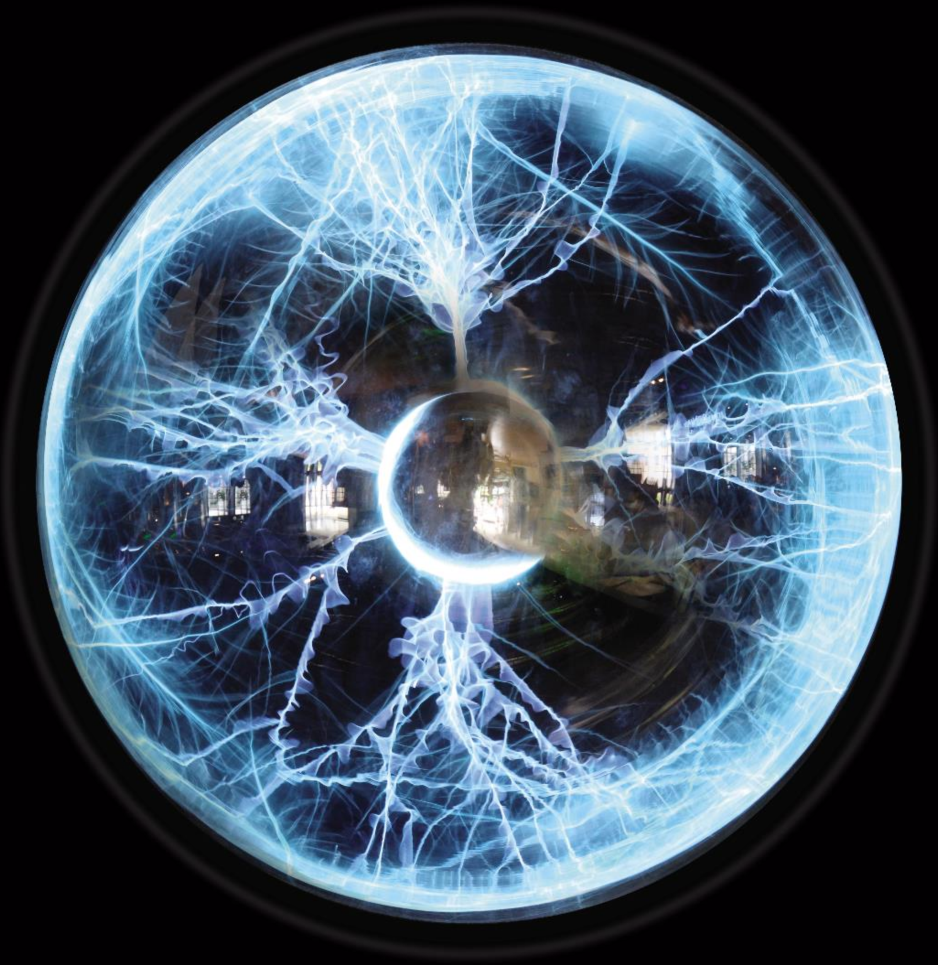
Deloitte.



Breaking down XVAs

A sensitivity-based approach for trade-level allocations

Financial Risk

Breaking down XVAs

A sensitivity-based approach for trade-level allocations

Introducing a “fair”, intuitive and fast XVA allocation methodology.

Contents

1. Introduction

Valuation Adjustments for OTC derivatives
In the post-crisis world, it has become a prevalent practise to include certain costs in the pricing of OTC derivatives that in many cases have previously been ignored, the so-called valuation adjustments or "XVAs". The crisis revealed that counterparty credit risks associated with OTC derivatives can be very substantial (e.g., CVA losses caused by US monoliners) and ought to be mitigated. Market participants incur costs associated with hedging counterparty credit risks through their CVA management activities. Other costs include capital, funding and liquidity costs.

XVAs are counterparty or netting set level metrics, and necessitate a combined assessment of a full range of trades, typically across a broad range of asset classes. For general OTC derivatives netting sets, the calculation of the valuation adjustments is highly complex, and requires the generation of exposure profiles over the lifetime of the trades. The complexity of the modelling is due to the fact that exposures are aggregated across a whole portfolio of trades (both long and short) covered by a master netting agreement, which requires a simultaneous projection of the mark-to-market values of all trades within the netting set.

From an accounting perspective, CVA, DVA and FVA provisions are often only set aside for a reduced set of netting sets that are uncollateralised or have a high margin threshold. Under the CVA rules introduced by Basel III, however, CVA risk needs to be assessed for all bilateral netting sets, even those that are closely collateralised with a low threshold.

Computational complexity
The estimation of the netting set level exposure profiles is typically performed using a Monte Carlo simulation. Within each individual Monte Carlo path, one generates a joint realisation of all risk factors that drive the market value of the netting set. For this "realisation of the future", one then performs a revaluation of all trades in order to obtain the corresponding exposure.

A large OTC derivative netting set can easily have in excess of 100 risk drivers, spanning different interest rates curves, foreign exchange rates, inflation curves, credit spreads, equity indices, etc. Estimating the valuation adjustment for a particular netting set is thus computationally very expensive.

XVA sensitivities
Aside from the XVA metric in itself, a quantity of interest is the sensitivity of the XVA to the netting set's key risk drivers. Indeed, in recent years, OTC valuation adjustments have become traded quantities (especially CVA). Managing a CVA desk hence not only requires a market value, but also creates a need for calculating CVA Greeks (mainly deltas and vegas), in order to adequately hedge the risks.

While the core mandate of a CVA desk is to manage the volatility of accounting CVA, the introduction of capital requirements for CVA risk under Basel III made banks focus on additionally managing CVA from a regulatory capital perspective. In fact, the standardised approach for CVA risk introduced in Basel IV (SA-CVA) explicitly requires the calculation of CVA sensitivities for all risk factors.

As previously mentioned, an XVA calculation in itself is computationally very expensive. Calculating sensitivities of XVAs to the relevant risk factors adds a new dimension to the computational costs. Indeed, if one would apply "brute force" to calculate first order XVA Greeks using finite differences, one would have to re-run the full XVA simulation hundreds of times (once for each risk factor). In most practical cases, this tends to be computationally unfeasible. Recent developments in quantitative finance, however, show that the calculation of XVA Greeks can be made computationally tractable by making use of algorithmic differentiation (AD).

Cost allocations and pre-trade assessments
In larger organisations, the allocation of XVAs to desks or trades is a prerequisite for efficient management of OTC derivatives trading businesses. Understanding how individual trades affect the XVAs is a key component in pre-trade assessments. Furthermore, a fair allocation of costs along an institution's business hierarchy ensures that everyone in the business is incentivise to deploy the firm's capital in the most efficient way.

Since the valuation adjustments are calculated at an aggregate level across all trades in a netting set, sophisticated methods have to be applied in order to break down the netting set-level XVAs into contributions by single desks or trades.

Our solution: sensitivity-based XVA allocations
In this article, we introduce an intuitive and fair XVA allocation technique based on risk factor sensitivities. The approach can be integrated as a stand-alone addition to any XVA calculation infrastructure, leveraging an existing XVA sensitivity framework (e.g., the CVA sensitivities used in the capital calculations for the Basel IV standardised approach for CVA risk[1]). We further discuss how XVA sensitivities can be calculated efficiently using algorithmic differentiation techniques.

The article concludes by providing practical example, in which CVA is allocated for a simple portfolio of equity derivatives. With this example we illustrate computational gains that can be made by applying algorithmic differentiation.

An efficient calculation of XVA sensitivities provides several useful applications, and can significantly improve OTC derivatives portfolio management

[1] Basel Committee. (2017) "Basel III: Finalising post-crisis reforms".

2. XVAs: a quick refresher

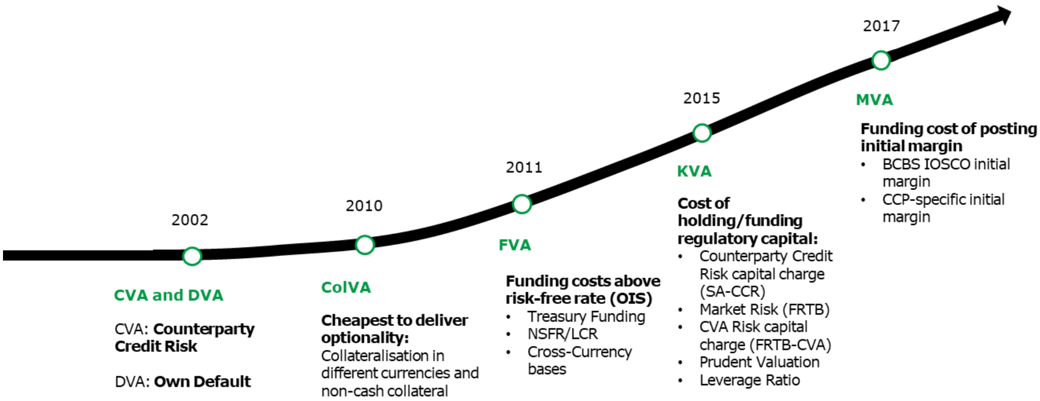
A brief overview of the different XVAs

XVAs is a collective term referring to a variety of adjustments in fair value of a derivatives netting set, reflecting the credit quality of the contractual counterparty, and the collateralisation and the regulatory capital costs associated with the netting set. When valuing derivatives, the valuation adjustments are applied to an "idealised" valuation that assumes a risk free counterparty, and ignores the costs of providing and benefits of receiving collateral as well as regulatory capital costs.

