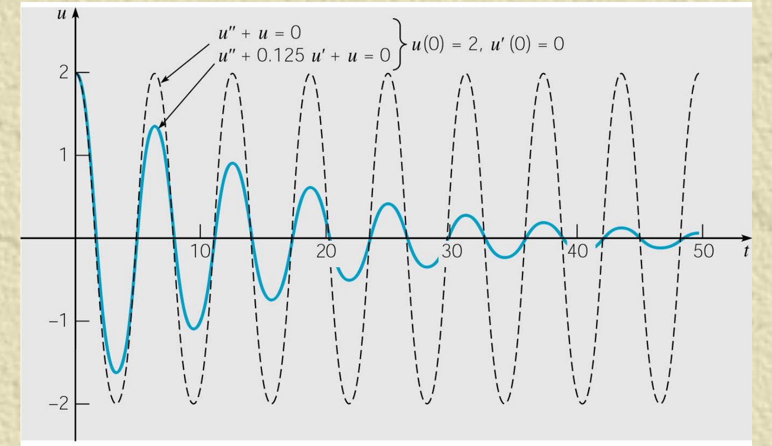
- If $\omega \cong 0$, then $\cos\delta \cong 1$, $\sin\delta \cong 0$, and hence $\delta \cong 0$. Thus the response is nearly in phase with the excitation.
- If $\omega = \omega_0$, then $\cos\delta = 0$, $\sin\delta = 1$, and hence $\delta \cong \pi/2$. Thus response lags behind excitation by nearly $\pi/2$ radians.
- If ω large, then $\cos\delta \cong -1$, $\sin\delta = 0$, and hence $\delta \cong \pi$. Thus response lags behind excitation by nearly π radians, and hence they are nearly out of phase with each other.

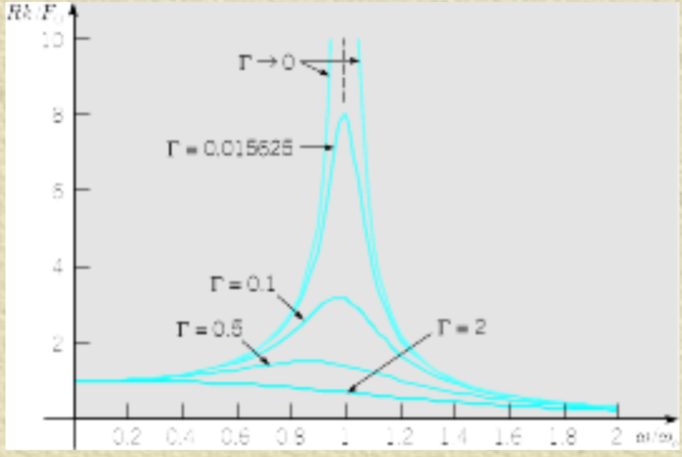
Example 2:
Forced Vibrations with Damping (1 of 4)

- Consider the initial value problem

$$u''(t) + 0.125u'(t) + u(t) = 3\cos\omega t, \;\; u(0) = 2,\; u'(0) = 0$$

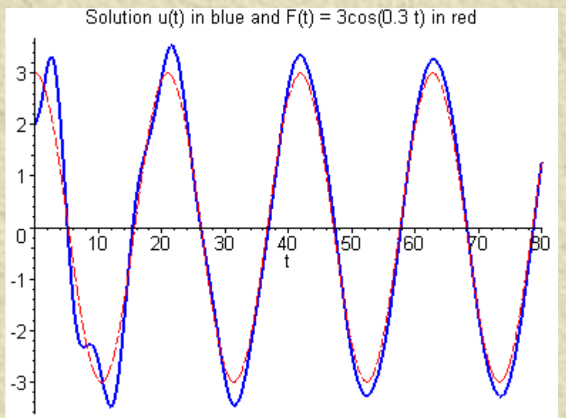
- Then $\omega_0 = 1$, $F_0 = 3$, and $\Gamma = \gamma^2/(mk) = 1/64 = 0.015625$.
- The unforced motion of this system was discussed in Ch 3.7, with the graph of the solution given below, along with the graph of the ratios $R/(F_0/k)$ vs. ω/ω_0 for different values of Γ.





