Brownian Motion and Stochastic Calculus

Chapters 0 to 7

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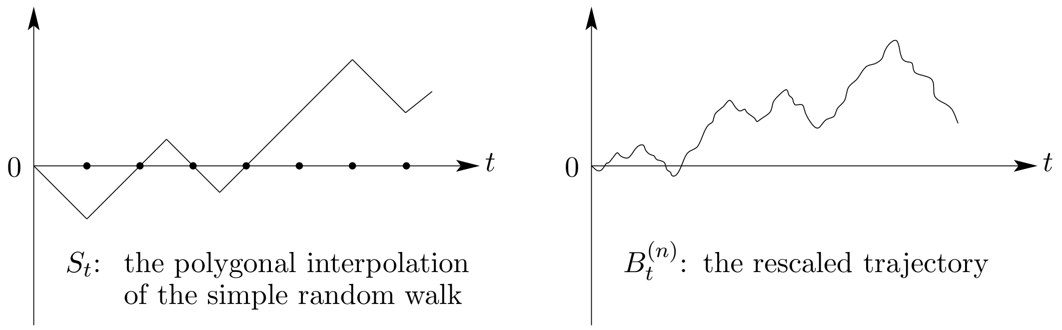
Chapter 0: Introduction

The object of this course is to present Brownian motion, develop the infinitesimal calculus attached to Brownian motion, and discuss various applications to diffusion processes.

The name "Brownian motion" comes from Robert Brown, who in 1827, director at the time of the British botanical museum, observed the disordered motion of "pollen grains suspended in water performing a continual swarming motion". Louis Bachelier in his thesis in 1900 used Brownian motion as a model of the stock market, and Albert Einstein considered it in 1905 when discussing the motion of small particles in suspension in a fluid, under the influence of shocks due to thermal agitation of molecules in the fluid. The mathematical theory of Brownian motion was then put on a firm basis by Norbert Wiener in 1923.

There are several ways to mathematically construct Brownian motion. One can for instance construct Brownian motion as the limit of rescaled polygonal interpolations of a simple random walk, choosing time units of order n^2 and space units of order n:



S_t: the polygonal interpolation of the simple random walk

$B_t^{(n)}$: the rescaled trajectory

$$
(0.1)\qquad
\begin{aligned}
&X_1, \dots, X_n, \dots, \text{ are i.i.d. with } P[X_i = 1] = P[X_i = -1] = \frac{1}{2}\,,\\
&S_m = X_1 + \dots + X_m,\ m \ge 1,\ S_0 = 0\,,\\
&S_t,\ t \ge 0, \text{ is the polygonal interpolation of } S_m,\ m \ge 0, \text{ and}\\
&B_t^{(n)} = \frac{1}{n}\, S_{tn^2},\ t \ge 0, \text{ is the rescaled (in time and space) trajectory.}
\end{aligned}
$$

From the central limit theorem, one knows that $B_1^{(n)}$ converges in law to a $\mathcal{N}(0,1)$-distribution, that is:

$$P[B_1^{(n)} \le a] \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}}\, dx, \text{ for } a \in \mathbb{R}\,.$$

In fact, much more is true, and the law of $B_{\cdot}^{(n)}$ viewed as a random continuous trajectory converges in a suitable sense to the law of Brownian motion (this is a special case of the so-called "invariance principle" of Donsker).

An important advantage of continuous models versus discrete models is the presence of the whole apparatus of "infinitesimal calculus". However, in the case of a typical realization of Brownian motion, the **trajectory** $t \ge 0 \to B_t(\omega) \in \mathbb{R}$, is **continuous, but very rough** (in particular nowhere differentiable, and of infinite variation on any proper interval).

The basic formula of calculus:

$$\frac{d}{dt}\, f(b(t)) = f'(b(t))\, b'(t), \text{ for } f \text{ and } b \text{ two } C^1\text{-functions,} \tag{0.2}$$

can still be given a meaning when b is continuous of finite variation, and f is C^1, namely:

$$f(b(t)) = f(b(0)) + \int_0^t f'(b(s))\, db(s), \text{ for } t \ge 0\,, \tag{0.3}$$

where $db(s)$ stands for the Stieltjes measure on $[0,\infty)$, such that $\int_{[0,a]} db(s) = b(a) - b(0)$, for $0 \le a < \infty$.

However, this extension is of little help in the case of Brownian motion since $t \to B_t$ is of infinite variation on any proper interval.

Nonetheless, we will develop an infinitesimal calculus based on a formula (Ito's formula), which brings into play an "extra term":

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s) ds, \text{ for } f \in C^2(\mathbb{R}),\ t \ge 0, \tag{0.4}$$

or in differential notation:

$$df(B_t) = f'(B_t)\, dB_t + \frac{1}{2}\, f''(B_t)\, dt\,.$$

Of course, part of the work has to do with defining what is meant by "$\int_0^t f'(B_s)\, dB_s$", since, as explained above, this expression has no meaning as a Stieltjes integral. This task will correspond to the construction of stochastic integrals.

Once this infinitesimal calculus is at our disposal, we will be able to solve certain differential equations with random perturbations, the so-called "stochastic differential equations" (SDEs):

$$dX_t = b(X_t)dt + \underbrace{\sigma(X_t)dB_t}_{\text{random perturbation}}\,. \tag{0.5}$$

There turns out to be a deep connection between solutions of such stochastic differential equations and certain partial differential equations (PDEs).

For instance, when $B_t = (B_t^1, \dots, B_t^d)$, where the $B_{\cdot}^i$ are independent real-valued Brownian motions, and $D \subseteq \mathbb{R}^d$ is a smooth bounded domain, e.g. a ball, one can consider the

Dirichlet problem: given $f \in C(\partial D)$, find u such that

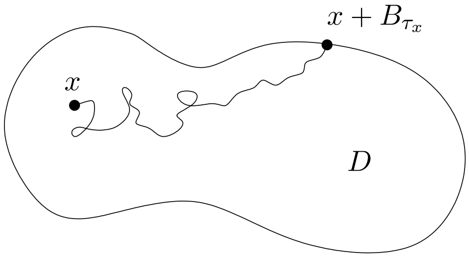
$$\begin{cases} \frac{1}{2}\,\Delta u = 0 \ \text{ in } \ D\,, \\ u|_{\partial D} = f\,, \end{cases} \tag{0.6}$$

or the

Poisson equation: for $g \in C^\alpha(\overline{D})$, find u such that

$$\begin{cases} \frac{1}{2}\,\Delta u = g \ \text{ in } \ D\,, \\ u|_{\partial D} = 0\,. \end{cases} \tag{0.7}$$

The two problems have solutions, which can be expressed in terms of Brownian motion:



Setting for $x \in D$,

$$\tau_x = \inf\{s \ge 0;\ x + B_s \in \partial D\}\,, \tag{0.8}$$

one has

$$u_{\text{Dirichlet}}\,(x) = E[f(x + B_{\tau_x})] \tag{0.9}$$

and

$$u_{\text{Poisson}}\,(x) = -E\Big[\int_0^{\tau_x} g(x + B_s)ds\Big]\,. \tag{0.10}$$

With stochastic differential equations, one is able to handle more general partial differential equations with $\frac{1}{2}\,\Delta$ replaced by:

$$L = \frac{1}{2}\,\sum_{i,j=1}^d (\sigma(x)\ {}^t\sigma(x))_{i,j}\ \partial^2_{i,j} + \sum_{i=1}^d b(x)_i\,\partial_i\,, \tag{0.11}$$

and, during this course, we will describe a number of applications of these ideas and concepts.

Chapter 1: Brownian Motion: Definition and Construction

We will see that Brownian motion plays a prominent role as a canonical example of three different notions:

- a **continuous Gaussian process**,
- a **continuous Markov process**,
- a **continuous martingale**.

In this chapter, we will mainly deal with the first of these three notions.

Definition 1.1. *Let $(\Omega, \mathcal{A}, P)$ be a probability space. A d-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ is an $\mathbb{R}^d$-valued stochastic process (i.e. for each $t \geq 0$, $B_t(\cdot)$ is an $\mathbb{R}^d$-valued random variable defined on $(\Omega, \mathcal{A}, P)$), such that:*

i) $B_0 = 0$, *P-a.s.*,

ii) *for any* $0 = t_0 < t_1 < \dots < t_n$, $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ *are independent random variables ("independent increments"),*

(1.1) iii) *for* $t > 0$, $s \geq 0$, $A \in \mathcal{B}(\mathbb{R}^d)$,
$$P[B_{t+s} - B_s \in A] = \int_A (2\pi t)^{-\frac{d}{2}}\, e^{-\frac{|x|^2}{2t}}\, dx,$$ *("$B_{t+s} - B_s$ is $N(0, tI)$-distributed"),*

iv) *P-a.s.,* $t \geq 0 \to B_t(\omega) \in \mathbb{R}^d$ *is continuous.*

In the above definition $(\Omega, \mathcal{A}, P)$ is "arbitrary". As we will see, there is a way to construct a "canonical Brownian motion", once we know that at least one Brownian motion in the sense of the above definition exists.

We take as a model the "canonical" space

(1.2) $$C = C(\mathbb{R}_+, \mathbb{R}^d) = \{\text{continuous functions } \mathbb{R}_+ \to \mathbb{R}^d\}.$$

On C we have the canonical coordinates:

(1.3) $$X_u\colon C \to \mathbb{R}^d,\ u \geq 0, \text{ such that } X_u(w) = w(u), \text{ for } w \in C,$$

and the σ-algebra generated by these coordinates:

(1.4) $$\mathcal{F} = \sigma(X_u, u \geq 0), \text{ (i.e. the smallest } \sigma\text{-algebra on } C \text{ for which all } X_u, u \geq 0, \text{ are measurable).}$$

Lemma 1.2. *(ψ: a map $\Omega \to C$)*

(1.5) $$(\Omega, \mathcal{A}) \xrightarrow{\psi} (C, \mathcal{F}) \text{ is measurable if and only if}$$
$$(\Omega, \mathcal{A}) \xrightarrow{X_u \circ \psi} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is measurable for each } u \geq 0.$$

Proof. $\Longrightarrow$: immediate (the composition of two measurable maps is measurable).

$\Longleftarrow$: the collection $\mathcal{S}$ of $B \subseteq C$ such that $\psi^{-1}(B) \in \mathcal{A}$ is a σ-algebra, which contains $X_u^{-1}(D)$ for $D \in \mathcal{B}(\mathbb{R}^d)$, and $u \geq 0$. Hence, $\mathcal{S}$ contains $\mathcal{F}$, the smallest σ-algebra for which all $X_u, u \geq 0$, are measurable. As a result for all $F \in \mathcal{F}$, $\psi^{-1}(F) \in \mathcal{A}$, and ψ is measurable. $\square$

We will later see that on a suitable $(\Omega, \mathcal{A}, P)$ we can construct a Brownian motion. For such a Brownian motion we can pick by (1.1) iv), a negligible set $N \in \mathcal{A}$ (i.e. $P(N) = 0$), and define

$$\big(\Omega\backslash N, \underbrace{\mathcal{A} \cap (\Omega\backslash N)}\big) \xrightarrow{B.} (C, \mathcal{F})$$

the notation means the collection of sets $A \cap (\Omega\backslash N)$, with $A \in \mathcal{A}$.

The above map is measurable, indeed:

for $u \geq 0$, $X_u \circ B.(\omega) = B_u(\omega)$ is measurable $(\Omega\backslash N, \mathcal{A} \cap (\Omega\backslash N)) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

and we can apply (1.5).

We can consider the image under $B.$ of the probability measure P restricted to $\Omega\backslash N$ (i.e. $P : \mathcal{A} \cap \Omega\backslash N \to [0,1]$). We denote by W this image probability.

Proposition 1.3.

(1.6) *The law W on $(C, \mathcal{F})$ is uniquely determined,*
*(it is the so-called **d-dimensional Wiener measure**).*

(1.7) *For $0 = t_0 < t_1 < \cdots < t_n$ and $h \in b\mathcal{B}((\mathbb{R}^d)^{n+1})$ (i.e. bounded measurable on $(\mathbb{R}^d)^{n+1}$)*

$$E^W[h(X_{t_0}, X_{t_1}, \dots, X_{t_n})] =$$
$$\int_{(\mathbb{R}^d)^n} h(0, x_1, \dots, x_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-\frac{d}{2}} \exp\Big\{ - \frac{|x_i - x_{i-1}|^2}{2(t_i - t_{i-1})}\Big\} \, dx_1 \dots dx_n \,.$$

(1.8) *$X_t(w), t \geq 0$, is a Brownian motion on $(C, \mathcal{F}, W)$,*
*(it is the **canonical d-dimensional Brownian motion**).*

Proof.

- (1.6): For $h \in b\mathcal{B}((\mathbb{R}^d)^{n+1})$ and $0 = t_0 < t_1 < \cdots < t_n$, we have

$$\begin{aligned} a &\overset{\text{def}}{=} E^W[h(X_{t_0}, X_{t_1}, \dots, X_{t_n})] = E^P[h(B_0, B_{t_1}, \dots, B_{t_n})] \\ &= E^P\big[h(B_0, B_0 + B_{t_1} - B_{t_0}, \dots, B_0 + B_{t_1} - B_{t_0} + B_{t_2} \\ &\qquad - B_{t_1} + \cdots + B_{t_n} - B_{t_{n-1}})\big] \,. \end{aligned} \tag{1.9}$$

By (1.1), the $B_{t_i} - B_{t_{i-1}}$, $1 \le i \le n$, are independent, respectively $N(0,(t_i - t_{i-1})I)$-distributed, and $B_0 = 0$, P-a.s.. Hence we find

$$(1.10) \qquad \begin{aligned} a = &\int_{(\mathbb{R}^d)^n} h(0, y_1, y_1 + y_2, \dots, y_1 + \dots + y_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-\frac{d}{2}} \\ &e^{-\frac{|y_i|^2}{2(t_i - t_{i-1})}} \, dy_1 \dots dy_n \,. \end{aligned}$$

Picking $h = 1_D$, where $D \in \mathcal{B}((\mathbb{R}^d)^{n+1})$, we see that (1.9), (1.10) determine

$$W(\{(X_0, X_{t_1}, \dots, X_{t_n}) \in D\}) \,.$$

The class of sets of the form $\{(X_0, X_{t_1}, \dots, X_{t_n}) \in D\}$, $n \ge 1$, $0 = t_0 < t_1 < \dots < t_n$, and $D \in \mathcal{B}((\mathbb{R}^d)^{n+1})$ arbitrary, is a π-system (i.e. is stable under intersection), which generates $\mathcal{F}$. From Dynkin's lemma, W is completely determined on $\mathcal{F}$, and in particular does not depend on the specific $(\Omega, \mathcal{A}, P, B)$ and N we used.

• (1.7): We perform the change of variables in (1.10)

$$(1.11) \qquad x_1 = y_1, \; x_2 = y_1 + y_2, \dots, x_n = y_1 + y_2 + \dots + y_n \,.$$

Note that $Jac(y_1, \dots, y_n | x_1, \dots, x_n) = 1$, so that

$$a \overset{(1.10)}{=} \int_{(\mathbb{R}^d)^n} h(0, x_1, \dots, x_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-\frac{d}{2}} \exp\Big\{ - \frac{|x_i - x_{i-1}|^2}{2(t_i - t_{i-1})} \Big\} \, dx_1 \dots dx_n \,,$$

and this proves (1.7).

• (1.8): We pick h in (1.9), (1.10) of the form

$$\begin{aligned} &h(x_0, x_1, \dots, x_n) = g(x_0, x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1}), \text{ so that} \\ &h(0, y_1, y_1 + y_2, \dots, y_1 + \dots + y_n) = g(0, y_1, y_2, \dots, y_n) \,. \end{aligned}$$

It then follows that $X_t, t \ge 0$, fulfills (1.1), i), ii), iii). Since for all $w \in C$, $t \ge 0 \to X_t(w) \in \mathbb{R}^d$ is continuous, (1.1) iv) holds as well, and $X_t, t \ge 0$, is a Brownian motion on $(C, \mathcal{F}, W)$. □

Definition 1.4.

• *An $\mathbb{R}^d$-valued process, X_t, $t \in T$, (T is some arbitrary non-empty set), defined on $(\Omega, \mathcal{A}, P)$ is a* ***centered Gaussian process*** *(when T is finite, one also speaks of a centered Gaussian vector), if for any $n \ge 1$, $t_1, \dots, t_n \in T$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}^d$, $\sum_{i=1}^n \lambda_i \cdot X_{t_i}$ is a real-valued centered Gaussian variable (possibly $\equiv 0$).*

↑
scalar product when $d \ge 2$

• *The $d \times d$-matrix valued function on T^2:*

$$\Gamma(u,v) = E[X_u\, {}^tX_v] = (E[X_u^i\, X_v^j])_{1\le i,j\le d},\ u,v\in T \quad (\text{note that } \Gamma(v,u) = {}^t\Gamma(u,v)), \tag{1.12}$$

is the ***covariance function*** *of the process (note that $E[X_t]=0$, for each $t\in T$).*

Lemma 1.5. ($(X_t)_{t\in T}$ *a centered Gaussian process with covariance function* Γ)

(1.13) *The function $\Gamma(u,v), u,v\in T$, completely determines all finite distributions $X_{t_1},\dots,X_{t_n}$ on $(\mathbb{R}^d)^n$, for any $n\ge 1$, and $t_1,\dots,t_n\in T$.*

Proof. For $\xi = (\lambda_1,\dots,\lambda_n)\in(\mathbb{R}^d)^n$, we set

$$\varphi(\xi) \overset{\text{def}}{=} E\Big[\exp\Big\{i\,\underbrace{\sum_{j=1}^n \lambda_j\cdot X_{t_j}}_{\text{real-valued centered Gaussian variable}}\Big\}\Big] = \exp\Big\{-\tfrac{1}{2}\,E\Big[\Big(\sum_{j=1}^n \lambda_j\cdot X_{t_j}\Big)^2\Big]\Big\}$$

$$= \exp\Big\{-\tfrac{1}{2}\sum_{i,j=1}^n E[(\lambda_i\cdot X_{t_i})(\lambda_j\cdot X_{t_j})]\Big\} \tag{1.14}$$

$$= \exp\Big\{-\tfrac{1}{2}\sum_{i,j=1}^n \overset{\substack{\text{row vector}\\ \downarrow}}{{}^t\lambda_i}\ \underbrace{E[X_{t_i}\,{}^tX_{t_j}]}_{d\times d\text{ matrix}}\ \overset{\substack{\text{column vector}\\ \downarrow}}{\lambda_j}\Big\} \overset{(1.12)}{=} \exp\Big\{-\tfrac{1}{2}\sum_{i,j=1}^n {}^t\lambda_i\,\Gamma(t_i,t_j)\,\lambda_j\Big\}.$$

But the characteristic function $\varphi(\cdot)$ completely determines the law of $(X_{t_1},\dots,X_{t_n})$ on $(\mathbb{R}^d)^n$. □

We will now provide a characterization of Brownian motion as a continuous centered Gaussian process.

Proposition 1.6. *Let B_t, $t\ge 0$, be an $\mathbb{R}^d$-valued process defined on $(\Omega,\mathcal{A},P)$ with P-a.s. continuous trajectories:*

$$B_t,\ t\ge 0 \text{ is a Brownian motion} \Longleftrightarrow B_t,\ t\ge 0 \text{ is a centered Gaussian process with } \Gamma(s,t) = (s\wedge t)\,\underset{\substack{\uparrow\\ \text{identity matrix}}}{I_{d\times d}}\,. \tag{1.15}$$

Proof.

- $\Longrightarrow$: For $n \geq 1$, $0 \leq t_1 < t_2 < \cdots < t_n$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}^d$, P-a.s.,

$$a \stackrel{\text{def}}{=} \sum_{i=1}^n \lambda_i \cdot B_{t_i} \stackrel{(1.1)\,i)}{=} \sum_{i=1}^n \lambda_i \cdot \Big(\sum_{j=1}^i (B_{t_j} - B_{t_{j-1}})\Big) = \sum_{j=1}^n \Big(\sum_{i=j}^n \lambda_i\Big) \cdot (B_{t_j} - B_{t_{j-1}})$$

(denoting $t_0 = 0$). By (1.1), the $(B_{t_j} - B_{t_{j-1}})$, $1 \leq j \leq n$, are independent respectively $N(0, (t_j - t_{j-1}) I_{d\times d})$-distributed. Therefore, a is a real valued centered Gaussian variable (use the characteristic function), and we have shown that B_t, $t \geq 0$, is a centered Gaussian process. Moreover for $0 \leq s \leq t$, we have

$$\begin{aligned} \Gamma(s,t) &= E[B_s\, {}^t B_t] = E[B_s\, {}^t B_s] + \underbrace{E[B_s\, {}^t(B_t - B_s)]}_{=0 \text{ (independent and centered)}} \\ &= s I_{d\times d} = (s \wedge t)\, I_{d\times d}\,. \end{aligned} \tag{1.16}$$

Then, for $t \leq s$, $\Gamma(s,t) = {}^t\Gamma(t,s) \stackrel{(1.16)}{=} (t \wedge s)\, {}^t I_{d\times d} = (s \wedge t)\, I_{d\times d}$.

- $\Longleftarrow$: If $0 < t_1 < \cdots < t_n$ are given, and on some auxiliary probability space Y_j, $1 \leq j \leq n$, are independent $N(0, (t_j - t_{j-1})\, I_{d\times d})$-distributed, we can define $X_j = \sum_{k=1}^j Y_k$.

A repetition of the argument above shows that $X_j, 1 \leq j \leq n$, is a centered Gaussian process, and we can calculate its covariance as follows. For $1 \leq i \leq j \leq n$, one has:

$$\begin{aligned} \Gamma(i,j) &\stackrel{\text{def}}{=} E[X_i\, {}^t X_j] = E[X_i\, {}^t X_i] + \underbrace{E[X_i\, {}^t(X_j - X_i)]}_{=0 \text{ independent and centered}} \\ &= E\Big[\Big(\sum_1^i Y_k\Big)\, {}^t\Big(\sum_1^i Y_k\Big)\Big] \\ &\stackrel{\text{indep. centered}}{=} \sum_{1 \leq k \leq i} E[Y_k\, {}^t Y_k] = t_i\, I_{d\times d}\,. \end{aligned} \tag{1.17}$$

As below (1.16), we thus see that:

$$\Gamma(i,j) = (t_i \wedge t_j)\, I_{d\times d}, \text{ for } 1 \leq i,j \leq n\,. \tag{1.18}$$

By (1.13), we thus see that $(X_1, \dots, X_n)$ has the same law as $(B_{t_1}, \dots, B_{t_n})$. Therefore, $(B_{t_1},\, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}})$ has the same law as $(X_1, X_2 - X_1, \dots, X_n - X_{n-1}) = (Y_1, \dots, Y_n)$. Hence (1.1) ii), iii) follow. Moreover (1.1) iv) holds by assumption and $E[B_0\, {}^t B_0] = 0$ implies by looking at the diagonal coefficients of this matrix that P-a.s., $B_0 = 0$. This proves that B_t, $t \geq 0$, is a Brownian motion. □

The above characterization is very helpful.

Examples:

1) *Invariance by scaling:*

Consider $B_t, t \ge 0$, an $\mathbb{R}^d$-valued Brownian motion, $\lambda > 0$, then

$$\lambda B_{t/\lambda^2},\ t \ge 0, \text{ is also an } \mathbb{R}^d\text{-valued Brownian motion}. \tag{1.19}$$

Indeed:

$B_t^\lambda \overset{\text{def}}{=} \lambda B_{t/\lambda^2}$, $t \ge 0$, is also a continuous centered Gaussian process, and for $s \ge 0$, $t \ge 0$:

$$E[B_s^\lambda\, {}^t B_t^\lambda] = \lambda^2\, E[B_{s/\lambda^2}\, {}^t B_{t/\lambda^2}] = \lambda^2 \Big(\frac{s}{\lambda^2} \wedge \frac{t}{\lambda^2}\Big)\, I_{d\times d} = (s \wedge t)\, I_{d\times d}\,,$$

and (1.19) follows from (1.15).

2) *Invariance by time inversion:*

Consider $B_t, t \ge 0$, an $\mathbb{R}^d$-valued Brownian motion, and define

$$\beta_0 = 0, \text{ and } \beta_s = s\, B_{1/s}, \text{ for } s > 0\,. \tag{1.20}$$

Then one has:

$$\beta_s, s \ge 0, \text{ is an } \mathbb{R}^d\text{-valued Brownian motion}. \tag{1.21}$$

Indeed:

$\beta_s, s \ge 0$, is a centered Gaussian process, and for $0 < s, t$:

$$\begin{aligned} E[\beta_s\, {}^t\beta_t] &= st\, E[B_{\frac{1}{s}}\, {}^t B_{\frac{1}{t}}] = st\,\Big(\frac{1}{s} \wedge \frac{1}{t}\Big)\, I_{d\times d} = \frac{st}{s \vee t}\, I_{d\times d} \\ &= (s \wedge t)\, I_{d\times d}\,, \end{aligned} \tag{1.22}$$

and this formula immediately extends to the case $0 \le s, t$.

There only remains to see that P-a.s., $s \ge 0 \to \beta_s$ is continuous to conclude (1.21). To this end, we note that by (1.22) and (1.13),

$$\text{the laws of } \beta_s, s > 0, \text{ and } B_t, t > 0, \text{ on } C(\underset{\substack{\uparrow \\ \text{open at 0!}}}{(0,\infty)}, \mathbb{R}^d) \text{ are identical.} \tag{1.23}$$

We let $X_u, u > 0$, denote the canonical process on $C((0,\infty), \mathbb{R}^d)$, and $\mathcal{G} = \sigma(X_u, u > 0)$ the canonical σ-algebra. If Q stands for the common law on $(C((0,\infty), \mathbb{R}^d), \mathcal{G})$ of $\beta_s, s > 0$, or $B_t, t > 0$, then

$$\big\{\lim_{u \to 0} X_u = 0\big\} \in \mathcal{G} \qquad \text{(it is an "event")}\,. \tag{1.24}$$

Indeed one has

$$\big\{\lim_{u \to 0} X_u = 0\big\} = \bigcap_{n \ge 1} \bigcup_{m \ge 1} \bigcap_{u \in \mathbb{Q} \cap (0, \frac{1}{m})} \Big\{|X_u| \le \frac{1}{n}\Big\} \in \mathcal{G}\,.$$

As a result we find

$$\underbrace{Q(\lim_{u\to 0} X_u = 0)}_{\overset{\shortparallel}{} \;(1.23)} = P\big[\lim_{u\to 0} B_u = 0\big] \overset{(1.1)\, i),iv)}{=} 1$$

$$P\big[\lim_{u\to 0} \beta_u = 0\big],$$

and hence $\beta_s, s \ge 0$, fulfills (1.1) iv) as well. □

Construction of Brownian motion:

We are now going to construct a Brownian motion on some $(\Omega, \mathcal{A}, P)$. It suffices to consider the case $d = 1$, since by taking d independent copies of a real-valued Brownian motion, we obtain a d-dimensional Brownian motion.

We follow a method of Paul Lévy (1948), later simplified by Z. Ciesielski (1961).

We recall the **Haar functions** on $\mathbb{R}_+$:

(1.25)
$$\begin{aligned} &\varphi_\ell(t) = 1_{[\ell,\ell+1)}(t),\ \ell \in \mathbb{N} \text{ (the set of non-negative integers)},\\ &\varphi_{m,k}(t) = 2^{\frac{m}{2}}\ 1_{\big[\frac{k}{2^m},\, \frac{k}{2^m}+\frac{1}{2^{m+1}}\big)}(t) - 2^{\frac{m}{2}}\ 1_{\big[\frac{k}{2^m}+\frac{1}{2^{m+1}},\, \frac{k+1}{2^m}\big)}(t) \ \text{ with } m,k \in \mathbb{N}\,. \end{aligned}$$

Fact:

(1.26) The $\varphi_\ell, \ell \ge 0, \varphi_{m,k}, m,k \ge 0$, form a complete orthonormal basis of $L^2(\mathbb{R}_+, dt)$.

Indeed:

- The functions $\varphi_\ell, \varphi_{m,k}$ have unit $L^2(\mathbb{R}_+, dt)$-norms.
- They are pairwise orthogonal in $L^2(\mathbb{R}_+, dt)$.
- The L^2-closure of the span of the $\varphi_\ell, \varphi_{m,k}$ is $L^2(\mathbb{R}_+, dt)$, because one sees by induction on $m \ge 0$, that all $1_{[\frac{j}{2^m}, \frac{j+1}{2^m})}, j \ge 0$, belong to the space generated by

  $$\varphi_\ell, \ell \ge 0, \text{ and } \varphi_{m',k},\ 0 \le m' < m,\ 0 \le k\,,$$

  and the above claim follows.

Heuristic (non-rigorous) description of the construction of Brownian motion

The idea is to use the formal development of $\dot{B}$ (the derivative of Brownian motion !!!) in the above Haar basis. Formally we have:

$$\dot{B}(\cdot) \text{ “=” } \sum_{\ell \ge 0} \varphi_\ell(\cdot)\Big(\int_0^\infty \dot{B}\,\varphi_\ell\, dt\Big) + \sum_{m,k\ge 0} \varphi_{m,k}(\cdot)\Big(\int_0^\infty \dot{B}\,\varphi_{m,k}\, dt\Big)\,.$$

We then write

$$
\begin{aligned}
B(t) \text{ ``=''} \int_0^t \dot{B}(u)du \text{ ``=''} \sum_{\ell\ge 0} \int_0^t \varphi_\ell(u)du \Big(\int_0^\infty \dot{B}\,\varphi_\ell\,dt\Big) + \\
\sum_{m,k\ge 0} \int_0^t \varphi_{m,k}(u)du \Big(\int_0^\infty \dot{B}\,\varphi_{m,k}\,dt\Big).
\end{aligned}
\tag{1.27}
$$

We now define

$$
\xi_\ell \text{ ``=''} \int_0^\infty \dot{B}\,\varphi_\ell\,dt \text{ ``=''} \int_\ell^{\ell+1} \dot{B}\,dt \text{ ``=''} B(\ell+1) - B(\ell), \tag{1.28}
$$

$$
\begin{aligned}
\xi_{m,k} \text{ ``=''} \int_0^\infty \dot{B}\,\varphi_{m,k}\,dt \text{ ``=''} 2^{\frac{m}{2}} \Big(\int_{\frac{k}{2^m}}^{\frac{k}{2^m}+\frac{1}{2^{m+1}}} \dot{B}\,dt - \int_{\frac{k}{2^m}+\frac{1}{2^{m+1}}}^{\frac{k+1}{2^m}} \dot{B}\,dt\Big) \\
\text{ ``=''} 2^{\frac{m}{2}} \Big(B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big) - B\Big(\frac{k}{2^m}\Big)\Big) - 2^{\frac{m}{2}} \Big(B\Big(\frac{k+1}{2^m}\Big) - B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big)\Big).
\end{aligned}
\tag{1.29}
$$

Now, if a Brownian motion exists, then the right-hand sides of (1.28) and (1.29) make sense, and the above $\xi_\ell, \xi_{m,k}$ are $N(0,1)$-variables, and form a centered Gaussian family (since they are linear combinations of $B(t), t \ge 0$). Moreover, the variables ξ_ℓ, $\ell \ge 0$, $\xi_{m,k}$, $m, k \ge 0$, are pairwise orthogonal:

$$
\begin{aligned}
&E[\xi_\ell\,\xi_{\ell'}] = 0, \text{ for } \ell \neq \ell' \ge 0, \; E[\xi_\ell\,\xi_{m,k}] = 0, \text{ for } \ell \ge 0, \; m,k \ge 0\,, \\
&E[\xi_{m,k}\,\xi_{m',k'}] = 0, \text{ for } (m,k) \neq (m',k')\,.
\end{aligned}
$$

This orthogonality follows from (1.1) ii), iii) (for instance $E[\xi_\ell\,\xi_{\ell'}] = 0$, for $\ell \neq \ell'$ is immediate to check, likewise $E[\xi_\ell\,\xi_{m,k}] = 0$, if $[\frac{k}{2^m}, \frac{k+1}{2^m}) \cap [\ell, \ell+1) = \emptyset$ is immediate, and on the other hand when the intersection is not empty, then $[\frac{k}{2^m}, \frac{k+1}{2^m}) \subseteq [\ell, \ell+1)$, and one has

$$
\begin{aligned}
-E[\xi_\ell\,\xi_{m,k}] = 2^{\frac{m}{2}}\, E\Big[(B(\ell+1) - B(\ell))\Big\{\Big(B\Big(\frac{k+1}{2^m}\Big) - B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big)\Big) \\
- \Big(B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big) - B\Big(\frac{k}{2^m}\Big)\Big)\Big\}\Big],
\end{aligned}
$$

and writing

$$
\begin{aligned}
B(\ell+1) - B(\ell) = \underbrace{B(\ell+1) - B\Big(\frac{k+1}{2^m}\Big)} + \underbrace{B\Big(\frac{k+1}{2^m}\Big) - B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big)} + \\
\underbrace{B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big) - B\Big(\frac{k}{2^m}\Big)} + \underbrace{B\Big(\frac{k}{2^m}\Big) - B(\ell)},
\end{aligned}
$$

we can use (1.1), ii), iii) to conclude that $E[\xi_\ell\,\xi_{m,k}] = 0$ as well, the last equality $E[\xi_{m,k}\,\xi_{m',k'}] = 0$ for $(m,k) \neq (m',k')$ is shown with analogous considerations).

Since the ξ_ℓ, $\xi_{m,k}$ form a centered Gaussian family, are $N(0,1)$-distributed, and are pairwise uncorrelated, they are in fact independent (cf. (1.13)). Hence, the formal formulas (1.27), (1.28), (1.29) tell us where we should "look for a Brownian motion". We will now see how the above non-rigorous considerations can be transformed into a real proof.

Mathematical construction:

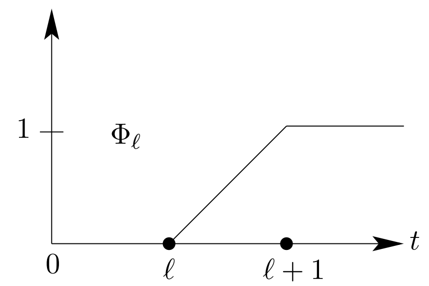
We consider on some suitable probability space $(\Omega, \mathcal{A}, P)$ a (countable) family of i.i.d. $N(0,1)$-distributed variables ξ_ℓ, $\xi_{m,k}$, $\ell \geq 0$, $m,k \geq 0$ (for instance $\Omega = (0,1)$, $\mathcal{A} = \mathcal{B}((0,1))$, $P =$ Lebesgue measure on $(0,1)$, will do the job).

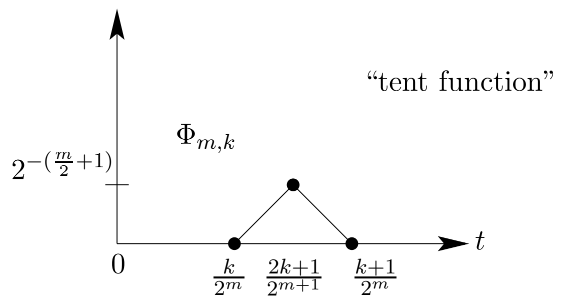
We then define for $n \geq 1$ and $t \geq 0$:

$$X_n(t) = \sum_{0 \leq \ell < n} \Phi_\ell(t)\, \xi_\ell + \sum_{0 \leq m < n} \Big(\sum_{0 \leq k < n2^m} \Phi_{m,k}(t)\, \xi_{m,k} \Big), \tag{1.30}$$

where

$$\Phi_\ell(t) = \int_0^t \varphi_\ell(u) du \;\text{ and }\; \Phi_{m,k}(t) = \int_0^t \varphi_{m,k}(u) du \tag{1.31}$$
$$\text{(these are called Schauder functions)}.$$





Lemma 1.7.

(1.32) *P-a.s., $X_n(\cdot,\omega)$ converges uniformly on compact intervals of $\mathbb{R}_+$ to a finite limit $X(\cdot,\omega)$.*

Proof. It suffices to prove that for any $n_0 \geq 1$, P-a.s., $X_n(\cdot,\omega)$ converges uniformly on $[0,n_0]$ to a finite limit (because we can then find $N \in \mathcal{A}$, with $P[N] = 0$, so that on N^c, $X_n(\cdot,\omega)$ converges uniformly on any $[0,n_0]$, and then define

$$\begin{aligned} X(t,\omega) &= \lim_n \, X_n(t,\omega),\; \omega \in N^c,\; t \geq 0\,, \\ &= 0, \;\text{ when } \omega \in N,\; t \geq 0). \end{aligned}$$

For $t \in [0,n_0]$, we have for $n \geq n_0$:

$$X_n(t) = \sum_{\ell=0}^{n_0-1} \Phi_\ell(t)\, \xi_\ell + \sum_{0 \leq m < n} \Big(\sum_{0 \leq k < n_0 2^m} \Phi_{m,k}(t)\, \xi_{m,k} \Big), \tag{1.33}$$

and for each $t \in [0, n_0]$, and each $m \geq 0$, there is at most one $k \geq 0$, such that $\Phi_{m,k}(t) \neq 0$. We then define:

$$a_m(\omega) = \sup_{t \in [0,n_0]} \Big| \sum_{k < n_0 2^m} \Phi_{m,k}(t)\, \xi_{m,k} \Big| \leq 2^{-(\frac{m}{2}+1)} \sup_{k < n_0 2^m} |\xi_{m,k}| \,. \tag{1.34}$$

We will control this supremum. To this end, we note that for ξ $N(0,1)$-distributed:

$$P[|\xi| > a] \leq \sqrt{\frac{2}{\pi}}\ \frac{1}{a}\ \exp\Big\{-\frac{a^2}{2}\Big\}, \ \text{ for } a > 0\,, \tag{1.35}$$

(indeed $\int_0^\infty \exp\{-\frac{(a+u)^2}{2}\}du \leq \int_0^\infty \exp\{-\frac{a^2}{2} - au\}du = \frac{1}{a}\ \exp\{-\frac{a^2}{2}\}$).

It follows that

$$\sum_{m \geq 1} P\Big[\sup_{0 \leq k < n_0 2^m} |\xi_{m,k}| > \sqrt{2m}\Big] \leq \sum_{m \geq 1} \sqrt{\frac{2}{\pi}}\ \frac{n_0 2^m}{\sqrt{2m}}\ e^{-m} < \infty\,.$$

Thus, Borel Cantelli's lemma implies that for P-almost every ω, there exists $m_0(\omega)$ such that for $m \geq m_0(\omega)$, $\sup_{k < n_0 2^m} |\xi_{m,k}(\omega)| \leq \sqrt{2m}$. As a result:

$$P\text{-a.s.},\ \sum_{m \geq m_0(\omega)} a_m(\omega) \leq \sum_{m \geq m_0(\omega)} 2^{-(\frac{m}{2}+1)} \sqrt{2m} < \infty\,. \tag{1.36}$$

It follows that P-a.s., $X_n(\cdot, \omega)$ converges uniformly on $[0, n_0]$ to a finite limit. $\square$

We will now see that the above defined $X(t,\omega)$, $t \geq 0$, is a Brownian motion. First, we observe that each $X_n(t)$, $t \geq 0$, is a centered Gaussian process (the $X_n(t)$ are finite linear combinations of the i.i.d. $N(0,1)$-distributed ξ_ℓ, $\ell \geq 0$, $\xi_{m,k}$, $m, k \geq 0$). Note also that

$$\text{for } t \geq 0,\ X_n(t,\omega) \overset{L^2(\Omega,\mathcal{A},P)}{\longrightarrow} X(t,\omega)\,. \tag{1.37}$$

Indeed the ξ_ℓ, $\xi_{m,k}$ are orthogonal in $L^2(P)$ so that, cf. (1.30), $X_n(t)$ and $X_{n+m}(t) - X_n(t)$ are orthogonal. So, for $1 \leq m < \ell$, $E[(X_\ell(t) - X_m(t))^2] = \sum_{m < k \leq \ell} E[(X_k(t) - X_{k-1}(t))^2]$, and to prove that $X_n(t)$, $n \geq 1$ is a Cauchy sequence in $L^2(\Omega, \mathcal{A}, P)$, it suffices to show that $\sum_{k \geq 1} E[(X_k(t) - X_{k-1}(t))^2] < \infty$ (with $X_0(t) = 0$, by convention). Since $E[X_n(t)^2] = \sum_{1 \leq k \leq n} E[(X_k(t) - X_{k-1}(t))^2]$, we only need to check that $\sup_n E[X_n^2(t)] < \infty$, that is:

$$\sum_{\ell \geq 0} \Phi_\ell(t)^2 + \sum_{m,k \geq 0} \Phi_{m,k}(t)^2 < \infty\,. \tag{1.38}$$

To check this last point, we observe that the above sum equals:

$$\begin{aligned} &\sum_{\ell \geq 0} \Big(\int_0^t \varphi_\ell(u)du\Big)^2 + \sum_{m,k \geq 0} \Big(\int_0^t \varphi_{m,k}(u)du\Big)^2 \underset{\text{Parseval relation}}{\overset{(1.26)}{=}} \\ &\|1_{[0,t]}\|^2_{L^2(\mathbb{R}_+,du)} = t < \infty\,. \end{aligned} \tag{1.39}$$

In a very similar vein, we calculate $E[X_n(s)\, X_n(t)]$ for $0 \le s, t$, as follows:

$$(1.40)\qquad \begin{aligned} E[X_n(s)\, X_n(t)] &\overset{(1.30)}{=} \sum_{0 \le \ell < n} \Phi_\ell(t)\, \Phi_\ell(s) + \sum_{\substack{0 \le m < n \\ 0 \le k < n2^m}} \Phi_{m,k}(t)\; \Phi_{m,k}(s) \\ &= \big\langle \pi_n(1_{[0,t]}),\; \pi_n(1_{[0,s]})\big\rangle_{L^2(\mathbb{R}_+, du)}\,, \end{aligned}$$

with π_n the orthogonal projection in $L^2(\mathbb{R}_+, du)$ on the space spanned by φ_ℓ, $0 \le \ell < n$, $\varphi_{m,k}$, $0 \le m < n$, $0 \le k < n2^m$. Combining (1.37) and (1.40), we find that

$$\begin{array}{ccc} E[X_n(s)\, X_n(t)] = \big\langle \pi_n(1_{[0,s]}),\; \pi_n(1_{[0,t]})\big\rangle_{L^2(\mathbb{R}_+, du)} \\ \downarrow^{\,n\to\infty} \qquad\qquad\qquad \downarrow^{\,n\to\infty} \\ E[X(s)\, X(t)] = \big\langle 1_{[0,s]}, 1_{[0,t]}\big\rangle_{L^2(\mathbb{R}_+, du)} = s \wedge t\,. \end{array}$$

Note that weak limits of centered Gaussian distributions are centered Gaussian (use characteristic functions). It follows that linear combinations of $X(t)$, which are limit, in $L^2(P)$ by (1.37), and thus in distribution, of linear combinations of $X_n(t)$, are centered Gaussian variables.

So $X(t,\omega)$, $t \ge 0$, is a centered Gaussian process. From the above calculation $\Gamma(s,t) = s \wedge t$, and due to (1.32) P-a.s., $t \ge 0 \to X(t,\omega)$ is continuous.

We have thus proved that $X(t,\omega)$ is a Brownian motion on the probability space $(\Omega, \mathcal{A}, P)$ selected above (1.30).

Complement:

We will now discuss another construction of Brownian motion B_t, $0 \le t \le 1$, on the time interval $[0,1]$, which gives a proof of a result of Paley and Wiener (1934). This approach will **bring into play some important methods on how to control the modulus of continuity of a stochastic process**.