The first XVA that became prevalent in dealers' risk management was the credit valuation adjustment (CVA), which gained popularity in the early 2000s. CVA reflects the cost of hedging a counterparty's credit risk associated to a derivative transaction (or netting set of derivative transactions). Around the same time, banks started taking into account their own default risk by means of a debit value adjustment (DVA). The calculation of DVA is closely related to the CVA calculation. Indeed, one institution's DVA is equal to its counterparty's CVA.

From 2011 onwards, financial institutions started to incorporate the funding cost that arises in hedging uncollateralised OTC derivatives: the funding value adjustment (FVA). FVA consists out of two components: the funding cost adjustment (FCA) and the funding benefit adjustment (FBA). Consider a situation in which an uncollateralised trade (or netting set) is hedged back-to-back with a collateralised trade. This setup is very common for the larger dealers, as there is mandatory clearing for vanilla trades, which involves full re-margining. A funding cost arises when the uncollateralised derivative has a positive mark-to-market. In this instance, the hedge has a negative MtM and thus requires the posting of variation margin. This variation margin needs to be funded at the institution's own funding rate, however, the margin account is only accruing interest at the risk-free rate. The FCA of an institution corresponds to the cost of funding variation margin. Conversely, a funding benefit occurs when the uncollateralised derivative trade has a negative MtM.

Financial institutions are required to hold regulatory capital for the business arising from their derivatives portfolios. Banks consider the cost of holding this capital for their derivatives trading business by means of a capital valuation adjustment (KVA) Other valuation adjustments include the margin valuation adjustment (MVA), which reflects the funding cost of posting initial margin over the lifetime of a transaction, or the collateral valuation adjustment (ColVA), which quantifies the costs and benefits from embedded optionalities in the collateral agreement (cheapest to deliver optionality) and other non-standard collateral terms.



The XVA landscape within a bank

From an accounting perspective, provisions are typically only set aside for CVA, DVA and FVA. In fact, some institutions only set aside these provisions for a reduced scope of uncollateralised or high threshold netting sets, but not for closely collateralised and low threshold netting sets.

In terms of regulatory capital requirements, only CVA risk has to be included in the risk-weighted asset calculation, and DVA gains are reversed from capital base. Institutions do not have to capitalise (or adjust for in the capital base) other XVAs than those two mentioned above.

The introduction of a CVA capital charge under Basel III was one of the main drivers of major increases in capital requirements for heavy derivatives users. The capital charge requires the calculation of CVA for all bilateral netting sets, even when fully collateralised, and hence might deviate in terms of scope from the accounting treatment. In terms of determining the requirements for CVA risk, Basel IV allows for two approaches: the basic approach (BA-CVA) and the standardised approach (SA-CVA). Since the latter approach involves the calculation of internal estimates of the CVA itself and the associated sensitivities to the relevant risk factors, it is subject to regulatory approval.

While from an accounting (and regulatory capital) perspective, banks focus on CVA, DVA and FVA, they aim to quantify the other costs such as ColVA, KVA and MVA as well. Priorities to mitigate these costs might vary across the different measures and across different banks. Whereas some banks closely manage all the additional XVAs, others might consider some of them as merely a "cost of doing business".

Key components of the XVA calculation

XVAs reflect costs or benefits arising over the lifetime of a trade (or netting set). They are typically expressed as an integral over time:

$$XVA = \int_0^T XVA_t \, dt,$$

for some quantity XVA_t. The quantity XVA_t takes different forms for the different XVAs, for example:

- $CVA_t = PD_t \cdot LGD_t \cdot EPE_t$
- $DVA_t = PD_t^{Own} \cdot LGD_t^{Own} \cdot ENE_t$
- $FCA_t = Funding\ Spread_t \cdot EPE_t$
- $FBA_t = Funding\ Spread_t^{Own} \cdot ENE_t$

where we have introduced the following key components:

- EPE_t — The risk-neutral discounted expected positive exposure
- ENE_t — The risk-neutral discounted expected negative exposure
- PD_t and PD_t^{Own} — The probability of default of the counterparty and the institution itself
- LGD_t and LGD_t^{Own} — The loss given default of the counterparty and the institution itself
- $Funding\ Spread_t$ and $Funding\ Spread_t^{Own}$ — The counterparty's and the institution's own funding spread (over OIS)

Calculation of XVAs in practice

XVA calculations can differ significantly across banks, depending on the calculation approach, the granularity of risk factors, calibration assumptions, different curves, etc. The most common way of calculating XVAs in practice is by means of a Monte Carlo simulation. In each iteration of the Monte Carlo simulation, one simulates a future (joint) realisations of the portfolio risk factors (in a risk-neutral world)[2]. Conditional on such a realisation of risk factors, one then re-values the future market value of the trades in the portfolio and calculates the positive exposure (CVA, FCA), negative exposure (DVA, FBA), initial margin requirements (MVA) or capital requirements (KVA). The final XVA is calculated by averaging the relevant quantities over the different Monte Carlo iterations.

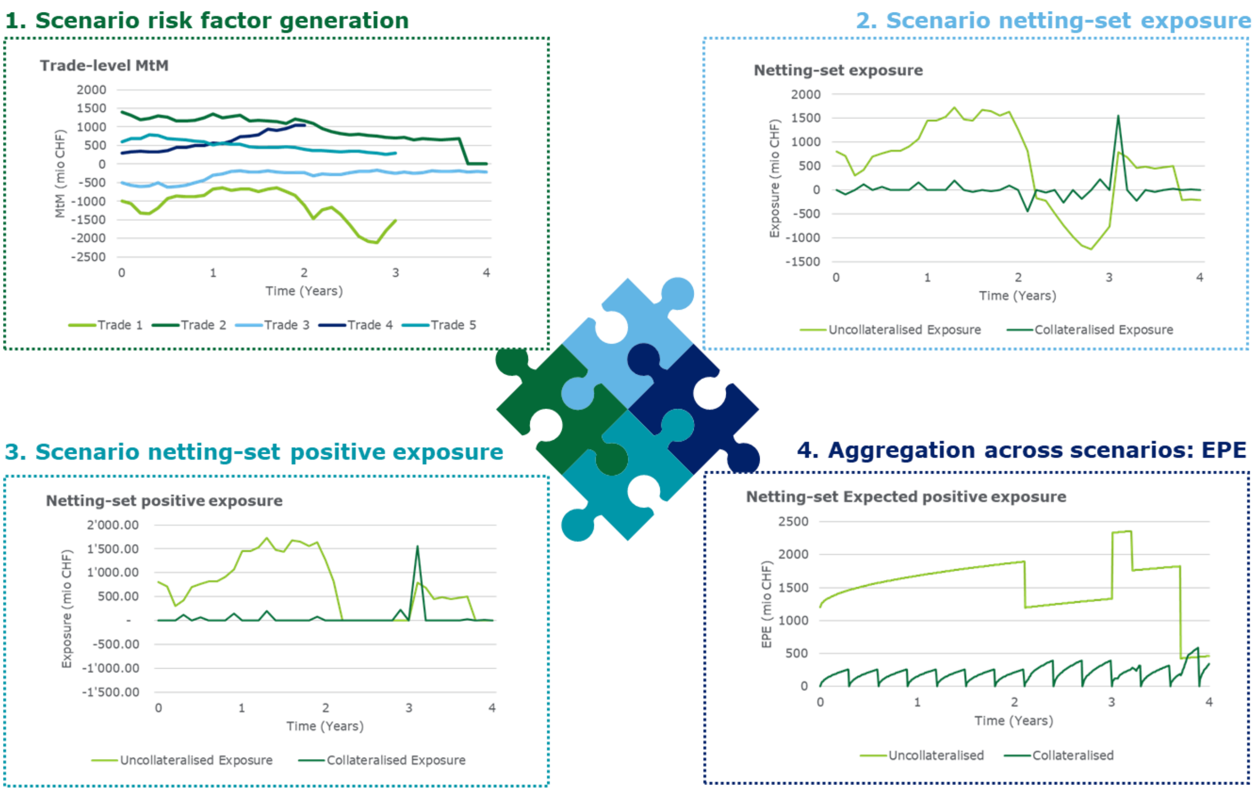


Figure 1: Overview of calculation steps for calculating the expected positive exposure (EPE), a key component of the CVA and FCA.