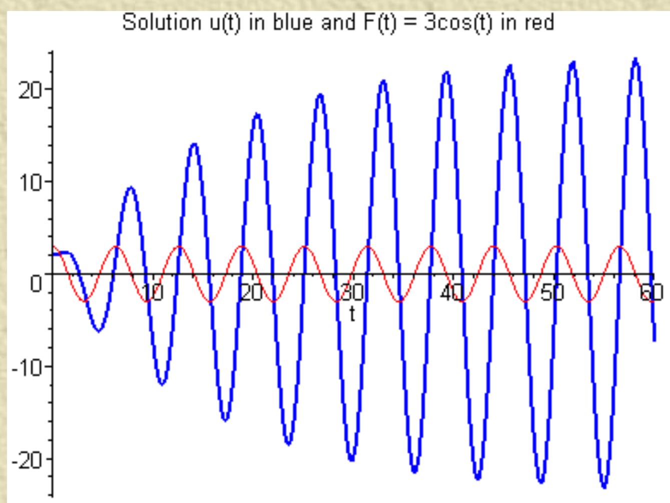
Example 2:
Forced Vibrations with Damping (2 of 4)

- Recall that $\omega_0 = 1$, $F_0 = 3$, and $\Gamma = \gamma^2/(mk) = 1/64 = 0.015625$.
- The solution for the low frequency case $\omega = 0.3$ is graphed below, along with the forcing function.
- After the transient response is substantially damped out, the steady-state response is essentially in phase with excitation, and response amplitude is larger than static displacement.
- Specifically, $R \cong 3.2939 > F_0/k = 3$, and $\delta \cong 0.041185$.



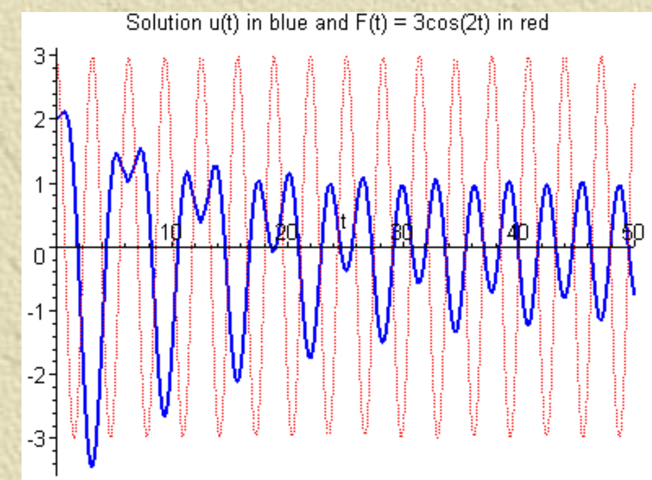
Example 2:
Forced Vibrations with Damping (3 of 4)

- Recall that $\omega_0 = 1$, $F_0 = 3$, and $\Gamma = \gamma^2/(mk) = 1/64 = 0.015625$.
- The solution for the resonant case $\omega = 1$ is graphed below, along with the forcing function.
- The steady-state response amplitude is eight times the static displacement, and the response lags excitation by $\pi/2$ radians, as predicted. Specifically, $R = 24 > F_0/k = 3$, and $\delta = \pi/2$.



Example 2: Forced Vibrations with Damping (4 of 4)

- Recall that $\omega_0 = 1$, $F_0 = 3$, and $\Gamma = \gamma^2/(mk) = 1/64 = 0.015625$.
- The solution for the relatively high frequency case $\omega = 2$ is graphed below, along with the forcing function.
- The steady-state response is out of phase with excitation, and response amplitude is about one third the static displacement.
- Specifically, $R \cong 0.99655 \cong F_0/k = 3$, and $\delta \cong 3.0585 \cong \pi$.