In place of the complete orthonormal basis of $L^2(\mathbb{R}_+, dt)$ given by the Haar functions, cf. (1.25), (1.26), we consider

(1.41) $\qquad \varphi_k(t)$, $0 \le t \le 1$, $k \ge 0$, an orthonormal basis of $L^2([0,1], dt)$,

as well as the sequence

$$(1.42)\qquad \Phi_k(t) = \int_0^t \varphi_k(u)du,\; 0 \le t \le 1,\; k \ge 0\,.$$

Example: (Paley and Wiener)

The concrete choice of Paley and Wiener (1934) corresponds to:

$$(1.43)\qquad \begin{aligned} &\varphi_0 \equiv 1,\ \varphi_k(t) = \sqrt{2}\, \cos(k\pi t),\ k \ge 1,\ 0 \le t \le 1, \text{ so that} \\ &\Phi_0(t) = t, \text{ and } \Phi_k(t) = \frac{\sqrt{2}}{k\pi}\, \sin(k\pi t),\ 0 \le t \le 1,\ k \ge 1\,. \end{aligned}$$

In the spirit of (1.30), we consider on some probability space $(\Omega, \mathcal{A}, P)$ (for instance $\Omega = (0,1)$, $\mathcal{A} = \mathcal{B}((0,1))$, $P =$ Lebesgue measure on $(0,1)$), a sequence ξ_k, $k \ge 0$, of i.i.d. $N(0,1)$-distributed variables.

Similarly to (1.30), we define for $n \ge 0$, $0 \le t \le 1$,

$$X_n(t) = \sum_{0 \le k \le n} \Phi_k(t)\, \xi_k\,. \tag{1.44}$$

(So, in the situation corresponding to the choice (1.43) of Paley and Wiener:

$$X_n(t) = t\xi_0 + \sum_{1 \le k \le n} \frac{\sqrt{2}}{k\pi}\, \sin(k\pi t)\xi_k, \text{ for } 0 \le t \le 1,\ n \ge 0\ ). \tag{1.45}$$

We will see that P-a.s., $X_n(t,\omega)$ converges uniformly on $[0,1]$ to $X(t,\omega)$ distributed as the time restriction to $[0,1]$ of a Brownian motion. Here again, the most delicate point has to do with the fact that the convergence is P-a.s. uniform on $[0,1]$. However, unlike for the proof of (1.32), we cannot make use of the special properties of the orthonormal basis (see for instance (1.34)). This will bring into play interesting considerations. We begin with a lemma concerning functions.

Lemma 1.8. $(T > 0)$

For $r, \gamma > 0$, and $f \in C([0,T], \mathbb{R}^d)$, define

$$I = \int_{[0,T]^2} \Big(\frac{|f(t) - f(s)|}{|t-s|^\gamma}\Big)^r\, ds\, dt\,. \tag{1.46}$$

Assume $I < \infty$. Then for $0 \le s < t \le T$, one has

$$|f(t) - f(s)| \le 8 \int_0^{t-s} \Big(\frac{4I}{u^2}\Big)^{\frac{1}{r}}\, \gamma\, u^{\gamma - 1} du \tag{1.47}$$

and hence, for $\frac{2}{r} < \gamma$, one has

$$|f(t) - f(s)| \le 8\, \frac{\gamma}{\gamma - \frac{2}{r}}\, (4I)^{\frac{1}{r}}\ |t-s|^{\gamma - \frac{2}{r}}, \ \textit{for}\ 0 \le s < t \le T\,. \tag{1.47'}$$

Proof. We only need to prove (1.47) when

$$t = 1 = T,\ s = 0, \text{ and } \frac{2}{r} < \gamma\,. \tag{1.48}$$

The restriction $\frac{2}{r} < \gamma$ is clear (otherwise the right-hand side is infinite) and note that given f and $0 \le s < t \le T$, as in Lemma 1.8, we can define

$$\overline{f}(\tau) = f(s + (t-s)\,\tau),\ 0 \le \tau \le 1\,,$$

so that if we know that (1.47) holds under (1.48), we find that:

$$
(1.49)\qquad \begin{aligned}
&|f(t)-f(s)) = |\overline{f}(1)-\overline{f}(0)| \le 8\,\frac{\gamma}{\gamma-\frac{2}{r}}\,(4\overline{I})^{\frac{1}{r}}, \text{ with}\\
&\overline{I} = \int_{[0,1]^2} \Big(\frac{|\overline{f}(u)-\overline{f}(v)|}{|u-v|^\gamma}\Big)^r du\,dv \overset{\substack{\text{change of}\\ \text{variable}}}{\le} (t-s)^{-2+r\gamma} I
\end{aligned}
$$

and inserting in (1.49) we find (1.47') (and (1.47) as well).

We thus assume (1.48). We define for $0 \le u \le 1$,

$$
(1.50)\qquad J(u) = \int_0^1 \Big(\frac{|f(u)-f(v)|}{|u-v|^\gamma}\Big)^r dv\,.
$$

Since $\int_0^1 J(u)du = I$, there is a $t_0 \in (0,1)$ s.t. $J(t_0) \le I$. We will show that

$$
(1.51)\qquad \begin{cases}
\text{There are } t_n,\ n \ge 0, \text{ in (0,1) such that with } d_n^\gamma = \frac{1}{2}\,t_n^\gamma,\\
t_{n+1} \in (0,d_n),\\
J(t_{n+1}) \le \dfrac{2I}{d_n}, \text{ and } \Big(\dfrac{|f(t_{n+1})-f(t_n)|}{|t_{n+1}-t_n|^\gamma}\Big)^r \le \dfrac{2J(t_n)}{d_n}\,.
\end{cases}
$$

Indeed given t_n, define d_n as indicated, and note that

$$
\underset{\substack{\uparrow\\ \text{Lebesgue measure}}}{\Big|\Big\{u \in (0,d_n);\ J(u) > \frac{2I}{d_n}\Big\}\Big|} < \frac{d_n}{2}\,, \text{ since } \int_0^1 J(u)du = I
$$

and

$$
\Big|\Big\{u \in (0,d_n);\ \Big(\frac{|f(u)-f(t_n)|}{|u-t_n|^\gamma}\Big)^r > \frac{2J(t_n)}{d_n}\Big\}\Big| < \frac{d_n}{2},
$$
$$
\text{since } \int \Big(\frac{|f(u)-f(t_n)|}{|u-t_n|^\gamma}\Big)^r du = J(t_n)\,.
$$

Hence we must be able to find t_{n+1} in $(0,d_n)$, for which the last line of (1.51) holds. This proves (1.51). We now have

$$
(1.52)\qquad 1 > t_0 > d_0 > t_1 > d_1 > \dots > t_n > d_n > \dots\,,
$$

and since $d_{n+1}^\gamma \le \frac{1}{2}\,d_n^\gamma$, $d_n \to 0$, as $n \to \infty$.

Note also that:

$$
(1.53)\qquad |t_n - t_{n+1}|^\gamma \le t_n^\gamma = 2d_n^\gamma = 4\Big(d_n^\gamma - \frac{1}{2}\,d_n^\gamma\Big) \le 4(d_n^\gamma - d_{n+1}^\gamma)\,.
$$

From the last line of (1.51), with the convention $d_{-1} = 1$, we find for $n \geq 0$:

$$\begin{aligned}|f(t_n) - f(t_{n+1})| &\overset{(1.51)}{\leq} \Big(\frac{2J(t_n)}{d_n}\Big)^{\frac{1}{r}} |t_n - t_{n+1}|^\gamma \overset{(1.51)}{\leq} \Big(\frac{4I}{d_n\, d_{n-1}}\Big)^{\frac{1}{r}} |t_n - t_{n+1}|^\gamma \\ &\overset{(1.53)}{\leq} 4\Big(\frac{4I}{d_n\, d_{n-1}}\Big)^{\frac{1}{r}} (d_n^\gamma - d_{n+1}^\gamma) \leq 4 \int_{d_{n+1}}^{d_n} \Big(\frac{4I}{u^2}\Big)^{\frac{1}{r}} \gamma u^{\gamma-1}\, du\,. \end{aligned} \tag{1.54}$$

Summing over n, we find:

$$|f(t_0) - f(0)| \leq 4 \int_0^{t_0} \Big(\frac{4I}{u^2}\Big)^{\frac{1}{r}} \gamma u^{\gamma-1}\, du\,. \tag{1.55}$$

If we introduce the function $\widetilde{f}(\cdot) = f(1-\cdot)$, for which the corresponding $\widetilde{I}$ of course equals I, we can pick $\widetilde{t_0} = 1 - t_0$, and obtain with (1.55) that

$$|\widetilde{f}(1-t_0) - \widetilde{f}(0)| = |f(t_0) - f(1)| \leq 4 \int_0^{\widetilde{t_0}} \Big(\frac{4I}{u^2}\Big)^{\frac{1}{r}} \gamma u^{\gamma-1}\, du\,,$$

and combining with (1.55) deduce that

$$|f(1) - f(0)| \leq 8 \int_0^1 \Big(\frac{4I}{u^2}\Big)^{\frac{1}{r}} \gamma u^{\gamma-1}\, du\,, \tag{1.56}$$

thus concluding the proof of Lemma 1.8. □

Remark 1.9. The above lemma is a special case of a more general result (see [12], p. 170). The interest of the lemma is that it permits to control the modulus of continuity of f in terms of an integral of $f(\cdot)$ corresponding to I. This will be handy when proving Kolmogorov's criterion below. The quantity I is also related to certain Besov norms (see for instance [1], p. 214). □

As an application of Lemma 1.8 we have

Theorem 1.10. *(Kolmogorov's criterion)*

If $X_n(t,\omega)$, $0 \leq t \leq T$, $n \geq 1$, are d-dimensional stochastic processes on $(\Omega, \mathcal{A}, P)$ with continuous trajectories such that for $r > 0$, $\alpha > 0$,

$$E\big[\sup_{n \geq 1} |X_n(t) - X_n(s)|^r\big] \leq C\, |t-s|^{1+\alpha}, \text{ for } 0 \leq s \leq t \leq T\,, \tag{1.57}$$

then for each $\beta \in (0, \frac{\alpha}{r})$ there is a $K(r, \alpha, \beta, T) > 0$, such that:

$$P\Big[\sup_{n \geq 1, 0 \leq s < t \leq T} \frac{|X_n(t) - X_n(s)|}{(t-s)^\beta} \geq R\Big] \leq \frac{KC}{R^r}, \text{ for all } R > 0\,. \tag{1.58}$$

(Note that the processes $X_n(\cdot)$ may very well all coincide with $X_1(\cdot)$.)

Proof. Consider $\beta \in (0, \frac{\alpha}{r})$ and set

$$\gamma \overset{\text{def}}{=} \frac{2}{r} + \beta < \frac{2+\alpha}{r}\,. \tag{1.59}$$

Then, observe that (1.57) and Fubini's theorem imply that

$$E\Big[\underbrace{\int_{[0,T]^2} \sup_{n\ge 1} \frac{|X_n(t) - X_n(s)|^r}{|t-s|^{r\gamma}}\, dt\, ds}_{\overset{\text{def}}{=}\, J(\omega)}\Big] \le \int_{[0,T]^2} C\,|t-s|^{\underbrace{1+\alpha-r\gamma}_{\uparrow\; > -1 \text{ by } (1.59)}} ds\, dt \tag{1.60}$$

$$\overset{\text{def}}{=} C\, K_1(r,\alpha,\beta,T) < \infty\,.$$

By Lemma 1.8, we know that for $0 \le s \le t \le T$,

$$\sup_{n\ge 1} |X_n(t,\omega) - X_n(s,\omega)| \overset{(1.47')}{\le} 8\, \frac{\gamma}{\beta}\, (4J(\omega))^{\frac{1}{r}} (t-s)^{\beta}\,. \tag{1.61}$$

It thus follows that for $R > 0$

$$P\Big[\sup_{n\ge 1} \frac{|X_n(t,\omega) - X_n(s,\omega)|}{|t-s|^{\beta}} \ge R\Big] \overset{(1.61)}{\le} P\Big[8^r\Big(\frac{\gamma}{\beta}\Big)^r\, 4\,J \ge R^r\Big] \underset{\text{inequality}}{\overset{\text{Markov}}{\le}}$$
$$4\cdot 8^r \Big(\frac{\gamma}{\beta}\Big)^r\, \frac{E[J]}{R^r} \overset{(1.60)}{\le} 4\cdot 8^r \Big(\frac{\gamma}{\beta}\Big)^r\, \frac{CK_1}{R^r}\,,$$

and (1.58) follows. □

Kolmogorov's criterion provides a powerful tool to estimate the modulus of continuity of stochastic processes.

The next result in the special case of (1.43), (1.45), recovers a famous result of Paley and Wiener (1934).

Theorem 1.11. *(under (1.41), (1.42))*

(1.62) *P-a.s., $X_n(t) = \sum_{0\le k\le n} \Phi_k(t)\,\xi_k$, $0 \le t \le 1$, converges uniformly on $[0,1]$ to $X(t,\omega)$, $0 \le t \le 1$, which has the same law on $C([0,1]),\mathbb{R})$ as B_t, $0 \le t \le 1$, if B_t, $t \ge 0$, is a Brownian motion.*

Proof. We introduce the filtration

$$\mathcal{F}_n = \sigma(\xi_0, \xi_1, \ldots, \xi_n),\; n \ge 0\,, \tag{1.63}$$

and note that for each $t \in [0,1]$, and $p \ge 1$:

(1.64) $X_n(t)$ is an $\mathcal{F}_n$-martingale, with finite L^p-norm.

It then follows from Doob's inequality (with $p = 4$), that for $0 \le s \le t \le 1$:

$$E\big[\sup_{n\ge 0} (X_n(t) - X_n(s))^4\big] \le \Big(\frac{4}{3}\Big)^4 \sup_{n \ge 0} E[(X_n(t) - X_n(s))^4]\,. \tag{1.65}$$

Note that $X_n(t) - X_n(s)$ is a centered Gaussian variable with variance

$$\begin{aligned} E\big[\big(X_n(t) - X_n(s)\big)^2\big] &\overset{(1.44)}{\underset{(1.42)}{=}} \sum_{0\le k\le n} \Big(\int_s^t \varphi_k(u)du\Big)^2 \le \sum_{k\ge 0} \Big(\int_s^t \varphi_k(u)du\Big)^2 \\ &\overset{\text{Parseval}}{\underset{\text{identity}}{=}} \|1_{[s,t]}\|^2_{L^2([0,1],du)} = t - s\,. \end{aligned} \tag{1.66}$$

Now for Z an $N(0,1)$ variable we have

$$E[Z^4] = 3\,,$$

$$\Big(\text{indeed: } \int_{\mathbb{R}} x^4\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} \overset{\text{integration}}{\underset{\text{by parts}}{=}} \Big[x^3\Big(-\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Big)\Big]_{-\infty}^{\infty} + \int_{\mathbb{R}} 3x^2\, e^{-\frac{x^2}{2}}\, \frac{dx}{\sqrt{2\pi}} = 3\Big)\,,$$

and hence, by scaling we find that for Z, $N(0,\sigma^2)$ distributed,

$$E[Z^4] = 3\sigma^4\,. \tag{1.67}$$

Combining this identity with (1.65), (1.66), we see that

$$E\big[\sup_{n\ge 0} (X_n(t) - X_n(s))^4\big] \le C(t-s)^2, \text{ for } 0 \le s \le t \le 1,\ \Big(\text{with } C = 3\Big(\frac{4}{3}\Big)^4\Big)\,. \tag{1.68}$$

We can then apply Kolmogorov's criterion, with $\alpha = 1, r = 4$, in (1.57) and choosing $\beta = \frac{1}{8} (< \frac{1}{4} = \frac{\alpha}{r})$, we deduce from (1.58) that

$$P\text{-a.s.,} \sup_{n\ge 1, 0\le s<t\le 1} \frac{|X_n(t) - X_n(s)|}{(t-s)^{1/8}} < \infty\,. \tag{1.69}$$

Since $X_n(0) = 0$, for each n, it follows from Ascoli-Arzela's theorem that

$$\begin{aligned} &P\text{-a.s., } X_n(\cdot,\omega),\ n\ge 0, \text{ is a relatively compact sequence in } C([0,1],\mathbb{R}) \\ &\qquad \text{endowed with the topology of uniform convergence}\,. \end{aligned} \tag{1.70}$$

By (1.68) with $s = 0$, (1.64) and the martingale convergence theorem we also see that

$$P\text{-a.s., for all } t \in \mathbb{Q}\cap(0,1),\ X_n(t,\omega) \text{ has a finite limit}\,. \tag{1.71}$$

Combining (1.70) and (1.71), we have thus shown that

$$P\text{-a.s., } X_n(t,\omega) \text{ converges uniformly on } [0,1] \tag{1.72}$$

(this plays the role of (1.32)).

We can thus define a stochastic process $X(t,\omega)$, $0 \le t \le 1$, such that

$$
\begin{aligned}
&t \in [0,1] \to X(t,\omega) \text{ is continuous for each } \omega \in \Omega, \text{ and} \\
&P\text{-a.s.}, \lim_n \sup_{t\in[0,1]} |X_n(t,\omega) - X(t,\omega)| = 0\,.
\end{aligned}
\tag{1.73}
$$

For each $t \in [0,1]$ the $(\mathcal{F}_n)$-martingale $X_n(t)$ also converges in $L^2(P)$, due to (1.66), with $s = 0$, see also below (1.37). Proceeding as in (1.40) (see also (1.66)), we find that for $0 \le s,\, t \le 1$:

$$
\begin{aligned}
&E[X_n(s)\,X_n(t)] \underset{n}{\longrightarrow} \big\langle 1_{[0,s]}, 1_{[0,t]}\big\rangle_{L^2([0,1],du)} = s \wedge t \\
&\qquad\quad \downarrow n \to \infty \\
&E[X(s)\,X(t)] = s \wedge t\,.
\end{aligned}
\tag{1.74}
$$

Moreover $X(t)$, $0 \le t \le 1$, by the same arguments as on page 15 is also a centered Gaussian process. Applying (1.13), we thus find that $X(t)$, $0 \le t \le 1$, has the same law on $C([0,1],\mathbb{R})$ as B_t, $0 \le t \le 1$, if B_t, $t \ge 0$, is a Brownian motion. $\square$

Chapter 2: Brownian Motion and Markov Property

We are going to successively discuss the "simple Markov property" and the "strong Markov property", and this chapter will revolve around the fact that Brownian motion is a canonical example of a continuous Markov process.

Heuristically the simple Markov property states that if one "knows" the trajectory of a Brownian motion $X_{\cdot}$ until time s, then the trajectory after time $s : X_{\cdot+s}$, given this information behaves like a Brownian motion starting from the random initial position X_s. In particular, only the knowledge of X_s matters, in this prediction of the future after time s given the past up to time s. The strong Markov property will extend this to stopping times in place of the fixed time s.

Notation: (as in Chapter 1)

$C = C(\mathbb{R}_+, \mathbb{R}^d)$,

$X_t : C \to \mathbb{R}^d,\ t \geq 0$ ("the canonical coordinates"),

$\mathcal{F} = \sigma(X_u, u \geq 0)$,

$W =$ Wiener measure on $(\Omega, \mathcal{F})$.

For $x \in \mathbb{R}^d$, "Brownian motion starting from x" is described by the probability:

$$W_x = \text{the image of } W \text{ under the map } w(\cdot) \in C \to w(\cdot) + x \in C, \tag{2.1}$$

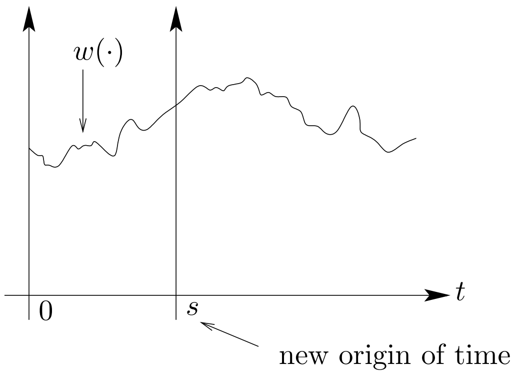
(we write E_x for the corresponding expectation).

In particular, for $h(x_0, \dots, x_n) \in b\mathcal{B}((\mathbb{R}^d)^{n+1})$, $0 = t_0 < t_1 < \dots < t_n$,

$$\begin{aligned} E_x[h(X_{t_0}, X_{t_1}, \dots, X_{t_n})] &= E^W[h(x + X_0, x + X_{t_1}, \dots, x + X_{t_n})] \\ &\overset{(1.7)}{=} \int_{(\mathbb{R}^d)^n} h(x, x_1, \dots, x_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-\frac{d}{2}} \times \\ &\exp\Big\{ - \sum_{i=1}^n \frac{|x_i - x_{i-1}|^2}{2(t_i - t_{i-1})} \Big\}\, dx_1 \dots dx_n \text{ (with } x_0 = x). \end{aligned} \tag{2.2}$$

On C we have the time-shift operators:

$$\text{for } s \geq 0,\ (C, \mathcal{F}) \underset{\substack{\uparrow \\ \text{measurable by (1.5)}}}{\xrightarrow{\theta_s}} (C, \mathcal{F}),\ \theta_s(w)(\cdot) = w(s + \cdot)\,. \tag{2.3}$$



Note that:

$$f(X_{t_0},\dots,X_{t_n})\circ\theta_s(w) = f(w(s+t_0),\ w(s+t_1),\dots,w(s+t_n)). \tag{2.4}$$

$\nwarrow$ concerns the trajectory after time s

The information contained in the part of the trajectory up to time s is described by:

$$\mathcal{F}_s = \sigma(X_u, u \le s), \text{ and} \tag{2.5}$$

$$\mathcal{F}_s^+ = \bigcap_{\varepsilon>0} \mathcal{F}_{s+\varepsilon}\ (\supseteq \mathcal{F}_s), \tag{2.6}$$

so that "**$\boldsymbol{\mathcal{F}_s^+}$ peeks infinitesimally into the future after time $\boldsymbol{s}$**". For instance the event "the trajectory immediately leaves its starting point":

$$A = \bigcap_{n\ge1}\Big(\bigcup_{r\in\mathbb{Q}\cap[0,\frac{1}{n}]} \{X_r \ne X_0\}\Big) \text{ is in } \mathcal{F}_0^+ \text{ but not in } \mathcal{F}_0\,.$$

Theorem 2.1. *(simple Markov property)*

$Y \in b\mathcal{F}$, $s \ge 0$, $x \in \mathbb{R}^d$, then

$$E_x[Y\circ\theta_s \,|\, \mathcal{F}_s^+] = E_{X_s}[Y],\ W_x\text{-a.s.}. \tag{2.7}$$

$$\text{Under } W_x,\ (X_{s+u}-X_s)_{u\ge0} \text{ is a Brownian motion independent of } \mathcal{F}_s^+\,. \tag{2.8}$$

Proof.

• (2.7): Note that

$$y \in \mathbb{R}^d \to E_y[Y] \text{ is in } b\mathcal{B}(\mathbb{R}^d) \text{ for } Y \in b\mathcal{F}\,. \tag{2.9}$$

Indeed this is true when $Y = 1_{A_0}\circ X_{t_0}\dots 1_{A_n}\circ X_{t_n}$, with $t_0 < t_1 < \dots < t_n$ and $A_0,\dots,A_n \in \mathcal{B}(\mathbb{R}^d)$, thanks to (2.2). We can then use Dynkin's lemma to conclude that this is also true when $Y = 1_F$, with $F \in \mathcal{F}$, and then approximate a general $Y \in b\mathcal{F}$ by step-functions of the form $\sum_{i=1}^m \lambda_i\, 1_{F_i}$ to obtain (2.9).

Then, note that for $u_0 = 0 < \cdots < u_n = s$, $t_0 = 0 < \cdots < t_k$, with f, g bounded measurable one has

$$
\begin{aligned}
&E_x[f(X_{u_0}, \dots, X_{u_n})\, g(X_{s+t_0}, \dots, X_{s+t_k})] = \\
&\int_{(\mathbb{R}^d)^{n+k}} f(x, x_1, \dots, x_n)\, g(x_n, \dots, x_{n+k}) \prod_{i=1}^n [2\pi(u_i - u_{i-1})]^{-\frac{d}{2}} \\
&\prod_{j=1}^k [2\pi(t_j - t_{j-1})]^{-\frac{d}{2}} \ \exp\Big\{ -\frac{1}{2} \sum_{i=1}^n \frac{|x_i - x_{i-1}|^2}{u_i - u_{i-1}} \\
&\qquad\qquad\qquad -\frac{1}{2} \sum_{j=1}^k \frac{|x_{n+j} - x_{n+j-1}|^2}{t_j - t_{j-1}} \Big\}\, dx_1 \dots dx_{n+k} \\
&\overset{(2.2)}{=} \int_{(\mathbb{R}^d)^n} f(x, x_1, \dots, x_n)\ E_{x_n}[g(X_{t_0}, \dots, X_{t_k})] \exp\Big\{ -\frac{1}{2} \sum_{i=1}^n \frac{|x_i - x_{i-1}|^2}{u_i - u_{i-1}} \Big\} \\
&\prod_{i=1}^n [2\pi(u_i - u_{i-1})]^{-\frac{d}{2}}\, dx_1 \dots dx_n \overset{(2.2)}{=} \\
&E_x[f(X_0(w), \dots, X_{u_n}(w))\ \underset{\substack{\uparrow \\ \text{this is the function: } w \to E_{X_s(w)}[g(X_{t_0}, \dots, X_{t_k})]\,.}}{E_{X_s(w)}}[g(X_{t_0}, \dots, X_{t_k})]]
\end{aligned}
\tag{2.10}
$$

Using Dynkin's lemma, we see that for $s \geq 0$, and $A \in \mathcal{F}_s$:

$$
E_x[1_A\, g(X_{s+t_0}, \dots, X_{s+t_k})] = E_x[1_A\, E_{X_s}[g(X_{t_0}, \dots, X_{t_k})]]\,. \tag{2.11}
$$

If we now pick g continuous and bounded, clearly

$$
y \in \mathbb{R}^d \to E_y[g(X_{t_0}, \dots, X_{t_k})] = E_0[g(X_{t_0} + y, \dots, X_{t_k} + y)] \in \mathbb{R} \tag{2.12}
$$

is a continuous bounded function thanks to dominated convergence. We can apply (2.11) with $s+\varepsilon$ in place of s, $A \in \mathcal{F}_s^+ \subseteq \mathcal{F}_{s+\varepsilon}$, and letting $\varepsilon \to 0$, see, by dominated convergence, that (2.11) holds for $s \geq 0$, $A \in \mathcal{F}_s^+$, and g bounded continuous. Then, by approximation, it holds for g of the form

$$
g(x_0, \dots, x_k) = \prod_{i=0}^k 1_{K_i}(x_i), \text{ with } K_i, i = 0, \dots, k, \text{ closed in } \mathbb{R}^d\,.
$$

Using Dynkin's lemma once more we see that for $s \geq 0$, $A \in \mathcal{F}_s^+$, $A' \in \mathcal{F}$, $Y = 1_{A'}$:

$$
E_x[1_A\, Y \circ \theta_s] = E_x[1_A\, E_{X_s}[Y]]\,. \tag{2.13}
$$

Then, using a uniform approximation by step functions, we see that (2.13) holds for $Y \in b\mathcal{F}$. This proves (2.7).

• (2.8): $(X_{s+u} - X_s)_{u \ge 0}$ has continuous trajectories, and for $f \in b\mathcal{B}((\mathbb{R}^d)^{n+1})$, $0 = t_0 < t_1 < \cdots < t_n$, W_x-a.s.,

$$\begin{aligned} &E_x[f(X_{s+t_0} - X_s, \dots, X_{s+t_n} - X_s) \,|\, \mathcal{F}_s^+] = \\ &E_x[f(X_{t_0} - X_0, \dots, X_{t_n} - X_0) \circ \theta_s \,|\, \mathcal{F}_s^+] \overset{(2.7)}{=} \\ &E_{X_s}[f(X_{t_0} - X_0, \dots, X_{t_n} - X_0)] \overset{(2.1)}{=} E_0[f(X_{t_0}, \dots, X_{t_n})] \,. \end{aligned} \tag{2.14}$$

It now readily follows that $(X_{s+u} - X_s)_{u \ge 0}$ fulfills (1.1), and is a Brownian motion on $(C, \mathcal{F}, W_x)$. Moreover it is straightforward from (2.14) (with Dynkin's lemma) to see that for any $F(\cdot) : C \to \mathbb{R}$, bounded measurable $F(X_{s+\cdot} - X_s)$ is independent of $\mathcal{F}_s^+$. This proves (2.8). □

Corollary 2.2. *(Blumenthal's $0-1$ law)*

$$\text{For any } x \in \mathbb{R}^d,\ W_x(A) \in \{0,1\},\ \text{when } A \in \mathcal{F}_0^+ \,. \tag{2.15}$$

Proof. $1_A \circ \theta_0 = 1_A$, since θ_0 is the identity map. Therefore, we find that

$$E_x[1_A \,|\, \mathcal{F}_0^+] = 1_A,\ W_x\text{-a.s. (since } A \in \mathcal{F}_0^+),$$

and

$$E_x[1_A \,|\, \mathcal{F}_0^+] = E_x[1_A \circ \theta_0 \,|\, \mathcal{F}_0^+] \overset{(2.7)}{=} E_{X_0}[1_A] = W_x(A),\ W_x\text{-a.s., since } W_x(X_0 = x) = 1 \,.$$

As a result, W_x-a.s., $1_A = W_x(A)$, and the claim (2.15) follows. □

As we now see, the σ-algebra $\mathcal{F}_0^+$ contains some interesting events and this explains the interest of Blumenthal's $0-1$ law.

Examples:

1) $d = 1$, let $\widetilde{H}_+ \overset{\text{def}}{=} \inf\{s > 0; X_s > 0\}$, $\widetilde{H}_- \overset{\text{def}}{=} \inf\{s > 0; X_s < 0\}$ denote the respective hitting times of $(0, \infty)$ and $(-\infty, 0)$.

Proposition 2.3.

$$W_0\text{-a.s., } \widetilde{H}_+ = \widetilde{H}_- = 0 \,. \tag{2.16}$$

Proof.
$$\{\widetilde{H}_+ = 0\} = \bigcap_{n\ge 1} \Big(\bigcup_{r\in(0,\frac{1}{n}]\cap\mathbb{Q}} \{X_r \in (0,\infty)\}\Big) \in \mathcal{F}_0^+$$

and for $t > 0$,
$$W_0(\widetilde{H}_+ \le t) \ge W_0(X_t > 0) = \frac{1}{2}$$
$$\swarrow \text{ decreases for } t \to 0$$
$$W_0(\widetilde{H}_+ = 0) \ge \frac{1}{2}\,.$$

Thus by (2.15), we find that $W_0(\widetilde{H}_+ = 0) = 1$.

Of course, in the same way $W_0(\widetilde{H}_- = 0) = 1$. □

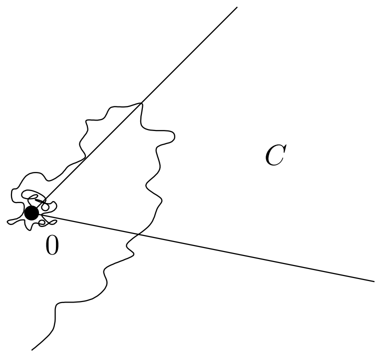
2) $d \ge 2$, C some open cone with tip 0 in $\mathbb{R}^d$
(i.e. C is open, and for $x \in \mathbb{R}^d$, $x \in C \Longleftrightarrow \lambda x \in C$, for all $\lambda > 0$).

Define the **hitting time** of C:

(2.17)
$$\widetilde{H}_C = \inf\{s > 0;\; X_s \in C\}\,.$$

Proposition 2.4.

(2.18)
$$W_0\text{-}a.s.,\ \widetilde{H}_C = 0\,.$$



Proof. The argument is similar to the proof of (2.16).

We use the fact that $\{\widetilde{H}_C = 0\} \in \mathcal{F}_0^+$ and
$$W_0(\widetilde{H}_C \le t) \ge W_0(X_t \in C) \overset{\text{scaling}}{=} W_0(\sqrt{t}\, X_1 \in C)$$
$$\underset{\text{with tip 0}}{\overset{C \text{ is a cone}}{=}} W_0(X_1 \in C) > 0\,.$$

One then concludes as for (2.16). □

3) $d = 1$, $t_n > 0$, $n \ge 1$, with $\lim t_n = 0$.

Proposition 2.5.

$$\text{(2.19)} \qquad W_0\text{-}a.s., \ \overline{\lim_n} \ \frac{X_{t_n}}{\sqrt{t_n}} = \infty\,.$$

Proof. For $c > 0$, note that $A \overset{\text{def}}{=} \limsup\limits_n \{X_{t_n} > c\sqrt{t_n}\} \in \mathcal{F}_0^+$. Indeed

$$A \overset{\text{def}}{=} \bigcap_{n \ge 1} A_n = \bigcap_{n \ge n_0} A_n \text{ with } A_n = \bigcup_{m \ge n} \{X_{t_m} > c\sqrt{t_m}\}\,,$$

so $A \in \mathcal{F}_\varepsilon$, if $t_m \le \varepsilon$, for $m \ge n_0$. But $\varepsilon > 0$ can be chosen arbitrary.

By (2.15), $W_0(A) = 0$ or 1, moreover A_n decreases with n, so that:

$$\text{(2.20)} \qquad \begin{aligned} W_0(A) = \lim_n \ W_0(A_n) &\ge \underline{\lim}_n \, W_0(X_{t_n} > c\sqrt{t_n}) \\ &\overset{\text{scaling}}{=} W_0(X_1 > c) > 0\,. \end{aligned}$$

As a result $W_0(A) = 1$, for arbitrary $c > 0$.

In particular, W_0-a.s., $\overline{\lim}_n \ \frac{X_{t_n}}{\sqrt{t_n}} \ge c$, and choosing $c = k$, $k \ge 1$, we obtain (2.19). □

Exercise 2.6. Show that:

Under W_0, the asymptotic σ-field $\mathcal{A} = \bigcap_{u < \infty} \sigma(X_v, v \ge u)$ is trivial, that is

$$W_0(A) \in \{0, 1\}, \text{ for any } A \in \mathcal{A}\,.$$

In fact more is true:

$$A \in \mathcal{A} \Longrightarrow W_x(A) = 0, \text{ for all } x \in \mathbb{R}^d, \text{ or } W_x(A) = 1, \text{ for all } x \in \mathbb{R}^d\,.$$

(Hint: use (1.21), Blumenthal's $0-1$ law and the Markov property). □

We continue our discussion of the simple Markov property, and will in the spirit of (1.15) (in the case of Gaussian processes), provide a Markovian characterization of Brownian motion. For this purpose, we introduce the **Brownian transition semigroup:**

$$\text{(2.21)} \qquad \begin{aligned} R_t\, f(x) &= E_x[f(X_t)], \ x \in \mathbb{R}^d, t \ge 0, f \in b\mathcal{B}(\mathbb{R}^d) \\ &= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} f(y) \exp\Big\{ - \frac{|y-x|^2}{2t} \Big\}\, dy, \text{ when } t > 0\,. \end{aligned}$$

R_t, $t \ge 0$, satisfies the **semigroup property:**

$$\text{(2.22)} \qquad R_{t+s} = R_t \circ R_s, \quad t, s \ge 0\,.$$

Indeed, one has with the help of the simple Markov property:

$$\begin{aligned} R_{t+s}\, f(x) &= E_x[f(X_{t+s})] = E_x[f(X_s) \circ \theta_t] \overset{(2.7)}{=} E_x[E_{X_t}[f(X_s)]] \\ &\overset{(2.21)}{=} E_x[R_s\, f(X_t)] \overset{(2.21)}{=} R_t(R_s\, f)(x)\,. \end{aligned}$$

One then has the following Markovian characterization of Brownian motion (compare with (1.15) for Gaussian processes):

Proposition 2.7. *Let B_t, $t \geq 0$, be an $\mathbb{R}^d$-valued process defined on $(\Omega, \mathcal{A}, P)$, with P-a.s., continuous trajectories, and $\mathcal{G}_s = \sigma(B_u, u \leq s)$. Then*

$$\text{(2.23)} \qquad \begin{aligned} &B_t, t \geq 0, \text{ is a Brownian motion} \iff B_0 = 0,\ P\text{-a.s., and} \\ &E[f(B_{t+s})|\mathcal{G}_s] = R_t\, f(B_s),\ P\text{-a.s., for } f \in b\mathcal{B}(\mathbb{R}^d) \text{ and } t, s \geq 0\,. \end{aligned}$$

Proof.

- $\Longrightarrow$: Using (1.7) and (2.21), if B_t, $t \geq 0$, is a Brownian motion, for $0 = t_0 < \cdots < t_n = s$, $t > 0$, and $f_0, \ldots, f_n, f \in b\mathcal{B}(\mathbb{R}^d)$:

  $$E[f_0(B_{t_0}) \ldots f_n(B_{t_n})\, f(B_{t+s})] \overset{(1.7)}{=} E[f_0(B_{t_0}) \ldots f_n(B_{t_{n=s}})\, R_t\, f(B_s)]$$

  and using Dynkin's lemma, for any $G \in \mathcal{G}_s$:

  $$E[1_G\, f(B_{t+s})] = E[1_G\, R_t\, f(B_s)]\,,$$

  from which we deduce that P-a.s., $E[f(B_{t+s})\,|\,\mathcal{G}_s] = R_t\, f(B_s)$. The fact that $B_0 = 0$, P-a.s., is automatic.

- $\Longleftarrow$: By induction we see that for $t_0 = 0 < \cdots < t_n$, $f_0, \ldots, f_n \in b\mathcal{B}(\mathbb{R}^d)$:

  $$E[f_0(B_{t_0}) \ldots f_n(B_{t_n})] =$$
  $$\int_{(\mathbb{R}^d)^n} f_0(0)\, f_1(x_1) \ldots f_n(x_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-\frac{d}{2}}\, e^{-\frac{1}{2} \sum\limits_{i=1}^n \frac{|x_i - x_{i-1}|^2}{t_i - t_{i-1}}}\, dx_1, \ldots dx_n\,.$$

  As a result B_t, $t \geq 0$, has the same finite marginal distributions as X_t, $t \geq 0$, under W (= Wiener measure), cf. (1.7), and it thus satisfies (1.1). Our claim follows.

□

Strong Markov property

In order to discuss the strong Markov property we need to introduce the notion of **stopping times**.

In the case of a discrete filtration $(\Omega, \mathcal{G}, (\mathcal{G}_n)_{n \geq 0})$ (i.e. the σ-algebras $\mathcal{G}_n, n \geq 0$, $\mathcal{G}$ satisfy $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_n \subseteq \cdots \subseteq \mathcal{G}$), a stopping time is defined as a map $T : \Omega \to \mathbb{N} \cup \{\infty\}$ (recall $\mathbb{N} = \{0, 1, 2, \dots\}$), such that $\{T = n\} \in \mathcal{G}_n$, for each $n \geq 0$.

In other words "the decision to stop at a certain time n is a function of the information known by time n".

In the case when time varies in $\mathbb{R}_+ = [0, \infty)$ in place of $\mathbb{N}$, the "right way" to interpret the above sentence comes in the next:

Definition 2.8. $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})$ *where* $(\mathcal{G}_t)_{t \geq 0}$ *is assumed to be a filtration (i.e. the* σ-*algebras* $\mathcal{G}_t$, $t \geq 0$, *satisfy* $\mathcal{G}_s \subseteq \mathcal{G}_t \subseteq \mathcal{G}$, *for* $0 \leq s \leq t$*), then* $T : \Omega \to [0, \infty]$ *is a* $(\mathcal{G}_t)$*-stopping time if:*

$$\{T \leq t\} \in \mathcal{G}_t, \text{ for all } t \geq 0 \,. \tag{2.24}$$

The "σ*-algebra of the past of* T*" is defined as:*

$$\begin{aligned} &\mathcal{G}_T = \{A \in \mathcal{G};\ A \cap \{T \leq t\} \in \mathcal{G}_t, \text{ for each } t \geq 0\} \\ &\text{(this is indeed a } \sigma\text{-algebra!).} \end{aligned} \tag{2.25}$$

Remark 2.9. Note that when T is a $(\mathcal{G}_t)$-stopping time, then for any $t \geq 0$,

$$\begin{aligned} \{T < t\} &= \bigcup_{s \in \mathbb{Q} \cap [0,t)} \{T \leq s\} \text{ belongs to } \mathcal{G}_t, \textit{and} \\ \{T = t\} &= \{T \leq t\} \backslash \{T < t\} \text{ belongs to } \mathcal{G}_t. \end{aligned}$$

□

Examples:

1) Entrance time in a closed set:

Consider the canonical space $(C, \mathcal{F})$ and $\mathcal{F}_t$, $t \geq 0$, as in (2.5), as well as A a closed subset of $\mathbb{R}^d$. The entrance time of $X_.$ in A is

$$H_A = \inf\{s \geq 0; X_s \in A\} \text{ (by convention, } H_A = \infty \text{ when } \{\dots\} = \emptyset) \,. \tag{2.26}$$

We will now see that

$$H_A \text{ is an } (\mathcal{F}_t)\text{-stopping time} \,. \tag{2.27}$$

closed

Indeed, for $w \in C$, $\{s \geq 0, X_s(w) \in A\}$ is a closed subset of $\mathbb{R}_+$, which thus contains $H_A(w)$ when it is finite.

continuous

Hence for $t \geq 0$:

$$H_A(w) > t \iff \forall s \in [0,t],\ \mathrm{dist}(X_s(w), A) > 0 \iff \inf_{[0,t]} \mathrm{dist}(X_s(w), A) > 0 \,.$$

Therefore we see that

$$\{H_A > t\} = \bigcup_{n\geq 1} \bigcap_{s\in\mathbb{Q}\cap[0,t]} \Big\{\mathrm{dist}(X_s(w), A) > \tfrac{1}{n}\Big\} \in \mathcal{F}_t\,,$$

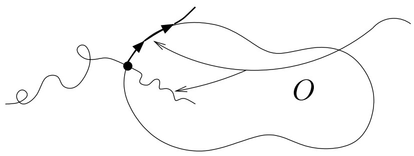
and (2.27) follows.

2) Entrance time in an open set:

We now replace A with O an open subset of $\mathbb{R}^d$, and of course set

$$H_O = \inf\{s \geq 0;\; X_s \in O\}\,. \tag{2.28}$$

Observe that in general $\{H_O = t\}$ is not $\mathcal{F}_t$-measurable (and hence H_O is not an $(\mathcal{F}_t)$-stopping time by Remark 2.9).



two possible trajectories which agree up to time t, one of which has $H_O = t$ and the other not.

One needs to "peek a little bit into the future" to decide whether $H_O = t$ or not. This motivates the use of the filtration $\mathcal{F}_t^+ (= \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon})$, $t \geq 0$.

Proposition 2.10.

(2.29) *If O is an open set of $\mathbb{R}^d$, H_O is an $(\mathcal{F}_t^+)$-stopping time.*

Proof.

$$\{H_O < s\} = \bigcup_{u\in\mathbb{Q}\cap[0,s)} \{X_u \in O\} \in \mathcal{F}_s, \text{ for } s \geq 0\,,$$

and hence

$$\{H_O \leq s\} = \bigcap_{\varepsilon>0} \{H_O < s+\varepsilon\} \in \mathcal{F}_s^+, \text{ for } s \geq 0\,.$$

□

Remark 2.11. By the above argument we also see that:

(2.30) **when** $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$ is such that the **filtration** $(\mathcal{G}_t)_{t\geq 0}$ is **right continuous** (i.e. $\mathcal{G}_t = \cap_{\varepsilon>0}\mathcal{G}_{t+\varepsilon}$, for all $t \geq 0$), then T is a $(\mathcal{G}_t)$-stopping time $\iff \{T < t\} \in \mathcal{G}_t$, for all $t > 0$.