The simulation of the realisation of the risk factors is generated by a stochastic process, for example:

$$\mathrm{d}\, RF_t = \mu \cdot RF_t \cdot \mathrm{d}t + \sigma \cdot RF_t \cdot \mathrm{d}W_t,$$

for some Brownian motion W_t. In application, the risk factor process is discretised, so that one obtains an iterative relationship: $RF_{t+\Delta t} = f(RF_t)$. In particular, in the case of the simple Brownian motion:

$$RF_{t+\Delta t} = f(RF_t) = RF_t \cdot \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma \cdot \sqrt{\Delta t} \cdot \mathrm{W_t}\right\}$$

[2] Both for accounting purposes as well as for calculating the CVA risk under Basel IV, one requires a market-implied simulation of risk factors. For the default risk RWA calculation, on the other hand, the risk factor generation for the EPE is in the physical world (i.e., historical calibration of risk factors).

3. Allocation of XVAs

The allocation of XVAs to desks and single trades is a prerequisite for efficient management of OTC derivatives. Understanding how individual trades affect the XVAs allows for a fair allocation of costs along a business hierarchy down to desks.

Large derivatives dealers seek to quantify the contribution of individual trades to the XVAs of the associated netting set, enabling them to:

- Reduce hedging costs by optimising accounting CVA
- Manage capital requirements for CVA risk (as well as default risk) efficiently by optimising EPE profiles
- Provide incentive for traders to hedge with the street (dealers and CCPs) in a way that optimises capital requirements and initial margin
- Have full transparency on the costs of doing uncollateralised business with derivatives end-user

In order to be capable of conducting pre-trade assessments of the XVA impacts, traders need to be have access to a fast trade-level allocation methodology. A fair allocation of the charges down to desks provides incentives to traders deploy the firm's financial resources (capital and funding) efficiently, and makes sure that, from an overall firm perspective, the different traders "run in the right direction".

What constitutes a "good" allocation approach?

Any "good" allocation approaches must, in the least, satisfy the following criteria:

- Fairness and incentive: the allocation approach incentivises "good trading". Trades that adversely impact a netting set's XVA should receive a high XVA allocation, whereas trades that decrease XVAs are rewarded.
- Intuitiveness: the allocation approach should be easily understood by the key stakeholders. It should provide a clear reasoning in explaining the extent to which trades impact the netting set's XVAs.

In addition to the criteria outlined above, the following points are desirable in terms of practical application:

- Versatility: the same approach can easily be applied to other XVAs, many different netting set characteristics and modelling assumptions. An ideal allocation approach would, in fact, be agnostic to any XVA modelling assumptions or netting set characteristics
- Easy to implement: the implementation of the XVA allocation leverages the existing XVA calculation framework
- Fast: the calculation approach is computationally fast in order to support pre-trade assessments of the XVA impacts

In the following sections we discuss two different allocation approaches: the "classic" continuous marginal contributions and a newly proposed sensitivity-based allocation, both of which satisfy the criteria relating to fairness and intuitiveness. Whereas continuous marginal contributions can lead to some complications when it comes to a practical implementation, the sensitivity-based allocation can be easily implemented in a stand-alone fashion and leverages outputs from existing calculation frameworks.

Continuous marginal contributions

Continuous marginal contributions (CMC), also known as Euler contributions, are a commonly used technique within portfolio economic risk capital (ERC) calculations, to determine the contributions of individual transactions towards the portfolio-level ERC measure. The continuous marginal contribution of a given trade to the portfolio ERC is determined by assessing infinitesimal increase of the transaction's weight in the portfolio, and assessing the resulting increment of the ERC (See Tasche (2008) for details).

Pykhtin & Rosen (2010) adapt the CMC methodology in order to calculate EPE contributions for both collateralised and non-collateralised counterparties. Their solution is very elegant, and yields an intuitive and fair allocation of CVA.

Applying the CMC approach requires storing throughout the Monte Carlo simulation a contribution for *each* trade at *each* point in time. For large netting sets with several thousands of trades this might become computationally very expensive. Furthermore, although the Pykhtin & Rosen (2010) provide a very elegant solution for allocating CVA for many different cases, the implementation becomes slightly more cumbersome when including PD-EAD right/wrong-way risks[3] or extending the concept to other XVAs. In practice, implementing the CMC approach would require altering an existing XVA engine, making it slightly less versatile in application.

In what follows, we propose an alternative XVA allocation approach: the "sensitivity-based allocation". In contrast to the CMC approach, the sensitivity-based allocation allows for a straightforward standalone implementation, and leverages outputs from existing calculation frameworks (in particular, CVA risk factor sensitivities, which are computed as part of the SA-CVA risk RWA calculation).
first order

Sensitivity-based XVA allocation

Prior to the financial crisis, the primary (and possibly even only) concern of banks in the XVA space was the management of their accounting CVA. In particular, banks focussed on managing the expected positive exposure profile for uncollateralised netting sets, as these drive the accounting provisions. The key driver of the EPE of an uncollateralised netting set is the current market value of the constituent trades. Hence, allocating accounting CVA provisions to the underlying trades based on their MtM is a reasonably intuitive and fair approach, and is commonly applied throughout the industry.

---

[3] PD-EAD wrong way risk arises when the exposure towards a counterparty is positively correlated with the counterparty's probability of default. Conversely, PD-EAD right way risk arises from a negative correlation between the EAD towards a counterparty and the counterparty's PD.

During the financial crisis, the then newly introduced Basel III framework required banks to hold capital for CVA stemming from all bilateral netting sets, even when fully collateralised. For heavy derivatives users, the introduction of the CVA capital charge led to major increases in capital requirements. In particular, large collateralised inter-dealer portfolios became significantly more costly to manage. For a collateralised portfolio, the key driver of the EPE (and hence CVA) is not the current MtM of the underlying trades, but rather the sensitivity of the portfolio MtM during the margin period of risk. Hence, applying a simple trade-level allocation based on MtM, is not appropriate for collateralised netting sets.

In this section we set out a simple, intuitive and fair trade-level XVA allocation approach for collateralised netting sets, and then further generalise this approach for non-collateralised and semi-collateralised portfolios (i.e., those with a significant margin threshold or minimum transfer amount).

Allocation for collateralised portfolios:
As previously discussed, for a collateralised netting set, one has that the XVA is driven by changes in the future market value of the trades, resulting from changes in risk factors. In this instance, the allocation approach consists of two steps. In a first step one distributes the XVA to each underlying risk factors according to two criteria:

1. A higher (absolute) risk factor sensitivity equates to a higher XVA allocation:
   The higher the sensitivity of the XVA to a small change in the risk factor, the more this risk factor is driving XVA, hence deserving a higher allocation.
2. A more volatile risk factor leads to a higher XVA allocation:
   The risk factor volatility is a measure of how likely a small change in a risk factor will occur. Hence, more volatile risk factors are more likely to affect the XVA, and should hence receive a higher XVA allocation.

Concretely, the XVA is prorated to the different risk factors by a volatility-weighted (absolute) risk factor delta. In a second step, the XVA allocation to each risk factor is further broken down to each underlying trade based on the trade-level deltas. Figure 2 Error! Reference source not found.below provides a summary of the key calculation steps.

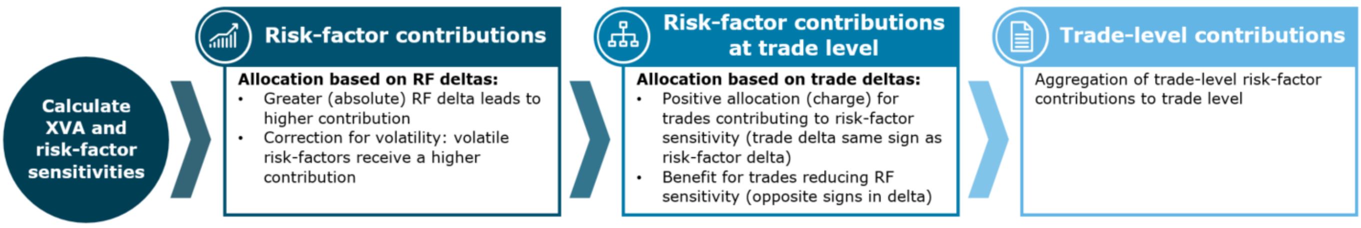


Figure 2: Overview of key calculation steps for the sensitivity-based XVA allocation for a collateralised netting set.