Undamped Equation: General Solution for the Case $\omega_0 \neq \omega$

- Suppose there is no damping term. Then our equation is

$$mu''(t) + ku(t) = F_0 \cos \omega t$$

- Assuming $\omega_0 \neq \omega$, then the method of undetermined coefficients can be use to show that the general solution is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

Undamped Equation: Mass Initially at Rest (1 of 3)

- If the mass is initially at rest, then the corresponding initial value problem is

$$mu''(t) + ku(t) = F_0 \cos \omega t, \;\; u(0) = 0, \; u'(0) = 0$$

- Recall that the general solution to the differential equation is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t$$

- Using the initial conditions to solve for c_1 and c_2, we obtain

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)}, \quad c_2 = 0$$

- Hence

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left(\cos \omega t - \cos \omega_0 t \right)$$

Undamped Equation: Solution to Initial Value Problem (2 of 3)

- Thus our solution is

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos\omega t - \cos\omega_0 t\right)$$

- To simplify the solution even further, let $A = (\omega_0 + \omega)/2$ and $B = (\omega_0 - \omega)/2$. Then $A + B = \omega_0 t$ and $A - B = \omega t$. Using the trigonometric identity

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B,$$

it follows that

$$\cos\omega t = \cos A \cos B + \sin A \sin B$$

$$\cos\omega_0 t = \cos A \cos B - \sin A \sin B$$

and hence

$$\cos\omega t - \cos\omega_0 t = 2\sin A \sin B$$

Undamped Equation: Beats (3 of 3)

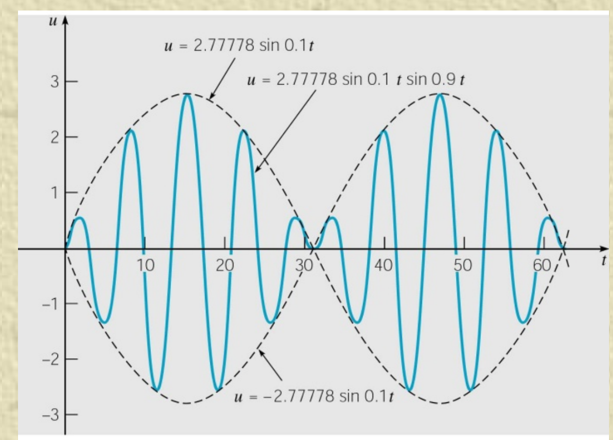
- Using the results of the previous slide, it follows that

$$u(t) = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2} \right] \sin \frac{(\omega_0 + \omega)t}{2}$$

- When $|\omega_0 - \omega| \cong 0$, $\omega_0 + \omega$ is much larger than $\omega_0 - \omega$, and $\sin[(\omega_0 + \omega)t/2]$ oscillates more rapidly than $\sin[(\omega_0 - \omega)t/2]$.
- Thus motion is a rapid oscillation with frequency $(\omega_0 + \omega)/2$, but with slowly varying sinusoidal amplitude given by

$$\frac{2F_0}{m\left|\omega_0^2 - \omega^2\right|} \left| \sin \frac{(\omega_0 - \omega)t}{2} \right|$$

- This phenomena is called a **beat**.
- Beats occur with two tuning forks of nearly equal frequency.



Example 3: Undamped Equation, Mass Initially at Rest

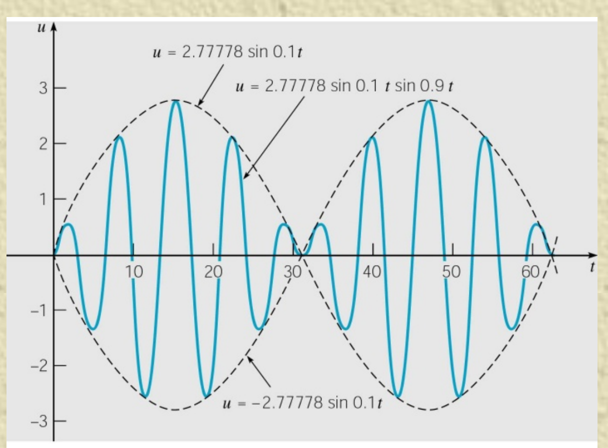
- Consider the initial value problem

$$u''(t) + u(t) = 0.5\cos 0.8t, \;\; u(0) = 0, \; u'(0) = 0$$

- Then $\omega_0 = 1$, $\omega = 0.8$, and $F_0 = 0.5$, and hence the solution is

$$u(t) = 2.77778\left(\sin 0.1t\right)\left(\sin 0.9t\right)$$

- The displacement of the spring–mass system oscillates with a frequency of 0.9, slightly less than natural frequency $\omega_0 = 1$.
- The amplitude variation has a slow frequency of 0.1 and period of 20π.
- A half-period of 10π corresponds to a single cycle of increasing and then decreasing amplitude.