(Indeed "$\Longrightarrow$" immediate and for "$\Longleftarrow$" : $\{T \leq s\} = \bigcap_{n\geq 1}\{\underbrace{T < s + \tfrac{1}{n}}_{\text{decreasing in } n}\}$ which belongs to $\cap_{\varepsilon>0}\mathcal{G}_{s+\varepsilon} = \mathcal{G}_s$). □

Here are now some simple useful properties:

Proposition 2.12. $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$

(2.31) $\quad$ *T stopping time $\Longrightarrow T$ is $\mathcal{G}_T$-measurable.*

(2.32) $\quad$ *S, T stopping times, then $T \wedge S$ $(= \min(T,S))$ and $T \vee S$ $(= \max(T,S))$ are stopping times.*

(2.33) $\quad$ *In the case of $(C, \mathcal{F})$, if T, S are $(\mathcal{F}_t^+)$-stopping times, then*

$$\begin{aligned} T + S \circ \theta_T &= T(w) + S(\theta_{T(w)}(w)), \text{ when } T(w) < \infty\,,\\ &= \infty, \text{ when } T(w) = \infty\,, \end{aligned}$$

is also an $(\mathcal{F}_t^+)$-stopping time.

Proof.

• (2.31):

It suffices to show that $\{T \leq u\} \in \mathcal{G}_T$, for $u \geq 0$, and indeed

$$\{T \leq u\} \cap \{T \leq t\} = \{T \leq u \wedge t\} \in \mathcal{G}_{u \wedge t} \subseteq \mathcal{G}_t, \text{ for all } t \geq 0\,,$$

and by (2.25), $\{T \leq u\} \in \mathcal{G}_T$, for all $u \geq 0$.

• (2.32):

$\{T \wedge S \leq t\} = \{T \leq t\} \cup \{S \leq t\} \in \mathcal{G}_t$, for all $t \geq 0$. Hence, $T \wedge S$ is a $(\mathcal{G}_t)$-stopping time.
$\{T \vee S \leq t\} = \{T \leq t\} \cap \{S \leq t\} \in \mathcal{G}_t$, for all $t \geq 0$. Hence, $T \vee S$ is a $(\mathcal{G}_t)$-stopping time.

• (2.33):

$(\mathcal{F}_t^+)_{t\geq 0}$ is a **right-continuous filtration** (check it!), and by (2.30) we only need to show that for $t > 0$:

$$\{T + S \circ \theta_T < t\} \in \mathcal{F}_t^+\,. \tag{2.34}$$

To this effect, note that

$$\{T + S \circ \theta_T < t\} = \bigcup_{\substack{u,v \in \mathbb{Q} \cap (0,\infty)\\ u+v<t}} \{T < u, S \circ \theta_T < v\}\,. \tag{2.35}$$

We will use the following **claim**:

(2.36) $\quad$ Assume $\{T < u\} \neq \emptyset$, then $\theta_T : (\{T < u\}, \mathcal{F}_{v+u} \cap \{T < u\}) \to (C, \mathcal{F}_v)$ is measurable.

Indeed, for $0 \leq s \leq v$, $X_s \circ \theta_T$, is measurable as a map:

$$(\{T < u\},\ \mathcal{F}_{v+u} \cap \{T < u\}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\,,$$

as follows from the equality valid for $w \in \{T < u\}$:

$$X_s \circ \theta_T(w) = X_{s+T(w)}(w) = \lim_{n\to\infty} \sum_{1\le k\le n} \underbrace{X_{s+\frac{k}{n}u}(w)}_{\substack{\text{measurable}\\ \mathcal{F}_{s+u}\cap\{T<u\}\\ \subseteq \mathcal{F}_{v+u}\cap\{T<u\}}} \; 1\underbrace{\Big\{ \tfrac{(k-1)}{n}u \le T < \tfrac{k}{n}u \Big\}}_{\substack{\in \mathcal{F}_u^+\cap\{T<u\}\\ \subseteq \mathcal{F}_{v+u}\cap\{T<u\}}} .$$

The claim (2.36) now follows from a similar argument as (1.5). On the other hand, $\{S < v\} = \bigcup_{r<v, r\in\mathbb{Q}\cap(0,\infty)} \{S < r\} \in \mathcal{F}_v$, and we see that the event in the union in the right-hand side of (2.35) satisfies:

$$\begin{aligned}&\{T < u, S\circ\theta_T < v\} = \\ &\{w \in \{T<u\},\ \theta_T(w) \in \{S < v\}\} \overset{(2.36)}{\in} \mathcal{F}_{u+v} \cap \overbrace{\{T < u\}}^{\in \mathcal{F}_u^+ \,(\text{even } \mathcal{F}_u)} \subseteq \mathcal{F}_{u+v} .\end{aligned}$$

Thus, coming back to (2.35) we have shown that $\{T + S\circ\theta_T < t\} \in \mathcal{F}_t \subseteq \mathcal{F}_t^+$, and (2.34) is proved, whence (2.33). □

Complement:

Special **characterization of** $\mathcal{F}_T$, when $\boldsymbol{T}$ **is an** $(\boldsymbol{\mathcal{F}_t})$**-stopping time** on the **canonical space** C. We have the identity:

(2.37) $$\mathcal{F}_T = \sigma(X_{T\wedge s}, s \ge 0)$$

(in other words $\mathcal{F}_T$ describes the information of the trajectory $X_.$ stopped at time T).

Proof.

• "$\supseteq$":

This is the easier direction. We need to show that

(2.38) $$X_{T\wedge s} \text{ is } \mathcal{F}_T\text{-measurable for each } s \ge 0 .$$

For this purpose we write

(2.39) $$X_{T\wedge s} = \lim_{n\to\infty} \sum_{k=0}^{\infty} X_{\frac{k}{2^n}\wedge s}\, 1\Big\{ \frac{k}{2^n} \le T < \frac{k+1}{2^n} \Big\} .$$

Observe now that for $A \in \mathcal{F}_{\frac{k}{2^n}\wedge s}$, and $u \ge 0$,

$$A \cap \Big\{ \frac{k}{2^n} \le T < \frac{k+1}{2^n} \Big\} \cap \{T \le u\}$$

is empty if $u < \frac{k}{2^n}$, and when $\frac{k}{2^n} \le u < \frac{k+1}{2^n}$, it coincides with $A \cap \{T < \frac{k}{2^n}\}^c \cap \{T \le u\} \in \mathcal{F}_u$, and when $u \ge \frac{k+1}{2^n}$, it equals $A \cap \{\frac{k}{2^n} \le T < \frac{k+1}{2^n}\} \in \mathcal{F}_u$.

It thus follows that $A \cap \{\frac{k}{2^n} \le T < \frac{k+1}{2^n}\} \in \mathcal{F}_T$. Then, using an approximation of $X_{\frac{k}{2^n}\wedge s}$ by step functions we conclude that $X_{\frac{k}{2^n}\wedge s}\, 1\{\frac{k}{2^n} \le T < \frac{k+1}{2^n}\}$ is $\mathcal{F}_T$-measurable, and (2.38) follows from (2.39).

• "$\subseteq$":

This step is more involved. We introduce the notation

(2.40) $$w_t(\cdot) \overset{\text{def}}{=} w(\cdot \wedge t) (\in C), \text{ for any } t \ge 0, \text{ and } w \in C\,.$$

We will use the following

Claim:

(2.41) $$f(w) = f(w_{T(w)}), \text{ for any } f \in b\mathcal{F}_T \text{ and } w \in C\,.$$

To see that the claim holds note that it is obviously true when $T(w) = \infty$, since $w_{T(w)} = w$ in this case, and we only need to check that

(2.42) $$f(w)\, 1\{T(w) = t\} = f(w_t)\, 1\{T(w) = t\}, \text{ for all } t \ge 0, \text{ and } f \in b\mathcal{F}_T\,.$$

To see this last point we argue as follows: Using Dynkin's lemma we find that for $t \ge 0$,

(2.43) $$Y(w) = Y(w_t), \text{ for any } Y \in b\mathcal{F}_t, \text{ and } w \in C\,.$$

and since T is an $(\mathcal{F}_t)$-stopping time, $\{T = t\} \in \mathcal{F}_t$, and hence with (2.43):

$$1\{T(w) = t\} = 1\{T(w_t) = t\}, \text{ for } t \ge 0,\ w \in C\,.$$

Similarly, when $f(\cdot)$ is $b\mathcal{F}_T$, using (2.25), (2.31), we see that

$$f(w)\, 1\{T(w) = t\} = \underbrace{f(w)\, 1\{T(w) = t\}}_{\in b\mathcal{F}_T}\, 1\{T(w) \le t\} \in b\mathcal{F}_t\,.$$

As a result, by (2.43) and the identity below (2.43), we find that for $w \in C$, $t \ge 0$,

$$f(w)\, 1\{T(w) = t\} = f(w_t)\, 1\{T(w_t) = t\} = f(w_t)\, 1\{T(w) = t\}\,,$$

and this proves (2.42) and completes the proof of (2.41).

Now, from Dynkin's lemma we see that for any $f \in b\mathcal{F}$, there exists F bounded measurable on $(\mathbb{R}^d)^{\mathbb{N}}$ and a sequence $(t_k)_{k\ge 0}$ in $[0,\infty)$ such that:

(2.44) $$f(w) = F(w(t_0), w(t_1), \dots, w(t_k), \dots)\,.$$

Applying the claim (2.41) we thus find that for $f \in b\mathcal{F}_T$

$$\begin{aligned} f(w) = f(w_{T(w)}) &\overset{(2.44)}{=} F(w_{T(w)}(t_0),\ w_{T(w)}(t_1), \dots, w_{T(w)}(t_k), \dots) \\ &\overset{(2.40)}{=} F(w(T(w) \wedge t_0),\ w(T(w) \wedge t_1), \dots, w(T(w) \wedge t_k), \dots) \\ &= F(X_{T\wedge t_0}(w),\ X_{T\wedge t_1}(w), \dots, X_{T\wedge t_k}(w), \dots) \end{aligned}$$

and this proves that $\mathcal{F}_T \subseteq \sigma(X_{T\wedge s}, s \ge 0)$. □

Exercise 2.13.

1) Show that $\mathcal{F}_T$ is generated by a countable collection of events (hint: use (2.37)).

2) Given $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$ and S, T two $(\mathcal{G}_t)$-stopping times, show that

a) if $S \leq T$, $\mathcal{G}_S \subseteq \mathcal{G}_T$,

b) $\{S < T\}$ and $\{S \leq T\}$ belong to $\mathcal{G}_S \cap \mathcal{G}_T$
(hint: write $\{S < T\} \cap \{T \leq t\} = \bigcup_{s\in\mathbb{Q}\cap[0,t]} \{S \leq s\} \cap \{T > s\} \cap \{T \leq t\}$, and $\{S < T\} \cap \{S \leq t\} = \{S \leq t\} \cap \{T > t\} \cup \{S < T\} \cap \{T \leq t\}$, and use that $\{S < T\} \in \mathcal{G}_T\}$),

c) for $A \in \mathcal{G}_S$, $A \cap \{S < T\}$ and $A \cap \{S \leq T\}$ belong to $\mathcal{G}_{S\wedge T}$.

$\square$

We continue with the discussion of the **strong Markov property**. We consider $(C, \mathcal{F})$. We recall that $(\mathcal{F}_t^+)_{t\geq 0}$ is a right-continuous filtration, see (2.30) and above (2.34). We further observe that:

$$\text{for } T \text{ an } (\mathcal{F}_t^+)\text{-stopping time, } \mathcal{F}_T^+ \overset{(2.25)}{=} \{A \in \mathcal{F};\ A \cap \{T \leq t\} \in \mathcal{F}_t^+,\ \forall t \geq 0\} \\ = \{A \in \mathcal{F};\ A \cap \{T < t\} \in \mathcal{F}_t^+,\ \forall t \geq 0\}. \tag{2.45}$$

Indeed "$\subseteq$" is immediate and for "$\supseteq$" when $A \cap \{T < t\} \in \mathcal{F}_t^+$, for all $t \geq 0$, then

$$A \cap \{T \leq t\} = \bigcap_{n\geq 1} A \cap \Big\{T < t + \frac{1}{n}\Big\} \in \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}^+ = \mathcal{F}_t^+, \text{ for } t \geq 0\,.$$

Theorem 2.14. *(strong Markov property)*

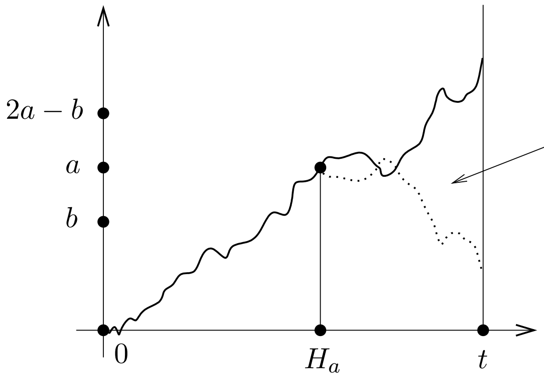
Let T be an $(\mathcal{F}_t^+)$-stopping time, $Y \in b\mathcal{F}$, $x \in \mathbb{R}^d$, then

$$E_x[Y \circ \theta_T \,|\, \mathcal{F}_T^+] = E_{X_T}[Y], \text{ on } \{T < \infty\},\ W_x\text{-a.s.} \tag{2.46}$$

(in other words, θ_T: $(\{T < \infty\}, \mathcal{F} \cap \{T < \infty\}) \to (C, \mathcal{F})$ is measurable, and the random variable $E_{X_T}[Y]$ defined on $\{T < \infty\}$ is $\mathcal{F}_T^+ \cap \{T < \infty\}$-measurable and for any $A \in \mathcal{F}_T^+ \cap \{T < \infty\}$, $E_x[\underbrace{Y \circ \theta_T\, 1_A}_{\substack{\uparrow \\ \text{well-defined since } A \subseteq \{T < \infty\}}}] = E_x[E_{X_T}[Y]\, 1_A]$).

Rather than discussing the proof right away we first give an application.

The reflection principle:



$a \geq 0$, $b \leq a$

reflection of the path after time H_a

$H_a = \inf\{s \geq 0, X_s = a\}$,

"entrance time in $\{a\}$".

Theorem 2.15. *($d = 1$, $a \geq 0$, $b \leq a$)*

$$W_0(X_t \leq b, \sup_{s \leq t} X_s \geq a) = W_0(X_t \geq 2a - b), \text{ for } t > 0, \text{ and} \tag{2.47}$$

$$W_0\big(\sup_{s \leq t} X_s \geq a\big) = 2W_0(X_t \geq a) \tag{2.48}$$

(in particular $\sup_{s \leq t} X_s$ *under* W_0 *has same law as* $|X_t|$*).*

Proof.

- (2.47):

$$\begin{aligned} & W_0(X_t \leq b, \sup_{s \leq t} X_s \geq a) = W_0(H_a \leq t, X_t \leq b) = \\ & W_0\big[\{w \in C;\ H_a(w) \leq t,\ X_{(t - H_a(w))_+}\big(\theta_{H_a(w)}(w)\big) \leq b\}\big]. \end{aligned} \tag{2.49}$$

We will use the following

Lemma 2.16. *(T an $(\mathcal{F}_t^+)$-stopping time)*

If $h(w_1, w_2)$ is $b\mathcal{F} \otimes \mathcal{F}_T^+$, then for any $x \in \mathbb{R}^d$, W_x-a.s. on $\{T < \infty\}$,

$$E_x[h(\theta_{T(w)}(w), w) \,|\, \mathcal{F}_T^+] = \int_C h(w_1, w) \, dW_{X_{T(w)}}(w_1). \tag{2.50}$$

variable of integration

Proof. For $h = 1_{A_1}(w_1)\, 1_{A_2}(w_2)$, $A_1 \in \mathcal{F}$, $A_2 \in \mathcal{F}_T^+$, (2.46) implies that for any $B \in \mathcal{F}_T^+ \cap \{T < \infty\}$, one has:

$$E_x[h(\theta_{T(w)}(w), w) \, 1_B] = E_x\Big[1_B \int_C h(w_1, w) \, dW_{X_T}(w_1)\Big]. \tag{2.51}$$

Then using Dynkin's lemma and approximation, (2.51) holds for any $h \in b\mathcal{F} \otimes \mathcal{F}_T^+$, and $\int_C h(w_1, w)\, dW_{X_T}(w_1)$ (defined on $\{T < \infty\}$) is $\mathcal{F}_T^+ \cap \{T < \infty\}$ measurable. Our claim follows. $\square$

We now apply the above lemma with $T = H_a$, and

$$h(w_1, w_2) = 1\{X_{(t-H_a(w_2))_+}(w_1) \le b\}\,,$$

which is $\mathcal{F} \otimes \mathcal{F}_{H_a}^+$-measurable, because

$$(w,t) \to X_t(w) = \lim_n \sum_{k \ge 0} X_{\frac{k}{2^n}}(w)\; 1_{[\frac{k}{2^n}, \frac{k+1}{2^n})}(t) \text{ is } \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)\text{-measurable}$$

and one can realize $h(w_1, w_2)$ in the following steps:

$$\begin{aligned}
&(w_1, w_2) \in C \times C \to (w_1, (t - H_a(w_2))_+) \in C \times \mathbb{R}_+ \\
&\to X_{(t-H_a(w_2))_+}(w_1) \in \mathbb{R} \to h(w_1, w_2) \in \mathbb{R}_+\,,
\end{aligned}$$

and each step is induced by a measurable transformation relative to the natural σ-algebra.

We can thus apply (2.50) to the last line of (2.49) and find:

$$\begin{aligned}
&W_0[H_a \le t, X_t \le b] = E_0\big[H_a \le t, \widetilde{W}_{X_{H_a}}[\widetilde{X}_{(t-H_a)_+} \le b]\big] \text{ (note } X_{H_a} = a \text{ on } \{H_a \le t\})\\
&\overset{\text{symmetry}}{=} E_0\big[H_a \le t, \widetilde{W}_{X_{H_a}}[\widetilde{X}_{(t-H_a)_+} \ge 2a - b]\big] \text{ and going backward}\\
&\overset{(2.50)}{=} W_0[H_a \le t, X_t \ge 2a - b] = W_0[X_t \ge 2a - b]\,.
\end{aligned}$$

We have thus shown that:

$$W_0(H_a \le t, X_t \le b) = W_0(X_t \ge 2a - b)\,, \tag{2.52}$$

and together with (2.49) this proves (2.47).

- (2.48):

$$\begin{aligned}
W_0[H_a \le t] &= W_0[H_a \le t, X_t \ge a] + W_0[H_a \le t, X_t \le a]\\
&= W_0[X_t \ge a] + W_0[H_a \le t, X_t \le a]\\
&\underset{\text{with } b=a}{\overset{(2.52)}{=}} W_0[X_t \ge a] + W_0[X_t \ge a] = 2W_0[X_t \ge a]\,,
\end{aligned} \tag{2.53}$$

and this proves (2.48). $\square$

Corollary 2.17. *For $a \in \mathbb{R}$, H_a is W_0-a.s. finite and has distribution:*

$$W_0(H_a \in ds) = \frac{1}{\sqrt{2\pi}}\, \frac{|a|}{s^{\frac{3}{2}}}\, \exp\Big\{-\frac{a^2}{2s}\Big\}\, 1_{(0,\infty)}(s)\, ds\,. \tag{2.54}$$

The joint law of X_t and $\sup_{s \le t} X_s$, for $t > 0$, is given by:

$$W_0(X_t \in db, \sup_{s \le t} X_s \in da) = \tag{2.55}$$
$$2\,\frac{2a-b}{\sqrt{2\pi\, t^3}}\ \exp\Big\{-\frac{(2a-b)^2}{2t}\Big\}\ 1\{a>0, b<a\}\, da\, db\,.$$

Proof.

• (2.54):

$$a > 0, \text{ then } W_0[H_a \le s] \overset{(2.48)}{=} W_0[|X_s| \ge a] \overset{\text{scaling}}{=} W_0[|X_1| \ge a/\sqrt{s}] \underset{s\to\infty}{\longrightarrow} 1\,,$$

and hence $W_0[H_a < \infty] = 1$. Moreover we have:

$$W_0[H_a \le s] = W_0[\sup_{u \le s} X_u \ge a] \overset{(2.48)}{=} 2W_0\Big[X_1 \ge \frac{a}{\sqrt{s}}\Big] = \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{s}}}^{\infty} e^{-\frac{u^2}{2}}\, du\,. \tag{2.56}$$

Differentiating in s we find (2.54).

• (2.55): Consider $0 < a, b < a$, we have

$$W_0[X_t \le b, \sup_{s\le t} X_s \ge a] \overset{(2.47)}{=} W_0[X_t \ge 2a-b] = \frac{1}{\sqrt{2\pi t}} \int_{2a-b}^{\infty} e^{-\frac{x^2}{2t}}\, dx$$

$$\underset{\downarrow}{\overset{\text{setting } x=2u-b}{}}\!\!= \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{(2u-b)^2}{2t}}\, du \overset{\text{derivating in } b}{\underset{\downarrow}{=}} \int_{[a,\infty)\times(-\infty,b]} f(u,v)\, du\, dv, \ \text{ if} \tag{2.57}$$

$$f(u,v) = 2\,\frac{2u-v}{\sqrt{2\pi t^3}}\ \exp\Big\{-\frac{(2u-v)^2}{2t}\Big\}\ 1\{u>0, v<u\}\,,$$

is the probability density that appears in the right-hand side of (2.55). This probability density is concentrated on the open set:

$$\Delta = \{(x,y) \in \mathbb{R}^2;\ x > 0, y < x\}\,.$$

Note that the same holds true for the joint law of $(\sup_{s\le t} X_s, X_t)$ under W_0. Indeed observe that

$$B_s \overset{\text{def}}{=} X_{t-s} - X_t,\ 0 \le s \le t\,, \tag{2.58}$$

is a Brownian motion with time parameter $[0,t]$ (because it is a centered Gaussian process with continuous trajectories and covariance $E_0[B_s\, B_{s'}] = s \wedge s'$, $0 \le s, s' \le t$). We know from (2.16) that the hitting time of $(0,\infty)$ by $B_.$ or by $X_.$ is a.s. equal to 0 (one can also see this from (2.47)). Therefore we have:

$$W_0\text{-a.s.}, \sup_{s\le t} X_s > X_t, \text{ and } \sup_{s \le t} X_s > 0\,. \tag{2.59}$$

As a result the joint law of $(\sup_{s\le t} X_s, X_t)$ under W_0 is supported by Δ. Now the collection of subsets of Δ of the form $[a,\infty)\times(-\infty,b]$, with $a>0$, $b<a$, is a π-system, which generates $\mathcal{B}(\Delta)$ (the Borel subsets of Δ). By (2.57) and Dynkin's lemma we can conclude that (2.55) holds. □

Remark 2.18. The collection of subsets $[a,\infty)\times(-\infty,b]$, of $\overline{\Delta}=\{(x,y)\in\mathbb{R}^2;\ x\ge 0, y\le x\}$, with $a\ge 0$, $b\le a$, is not rich enough to generate $\mathcal{B}(\overline{\Delta})$ (their trace on $\{(x,y)\in\mathbb{R}^2, x=y\ge 0\}\subseteq\partial\Delta$ consists at most of a point). This is why we work with Δ and not $\overline{\Delta}$. □

As a preparation for the proof of the strong Markov property, we introduce the following

Definition 2.19. *Given $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\ge 0})$, an $\mathbb{R}^d$-valued process $Z_u(\omega)$, $u\ge 0$, $\omega\in\Omega$, is called* **progressively measurable** *if the restriction of Z to $\Omega\times[0,t]$ is $\mathcal{G}_t\otimes\mathcal{B}([0,t])$-measurable for each $t\ge 0$.*

Example:

A process $Z_u(\omega)$ **right-continuous** in u, **adapted**, (i.e. $Z_u(\cdot)$ is $\mathcal{G}_u$-measurable for each $u\ge 0$), **is progressively measurable** because on $[0,t]\times\Omega$:

$$(2.60)\qquad Z_s(\omega)=\lim_{n\to\infty}\underbrace{\sum_{k=1}^{n} Z_{\frac{k}{n}t}\,1\Big\{\frac{k-1}{n}\,t\le s<\frac{k}{n}\,t\Big\}+Z_t(\omega)\,1_{\{s=t\}}}_{\mathcal{B}([0,t])\otimes\mathcal{G}_t-\text{measurable}}\,.$$

□

The interest of this notion in our context comes from the next

Lemma 2.20. *Given $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\ge 0})$, Z a progressively measurable process, and T a $(\mathcal{G}_t)$-stopping time one has*

$$(2.61)\qquad Z_T\ \textit{(i.e. } \omega\in\{T<\infty\}\to Z_{T(\omega)}(\omega)\in\mathbb{R}^d\textit{) is } \mathcal{G}_T\cap\{T<\infty\}\textit{-measurable}.$$

Proof. It suffices to show that for any $t\ge 0$,

$$(2.62)\qquad (\{T\le t\},\ \mathcal{G}_t\cap\{T\le t\})\ \xrightarrow{Z_T}\ (\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))\ \text{is measurable}.$$

But the above map is the composition of the measurable maps:

$$\begin{array}{ccccc}\omega\in\{T\le t\} & \longrightarrow & \big(\omega,T(\omega)\big)\in\Omega\times[0,t]\ni(\omega,u) & \longrightarrow & Z_u(\omega)\in\mathbb{R}^d\,.\\ \mathcal{G}_t\cap\{T\le t\} & & \mathcal{G}_t\otimes\mathcal{B}([0,t]) & & \mathcal{B}(\mathbb{R}^d)\end{array}$$

The first map is measurable because both maps

$$(\{T\le t\},\ \mathcal{G}_t\cap\{T\le t\})\xrightarrow{Id}(\Omega,\mathcal{G}_t)\ \text{and}$$

$$(\{T\le t\},\ \mathcal{G}_t\cap\{T\le t\})\xrightarrow{T}([0,t],\ \mathcal{B}([0,t])\ \text{are measurable},$$

and the second map is measurable because Z is progressively measurable. □

We now turn to the **proof of the strong Markov property:**

Proof. We will prove Theorem 2.14 in a number of steps. The **first step** is to show that:

$$\theta_T : (\{T < \infty\},\ \mathcal{F} \cap \{T < \infty\}) \longrightarrow (C, \mathcal{F}) \text{ is measurable}. \tag{2.63}$$

Due to (1.5) we only need to show that for $s \geq 0$,

$$X_s \circ \theta_T : (\{T < \infty\},\ \mathcal{F} \cap \{T < \infty\}) \longrightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is measurable}, \tag{2.64}$$

and in the spirit of the proof in (2.36), we write for $w \in \{T < \infty\}$

$$X_s \circ \theta_T(w) = X_{s+T(w)}(w) = \lim_{n\to\infty} \sum_{k\geq 1} \underbrace{X_{s+\frac{k}{n}}(w)}_{\text{meas. } \mathcal{F}\cap\{T<\infty\}} \quad 1\underbrace{\Big\{\frac{(k-1)}{n} \leq T < \frac{k}{n}\Big\}}_{\in \mathcal{F}\cap\{T<\infty\}}$$

whence (2.64) and therefore (2.63).

The **second step** is then to show that:

$$X_T : (\{T < \infty\},\ \mathcal{F}_T^+ \cap \{T < \infty\}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is measurable}. \tag{2.65}$$

To this effect we note that $X_t(\omega)$ is a progressively measurable process due to (2.60) and the claim (2.65) now follows from (2.61).

Note that $y \in \mathbb{R}^d \to E_y[Y]$ is measurable for any $Y \in b\mathcal{F}$, as shown in (2.9) (or in other words $y \in \mathbb{R}^d$, $A \in \mathcal{F} \to W_y(A) \in [0,1]$, is a probability kernel). Combining this observation with (2.63) and (2.65), the statement in (2.46) now makes sense.

The **third step** is to show that:

(2.66) when T takes an at most denumerable set of values in $\mathbb{R}_+ \cup \{\infty\}$, then (2.46) is true.

This step will follow from a direct application of the simple Markov property. We write a_n, $0 \leq n < N(\leq \infty)$ for the set of values of T in $[0, \infty)$.

Then for $A \in \mathcal{F}_T^+ \cap \{T < \infty\}$, $0 = t_0 < \cdots < t_k$, $f \in b\mathcal{B}((\mathbb{R}^d)^{k+1})$ we find:

(2.67)

$$E_x[f(X_{t_0}, \dots, X_{t_k}) \circ \theta_T\, 1_A] = E_x[f(X_{T+t_0}, \dots, X_{T+t_k})\, 1_A] =$$

$$\sum_n E_x\Big[f(X_{a_n+t_0}, \dots, X_{a_n+t_k})\, 1_{\underbrace{A \cap \{T = a_n\}}_{\in \mathcal{F}_{a_n}^+}}\Big] \overset{(2.7)}{=}$$

and by the simple Markov property (2.7):

$$\sum_n E_x\big[E_{X_{a_n}}[f(X_{t_0}, \dots, X_{t_k})]\, 1_{A\cap\{T=a_n\}}\big] =$$

$$E_x[E_{X_T}[f(X_{t_0}, \dots, X_{t_k})]\, 1_A],$$

where the summation runs over the set $0 \le n < N$ (such that $a_n \in [0,\infty)$) in lines two and four of (2.67). We can now use Dynkin's lemma and approximation to deduce that for $Y \in b\mathcal{F}$, one has

$$E_x[Y \circ \theta_T \, 1_A] = E_x[E_{X_T}[Y] \, 1_A]\,,$$

and obtain (2.66).

The **last step** of the proof will be:

$$\text{the claim (2.46) holds for } T \text{ a general } (\mathcal{F}_t^+)\text{-stopping time}\,. \tag{2.68}$$

For this purpose, we use the **discrete skeleton approximation** of T:

$$T_n = \sum_{k \ge 0} \frac{k+1}{2^n} \, 1\Big\{\frac{k}{2^n} \le T < \frac{k+1}{2^n}\Big\} + \infty 1\{T = \infty\}\,. \tag{2.69}$$

The key observation is that:

$$\text{for each } n,\ T_n \text{ is an } (\mathcal{F}_t^+)\text{-stopping time, and } T_n \downarrow T \text{ as } n \to \infty\,. \tag{2.70}$$

Indeed the fact that $T_n \ge T$ and $T_n \downarrow T$ as $n \to \infty$ is obvious from (2.69). In addition, for $k, n \ge 0$,

$$\Big\{T_n \le \frac{k+1}{2^n}\Big\} \overset{(2.69)}{=} \Big\{T < \frac{k+1}{2^n}\Big\} \in \mathcal{F}^+_{\frac{k+1}{2^n}}\,,$$

and for $t \in [\frac{k}{2^n}, \ \frac{k+1}{2^n})$ we have

$$\{T_n \le t\} = \Big\{T_n \le \frac{k}{2^n}\Big\} \in \mathcal{F}^+_{\frac{k}{2^n}} \subseteq \mathcal{F}_t^+\,,$$

so that T_n is an $(\mathcal{F}_t^+)$-stopping time.

Since $T \le T_n$, it follows, cf. Exercise 2.13 2) a), that

$$\mathcal{F}_T^+ \subseteq \mathcal{F}_{T_n}^+, \text{ for } n \ge 0\,. \tag{2.71}$$

Consider $A \in \mathcal{F}_T^+ \cap \{T < \infty\}$. Since $\{T < \infty\} = \{T_n < \infty\}$, we also have $A \in \mathcal{F}_{T_n}^+ \cap \{T_n < \infty\}$, and applying (2.66) we see that for $Y \in b\mathcal{F}$:

$$E_x[Y \circ \theta_{T_n} \, 1_A] = E_x[E_{X_{T_n}}[Y] \, 1_A]\,.$$

Specializing to the case where $0 = t_0 < \dots < t_k$ and $Y = f(X_{t_0}, \dots, X_{t_k})$, with f bounded continuous on $(\mathbb{R}^d)^{k+1}$, we obtain that for $n \ge 0$:

$$E_x[f(X_{T_n+t_0}, \dots, X_{T_n+t_k}) \, 1_A] = E_x[E_{X_{T_n}}[f(X_{t_0}, \dots, X_{t_k})] \, 1_A]\,. \tag{2.72}$$

We also know from (2.12) that:

$$y \in \mathbb{R}^d \longrightarrow E_y[f(X_{t_0}, \dots, X_{t_k})] \text{ is a bounded continuous function}\,.$$

Therefore, letting n tend to infinity in (2.72) we find that

$$E_x[f(X_{T+t_0},\dots,X_{T+t_k})\,1_A] = E_x[E_{X_T}[f(X_{t_0},\dots,X_{t_k})]\,1_A]\,. \tag{2.73}$$

By the same argument as below (2.12), we then find that (2.73) holds for $f(x_0,\dots,x_k) = \prod_{i=0}^k 1_{K_i}(x_i)$, with K_i, $i=0,\dots,k$, closed subsets of $\mathbb{R}^d$, and then by Dynkin's lemma and approximation we obtain that

$$E_x[Y\circ\theta_T\,1_A] = E_x[E_{X_T}[Y]\,1_A]\,,$$

for $Y \in b\mathcal{F}$ and $A \in \mathcal{F}_T^+ \cap \{T<\infty\}$. This identity, recall (2.65), (2.9), now completes the proof of (2.46). □

Complement:

What can go wrong when going from the simple to the strong Markov property:

A typical example is given by the following process:

state space is $\mathbb{R}_+$.

The process waits an exponential time of parameter 1 in 0, and afterwards moves with unit speed to the right. If it starts in $x>0$, it simply moves to the right with unit speed.

We denote by P_x, $x\ge 0$, the law on $(C(\mathbb{R}_+,\mathbb{R}_+),\mathcal{F})$ of the process starting at $x\ge 0$, with $\mathcal{F}$ the canonical σ-algebra on $C(\mathbb{R}_+,\mathbb{R}_+)$.

For $t\ge 0$, one defines the operator $R_t : b\mathcal{B}(\mathbb{R}_+)\to b\mathcal{B}(\mathbb{R}_+)$ in analogy with (2.21), via:

$$\begin{aligned} R_t\,f(x) &\overset{\text{def}}{=} E_x[f(X_t)] = f(x+t), \text{ if } x>0,\\ &= e^{-t}\,f(0) + \int_0^t e^{-u}\,f(t-u)du, \text{ if } x=0\,. \end{aligned} \tag{2.74}$$

Note that R_t, $t\ge 0$, has the semigroup property:

$$R_{t+s} = R_t\circ R_s, \text{ for } s,t\ge 0\,. \tag{2.75}$$

Indeed, when $f\in b\mathcal{B}(\mathbb{R}_+)$ and $x>0$,

$$R_t(R_s\,f)(x) = (R_s\,f)(x+t) = f(x+t+s) = R_{t+s}\,f(x)\,,$$

and when $x = 0$,

$$\begin{aligned}
R_t(R_s\, f)(0) &= e^{-t}\, R_s\, f(0) + \int_0^t e^{-u}\, R_s\, f(t-u)\, du \\
&= e^{-t-s}\, f(0) + e^{-t} \int_0^s e^{-v}\, f(s-v)\, dv + \int_0^t e^{-u}\, f(t+s-u)\, du \\
&= e^{-(t+s)}\, f(0) + \int_0^s e^{-(t+v)}\, f(s-v)\, dv + \int_0^t e^{-u} f(t+s-u)\, du \\
&\overset{\text{setting } t+v=u}{=} e^{-(t+s)}\, f(0) + \int_t^{t+s} e^{-u} f(t+s-u)\, du + \int_0^t e^{-u}\, f(t+s-u)\, du \\
&= R_{t+s}\, f(0), \ \text{whence (2.75)}\,.
\end{aligned}$$

Moreover, one has the regularity:

$$R_t\, f(x) \underset{t\to 0}{\longrightarrow} f(x), \text{ for } x \ge 0, \text{ when } f \text{ is continuous bounded (by direct inspection of (2.74))}\,. \tag{2.76}$$

One checks that $E_x[f(X_{t+s})|\mathcal{F}_s] = R_t f(X_s)$, P_x-a.s., for t, $s \ge 0$, $f \in b\mathcal{B}(\mathbb{R}_+)$, and $x \ge 0$ (one looks separately at the events $\{X_s > 0\}$ and $\{X_s = 0\}$). From this identity one can deduce that $X_.$ has the simple Markov property with respect to $(\mathcal{F}_t)_{t\ge 0}$. One can further check that:

$$X_. \text{ has the simple Markov property with respect to } (\mathcal{F}_t^+)_{t\ge 0}\,. \tag{2.77}$$

In essence as below (2.12) one uses the fact that for g continuous bounded, $x \ge 0$,

$$E_{X_{s+\varepsilon}}[g(X_{t_0}, \dots, X_{t_k})] \overset{\varepsilon\to 0}{\longrightarrow} E_{X_s}[g(X_{t_0}, \dots, X_{t_k})], \ P_x\text{-a.s.}$$

and this is done by looking separately at the events $\{X_s > 0\}$ and $\{X_s = 0\}$.

However, the process is **not strong Markov**! For instance, $H_{(0,\infty)}$ the entrance time in $(0,\infty)$ is an $(\mathcal{F}_t^+)$-stopping time, cf. (2.29), and P_0-a.s. $H_{(0,\infty)} > 0$, but on the other hand, $H_{(0,\infty)} \circ \theta_{H_{(0,\infty)}} = 0$, P_0-a.s., so that:

$$0 = E_0\big[1\{H_{(0,\infty)} > 0\} \circ \theta_{H_{(0,\infty)}}\big] \ne E[E_{X_{H_{(0,\infty)}}}[1\{H_{(0,\infty)} > 0\}]] = 1 \quad (\text{since } X_{H_{(0,\infty)}} = 0,\ P_0\text{-a.s.}). \tag{2.78}$$

Roughly speaking the problem is that P_0 does not describe the motion of $X_{H_{(0,\infty)}+\cdot}$, i.e. of $X_.$ after time $H_{(0,\infty)}$. Note that even when f is smooth one can have for $t > 0$,

$$\lim_{x\to 0_+} R_t\, f(x) = f(t) \ne R_t\, f(0): \ R_t\, f \text{ is not continuous!} \tag{2.79}$$

So, the crucial property (2.12) in the Brownian case, which was used below (2.72), is not satisfied in the present example. Indeed, if $H^n_{(0,\infty)}$ denotes the discrete skeleton of $H_{(0,\infty)}$, cf. (2.69), for bounded continuous g,

$$\begin{array}{l} E_{X_{H^n_{(0,\infty)}}}[g(X_{t_0},\dots,X_{t_k})] \text{ need not } P_0\text{-a.s., converge for } n\to\infty, \text{ to} \\ E_{X_{H_{(0,\infty)}}}[g(X_{t_0},\dots,X_{t_k})] \underset{P_0-\text{a.s.}}{=} E_0[g(X_{t_0},\dots,X_{t_k})]\,. \end{array} \tag{2.80}$$

This point should be contrasted with (2.77). □

Chapter 3: Some Properties of the Brownian Sample Path

We will now discuss some typical properties of the Brownian sample paths. From this discussion the "**roughness**" of the typical sample path will be apparent. We begin with the **quadratic variation** and the variation of the sample path.

Theorem 3.1. *($d = 1$, on the canonical space $(C, \mathcal{F}, W_0)$)*

For $t > 0$, W_0-a.s., and in $L^2(W_0)$,

$$\lim_{n\to\infty} \sum_{k\geq 0, \frac{k+1}{2^n}\leq t} \big|X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\big|^2 = t\,, \tag{3.1}$$

(3.2) *W_0-a.s., the map $t \geq 0 \to X_t(w) \in \mathbb{R}$, has infinite variation on any $[a,b]$, $0 \leq a < b$.*

Proof.

• (3.1): We set

$$\Delta_{k,n} \overset{\text{def}}{=} \big|X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\big|^2, \text{ for } k, n \geq 0\,. \tag{3.3}$$

For fixed n, by (1.1), the $\Delta_{k,n}$, $k \geq 0$, are i.i.d. under W_0, with mean 2^{-n}. Moreover, we find that

$$\begin{aligned} &E_0\Big[\Big(\sum_{\frac{k+1}{2^n}\leq t} \Delta_{k,n} - t\Big)^2\Big] = E_0\Big[\Big\{\sum_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n}) - \underbrace{\Big(t - \frac{[t2^n]}{2^n}\Big)}_{=a_n}\Big\}^2\Big] = \\ &a_n^2 - 2a_n \underbrace{E_0\Big[\sum_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n})\Big]}_{=0} + E_0\Big[\Big(\sum_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n})\Big)^2\Big]. \end{aligned} \tag{3.4}$$

Since we have to do with the variance of a sum of i.i.d. centered variables, we find:

$$\begin{aligned} E_0\Big[\Big(\sum_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n})\Big)^2\Big] &= [2^n t]\, E_0[(\Delta_{0,n} - 2^{-n})^2] \\ &\quad \text{and since } \Delta_{0,n} \text{ is distributed as } 2^{-n}X_1^2 \text{ under } W_0 \\ &= \frac{[2^n t]}{2^{2n}}\, E_0[(X_1^2 - 1)^2]\,. \end{aligned} \tag{3.5}$$

We have thus found that

$$E_0\Big[\Big(\sum_{\frac{k+1}{2^n}\leq t} \Delta_{k,n} - t\Big)^2\Big] = a_n^2 + \frac{[2^n t]}{2^{2n}}\, E_0[(X_1^2 - 1)^2] \text{ is summable in } n\,. \tag{3.6}$$

From this we deduce that $(\sum_{\frac{k+1}{2^n}\leq t} \Delta_{k,n} - t)$ converges a.s. and in $L^2(W_0)$ to 0 as $n \to \infty$. The claim (3.1) now follows.