Generalisation for non-collateralised or partially collateralised netting sets:
The above approach for collateralised netting sets can also be used to refine the classic MtM-based allocation for uncollateralised netting sets. For an uncollateralised netting set, we propose to divide the XVA (equivalently EPE, ENE, etc.) into a component due to the current MtM of the netting set, and a component due to risk factor sensitivities. The former component would be simply allocated to the trade-level based on their MtM, whereas the latter would be allocated based on the risk factor sensitivities.

For collateralised netting sets with a significant margin threshold, one would apply a similar hybrid approach. For example in the case CVA, DVA and FVA, one would allocate the EPE/ENE below the threshold on an MtM basis, whereas any positive/negative exposure in excess of the threshold would be allocated using the sensitivity-based approach.

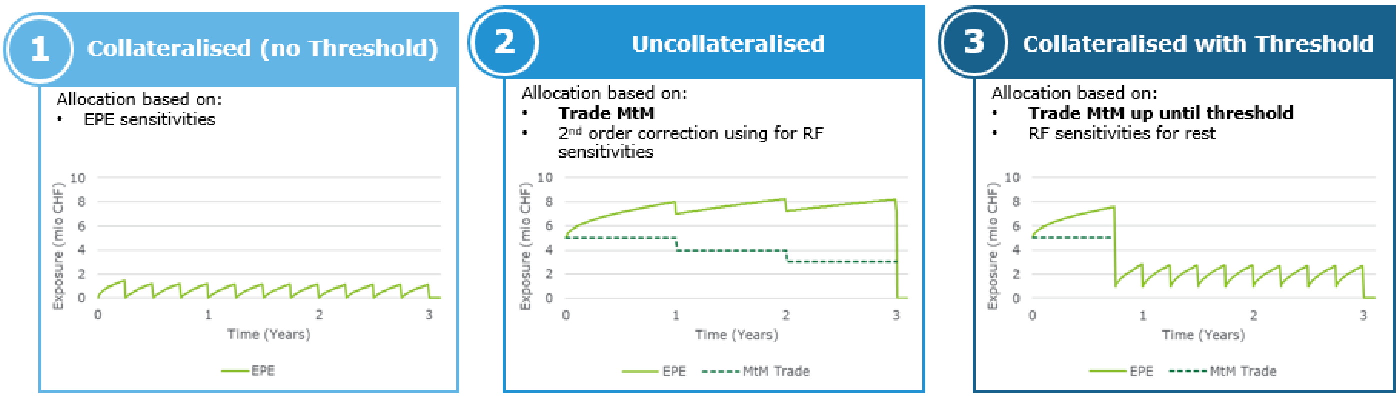


Figure 3: EPE allocation for collateralised, uncollateralised and partially collateralised netting sets.

The XVA allocation approach proposed above is based solely on trade-level information (MtM and deltas), and XVA sensitivities, which are typically outputs of existing calculation frameworks (e.g. CVA sensitivities are used in the SA-CVA risk RWA calculation). The allocation approach can be implemented in a straightforward manner alongside current XVA infrastructure and is computationally fast, enabling pre-trade XVA assessments.

straightforward
Below, we illustrate the sensitivity-based allocation approach by going through a CVA example. The same idea, however, can be easily extended to other XVAs.

Example allocation for CVA

For the purposes of this illustrative example, we assume that the CVA is given by:

$$CVA = \int_0^T CVA_t \, dt,$$

$$CVA_t = PD_t^{cpty} \cdot LGD_t^{cpty} \cdot DF_t \cdot EPE_t.$$

Furthermore, we assume that PD_t^{cpty} and LGD_t^{cpty} are functions of time only (i.e., no "PD-EAD wrong-way risk"), which is common market practice. In this case, the risk factor sensitivities of CVA are given by the risk factor sensitivities of the exposure profile (EPE_t). That is, for any risk factor RF_i one has that:

$$\frac{\partial CVA}{\partial RF_i} = \int_0^T PD_t^{cpty} \cdot LGD_t^{coty} \cdot \frac{\partial EPE_t}{\partial RF_i} \, dt.$$

The core driver of the CVA is the Expected Positive Exposure profile (EPE_t). One has that the exposure profile of a netting set is driven by two components:

- Current exposure against the counterparty, which is the current market value of the netting set, minus any posted collateral.
- Changes in the future market value of the trades, as a result of changes in risk factors.

For a typical collateralised netting set, the current exposure against a counterparty is limited. In this instance, the CVA is primarily driven by changes in the market value of trades since the last collateralisation date, as well as during the *margin period of risk*. We first outline a CVA allocation approach for a general collateralised netting set. Later, we extend this allocation approach to uncollateralised netting sets.

The core assumption of the allocation algorithm is that the changes in CVA as well as changes in the underlying trades can be adequately captured by their first order derivatives (deltas). Before applying the allocation algorithm, one hence requires the calculation of the following sensitivities:

- Deltas of the Expected Positive Exposure profile (EPE_t) with respect to all relevant risk factors. That is, we calculate $\delta_{RF_i}^{EPE_t} = \frac{\partial EPE_t}{\partial RF_i}$ for every time t and every risk factor RF_i (these can, for example, be calculated using algorithmic differentiation – see chapter 4).
- Deltas for each trade k, $\delta_{RF_i}^{trade_k}$, with respect to every relevant risk factor RF_i.

Sensitivity-based CVA allocation for a collateralised netting set

As mentioned above, the CVA of a collateralised netting set is predominantly driven by changes in mark-to-market (MtM) between two collateralisation dates as well as MtM changes during the margin period of risk (MPOR). Changes in MtM are in turn driven by changes in the netting set's key risk factors. As a result, one has that CVA is predominantly driven by its sensitivities to the key risk factors.

Once the first order CVA and trade-level sensitivities are calculated, one can allocate the expected positive exposure profile to each individual trade using the following steps:

Step 1: Allocation of EPE to risk factors
In a first step, the EPE is contributed to each risk factor based on the risk factorrisk factor deltas. We introduce the notation: $Cont(EPE_t|RF_i)$, denoting the contribution of EPE_t to the risk factor RF_i:

$$Cont(EPE_t|RF_i) = \frac{EPE_t \cdot \left|\delta_{RF_i}^{EPE_t}\right| \cdot \sigma_{RF_i}}{\sum_j \left|\delta_{RF_j}^{EPE_t}\right| \cdot \sigma_{RF_i}}.$$

Here σ_{RF_i} denotes a measure of volatility of the risk factor RF_i.[4] This volatility component is included to capture the fact that more volatile risk factorrisk factors have a greater impact on the market value fluctuations of the portfolio, and hence the EPE.

Step 2: Assigning risk factor contributions to trades
Once the EPE has been allocated to each to each risk factor, we allocate this risk factor contribution to each individual trade using the trade-level risk factor deltas:

$$Cont(EPE_t|RF_i, trade_k) = \frac{Cont(EPE_t|RF_i) \cdot sign\left(\delta_{RF_i}^{EPE_t}\right) \cdot \delta_{RF_i}^{trade_k} \cdot 1_{t<Maturity_{trade_k}}}{\sum_l sign\left(\delta_{RF_i}^{EPE_t}\right) \cdot \delta_{RF_i}^{trade_l} \cdot 1_{t<Maturity_{trade_l}}}$$

Here we introduced the following notation:

- $sign(\ldots)$ is a function that equals 1 if the argument is non-negative, and -1 otherwise.
- $1_{t<Maturity_{trade_k}}$ is an indicator that equals 1 if t is less than the trade maturity and 0 otherwise.

[4] The volatility measure ought to be more or less consistent with the volatility used in the stochastic process driving the risk factor in the EPE calculation.

The rationale behind the above formula is as follows: trades whose sensitivity points in the same direction as the EPE-sensitivity are driving the EPE, and hence are contributing towards a higher EPE. These trades are therefore assigned a positive EPE contribution (and consequently a positive CVA charge). On the other hand, the trades whose deltas have the opposite sign to the EPE delta are reducing the EPE. These trades will receive a benefit (i.e., a negative contribution). This approach incentivises traders to reduce the EPE and consequently CVA. A similar incentive can be observed when applying the continuous marginal contribution approach outlined Pykhtin & Rosen (2010).