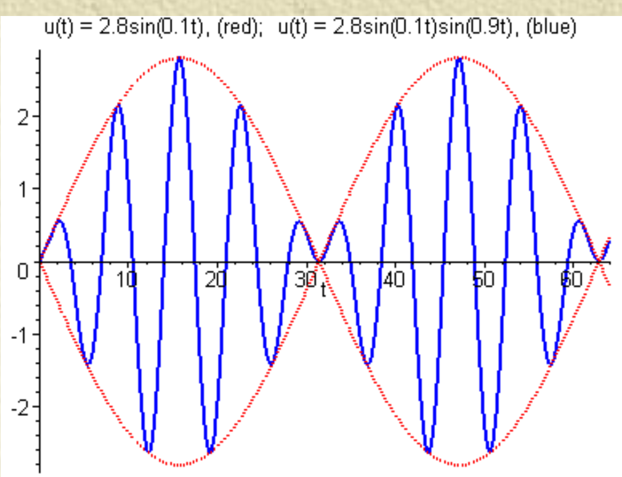


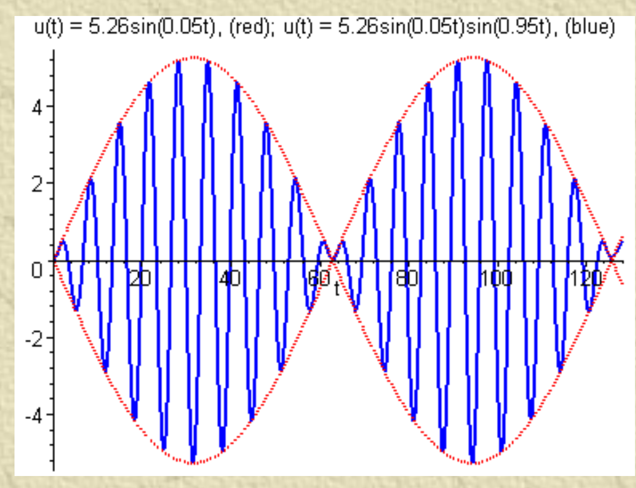
Example 3: Increased Frequency

- Recall our initial value problem

$$u''(t) + u(t) = 0.5\cos 0.8t, \ \ u(0) = 0, \ u'(0) = 0$$

- If driving frequency ω is increased to $\omega = 0.9$, then the slow frequency is halved to 0.05 with half-period doubled to 20π.
- The multiplier 2.77778 is increased to 5.2632, and the fast frequency only marginally increased, to 0.095.





Undamped Equation: General Solution for the Case $\omega_0 = \omega$ (1 of 2)

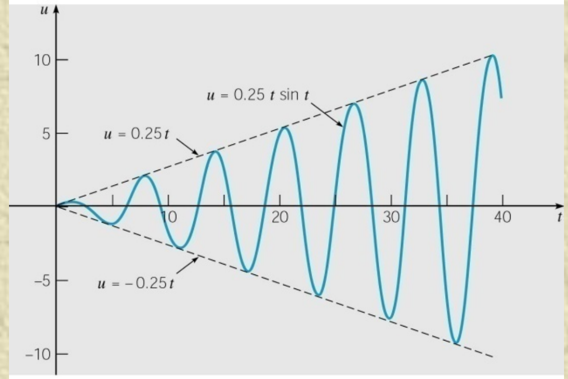
- Recall our equation for the undamped case:

$$mu''(t) + ku(t) = F_0 \cos \omega t$$

- If forcing frequency equals natural frequency of system, i.e., $\omega = \omega_0$, then nonhomogeneous term $F_0 \cos \omega t$ is a solution of homogeneous equation. It can then be shown that

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- Thus solution u becomes unbounded as $t \to \infty$.
- Note: Model invalid when u gets large, since we assume small oscillations u.



Undamped Equation: Resonance (2 of 2)

- If forcing frequency equals natural frequency of system, i.e., $\omega = \omega_0$, then our solution is

$$u(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

- Motion u remains bounded if damping present. However, response u to input $F_0 \cos \omega t$ may be large if damping is small and $|\omega_0 - \omega| \cong 0$, in which case we have resonance.