• (3.2):

The set of $w \in C$ for which there exists $0 \le a < b < \infty$, such that $t \to X_t(w)$ has finite variation on $[a,b]$ equals the event

$$\bigcup_{r<s \text{ in } \mathbb{Q}\cap[0,\infty)} \{w \in C:\ V_{r,s}(w) < \infty\}\,, \tag{3.7}$$

where $V_{r,s}(w)$ denotes the random variable:

$$V_{r,s}(w) = \sup_{\substack{r=t_0<\dots<t_k=s\\ \text{rationals}}} \sum_{i=1}^{k} |X_{t_i}(w) - X_{t_{i-1}}(w)|\,. \tag{3.8}$$

If (3.2) did not hold, then for some $0 \le r_0 < s_0 \in \mathbb{Q} \cap [0,\infty)$ one would have

$$W_0[V_{r_0,s_0} < \infty] > 0\,. \tag{3.9}$$

However on the event $\{V_{r_0,s_0} < \infty\}$,

$$\begin{aligned}\sum_{r_0 \le \frac{k}{2^n},\, \frac{k+1}{2^n} \le s_0} \big|X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\big|^2 &\le \sup_{\substack{|u-v|\le 2^{-n}\\ u,v\le s_0}} |X_u - X_v| \sum_{r_0 \le \frac{k}{2^n},\, \frac{k+1}{2^n} \le s_0} \big|X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\big| \\ &\le \sup_{\substack{|u-v|\le 2^{-n}\\ u,v\le s_0}} |X_u - X_v|\, V_{r_0,s_0} \underset{n\to\infty}{\longrightarrow} 0, \ \text{thanks to the continuity of the trajectory } t \to X_t\,.\end{aligned} \tag{3.10}$$

On the other hand by (3.1) and the continuity of the trajectory, we find that W_0-a.s.,

$$\sum_{r_0 \le \frac{k}{2^n},\, \frac{k+1}{2^n} \le s_0} \big|X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\big|^2 \underset{n\to\infty}{\longrightarrow} s_0 - r_0 > 0, \ \text{thus contradicting (3.10)}\,. \tag{3.11}$$

This proves (3.2). □

Remark 3.2. Using Dini's second theorem (i.e. a sequence of non-decreasing functions on a compact interval $I \subseteq \mathbb{R}$, converging to a continuous function, converges uniformly on I to this function), we deduce from (3.1) that

$$W_0\text{-a.s., for any } N \ge 1,\ \lim_{n\to\infty} \sup_{0\le t\le N} \Big| \sum_{0 \le \frac{k+1}{2^n} \le t} \big|X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}\big|^2 - t\Big| = 0\,. \tag{3.12}$$

Exercise 3.3. Consider for $0 \le r < s$ and $w \in C$, the function theoretic quadratic variation of w on $[r,s]$

$$V_{2,r,s}(w) = \sup_{\substack{r\le t_0<\dots<t_k\le s\\ \text{rationals}}} \sum_{i=1}^{k} |X_{t_i}(w) - X_{t_{i-1}}(w)|^2\,.$$

Show that (in spite of (3.1)):

$$W_0\text{-a.s., } V_{2,r,s} = \infty \text{ for all } 0 \le r < s < \infty \text{ in } \mathbb{Q} \cap [0,\infty)\,.$$

(Hint: Take advantage of (2.19) to construct partitions of $[r,s]$ for which $|X_{t_i} - X_{t_{i-1}}| \ge K\sqrt{t_i - t_{i-1}}$ occurs often. See also [5], Exercise 2.4, p. 345).

□

Our next objective is the **law of the iterated logarithm**.

Theorem 3.4. *(A. Khinchin, 1933).*

$$\text{(3.13)} \qquad \begin{aligned} &\text{i)} \quad W_0\text{-}a.s., \ \overline{\lim_{t\to 0}}\, X_t \Big/ \Big(\sqrt{2t \log\log \tfrac{1}{t}}\Big) = 1, \qquad \text{“small time behavior”} \\ &\text{ii)} \quad W_0\text{-}a.s., \ \underline{\lim_{t\to 0}}\, X_t \Big/ \Big(\sqrt{2t \log\log \tfrac{1}{t}}\Big) = -1\,, \end{aligned}$$

and

$$\text{(3.14)} \qquad \begin{aligned} &\text{i)} \quad W_0\text{-}a.s., \ \overline{\lim_{t\to \infty}}\, X_t \Big/ \Big(\sqrt{2t \log\log t}\Big) = 1, \qquad \text{“large time behavior”} \\ &\text{ii)} \quad W_0\text{-}a.s., \ \underline{\lim_{t\to \infty}}\, X_t \Big/ \Big(\sqrt{2t \log\log t}\Big) = -1\,. \end{aligned}$$

Proof. Under W_0, $(-X_t)_{t\ge 0}$ is also a Brownian motion, so that we only need to prove (3.13) i) and (3.14) i).

Moreover, we know from (1.20), (1.21), that

$$\text{(3.15)} \qquad \begin{aligned} \beta_s &= s\, X_{1/s},\ s > 0 \\ &= 0,\ s = 0 \end{aligned}$$

is a Brownian motion. Thus, if we can prove (3.13) i), it follows that W_0-a.s.

$$1 = \overline{\lim_{s\to 0}}\, s\, X_{1/s} \Big/ \sqrt{2s \log\log \frac{1}{s}} = \overline{\lim_{s\to 0}}\, X_{1/s} \Big/ \sqrt{\frac{2}{s} \log\log \frac{1}{s}}\,.$$

Setting $t = \frac{1}{s}$, we then find (3.14) i).

As a result, we only need to prove (3.13) i).

First step: “the upper bound”.

We set $\varphi(t) \overset{\text{def}}{=} \sqrt{2t \log\log \frac{1}{t}}$, our goal is to prove that

$$\text{(3.16)} \qquad W_0\text{-a.s., } \overline{\lim_{t\to 0}}\, \frac{X_t}{\varphi(t)} \le 1\,.$$

The idea is to use Borel-Cantelli's lemma, and to produce some decoupling, we look at geometrically decreasing times. Indeed, we choose $\delta > 0$ and $q \in (0,1)$ (δ will be small and q close to 1), so that

$$(1+\delta)^2\, q > 1, \quad \text{and define} \tag{3.17}$$

$$t_n = q^n,\ n \ge 0,\ (\text{note that } t_n \downarrow 0)\,, \tag{3.18}$$

$$A_n = \{w \in C;\ \text{for some } t \in [t_{n+1}, t_n],\ X_t > (1+\delta)\,\varphi(t)\},\ n \ge 0\,. \tag{3.19}$$

Note that φ is non-decreasing on $[0,T]$, when T is small and positive, because

$$\begin{aligned}
\psi(t) &\overset{\text{def}}{=} \frac{\varphi^2(t)}{2} = t \log\log \tfrac{1}{t}, \text{ so that} \\
\psi'(t) &= \log\log \tfrac{1}{t} + \frac{t}{\log \frac{1}{t}} \times -\tfrac{1}{t} = \log\log \tfrac{1}{t} - \frac{1}{\log \frac{1}{t}} > 0, \text{ for } t \text{ small}\,.
\end{aligned}$$

As a result, we see that for large enough n

$$\begin{aligned}
W_0(A_n) &\le W_0\big(\sup_{0 \le s \le t_n} X_s > (1+\delta)\,\varphi(t_{n+1})\big) \ (\text{recall } t_{n+1} < t_n) \\
&\overset{(2.48)}{=} 2W_0(X_{t_n} > (1+\delta)\,\varphi(t_{n+1})) \\
&\overset{(1.35)}{\le} \sqrt{\frac{2}{\pi}}\, \frac{1}{x_n}\ \exp\Big\{-\frac{x_n^2}{2}\Big\}, \ \text{with } x_n \overset{\text{def}}{=} (1+\delta)\,\varphi(t_{n+1})/\sqrt{t_n}\,.
\end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned}
x_n &= (1+\delta)\ \sqrt{2q^{n+1-n} \log\log q^{-n-1}} \\
&= (1+\delta)\ \Big[2q \log\Big((n+1)\log \tfrac{1}{q}\Big)\Big]^{\frac{1}{2}} \\
&= [2\log\{(\alpha(n+1))^\lambda\}]^{1/2} \ \text{ with } \ \alpha = \log(1/q) \\
&\qquad\qquad\qquad\qquad\qquad \lambda = q(1+\delta)^2 \overset{(3.17)}{>} 1\,.
\end{aligned}$$

Coming back to the last line of (3.20), we find that

$$W_0(A_n) \le \sqrt{\frac{2}{\pi}}\ \frac{1}{\alpha^\lambda}\ \frac{1}{(n+1)^\lambda} \ \text{ for large } n\,. \tag{3.21}$$

Since $\lambda > 1$, by (3.17), it follows that:

$$\sum_n W_0(A_n) < \infty\,, \tag{3.22}$$

and by the first lemma of Borel-Cantelli, we see that

$$W_0\text{-a.s., } A_n \text{ occurs only finitely many times}\,. \tag{3.23}$$

As a result we obtain W_0-a.s., $\overline{\lim}_{t \to 0} X_t/\varphi(t) \le 1 + \delta$. Letting $\delta \to 0$ (this is possible, cf. (3.17)), we obtain (3.16).

Second step: "the lower bound"

$$W_0\text{-a.s.}, \ \overline{\lim_{t \to 0}} \ \frac{X_t}{\varphi(t)} \ge 1 \,. \tag{3.24}$$

To this end we choose $q \in (0,1)$, $\varepsilon \in (0, \frac{1}{2})$, and define t_n, $n \ge 0$, as in (3.18). Here both ε and q will be chosen small, see (3.29) below.

We will use the lower bound (in the spirit of (1.35)):

$$P[\xi > x] \ge \frac{x}{x^2+1} \ \frac{1}{\sqrt{2\pi}} \ \exp\Big\{ - \frac{x^2}{2}\Big\}, \text{ where } x > 0 \text{ and } \xi \text{ is } N(0,1)\text{-distributed} \tag{3.25}$$

(indeed $x^{-1}\, e^{-\frac{x^2}{2}} = \int_x^{+\infty} (1 + z^{-2})\, e^{-\frac{z^2}{2}}\, dz \le (1 + x^{-2}) \int_x^\infty e^{-\frac{z^2}{2}}\, dz$, whence (3.25)).

As a result, setting now $x_n = (1-\varepsilon)\, \varphi(t_n)(t_n - t_{n+1})^{-1/2}$, we find for large n that:

$$\begin{aligned} &W_0[X_{t_n} - X_{t_{n+1}} > (1-\varepsilon)\, \varphi(t_n)] \overset{(1.1)}{=} W_0[X_1 > x_n] \overset{(3.25)}{\ge} \\ &\sqrt{2\pi}^{-1} \ x_n(1+x_n^2)^{-1} \ \exp\Big\{ - \frac{x_n^2}{2}\Big\} \ge \sqrt{2\pi}^{-1} \ (2x_n)^{-1} \ \exp\Big\{ - \frac{x_n^2}{2}\Big\} \,. \end{aligned} \tag{3.26}$$

Moreover, we have

$$x_n = \frac{1-\varepsilon}{\sqrt{1-q}} \sqrt{2 \log \Big(n \log \frac{1}{q}\Big)} = \sqrt{\beta \log(\alpha n)}, \ \text{ with } \alpha = \log \tfrac{1}{q} \text{ and} \tag{3.27}$$

$$\beta = 2 \frac{(1-\varepsilon)^2}{1-q} \,. \tag{3.28}$$

We assume that q is small enough so that

$$q < \frac{\varepsilon^2}{4} \ \text{ (and in particular, as a result } \beta < 2) \,. \tag{3.29}$$

Then the variables $X_{t_n} - X_{t_{n+1}}$, $n \ge 0$, are independent and

$$W_0(X_{t_n} - X_{t_{n+1}} > (1-\varepsilon)\, \varphi(t_n)] \ge \frac{c}{(\log n)^{1/2}} \ n^{-\beta/2}, \ \text{ for large } n \,. \tag{3.30}$$

The above expression is the general term of a divergent series. Hence, the second lemma of Borel-Cantelli yields that

$$W_0\text{-a.s., for infinitely many } n, \ X_{t_n} - X_{t_{n+1}} > (1-\varepsilon)\, \varphi(t_n) \,. \tag{3.31}$$

From the upper bound (3.16) applied to $(-X_t)_{t\ge 0}$, we see that W_0-a.s., for large n, $X_{t_n} \ge -(1+\varepsilon)\,\varphi(t_n)$, and therefore

$$
\text{(3.32)}\qquad
\begin{aligned}
&W_0\text{-a.s., for infinitely many } n\\
&X_{t_n} = X_{t_n} - X_{t_{n+1}} + X_{t_{n+1}} \ge (1-\varepsilon)\,\varphi(t_n) - (1+\varepsilon)\,\varphi(t_{n+1})\\
&\quad\;\; = \varphi(t_n)[1-\varepsilon-(1+\varepsilon)\,\varphi(t_{n+1})/\varphi(t_n)]\,.
\end{aligned}
$$

Note that by (3.29):

$$
\text{(3.33)}\qquad \lim_n \varphi(t_{n+1})/\varphi(t_n) = \sqrt{q} \overset{(3.29)}{<} \frac{\varepsilon}{2}\,,
$$

and it follows from (3.32) that

$$
\text{(3.34)}\qquad W_0\text{-a.s., for infinitely many } n,\ X_{t_n} \ge \varphi(t_n)(1-2\varepsilon)\,,
$$

so that W_0-a.s., $\overline{\lim}_{t\to 0}\, X_t/\varphi(t) \ge 1-2\varepsilon$. Letting ε tend to zero along some sequence, we deduce (3.24). $\square$

Remark 3.5. (further extensions and related results)

1) There is a "functional" extension of the law of the iterated logarithm due to V. Strassen (1964). Given $w \in C$, one considers the subset of $C([0,1];\mathbb{R})$ (endowed with the sup-norm):

$$
F_w = \Big\{f \in C([0,1];\mathbb{R}),\ \text{for some } t \ge 10,\ f(u) = \frac{X_{ut}(w)}{\sqrt{2t\log\log t}}\,,\ 0 \le u \le 1\Big\}\,.
$$

Theorem 3.6.

(3.35) *W_0-a.s., F_w is relatively compact, and the set of limit points of $(X_{ut}/\sqrt{2t\log\log t})_{0\le u\le 1}$, as $t\to\infty$, coincides with:*

$$
\text{(3.36)}\qquad
\begin{aligned}
K = \Big\{f \in C([0,1];\mathbb{R});\ f(u) = \int_0^u g(s)\,ds \text{ for some } g \in L^2([0,1],ds)\\
\text{with } \int_0^1 g^2(s)\,ds \le 1\Big\}\ \text{(of course } K \text{ is compact)}\,.
\end{aligned}
$$

For the proof see [2], p. 21. Note that when f runs over K, $f(1)$ runs over $[-1,1]$ (indeed $|f(1)| \le (\int_0^1 g^2(s)ds)^{1/2}) \le 1$, and $f(u) = au$, with $|a| \le 1$ belongs to K). From this one recovers that:

$$
\text{(3.37)}\qquad W_0\text{-a.s., the set of limit points of } \frac{X_t}{\sqrt{2t\log\log t}},\ \text{as } t\to\infty,\ \text{equals } [-1,1]\,,
$$

which in essence is a restatement of (3.14).

Exercise 3.7. Given $T > 0$, what is the W_0-a.s. set of limit points as $t\to\infty$, of $(X_{ut}/\sqrt{2t\log\log t})_{0\le u\le T}$ in $C([0,T];\mathbb{R})$ endowed with the sup-norm? $\square$

2) Another related result is **Lévy's modulus of continuity** for Brownian motion.

Theorem 3.8. *(P. Lévy, 1937)*

$$W_0\text{-}a.s.,\ \overline{\lim_{u\to 0}}\ \frac{1}{\sqrt{2u\log\frac{1}{u}}}\ \sup_{\substack{0\le s<t\le 1\\ t-s\le u}} |X_t - X_s| = 1\,. \tag{3.38}$$

For the proof, which has a similar flavour as the proof of the law of the iterated logarithm, see for instance [8], p. 114.

Note that in (3.38), $\sqrt{2t\log\log\frac{1}{t}}$ in (3.13) is replaced with the "bigger" function $\sqrt{2t\log\frac{1}{t}}$. This has to do with the fact that in (3.38) one also takes the supremum over the "starting point X_s", whereas for fixed s, W_0-a.s., $\overline{\lim}_{u\to 0}\frac{|X_{s+u}-X_s|}{\sqrt{2u\log\log\frac{1}{u}}} = 1$.

3) A further law of the iterated logarithm was proved by K.L. Chung (1948). It governs the small values of $\sup_{0\le s\le t}|X_s|$.

Theorem 3.9.

$$W_0\text{-}a.s.,\ \underline{\lim_{t\to\infty}}\ \Big(\frac{\log\log t}{t}\Big)^{\frac{1}{2}}\ \sup_{0\le s\le t}|X_s| = \frac{\pi}{\sqrt{8}}\,. \tag{3.39}$$

This shows that $\sup_{0\le s\le t}|X_s|$ cannot grow too slowly. On the other hand it follows from (3.35), (3.36) that it cannot grow too fast and

$$W_0\text{-a.s.},\ \overline{\lim_{t\to\infty}}\ \Big(\frac{1}{2t\log\log t}\Big)^{\frac{1}{2}}\ \sup_{0\le s\le t}|X_s| = 1\,. \tag{3.40}$$

□

We will now conclude this short chapter with a discussion of the **Hölder property** of the Brownian path.

Proposition 3.10. *For $\gamma\in(0,\frac{1}{2})$,*

$$W_0\text{-}a.s.,\ \textit{for any } T>0,\ \sup_{0\le s<t\le T}\frac{|X_t-X_s|}{(t-s)^\gamma} < \infty\,, \tag{3.41}$$

and if $\gamma\ge\frac{1}{2}$,

$$W_0\text{-}a.s.,\ \textit{for any } 0\le a<b<\infty,\ \sup_{a\le s<t\le b}\frac{|X_t-X_s|}{(t-s)^\gamma} = \infty \tag{3.42}$$

(so, the Brownian path is Hölder continuous with exponent γ for $\gamma<\frac{1}{2}$, but not for larger γ).

Proof.

• (3.41):

We use Kolmogorov's criterion, cf. (1.57), (1.58). Indeed for $0 \le s < t$, $m > 1$:

$$E_0[(X_t - X_s)^{2m}] \stackrel{\text{scaling}}{=} (t-s)^m \, E_0[X_1^{2m}] \, .$$

It now follows from (1.57), with the choices $r = 2m$, $\alpha = m - 1$, $\beta \in (0, \frac{m-1}{2m})$, that for $N \ge 1$, $R > 0$,

$$W_0\Big[\sup_{0 \le s < t \le N} \frac{|X_t - X_s|}{(t-s)^\beta} \ge R\Big] \le \frac{K(m,\beta,N)}{R^{2m}} \; E_0[X_1^{2m}] \, ,$$

so that for $\beta \in (0, \frac{1}{2} - \frac{1}{2m})$ one finds

$$\text{(3.43)} \qquad W_0\text{-a.s.}, \sup_{0 \le s < t \le N} \frac{|X_t - X_s|}{(t-s)^\beta} < \infty, \text{ for all } N \ge 1 \, .$$

Picking m large enough, (3.41) follows.

• (3.42):

From (3.13), we see that

$$W_0\text{-a.s., for all } s \in \mathbb{Q} \cap [0,\infty), \quad \overline{\lim_{\substack{u \to 0 \\ u > 0}}} \frac{|X_{s+u} - X_s|}{\sqrt{2u \log\log \frac{1}{u}}} = 1 \, ,$$

so that for $\gamma \ge \frac{1}{2}$

$$\begin{aligned} &W_0\text{-a.s., for all } s \in \mathbb{Q} \cap [0,\infty), \\ \text{(3.44)} \qquad &\overline{\lim_{\substack{u \to 0 \\ u > 0}}} \frac{|X_{s+u} - X_s|}{u^\gamma} = \overline{\lim_{\substack{u \to 0 \\ u > 0}}} \frac{|X_{s+u} - X_s|}{\sqrt{2u \log\log \frac{1}{u}}} \; \frac{\sqrt{2u \log\log \frac{1}{u}}}{u^\gamma} = \infty \, . \end{aligned}$$

The claim (3.42) immediately follows. □

Remark 3.11. Of course the above proposition offers a weaker result than the aforementioned Lévy's modulus of continuity (3.38):

$$W_0\text{-a.s.}, \lim_{u \to 0} \frac{1}{\sqrt{2u \log \frac{1}{u}}} \sup_{\substack{0 \le s < t \le T \\ t - s \le u}} |X_t - X_s| = 1 \, .$$

□

Chapter 4: Stochastic Integrals

The fact that Brownian motion is a **continuous martingale** will now play a major role in this chapter. We know from (3.2) that W_0-a.s., $t \to X_t(w)$ has infinite variation on any non-trivial interval of $\mathbb{R}_+$. As explained in the introduction, this precludes the definition of a Stieltjes-type integral "$dX_s(w)$", because "$dX_s(w)$ is not a signed measure". The next proposition will play a key role.

Proposition 4.1. *($d = 1$, on the canonical space $(C, \mathcal{F}, W_0)$)*

$$X_t \text{ is an } (\mathcal{F}_t^+)\text{-martingale,} \tag{4.1}$$

$$X_t^2 - t \text{ is an } (\mathcal{F}_t^+)\text{-martingale.} \tag{4.2}$$

Proof. For $0 \le s < t$:

$$\begin{aligned} E_0[X_t - X_s \,|\, \mathcal{F}_s^+] &= E_0[(X_{t-s} - X_0) \circ \theta_s \,|\, \mathcal{F}_s^+] \\ &\overset{(2.7)}{=} E_{X_s}[X_{t-s} - X_0] = 0, \;\; W_0\text{-a.s.,} \end{aligned} \tag{4.3}$$

an likewise:

$$\begin{aligned} &E_0[X_t^2 - X_s^2 - (t-s) \,|\, \mathcal{F}_s^+] = E_0[2(X_t - X_s)\, X_s + (X_t - X_s)^2 - (t-s) \,|\, \mathcal{F}_s^+] = \\ &2X_s\, E_0[X_t - X_s \,|\, \mathcal{F}_s^+] + E_0[(X_{t-s} - X_0)^2 \circ \theta_s \,|\, \mathcal{F}_s^+] - (t-s) \underset{(2.7)}{\overset{(4.3)}{=}} \\ &E_{X_s}[(X_{t-s} - X_0)^2] - (t-s) = 0, \;\; W_0\text{-a.s.}\,. \end{aligned} \tag{4.4}$$

The claims (4.1), (4.2) now follow since X_t and $X_t^2 - t$ are $(\mathcal{F}_t^+)$-adapted. □

We will later see that **the above two continuous martingales characterize Brownian motion!** (a fact due to Paul Lévy). The increasing process t that appears in (4.2) coincides with the limit of the quadratic variation of the Brownian path as discussed in (3.12).

Before discussing the construction of stochastic integrals, we introduce the following

Definition 4.2. *We say that a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \ge 0}, P)$ satisfies the usual conditions if:*

$$\mathcal{G}_0 \text{ contains all sets } N \in \mathcal{G} \text{ with } P(N) = 0, \text{ and} \tag{4.5}$$

$$(\mathcal{G}_t)_{t \ge 0} \text{ is right-continuous (i.e. } \mathcal{G}_t = \textstyle\bigcap_{\varepsilon > 0} \mathcal{G}_{t+\varepsilon}, \text{ for } t \ge 0). \tag{4.6}$$

Example:

We consider the canonical space $(C, \mathcal{F}, W_0)$ for the d-dimensional Brownian motion and define for $t \ge 0$ the σ-algebra:

$$F_t \overset{\text{def}}{=} \{A \in \mathcal{F};\; \exists B \in \mathcal{F}_t \text{ with } 1_A = 1_B,\; W_0\text{-a.s.}\}\,. \tag{4.7}$$

Proposition 4.3.

(4.8) $\mathcal{F}_t^+ \subseteq F_t$, *for* $t \geq 0$,

(4.9) $(F_t)_{t\geq 0}$, *is right-continuous,*

(4.10) $(C, \mathcal{F}, (F_t)_{t\geq 0}, W_0)$ *satisfies the usual conditions.*

Proof.

• (4.8): Observe that for $t \geq 0$,

(4.11) for any $A \in \mathcal{F}$, there exists a $Y \in b\mathcal{F}_t$, such that $E_0[1_A \,|\, \mathcal{F}_t^+] = Y$, W_0-a.s..

Indeed, when A is of the form, $t_0 = 0 < \cdots < t_k = t$, $0 < s_1 < \cdots < s_m$, $D_i \in \mathcal{B}(\mathbb{R}^d)$, for $0 \leq i \leq k+m$,

$$A = \{X_{t_0} \in D_0, \ldots, X_{t_k} \in D_k,\ X_{t_k+s_1} \in D_{k+1}, \ldots, X_{t_k+s_m} \in D_{k+m}\}\,,$$

it follows from (2.7) that

$$E_0[1_A \,|\, \mathcal{F}_t^+] \overset{W_0-\text{a.s.}}{=} 1\{X_{t_0} \in D_0, \ldots, X_{t_k} \in D_k\}\; W_{X_t}\left[X_{s_1} \in D_{k+1}, \ldots, X_{s_m} \in D_{k+m}\right],$$

which is $\mathcal{F}_t$-measurable.

The claim (4.11) now follows from Dynkin's lemma.

As a result of (4.11), we see that for $A \in \mathcal{F}_t^+$,

(4.12) $$1_A = E_0[1_A \,|\, \mathcal{F}_t^+] = Y,\ W_0\text{-a.s., for some } Y \in b\mathcal{F}_t\,.$$

It follows that $W_0(Y \notin \{0,1\}) = 0$, and hence $\widetilde{Y} \overset{\text{def}}{=} 1\{Y = 1\} = Y$, W_0-a.s.. Therefore, we have

(4.13) $$1_A = 1_{\{Y=1\}},\ W_0\text{-a.s., with } \{Y=1\} \in \mathcal{F}_t\,,$$

and (4.8) follows in view of (4.7).

• (4.9):

Let $A \in F_t^+ (\overset{\text{def}}{=} \bigcap_{\varepsilon>0} F_{t+\varepsilon})$, then for each $n \geq 1$, by (4.7) we can find $B_n \in \mathcal{F}_{t+\frac{1}{n}}$, with

(4.14) $$1_A = 1_{B_n},\ W_0\text{-a.s.}.$$

We now define $B \in \mathcal{F}_t^+$ via:

(4.15) $$1_B = \limsup_n\ 1_{B_n}$$

(B is indeed $\mathcal{F}_t^+$-measurable because it belongs to $\mathcal{F}_{t+\varepsilon}$ for each $\varepsilon > 0$). By (4.14) we find that

(4.16) $$1_B = 1_A,\ W_0\text{-a.s.},$$

and by (4.8) since $B \in \mathcal{F}_t^+$, we can find $C \in \mathcal{F}_t$ such that

$$1_C = 1_B = 1_A, \; W_0\text{-a.s.}.$$

This proves that $A \in F_t$, and hence (4.9) holds.

• (4.10):

From (4.7) we see that $N \in \mathcal{F}$ with $W_0(N) = 0$, belongs to F_0, so (4.5) holds. With (4.9) it follows that $(C, \mathcal{F}, (F_t)_{t \geq 0}, W_0)$ satisfies the usual conditions. □

Remark 4.4.

1) Note that (4.8) can be seen as a generalization of Blumenthal's $0-1$ law (2.15). Indeed, when $t = 0$, (4.8) implies that for any $A \in \mathcal{F}_0^+$ one can find a $B \in \mathcal{F}_0$ such that $1_A = 1_B$, W_0-a.s.. Moreover, $B \in \mathcal{F}_0$ is of the form $B = \{X_0 \in C\}$, for some $C \in \mathcal{B}(\mathbb{R})$. Hence, $W_0(B) = 1$, if $0 \in C$, and $W_0(B) = 0$, if $0 \notin C$. This shows that $W_0(A) = W_0(B) \in \{0, 1\}$, and we recover (2.15).

2) From (4.1), (4.2) it naturally follows that

(4.17) $\quad X_t$ is an $(F_t)_{t \geq 0}$-martingale,

(4.18) $\quad X_t^2 - t$ is an $(F_t)_{t \geq 0}$ martingale,

when one considers $(C, \mathcal{F}, (F_t)_{t \geq 0}, W_0)$, $(d = 1)$. □

We will now begin the **discussion of stochastic integrals**. We assume that

(4.19) $\quad (\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$ is a filtered probability space satisfying the usual conditions (4.5), (4.6),

(4.20) $\quad X_t$, $t \geq 0$, is a continuous square integrable $(\mathcal{G}_t)$-martingale,

(4.21) $\quad X_t^2 - t$, $t \geq 0$, is a $(\mathcal{G}_t)$-martingale.

A concrete example of this situation occurs for instance in (4.10), (4.17), (4.18), when considering $(C, \mathcal{F}, (F_t)_{t \geq 0}, W_0)$ and the canonical process $(X_t)_{t \geq 0}$.

Remark 4.5. We will later see, cf. Theorem 5.2 in Chapter 5, that when $(M_t)_{t \geq 0}$ is a continuous square integrable martingale on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0}, P)$, as in (4.19) (i.e. for each $t \geq 0$, $E[M_t^2] < \infty$), one can construct a process $\langle M \rangle_t$, $t \geq 0$, such that

(4.22) $\quad t \to \langle M \rangle_t(\omega)$ is non-decreasing continuous, for each $\omega \in \Omega$,

(4.23) $\quad \langle M \rangle_0 = 0$,

(4.24) $\quad (\langle M \rangle_t)_{t \geq 0}$, is $(\mathcal{G}_t)$-adapted, integrable,

(4.25) $\quad M_t^2 - \langle M \rangle_t$, $t \geq 0$ is a $(\mathcal{G}_t)$-martingale.

Moreover, $(\langle M\rangle_t)_{t\ge 0}$ is **essentially unique** (i.e. two such processes agree for all $t \ge 0$, except maybe on a negligible set of $\omega \in \Omega$), and it is called the "quadratic variation process'. The terminology stems from the fact that for $t \ge 0$,

$$\sum_{\frac{k+1}{2^n}\le t} \Big(M_{\frac{k+1}{2^n}} - M_{\frac{k}{2^n}}\Big)^2 \underset{n\to\infty}{\longrightarrow} \langle M\rangle_t \text{ in } P\text{-probability},$$

see [8], Section 1.5. □

We are going to define the integral $\int_0^t H_s\,dX_s$ for some suitable basic integrands, for which the definition is "natural", and then we will use an isometry property to extend the class of processes we integrate. Later, we will further extend the class of integrands by a so-called "localization argument".

Our building blocks are the **basic processes**:

$$H_s(\omega) = C(\omega)\,1\{a < s \le b\},\ s \ge 0,\ \omega \in \Omega,\ \text{with } C \in b\mathcal{G}_a,\ 0 \le a \le b\,. \tag{4.26}$$

For such an H as in (4.26), we define:

$$\int_0^\infty H_s\,dX_s \overset{\text{def}}{=} C(\omega)\big(X_b(\omega) - X_a(\omega)\big) \in L^2(P)\,. \tag{4.27}$$

The restriction $C(\omega) \in b\mathcal{G}_a$ is not a priori natural. It is motivated by the fact that if we define

$$\underbrace{(H.X)_t}_{\text{also denoted } \int_0^t H_s\,dX_s} \overset{\text{def}}{=} \int_0^\infty H_s\,1_{[0,t]}(s)\,dX_s \overset{(4.27)}{=} C(\omega)\big(X_{b\wedge t}(\omega) - X_{a\wedge t}(\omega)\big) \tag{4.28}$$

we have the

Proposition 4.6.

$$M_t = (H.X)_t \text{ is a continuous square integrable } (\mathcal{G}_t)\text{-martingale}\,. \tag{4.29}$$

Proof.

$$\begin{aligned} (H.X)_t &= C(\omega)(X_{b\wedge t} - X_{a\wedge t}) \\ &= 0, && \text{if } 0 \le t \le a\,, \\ &= C(X_t - X_a), && \text{if } a \le t \le b\,, \\ &= C(X_b - X_a), && \text{if } b \le t \le \infty\,, \end{aligned}$$

clearly defines a continuous adapted process which is square integrable. Considering the case $a \le s < t \le b$ (the other cases are simpler), we see that

$$\begin{aligned} E[M_t - M_s \,|\, \mathcal{G}_s] &= E[\underbrace{C}_{b\mathcal{G}_a\subseteq b\mathcal{G}_s}(X_t - X_s)\,|\,\mathcal{G}_s] \overset{P\text{-a.s.}}{=} C\,E[X_t - X_s\,|\,\mathcal{G}_s] \\ &= 0\,. \end{aligned}$$

Our claim follows. □

The next step is the following

Proposition 4.7. *If H, K are basic processes, then*

$$E[(H.X)_t\ (K.X)_t] = E\Big[\int_0^t H_s(\omega)\, K_s(\omega) ds\Big], \text{ for } 0 \le t \le \infty\,. \tag{4.30}$$

Proof. It suffices to consider the case $t = \infty$, because

$$(H.X)_t = (H\, 1_{[0,t]}.X)_\infty\,.$$

It also suffices to check (4.30) when $(a, b] = (c, d]$ or $(a, b]$ "<" $(c, d]$ (i.e. $a \le b \le c \le d$), and $H = C\, 1_{(a,b]}$, $K = D\, 1_{(c,d]}$.

Indeed, one makes repeated use of identities such as

$$\begin{aligned} &\text{for } 0 \le \alpha \le \beta \le \gamma,\ \ H = C\, 1_{(\alpha,\gamma]},\ \ H^1 = C\, 1_{(\alpha,\beta]},\ \ H^2 = C\, 1_{(\beta,\gamma]},\\ &\int_0^\infty H_s\, dX_s = \int_0^\infty H_s^1\, dX_s + \int_0^\infty H_s^2\, dX_s\,. \end{aligned} \tag{4.31}$$

We will thus only need to check (4.30) in two cases:

- Case $(a, b] = (c, d]$:

$$\begin{aligned} &E[(H.X)_\infty\, (K.X)_\infty] = E[\overbrace{CD}^{\in b\mathcal{G}_a}(X_b - X_a)^2] =\\ &E[CDE[(X_b - X_a)^2 \,|\, \mathcal{G}_a]]\,. \end{aligned} \tag{4.32}$$

Note that:

$$\begin{aligned} &E[(X_b - X_a)^2 \,|\, \mathcal{G}_a] = E[X_b^2 - 2X_b\, X_a + X_a^2 \,|\, \mathcal{G}_a] =\\ &E[X_b^2 \,|\, \mathcal{G}_a] - 2X_a\, E[X_b \,|\, \mathcal{G}_a] + X_a^2 \overset{(4.20)}{=} E[X_b^2 \,|\, \mathcal{G}_a] - X_a^2 =\\ &E[X_b^2 - b \,|\, \mathcal{G}_a] + b - X_a^2 \overset{(4.21)}{=} X_a^2 + b - X_a^2 = b - a\,. \end{aligned} \tag{4.33}$$

Thus coming back to (4.32) we have shown that

$$E[(H.X)_\infty\, (K.X)_\infty] = E[CD(b - a)] = E\Big[\int_0^\infty H_s(\omega)\, K_s(\omega) ds\Big]\,,$$

i.e. (4.30) holds.

- Case $(a, b]$ "<" $(c, d]$:

$$\begin{aligned} E[\underbrace{(H.X)_\infty}_{\mathcal{G}_b-\text{meas}} (K.X)_\infty] &= E[(H.X)_\infty\, E[(K.X)_\infty \,|\, \mathcal{G}_b]]\\ &\overset{(4.29)}{=} E[(H.X)_\infty\, \underbrace{(K.X)_b}_{=0}] = 0\,. \end{aligned}$$

Analogously we have

$$E\Big[\int_0^\infty H_s(\omega)\, K_s(\omega)\, ds\Big] = 0, \text{ and (4.30) holds.}$$

□

Remark 4.8. If one replaces $(X_t)_{t\geq 0}$ satisfying (4.20), (4.21), with $(M_t)_{t\geq 0}$, a continuous square integrable martingale, and defines for basic processes $H_s(\omega) = C(\omega)\,1\{a < s \leq b\}$, with $C \in b\mathcal{G}_a$,

$$\int_0^\infty H_s\,dM_s = C(M_b - M_a), \text{ and } \int_0^t H_s\,dM_s = \int_0^\infty H_s\,1_{[0,t]}(s)\,dM_s,\ 0 \leq t \leq \infty\,, \tag{4.34}$$

then (4.30) is replaced by

$$\begin{aligned} &E[(H.M)_t\,(K.M)_t] = E\Big[\int_0^t H_s(\omega)\,K_s(\omega)\,d\langle M\rangle_s(\omega)\Big]\,, \\ &\text{with } \langle M\rangle_{.} \text{ the quadratic variation of } M, \text{ and } (H.M)_{.} \\ &\text{defined as in (4.28) with } M \text{ replacing } X_{.}. \end{aligned} \tag{4.35}$$

□

We now define the class Λ_1 of **simple processes**:

$$\Lambda_1 = \{H_s(\omega) = H_s^1(\omega) + \dots + H_s^n(\omega),\ H^i \text{ are basic}\}\,. \tag{4.36}$$

Proposition and definition:

$$\text{For } H \in \Lambda_1,\ \int_0^\infty H_s\,dX_s \overset{\text{def}}{=} \sum_{i=1}^n \int_0^\infty H_s^i\,dX_s, \text{ is well defined.} \tag{4.37}$$

Proof. The only point to check is that when $H^1, \dots, H^p$ are basic processes such that $H^1 + \dots + H^p = 0$, then $\sum_{i=1}^p \int_0^\infty H_s^i\,dX_s = 0$.

Making repeated use of (4.31) and

$$\begin{aligned} &H = C\,1_{(\alpha,\beta]},\ K = D\,1_{(\alpha,\beta]} \text{ basic processes, then} \\ &\int_0^\infty H_s\,dX_s + \int_0^\infty K_s\,dX_s = \int_0^\infty L_s\,dX_s, \text{ with} \\ &L = (C + D)\,1_{(\alpha,\beta]} \text{ basic process,} \end{aligned} \tag{4.38}$$

we can assume that $H^i = C_i\,1_{I^i}$, $1 \leq i \leq p$, with $I^i \cap I^j = \emptyset$, when $i \neq j$. In this case $H^1 + \dots + H^p = 0$ implies $H^1 = H^2 = \dots = H^p = 0$, and hence $\sum_{i=1}^p \int_0^\infty H_s^i\,dX_s = 0$. Our claim is thus proved. □

As a consequence of (4.29) and (4.37) we see that for H, $K \in \Lambda_1$,

$$\begin{aligned} &\int_0^t H_s\,dX_s\Big(\overset{\text{def}}{=} \int_0^\infty H_s\,1_{[0,t]}(s)\,dX_s\Big) \text{ is a continuous square} \\ &\text{integrable martingale, and} \\ &E\Big[\int_0^\infty H_s\,dX_s \int_0^\infty K_s\,dX_s\Big] = E\Big[\int_0^\infty H_s\,K_s\,ds\Big]\,. \end{aligned} \tag{4.39}$$

Remark 4.9. In the case of a general continuous square integrable $(\mathcal{G}_t)$-martingale $(M_t)_{t\geq 0}$, in place of $(X_t)_{t\geq 0}$, we can use the same construction as above. The role of ds is simply replaced by $d\langle M\rangle_s(\omega)$ so that for $H, K \in \Lambda_1$ one has

$$(4.40)\qquad E\Big[\int_0^\infty H_s\, dM_s \int_0^\infty K_s\, dM_s\Big] = E\Big[\int_0^\infty H_s(\omega)\, K_s(\omega)\, d\langle M\rangle_s(\omega)\Big]\,.$$

□

As a result of (4.39), we see that

$$(4.41)\qquad \begin{array}{l} K \in \Lambda_1 \longrightarrow \displaystyle\int_0^\infty K_s\, dX_s \in L^2(\Omega, \mathcal{G}, P) \text{ is an isometry,} \\ \text{if } \Lambda_1 \text{ is viewed as a subspace of } L^2(\Omega \times \mathbb{R}_+,\, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+), dP \otimes ds)\,. \end{array}$$

Note that Λ_1 is in general not dense in $L^2(\Omega \times \mathbb{R}_+,\, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+), dP \otimes ds)$ since all K in Λ_1 are progressively measurable processes. We hence consider

$$(4.42)\qquad \begin{array}{l} \mathcal{P} = \text{the } \sigma\text{-algebra of progressively measurable sets in } \Omega \times \mathbb{R}_+, \\ \qquad \text{(i.e. of } A \in \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+) \text{ such that for all } t \geq 0, \\ \qquad A \cap (\Omega \times [0,t]) \in \mathcal{G}_t \otimes \mathcal{B}([0,t]))\,. \end{array}$$

Remark 4.10. A process $Z_u(\omega)$ on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ is progressively measurable in the sense of the definition below (2.61) exactly when

$$(4.43)\qquad (\Omega \times \mathbb{R}_+, \mathcal{P}) \xrightarrow{Z} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is measurable}\,.$$

□

We then define:

$$(4.44)\qquad \Lambda_2 = L^2(\Omega \times \mathbb{R}_+,\, \mathcal{P},\, dP \otimes ds)\,,$$

the set of progressively measurable processes $H_s(\omega)$ for which $E[\int_0^\infty H_s^2(\omega) ds] < \infty$. The interest of this definition comes from the next

Proposition 4.11.

(4.45) Λ_1 *is a dense subset of* Λ_2 *for the* $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP \otimes ds)$*-distance,*

(4.46) $H \to \displaystyle\int_0^\infty H_s\, dX_s$ *extends uniquely into an isometry from* Λ_2 *into* $L^2(\Omega, \mathcal{G}, P)$.

(we will also write $\int_0^t H_s\, dX_s$ for $\int_0^\infty H_s\, 1_{[0,t]}(s)\, dX_s$, for $0 \leq t \leq \infty$).

Proof. In view of (4.41) we see that (4.46) immediately follows from (4.45).

The proof of (4.45) will in fact rely on a lemma, which is more general than what is needed to prove (4.45), but applies as well to the subsequent discussion of stochastic integrals with respect to continuous square integrable martingales. The non-decreasing process $t \to A_t(\omega)$ in the next lemma plays the role of $t \to \langle M\rangle_t(\omega)$, cf. (4.22).

Lemma 4.12. *Suppose that A_t, $t \ge 0$, is a continuous $(\mathcal{G}_t)$-adapted process, non-decreasing in t, with $A_0 = 0$, and $E[A_t] < \infty$, for every $t \ge 0$, then*

(4.47) Λ_1 *is a dense subset of $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP \times dA_s)$ for the $L^2(dP \times dA_s)$-distance.*

(The σ-finite measure $d\mu = dP \times dA_s$ is defined via

$$\mu(B) = \int_\Omega \Big(\int_0^\infty 1_B(\omega, u)\, dA_u(\omega)\Big)\, dP(\omega), \text{ for } B \in \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+) \supset \mathcal{P}\ ).$$

With the above lemma, (4.45) clearly follows.

We have thus reduced the proof of the important Proposition 4.11, to the

Proof of Lemma 4.12: Since $E[A_t] < \infty$, for each $t \ge 0$, it follows that indeed $\Lambda_1 \subset \Lambda_2 \stackrel{\text{def}}{=} L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP \times dA_s)$.

We further observe that

$$\widetilde{A}_t = t + A_t,\ t \ge 0\,,$$

satisfies the same assumptions as A_t, $t \ge 0$, and proving (4.47) for $\widetilde{A}_t$, $t \ge 0$, implies (4.47) for A_t, $t \ge 0$. We thus assume that for $\omega \in \Omega$:

$$\begin{array}{l} t \in [0,\infty) \longrightarrow A_t(\omega) \in [0,\infty) \text{ is an increasing bijection,} \\ \text{and for } 0 \le s \le t,\ t - s \le A_t(\omega) - A_s(\omega)\,. \end{array} \tag{4.48}$$

We define for $H \in \Lambda_2$,

$$H^n = 1_{[0,n]} \times \{(-n) \vee (H \wedge n)\} \in \Lambda_2\,, \tag{4.49}$$

and we find that by dominated convergence:

$$\|H - H^n\|_{L^2(dP\times dA)} \underset{n\to\infty}{\longrightarrow} 0\,. \tag{4.50}$$

We then introduce the inverse function of A.

$$\tau_u = \inf\{t \ge 0;\ A_t > u\}, \ \text{ for } u \ge 0 \tag{4.51}$$

(we will sometimes write $\tau(u)$ in place of τ_u).

Note that for $f \ge 0$, $\mathcal{B}(\mathbb{R}_+)$-measurable and $\omega \in \Omega$:

$$\int_0^\infty f(t)\, dA_t = \int_0^\infty f(\tau_u)\, du \quad \text{"change of variable formula".} \tag{4.52}$$

(Indeed this identity holds when $f = 1_{[a,b]}$ with $a \le b$, since $\tau_u \in [a,b]$ is equivalent to $u \in [A_a, A_b]$. Then, by Dynkin's lemma, (4.52) holds for any $f = 1_C$, with $C \in \mathcal{B}([0,T])$, $T > 0$, arbitrary, and the general case follows by approximation).