Step 3: Aggregation of EPE contribution across risk factors at trade level
In a final stage, the contributions from step 2 are aggregated to the trade level by simply summing across the different risk factors:

$$Cont(EPE_t|trade_k) = \sum_i Cont(EPE_t|RF_i, trade_k).$$

Step 3: Final CVA allocation
Once the EPE has been allocated to each trade, the CVA contribution for trade k is calculated as:

$$Cont(CVA|trade_k) = \int_0^T PD_t^{counterparty} \cdot LGD_t^{counterparty} \cdot Cont(EPE_t|trade_k)\, dt$$

Extension: CVA allocation for an uncollateralised netting set
In the case of an uncollateralised netting set, one can have that a significant CVA contribution comes from the current exposure towards the counterparty. Consider the trivial case of an uncollateralised netting set consisting of a single (long) call option. In this case, the (risk-neutral discounted) EPE profile ought to be flat, and equal to the current market value of the option.[5] In other words, the CVA is completely determined by the current trades' market value, and the counterparty PD and LGD.

For an uncollateralised netting set, we hence split the EPE into two components:
- The an *expected exposure* component, which captures the current level of the MtM of the trades
- A *residual component*, which is driven by the key risk factor sensitivities

We proceed by allocating the expected exposure component and the residual EPE component separately.

Allocation of the expected exposure component
The expected exposure profile of the netting set is given by:

$$\begin{aligned} EE_t &= \mathbb{E}\left[MtM_t^{netting\ set}\right] \\ &= \mathbb{E}\left[\sum_k MtM_t^{trade_k} \cdot 1_{t<Maturity_{trade_k}}\right] \\ &= \sum_k MtM_0^{trade_k} \cdot 1_{t<Maturity_{trade_k}} \end{aligned}$$

where in the last step we use the fact that the expectation of the future market value is today's market value (i.e., no arbitrage). We define the expected exposure component of the EPE as $\max(0; EE_t)$.

This component of the EPE is allocated to the trades based on their current market values:

$$Cont(\max(0; EE_t)\,|trade_k) = \frac{\max(0; EE_t) \cdot MtM_0^{trade_k} \cdot 1_{t<Maturity_{trade_k}}}{\sum_j MtM_0^{trade_j} \cdot 1_{t<Maturity_{trade_j}}}.$$

Once again, one has that the trades that are adversely affecting the EE (i.e., positive MtM) will receive a positive CVA contribution, whereas treads leading to a reduction in the EE (and hence EPE) are assigned a negative contribution (i.e., a benefit).

Allocation of the residual EPE component
The Residual EPE is then defined by:

$$EPE_t^{Residual} = EPE_t - \max(0; EE_t).$$

Note that, $EPE_t \geq \max(0; EE_t)$, so that the residual EPE is non-negative. Furthermore, we have that:

$$\frac{\partial EPE_t^{Residual}}{\partial RF_i} = \frac{\partial EPE_t}{\partial RF_i} - 1_{EE_t>0} \cdot \frac{\partial EE_t}{\partial RF_i}$$

[5] The market value of a call option is always positive. The (discounted) EPE of the netting set is hence equal to the (discounted) expected market value of the call option under the risk-neutral measure (recall that the generation of risk factors for the purpose of CVA calculation is market-implied). Under the risk-neutral measure, this expectation is exactly today's market value.

$$= \frac{\partial EPE_t}{\partial RF_i} - 1_{EE_t>0} \cdot \sum_k \frac{\partial MtM_0^{trade_k}}{\partial RF_i} \cdot 1_{t<Maturity_{trade_k}}$$
$$= \frac{\partial EPE_t}{\partial RF_i} - 1_{EE_t>0} \cdot \sum_k \delta_{RF_i}^{trade_k} \cdot 1_{t<Maturity_{trade_k}}$$

Similar to the EPE in the collateralised netting set, the residual EPE is primarily driven by the sensitivities of the netting set's trades to their underlying risk factors. Hence, one can apply the contribution approach outlined for the collateralised case to estimate trade-level contributions of the residual EPE: $Cont\left(\max(0; EPE_t^{Residual}) \,|trade_k\right)$.

Final CVA allocation

The total EPE contribution for $trade_k$ is then calculated as:

$$Cont(EPE_t|trade_k) = Cont(\max(0; EE_t)\,|trade_k) + Cont\left(\max(0; EPE_t^{Residual})\,|trade_k\right)$$

The final trade-level CVA contribution is then calculated as:

$$Cont(CVA|trade_k) = \int_0^T PD_t^{counterparty} \cdot LGD_t^{counterparty} \cdot Cont(EPE_t|trade_k)\, dt$$

Extension: including wrong-way risk

In the allocation approach outlined above, one can in fact relax the independence assumption of $PD_t^{counterparty}$ and $LGD_t^{counterparty}$. In this instance, one would simply perform the allocation directly using the CVA_t risk factor sensitivities as opposed to the EPE_t risk factor sensitivities. We point out, however, that including "PD-EAD wrong-way risk" in the CVA calculation requires slightly more care when calculating the risk factor sensitivities of CVA_t.

Properties of the trade-level allocation approach

The allocation approach outlined above has a number of nice properties:

- The approach is based on the key risk drivers of the netting set's CVA, and is hence intuitive to understand by key stakeholders.
- It is fair: it allocates higher costs to the trades that lead to higher CVAs. The approach hence incentivises trades that lead to a reduction in CVA.
- The approach is agnostic to any particular CVA assumptions – it requires only EPE sensitivities. Furthermore, the approach can easily be extended to accommodate other XVAs contributions.
- Once the CVA sensitivities are calculated, the approach is quick, and hence enables a preliminary CVA contribution estimate for new trades, ideal for pre-trade assessments.

The sensitivity-based XVA allocation allows for a fair allocation of costs along an institution's business hierarchy, ensuring that everyone in the business is "running in the right direction".

4. XVA sensitivities using algorithmic differentiation

In application, the most straightforward and most commonly used approach to calculate Greeks of complex derivatives or XVAs is a "finite difference" approximation – also known as "bumping"; one makes an incremental adjustment to the input risk factor of interest, and re-values the derivative or valuation adjustment. The derivative is then approximated by the difference between the "base valuation" and the "incrementally adjusted valuation". A general OTC derivative netting set can easily have in excess of 100 risk drivers. Calculating all XVA risk factor sensitivities using the "bumping" approach, would result in re-running the XVA simulation hundreds of times (once for each risk factor), and can hence computationally very expensive.

Algorithmic Differentiation (AD) allows for *simultaneous* computation of several derivatives of a function with respect to its arguments. This is made possible by exploiting a key mathematical property: the *chain rule of differentiation*, which links the derivatives of parts of a function to the derivative of the whole. In the context of XVAs, algorithmic differentiation allows for calculating accurate XVA sensitivities alongside the XVA metric, within a single Monte Carlo simulation. In other words, AD removes the need for several re-runs of an XVA engine, and can significantly reduce computation time.

Overview of key calculation steps

The most interesting XVA sensitivities are the risk factor deltas. That is, the derivative of the XVA with respect to the today's risk factor value:

$$\frac{\partial XVA}{\partial \mathrm{RF}_0}.$$

Recall that the XVAs are an integral over the lifetime of a netting set.

$$XVA = \int_0^T XVA_t \, dt.$$

Changing the order of integration and differentiation, one has that the quantity of interest is thus:

$$\frac{\partial XVA_t}{\partial \mathrm{RF}_0}.$$

Below we outline the key calculation steps of calculating XVA sensitivities using AD in the context of CVA (in particular, for the EPE calculation). A more detailed step-by-step example is provided in the Appendix. Note that the calculation steps are similar for all other XVAs.

Step 1: Changing order of differentiation and Monte Carlo averaging
First, one makes use of the fact that one can change the order differentiation and averaging over Monte Carlo scenarios. For the EPE we have:

$$\frac{\partial EPE_t}{\partial \mathrm{RF}_0} = \frac{\partial}{\partial RF_0}\left(\frac{1}{\#\, scen}\sum_{scen} PE_t^{scen}\right) = \frac{1}{\#\, scen}\sum_{scen}\frac{\partial PE_t^{scen}}{\partial \mathrm{RF}_0},$$

where PE_t^{scen} denotes the positive portfolio exposure simulated at time t under Monte Carlo scenario $scen$.

Step 2: Application of the chain rule
We wish to calculate the derivative of the PE_t^{scen} with respect to RF_0. One has, however, that PE_t^{scen} is a function of the risk factor realisation at time t of scenario $scen$, that is: $\mathrm{RF}_\mathrm{t}^{scen}$. Applying the chain rule, one has that:

$$\frac{\partial PE_t^{scen}}{\partial \mathrm{RF}_0^{scen}.} = \frac{\partial PE_t^{scen}}{\partial \mathrm{RF}_\mathrm{t}^{scen}.} \cdot \frac{\partial \mathrm{RF}_\mathrm{t}^{scen}.}{\partial \mathrm{RF}_{\mathrm{t-\Delta t}}^{scen}.} \cdot \frac{\partial \mathrm{RF}_{\mathrm{t-\Delta t}}^{scen}.}{\partial \mathrm{RF}_{\mathrm{t-2\Delta t}}^{scen}.} \cdot \ldots \frac{\partial \mathrm{RF}_{\Delta \mathrm{t}}^{scen}.}{\partial \mathrm{RF}_0}$$

Recall from Chapter 2 that the discretisation of the risk factor leads to a relationship $RF_{t+\Delta t} = f(RF_t)$. Hence, one can easily calculate the derivative of $\mathrm{RF}_\mathrm{t}^{scen}$ with respect to $\partial \mathrm{RF}_{\mathrm{t-\Delta t}}^{scen}$ etc. On the other hand, PE_t^{scen} depends on $\mathrm{RF}_\mathrm{t}^{scen}$ through the market values of the underlying trades. This quantity can easily be expressed in terms of trade-level deltas.

Algorithmic differentiation allows for a simultaneous calculation of all XVA sensitivities, within a single Monte Carlo run, significantly reducing computation time.

5. Practical Implementation

In this section we show the results from a practical implementation of a CVA allocation engine. First, we illustrate the sensitivity-based XVA allocation approach for a number of simple proxy portfolio of equity options. We conclude by illustrating the computational gains that can be made by calculating CVA sensitivities using algorithmic differentiation.

Illustrative examples

The following sample portfolios are considered:

- Sample portfolio 1: A simple portfolio of long and short options all referencing the same equity
- Sample portfolio 2: A simple portfolio of two call options each referencing a different underlying equities
- Sample portfolio 3: A combined portfolio of put and call options referencing different underlyings

In each case, we assume the following dynamics of the underlyings:

Underlying	Current Stock Price (CHF)	Drift	Volatility
A	100	1%	30%
B	100	1%	45%

Table 1: Characteristics of sample portfolio risk factors

In addition, a correlation of 20% is assumed between the stocks A and B.

In each case, we assess the CVA for both a collateralised and an uncollateralised portfolio. Both for the collateralised and uncollateralised portfolio, we assume a Margin Period of Risk (MPOR) of 10 business days. For the CVA calculation, we assume the counterparty has a constant PD of 1% and a recovery rate of 60%.

Sample portfolio 1: Impact of different trade characteristics

In this example we illustrate how different trade characteristics impact the final CVA allocation. The first portfolio consists of a number of equity options, all referencing the same underlying stock:

ID	Notional[6] (mio CHF)	Underlying	Option Type	Strike (CHF)	Maturity (Years)	MtM (mio CHF)	Delta (mio CHF)
1	1.00	A	Call	100	1	12.12	0.58
2	0.50	A	Put	100	2	7.52	-0.20
3	1.00	A	Call	100	3	20.44	0.64
4	-1.00	A	Put	90	3	-12.72	0.31

Table 2: Overview of sample portfolio 1

Figure 4 below shows the EPE profile for both the collateralised and uncollateralised case. The drop in the uncollateralised EPE after 1 year is due to the maturing of trade 1.[7]

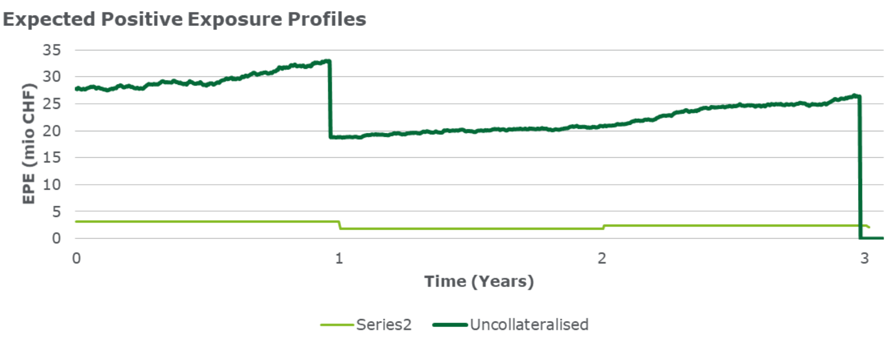


Figure 4: Collateralised and uncollateralised EPE profile for sample portfolio 1

In Table 3, we present the CVA contributions for the sample portfolio both under the sensitivity-based approach outlined in this article, as well as the CMC approach outlined by Pykhtin & Rosen (2010). For both the collateralised and uncollateralised case, we observe that both approaches

[6] Note: a negative notional corresponds to a short position.

[7] Note that the maturing of trade 2 (at 2 years) does impact the EPE. This is due to the fact that, in scenarios in which the stock price goes down, trade 2 and 4 dominate the portfolio, and hence yields an overall negative portfolio MtM. Such scenarios do not add towards the EPE. In scenarios in which the stock price is above 100 at the 2 year mark, trade 2 is worthless.

yield similar CVA contributions. The contributions below illustrate how different trade characteristics impact the CVA contribution, and how this differs between a collateralised and uncollateralised netting set. Recall that, for a collateralised netting set, the driver of the EPE is the changes in the portfolio values. In the case of our sample portfolio, we have that trade 2 moves in the opposite direction to the other trades (it has a negative delta), hence reducing volatility in the portfolio. As a consequence, trade 2 receives a negative CVA charge (i.e., a benefit). In the uncollateralised case, we have that the key driver is the market value of the trades. In the sample portfolio, we see that trade 4 is reducing the overall portfolio exposure to the counterparty, and is hence one would expect it to receive a negative CVA charge.

	Collateralised (Daily)		Uncollateralised	
ID	Sensitivity-based Contribution (kCHF)	Continuous Marginal Contribution (kCHF)	Sensitivity-based Contribution (kCHF)	Continuous Marginal Contribution (kCHF)
1	5.44	5.68	51.73	49.44
2	-3.85	-3.54	54.16	29.45
3	18.58	18.80	306.24	266.43
4	8.97	8.19	-123.56	-56.75
Total	**29.14**	**29.14**	**288.57**	**288.57**

Table 3: Trade-level CVA contributions for sample portfolio 1 (Collateralised and Uncollateralised)

Sample portfolio 2: Impact of different underlyings

We illustrate the effect of different having different underlyings in the portfolio. Consider the following portfolio:

ID	Notional[6] (mio CHF)	Underlying	Option Type	Strike (CHF)	Maturity (Years)	MtM (mio CHF)	Delta (mio CHF)
1	1.00	A	Call	100	1	12.12	0.58
2	1.00	B	Call	100	1	17.86	0.60

Table 4: Overview of sample portfolio 2

Figure 5 below shows the EPE profile for this sample portfolio. We note that, in the uncollateralised case, the EPE profile is flat. This is due to the fact that all trades have a positive MtM at all times. Hence, the EPE is equal to the expected exposure, which is constant.

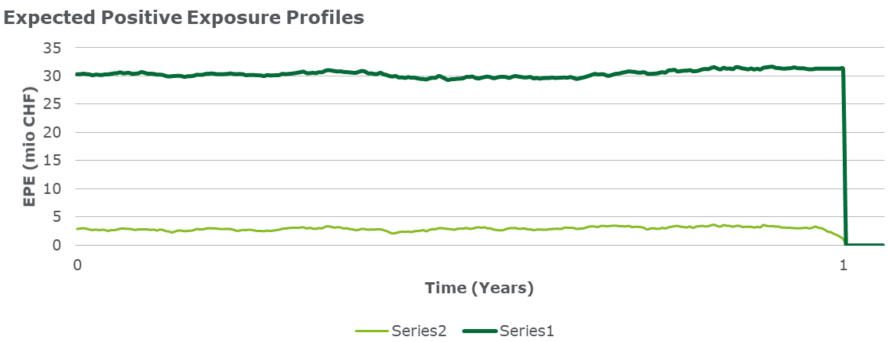


Figure 5: Collateralised and uncollateralised EPE profile for sample portfolio 2

Table 5 below shows the CVA contributions for the collateralised and for the uncollateralised portfolio under both the sensitivity-based approach and the CMC approach. We see that, in all cases, the call option with underlying B leads to higher contributions. In the collateralised case, this is due to the fact that underlying B has a higher volatility. Indeed, a higher volatility means that trade 2 will contribute more to the changes in the portfolio MtM, and hence to the changes in the collateralised EPE. In the uncollateralised case, the higher contribution of the second trade is due to the higher MtM (which is also driven by the higher volatility of underlying B). Furthermore, we note that both the sensitivity-based approach and the CMC approach lead to similar CVA contributions.