We then define for $n \geq 1$, $\ell \geq 0$,

$$
(4.53) \qquad \begin{array}{l} H_t^{n,\ell}(\omega) = 2^\ell \displaystyle\int_{\tau(A_t - 2^{-\ell})}^t H_s^n(\omega)\, dA_s(\omega), \text{ for } t \geq 0,\, \omega \in \Omega\,, \\ \text{where by convention } A_t = 0, \text{ for } t \leq 0,\ \tau(u) = 0, \text{ for } u \leq 0\,. \end{array}
$$

Clearly $H_\cdot^{n,\ell}$ is bounded in absolute value by $\|H^n\|_\infty$. It is a continuous function of t, and it vanishes when $t > n + 2^{-\ell}$ (indeed $A_{n+2^{-\ell}} \overset{(4.48)}{\geq} A_n + 2^{-\ell}$, so that for $t > n + 2^{-\ell}$, $A_t - 2^{-\ell} > A_n$, and hence $\tau(A_t - 2^{-\ell}) > n$, which in turns implies that the integral in (4.53) vanishes in view of (4.49). Moreover

$$
(4.54) \qquad H^{n,\ell} \text{ is } (\mathcal{G}_t)\text{-adapted.}
$$

Indeed $\tau(A_t - 2^{-\ell}) = \inf\{s \geq 0; A_s > A_t - 2^{-\ell}\}$ is $\mathcal{G}_t$-measurable (simply observe that for $u \leq t$, $\{\tau(A_t - 2^{-\ell}) < u\} = \{\text{for some } v \in \mathbb{Q} \cap [0,u),\ A_v > A_t - 2^{-\ell}\} \in \mathcal{G}_t$, and it equals $\Omega \in \mathcal{G}_t$, when $u > t$). Moreover for any $F \in b\mathcal{G}_t \otimes \mathcal{B}([0,t])$, $\int_0^t F_s(\omega)\, dA_s(\omega)$ is $\mathcal{G}_t$-measurable, as follows from Dynkin's lemma, approximation, and consideration of functions of the form $F = 1_{D \times [a,b]}$, with $D \in \mathcal{G}_t$, $0 \leq a \leq b \leq t$. Coming back to (4.53), the claim (4.54) follows.

Now, as a result of (4.52) we find that:

$$
(4.55) \qquad \int_0^\infty (H_t^n(\omega) - H_t^{n,\ell}(\omega))^2\, dA_t = \int_0^\infty (H_{\tau_u}^n(\omega) - H_{\tau_u}^{n,\ell}(\omega))^2\, du
$$

and for $u \geq 0$,

$$
(4.56) \qquad \begin{aligned} H_{\tau_u}^{n,\ell}(\omega) &\overset{(4.53)}{=} 2^\ell \int_{\tau(\underbrace{A_{\tau_u}}_{\| \atop u} - 2^{-\ell})}^{\tau_u} H_s^n(\omega)\, dA_s = 2^\ell \int_{\tau(u - 2^{-\ell})}^{\tau_u} H_s^n(\omega)\, dA_s \\ &= 2^\ell \int_0^\infty 1\{\tau(u - 2^{-\ell}) \leq s \leq \tau_u\}\, H_s^n(\omega) dA_s \\ &\overset{(4.52)}{=} 2^\ell \int_0^\infty 1\{\tau(u - 2^{-\ell}) \leq \tau_v \leq \tau_u\}\, H_{\tau_v}^n(\omega)\, dv \\ &= 2^\ell \int_{(u - 2^{-\ell})_+}^u H_{\tau_v}^n(\omega)\, dv\,. \end{aligned}
$$

Note that for any $g \in L^2(\mathbb{R}_+, du)$

$$
g_\ell(u) = 2^\ell \int_0^\infty 1\{u - 2^{-\ell} \leq v \leq u\}\, g(v)\, dv \ \underset{\ell \to \infty}{\overset{L^2(du)}{\longrightarrow}}\ g(u)
$$

(this follows directly from the continuity of translations in $L^2(\mathbb{R}, du)$).

Thus, combining (4.55) and (4.56), it follows by dominated convergence that

$$
(4.57)\qquad \begin{aligned} &\|H^n - H^{n,\ell}\|^2_{L^2(dP\times dA)} = \\ &E\Big[\int_0^\infty (H_t^n(\omega) - H_t^{n,\ell}(\omega))^2\, dA_t(\omega)\Big] \underset{\ell\to\infty}{\longrightarrow} 0, \text{ for any } n \geq 1\,. \end{aligned}
$$

We can now define for $n \geq 1$, ℓ, $m \geq 0$:

$$
(4.58)\qquad H_t^{n,\ell,m}(\omega) = \sum_{k\geq 0} H^{n,\ell}_{\frac{k}{2^m}}(\omega)\, 1_{(\frac{k}{2^m},\frac{k+1}{2^m}]}(t), \text{ for } t \geq 0,\ \omega \in \Omega\,.
$$

Clearly $H^{n,\ell,m} \in \Lambda_1$ are uniformly bounded in m, and for $t > 0$, $\omega \in \Omega$, thanks to the continuity of $H_{.}^{n,\ell}(\omega)$, $H_t^{n,\ell,m}(\omega) \underset{m\to\infty}{\longrightarrow} H_t^{n,\ell}(\omega)$. Since dA_t does not give positive mass to $\{0\}$, we find that:

$$
(4.59)\qquad \|H^{n,\ell} - H^{n,\ell,m}\|^2_{L^2(dP\times dA)} \underset{m\to\infty}{\longrightarrow} 0, \text{ for } n \geq 1,\ \ell \geq 0\,.
$$

Combining (4.50), (4.57), (4.59) we have proved (4.45). □

This concludes the proof of the Proposition 4.11. □

Remark 4.13.

1) Reconstructing some trajectorial character to the stochastic integral.

Note that when H and K belong to Λ_2, and $G \in \mathcal{G}$ are such that

$$
(4.60)\qquad H_s(\omega) = K_s(\omega) \text{ for all } s \geq 0, \text{ and } \omega \in G,
$$

then we see from (4.49) that a similar identity holds for H^n and K^n, from (4.53), that the same holds for $H^{n,\ell}$ and $K^{n,\ell}$, and finally from (4.58), that the same holds for $H^{n,\ell,m}$ and $K^{n,\ell,m}$. As a result we can find $H^{(i)}$ and $K^{(i)}$ in Λ_1, $i \geq 1$, with the property:

$$
(4.61)\qquad \begin{aligned} &H^{(i)} \to H \text{ in } L^2(dP\otimes ds),\ K^{(i)} \to K \text{ in } L^2(dP\otimes ds), \text{ and for all } i \geq 1, \\ &H_s^{(i)}(\omega) = K_s^{(i)}(\omega), \text{ for all } s \geq 0 \text{ and } \omega \in G\,. \end{aligned}
$$

On the other hand when $H, K \in \Lambda_1$ are such that $H_{.}(\omega) = K_{.}(\omega)$ for $\omega \in G$, one checks from (4.27), (4.28), (4.37), (4.39) that

$$
(4.62)\qquad (H.X)_t(\omega) = (K.X)_t(\omega), \text{ for } 0 \leq t \leq \infty, \text{ and } \omega \in G\,.
$$

Combining this observation with (4.61) and (4.46), we see that:

$$
(4.63)\qquad \text{when } H, K \in \Lambda_2 \text{ satisfy (4.60), then } \int_0^\infty H_s\, dX_s = \int_0^\infty K_s\, dX_s,\ P\text{-a.s. on } G\,.
$$

This somehow reconstructs some trajectorial character to the stochastic integral.

2) The class of processes we can integrate has severe limitations.

If we consider the canonical space $(C, \mathcal{F}, (F_t)_{t\geq 0}, W_0)$ with $(X_t)_{t\geq 0}$, the canonical process, we can now consider

$$\int_0^1 e^{\alpha X_s} dX_s \Big(= \int_0^\infty 1_{[0,1]}(s)\, e^{\alpha X_s} dX_s \Big), \text{ for } \alpha \in \mathbb{R}$$

because $e^{\alpha X_s}$ is progressively measurable and

$$E_0\Big[\int_0^\infty 1_{[0,1]}(s)\, e^{2\alpha X_s} ds\Big] = \int_0^1 E_0[e^{2\alpha X_s}]\, ds = \int_0^1 e^{2\alpha^2 s}\, ds < \infty\,,$$

so that $1_{[0,1]}(s)\, e^{\alpha X_s}$ belongs to Λ_2, for all $\alpha \in \mathbb{R}$.

On the other hand, if we consider $\alpha \in \mathbb{R}$ and

$$\int_0^1 e^{\alpha X_s^2} dX_s\,, \tag{4.64}$$

then, we observe that

$$E_0\Big[\int_0^\infty 1_{[0,1]}(s)\, e^{2\alpha X_s^2} ds\Big] = \int_0^1 \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}}\, e^{(2\alpha - \frac{1}{2s})x^2} dx\, ds =$$

$$\int_0^1 (1-4\alpha s)_+^{-\frac{1}{2}} ds < \infty \text{ when } \alpha < \frac{1}{4}\,,$$

$$= \infty \text{ when } \alpha \geq \frac{1}{4}\,.$$

Thus, at the present state of the construction of stochastic integrals, $\int_0^1 e^{\frac{1}{10} X_s^2}\, dX_s$ is meaningful, but $\int_0^1 e^{X_s^2}\, dX_s$ is not!

We will later extend the definition of stochastic integrals so that $\int_0^1 e^{X_s^2}\, dX_s$ (or even $\int_0^1 e^{(e^{X_s})} dX_s$!) are well-defined. However, in the theory we develop

$$\int_0^1 X_1\, dX_s \text{ will not be defined because } 1_{[0,1]} X_1 \text{ is not } \mathcal{P}\text{-measurable}\,. \tag{4.65}$$

$\square$

Observe that given $H \in \Lambda_2$ and $\|H^n - H\|_{L^2(dP\otimes ds)} \underset{n\to\infty}{\longrightarrow} 0$, with $H^n \in \Lambda_1$ for each n, we know that for each $t \geq 0$, $(H^n.X)_t \underset{n\to\infty}{\longrightarrow} (H.X)_t$ in $L^2(\Omega, \mathcal{G}, P)$ and in fact $(H.X)_t \in L^2(\Omega, \mathcal{G}_t, P)$. We are now going to **select a nice version of the process $(H.X)_t$, $t \geq 0$**, so that it defines a continuous square integrable $(\mathcal{G}_t)$-martingale. We recall Doob's inequality in the discrete setting:

Proposition 4.14. *Consider a filtered probability space $(\Omega, F, (F_m)_{m\ge 0}, P)$ and $(X_m)_{m\ge 0}$, an (F_m)-submartingale (i.e. X_m is F_m-measurable and integrable, and $E[X_{m+1} \,|\, F_m] \ge X_m$, for $m \ge 0$). Then for $\lambda > 0$, $n \ge 0$, $A = \{\omega \in \Omega;\ \sup_{0\le m\le n} X_m(\omega) \ge \lambda\}$, one has*

$$\lambda P[A] \le E[X_n \, 1_A] \le E[X_n^+] \tag{4.66}$$

(see [5], p. 215).

In the continuous time set-up we obtain:

Proposition 4.15. *Consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ and $(X_t)_{t\ge 0}$, a continuous $(\mathcal{G}_t)$-submartingale. Then for $\lambda > 0$, $t \ge 0$, and $A = \{\sup_{0\le u\le t} X_u \ge \lambda\}$ one has*

$$\lambda P\big[\sup_{0\le u\le t} X_u \ge \lambda\big] \le E[X_t \, 1_A] \le E[X_t^+]\,. \tag{4.67}$$

Proof. It suffices to prove that for $\lambda > 0$:

$$\lambda P\big[\sup_{0\le u\le t} X_u > \lambda\big] \le E[X_t \, 1\{\sup_{0\le u\le t} X_u > \lambda\}\big]\,. \tag{4.68}$$

One then applies (4.68) to $\lambda_n \uparrow \lambda$ and obtains (4.67). By the same argument with $\lambda_n \downarrow \lambda$, we deduce from (4.66) that for $\lambda > 0$, one has:

$$\lambda P\big[\sup_{0\le m\le 2^\ell} X_{\frac{mt}{2^\ell}} > \lambda\big] \le E\big[X_t \, 1\{\sup_{0\le m\le 2^\ell} X_{\frac{mt}{2^\ell}} > \lambda\}\big]\,.$$

Letting $\ell \uparrow \infty$, since $\{\sup_{0\le m\le 2^\ell} X_{\frac{mt}{2^\ell}} > \lambda\} \uparrow \{\sup_{0\le u\le t} X_u > \lambda\}$, as $\ell \uparrow \infty$, we obtain (4.68), and our claim is proved. □

Doob's inequality will be a key tool for the construction of a good version of $\int_0^t H_s \, dX_s$, when $H \in \Lambda_2$.

We now proceed to the **construction of a good version of the stochastic integral** $\int_0^t H_s \, dX_s$, for $H \in \Lambda_2$, cf. (4.44). We recall our standing assumptions (4.19), (4.20), (4.21).

Theorem 4.16. *For $H_s(\omega) \in \Lambda_2$, there is a process $(I_t)_{0\le t\le\infty}$, essentially unique (i.e. two such processes, except on a P-negligible set, agree for all $t \ge 0$), continuous, $(\mathcal{G}_t)$-adapted, such that:*

$$\text{for each } 0 \le t \le \infty,\ I_t = \int_0^t H_s \, dX_s,\ P\text{-a.s.}, \tag{4.69}$$

$$\begin{aligned} &(I_t)_{0\le t\le\infty} \text{ is a continuous square integrable } (\mathcal{G}_t)\text{-martingale},\\ &\text{(and of course } E[I_t^2] \overset{(4.46)}{=} E\Big[\int_0^t H_s^2(\omega) ds\Big], \text{ for } 0 \le t \le \infty). \end{aligned} \tag{4.70}$$

Proof. When $H \in \Lambda_1$, our definition of $\int_0^t H_s\, dX_s$ satisfies the above properties, see (4.37), (4.28), (4.29). When $H \in \Lambda_2$, we pick $H^n \in \Lambda_1$, $n \geq 0$, with $\lim_n \|H - H^n\|_{L^2(dP \otimes ds)} = 0$. As a result of (4.46), for $0 \leq s \leq t$, $A \in \mathcal{G}_s$,

$$\begin{array}{ccc} E\Big[\displaystyle\int_0^t H_u^n\, dX_u\, 1_A\Big] & = & E\Big[\displaystyle\int_0^s H_u^n\, dX_u\, 1_A\Big] \\ \downarrow n \to \infty & & \downarrow n \to \infty \\ E\Big[\displaystyle\int_0^t H_u\, dX_u\, 1_A\Big] & = & E\Big[\displaystyle\int_0^s H_u\, dX_u\, 1_A\Big] \end{array}$$

and by the discussion below (4.65), we thus find that:

$$E[(H.X)_t \,|\, \mathcal{G}_s] = (H.X)_s,\ P\text{-a.s.}, \tag{4.71}$$

so the martingale property comes for free. We thus only need to find $I_t(\omega)$ a continuous $(\mathcal{G}_t)$-adapted process for which (4.69) holds. We choose $n_k \to \infty$ such that

$$\sum_k k^4\, \|H^{n_k} - H^{n_{k+1}}\|^2_{L^2(\mathcal{P}, dP \otimes ds)} < \infty\,. \tag{4.72}$$

Then for each $k \geq 0$, $((H^{n_k} - H^{n_{k+1}}).X)_t^2$ is a continuous submartingale and by Doob's inequality (4.67), for $\lambda > 0$:

$$\begin{aligned} \lambda^2\, P\big[\sup_{u \geq 0} |(H^{n_k}.X)_u - (H^{n_{k+1}}.X)_u| \geq \lambda\big] &\leq E\Big[\Big(\int_0^\infty \big(H_s^{n_k} - H_s^{n_{k+1}}\big)\, dX_s\Big)^2\Big] \\ &= \|H^{n_k} - H^{n_{k+1}}\|^2_{L^2(\mathcal{P}, dP \otimes ds)}\,. \end{aligned} \tag{4.73}$$

Choosing $\lambda = k^{-2}$, we obtain

$$P\big[\sup_{u \geq 0} |\,(H^{n_k}.X)_u - (H^{n_{k+1}}.X)_u\,| \geq k^{-2}\big] \leq k^4 \|H^{n_k} - H^{n_{k+1}}\|^2_{L^2(\mathcal{P}, dP \otimes ds)}\,.$$

Applying Borel-Cantelli's lemma, we can find $N \in \mathcal{G}$ with $P(N) = 0$, such that for $\omega \notin N$, we have $k_0(\omega) < \infty$, such that

$$\sup_{u \geq 0} |(H^{n_k}.X)_u(\omega) - (H^{n_{k+1}}.X)_u(\omega)| \leq \frac{1}{k^2},\ \text{ for } k \geq k_0(\omega)\,. \tag{4.74}$$

As a result for $\omega \notin N$, $(H^{n_k}.X)_u(\omega)$ converges uniformly on $[0, \infty]$. We thus define:

$$\begin{aligned} I_u(\omega) &= \lim_k (H^{n_k}.X)_u(\omega),\ \text{ for } \omega \notin N\,, \\ &= 0 \qquad\qquad\qquad\quad\ ,\ \text{ for } \omega \in N\,, \end{aligned} \tag{4.75}$$

so that $u \in [0, \infty] \to I_u(\omega)$ is continuous for all $\omega \in \Omega$, and $I_u(\cdot)$ is $\mathcal{G}_u$-measurable (we use here the fact that $\mathcal{G}_u$ contains all negligible sets of $\mathcal{G}$, see (4.5)).

Observe that $(H^{n_k}.X)_u \overset{L^2(P)}{\longrightarrow} (H.X)_u$, for $0 \leq u \leq \infty$, and P-a.s., $(H^{n_k}.X)_u \to I_u$. As a result $I_u \overset{P\text{-a.s.}}{=} (H.X)_u$, and (4.69) holds. The theorem is proved. $\square$

From now on $(H.X)_t$ will denote the essentially unique regular version I_t. We will use the following inequality:

Proposition 4.17. *If $(X_t)_{t\geq 0}$ is a continuous non-negative submartingale on a filtered probability space, then for $0 \leq t < \infty$, $p \in (1,\infty)$*

$$E\big[\sup_{s\leq t} X_s^p\big]^{1/p} \leq \frac{p}{p-1}\, E[X_t^p]^{1/p} . \tag{4.76}$$

Proof. We apply the discrete time inequality to $X_{\frac{kt}{2^n}}$, $0 \leq k \leq 2^n$, and let $n \to \infty$ (see for instance [5], p. 216 for the discrete time inequality). $\square$

As an immediate application we have:

$$E\big[\sup_{t\geq 0} (H.X)_t^2\big]^{1/2} \leq 2\, \|H\|_{L^2(\mathcal{P},dP\otimes ds)}, \text{ for } H \in \Lambda_2 . \tag{4.77}$$

We now proceed to the next

Proposition 4.18. *For $H, K \in \Lambda_2$, the essentially uniquely defined process*

$$N_t \overset{\text{def}}{=} (H.X)_t (K.X)_t - \int_0^t H_s(\omega)\, K_s(\omega)\, ds, \;\; 0 \leq t \leq \infty , \tag{4.78}$$

is a continuous $(\mathcal{G}_t)$-martingale and

$$\sup_{t\geq 0} |N_t| \in L^1(\Omega, \mathcal{G}, P) . \tag{4.79}$$

Proof. Note that by the Cauchy-Schwarz inequality $E[\int_0^\infty |H_s(\omega)|\, |K_s(\omega)| ds] \leq \|H\|_{L^2(dP\otimes ds)} \|K\|_{L^2(dP\otimes ds)} < \infty$, so that (4.78) is well-defined for all $0 \leq t \leq \infty$, and ω outside the negligible set N where $\int_0^\infty |H_s(\omega)|\, |K_s(\omega)| ds = \infty$. It also defines a process with continuous trajectories outside N, and setting for instance $\int_0^t H_s(\omega)\, K_s(\omega) ds \equiv 0$, for $\omega \in N$, the property (4.79) is an immediate consequence of (4.77), and the above inequality. We thus only need to check that N_t is a $(\mathcal{G}_t)$-martingale.

- 1st case: H, K are basic (similar to (4.30)):

We only need to treat the case of $H = C\, 1_{(a,b]}$, $K = D 1_{(c,d]}$, with either $(a,b] = (c,d]$ or "$(a,b] < (c,d]$".

If $(a,b] = (c,d]$, then for $t \geq 0$,

$$N_t = CD\{(X_{t\wedge b} - X_{t\wedge a})^2 - (t \wedge b - t \wedge a)\} \quad ((\mathcal{G}_t)\text{-measurable}) \tag{4.80}$$

is a martingale because it is adapted, and when for instance $a \leq s < t \leq b$:

$$\begin{aligned} E[N_t \,|\, \mathcal{G}_s] &= E[(X_t - X_a)^2 - (t-a) \,|\, \mathcal{G}_s]\, CD \\ &= E[X_t^2 - 2X_t\, X_a + X_a^2 - (t-a) \,|\, \mathcal{G}_s]\, CD \\ &\overset{(4.20),(4.21)}{=} (X_s^2 - s - 2X_s\, X_a + X_a^2 + a)\, CD \\ &= \{(X_s - X_a)^2 - (s-a)\}\, CD = N_s , \end{aligned} \tag{4.81}$$

and the other cases are easier to check.

If "$(a,b] < (c,d]$":

$$N_t = (X_{t\wedge b} - X_{t\wedge a})(X_{t\wedge d} - X_{t\wedge c})\, CD,\ t \ge 0\,, \tag{4.82}$$

is a martingale because it is adapted, and when for instance $c \le s < t \le d$:

$$E[N_t \,|\, \mathcal{G}_s] = CD(X_b - X_a)\, E[X_t - X_c \,|\, \mathcal{G}_s] \overset{(4.20)}{=} CD(X_b - X_a)(X_s - X_c) = N_s\,, \tag{4.83}$$

and the other cases are simpler to check.

• 2^{nd} case: $H, K \in \Lambda_1$:

Immediate from the previous case by bilinearity.

• General case: $H, K \in \Lambda_2$:

We choose $H^n, K^n, n \ge 0$, in Λ_1, respectively converging to H and K in $L^2(\mathcal{P}, dP \otimes ds)$. By (4.77) we see that

$$E\big[\sup_{t\ge 0} |(H^n.X)_t - (H.X)_t|^2\big] \le 4\, \|H^n - H\|^2_{L^2(\mathcal{P}, dP\otimes ds)} \to 0, \tag{4.84}$$

and a similar inequality for K. Note also that

$$\Big|\int_0^t H_s\, K_s\, ds - \int_0^t H^n_s\, K^n_s\, ds\Big| \le \int_0^\infty |H_s - H^n_s|\; |K_s|\, ds + \int_0^\infty |H^n_s|\; |K_s - K^n_s|\, ds\,,$$

so taking expectations and using Cauchy-Schwarz's inequality, we find that

$$\begin{aligned} E\Big[\sup_{t\ge 0}\Big|\int_0^t H_s\, K_s\, ds - \int_0^t H^n_s\, K^n_s\, ds\Big|\Big] &\le \|H - H^n\|_{L^2(\mathcal{P}, dP\otimes ds)} \|K\|_{L^2(\mathcal{P}, dP\otimes ds)} \\ &+ \|H^n\|_{L^2(\mathcal{P}, dP\otimes ds)} \|K - K^n\|_{L^2(\mathcal{P}, dP\otimes ds)} \to 0\,. \end{aligned} \tag{4.85}$$

As a result, we find that

$$\sup_{t\ge 0} |N^{(n)}_t - N_t| \to 0 \ \text{ in } \ L^1(\Omega, \mathcal{G}, P)\,, \tag{4.86}$$

if $N^{(n)}_t$ denotes the martingale attached to H^n, K^n via (4.78). This is more than enough to conclude that N_t, $t \ge 0$, satisfies the martingale property, and this concludes the proof of the Proposition. □

Remark 4.19. Note that the above proposition shows that for $H \in \Lambda_2$,

$$(H.X)^2_t - \int_0^t H^2_s(\omega)\, ds \text{ is a continuous } (\mathcal{G}_t)\text{-martingale}\,,$$

and the non-decreasing adapted process:

$$t \ge 0 \longrightarrow \int_0^t H^2_s(\omega)\, ds$$

fulfills the properties (4.22) - (4.25) relative to $M_t = (H.X)_t$. We have thus constructed by "bare hands"

$$\langle (H.X)\rangle_t = \int_0^t H_s^2(\omega)\, ds,\ t \ge 0 \tag{4.87}$$

(as mentioned below (4.25), the process satisfying (4.22) - (4.25) is essentially unique). □

The good version of the stochastic integral, which we have produced, is, in essence, based on an isometry. We will now **reconstruct some trajectorial property of the integral**.

When T is a $(\mathcal{G}_t)$-stopping time, the process

$$(\omega, s) \to 1_{[0,T]}(\omega, s) \overset{\text{def}}{=} 1\{s \le T(\omega)\} \tag{4.88}$$

is progressively measurable (it is adapted, left-continuous in s, and a simple variant of (2.60) yields the claim). For such a T we have two "natural ways" to define "$\int_0^T H_s\, dX_s$", when $H \in \Lambda_2$:

- We can for instance use the continuous version $(H.X)_t(\omega)$ and replace t by $T(\omega)$.

Observe that the essential uniqueness of the continuous version of $(H.X)_t(\omega)$ ensures that two different continuous versions of the stochastic integral give rise to resulting random variables which differ on an at most negligible set. In other words:

$$(H.X)_{T(\omega)}(\omega) \text{ is uniquely defined up to a negligible set.} \tag{4.89}$$

- Alternatively we can use the definition

$$\int_0^\infty (1_{[0,T]} H)_s\, dX_s\,, \tag{4.90}$$

once we note that $1_{[0,T]} H \in \Lambda_2$.

As we now explain both definitions coincide.

Proposition 4.20. *(stopping theorem for stochastic integrals)*

Let T be a $(\mathcal{G}_t)$-stopping time, and $H \in \Lambda_2$, then P-a.s.,

$$\int_0^{t\wedge T} H_s\, dX_s = \int_0^t (1_{[0,T]}H)_s\, dX_s,\ \textit{for } 0 \le t \le \infty\,. \tag{4.91}$$

Proof. We consider $(H.X)_t$, $t \ge 0$, and $(H1_{[0,T]}.X)_{t\ge 0}$. For a given $u \ge 0$,

$$(H1_{[0,u]})_s(\omega) = (H1_{[0,T]}\, 1_{[0,u]})_s(\omega),\ \text{for all } s \ge 0, \text{ on } G \overset{\text{def}}{=} \{u \le T\}\,.$$

It now follows from (4.63) that for $u \ge 0$, P-a.s. on $\{u \le T\}$,

$$(H.X)_u = (H1_{[0,u]}.X)_\infty = (H1_{[0,T]}\, 1_{[0,u]}.X)_\infty = (H1_{[0,T]}.X)_u\,. \tag{4.92}$$

As a result we see that

$$P\text{-a.s., for all } u \in \mathbb{Q} \cap [0,\infty),\ u \le T(\omega) \Longrightarrow (H.X)_u(\omega) = (H1_{[0,T]}.X)_u(\omega)\,,$$

and using continuity that

$$P\text{-a.s. for all } 0 \le t \le T(\omega),\ (H.X)_t(\omega) = (H1_{[0,T]}.X)_t(\omega)\,. \tag{4.93}$$

Analogously for $u \ge 0$:

$$H1_{[0,T]}\,1_{[0,u]} = H1_{[0,T]} \text{ on } \widetilde{G} = \{T \le u\}\,,$$

so that for $u \ge 0$, P-a.s., on $\{T \le u\}$

$$(H1_{[0,T]}.X)_u = (H1_{[0,T]}.X)_\infty\,.$$

From this we deduce that

$$P\text{-a.s., for all } u \in \mathbb{Q} \cap [0,\infty),\ T(\omega) \le u \Longrightarrow (H1_{[0,T]}.X)_u(\omega) = (H1_{[0,T]}.X)_\infty(\omega)\,.$$

Using continuity as above, we thus find that

$$P\text{-a.s., for all } t \ge T(\omega),\ (H1_{[0,T]}.X)_t(\omega) = (H1_{[0,T]}.X)_\infty(\omega)\,. \tag{4.94}$$

Combining (4.93) and (4.94), we see that

$$P\text{-a.s., for all } t \ge 0,\ (H.X)_{t\wedge T(\omega)}(\omega) = (H1_{[0,T]}.X)_{t\wedge T(\omega)}(\omega) = (H1_{[0,T]}.X)_t(\omega)\,,$$

and this proves (4.91). □

We then have the following

Corollary 4.21. *Given $H, K \in \Lambda_2$, T a $(\mathcal{G}_t)$-stopping time such that "$H = K$ on the random interval $[0,T]$" (i.e. $H1_{[0,T]} = K1_{[0,T]}$), then one has*

$$P\text{-}a.s., \int_0^t H_s\,dX_s = \int_0^t K_s\,dX_s,\ \textit{for } 0 \le t \le T(\omega)\,. \tag{4.95}$$

Proof.

$$\begin{aligned} P\text{-a.s., for } t \ge 0,\ \int_0^{t\wedge T} H_s\,dX_s &\overset{(4.91)}{=} \int_0^t (1_{[0,T]}H)_s\,dX_s = \int_0^t (1_{[0,T]}K)_s\,dX_s \\ &\overset{(4.91)}{=} \int_0^{t\wedge T} K_s\,dX_s\,, \end{aligned}$$

and the claim follows. □

The above corollary provides some "pathwise feeling" to the stochastic integral and also has important consequences.

Our next item of discussion is the "**localization of stochastic integrals**". We are going to relax the integrability condition $H \in \Lambda_2$ (i.e. $H \in L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP \otimes ds)$) in the definition of stochastic integrals. As mentioned previously, cf. (4.64), presently $\int_0^1 e^{\alpha X_s^2} dX_s$ has no meaning when $\alpha \geq \frac{1}{4}$ (for the sake of definiteness we consider the canonical space $(C, \mathcal{F}, (F_t)_{t \geq 0}, W_0)$ and the canonical process $X_t, t \geq 0$). We are going to remedy this feature and $\int_0^1 e^{\alpha X_s^2} dX_s$ will become well-defined for any $\alpha \in \mathbb{R}$, as a result of the construction below (together with many other stochastic integrals!).

We introduce

$$
\begin{aligned}
\Lambda_3 = \Big\{ K : \mathcal{P}\text{-measurable functions on } \Omega \times [0,\infty), \text{ such that} \\
P\text{-a.s., for all } t \geq 0, \int_0^t K_s^2(\omega)\, ds < \infty \Big\}.
\end{aligned}
\tag{4.96}
$$

Remark 4.22.

1) Note that when $K_s(\omega)$ is $(\mathcal{G}_s)$-adapted, for each $s \geq 0$, and continuous in s, for each ω, then automatically $K \in \Lambda_3$. In particular $\exp\{\alpha X_s^2\}$, or $\exp\{\exp\{X_s^2\}\}$ belong to Λ_3!

2) In the case where we consider a continuous square integrable martingale $M_.$ in place of $X_.$, the relevant condition will be that outside a P-negligible set of ω, one has $\int_0^t K_s^2(\omega)\, d\langle M \rangle_s(\omega) < \infty$, for all $t \geq 0$. □

Lemma 4.23. *When $H \in \Lambda_3$, there exists a non-decreasing sequence of $(\mathcal{G}_t)$-stopping times $S_n, n \geq 0$, which is P-a.s. tending to $+\infty$, such that for each $n \geq 0$:*

$$
H 1_{[0,S_n]} \in \Lambda_2\,. \tag{4.97}
$$

Proof. Note that $(\omega, t) \to \int_0^t H_s^2(\omega) ds \in [0,\infty]$ is a continuous, non-decreasing, $(\mathcal{G}_t)$-adapted stochastic process. As a result

$$
S_n \overset{\text{def}}{=} \inf\{t \geq 0; \int_0^t H_s^2(\omega)\, ds \geq n\} \leq \infty \tag{4.98}
$$

is a $(\mathcal{G}_t)$-stopping time (cf. (2.26), (2.27), in fact the proof is simpler here because $\{S_n > t\} = \{\int_0^t H_s^2(\omega)\, ds < n\} \in \mathcal{G}_t$, for each $t \geq 0$). In addition we have:

$$
E\Big[\int_0^\infty (H^2 1_{[0,S_n]})_s(\omega)\, ds\Big] \leq n < \infty\,,
$$

and (4.97) holds. Moreover since $H \in \Lambda_3$, it follows that $S_n(\omega) \uparrow \infty$ for P-a.e. ω. This proves our claim. □

We are now ready to extend the definition of the stochastic integral to all integrands in Λ_3.

Definition and Theorem 4.24. *Let $H \in \Lambda_3$ and S_n, $n \ge 0$, be any sequence of $(\mathcal{G}_t)$-stopping times, non-decreasing in n, P-a.s. tending to $+\infty$, and such that (4.97) holds. Then the event*

$$\begin{aligned} N = \bigcup_{n \ge 0} & \ \{\omega \in \Omega; \exists t \in \mathbb{Q}_+, (H1_{[0,S_n]}.X)_{t \wedge S_n} \neq (H1_{[0,S_{n+1}]}.X)_{t \wedge S_n}\} \\ & \cup \{\omega \in \Omega; \lim_n S_n(\omega) < \infty\} \ \textit{is } P\textit{-negligible,} \end{aligned} \tag{4.99}$$

and the process

$$\begin{aligned} (H.X)_t(\omega) & \overset{\text{def}}{=} (H1_{[0,S_n]}.X)_t(\omega), \ \textit{for } \omega \notin N, \textit{ and } t \le S_n(\omega), \\ & \overset{\text{def}}{=} 0, \ \textit{if } \omega \in N\,, \end{aligned} \tag{4.100}$$

is well-defined, continuous, adapted. Two such processes arising from two possible choices of sequences of S_n, $n \ge 0$, and versions of $(H1_{[0,S_n]}.X)_t(\omega)$, agree for all $t \ge 0$, except maybe on a negligible set (i.e. (4.100) defines $(H.X)$ in an essentially unique fashion).

Proof. Note that $H1_{[0,S_n]}$ and $H1_{[0,S_{n+1}]}$ agree on $[0, S_n]$. As a result of (4.95), the event N in (4.99) is P-negligible. Note that

$$\begin{aligned} & \text{for } \omega \in N^c, \ (H1_{[0,S_n]}.X)_t(\omega) = (H1_{[0,S_{n'}]}.X)_t(\omega), \\ & \qquad \text{for } n, n' \ge 0, \text{ and for } 0 \le t \le S_n(\omega) \wedge S_{n'}(\omega)\,. \end{aligned} \tag{4.101}$$

Hence $(H.X)_t(\omega)$ in (4.100) is well-defined. Moreover for $t \ge 0$,

$$(H.X)_t(\omega) = \overline{\lim_n}\ (H1_{[0,S_n]}.X)_t(\omega), \ \text{if } \omega \notin N, \ = 0, \ \text{ if } \omega \in N\,.$$

Since $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \ge 0}, P)$ satisfies the usual conditions, cf. (4.5), (4.6), $(H.X)_t$, $t \ge 0$, is $(\mathcal{G}_t)$-adapted. Further $t \to (H.X)_t(\omega)$ is continuous, for each $\omega \in \Omega$.

If S_n, S'_n, $n \ge 0$, are two sequences satisfying the assumptions of the theorem, the same holds for $T_n \overset{\text{def}}{=} S_n \wedge S'_n$. From (4.95) we thus find that

$$\begin{aligned} P\text{-a.s., for } 0 \le t \le T_n(\omega), \ (H1_{[0,S_n]}.X)_t(\omega) & = (H1_{[0,S'_n]}.X)_t(\omega), \\ & = (H1_{[0,T_n]}.X)_t(\omega)\,. \end{aligned} \tag{4.102}$$

The claim about the essential uniqueness in the claim (4.100) easily follows. □

Remark 4.25. Of course $\Lambda_2 \subseteq \Lambda_3$, and for $H \in \Lambda_2$, we can choose $S_n \equiv \infty$, for all $n \ge 0$, so that (4.97) holds. Noting that $(H1_{[0,\infty]}.X)_t$, $t \ge 0$, and $(H.X)_t$, $t \ge 0$, are indistinguishable, we see that the definition (4.100) is consistent when $H \in \Lambda_2 \subseteq \Lambda_3$. □

We now have given a meaning to expressions like $\int_0^t \exp\{\exp\{X_s^2\}\} dX_s$, and of course we should not expect that we still keep the martingale property for $H \in \Lambda_3$ (an indication of this feature appeared below (4.64)). The adequate notion comes in the next Definition.

Definition 4.26. *A process $(M_t)_{t\ge 0}$, such that there exists an increasing sequence of $(\mathcal{G}_t)$-stopping times S_n, P-a.s. tending to ∞, such that for each n, $(M_{t\wedge S_n})_{t\ge 0}$ is a continuous square integrable martingale, is called a* **continuous $(\mathcal{G}_t)$-local martingale**.

Remark 4.27. When M_0 is bounded, one can replace "continuous square integrable" with "continuous bounded". Indeed for such an $(M_t)_{t\ge 0}$ as above, with M_0 bounded, one defines the sequence of $(\mathcal{G}_t)$-stopping times:

$$T_m = \inf\{s \ge 0;\ |M_s| \ge m\} \le \infty\,, \tag{4.103}$$

so that $T_m \uparrow \infty$, as $m \to \infty$. Then, for fixed $m \ge \|M_0\|_\infty$, we have $|M_{t\wedge T_m}| \le m$, for all $t \ge 0$. Hence, when $0 \le s < t$, $A \in \mathcal{G}_s$, we have

$$E[M_{t\wedge T_m} 1_A] \overset{\text{dom. conv.}}{=} \lim_n E[M_{t\wedge T_m \wedge S_n}\, 1_A]\,. \tag{4.104}$$

By the stopping theorem, $\widetilde{M}_{t\wedge T_m}$ is a continuous martingale if $\widetilde{M}_t$ is a continuous martingale, and applying this to $\widetilde{M}_t \overset{\text{def}}{=} M_{t\wedge S_n}$, we find that the last term of (4.104) equals

$$\lim_n E[M_{s\wedge T_m\wedge S_n} 1_A] \overset{\text{dom. conv.}}{=} E[M_{s\wedge T_m} 1_A]\,.$$

In other words $(M_{t\wedge T_m})_{t\ge 0}$ is a $(\mathcal{G}_t)$-martingale, which is bounded and continuous, and our claim follows. □

Exercise 4.28.

1) Deduce the continuous time stopping theorem we used above from the discrete time version (see also [8], p. 19).

2) Show that a bounded continuous local martingale is a martingale. □

Continuous local martingales naturally arise in our context as shown by the next

Proposition 4.29.

$$\text{For } H \in \Lambda_3,\ (H.X)_t,\ t \ge 0, \text{ is a continuous } (\mathcal{G}_t)\text{-local martingale.} \tag{4.105}$$

Proof. Consider an increasing sequence of stopping times $S_n \uparrow \infty$, P-a.s., such that for each n, $H1_{[0,S_n]} \in \Lambda_2$, then

$$\begin{aligned} P\text{-a.s., for } t \ge 0,\ (H.X)_{t\wedge S_n} &\overset{(4.100)}{=} (H1_{[0,S_n]}.X)_{t\wedge S_n} \\ &\overset{(4.91)}{=} \underset{\substack{\uparrow \\ \text{continuous square integrable martingale}}}{(H1_{[0,S_n]}.X)_t}\,. \end{aligned} \tag{4.106}$$

Our claim follows. □

5 Stochastic Integrals for Continuous Local Martingales

In this chapter we are going to define the stochastic integral $\int_0^t H_s \, dM_s$ when the integrator M is a continuous local martingale, and H is progressively measurable and such that $\int_0^t H_s^2(\omega) \, d\langle M\rangle_s(\omega) < \infty$, where $\langle M\rangle$ is the so-called "quadratic variation of the local martingale M". As in the previous chapter the filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ satisfies the "usual conditions", see (4.5), (4.6). Our first task will be the construction of $\langle M\rangle$. We begin with the

Lemma 5.1. *Suppose A_t, $t \ge 0$, B_t, $t \ge 0$, are continuous, $(\mathcal{G}_t)$-adapted, non-decreasing processes such that $A_0 = B_0 = 0$, and*

$$A_t - B_t \text{ is a } (\mathcal{G}_t)\text{-local martingale,} \tag{5.1}$$

then

$$P\text{-a.s.}(\omega), \text{ for all } t \ge 0, \ A_t(\omega) = B_t(\omega)\,. \tag{5.2}$$

Proof. Introduce the non-decreasing sequence of $(\mathcal{G}_t)$-stopping times

$$S_n = \inf\{s \ge 0, \ A_s \text{ or } B_s \ge n\}\,, \tag{5.3}$$

and note that $S_n \uparrow \infty$ as $n \to \infty$. As in (4.103) we see that

$$A_{t\wedge S_n} - B_{t\wedge S_n}, \ t \ge 0, \text{ is a bounded martingale, for each } n \ge 0\,. \tag{5.4}$$

It thus suffices to prove the theorem in the case where A_t, $t \ge 0$, and B_t, $t \ge 0$, are uniformly bounded, and

$$M_t = A_t - B_t, \ t \ge 0, \text{ is a bounded continuous martingale.} \tag{5.5}$$

We now observe that for $t \ge 0$:

$$E[M_t^2] = E\Big[\Big(\sum_{k=0}^{2^m-1} M_{\frac{k+1}{2^m}\,t} - M_{\frac{k}{2^m}\,t}\Big)^2\Big]$$

and expanding the square, the cross terms disappear by the martingale property. So we find

$$\begin{aligned}
E[M_t^2] &= \sum_{k=0}^{2^m-1} E\Big[\Big(M_{\frac{k+1}{2^m}\,t} - M_{\frac{k}{2^m}\,t}\Big)^2\Big] = E\Big[\sum_{0\le k<2^m} \Big(M_{\frac{k+1}{2^m}\,t} - M_{\frac{k}{2^m}\,t}\Big)^2\Big] \\
&\le E\Big[\underbrace{\sup_{0\le k<2^m} \big|M_{\frac{k+1}{2^m}\,t} - M_{\frac{k}{2^m}\,t}\big|}_{\text{by continuity} \downarrow_0 \, m\to\infty} \times \underbrace{\sum_{0\le k<2^m} \big|M_{\frac{k+1}{2^m}\,t} - M_{\frac{k}{2^m}\,t}\big|}_{\le A_\infty + B_\infty \le \text{Const} < \infty}\Big] \underset{m\to\infty}{\overset{\text{dom. conv.}}{\longrightarrow}} 0\,.
\end{aligned}$$

We have thus shown that for $t \geq 0$:

$$E[M_t^2] = 0\,, \tag{5.6}$$

and hence P-a.s., for all $t \in \mathbb{Q}_+$, $M_t = A_t - B_t = 0$. But by continuity we have P-a.s.(ω), for all $t \geq 0$, $A_t(\omega) = B_t(\omega)$. This completes the proof of the lemma. $\square$

We now proceed with the construction of $\langle M \rangle$, when M is a continuous square integrable martingale. The result is a special case of the so-called Doob-Meyer decomposition (see for instance [8], p. 24).