	Collateralised (Daily)		Uncollateralised	
ID	Sensitivity-based Contribution (kCHF)	Continuous Marginal Contribution (kCHF)	Sensitivity-based Contribution (kCHF)	Continuous Marginal Contribution (kCHF)
1	3.84	4.30	47.95	49.50
2	7.82	7.36	68.65	67.10
Total	**11.66**	**11.66**	**116.60**	**116.60**

Table 5: Trade-level CVA contributions for sample portfolio 3 (Collateralised and Uncollateralised)

Sample portfolio 3: Combined impact

Finally, we show the results for a portfolio consisting of various options referencing different underlyings:

ID	Notional[6] (mio CHF)	Underlying	Option Type	Strike (CHF)	Maturity (Years)	MtM (mio CHF)	Delta (mio CHF)
1	1.00	A	Call	100	1	12.12	0.58
2	1.00	B	Put	100	2	22.81	-0.37
3	-0.50	A	Call	100	3	-10.22	-0.32
4	1.00	A	Call	90	3	24.64	0.72
5	-0.50	B	Put	110	2	-14.43	0.21
6	0.50	B	Call	100	1	8.93	0.30
7	-0.70	B	Call	90	3	-23.17	-0.51

Table 6: Overview of sample portfolio 3

Figure 6 below shows the EPE profiles in the collateralised and uncollateralised portfolio.

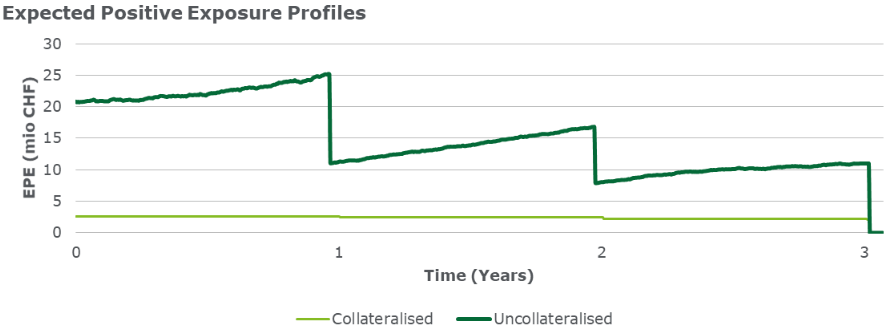


Figure 6: Collateralised and uncollateralised EPE profile for sample portfolio 3

Table 7 below shows the CVA contributions both for a collateralised and uncollateralised CSA agreement. We see that, once again, the sensitivity-based contribution approach leads to very similar contributions compared to the CMC approach. Most importantly, the rank-ordering of the different trades is pertained, hence indicating that the sensitivity-based CVA allocation is "fair".

	Collateralised (Daily)		Uncollateralised	
ID	Sensitivity-based Contribution (kCHF)	Continuous Marginal Contribution (kCHF)	Sensitivity-based Contribution (kCHF)	Continuous Marginal Contribution (kCHF)
1	5.74	4.91	49.28	47.19
2	4.92	6.61	113.44	167.24
3	-5.89	-5.85	-80.33	-105.58
4	13.17	12.83	186.83	250.31
5	-2.85	-3.84	-70.58	-104.34
6	-0.25	-1.39	32.77	24.44
7	13.60	15.19	-50.67	-98.51
Total	**28.46**	**28.46**	**180.74**	**180.74**

Table 7: Trade-level CVA contributions for sample portfolio 2 (Collateralised and Uncollateralised)

Computational gains of using AD

The CVA sensitivities used in the examples above were calculated using the algorithmic differentiation (AD) approach outlined in this article. In this section, we assess the computational gains that AD can bring to the sensitivity calculation. We calculate the CVA sensitivities with both the AD approach and the finite difference approach, for a portfolio with an increasing number of underlying risk factors.[8] Figure 7 below shows that the calculation time is significantly higher for the finite difference approach, especially for portfolios with many risk factors.

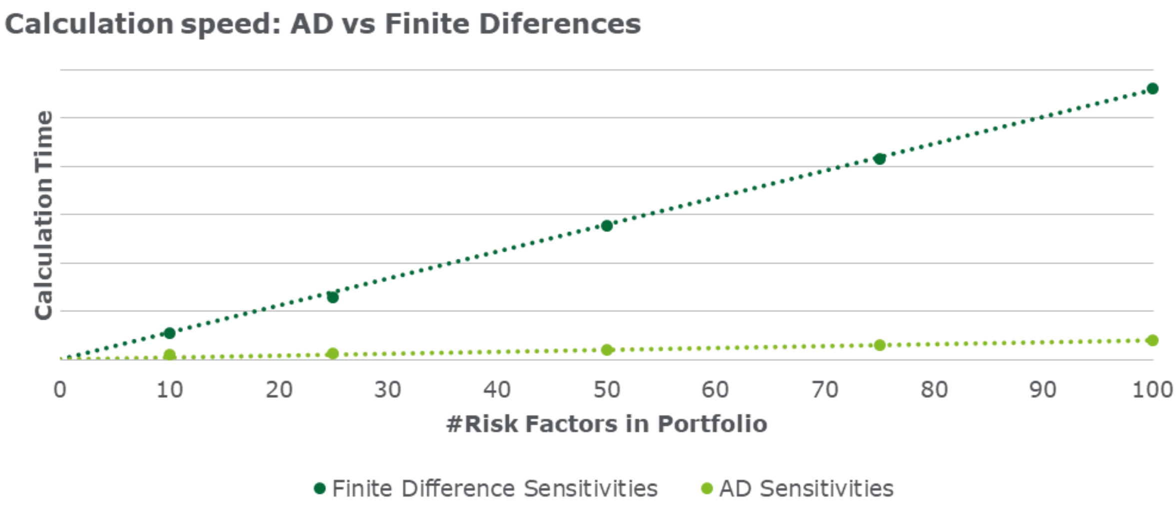


Figure 7: Comparison of calculation time: calculating sensitivities with Finite Difference versus Algorithmic Differentiation (AD)

[8] The number of trades in the portfolio is assumed to be fixed.

6. Conclusion

Evolving XVA landscape

Over the last two decades, XVAs have become a prevalent feature across various departments within a bank; be it setting aside accounting provisions for CVA, DVA and FVA, managing regulatory capital requirements for CVA risk, or mitigating costs associated to derivatives transactions (ColVA, KVA, MVA) in the front office.

Regulatory challenges

The introduction of a CVA capital charge under Basel III was one of the main drivers of major increases in capital requirements for heavy derivatives users, and required the calculation of CVA for all bilateral netting sets, even when fully collateralised. Under the new Basel IV rules, banks will be allowed choose between two approaches for calculating their regulatory capital for CVA risk: the basic approach and the standardised approach (subject to regulatory approval). In addition to calculating an internal estimate of the CVA itself, the standardised approach requires to calculate CVA sensitivities to all relevant risk factors.

Computational complexity and application of algorithmic differentiation

The increasing presence of XVAs has led to a number of modelling challenges for quants. Indeed, the calculation of XVAs is a highly complex topic, combining both the intricacies of derivative pricing with the computational challenges of simulating a full universe of risk factors. The XVA calculation in itself is already computationally very expensive. The Basel IV requirement to calculate XVA sensitivities to the relevant risk factors, adds an additional dimension to the computational costs. "Brute-force" finite-difference XVA Greeks will often be computationally unfeasible. In this article, we illustrate how the calculation of XVA Greeks can be made computationally tractable by making use of algorithmic differentiation (AD).

Allocating XVAs – a must for efficient risk-management

Given the complexity of the XVA calculations, it is often not straightforward to identify the key drivers behind the XVA measures. Providing transparency on how individual trades affect the XVA metrics, is a prerequisite for efficient management of OTC derivatives portfolios. This is particularly important for large inter-dealer portfolios, which without such transparency become practically impossible to manage. A fair allocation of costs along a business hierarchy enables:

- A reduction in hedging costs by optimising accounting CVA,
- An efficient management of CVA risk capital requirements by optimising EPE profiles,
- An allocation of the full costs of doing uncollateralised business with end users (FVA),
- An Incentives for traders to take into account the wider firm's capital efficiency in their transactions.

Sensitivity-based allocations

This article presents an intuitive trade-level XVA allocation approach based on the sensitivities to the key risk-drivers. For collateralised netting sets, the allocation approach consists of two steps. In a first step the XVA is distributed to each underlying risk factors according to a volatility-weighted (absolute) risk factor delta. In a second step, the XVA allocation to each risk factor is further broken down to each underlying trade based on the trade-level deltas.

For uncollateralised netting sets, one splits the XVA into a component due to the current MtM of the netting set, and a component due to risk factor sensitivities. The former component is allocated to the trade-level based on their MtM, whereas the latter "sensitivity component" is allocated using the procedure for collateralised netting sets outlined above.