Theorem 5.2. *Let $M_t, t \geq 0$, be a continuous square integrable $(\mathcal{G}_t)$-martingale. Then there exists a continuous, non-decreasing, $(\mathcal{G}_t)$-adapted process A_t, $t \geq 0$, such that*

$$A_0 = 0\,, \tag{5.7}$$

$$A_t \text{ is integrable for each } t \geq 0\,, \tag{5.8}$$

$$M_t^2 - A_t \text{ is a } (\mathcal{G}_t)\text{-martingale}\,, \tag{5.9}$$

and $A_t, t \geq 0$ is essentially unique.

Proof. The essential uniqueness follows from (5.2). We only need to prove the existence of A_t, $t \geq 0$. Without loss of generality we assume that $M_0 = 0$ (otherwise we replace M_t with $\widetilde{M_t} = M_t - M_0$).

We are going to construct $A_.$ as a suitable limit of discrete quadratic variations of $M_.$, along certain random grids with mesh tending to 0. For this purpose we define, for each $n \geq 0$ (n controls the mesh of the discrete grid), a sequence τ_k^n, $k \geq 0$, of stopping times as follows:

$$\tau_k^0 = k, \text{ for } k \geq 0\,, \tag{5.10}$$

and for $n \geq 1$, by induction:

$$\begin{aligned} &\tau_0^n = 0, \text{ and for } \ell \geq 0, \text{ on the event } \{\tau_k^{n-1} \leq \tau_\ell^n < \tau_{k+1}^{n-1}\},\ k \geq 0, \\ &\tau_{\ell+1}^n = \inf\Big\{t \geq \tau_\ell^n;\ |M_t - M_{\tau_\ell^n}| \geq \frac{1}{n}\Big\} \wedge \Big(\tau_\ell^n + \frac{1}{n}\Big) \wedge \tau_{k+1}^{n-1}\,. \end{aligned} \tag{5.11}$$

Using the continuity of $M_.$, we see that for $\omega \in \Omega$,

$$\left\{ \begin{array}{l} \tau_k^n(\omega) < \tau_{k+1}^n(\omega), \text{ for } n, k \geq 0\,, \\ \{\tau_0^n(\omega),\ \tau_1^n(\omega), \dots\} \subseteq \{\tau_0^{n+1}(\omega),\ \tau_1^{n+1}(\omega), \dots\}\,, \\ \qquad\qquad\qquad \nwarrow \qquad\qquad \nearrow \\ \qquad\qquad\qquad\quad \text{subsets of } \mathbb{R}_+ \\ \tau_k^n(\omega) \to \infty, \text{ as } k \to \infty\,, \\ \big|M_{\tau_{k+1}^n(\omega)}(\omega) - M_{\tau_k^n(\omega)}(\omega)\big| \leq \dfrac{1}{n}\,. \end{array} \right. \tag{5.12}$$

We then choose $K_0 < K_1 < \dots < K_n < \dots$ in $\mathbb{N}$ so that

$$P(\tau^n_{K_n} \le n) \le \frac{1}{n}\,, \text{ for } n \ge 0\,, \tag{5.13}$$

and define

$$I_n(t) = \sum_{k=0}^{K_n - 1} M_{\tau_k^n}(M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t}) \tag{5.14}$$

as well as

$$A_n(t) = \sum_{k=0}^{K_n - 1} (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t})^2\,. \tag{5.15}$$

Note that $I_n(0) = 0$, $A_n(0) = 0$, that $I_n(\cdot)$, $A_n(\cdot)$ are continuous and adapted (for instance in the case of (5.14), the generic term vanishes on $\{\tau_k^n > t\}$ and one has

$$\underbrace{1\{\tau_k^n \le t\} M_{\tau_k^n}}_{\substack{\uparrow \\ \mathcal{G}_t-\text{measurable}}} \Big(\underbrace{M_{\tau^n_{k+1}\wedge t}}_{\substack{\uparrow \\ \mathcal{G}_{\tau^n_{k+1}\wedge t} \subseteq \mathcal{G}_t-\text{meas.}}} - \underbrace{M_{\tau^n_k \wedge t}}_{\substack{\uparrow \\ \mathcal{G}_{\tau^n_k \wedge t} \subseteq \mathcal{G}_t-\text{meas.}}} \Big) \text{ is } \mathcal{G}_t\text{-measurable}\,,$$

and the case of (5.15) is easier). In fact one has

(5.16) $\quad I_n(t)$, $t \ge 0$, is a continuous $(\mathcal{G}_t)$-martingale, bounded for each t.

Here we only need to check that for $s \le t$:

$$E[M_{\tau_k^n}(M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t}) \,|\, \mathcal{G}_s] = M_{\tau_k^n}(M_{\tau^n_{k+1}\wedge s} - M_{\tau^n_k \wedge s})\,. \tag{5.17}$$

But using the stopping theorem and the observation above (5.16), we have

$$E[\underbrace{1\{\tau_k^n \le s\}\, M_{\tau_k^n}}_{\in \mathcal{G}_s} (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t}) \,|\, \mathcal{G}_s] = \text{right-hand side of (5.17)}$$

(note that the right-hand side of (5.17) equals 0 on $\{\tau_k^n > s\}$).

On the other hand $1\{s < \tau_k^n \le t\}\, M_{\tau_k^n}$ is $\mathcal{G}_{\tau_k^n}$-measurable and for $A \in \mathcal{G}_s$, $1_A\, 1\{s < \tau_k^n\}$ is also $\mathcal{G}_{\tau_k^n}$-measurable and on this set $\tau_k^n \le \tau_{k+1}^n \wedge t$. Hence,

$$\begin{aligned} &E[1_A\, 1\{s < \tau_k^n \le t\}\, M_{\tau_k^n}\, (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t})] = \\ &E[1_A\, 1\{s < \tau_k^n \le t\}\, M_{\tau_k^n} (M_{\tau_k^n \vee (\tau^n_{k+1}\wedge t)} - M_{\tau^n_k})] = 0, \end{aligned}$$

using the stopping theorem, cf. [8], p. 19, for the last equality.

As a result, $E[1\{s < \tau_k^n \le t\}\, M_{\tau_k^n}\, (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t}) \,|\, \mathcal{G}_s] = 0$, and (5.17) now easily follows since $1\{\tau_k^n > t\}\, (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t}) = 0$. This proves (5.16).

By direct inspection of (5.15) and (5.11) we see that

$$\text{for } t \geq s + \frac{1}{n}, \;\; A_n(t) + \frac{1}{n^2} \geq A_n(s)\,. \tag{5.18}$$

Moreover, for $n \geq 1$, on $\{\tau^n_{K_n} > n\}$, $M_t = \sum_{0 \leq k < K_n} (M_{\tau^n_{k+1} \wedge t} - M_{\tau^n_k \wedge t})$, for $0 \leq t \leq n$, so that expanding the square and regrouping terms, we see that, cf. (5.14), (5.15):

$$M_t^2 = 2I_n(t) + A_n(t), \text{ for } 0 \leq t \leq n, \text{ on } \{\tau^n_{K_n} > n\}\,. \tag{5.19}$$

The next step is to prove the P-a.s. uniform convergence on compact intervals of $I_{n_\ell}(\cdot)$ for a suitably chosen subsequence n_ℓ. To this end we will use the next

Lemma 5.3. $(T > 0, \varepsilon > 0)$

$$\lim_{m \to \infty} \sup_{n \geq m} P[\sup_{0 \leq t \leq T} |I_n(t) - I_m(t)| \geq \varepsilon] = 0\,. \tag{5.20}$$

Proof. Choose $n \geq m \geq T$, and define $S = T \wedge \tau^m_{K_m} \wedge \tau^n_{K_n}$. By Doob's inequality, cf. (4.67), we find that

$$\begin{aligned} &P\big[\sup_{0 \leq t \leq T} |I_n(t) - I_m(t)| \geq \varepsilon\big] \leq P[\tau^m_{K_m} \leq m \text{ or } \tau^n_{K_n} \leq n] + \\ &P\big[\sup_{0 \leq t \leq T} |I_n(t \wedge S) - I_m(t \wedge S)| \geq \varepsilon\big] \underset{(5.13)}{\overset{(4.67)}{\leq}} \frac{2}{m} + \frac{1}{\varepsilon^2}\, E[(I_n(S) - I_m(S))^2]\,, \end{aligned} \tag{5.21}$$

where we have used (5.16) and the stopping theorem.

If we now define for $k, \ell \geq 0$,

$$\rho_k = \tau^m_k \wedge S, \quad \sigma_\ell = \tau^n_\ell \wedge S\,, \tag{5.22}$$

it follows from the second line of (5.12) that

$$\{\rho_0(\omega), \dots, \rho_k(\omega), \dots\} \subseteq \{\sigma_0(\omega), \sigma_1(\omega), \dots, \sigma_\ell(\omega), \dots\}\,,$$

and from (5.14) that

$$\begin{aligned} &I_n(S) - I_m(S) = \sum_{\ell \geq 0} M_{\sigma_\ell}(M_{\sigma_{\ell+1}} - M_{\sigma_\ell}) - \sum_{k \geq 0} M_{\rho_k}(M_{\rho_{k+1}} - M_{\rho_k}) = \\ &\sum_{k,\ell \geq 0} 1\{\rho_k \leq \sigma_\ell < \rho_{k+1}\}\, M_{\sigma_\ell}(M_{\sigma_{\ell+1}} - M_{\sigma_\ell}) - \\ &\sum_{k,\ell \geq 0} 1\{\rho_k \leq \sigma_\ell < \rho_{k+1}\}\, M_{\rho_k}(M_{\sigma_{\ell+1}} - M_{\sigma_\ell}) = \\ &\sum_{k,\ell \geq 0} 1\{\rho_k \leq \sigma_\ell < \rho_{k+1}\}\, (M_{\sigma_\ell} - M_{\rho_k})(M_{\sigma_{\ell+1}} - M_{\sigma_\ell}) \overset{\text{def}}{=} \sum_{k,\ell \geq 0} a_{k,\ell}(\omega)\,. \end{aligned} \tag{5.23}$$

Note that

$$\text{the } a_{k,\ell},\ k,\ell \ge 0, \text{ are pairwise orthogonal in } L^2(P)\,. \tag{5.24}$$

Indeed for $\ell < \ell'$, cf. Exercise 2.13 2)

$$a_{k,\ell} = 1\{\rho_k \le \sigma_\ell < \rho_{k+1}\}(M_{\sigma_\ell} - M_{\rho_k})(M_{\sigma_{\ell+1}} - M_{\sigma_\ell}) \text{ is } \mathcal{G}_{\sigma_{\ell+1}} \subseteq \mathcal{G}_{\sigma_{\ell'}}\text{-measurable}$$

and:

$$1\{\rho_{k'} \le \sigma_{\ell'} < \rho_{k'+1}\}(M_{\sigma_{\ell'}} - M_{\rho_{k'}}) \text{ is } \mathcal{G}_{\sigma_{\ell'}}\text{-measurable as well.}$$

Since $E[M_{\sigma_{\ell'+1}}\,|\,\mathcal{G}_{\sigma_{\ell'}}] = M_{\sigma_{\ell'}}$ (see for instance [8], p. 19), it follows that $E[a_{k,\ell}\, a_{k',\ell'}] = 0$, for $\ell < \ell'$. To obtain (5.24), one simply notes that for $\ell \ge 0$, $k < k'$, $a_{k,\ell}(\omega)\, a_{k',\ell}(\omega) = 0$.

Coming back to (5.23), we conclude by (5.24) that

$$\begin{aligned}
&E[(I_n(S) - I_m(S))^2] = \sum_{k,\ell\ge 0} E[a^2_{k,\ell}] \overset{(5.11)}{\le} \\
&\frac{1}{m^2}\, E\Big[\sum_{k,\ell\ge 0} 1\{\rho_k \le \sigma_\ell < \rho_{k+1}\}(M_{\sigma_{\ell+1}} - M_{\sigma_\ell})^2\Big] = \frac{1}{m^2}\, E\Big[\sum_{\ell\ge 0}(M_{\sigma_{\ell+1}} - M_{\sigma_\ell})^2\Big] \\
&\text{and since the above increments are pairwise orthogonal} \\
&= \frac{1}{m^2}\, E[M_S^2] \le \frac{1}{m^2}\, E[M_T^2]\,, \\
&\text{since } M_t^2,\ t \ge 0, \text{ is a continuous submartingale, and } S \le T.
\end{aligned} \tag{5.25}$$

Inserting this inequality in (5.21) yields that

$$P\Big[\sup_{0\le t\le T} |I_n(t) - I_m(t)| \ge \varepsilon\Big] \le \frac{2}{m} + \frac{1}{m^2\varepsilon^2}\, E[M_T^2]\,. \tag{5.26}$$

In particular (5.20) follows, and the lemma is proved. □

We can now extract $n_\ell \ge \ell^2$, such that

$$\sup_{n\ge n_\ell} P\Big[\sup_{0\le t\le \ell} |I_n(t) - I_{n_\ell}(t)| \ge \frac{1}{\ell^2}\Big] \le \frac{1}{2^\ell}\,,$$

so that

$$P\Big[\sup_{0\le t\le \ell} |I_{n_{\ell+1}}(t) - I_{n_\ell}(t)| \ge \frac{1}{\ell^2}\Big] \le \frac{1}{2^\ell}\,. \tag{5.27}$$

By (5.13) and Borel-Cantelli's lemma we can choose $N \in \mathcal{G}$ with $P(N) = 0$, so that

$$\sup_{0\le t\le \ell} |I_{n_{\ell+1}}(t) - I_{n_\ell}(t)| \le \frac{1}{\ell^2},\ \tau^{n_\ell}_{K_{n_\ell}} > \ell^2, \text{ for } \ell \ge \ell_0(\omega), \text{ when } \omega \notin N\,. \tag{5.28}$$

Thus $I_{n_\ell}(\cdot,\omega)$ converges uniformly on compact intervals when $\omega \notin N$, and we define

$$\begin{aligned} I(t,\omega) &= \lim_\ell I_{n_\ell}(t,\omega), \text{ for } \omega \notin N, \\ &= \frac{1}{2}\, M_t^2(\omega), \text{ for } \omega \in N\,. \end{aligned} \tag{5.29}$$

Therefore $t \in \mathbb{R}_+ \to I(t,\omega)$ is continuous for $\omega \in \Omega$, and $I(t,\omega)$ is $(\mathcal{G}_t)$-measurable for each $t \geq 0$ (we use the fact that $\mathcal{G}_0$ contains all negligible sets of $\mathcal{G}$).

By (5.19) and the fact that $\tau^{n_\ell}_{K_{n_\ell}} \to \infty$, when $\omega \notin N$, we see that when $\omega \notin N$, $A_{n_\ell}(\cdot,\omega)$ converges uniformly on compact intervals of $\mathbb{R}_+$ to

$$A_t(\omega) \overset{\text{def}}{=} M_t^2(\omega) - 2I(t,\omega), \quad \omega \in \Omega\,. \tag{5.30}$$

Thus $A_t(\omega)$ is $(\mathcal{G}_t)$-measurable, for all $t \geq 0$, continuous in t for all $\omega \in \Omega$. Note that $A_0 = 0$, and due to (5.18) when $\omega \notin N$, and to (5.29), (5.30) when $\omega \in N$, $t \to A_t(\omega)$ is non-decreasing in t for all $\omega \in \Omega$.

We will now prove that for $n_0 \geq 1$, $k_0 \geq 1$, $t \geq 0$,

$$I_{n_\ell}(\tau^{n_0}_{k_0} \wedge t) \underset{\ell\to\infty}{\longrightarrow} I(\tau^{n_0}_{k_0} \wedge t) \text{ in } L^1(P)\,. \tag{5.31}$$

We already know the P-a.s. convergence, cf. (5.29). It thus suffices to prove that $I_{n_\ell}(\tau^{n_0}_{k_0} \wedge t)$, $\ell \geq 0$, are uniformly integrable. However writing for $m \geq n_0$,

$$\nu_k = \tau_k^m \wedge t \wedge \tau^{n_0}_{k_0}\,,$$

we see as in (5.23) that

$$I_m(\tau^{n_0}_{k_0} \wedge t) = \sum_{k\geq 0} M_{\nu_k}(M_{\nu_{k+1}} - M_{\nu_k})\,. \tag{5.32}$$

Since we have the bound

$$|M_{\nu_k}| \leq \sup_{0\leq u\leq \tau^{n_0}_{k_0}} |M_u| \overset{(5.11)}{\leq} \frac{k_0}{n_0}\,,$$

a similar (but easier) calculation as in (5.25) yields that for $m \geq n_0$:

$$E[I_m(\tau^{n_0}_{k_0} \wedge t)^2] \leq \Big(\frac{k_0}{n_0}\Big)^2 \sum_{k\geq 0} E[(M_{\nu_{k+1}} - M_{\nu_k})^2] \leq \Big(\frac{k_0}{n_0}\Big)^2 E[M_t^2]\,. \tag{5.33}$$

This proves the asserted uniform integrability and (5.31) follows.

By the stopping theorem, $I_{n_\ell}(\tau_{k_0}^{n_0} \wedge t)$, $t \ge 0$, are martingales, and by (5.31) we deduce that $I(\tau_{k_0}^{n_0} \wedge t)$, $t \ge 0$, is a (continuous) martingale. By (5.30) we now find that

$$\begin{array}{ccc} \underbrace{E[A(\tau_{k_0}^{n_0} \wedge t)]} & = & \underbrace{E[M^2(\tau_{k_0}^{n_0} \wedge t)]} \\ \text{monotone convergence} \Big\downarrow k_0 \to \infty & & \text{dominated convergence} \Big\downarrow k_0 \to \infty \\ E[A(t)] & = & E[M^2(t)]\,. \end{array} \qquad \text{(recall } E[\sup_{s \le t} |M_s|^2] \overset{(4.76)}{\le} 4E[M_t^2])$$

This proves (5.8), and it also follows that

$$M^2(\tau_{k_0}^{n_0} \wedge t)^2 - A(\tau_{k_0}^{n_0} \wedge t) \underset{k_0 \to \infty}{\overset{L^1(P)}{\longrightarrow}} M^2(t) - A(t), \text{ for } t \ge 0\,. \tag{5.34}$$

The claim (5.9) follows and the theorem is proved. □

Notation:

When $M_.$ is a continuous square integrable martingale, the essentially unique process $A_.$ constructed in the above theorem is denoted by $\langle M \rangle$, it is the so-called "quadratic variation" of M (in some sense (5.15) explains the terminology).

When $(Z_t)_{t \ge 0}$, is a stochastic process and T a random time, one introduces:

$$Z_t^T \overset{\text{def}}{=} Z_{t \wedge T},\ t \ge 0, \text{ the so-called stopped process}\,. \tag{5.35}$$

Corollary 5.4. *Let $(M_t)_{t \ge 0}$, be a continuous local martingale. Then, there exists an essentially unique, continuous, non-decreasing, $(\mathcal{G}_t)$-adapted process $\langle M \rangle_t$, $t \ge 0$, such that*

$$\langle M \rangle_0 = 0\,, \tag{5.36}$$

$$M_t^2 - M_0^2 - \langle M \rangle_t,\ t \ge 0, \textit{ is a continuous local martingale.} \tag{5.37}$$

Moreover, when T is a $(\mathcal{G}_t)$-stopping time, one has

$$P\text{-a.s., for all } t \ge 0,\ \langle M^T \rangle_t = \langle M \rangle_{T \wedge t}\ (= \langle M \rangle_t^T)\,. \tag{5.38}$$

Proof. The uniqueness part of the statement follows from (5.2). As for the existence part, choose stopping times $T_n \uparrow \infty$, P-a.s., so that $M_.^{T_n}$ is a continuous square integrable martingale. Note that by the stopping theorem,

$$\big(M_{t \wedge T_n}^{T_{n+1}}\big)^2 - \langle M^{T_{n+1}} \rangle_{t \wedge T_n} = M_{t \wedge T_n}^2 - \langle M^{T_{n+1}} \rangle_{t \wedge T_n},\ t \ge 0\,,$$

is a $(\mathcal{G}_t)$-martingale. From (5.2) it follows that

$$P\text{-a.s., for all } t \ge 0,\ \langle M^{T_{n+1}} \rangle_{t \wedge T_n} = \langle M^{T_n} \rangle_t\,. \tag{5.39}$$

As a result we can find N in $\mathcal{G}$ with $P(N) = 0$, so that

$$\text{(5.40)} \qquad \text{when } \omega \notin N, \text{ for all } m \geq n \geq 0,\ 0 \leq t \leq T_n(\omega),\ \langle M^{T_m}\rangle_t(\omega) = \langle M^{T_n}\rangle_t(\omega)\,.$$

We thus define

$$\text{(5.41)} \qquad \begin{aligned} \langle M\rangle_t(\omega) &= \langle M^{T_n}\rangle_t(\omega), \text{ for any } n \geq 0, \text{ with } T_n(\omega) \geq t, \text{ if } \omega \notin N\,, \\ &= 0, \text{ if } \omega \in N\,. \end{aligned}$$

Note that $\langle M\rangle_t$, $t \geq 0$, is continuous, non-decreasing, $(\mathcal{G}_t)$-adapted, and (5.36) holds. Moreover $\langle M\rangle_{\cdot}^{T_n}$ and $\langle M^{T_n}\rangle_{\cdot}$ are indistinguishable so that $M^2_{t\wedge T_n} - M_0^2 - \langle M\rangle_{t\wedge T_n}$ is a continuous martingale with value 0 at time 0. The argument below (4.104) shows that (5.37) holds. As for (5.38) it directly follows from the previous existence and uniqueness result, and the fact that $M^2_{t\wedge T} - M_0^2 - \langle M\rangle_{t\wedge T}$ is a continuous local martingale. □

Notation:

When M, N are continuous local martingales one writes:

$$\text{(5.42)} \qquad \langle M, N\rangle_t = \frac{1}{4}(\langle M+N\rangle_t - \langle M-N\rangle_t),\ t \geq 0 \textit{ (Polarization identity)}.$$

Corollary 5.5. *When M, N are continuous local martingales, $\langle M, N\rangle_t$, $t \geq 0$, is a continuous adapted process with bounded variation on finite intervals, essentially unique such that*

$$\text{(5.43)} \qquad \langle M, N\rangle_0 = 0\,,$$

$$\text{(5.44)} \qquad M_t\, N_t - M_0\, N_0 - \langle M, N\rangle_t,\ t \geq 0, \textit{ is a continuous local martingale.}$$

Proof. We only have to prove the uniqueness, the other properties being immediate. To this end observe that when C_t, $t \geq 0$, is a continuous adapted process with finite variation on finite intervals, then

$$\text{(5.45)} \qquad V_t = \lim_{n\to\infty} \sum_{\frac{k+1}{2^n} \leq t} \big|C_{\frac{k+1}{2^n}} - C_{\frac{k}{2^n}}\big|,\ t \geq 0\,,$$

is a continuous, non-decreasing, adapted process, and

$$\text{(5.46)} \qquad V_t - C_t,\ t \geq 0, \text{ is non-decreasing as well}\,.$$

We apply this observation to the difference of $\langle M, N\rangle_t$ with D_t, some other continuous adapted process, with finite variation on finite intervals, satisfying similar conditions as in (5.43), (5.44). By (5.2) we conclude that

$$\text{(5.47)} \qquad P\text{-a.s., for all } t \geq 0,\ \langle M, N\rangle_t = D_t\,.$$

□

We now turn to the **construction of the stochastic integrals with respect to continuous local martingales**. This construction involves several steps, which often are very similar to what has been done in the previous chapter (such steps will be merely briefly discussed below).

For $H_s(\omega)$ a **basic process** (i.e. $H_s(\omega) = C(\omega)\, 1\{a < s \le b\}$, with $C \in b\mathcal{G}_a$, cf. (4.26)), and M_s, $s \ge 0$, a **continuous square integrable martingale** one defines in the spirit of (4.27):

$$\int_0^\infty H_s\, dM_s \overset{\text{def}}{=} C(\omega)(M_b(\omega) - M_a(\omega))\,, \tag{5.48}$$

and for $0 \le t \le \infty$:

$$\int_0^t H_s\, dM_s \overset{\text{def}}{=} \int_0^\infty (H 1_{[0,t]})_s\, dM_s \overset{(5.48)}{=} C(\omega)(M_{b\wedge t}(\omega) - M_{a\wedge t}(\omega))\,. \tag{5.49}$$

One immediately extends the definition to $H \in \Lambda_1$, i.e. $H = H^1 + \dots + H^n$, with H^i basic processes, for $1 \le i \le n$, cf. (4.36), by the formula

$$\int_0^\infty H_s\, dM_s \overset{\text{def}}{=} \sum_{i=1}^n \int_0^\infty H_s^i\, dM_s\,, \tag{5.50}$$

and one checks that this is well-defined and that, as in (4.39), one has

$$E\Big[\int_0^\infty H_s\, dM_s \int_0^\infty K_s\, dM_s\Big] = E\Big[\int_0^\infty H_s(\omega)\, K_s(\omega)\, d\langle M\rangle_s(\omega)\Big], \text{ for } H, K \in \Lambda_1\,. \tag{5.51}$$

With the help of (4.47) one extends the definition of $\int_0^\infty H_s\, dM_s$ to H in

$$\Lambda_2(M) = L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP \times d\langle M\rangle_s)\,, \tag{5.52}$$

so that

$$H \in \Lambda_2(M) \to \int_0^\infty H_s\, dM_s \in L^2(P) \text{ is an isometry}\,. \tag{5.53}$$

One chooses a "good version" of $\int_0^t H_s\, dM_s$, $t \ge 0$, with similar arguments as in the proof of (4.69), (4.70), denoted by $(H \cdot M)_t$, $t \ge 0$, such that

(5.54) $(H.M)_t$, $t \ge 0$, is a continuous, square integrable $(\mathcal{G}_t)$-martingale with value 0 at time 0,

(5.55) for each $t \ge 0$, P-a.s., $(H.M)_t = \int_0^t H_s\, dM_s$,

(5.56) $N_t = (H.M)_t^2 - \int_0^t H_s^2\, d\langle M\rangle_s,\ t \ge 0,$ is a continuous martingale, with $\sup_{t\ge 0} |N_t| \in L^1(P)$ (and value 0 at time 0).

In particular, cf. (5.9), (5.2), (5.56),

$$\langle H.M\rangle_t = \int_0^t H_s^2 \, d\langle M\rangle_s, \; t \ge 0 \,. \tag{5.57}$$

The above defined stochastic integral has the following property (with a similar argument as for the proof of (4.62)): when $H, K \in \Lambda_2(M)$, $G \in \mathcal{G}$ are such that

$$H_s(\omega) = K_s(\omega), \text{ for all } s \ge 0, \text{ when } \omega \in G\,, \tag{5.58}$$

then

$$P\text{-a.s., for } 0 \le t \le \infty, \; (H.M)_t(\omega) = (K.M)_t(\omega), \text{ for } \omega \in G\,. \tag{5.59}$$

Then, one has (in the notation of (5.35)):

Theorem 5.6. *(stopping theorem for stochastic integrals)*

Let T be a $(\mathcal{G}_t)$-stopping time and $H \in \Lambda_2(M)$, then

$$\begin{aligned} &P\text{-}a.s., \textit{ for } 0 \le t \le \infty, \\ &((1_{[0,T]}H).M)_t = (H.M)_{t\wedge T} = (H.M^T)_t = ((1_{[0,T]}H).M^T)_t \,. \end{aligned} \tag{5.60}$$

Proof. Note that $M_t^T = M_{t\wedge T}$, $t \ge 0$, is also a continuous square integrable martingale and, cf. (5.38), $\langle M^T\rangle_. = \langle M\rangle_.^T$, so that $H \in \Lambda_2(M^T)$ as well. Then we find just as in (4.91) that

$$P\text{-a.s., for } 0 \le t \le \infty, \; ((1_{[0,T]}H).M)_t = (H.M)_{t\wedge T} \,. \tag{5.61}$$

Moreover since $\langle M^T\rangle_. = \langle M\rangle_.^T$, we see that

$$H = 1_{[0,T]}H \text{ in } L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP \times d\langle M^T\rangle)\,, \tag{5.62}$$

so that

$$P\text{-a.s., for } 0 \le t \le \infty, \; (H.M^T)_t = ((1_{[0,T]}H).M^T)_t \,. \tag{5.63}$$

Then, observe by coming back to (5.49) that for K basic process, and then K in Λ_1, one has

$$\text{for } 0 \le t \le \infty, \; (K.M)_{t\wedge T} = (K.M^T)_t \,. \tag{5.64}$$

Then by approximation of $H \in \Lambda_2(M)$ (and hence in $\Lambda_2(M^T)$) one finds that

$$P\text{-a.s., for } 0 \le t \le \infty, \; (H.M)_{t\wedge T} = (H.M^T)_t \,. \tag{5.65}$$

Combining (5.61), (5.63), (5.65), we obtain (5.60). $\square$

With the help of the stopping theorem for stochastic integrals, cf. (5.60), we will now extend the definition of stochastic integrals.

For M a **continuous local martingale with value 0 at time 0**, we define

$$\Lambda_3(M) = \Big\{H : \mathcal{P}\text{-measurable functions on } \Omega \times \mathbb{R}_+ \text{ such that } P\text{-a.s., } \forall t \geq 0,\ \int_0^t H_s^2(\omega)\, d\langle M\rangle_s(\omega) < \infty\Big\}. \tag{5.66}$$

One then considers a non-decreasing sequence of stopping times T_n, $n \geq 0$, P-a.s. tending to ∞, such that

$$M^{T_n} \text{ is a continuous square integrable martingale for each } n \geq 0\,, \tag{5.67}$$

$$1_{[0,T_n]} H \in \Lambda_2(M^{T_n})\,. \tag{5.68}$$

One such sequence is for instance obtained by setting:

$$T_n(\omega) = \inf\Big\{s \geq 0,\ |M_s(\omega)| \geq n \text{ or } \int_0^s H_s^2(\omega)\, d\langle M\rangle_s(\omega) \geq n\Big\}. \tag{5.69}$$

Definition and Theorem 5.7. *If $T_n \uparrow \infty$, P-a.s., is a sequence of stopping times satisfying (5.67), (5.68), then, the event*

$$N = \bigcup_{n \geq 0} \{\omega \in \Omega;\ \exists t \in \mathbb{Q}_+,\ ((H1_{[0,T_{n+1}]}).M^{T_{n+1}})_{t\wedge T_n} \neq ((H1_{[0,T_n]}).M^{T_n})_{t \wedge T_n}\} \cup \{\omega \in \Omega;\ \lim_n T_n(\omega) < \infty\} \tag{5.70}$$

is P-negligible, and

$$\begin{aligned}(H.M)_t(\omega) &\overset{\text{def}}{=} ((H1_{[0,T_n]}).M^{T_n})_t(\omega),\ \textit{for } 0 \leq t \leq T_n(\omega),\ \textit{if } \omega \notin N\,,\\ &= 0,\ \textit{if } \omega \in N\,,\end{aligned} \tag{5.71}$$

is a continuous local martingale. It is defined in an essentially unique way if one uses different choices of T_n, $n \geq 0$, and of $((H1_{[0,T_n]}).M^{T_n})_t$.

Proof. By (5.60), letting $M^{T_{n+1}}$ play the role of M, and $H1_{[0,T_{n+1}]}$ of H, we find that P-a.s., for $0 \leq t \leq \infty$,

$$((H1_{[0,T_{n+1}]}).M^{T_{n+1}})_{t\wedge T_n} = ((H1_{[0,T_n]}).M^{T_n})_t = ((H1_{[0,T_n]}).M^{T_n})_{t\wedge T_n}\,, \tag{5.72}$$

where the last equality follows from (5.60), with M replaced by M^{T_n} and H by $H1_{[0,T_n]}$. We thus find that $P(N) = 0$, and it is immediate from (5.71) that $(H.M)_t$ defines a continuous local martingale. Now, when T_n, T_n', $n \geq 0$, are two sequences of stopping times satisfying the assumptions of the theorem, setting $S_n = T_n \wedge T_n' \uparrow \infty$, P-a.s., we find that P-a.s., for $0 \leq t \leq S_n(\omega)$:

$$((H1_{[0,T_n]}).M^{T_n})_t(\omega) \overset{(5.60)}{=} ((H1_{[0,S_n]}).M^{S_n})_t(\omega) \overset{(5.60)}{=} ((H1_{[0,T_n']}).M^{T_n'})_t(\omega)\,, \tag{5.73}$$

and the claim about the essential uniqueness follows. □

Remark 5.8. When M is a continuous square integrable martingale, with $M_0 = 0$, and $H \in \Lambda_2(M)$, we can take $T_n \equiv \infty$ in the previous definition, so that (5.71) agrees with our previous definition of the stochastic integral. □

We will now give an alternative characterization of $(H.M)_.$, by means of its bracket $\langle (H.M), N\rangle$ with other continuous local martingales. The following will be helpful.

Proposition 5.9. *(Kunita-Watanabe's inequality, 1967)*

If H, K are progressively measurable, M, N are continuous local martingales, and

$$P\text{-a.s.},\ \int_0^\infty H_s^2(\omega)\, d\langle M\rangle_s(\omega) < \infty,\ \int_0^\infty K_s^2(\omega)\, d\langle N\rangle_s(\omega) < \infty\,, \tag{5.74}$$

then

$$\begin{aligned} P\text{-a.s.},\ \int_0^\infty H_s(\omega)\, K_s(\omega)\, d|\langle M, N\rangle|_s(\omega) \le \Big(\int_0^\infty H_s^2(\omega)\, d\langle M\rangle_s(\omega)\Big)^{\frac{1}{2}} \\ \cdot \Big(\int_0^\infty K_s^2(\omega)\, d\langle N\rangle_s(\omega)\Big)^{\frac{1}{2}} \end{aligned} \tag{5.75}$$

(with $|\langle M, N\rangle|_s$ denoting the total variation process of $\langle M, N\rangle_s$, cf. (5.45)).

Proof. From (5.43), (5.44), we see that

$$P\text{-a.s., for all } \lambda \in \mathbb{Q}, \text{ all } t \ge 0,\ \langle M + \lambda N\rangle_t = \langle M\rangle_t + 2\lambda\langle M, N\rangle_t + \lambda^2\langle N\rangle_t\,. \tag{5.76}$$

Hence, P-a.s., for $s \le t$, $\lambda \in \mathbb{Q}$ (with $\langle\ \rangle_s^t \stackrel{\text{def}}{=} \langle\ \rangle_t - \langle\ \rangle_s$)

$$\langle M\rangle_s^t + 2\lambda\langle M, N\rangle_s^t + \lambda^2\langle N\rangle_s^t \ge 0\,, \tag{5.77}$$

and thus, looking at the discriminant (in λ), we find that

$$\begin{aligned} P\text{-a.s., for } 0 \le s \le t,\ |\langle M, N\rangle_s^t(\omega)| &\le \big(\langle M\rangle_s^t(\omega)\big)^{\frac{1}{2}}(\langle N\rangle_s^t(\omega))^{\frac{1}{2}} \\ &\le \frac{1}{2}\langle M\rangle_s^t(\omega) + \frac{1}{2}\langle N\rangle_s^t(\omega)\,. \end{aligned} \tag{5.78}$$

This shows that on the above set of full P-measure

$$d\,|\langle M, N\rangle|_s \le \frac{1}{2}\, d\langle M\rangle_s + \frac{1}{2}\, d\langle N\rangle_s \stackrel{\text{def}}{=} d\nu_\omega(s) \tag{5.79}$$

and we can then introduce $f_s^M(\omega)$, $f_s^N(\omega)$, $f_s^{M,N}(\omega)$ the respective densities of $d\langle M\rangle_s(\omega)$, $d\langle N\rangle_s(\omega)$ and $d\langle M, N\rangle_s(\omega)$ with respect to $d\nu_\omega(s)$.

Coming back to (5.76), and expressing the density of $d\langle M+\lambda N\rangle_s(\omega)$ with respect to $d\nu_\omega(s)$, we see that P-a.s., for ν_ω-a.e. s,

$$f_s^M(\omega)+2\lambda\, f_s^{M,N}(\omega)+\lambda^2\, f_s^N(\omega)\ge 0, \text{ for } \lambda\in\mathbb{Q}, \text{ and hence } \lambda\in\mathbb{R}\,. \tag{5.80}$$

We first assume, $H,K\ge 0$, bounded and compactly supported in s. Setting $\mathrm{sign}(x)=1\{x\ge 0\}-1\{x<0\}$, $\widetilde{H}_s=H_s\,\mathrm{sign}(f_s^{M,N})$, and $\lambda=\gamma K_s\,\widetilde{H}_s^{-1}\,1\{H_s\neq 0\}$, we see multiplying (5.80) by H_s^2 and considering the set of s where $H_s=0$ separately that,

$$\begin{aligned}&P\text{-a.s., for } \nu_\omega\text{-a.e. } s, \text{ for all } \gamma\in\mathbb{R}\,,\\ &H_s^2(\omega)\, f_s^M(\omega)+2\gamma\,\widetilde{H}_s(\omega)\,K_s(\omega)\,f_s^{M,N}(\omega)+\gamma^2\,K_s^2(\omega)\,f_s^N(\omega)\ge 0\,.\end{aligned}$$

Integrating over s, with respect to $d\nu_\omega(s)$, we find that P-a.s. for all $\gamma\in\mathbb{R}$,

$$\int_0^\infty H_s^2(\omega)\,d\langle M\rangle_s+2\gamma\int_0^\infty H_s(\omega)\,K_s(\omega)\,d\,|\langle M,N\rangle|_s+\gamma^2\int_0^\infty K_s^2(\omega)\,d\langle N\rangle_s\ge 0\,, \tag{5.81}$$

and looking at the discriminant in γ we find that P-a.s.,

$$\int_0^\infty H_s\,K_s\,d\,|\langle M,N\rangle|_s\le\Big(\int_0^\infty H_s^2(\omega)\,d\langle M\rangle_s\Big)^{\frac12}\Big(\int_0^\infty K_s\,d\langle N\rangle_s\Big)^{\frac12}.$$

The case of non-negative H,K satisfying (5.74) follows by truncation and monotone convergence, and then the general case is immediate. $\square$

We now have the following characterization of $(H.M)$ for $H\in\Lambda_3(M)$:

Theorem 5.10. *Let M be a continuous local martingale with $M_0=0$, $H\in\Lambda_3(M)$, then $(H.M)$ is the unique continuous local martingale vanishing in 0, such that for all continuous local martingales N, P-a.s.*

$$\langle (H.M),N\rangle_t=\int_0^t H_s(\omega)\,d\langle M,N\rangle_s, \textit{ for all } t\ge 0 \tag{5.82}$$

(note that (5.75) implies that P-a.s. the right-hand side is well-defined).

Proof.

- Uniqueness:

If $I,\widetilde{I}$ are continuous local martingales vanishing at time 0 such that $\langle I,N\rangle=\langle\widetilde{I},N\rangle$ for all continuous local martingales N, we have $\langle I-\widetilde{I}\rangle=0$. Hence, we can find stopping times $T_n\uparrow\infty$, P-a.s., such that $(I-\widetilde{I})^2_{t\wedge T_n}$, $t\ge 0$, are bounded continuous martingales, cf. Remark 4.27. Hence, we see that

$$E[(I-\widetilde{I})^2_{t\wedge T_n}]=0,\ \ t\ge 0,\, n\ge 0\,, \tag{5.83}$$

so that P-a.s., for $t \in \mathbb{Q} \cap [0, \infty)$, $n \geq 0$, $I_{t \wedge T_n} = \widetilde{I}_{t \wedge T_n}$. Since P-a.s., $T_n \uparrow \infty$, using continuity, we find that

$$P\text{-a.s., for all } t \geq 0,\ I_t = \widetilde{I}_t\,. \tag{5.84}$$

• (5.82):

When $H = C1_{(a,b]}$, with $0 \leq a < b$, $C \in b\mathcal{G}_a$, is a basic process and M, N are continuous square integrable martingales,

$$\begin{aligned} &J_t = (H.M)_t\, N_t - \int_0^t H_s\, d\langle M, N\rangle_s = \\ &C\big\{(M_{b\wedge t} - M_{a\wedge t})\, N_t - \langle M, N\rangle_{a\wedge t}^{b\wedge t}\big\} \text{ is a continuous martingale.} \end{aligned} \tag{5.85}$$

For instance, when $a \leq s \leq b$, $s < t$:

$$\begin{aligned} E[J_t \,|\, \mathcal{G}_s] &= CE[E[(M_{b\wedge t} - M_a)\, N_t - \langle M, N\rangle_a^{b\wedge t} |\, \mathcal{G}_{b\wedge t}] \,|\, \mathcal{G}_s] \\ &= CE[(M_{b\wedge t} - M_a)\, N_{b\wedge t} - \langle M, N\rangle_a^{b\wedge t} \,|\, \mathcal{G}_s] \end{aligned} \tag{5.86}$$

since $N_{t\wedge b}$, $t \geq 0$, and $M_{b\wedge t}\, N_{b\wedge t} - \langle M, N\rangle_{b\wedge t}$ are martingales

$$= C\{(M_{b\wedge s} - M_a)\, N_{b\wedge s} - \langle M, N\rangle_a^{b\wedge s}\} = J_s\,,$$

and the other cases are easier to check. We then find that (5.82) holds for $H \in \Lambda_1$, M, N continuous square integrable martingales. Then, keeping M, N as above, for $H \in \Lambda_2(M)$ we can choose H^n in Λ_1, approximating H in $\Lambda_2(M)$, so that $(H^n.M)_t \overset{L^2(P)}{\longrightarrow} (H.M)_t$, for $t \geq 0$. Then, as a result of (5.75), for $t \geq 0$,

$$\begin{aligned} &E\Big[\int_0^t |H_s(\omega) - H_s^n(\omega)|\, d\,|\langle M, N\rangle|_s\Big] \leq \\ &E\Big[\Big(\int_0^\infty (H - H^n)_s^2(\omega)\, d\langle M\rangle_s(\omega)\Big)^{\frac{1}{2}} \langle N\rangle_t^{\frac{1}{2}}\Big] \overset{\text{Cauchy-Schwarz}}{\leq} \\ &\|H - H^n\|_{L^2(dP\times d\langle M\rangle)}\, E[\langle N\rangle_t]^{\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} 0\,. \end{aligned} \tag{5.87}$$

As a result, we see that for $t \geq 0$,

$$(H^n.M)_t\, N_t - \int_0^t H_s^n\, d\langle M, N\rangle_s \underset{n\to\infty}{\overset{L^1(P)}{\longrightarrow}} (H.M)_t\, N_t - \int_0^t H_s\, d\langle M, N\rangle_s\,, \tag{5.88}$$

and the limit is a martingale as well.

Thus we have proved that (5.82) holds when M, N are continuous square integrable martingales, and $H \in \Lambda_2(M)$.

Now, in the general case of the theorem, when $H \in \Lambda_3(M)$, we choose stopping times $T_n \uparrow \infty$, P-a.s., so that M^{T_n}, N^{T_n} are continuous square integrable martingales, and $H1_{[0,T_n]} \in \Lambda_2(M^{T_n})$, for each $n \geq 0$. Then we find from (5.71) that P-a.s., for all $t \geq 0$,

$$\begin{aligned}
&(H.M)_{t\wedge T_n}\, N_{t\wedge T_n} - \int_0^{t\wedge T_n} H_s(\omega)\, d\langle M, N\rangle_s = \\
&((H1_{[0,T_n]}).M^{T_n})_t\, N_t^{T_n} - \int_0^t (H1_{[0,T_n]})_s\, d\langle M, N\rangle_{s\wedge T_n} \underset{(5.42)}{\overset{(5.38)}{=}} \\
&((H1_{[0,T_n]}).M^{T_n})_t\, N_t^{T_n} - \int_0^t (H1_{[0,T_n]})_s\, d\langle M^{T_n}, N^{T_n}\rangle_s\,,
\end{aligned} \tag{5.89}$$

which is a continuous martingale.