The sensitivity-based XVA allocation approach is agnostic to any of the XVA modelling assumptions or netting set CSA characteristics and can be integrated as a stand-alone addition to any XVA calculation infrastructure, leveraging an existing XVA sensitivity framework (e.g., the CVA sensitivities used in the capital calculations for the Basel IV standardised approach for CVA risk). Given the XVA sensitivities as an input, the approach is computationally fast, hence enabling pre-trade assessments of XVA.

7. References

1. Giles, Mike, & Glasserman, Paul. (2006) "Smoking Adjoints: Fast Monte Carlo Greeks." Risk 19.1 (2006): 88-92.

2. Pykhtin, Michael & Rosen, Dan. (2010). "Pricing Counterparty Risk at the Trade Level and CVA Allocations", Journal of Credit Risk, vol. 6, 3-38

3. Tasche, Dirk. (2008) "Capital allocation to business units and sub-portfolios: the Euler principle", Pillar II in the New Basel Accord: The Challenge of Economic Capital, Risk Books, 423-453.

4. Basel Committee. (2017) "Basel III: Finalising post-crisis reforms". Basel Committee on Banking Supervision, Basel.

Appendix: Example CVA delta calculation using algorithmic differentiation

In this appendix we outline a step-by-step calculation of the CVA sensitivities using algorithmic differentiation. Recall that:

$$CVA = \int_0^T PD_t \cdot LGD_t \cdot EPE_t^* \, dt,$$

where EPE_t^* denotes the risk-neutral discounted expected positive exposure. In practice, the value of CVA is calculated numerically using discrete time-steps. Consider small time increments Δt, and let $\mathrm{T} = N \cdot \Delta t$ for some integer N. We then write:

$$CVA \approx \sum_{i=0}^{N} PD_{i \cdot \Delta_t} \cdot LGD_{i \cdot \Delta_t} \cdot EPE_{i \cdot \Delta_t}^* \cdot \Delta t$$

In our example, we consider a portfolio of equity derivatives. In this case, we can express:

$$EPE_{i \cdot \Delta_t}^* = DF(i \cdot \Delta t) \cdot EPE_{i \cdot \Delta_t},$$

where $DF(i \cdot \Delta t)$ denotes the value of a zero coupon bond (at $t = 0$) maturing at time $t = \ i \cdot \Delta t$. We further assume that $PD_{i \cdot \Delta_t}$ and $LGD_{i \cdot \Delta_t}$ are functions of time only (i.e., no "wrong-way risk"). Including "wrong-way risk" in the CVA calculation can be performed in a very similar the same way, and just requires a more intricate treatment of the calculation of the derivatives of CVA_t (i.e., some additional terms are included in addition to the derivative of EPE).

We wish to calculate the derivative of the CVA with respect to the risk factor: today's stock price S_0. We have that:

$$\frac{\partial CVA}{\partial S_0} = \sum_{i=0}^{N} PD_{i \cdot \Delta_t} \cdot LGD_{i \cdot \Delta_t} \cdot DF(i \cdot \Delta t) \cdot \frac{\partial EPE_{i \cdot \Delta_t}}{\partial S_0} \cdot \Delta t$$

The expected positive exposure EPE_t is calculated as an average of the positive exposures PE_t^{scen} across the different Monte Carlo scenarios $scen$:

$$EPE_t = \frac{1}{\# \, scen} \sum_{scen} PE_t^{scen}$$

We can change the order of differentiation and summation so that:

$$\frac{\partial EPE_t}{\partial S_0} = \frac{1}{\# \, scen} \sum_{scen} \frac{\partial PE_t^{scen}}{\partial S_0}$$

The scenario positive exposure PE_t^{scen} is a function of the scenario netting set MtM (MtM_t^{scen}), and depends on the CSA collateralisation rules applied. Different examples are:

- No collateralisation (most basic example): $PE_t^{\mathrm{scen}} = \max(0\ ;\ MtM_t^{scen})$
- Collateralisation at a certain frequency: $PE_t^{\mathrm{scen}} = \max(0\ ;\ MtM_t^{scen} - MtM_{last\ coll.date(t)}^{scen})$
- Collateralisation at a certain frequency including MPOR: $PE_t^{\mathrm{scen}} = \max(0\ ;\ MtM_{t+MPOR}^{\mathrm{scen}} - MtM_{last\ coll.date(t)}^{scen})$

For illustration purposes, we will proceed by presenting the no collateralisation example. The derivation for the other cases works similarly. In what follows, we drop the suffix $scen$ for clarity.

For no-collateralised netting set we have that:

$$\frac{\partial PE_t}{\partial S_0} = \begin{cases} 0 & \text{If } MtM_t < 0 \\ \dfrac{\partial MtM_t}{\partial S_0} & \text{If } MtM_t \geq 0 \end{cases}$$

Using the chain rule multiple times, we have that, for any $i > 0$:

$$\frac{\partial MtM_{i \cdot \Delta_t}}{\partial S_0} = \frac{\partial MtM_{i \cdot \Delta_t}}{\partial S_{i \cdot \Delta_t}} \cdot \frac{\partial S_{i \cdot \Delta_t}}{\partial S_{(i-1) \cdot \Delta t}} \cdot \frac{\partial S_{(i-1) \cdot \Delta t}}{\partial S_{(i-2) \cdot \Delta t}} \cdot \ldots \frac{\partial S_{\Delta t}}{\partial S_0} = \frac{\partial MtM_{i \cdot \Delta_t}}{\partial S_t} \cdot \prod_{j=0}^{i-1} \frac{\partial S_{(\mathrm{j}+1) \cdot \Delta t}}{\partial S_{j \cdot \Delta t}} = \frac{\partial MtM_{i \cdot \Delta_t}}{\partial S_t} \cdot D_i,$$

Where we have defined:

$$D_i = \prod_{j=0}^{i-1} \frac{\partial S_{(\mathrm{j}+1) \cdot \Delta t}}{\partial S_{j \cdot \Delta t}}$$

The calculation of D_i, depends on the stochastic process chosen to project the risk factor. For the purposes of this exercise, we consider a simple geometric Brownian motion:

$$S_{(j+1)\cdot\Delta t} = S_{j\cdot\Delta t} \cdot \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma \cdot \sqrt{\Delta t} \cdot \mathrm{Z}_{\mathrm{j+1}}\right\},$$

where $\mathrm{Z}_{\mathrm{j+1}} \sim N(0,1)$. In this instance, we would have:

$$\frac{\partial S_{(\mathrm{j+1})\cdot\Delta t}}{\partial S_{j\cdot\Delta t}} = \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma \cdot \sqrt{\Delta t} \cdot \mathrm{Z}_{\mathrm{j+1}}\right\}.$$

Similar expressions can be obtained for more complex models (the equivalent expression for the Libor Market model is given in Giles and Glasserman (2006)).

Once we have determined D_i, we can write:

$$\frac{\partial EPE_{i\cdot\Delta_t}}{\partial S_0} = \begin{cases} 0 & \text{If } MtM_{i\cdot\Delta_t} < 0 \\ \dfrac{\partial MtM_{i\cdot\Delta_t}}{\partial S_{i\cdot\Delta_t}} \cdot D_i & \text{If } MtM_{i\cdot\Delta_t} \geq 0 \end{cases}$$

The mark-to-market of a portfolio (MtM_t) is simply the sum of the market values of the underlying products:

$$\frac{\partial MtM_{i\cdot\Delta_t}}{\partial S_{i\cdot\Delta_t}} = \sum_{prod\in PF} \frac{\partial MtM_{prod_{i\cdot\Delta_t}}}{\partial S_{i\cdot\Delta_t}} = \sum_{prod\in PF} \delta_{prod_{i\cdot\Delta_t}}$$

The quantity $\frac{\partial MtM_{i\cdot\Delta_t}}{\partial S_{i\cdot\Delta_t}}$ can be calculated in a straightforward fashion using the product-level deltas. In case of a portfolio of vanilla options, these deltas have closed-form formulas, and are hence easy to evaluate.

Combining the above equations gives:

$$\frac{\partial EPE_{i\cdot\Delta_t}}{\partial S_0} = \frac{1}{\#\, scen} \sum_{scen} 1_{MtM_{i\cdot\Delta_t}>0} \cdot D_i^{scen} \sum_{prod\in PF} \delta^{scen}_{prod_{i\cdot\Delta_t}}$$

Contacts

Philipp Flockermann
Partner
Risk Advisory

Tel: +41 58 279 6026
pflockermann@deloitte.ch

George Garston
Manager
Risk Advisory

Tel: +41 58 279 7199
gggarston@deloitte.ch

Deloitte.

Designed and produced by The CoRe Creative Services at Deloitte, Rzeszow RITM180735