This proves that $(H.M)_t\, N_t - \int_0^t H_s\, d\langle M, N\rangle_s$, $t \geq 0$, is a continuous local martingale, and by (5.43), (5.44) (recall that $(H.M)_0 = 0$), the claim (5.82) follows in the general case. □

Corollary 5.11. *For M, N continuous local martingales, vanishing at time* 0*, $H \in \Lambda_3(M)$, $K \in \Lambda_3(N)$,*

$$P\text{-a.s.}, \ \langle (H.M),\ (K.N)\rangle_t = \int_0^t H_s(\omega)\, K_s(\omega)\, d\langle M, N\rangle_s, \text{ for } t \geq 0\,. \tag{5.90}$$

Proof. By (5.82) we find that P-a.s.

$$d\langle (H.M), (K.N)\rangle = Hd\langle M, (K.N)\rangle = HKd\langle M, N\rangle\,.$$

□

We have the following very useful consequence of this result:

Corollary 5.12. *For M continuous local martingale with $M_0 = 0$, $H \in \Lambda_3(M)$, $K \in \Lambda_3((H.M))$ one has*

$$HK \in \Lambda_3(M) \text{ and} \tag{5.91}$$

$$P\text{-a.s., for } t \geq 0,\ (K.(H.M))_t = ((K \cdot H).M)_t\,. \tag{5.92}$$

Proof.

• (5.91):

By (5.90), we have P-a.s., $d\langle (H.M)\rangle = H^2\, d\langle M\rangle$, so that $K \in \Lambda_3((H.M))$ means that K is $\mathcal{P}$ measurable and P-a.s., $\int_0^t K_s^2\, H_s^2\, d\langle M\rangle_s < \infty$, and therefore $HK \in \Lambda_3(M)$.

• (5.92):

By (5.82), for N continuous local martingale, P-a.s.

$$d\langle (K.(H.M)), N\rangle = K\, d\langle (H.M), N\rangle = KH\, d\langle M, N\rangle$$

and (5.92) now follows from the uniqueness part of (5.82). $\square$

6 Ito's formula and first applications

In this chapter we will prove Ito's formula, which is a fundamental "change of variable formula" for stochastic integrals, and the source of many explicit calculations. We will discuss some of its applications. Throughout this chapter $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ will denote a filtered probability space, which satisfies the "usual conditions", cf. (4.5), (4.6).

Definition 6.1. *A* **continuous semimartingale** $(Y_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ *is a continuous adapted process, which admits the decomposition*

$$Y_t = Y_0 + M_t + A_t,\ t \geq 0\,, \tag{6.1}$$

where M_t, $t \geq 0$, is a continuous local martingale such that $M_0 = 0$, and A_t, $t \geq 0$, is a continuous adapted process with bounded variation on finite intervals, such that $A_0 = 0$.

Remark 6.2. The same argument used in the proof of (5.43), (5.44) shows that when $(Y_t)_{t\geq 0}$ is a continuous semimartingale,

(6.2) the decomposition (6.1) is essentially unique.

Notation:

For $(Y_t)_{t\geq 0}$ a continuous semimartingale we will write

$$\begin{aligned}\Lambda(Y) = \Big\{ H &: \mathcal{P}\text{-measurable on } \Omega \times \mathbb{R}_+ \text{ such that } P\text{-a.s., for } t \geq 0, \\ &\int_0^t H_s^2\, d\langle M\rangle_s < \infty \text{ and } \int_0^t |H_s|\, d|A|_s < \infty \Big\}\,,\end{aligned} \tag{6.3}$$

where M and A are as in (6.1) and $|A|_.$ denotes the total variation process of A. Then, for $H \in \Lambda(Y)$, we will use the notation

$$\int_0^t H_s\, dY_s = \int_0^t H_s\, dM_s + \int_0^t H_s\, dA_s,\ t \geq 0\,, \tag{6.4}$$

so that (6.4) defines $\int_0^t H_s\, dY_s$, $t \geq 0$, in an essentially unique fashion (with respect to the various versions and decompositions in (6.1)).

Example:

Any continuous adapted process $H_s(\omega)$ is automatically in $\Lambda(Y)$. In particular an expression such as $\int_0^t \exp\{\exp(Y_s^2 + s^2)\}\, dY_s$ (for instance) is well defined. □

An important first step towards Ito's formula will be the next

Proposition 6.3. *(Integration by parts formula)*

If Y_t, $t \geq 0$, and Z_t, $t \geq 0$, are continuous semimartingales on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$, then P-a.s., for all $t \geq 0$,

$$Y_t\, Z_t = Y_0\, Z_0 + \int_0^t Y_s\, dZ_s + \int_0^t Z_s\, dY_s + \langle Y, Z\rangle_t\,, \tag{6.5}$$

where $\langle Y, Z\rangle_t$, $t \geq 0$, denotes the bracket of the local martingale parts of Y and Z.

We begin with several reductions.

It is enough to prove for Y as above, that P-a.s.,

$$Y_t^2 = Y_0^2 + 2\int_0^t Y_s\,dY_s + \langle Y\rangle_t,\ \text{for } t \ge 0 \tag{6.6}$$

(i.e. prove (6.5) when $Y = Z$). Indeed one then applies (6.6) to $(Y+Z)^2$ and $(Y-Z)^2$ and recovers (6.5).

Moreover, we can also replace Y by $Y^n = 1\{T_n > 0\}\,Y^{T_n}$, where $Y^{T_n}_{\cdot} = Y_{\cdot\wedge T_n}$, $n \ge 1$, and $T_n \uparrow \infty$ is the sequence of stopping times

$$T_n = \inf\{s \ge 0,\ |Y_s| \ge n,\ |M_s| \ge n,\ |A|_s \ge n\ \text{ or }\ \langle M\rangle_s \ge n\},\ n \ge 1\,, \tag{6.7}$$

and, in this fashion, we can assume that $Y_{\cdot}$, $M_{\cdot}$, $|A|_{\cdot}$, $\langle Y\rangle_{\cdot}$ are continuous bounded processes (so $M_{\cdot}$ is in fact a martingale). Since all processes, which appear in (6.6) are continuous, it is also sufficient to prove (6.6) for fixed t.

Proof. We thus pick a fixed $t \ge 0$, and define for $m \ge 1$,

$$t_i = \frac{it}{m},\ \ 0 \le i \le m\,. \tag{6.8}$$

We then write:

$$Y_t^2 = \Big(Y_0 + \sum_{i=0}^{m-1}(Y_{t_{i+1}} - Y_{t_i})\Big)^2 = Y_0^2 + \sum_{i<m}(Y_{t_{i+1}} - Y_{t_i})^2 + 2\sum_{i<m} Y_{t_i}(Y_{t_{i+1}} - Y_{t_i})\,. \tag{6.9}$$

We now analyze the convergence of the last two terms in the right-hand side of (6.9), as $m \to \infty$. We have:

$$\begin{aligned}\sum_{i<m} Y_{t_i}(Y_{t_{i+1}} - Y_{t_i}) &= \sum_{i<m} Y_{t_i}(M_{t_{i+1}} - M_{t_i}) + \sum_{i<m} Y_{t_i}(A_{t_{i+1}} - A_{t_i}) \\ &= \int_0^t Y_s^m\,dM_s + \int_0^t Y_s^m\,dA_s\,,\end{aligned} \tag{6.10}$$

where

$$Y_s^m(\omega) = \sum_{0 \le i < m} Y_{t_i}(\omega)\,1_{(t_i,t_{i+1}]}(s)\,. \tag{6.11}$$

Clearly, using dominated convergence, we find that:

$$\begin{aligned}&E\Big[\int_0^t (Y_s - Y_s^m)^2\,d\langle Y\rangle_s\Big] \underset{m\to\infty}{\longrightarrow} 0,\ \text{and}\\ &E\Big[\int_0^t |Y_s - Y_s^m|\,d|A|_s\Big] \underset{m\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

It thus follows that

$$\text{(6.12)} \qquad \int_0^t Y_s^m \, dM_s \underset{m\to\infty}{\overset{L^2(P)}{\longrightarrow}} \int_0^t Y_s \, dM_s \text{ and } \int_0^t Y_s^m \, dA_s \underset{m\to\infty}{\overset{L^1(P)}{\longrightarrow}} \int_0^t Y_s \, dA_s \, .$$

This shows that

$$\text{(6.13)} \qquad \sum_{i<m} Y_{t_i}(Y_{t_{i+1}} - Y_{t_i}) \underset{m\to\infty}{\overset{L^1(P)}{\longrightarrow}} \int_0^t Y_s \, dY_s \, .$$

We now come back to the second term of the right-hand side of (6.9). We write for $0 \le i < m$,

$$\text{(6.14)} \qquad \Delta_i^m \overset{\text{def}}{=} (M_{t_{i+1}} - M_{t_i})^2 - (\langle Y \rangle_{t_{i+1}} - \langle Y \rangle_{t_i}) \, .$$

The calculation below resembles what we did in (3.4). For $m \ge 1$,

$$\text{(6.15)} \qquad \begin{aligned} A_m \overset{\text{def}}{=} E\Big[\Big(\sum_{i<m} (M_{t_{i+1}} - M_{t_i})^2 - \langle Y \rangle_t\Big)^2\Big] = E\Big[\Big(\sum_{i<m} \Delta_i^m\Big)^2\Big] = \\ \sum_{i<m} E[(\Delta_i^m)^2] + 2 \sum_{i<j<m} E[\Delta_i^m \, \Delta_j^m] \, . \end{aligned}$$

As we now explain, the Δ_i^m, $1 \le i < m$, are pairwise orthogonal.

Indeed, by (5.9), we have for $j < m$

$$\text{(6.16)} \qquad \begin{aligned} & E[\Delta_j^m \,|\, \mathcal{G}_{t_j}] = E[M_{t_{j+1}}^2 - 2M_{t_{j+1}} \, M_{t_j} + M_{t_j}^2 - (\langle Y \rangle_{t_{j+1}} - \langle Y \rangle_{t_j}) \,|\, \mathcal{G}_{t_j}] = \\ & E[M_{t_{j+1}}^2 - \langle M \rangle_{t_{j+1}} \,|\, \mathcal{G}_{t_j}] - 2M_{t_j} \, E[M_{t_{j+1}} \,|\, \mathcal{G}_{t_j}] + M_{t_j}^2 + \langle Y \rangle_{t_j} \overset{(5.9)}{=} \\ & M_{t_j}^2 - \langle M \rangle_{t_j} - 2M_{t_j}^2 + M_{t_j}^2 + \langle Y \rangle_{t_j} = 0 \, . \end{aligned}$$

Thus, the last term of (6.15) vanishes (Δ_i^m is $\mathcal{G}_{t_j}$-measurable for $i < j$). One has

$$\begin{aligned} A_m = \sum_{i<m} E[(\Delta_i^m)^2] & \le \sum_{i<m} 2E[(M_{t_{i+1}} - M_{t_i})^4] + 2E[(\langle Y \rangle_{t_{i+1}} - \langle Y \rangle_{t_i})^2] \\ & \le 2E\Big[\sup_{i<m} |\, M_{t_{i+1}} - M_{t_i}|^2 \,\Big|\sum_{i<m} \Delta_i^m\Big|\Big] \\ & \quad + 2E\big[\big(\sup_{i<m} |M_{t_{i+1}} - M_{t_i}|^2 + \sup_{i<m} |\, \langle Y \rangle_{t_{i+1}} - \langle Y \rangle_{t_i}|\big) \langle Y \rangle_t\big] \, . \\ & \qquad\qquad\qquad\qquad\qquad\qquad \underset{m\to\infty}{\swarrow} \\ & \qquad\qquad\qquad\qquad\qquad 0 \ \text{dominated convergence} \end{aligned}$$

Further, note that $2ab \le 2a^2 + \frac{b^2}{2}$, for a, b in $\mathbb{R}$. So,

$$\text{(6.17)} \qquad \begin{aligned} E\Big[\sup_{i<m} |M_{t_{i+1}} - M_{t_i}|^2 \,\Big|\sum_{i<m} \Delta_i^m\Big|\Big] \overset{(6.15)}{\le} & \ E\big[\sup_{i<m} |M_{t_{i+1}} - M_{t_i}|^4\big] + \frac{A_m}{4}. \\ & \ \downarrow{\scriptstyle m\to\infty} \\ & \ 0 \ \text{dominated convergence} \end{aligned}$$

Thus, coming back to the line above, we have shown that

$$A_m = \sum_{i<m} E[(\Delta_i^m)^2] \underset{m\to\infty}{\longrightarrow} 0\,. \tag{6.18}$$

By (6.15), it means that

$$\sum_{i<m} (M_{t_{i+1}} - M_{t_i})^2 \underset{m\to\infty}{\overset{L^2(P)}{\longrightarrow}} \langle Y \rangle_t\,. \tag{6.19}$$

To prove an analogous statement for $\sum_{i<m}(Y_{t_{i+1}} - Y_{t_i})^2$, which is our main object of interest in view of (6.9), we write:

$$\begin{aligned}
&\Big| \sum_{i<m} (Y_{t_{i+1}} - Y_{t_i})^2 - \sum_{i<m} (M_{t_{i+1}} - M_{t_i})^2 \Big| \le \\
&\sum_{i<m} \Big(2|A_{t_{i+1}} - A_{t_i}|\,|M_{t_{i+1}} - M_{t_i}| + |A_{t_{i+1}} - A_{t_i}|^2 \Big) \le \\
&2\Big(\sum_{i<m} |A_{t_{i+1}} - A_{t_i}|^2 \Big)^{\frac{1}{2}} \times \Big(\sum_{i<m} (M_{t_{i+1}} - M_{t_i})^2 \Big)^{\frac{1}{2}} + \sum_{i<m} (A_{t_{i+1}} - A_{t_i})^2
\end{aligned} \tag{6.20}$$

and observe that by dominated convergence and continuity,

$$\sum_{i<m} (A_{t_{i+1}} - A_{t_i})^2 \ \le \sup_{i<m} |A_{t_{i+1}} - A_{t_i}|\, |A|_t \underset{m\to\infty}{\overset{L^2(P)}{\longrightarrow}} 0\,.$$

Thus, coming back to the last line of (6.20), we see using Cauchy-Schwarz's inequality for the first term and (6.19) that all terms converge to 0 in $L^2(P)$. We have shown that

$$\sum_{i<m} (Y_{t_{i+1}} - Y_{t_i})^2 \underset{m\to\infty}{\overset{L^2(P)}{\longrightarrow}} \langle Y \rangle_t\,. \tag{6.21}$$

Together with (6.13) this concludes the proof of (6.6) for fixed t, and thus our general claim (6.5) has been established. □

We now turn to the main result.

Theorem 6.4. *(Ito's formula)*

Let F be a C^2-function on $\mathbb{R}^d$, and $Y^1_{\cdot}, \dots, Y^d_{\cdot}$ be continuous semimartingales on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$. Then, writing $Y_{\cdot} = (Y^1_{\cdot}, \dots, Y^d_{\cdot})$, the real-valued process $F(Y_t)$, $t \ge 0$, is a continuous semimartingale and P-a.s., for all $t \ge 0$,

$$F(Y_t) = F(Y_0) + \sum_{i=1}^d \int_0^t \partial_i F(Y_s)\, dY_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial^2_{i,j} F(Y_s)\, d\langle Y^i, Y^j \rangle_s\,. \tag{6.22}$$

Proof. The formula (6.22) shows that $F(X_t)$, $t \geq 0$, is a continuous semimartingale. We use several reductions to prove (6.22).

• First reduction:

Using "localization", similarly as explained above (6.7), we can assume that Y_t^i, $|\langle Y^i, Y^j \rangle|_t$ and the total variation of the bounded variation processes entering the decomposition (6.1) of the Y_t^i, $t \geq 0$, are uniformly bounded processes.

• Second reduction:

We can assume that $F(\cdot)$ is C^2 with compact support.

• Third reduction:

We can assume that $F(\cdot)$ is a polynomial.

Indeed note that it suffices now to prove (6.22) for $F \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$, since any $F \in C_c^2(\mathbb{R}^d, \mathbb{R})$ is approximated, for instance by convolution, in C^2-topology by such functions, and (6.22) remains true in the limit. But $F \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ is approximated in C^2-topology on any compact set by linear combinations of $e^{i\xi \cdot x}$, with $\xi \in \mathbb{R}^d$ (for instance using Fourier series, when L is large enough so that $support(f) \subset (-\frac{L}{2}, \frac{L}{2})^d$, one has:

$$F(x) = \sum_{k \in \mathbb{Z}^d} a_k \, e^{i2\pi \frac{k}{L} \cdot x}, \text{ for } x \in \Big(-\frac{L}{2}, \frac{L}{2}\Big)^d, \text{ with } a_k = \frac{1}{L^d} \int_{\left(-\frac{L}{2}, \frac{L}{2}\right)^d} f(z) \, e^{-i2\pi \frac{k}{L} \cdot z} dz \,.$$

Now, we have the expansion $e^{i\xi \cdot x} = \sum_{n \geq 0} \frac{1}{n!} (i\xi \cdot x)^n$, which shows that $e^{i\xi \cdot x}$ is approximated in C^2-topology on compact sets by polynomials and the third reduction follows.

We are thus reduced to proving (6.22) for $F(\cdot)$ a polynomial in the coordinate variables. We will now prove that:

(6.23) the validity of (6.22) for the polynomial F implies the validity of (6.22) for the polynomials $G(x_1, \dots, x_d) = x_{i_0} F(x_1, \dots x_d)$, for any $1 \leq i_0 \leq d$.

Indeed, we apply (6.5) to Y^{i_0} and $F(Y)$, and find that

$$G(Y_t) = G(Y_0) + \int_0^t Y_s^{i_0} \, dF(Y)_s + \int_0^t F(Y_s) \, dY_s^{i_0} + \langle Y^{i_0}, F(Y) \rangle_t$$

and since (6.22) holds true for $F(Y)$

$$\begin{aligned} G(Y_t) = G(Y_0) &+ \sum_{i=1}^d \int_0^t Y_s^{i_0} \, \partial_i \, F(Y_s) \, dY_s^i + \int_0^t F(Y_s) \, dY_s^{i_0} \\ &+ \frac{1}{2} \sum_{i,j=1}^d \int_0^t Y_s^{i_0} \, \partial_{i,j}^2 \, F(Y_s) \, d\langle Y^i, Y^j \rangle_s + \langle Y^{i_0}, F(Y) \rangle_t \,. \end{aligned}$$

Now by (5.82) and (6.22) we also have

$$\langle Y^{i_0}, F(Y)\rangle_t = \sum_{i=1}^{d} \int_0^t \partial_i\, F(Y_s)\, d\langle Y^{i_0}, Y^i\rangle_s\,.$$

Inserting this identity in the last term of the previous formula, we find that:

$$G(Y_t) = G(Y_0) + \sum_{i=1}^{d} \int_0^t \partial_i\, G(Y_s)\, dY_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial_{i,j}^2\, G(Y_s)\, d\langle Y^i, Y^j\rangle_s\,,$$

and (6.23) is proved.

Since (6.22) clearly holds when F = constant, it follows by (6.23), that (6.22) holds for all polynomials F. This, as explained above, yields the general claim. $\square$

Example: (Canonical d-dimensional Brownian motion)

We can apply the above theorem in the special case when $Y_. = X_.(= (X_.^1, \dots, X_.^d))$ is the canonical d-dimensional Brownian motion, and $(C, \mathcal{F}, (F_t)_{t\ge 0}, W_0)$, cf. (4.7), is the filtered probability space (which as we have seen satisfies the usual conditions). In this example each X_t^i, $t \ge 0$, are (F_t)-martingales, and

(6.24) $$X_t^i\, X_t^j - \underset{\substack{\uparrow \\ \text{Kronecker's symbol}}}{\delta_{ij}}\, t,\ t \ge 0,\ \text{are } (F_t)\text{-martingales, for } 1 \le i, j \le d\,,$$

(when $i = j$, see (4.2), (4.18), the case $i \neq j$ simply uses a simple modification of (4.4)). In view of (5.43), (5.44), this means that

(6.25) $$\langle X^i, X^j\rangle_t = \delta_{i,j}\, t,\ t \ge 0,\ \text{for } 1 \le i, j \le d\,.$$

In particular, in the rightmost term of Ito's formula, only the terms with $i = j$ are present, and hence for $F \in C^2(\mathbb{R}^d, \mathbb{R})$, W_0-a.s., for all $t \ge 0$,

(6.26) $$F(X_t) = F(0) + \sum_{i=1}^{d} \int_0^t \partial_i\, F(X_s)\, dX_s^i + \frac{1}{2} \int_0^t \Delta F(X_s)\, ds\,,$$

where $\Delta F(x) \overset{\text{def}}{=} \sum_{i=1}^d \partial_i^2\, F(x)$ is the Laplacian of F. $\square$

We will now describe some first applications of Ito's formula. We recall that $(\Omega, \mathcal{G}, (\mathcal{G})_{t\ge 0}, P)$ is a filtered probability space satisfying the "usual conditions", cf. (4.5), (4.6).

Exponential Martingales:

Theorem 6.5. *Let M_t, $t \ge 0$, be a continuous $(\mathcal{G}_t)$-local martingale, with $M_0 = 0$.*

$$Z_t = \exp\Big\{M_t - \tfrac{1}{2}\,\langle M\rangle_t\Big\},\ t \ge 0, \text{ is a continuous } (\mathcal{G}_t)\text{-local martingale,} \tag{6.27}$$

which satisfies the "stochastic differential equation":

$$P\text{-a.s., for all } t \ge 0,\ Z_t = 1 + \int_0^t Z_s\, dM_s\,. \tag{6.28}$$

Moreover, if for some $\varepsilon > 0$ and $0 < T \le \infty$,

$$E\Big[\exp\Big\{\frac{(1+\varepsilon)}{2}\,\langle M\rangle_T\Big\}\Big] < \infty, \text{ then} \tag{6.29}$$

$$Z_t,\ t \le T, \text{ is a continuous } (\mathcal{G}_t)\text{-martingale.} \tag{6.30}$$

Remark 6.6.

1) We will later see that (6.30) is still valid when (6.29) holds with $\varepsilon = 0$. This is the so-called Novikov condition, cf. [8], [9]. For the time being we discuss this simpler result which has an elementary proof and can be helpful in a number of situations.

2) If (6.29) holds with $T = \infty$, then the proof below will show that $Z_\infty = \lim_{t\to\infty} Z_t$ exists P-a.s., and for small $\eta > 0$,

$$Z_t,\ 0 \le t \le \infty, \text{ is a continuous } (\mathcal{G}_t)\text{-martingale bounded in } L^{1+\eta}\,. \tag{6.30'}$$

$\square$

Exercise 6.7. Show that when $E[\langle M\rangle_T] < \infty$, ($M_t, t \ge 0$, as above), then M_t, $t \le T$, is a continuous square integrable martingale.

Proof of Theorem 6.5.

• (6.28): We introduce the function

$$f(x,t) = \exp\Big\{x - \tfrac{1}{2}\,t\Big\},\quad x,t \in \mathbb{R}\,. \tag{6.31}$$

This function satisfies the equation:

$$\frac{\partial f}{\partial t}\,(x,t) + \frac{1}{2}\,\frac{\partial^2 f}{\partial x^2}\,(x,t) = 0\,. \tag{6.32}$$

We will now apply Ito's formula (6.22) to $Y = (Y^1, Y^2) = (M, \langle M\rangle)$. We first note that

$$\langle Y^1, Y^2\rangle = 0 = \langle Y^2\rangle, \text{ and } \langle Y^1\rangle = \langle M\rangle\,.$$

We thus find that P-a.s., for $t \ge 0$:

$$
\begin{aligned}
f(M_t, \langle M\rangle_t) = f(0,0) &+ \int_0^t \partial_x f(M_s,\langle M\rangle_s)\, dM_s + \int_0^t \partial_t f(M_s,\langle M\rangle_s)\, d\langle M\rangle_s \\
&+ \tfrac{1}{2} \int_0^t \partial_x^2 f(M_s,\langle M\rangle_s)\, d\langle M\rangle_s \\
&\overset{(6.32)}{=} 1 + \int_0^t f(M_s,\langle M\rangle_s)\, dM_s, \quad (\partial_x f = f)\,.
\end{aligned}
\tag{6.33}
$$

Since $Z_t = f(M_t, \langle M\rangle_t)$, this proves (6.28), as well as the fact that Z_t, $t \ge 0$, is a continuous local martingale.

• (6.30):

We consider a sequence of finite stopping times $T_n \uparrow \infty$, P-a.s., such that

$$
M_{t\wedge T_n},\ 0 \le t \le T, \text{ is a bounded martingale for each } n\,. \tag{6.34}
$$

Observe that $Z_{t\wedge T_n}$ is a bounded continuous local martingale and hence, cf. (4.104),

$$
Z_{t\wedge T_n},\ 0 \le t \le T, \text{ is a bounded martingale for each } n\,. \tag{6.35}
$$

We will now see that

$$
\text{for some } q > 1,\ \sup_{n\ge 0} E[Z^q_{T\wedge T_n}] < \infty\,. \tag{6.36}
$$

From Doob's inequality (4.76), it will then follow that:

$$
E\big[\big(\sup_{t\le T} Z_t\big)^q\big] = \lim_{n\to\infty} E\big[\sup_{t\le T\wedge T_n} Z_t^q\big] \le \overline{\lim_n} \Big(\frac{q}{q-1}\Big)^q E[Z^q_{T\wedge T_n}] < \infty\,. \tag{6.37}
$$

Together with (6.35), this will imply by dominated convergence that:

$$
Z_t,\ 0 \le t \le T, \text{ is a continuous martingale bounded in } L^q\,. \tag{6.38}
$$

This will prove (6.30) (and also (6.30')) in the case $T = \infty$; in this case Z_∞ exists as a P-a.s. limit, by the martingale convergence theorem, see Theorem 3.15, p. 17 of [8], and in addition $Z_\infty = \lim_{t\to\infty} Z_t$ in $L^q(P)$ as well, by dominated convergence, thanks to (6.37)).

There remains to prove (6.36). We pick $q, \alpha > 1$, and write:

$$
\begin{aligned}
&E[Z^q_{T\wedge T_n}] = E\Big[\exp\Big\{q\, M_{T\wedge T_n} - \frac{q}{2}\, \langle M\rangle_{T\wedge T_n}\Big\}\Big] = \\
&E\Big[\exp\Big\{q\, M_{T\wedge T_n} - \frac{1}{2}\, \alpha q^2 \langle M\rangle_{T\wedge T_n} + \frac{1}{2}\, q(\alpha q - 1)\langle M\rangle_{T\wedge T_n}\Big\}\Big] \overset{\text{Hölder}}{\le} \\
&E\Big[\exp\Big\{\alpha q\, M_{T\wedge T_n} - \frac{1}{2}\, \alpha^2 q^2 \langle M\rangle_{T\wedge T_n}\Big\}\Big]^{\frac{1}{\alpha}} \times \\
&E\Big[\exp\Big\{\frac{1}{2}\, \frac{\alpha}{\alpha-1}\, q(\alpha q - 1)\langle M\rangle_{T\wedge T_n}\Big\}\Big]^{\frac{\alpha-1}{\alpha}}.
\end{aligned}
\tag{6.39}
$$

Note that $\exp\{\alpha q M_{t\wedge T_n} - \frac{1}{2}\,\alpha^2 q^2 \langle M\rangle_{t\wedge T_n}\}$ is a bounded martingale just as in (6.35), and the first term in the last line of (6.39) equals 1. So we see that for any $n \ge 0$,

$$E[Z^q_{T\wedge T_n}] \le E\Big[\exp\Big\{\frac{1}{2}\,\frac{\alpha}{\alpha-1}\;q(\alpha q-1)\langle M\rangle_T\Big\}\Big]^{\frac{\alpha-1}{\alpha}}\,.$$

Note that

$$\lim_{\alpha\downarrow 1}\;\lim_{q\downarrow 1}\;\frac{\alpha}{\alpha-1}\;q(\alpha q-1) = 1\,,$$

and hence we can choose $q, \alpha > 1$, so that

$$\frac{\alpha}{\alpha-1}\;q(\alpha q-1) < 1+\varepsilon\,.$$

The combination of (6.29) and the above inequality yields (6.36). This concludes the proof of (6.30). □

Example:

$(X_t)_{t\ge 0}$, Brownian motion on $\mathbb{R}$. Then for $\lambda \in \mathbb{R}$,

$$\exp\Big\{\lambda X_t - \frac{\lambda^2}{2}\,t\Big\} \text{ is a martingale.} \tag{6.40}$$

Consider $a > 0$ and the entrance time of X in $\{a\}$:

$$H_a = \inf\{s \ge 0; X_s = a\},\ \text{(the distribution of } H_a \text{ under } W_0 \text{ appears in (2.54))}\,.$$

Then, using the stopping theorem, we see that when $\lambda > 0$,

$$\exp\Big\{\lambda X_{t\wedge H_a} - \frac{\lambda^2}{2}\,(t\wedge H_a)\Big\} \text{ is a martingale, which is bounded by } e^{\lambda a}\,. \tag{6.41}$$

As a result we obtain that

$$1 = E_0\Big[\exp\Big\{\lambda X_{t\wedge H_a} - \frac{\lambda^2}{2}\,(t\wedge H_a)\Big\}\Big],\ \text{and since } H_a < \infty,\ W_0\text{-a.s.,}$$

$$\text{dominated convergence}\;\Big\downarrow\; t\to\infty$$

$$E_0\Big[\exp\Big\{\lambda X_{H_a} - \frac{\lambda^2}{2}\,H_a\Big\}\Big] = e^{\lambda a} E_0\big[e^{-\frac{\lambda^2}{2}\,H_a}\big]\,.$$

Setting $u = \lambda^2/2$, we obtain (by symmetry when a is negative):

$$E_0[\exp\{-uH_a\}] = \exp\{-|a|\,\sqrt{2u}\},\ \text{for } a\in\mathbb{R},\ u\ge 0\,. \tag{6.42}$$

□

As an application of exponential martingales, we will prove **Paul Lévy's characterization of Brownian motion**. We introduce the following

Definition 6.8. *A continuous adapted $\mathbb{R}^d$-valued process $(X_t)_{t\geq 0}$, with $X_0 = 0$, is called $(\mathcal{G}_t)$-Brownian motion if for $0 \leq s < t$,*

$$X_t - X_s \text{ is independent of } \mathcal{G}_s \text{ and } N(0,(t-s)I)\text{-distributed.} \tag{6.43}$$

Remark 6.9. A $(\mathcal{G}_t)$-Brownian motion is then of course in particular a d-dimensional Brownian motion in the sense of the definition (1.1). However, the independence assumption (6.43) is a (possibly) more stringent requirement ($\mathcal{G}_s$ may be strictly bigger than $\sigma(X_u, u \leq s)$).

Theorem 6.10. *(P. Lévy's characterization of Brownian motion)*

If $(X_t)_{t\geq 0}$, is a d-dimensional continuous $(\mathcal{G}_t)$-local martingale, such that $X_0 = 0$, and

$$\langle X^i, X^j \rangle_t = \delta_{ij}\, t, \text{ for } 1 \leq i,j \leq d\,, \tag{6.44}$$

then

$$X_t,\ t \geq 0, \text{ is a } d\text{-dimensional } (\mathcal{G}_t)\text{-Brownian motion.} \tag{6.45}$$

Proof. The same calculation as in (6.33) shows that for $\xi \in \mathbb{R}^d$,

$$Z_t = \exp\Big\{i\xi \cdot X_t + \tfrac{1}{2}\, |\xi|^2 t\Big\},\ t \geq 0\,, \tag{6.46}$$

is a complex-valued, continuous $(\mathcal{G}_t)$-local martingale, which is bounded when t remains bounded. Hence it is a continuous martingale and for $0 \leq s \leq t$:

$$\begin{aligned}
&E[Z_t \,|\, \mathcal{G}_s] \overset{P\text{-a.s.}}{=} Z_s, \text{ so that (since } |Z_s| > 0) \\
&1 \overset{P\text{-a.s.}}{=} E[Z_t\, Z_s^{-1} \,|\, \mathcal{G}_s] = E\Big[\exp\Big\{i\xi \cdot (X_t - X_s) + \frac{|\xi|^2}{2}(t-s)\Big\} \Big|\, \mathcal{G}_s\Big]\,.
\end{aligned}$$

As a result, for $0 \leq s < t$, $\xi \in \mathbb{R}^d$,

$$E\Big[\exp\Big\{i\xi \cdot (X_t - X_s)\Big\} \Big|\, \mathcal{G}_s\Big] \overset{P\text{-a.s.}}{=} \exp\Big\{-\tfrac{1}{2}\, |\xi|^2 (t-s)\Big\}\,. \tag{6.47}$$

This implies that for $0 \leq s < t$

$$X_t - X_s \text{ is independent of } \mathcal{G}_s \text{ and } N(0,(t-s)I)\text{-distributed.}$$

The claim (6.45) now readily follows. □

Remark 6.11. If we now look again at the assumptions (4.20), (4.21), when we began the discussion of stochastic integrals, we see that they are equivalent to the fact that $X_t - X_0$ is a $(\mathcal{G}_t)$-Brownian motion and $X_0 \in L^2(\Omega, \mathcal{G}_0, dP)$. This link with Brownian motion was not clear at the time we introduced (4.20), (4.21). □

We will now give a **further application of exponential martingales**.

Proposition 6.12. *(Bernstein's inequality)*

If M_t, $t \ge 0$, is a continuous local martingale with $M_0 = 0$, and $\langle M \rangle_t \le ct$, for $t \ge 0$, then $M_t, t \ge 0$, is a martingale and

$$P\big[\sup_{t \le T} M_t \ge a\big] \le \exp\big\{-\frac{a^2}{2cT}\big\}, \text{ for all } a, T > 0\,, \tag{6.48}$$

(and hence $P[\sup_{t \le T} |M_t| \ge a] \le 2\exp\{-\frac{a^2}{2cT}\}$, for all $a, T > 0$).

Proof. For $\lambda \in \mathbb{R}$, by (6.27), (6.30), we know that

$$Z_t = \exp\big\{\lambda M_t - \frac{\lambda^2}{2}\,\langle M \rangle_t\big\},\ t \ge 0\,,$$

is a continuous martingale. We can pick $\lambda > 0$, and we have by Doob's inequality, cf (4.67):

$$\begin{aligned} P\big[\sup_{t \le T} M_t \ge a\big] &\le P\Big[\sup_{t \le T} Z_t \ge \exp\Big\{\lambda a - \frac{\lambda^2}{2}\,cT\Big\}\Big] \\ &\overset{(4.67)}{\le} \exp\Big\{-\lambda a + \frac{\lambda^2}{2}\,cT\Big\}\ \underbrace{E[Z_T]}_{\substack{\| \\ E[Z_0]=1}}\,. \end{aligned} \tag{6.49}$$

We can optimize over $\lambda > 0$, and choose $\lambda = \frac{a}{cT}$, so that $-\lambda a + \frac{\lambda^2}{2}\,cT = -\frac{a^2}{2cT}$. As a result we find that

$$P\big[\sup_{t \le T} M_t \ge a\big] \le \exp\Big\{-\frac{a^2}{2cT}\Big\},$$

that is, (6.48) holds. Applying this inequality to $-M$, we thus see that $(M_t)_{t \ge 0}$, is a continuous local martingale, which is square integrable. It is therefore a martingale, cf. (4.104) (alternatively use the exercise below (6.30')). $\square$

We continue our discussion of some first applications of Ito's formula.

Harmonic functions and Brownian motion

When $U \subseteq \mathbb{R}^d$ is non-empty open set, a C^2-function on U is said **harmonic** (in U) when $\Delta f(x) (= \sum_1^d \partial_i^2 f(x)) = 0$, $x \in U$. In fact no regularity requirement on f is necessary in the sense that the equation $\Delta f = 0$ on U in the **distribution sense** implies that f is C^∞ on U and satisfies $\Delta f(x) = 0$, $x \in U$, **in the classical sense**, cf. [3], p. 127. Harmonic functions play a very important role in the study of Brownian motion. Here is an example.

Proposition 6.13. *When $d \ge 2$ and $x \ne 0$ is a point of $\mathbb{R}^d$, then*

$$W_x\text{-a.s., } X_t \ne 0, \text{ for all } t \ge 0\,. \tag{6.50}$$

(in other words, "Brownian motion does not hit points when $d \ge 2$").

Proof. When g is a C^2-function on $(0,\infty)$, one can define the radial function

$$f(x) = g(|x|) = g\Big(\sqrt{x_1^2 + \cdots + x_d^2}\Big),\ x \in \mathbb{R}^d\backslash\{0\}\,,$$

and one has the identity (exercise!)

$$\Delta f(x) = g''(r) + \frac{d-1}{r}\, g'(r),\ \text{with } r = |x|, \text{ for } x \in \mathbb{R}^d\backslash\{0\}\,. \tag{6.51}$$

When $d \ge 3$, we choose

$$g(r) = r^{2-d}\,, \tag{6.52}$$

so that

$$g''(r) + \frac{d-1}{r}\, g'(r) = (2-d)(1-d)\, r^{-d} + (d-1)(2-d)\, r^{-d} = 0\,,$$

and therefore

$$f(x) = \frac{1}{|x|^{d-2}},\ x \neq 0,\ \text{is harmonic in } \mathbb{R}^d\backslash\{0\}\,. \tag{6.53}$$

If $x \neq 0$, and $a < |x| < b$, we choose f_a, a smooth radial function, equal to $f(x)$ on $\{y \in \mathbb{R}^d;\ |y| \ge a\}$, so that applying Ito's formula, cf. (6.26), one finds that:
W_x-a.s., for all $t \ge 0$:

$$f_a(X_t) = f_a(x) + \int_0^t \nabla f_a(X_s)\cdot dX_s + \frac{1}{2}\int_0^t \Delta f_a(X_s)\, ds\,. \tag{6.54}$$

We then introduce the stopping time

$$\tau = \inf\{u \ge 0;\ |X_u| \le a \text{ or } |X_u| \ge b\}\,, \tag{6.55}$$

and see that W_x-a.s., for all $t \ge 0$,

$$\begin{aligned} |X_{t\wedge\tau}|^{-(d-2)} = f_a(X_{t\wedge\tau}) \overset{(6.54)}{=} |x|^{-(d-2)} &+ \int_0^{t\wedge\tau} \nabla f_a(X_s)\cdot dX_s \\ &+ \frac{1}{2}\int_0^{t\wedge\tau} \Delta f_a(X_s)\, ds\,, \end{aligned}$$

and the last term vanishes, since $\Delta f_a(X_s) = 0$, for $0 < s < \tau$. As a result $|X_{t\wedge\tau}|^{-(d-2)}$, $t \ge 0$, is a local martingale, which is bounded, and hence:

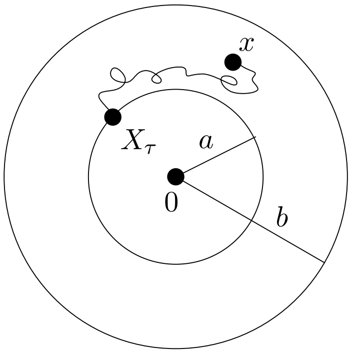
$$|X_{t\wedge\tau}|^{2-d},\ t \ge 0,\ \text{is a martingale.} \tag{6.56}$$

As a result, we find

$$E_x[|X_{t\wedge\tau}|^{2-d}] = |x|^{2-d},\ \text{for } t \ge 0\,. \tag{6.57}$$

Note that τ is W_x-a.s. finite, cf. Corollary 2.17. Letting $t \to \infty$, and using dominated convergence, we find that

$$|x|^{2-d} = E[|X_\tau|^{2-d}] = a^{2-d}\, W_x(|X_\tau| = a) + b^{2-d}\, W_x(|X_\tau| = b)\,.$$



An illustration of
$X_s, 0 \le s \le \tau$, under W_x.

Since $W_x(|X_\tau| = a) + W_x(|X_\tau| = b) = 1$, we obtain for $a < |x| < b$

$$(6.58) \qquad W_x(|X_\tau| = a) = \frac{|x|^{2-d} - b^{2-d}}{a^{2-d} - b^{2-d}},\ W_x(|X_\tau| = b) = \frac{a^{2-d} - |x|^{2-d}}{a^{2-d} - b^{2-d}}\,.$$

Letting $a \to 0$, with b fixed, we see that

$$(6.59) \qquad W_x(H_{\{0\}} < T_{B(0,b)}) = 0,\ \text{for } 0 < b,$$

with the notation $T_U = \inf\{s \ge 0; X_s \notin U\}$, the "exit time from U". It now follows that

$$(6.60) \qquad \begin{aligned} &W_x(H_{\{0\}} < \infty) = W_x(X_t = 0,\ \text{for some } t \ge 0) = \\ &\lim_{b\to\infty} W_x(H_{\{0\}} < T_{B(0,b)}) = 0\,, \end{aligned}$$

and this proves (6.50) when $d \ge 3$.

When $d = 2$, we choose instead

$$(6.61) \qquad g(r) = \log\ \frac{1}{r}\ ,$$

so that

$$g''(r) + \frac{d-1}{r}\ g'(r) = \frac{1}{r^2} - \frac{1}{r^2} = 0\,,$$

and

$$(6.62) \qquad f(x) = \log\ \frac{1}{|x|},\ x \ne 0,\ \text{is harmonic in } \mathbb{R}^2\backslash\{0\}\,.$$

The repetition of the above proof now yields that for $a < |x| < b$,

$$(6.63) \qquad W_x(|X_\tau| = a) = \frac{\log \frac{b}{|x|}}{\log \frac{b}{a}}\ ,\ \ W_x(|X_\tau| = b) = \frac{\log \frac{|x|}{a}}{\log \frac{b}{a}}\ ,$$

and one concludes as above, by letting $a \to 0$, with b fixed, and then $b \to \infty$. $\square$

As an application of the same circle of ideas we will discuss **recurrence and transience properties of Brownian motion** in $\mathbb{R}^d$, when $d \geq 2$.

Theorem 6.14. *(transience of Brownian motion in $\mathbb{R}^d$, $d \geq 3$)*

When $d \geq 3$, then for $x \in \mathbb{R}^d$,

$$W_x\text{-a.s.}, \ \lim_{t\to\infty} |X_t| = \infty \,. \tag{6.64}$$

Proof. Since under W_x, $(X_t + z)_{t \geq 0}$ is a Brownian motion starting from $x + z$, it suffices to prove (6.64) for some $x \neq 0$. By (6.54) we know that, letting $H_{\overline{B}(0,a)}$ stand for the entrance time of X in $\overline{B}(0,a)$,

$$|X_{t\wedge H_{\overline{B}(0,a)}}|^{2-d} \text{ is a continuous bounded martingale under } W_x \,. \tag{6.65}$$

Using Fatou's lemma for conditional expectations, we find that for $s \leq t$, W_x-a.s.,

$$\begin{array}{l} E_x[|X_t|^{2-d} \,|\, F_s] \overset{(6.50)}{=} E_x\big[\liminf\limits_n |X_{t\wedge H_{\overline{B}(0,\frac{1}{n})}}|^{2-d} | \, F_s\big] \overset{\text{Fatou}}{\leq} \\ \liminf\limits_n \ E_x\big[|X_{t\wedge H_{\overline{B}(0,\frac{1}{n})}}|^{2-d} | \, F_s\big] \overset{(6.65)}{=} \liminf\limits_n |X_{s\wedge H_{\overline{B}(0,\frac{1}{n})}}|^{2-d} \overset{(6.50)}{=} |X_s|^{2-d} \,. \end{array}$$

In other words we have proved that

$$|X_t|^{2-d},\ t \geq 0, \text{ is a continuous supermartingale under } W_x \,. \tag{6.66}$$

Since this supermartingale is non-negative, it follows from the convergence theorem, see [8], p. 17, that

$$W_x\text{-a.s.}, \ |X_t|^{2-d} \text{ has a finite limit as } t \to \infty \,. \tag{6.67}$$

On the other hand, looking at one of the components of X_t, we already know that

$$W_x\text{-a.s.}, \ \limsup_t |X_t| = \infty \,.$$

This observation combined with (6.67) implies that the finite limit in (6.67) is 0. $\square$

Exercise 6.15. Show that a non-negative continuous local martingale is a supermartingale. $\square$

We now turn to the two-dimensional situation.

Theorem 6.16. *(recurrence of Brownian motion in $\mathbb{R}^2$)*

When $d = 2$, for any $x \in \mathbb{R}^2$,

$$W_x\text{-a.s., for any non-empty open set } O \subseteq \mathbb{R}^2,\ \{t \geq 0; X_t \in O\} \text{ is unbounded.} \tag{6.68}$$

Proof. By (6.63), we see, letting $b \to \infty$, that when $a < |x|$, W_x-a.s., $H_{\overline{B}(0,a)} < \infty$. Of course this remains true when $|x| \le a$, so that

$$\text{for any } x \in \mathbb{R}^2,\ a > 0,\ W_x\text{-a.s., } H_{\overline{B}(0,a)} < \infty\,. \tag{6.69}$$

One can then define the sequence of $(\mathcal{F}_t^+)$-stopping times, cf. (2.33),

$$\begin{aligned} &S_1 = H_{\overline{B}(0,a)},\ S_2 = S_1 \circ \theta_{S_1+1} + S_1 + 1, \text{ and by induction for } i \ge 1: \\ &S_{i+1} = S_1 \circ \theta_{S_i+1} + S_i + 1, \text{ so that } S_i \uparrow \infty\,. \end{aligned}$$

Using the strong Markov property, cf. (2.46), we see that for any $y \in \mathbb{R}^2$, for $i \ge 1$,

$$\begin{aligned} W_y[S_{i+1} < \infty] &= W_y[S_i < \infty \text{ and } \theta^{-1}_{S_i+1}(S_1 < \infty)] \\ &\overset{(2.46)}{=} E_y\big[S_i < \infty,\ \underbrace{W_{X_{S_i+1}}[S_1 < \infty]}_{\substack{\parallel \\ 1 \text{ by } (6.69)}}\big] = W_y[S_i < \infty] \\ &\overset{\text{induction}}{=} W_y[S_1 < \infty] = 1\,. \end{aligned} \tag{6.70}$$

Note also that by construction, for $i \ge 1$,

$$W_y\text{-a.s., on } \{S_i < \infty\},\ X_{S_i} \in \overline{B}(0,a)\,.$$

It thus follows from (6.70) and $S_i \uparrow \infty$, that for any $y \in \mathbb{R}^2$, W_y-a.s., for any $a = \frac{1}{n}$, $\{t \ge 0;\ X_t \in \overline{B}(0,\frac{1}{n})\}$ is unbounded.

Since W_y is the law of $(X_t + y)_{t \ge 0}$, under W_0, the above property implies that (setting $z = -y$):

$$W_0\text{-a.s., for all } z \in \mathbb{Q}^2,\ n \ge 1,\ \Big\{t \ge 0;\ X_t \in \overline{B}\Big(z, \frac{1}{n}\Big)\Big\} \text{ is unbounded}\,. \tag{6.71}$$

This proves (6.68) when $x = 0$. The case of a general x follows since $(X_t)_{t\ge 0}$ under W_x has the law of $(X_t + x)_{t \ge 0}$ under W_0, as was already used in the proof. $\square$

Exercise 6.17. Give a proof of (6.68) using (6.69) and (6.50) (without the introduction of the stopping times $S_i, i \ge 1$).

Complement

We will now present **Nivokov's criterion**, which refines the condition we gave in (6.29) to ensure that Z_t is a martingale (and not merely a continuous local martingale).

Theorem 6.18. *(Novikov's criterion)*

Let $(M_t)_{t\ge 0}$, be a continuous local martingale with $M_0 = 0$, such that

$$E\Big[\exp\Big\{\tfrac{1}{2}\,\langle M\rangle_\infty\Big\}\Big] < \infty\,. \tag{6.72}$$

Then,

$$E\Big[\exp\Big\{\frac{1}{2}\,\sup_{t\ge 0}\,|M_t|\Big\}\Big] < \infty\,, \tag{6.73}$$

and

(6.74) $Z_t = \exp\Big\{M_t - \frac{1}{2}\,\langle M\rangle_t\Big\}$, $t \ge 0$, *is a uniformly integrable continuous martingale*

(and of course, $(M_t)_{t\ge 0}$, is a continuous martingale as well).

Remark 6.19. If instead of (6.72) we assume that for some $T > 0$,

$$E\Big[\exp\Big\{\frac{1}{2}\,\langle M\rangle_T\Big\}\Big] < \infty\,, \tag{6.75}$$

the above theorem can be applied to $M_{t\wedge T}$, $t \ge 0$, and we find that

$$Z_t = \exp\Big\{M_t - \frac{1}{2}\,\langle M\rangle_t\Big\},\ t \le T, \text{ is a continuous martingale.} \tag{6.76}$$

□

Proof. We first observe that $E[\langle M\rangle_\infty] < \infty$ implies that

$$(M_t)_{t\ge 0}, \text{ is a continuous martingale bounded in } L^2\,. \tag{6.77}$$

Indeed (this is just as in the exercise below (6.30')), one chooses a sequence $T_n \uparrow \infty$ of stopping times so that $(M_{t\wedge T_n})_{t\ge 0}$, are bounded martingales. Then, one has

$$E[M^2_{t\wedge T_n}] \overset{(5.36)}{=} E[\langle M\rangle_{t\wedge T_n}] \le E[\langle M\rangle_\infty], \text{ and by Fatou's lemma}$$
$$E[M_t^2] \le \liminf_n\ E[M^2_{t\wedge T_n}]\ \le E[\langle M\rangle_\infty]\,.$$

It now also follows with similar considerations as in (4.104) and Doob's inequality (4.76) that $(M_t)_{t\ge 0}$, is a continuous martingale, with $E[\sup_{t\ge 0}|M_t|^2] < \infty$ and that $M_\infty \overset{P\text{-a.s.}}{=} \lim_{t\to\infty} M_t$ is well-defined, by the martingale convergence theorem, cf. [8], p. 17.

• (6.73): Note that for $0 \le t \le \infty$,

$$\begin{aligned} E\Big[\exp\Big\{\frac{1}{2}\,M_t\Big\}\Big] &= E\Big[\exp\Big\{\frac{1}{2}\,M_t - \frac{1}{4}\,\langle M\rangle_t\Big\}\ \exp\Big\{\frac{1}{4}\,\langle M\rangle_t\Big\}\Big] \\ &\overset{\text{Cauchy-Schwarz}}{\le} E\Big[\exp\Big\{M_t - \frac{1}{2}\,\langle M\rangle_t\Big\}\Big]^{\frac{1}{2}}\ \exp\Big\{\frac{1}{2}\,\langle M\rangle_\infty\Big\}\Big]^{\frac{1}{2}}\,. \end{aligned} \tag{6.78}$$

Since $(Z_t)_{t\ge 0}$, is a non-negative local martingale, it is also a supermartingale (see also exercise below (6.67)). Indeed, for $0 \le s \le t$,

$$\begin{aligned} E[Z_t\,|\,\mathcal{G}_s] = E[\lim_n\ Z_{t\wedge T_n}\,|\,\mathcal{G}_s] &\overset{\text{Fatou}}{\le} \liminf_n\ E[Z_{t\wedge T_n}\,|\,\mathcal{G}_s] \\ &= \liminf_n\ Z_{s\wedge T_n} = Z_s, \text{ thus proving the claim}\,. \end{aligned}$$

As a result, $E[Z_t] \le E[Z_0] = 1$, and coming back to (6.78),

$$E\Big[\exp\Big\{\tfrac{1}{2}\, M_t\Big\}\Big] \le E\Big[\exp\Big\{\tfrac{1}{2}\, \langle M\rangle_\infty\Big\}\Big]^{\frac{1}{2}}, \;\; 0 \le t \le \infty\,. \tag{6.79}$$

The same argument applied to $-M$ yields that

$$\begin{aligned} &\sup_{t\ge 0}\, E\Big[\cosh\Big(\tfrac{1}{2}\, M_t\Big)\Big] \le E\Big[\exp\Big\{\tfrac{1}{2}\, \langle M\rangle_\infty\Big\}\Big]^{\frac{1}{2}}\,, \text{ so that} \\ &\qquad E[\cosh(cM_t)] \le E\Big[\exp\Big\{\tfrac{1}{2}\, \langle M\rangle_\infty\Big\}\Big]^{\frac{1}{2}}\,, \text{ for } 0 \le c \le \frac{1}{2},\, 0 \le t \le \infty\,. \end{aligned} \tag{6.80}$$

In particular, since $\cosh x \ge \frac{1}{2}\, e^x$,

$$\sup_{0\le t\le\infty}\; E[e^{cM_t}] < \infty, \text{ for } 0 \le c \le \frac{1}{2}\,. \tag{6.81}$$

Jensen's inequality implies that e^{cM_t}, $t \ge 0$, is a non-negative sub-martingale. It then follows from Doob's inequality (4.76) with $p = \frac{1}{2c}$, and $0 < c < \frac{1}{2}$, that

$$\begin{aligned} E\Big[\sup_{t\ge 0}\, \exp\Big\{\tfrac{1}{2}\, M_t\Big\}\Big] &= E\big[\big(\sup_{t\ge 0}\, \exp\{cM_t\}\big)^p\big] \\ &\overset{(4.76)}{\le} \Big(\frac{p}{p-1}\Big)^p \sup_{t\ge 0}\, E[\exp\{pcM_t\}] \\ &= \Big(\frac{p}{p-1}\Big)^p \sup_{t\ge 0}\, E\Big[\exp\Big\{\tfrac{1}{2}\, M_t\Big\}\Big] \overset{(6.81)}{<} \infty\,. \end{aligned} \tag{6.82}$$

Of course, a similar bound holds for $-M$ in place of M. Note also that $\sup_{t\ge 0}(\cosh(\frac{1}{2}\, M_t)) \le \frac{1}{2}\, \sup_{t\ge 0}\, e^{\frac{1}{2}\, M_t} + \frac{1}{2}\, \sup_{t\ge 0}\, e^{-\frac{1}{2}\, M_t}$, and hence

$$E\Big[\sup_{t\ge 0}\, \cosh\Big(\tfrac{1}{2}\, M_t\Big)\Big] < \infty\,. \tag{6.83}$$

This implies that $E[\exp\{\frac{1}{2}\, \sup_{t\ge 0} |M_t|\}] < \infty$, and (6.73) holds.

• (6.74):

We will use the next

Lemma 6.20.

(6.84) *If $E[Z_\infty] = 1$, then Z_t, $0 \le t \le \infty$, is a uniformly integrable martingale.*

Proof. By the supermartingale property of $(Z_t)_{t\ge 0}$, it follows that $1 = E[Z_\infty] \le E[Z_t] \le E[Z_0] = 1$, for $0 \le t \le \infty$, so that $E[Z_t] = 1$, for $0 \le t \le \infty$.

Note that P-a.s., $Z_{t\wedge T_n} \overset{n\to\infty}{\longrightarrow} Z_t$, and $E[Z_{t\wedge T_n}] = E[Z_t] = 1$ and these variables are non-negative.

It now follows that they are uniformly integrable and

$$Z_{t\wedge T_n} \xrightarrow[n\to\infty]{L^1} Z_t, \;\; \text{for } 0 \le t \le \infty\,, \tag{6.85}$$

see for instance [5], p. 224. This now implies that Z_t, $t \ge 0$, is a martingale. Moreover, since P-a.s., $Z_t \to Z_\infty$, as $t \to \infty$, and $E[Z_t] = E[Z_\infty] = 1$, the same argument shows that

$$Z_t \xrightarrow{L^1} Z_\infty, \;\; \text{as } t \to \infty\,, \tag{6.86}$$

so that

$$Z_t = E[Z_\infty \,|\, \mathcal{G}_t], \;\; \text{for } t \ge 0\,, \tag{6.87}$$

and the conclusion of the lemma follows, cf. [5], p. 223. □

We will now show that

$$E[Z_\infty] \ge 1\,. \tag{6.88}$$

Since we already know that $E[Z_\infty] \le 1$, the claim (6.74) will now follow from the lemma.

By (6.29), (6.30), with $T = \infty$, we know from (6.72) that

$$E\Big[\exp\Big\{a\,M_\infty - \frac{a^2}{2}\,\langle M\rangle_\infty\Big\}\Big] = 1, \;\text{for } a \in [0,1)\,.$$

Note that the following equality holds:

$$\exp\Big\{a\,M_\infty - \frac{a^2}{2}\,\langle M\rangle_\infty\Big\} = \exp\Big\{M_\infty - \tfrac{1}{2}\,\langle M\rangle_\infty\Big\}^{a^2}\;\exp\Big\{\frac{a}{1+a}\,M_\infty\Big\}^{1-a^2}\,.$$

Using Hölder's inequality with $p = a^{-2}$ and $q = (1-a^2)^{-1}$, we find:

$$1 \le E[Z_\infty]^{a^2}\;E\Big[\exp\Big\{\frac{a}{1+a}\,M_\infty\Big\}\Big]^{1-a^2}\,. \tag{6.89}$$

Using (6.73), we can use dominated convergence to argue that

$$\lim_{a\to 1} E\Big[\exp\Big\{\frac{a}{1+a}\,M_\infty\Big\}\Big] = E\Big[\exp\Big\{\tfrac{1}{2}\,M_\infty\Big\}\Big] \in (0,\infty)\,.$$

As a result we obtain that

$$\lim_{a\to 1} E\Big[\exp\Big\{\frac{a}{1+a}\,M_\infty\Big\}\Big]^{1-a^2} = 1\,, \tag{6.90}$$

and the claim (6.88) now follows from (6.89) and (6.90). This concludes the proof of (6.74). □

7 Stochastic differential equations and Martingale problems

We begin with some heuristic considerations.

In this chapter we want to construct processes, which locally, near each point x in $\mathbb{R}^d$, move like

$$x + tb(x) + \sigma(x)\, B_t\,,$$

where $b(x) \in \mathbb{R}^d$, $\sigma(x)$ is a $(d \times n)$-matrix and B_t is an n-dimensional Brownian motion. We want that "infinitesimally", the increment of the process we construct behaves as a Gaussian variable with

mean:	$b(x)dt$, and
covariance matrix:	$a(x)dt$,

where for $\xi \in \mathbb{R}^d$

$$\begin{aligned} {}^t\xi a(x)\,\xi dt \ \text{“}=\text{”}\ & E[({}^t\xi \cdot \sigma(x) \cdot (B_{t+dt} - B_t))^2] \\ =\ & E[({}^t({}^t\sigma(x)\xi) \cdot (B_{t+dt} - B_t))^2] = |{}^t\sigma(x)\xi|^2 dt \\ =\ & {}^t\xi\, \sigma(x)\, {}^t\sigma(x)\, \xi\, dt\,, \end{aligned}$$

in other words $a(x) = \sigma(x)\, {}^t\sigma(x)$ $\quad (d \times d\text{-matrix})$.

We will consider **two approaches** to build such processes. The first approach will rely on solving **stochastic differential equations** (SDE):

$$X_t^i = X_0^i + \int_0^t b_i(X_s)\, ds + \sum_{j=1}^n \int_0^t \sigma_{i,j}(X_s) \cdot dB_s^j, \ \ i = 1, \dots, d\,,$$

or in vector notation:

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) \cdot dB_s, \ x \in \mathbb{R}^d\,. \tag{7.1}$$

The second approach will be based on a **martingale problem**, i.e. finding on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ a probability P_x, such that

$$\begin{aligned} & M_t^f \overset{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds, \ t \geq 0, \text{ is an } (\mathcal{F}_t)\text{-martingale under } P_x, \\ & \text{when } f \in C_c^2(\mathbb{R}^d) \text{ with } Lf(y) \overset{\text{def}}{=} \tfrac{1}{2} \sum_{i,j=1}^d a_{i,j}(y) \partial_{i,j}^2 f(y) + \sum_{i=1}^d b_i(y) \partial_i f(y), y \in \mathbb{R}^d, \\ & P_x[X_0 = x] = 1\,. \end{aligned} \tag{7.2}$$

This latter approach shares the same spirit as Lévy's characterization of Brownian motion, cf. (6.44), (6.45).

Notation:

- For $b \in \mathbb{R}^d$, $|b| = \Big(\sum\limits_{i=1}^d b_i^2\Big)^{\frac{1}{2}}$.

- For $\sigma \in M_{d\times n}$, $|\sigma| = \Big(\sum\limits_{\substack{1\le i\le d\\ 1\le j\le n}} \sigma_{i,j}^2\Big)^{1/2} = \Big\{\text{Trace}\,\underbrace{(\sigma\,{}^t\sigma)}_{\in M_{d\times d}}\Big\}^{\frac{1}{2}} = \Big\{\text{Trace}\,\underbrace{({}^t\,\sigma\sigma)}_{\in M_{n\times n}}\Big\}^{\frac{1}{2}}$.

- $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ a probability space satisfying the usual conditions, cf. (4.5), (4.6).

- B_t, $t \ge 0$, an n-dimensional $(\mathcal{G}_t)$-Brownian motion, or equivalently in view of (6.44), (6.45) for all $1 \le i, j \le n$, B_t^i, $t \ge 0$, are continuous local martingales with $B_0^i = 0$, and $B_t^i\, B_t^j - \delta_{ij}\, t$, $t \ge 0$, are continuous local martingales.

We now begin with the discussion of **stochastic differential equations**. The next theorem provides a basic result.

Theorem 7.1. *(Picard's iteration method)*

Assume that $b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, and $\sigma(\cdot)$: $\mathbb{R}^d \to M_{d\times n}$ satisfy the Lipschitz condition

$$|b(y) - b(z)| + |\sigma(y) - \sigma(z)| \le K\,|y - z|, \text{ for } y, z \in \mathbb{R}^d\,. \tag{7.3}$$

Then, for any $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ and B_t, $t \ge 0$, as above, and any x in $\mathbb{R}^d$, there exists an essentially unique continuous $(\mathcal{G}_t)$-adapted $(X_t)_{t\ge 0}$ with values in $\mathbb{R}^d$, such that P-a.s., for $t \ge 0$,

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s)\cdot dB_s\,. \tag{7.4}$$

Proof.

- Uniqueness:

Consider $X_., Y_.$ two solutions. For $M > |x|$, we define

$$T = \inf\{u \ge 0;\ |X_u| \text{ or } |Y_u| \ge M\}\,, \tag{7.5}$$

so that P-a.s., for all $t \ge 0$,

$$X_{t\wedge T} - Y_{t\wedge T} = \int_0^{t\wedge T} (b(X_u) - b(Y_u))du + \int_0^{t\wedge T} (\sigma(X_u) - \sigma(Y_u))\cdot dB_u\,.$$

As a result, we see that for $t_0 > 0$:

$$E\big[\sup_{t\le t_0} |X_{t\wedge T} - Y_{t\wedge T}|^2\big] \le 2E\Big[\sup_{t\le t_0}\Big|\int_0^{t\wedge T} (\sigma(X_u) - \sigma(Y_u))\cdot dB_u\Big|^2\Big]$$

$$+\, 2t_0\, E\Big[\int_0^{t_0\wedge T} |b(X_u) - b(Y_u)|^2\, du\Big]\,.$$

Using Doob's inequality (4.76), with $p = 2$, for each component of the $\mathbb{R}^d$-valued stochastic integral we find that

$$
\begin{aligned}
E\Big[\sup_{t \le t_0} |X_{t\wedge T} - Y_{t\wedge T}|^2\Big] \le 8E\Big[\Big|\int_0^{t_0\wedge T} (\sigma(X_u) - \sigma(Y_u))\cdot dB_u\Big|^2\Big] \\
+ 2t_0\, E\Big[\int_0^{t_0\wedge T} |b(X_u) - b(Y_u)|^2 du\Big]\,.
\end{aligned}
\tag{7.6}
$$

On the other hand one has

$$
\begin{aligned}
& E\Big[\Big|\int_0^{t_0\wedge T} (\sigma(X_u) - \sigma(Y_u))\cdot dB_u\Big|^2\Big] = \\
& \sum_{i=1}^d E\Big[\Big(\sum_{j=1}^n \int_0^{t_0\wedge T} (\sigma_{i,j}(X_u) - \sigma_{i,j}(Y_u))\, dB_u^j\Big)^2\Big] = \\
& \sum_{i=1}^d E\Big[\sum_{1\le j,k\le n} \int_0^{t_0\wedge T} (\sigma_{i,j}(X_u) - \sigma_{i,j}(Y_u))\, dB_u^j \int_0^{t_0\wedge T} (\sigma_{i,k}(X_u) - \sigma_{i,k}(Y_u))\, dB_u^k\Big] \\
& \overset{(5.90)}{=} \sum_{i=1}^d \sum_{1\le j,k\le n} E\Big[\int_0^{t_0\wedge T} (\sigma_{i,j}(X_u) - \sigma_{i,j}(Y_u))(\sigma_{i,k}(X_u) - \sigma_{i,k}(Y_u))\, d\underbrace{\langle B^j, B^k\rangle_u}_{\underset{\delta_{j,k} du}{\|}}\Big] = \\
& \sum_{i=1}^d \sum_{j=1}^n E\Big[\int_0^{t_0\wedge T} (\sigma_{i,j}(X_u) - \sigma_{i,j}(Y_u))^2 du\Big] = E\Big[\int_0^{t_0\wedge T} |\sigma(X_u) - \sigma(Y_u)|^2 du\Big]\,.
\end{aligned}
\tag{7.7}
$$

Inserting (7.7) in the right-hand side of (7.6), and taking (7.3) into account we find that for any $t_0 \ge 0$:

$$
E\Big[\sup_{s\le t_0} |X_{s\wedge T} - Y_{s\wedge T}|^2\Big] \le (8K^2 + 2t_0K^2) \int_0^{t_0} E[|X_{u\wedge T} - Y_{u\wedge T}|^2]\, du\,.
\tag{7.8}
$$

The next result will be helpful.

Lemma 7.2. *(Gronwall's lemma)*

Let f be a non-negative integrable function on $[0, t]$ such that for some $a, b \ge 0$, and all $0 \le u \le t$:

$$
f(u) \le a + b\int_0^u f(s)\, ds\,,
$$

then

$$
f(u) \le a\, e^{bu}, \quad \text{for all} \quad 0 \le u \le t\,.
\tag{7.9}
$$

Proof. Iterating the inequality satisfied by f, we see that for $0 \le u \le t$,

$$\begin{aligned} f(u) &\le a + b \int_0^u f(s)\, ds \le a + bau + b^2 \int_0^u ds_1 \int_0^{s_1} f(s_2)\, ds_2 \le \dots \\ &\le a + bau + \frac{1}{2}\, b^2 a\, u^2 + \dots + \frac{1}{n!}\, b^n\, a\, u^n + \\ &\quad b^{n+1} \int_0^u ds_1 \int_0^{s_1} ds_2 \dots \int_0^{s_n} f(s_{n+1})\, ds_{n+1} \\ &\le a\, e^{bu} + b^{n+1} \int_0^u \frac{(u-s)^n}{n!}\, f(s)\, ds \le a\, e^{bu} + b^{n+1}\, \frac{u^n}{n!} \int_0^u f(s)\, ds\,. \end{aligned}$$

Letting $n \to \infty$, we find (7.9). $\square$

We will now apply the above lemma with the choice $f(u) = E[\sup_{s \le u} |X_{s\wedge T} - Y_{s\wedge T}|^2]$ ($\ge E[|X_{u\wedge T} - Y_{u\wedge T}|^2]$), $0 \le u \le t$, $t > 0$, $a = 0$, $b = K^2(8 + 2t)$, and find that (with t some positive number)

$$f(u) = 0, \ \text{ for } 0 \le u \le t\,. \tag{7.10}$$

Letting M in (7.5) tend to infinity, and then $t \to \infty$,

$$P\text{-a.s.}, \ X_u = Y_u, \text{ for all } u \ge 0. \tag{7.11}$$

Such a statement is called a **strong uniqueness** result (sometimes it is also called **path-wise uniqueness**).

• **Existence:**

We iteratively define for $m \ge 0$, $t \ge 0$,

$$\begin{cases} X_t^0 \equiv x\,, \\ X_t^1 = x + \int_0^t b(X_s^0)\, ds + \int_0^t \sigma(X_s^0) \cdot dB_s \\ X_t^{m+1} = x + \int_0^t b(X_s^m)\, ds + \int_0^t \sigma(X_s^m) \cdot dB_s \end{cases} . \tag{7.12}$$

Then, for $m \ge 1$:

$$X_t^{m+1} - X_t^m = \int_0^t \big(b(X_s^m) - b(X_s^{m-1})\big)\, ds + \int_0^t (\sigma(X_s^m) - \sigma(X_s^{m-1})) \cdot dB_s, \ \text{for } t \ge 0\,. \tag{7.13}$$

If we now pick $M > |x|$, and define

$$T_M = \inf\{u \ge 0;\ |X_u^m| \text{ or } |X_u^{m+1}| \ge M\}\,,$$

the same calculation as for (7.8) yields that for $0 \le t_0 \le t$,

$$E\big[\sup_{s \le t_0 \wedge T_M} |X_s^{m+1} - X_s^m|^2\big] \le (8 + 2t)\, K^2 \int_0^{t_0} E[|X_{u\wedge T_M}^m - X_{u\wedge T_M}^{m-1}|^2]\, du\,. \tag{7.14}$$

Now $\sup_{s\le t} |X_s^0| = |x|$, $\sup_{s\le t} |X_s^1| \in L^2(P)$, by (7.12).

We now see from (7.14), with $m = 1$, letting $M \to \infty$, that $\sup_{s\le t} |X_s^2| \in L^2(P)$, and repeating the argument that

$$\sup_{s\le t} |X_s^m| \in L^2(P), \text{ for any } m \ge 0, \text{ and } t \ge 0\,. \tag{7.15}$$

Coming back to (7.14) we can thus let $M \to \infty$ and find that for $0 \le t_0 \le t$:

$$\begin{aligned} &E\big[\sup_{s\le t_0} |X_s^{m+1} - X_s^m|^2\big] \le K^2(8+2t) \int_0^{t_0} E[|X_u^m - X_u^{m-1}|^2]\,du, \;\text{ and iterating} \\ &\le \{K^2(8+2t)\}^m \int_0^{t_0} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{m-1}} dt_m\, E[|X_{t_m}^1 - X_{t_m}^0|^2]\,. \end{aligned} \tag{7.16}$$

With (7.12) we also have:

$$\begin{aligned} E[|X_{t_m}^1 - X_{t_m}^0|^2] &\le 2\,|b(x)|^2\, t_m^2 + 2E[|\sigma(x)\cdot B_{t_m}|^2] \\ &\le K^1(x,t)\, t_m, \;\text{ for } 0 \le t_m \le t\,. \end{aligned} \tag{7.17}$$

Hence with (7.16) we obtain that

$$E\big[\sup_{s\le t_0} |X_s^{m+1} - X_s^m|^2\big] \le K^1(x,t) \frac{(8K^2 + 2t\,K^2)^m\, t^{m+1}}{(m+1)!} \;\text{ for } t > 0,\; m \ge 0\,. \tag{7.18}$$

We have thus proved that for $t > 0$,

$$E\Big[\sum_{m\ge 0} \sup_{s\le t} |X_s^{m+1} - X_s^m|\Big] \le \sum_{m\ge 0} E\big[\sup_{s\le t} |X_s^{m+1} - X_s^m|^2\big]^{\frac12} < \infty\,. \tag{7.19}$$

As a consequence of the finiteness of the expectation on the left-hand side, P-a.s., $X_\cdot^m$ converges uniformly on bounded time intervals to $X_\cdot^\infty$, which can be chosen $(\mathcal{G}_t)$-adapted continuous (see for instance (4.75)), and

$$\begin{aligned} &E\big[\sup_{s\le t} |X_s^\infty - X_s^m|^2\big]^{1/2} \overset{\text{Fatou}}{\le} \liminf_{p\to\infty} E\big[\sup_{s\le t} |X_s^p - X_s^m|^2\big]^{\frac12} \le \\ &\sum_{k=m}^\infty E[\sup_{s\le t} |X_s^{k+1} - X_s^k|^2]^{\frac12} \underset{m\to\infty}{\longrightarrow} 0, \;\text{ for } t > 0\,. \end{aligned} \tag{7.20}$$

Hence, by (7.20) and (7.3), we see that for $t > 0$, P-a.s.,

$$\begin{array}{ccccc} X_t^{m+1} = x & + & \displaystyle\int_0^t b(X_u^m)\,du & + & \displaystyle\int_0^t \sigma(X_u^m)\cdot dB_u \\ L^2 \Big\downarrow m\to\infty & & m\to\infty \Big\downarrow L^2 & & m\to\infty \Big\downarrow L^2 \\ X_t^\infty = x & + & \displaystyle\int_0^t b(X_u^\infty)\,du & + & \displaystyle\int_0^t \sigma(X_u^\infty)\cdot dB_u\,, \end{array}$$

and in view of the continuity of $X^\infty_.$, we see that P-a.s.,

$$X^\infty_t = x + \int_0^t b(X^\infty_u)\, du + \int_0^t \sigma(X^\infty_u) \cdot dB_u, \ \text{ for all } t \ge 0\,. \tag{7.21}$$

Therefore, $X^\infty_.$ is a solution of (7.4). □

Remark 7.3. From the definition of the $X^m_.$ in (7.12), and the fact that P-a.s., $X^m_.$ converges uniformly to $X^\infty_.$ on compact time intervals, we see that for each $t \ge 0$:

$$\begin{aligned} F_t^{X^\infty_.} &\overset{\text{def}}{=} \text{the smallest } \sigma\text{-algebra containing all negligible sets of} \\ &\qquad \mathcal{G} \text{ and making } X^\infty_s \text{ measurable for } s \le t \\ &\subseteq F_t^{B_.}, \ \text{ defined analogously (with } B_. \text{ in place of } X^\infty_.)\,. \end{aligned} \tag{7.22}$$

Due to (7.22), $X^\infty_.$ is called a **strong solution** of (7.4) (intuitively $X^\infty_.$ is a function of the "noise" $B_.$). The above theorem shows that for any $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$, $(B_t)_{t\ge 0}$, we have a strong solution of (7.4), which is strongly unique, cf. (7.11). □

We will now see that **solutions of** stochastic differential equations **(SDE's)** can be used to **represent solutions of** certain partial differential equations **(PDE's)**.

We begin with a result which will also be helpful in the subsequent discussion of martingale problems.

Proposition 7.4. *Assume that $b(\cdot)$: $\mathbb{R}^d \to \mathbb{R}^d$, $\sigma(\cdot)$: $\mathbb{R}^d \to M_{d\times n}$ are measurable, locally bounded functions, $x \in \mathbb{R}^d$, and on some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$, endowed with an n-dimensional $(\mathcal{G}_t)$-Brownian motion $(B_t)_{t\ge 0}$, a continuous adapted $\mathbb{R}^d$-valued, $(X_t)_{t\ge 0}$, satisfies P-a.s., for $t \ge 0$:*

$$X_t = x + \int_0^t b(X_s)\, ds + \int_0^t \sigma(X_s) \cdot dB_s\,, \tag{7.23}$$

then for any $f \in C^2(\mathbb{R}^d, \mathbb{R})$,

$$M_t^f \overset{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Lf(X_s)\, ds, \ t \ge 0, \ \ \textit{is a continuous local martingale} \tag{7.24}$$

where we used the notation

$$\begin{aligned} Lf(y) &= \tfrac{1}{2} \sum_{1\le i,j\le d} a_{i,j}(y)\, \partial^2_{i,j} f(y) + \sum_{1\le i\le d} b_i(y)\, \partial_i f(y), \ \textit{and} \\ a(y) &= \underbrace{\sigma(y)\, {}^t\sigma(y)}_{\in M_{d\times d}}, \ \ \textit{for } y \in \mathbb{R}^d\,. \end{aligned} \tag{7.25}$$

Proof. We apply Ito's formula and find that P-a.s., for $t \ge 0$:

$$f(X_t) = f(X_0) + \sum_{i=1}^d \int_0^t \partial_i f(X_s)\, dX_s^i + \tfrac{1}{2} \sum_{i,j=1}^d \int_0^t \partial^2_{i,j} f(X_s)\, d\langle X^i, X^j\rangle_s\,. \tag{7.26}$$

Note that by (5.90) and (7.23), it follows that

$$
\begin{aligned}
\langle X^i, X^j\rangle_t &= \Big\langle \sum_{k=1}^n \int_0^t \sigma_{i,k}(X_u)\, dB_u^k,\ \sum_{\ell=1}^n \int_0^t \sigma_{j,\ell}(X_u) \cdot dB_u^\ell \Big\rangle \\
&= \sum_{k=1}^n \int_0^t \sigma_{i,k}(X_u)\, \sigma_{j,k}(X_u)\, du = \int_0^t a_{i,j}(X_u)\, du\,.
\end{aligned}
\tag{7.27}
$$

Hence, coming back to (7.26), we find that P-a.s., for all $t \ge 0$,

$$
\begin{aligned}
f(X_t) &= f(X_0) + \sum_{i=1}^d \int_0^t \partial_i\, f(X_s)\, b_i(X_s)\, ds + \sum_{i=1}^d \sum_{k=1}^n \int_0^t \partial_i\, f(X_s)\, \sigma_{i,k}(X_s)\, dB_s^k \\
&\quad + \frac{1}{2} \sum_{1\le i,j\le d} \int_0^t a_{i,j}(X_s)\, \partial^2_{i,j}\, f(X_s)\, ds \\
&= f(X_0) + \int_0^t Lf(X_s)\, ds + \int_0^t \nabla f(X_s) \underset{\substack{\uparrow \\ \text{scalar product}}}{\cdot} \underbrace{\sigma(X_s) \cdot dB_s}_{\substack{\uparrow \\ d\text{-vector}}}
\end{aligned}
\tag{7.28}
$$

The claim (7.24) now follows. $\square$

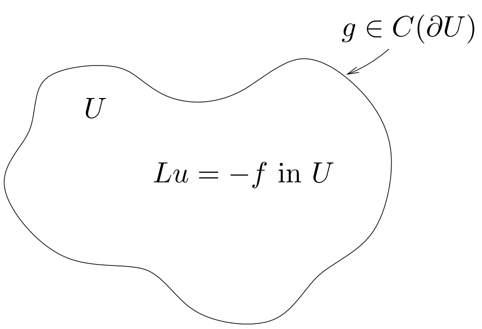
We will now see that the solutions of stochastic differential equations can be used to provide probabilistic representation formulas for the solutions of certain second order partial differential equations.

We consider the following **Dirichlet-Poisson problem:**

$U \neq \emptyset$ is a bounded open subset of $\mathbb{R}^d$, $f \in C_b(U)$, $g \in C(\partial U)$,

and we look for $u \in C^2(U) \cap C(\overline{U})$ such that, see (7.25) for the notation:

$$
\begin{cases} Lu(x) = -f(x), & \text{for } x \in U, \\ u(x) = g(x), & \text{for } x \in \partial U\,. \end{cases}
\tag{7.29}
$$



The Dirichlet problem corresponds to $f = 0$, and the Poisson equation to $g = 0$ in (7.29).

In addition to the local boundedness and measurability of $b(\cdot)$, $\sigma(\cdot)$, we assume the following **elliptic condition**:

$$\text{there is } c > 0, \text{ so that } {}^t\xi\, a(x)\, \xi \ge c|\xi|^2, \ \text{ for } \xi \in \mathbb{R}^d,\ x \in \overline{U}\,. \tag{7.30}$$

It is known that when $\sigma(\cdot)$, $b(\cdot)$ in addition satisfy (7.3) (in fact a Hölder condition is good enough), when f is bounded Hölder continuous in U, and U satisfies an exterior sphere condition:

$$\forall z \in \partial U, \text{ there is an open ball } B, \text{ with } \overline{B} \cap \overline{U} = \{z\}\,,$$

the problem (7.29) has a solution, cf. [6], p. 106.

Theorem 7.5. *($b(\cdot), \sigma(\cdot)$, measurable, locally bounded, and (7.30))*

If u is a solution of (7.29), and $(X_t)_{t\ge 0}$ satisfies (7.23), for some $x \in U$, then the exit time of $X_.$ from U

$$T_U = \inf\{s \ge 0;\ X_s \notin U\} \text{ is } P\text{-integrable,} \tag{7.31}$$

and

$$u(x) = E\Big[g(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds\Big]\,. \tag{7.32}$$

Proof.

• (7.31):

Pick $\varphi(y) = C(e^{\alpha R} - e^{\alpha y_1})$, where $y = (y_1, \dots, y_d) \in \overline{U}$, then

$$\begin{aligned} L\varphi(y) &= -C\, e^{\alpha y_1}\big(\tfrac{\alpha^2}{2}\, a_{1,1}(y) + \alpha\, b_1(y)\big) \\ &\overset{(7.30)}{\le} -C\, e^{\alpha y_1}\big(\tfrac{\alpha^2}{2}\, c - \alpha M\big), \ \text{ with } M = \sup_{\overline{U}} |b_1(\cdot)|\,. \end{aligned}$$

Choosing α, R large and then C large enough, we can make sure that

$$L\varphi \le -1, \ \text{ on } \overline{U}, \tag{7.33}$$

$$\varphi > 0, \ \text{ on } \overline{U}\,. \tag{7.34}$$

By (7.24) we find that under P,

$$\varphi(X_{t\wedge T_U}) - \varphi(x) - \int_0^{t\wedge T_U} L\varphi(X_s)\, ds, \ \ t \ge 0\,,$$

is a local martingale, which is bounded. Hence it is a martingale. Taking expectation, we find that

$$E[\varphi(X_{t\wedge T_U})] - \varphi(x) - E\Big[\int_0^{t\wedge T_U} L\varphi(X_u)\, du\Big] = 0\,,$$

and keeping in mind (7.33), (7.34), we thus find that

$$\sup_{\overline{U}} \varphi \geq E[\varphi(X_{t\wedge T_u})] - E\Big[\int_0^{t\wedge T_U} L\varphi(X_u)\, du\Big] \geq E[t \wedge T_U]\,. \tag{7.35}$$

Letting $t \to \infty$, we obtain (7.31) and in fact the more precise estimate

$$E[T_U] \leq \sup_{\overline{U}}\ \varphi\,. \tag{7.36}$$

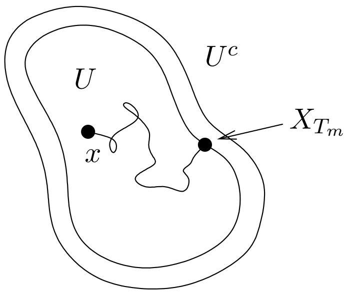
• (7.32):

Recall that $x \in U$ by assumption. For $m \geq 1$ large enough so that $\frac{1}{m} < d(x, U^c)$, we define

$$T_m = \inf\Big\{s \geq 0;\ d(X_s, U^c) \leq \frac{1}{m}\Big\}$$

and construct $u_m \in C_c^2(\mathbb{R}^d, \mathbb{R})$ such that

$$u = u_m \ \text{ on } \ \Big\{z \in U;\ d(z, U^c) \geq \frac{1}{m}\Big\}\,. \tag{7.37}$$



By (7.24), we see that under P,

$$u_m(X_{t\wedge T_m}) - u_m(x) - \int_0^{t\wedge T_m} Lu_m(X_s)\, ds = u(X_{t\wedge T_m}) - u(x) + \int_0^{t\wedge T_m} f(X_s)\, ds \tag{7.38}$$

is a bounded continuous local martingale, and hence a martingale. Taking expectation we conclude that

$$E[u(X_{t\wedge T_m})] - u(x) + E\Big[\int_0^{t\wedge T_m} f(X_s)\, ds\Big] = 0\,.$$

Since P-a.s., $T_m \uparrow T_U < \infty$, and T_U is integrable, we can let $t \to \infty$, and then $m \to \infty$, and conclude that

$$\begin{aligned} u(x) =&\ E[u(X_{T_U})] + E\Big[\int_0^{T_U} f(X_s)\, ds\Big] \\ \overset{(7.29)}{=}&\ E\Big[g(X_{T_U}) + \int_0^{T_U} f(X_s)\, ds\Big]\,, \end{aligned} \tag{7.39}$$

whence our claim (7.32). □

We will now discuss some features of the **martingale problem** (7.2), and its link with SDE's.

Assumptions and notation:

$b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma(\cdot)$: $\mathbb{R}^d \to M_{d\times n}$ are measurable, locally bounded, $a(\cdot) = (\sigma\, {}^t\sigma)(\cdot)$, and for $f \in C^2$

$$Lf(y) = \frac{1}{2} \sum_{i,j=1}^{d} a_{i,j}(y)\, \partial^2_{i,j} f(y) + \sum_{i=1}^{d} b_i(y)\, \partial_i f(y),\ y \in \mathbb{R}^d\,. \tag{7.40}$$

Theorem 7.6. *If on some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ endowed with an n-dimensional $(\mathcal{G}_t)$-Brownian motion B_t, $t \ge 0$, a continuous adapted process $(Y_t)_{t\ge 0}$ satisfies P-a.s., for all $t \ge 0$:*

$$Y_t = x + \int_0^t b(Y_s)\, ds + \int_0^t \sigma(Y_s) \cdot dB_s\,, \tag{7.41}$$

then

(7.42) *the law P_x of $(Y_t)_{t\ge 0}$, on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ is a solution of the martingale problem (7.2).*

Conversely, if P_x is a solution of the martingale problem (7.2), then there exists an $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ endowed with an n-dimensional Brownian motion $(\beta_t)_{t\ge 0}$, and a continuous adapted process $(Z_t)_{t\ge 0}$, such that P-a.s.,

$$Z_t = x + \int_0^t b(Z_s)\, ds + \int_0^t \sigma(Z_s)\, d\beta_s, \ \ \textit{for all } t \ge 0\,, \tag{7.43}$$

and the law of Z (on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$) is P_x.

Remark 7.7. One should not expect $Z_.$ (or $Y_.$) to be strong solutions of the SDE, as in (7.22). An example of this feature comes for instance when considering $d = 1 = n$,

$$\begin{aligned} \sigma(x) = \mathrm{sign}(x) &= 1, \ \text{ when } x \ge 0 \\ &= -1, \ \text{ when } x < 0\,, \end{aligned} \tag{7.44}$$

and $Y_. = X_.$ the canonical Brownian motion on $(C(\mathbb{R}_+, \mathbb{R}), \mathcal{F}, (F_t)_{t\ge 0}, W_0)$. Then, in this case

$$B_t = \int_0^t \mathrm{sign}(X_s)\, dX_s,\ t \ge 0\,, \tag{7.45}$$

is thanks to Lévy's characterization (6.44), (6.45) a Brownian motion. Moreover one has the identity: P-a.s., for all $t \ge 0$,

$$X_t = \int_0^t \mathrm{sign}(X_s)^2\, dX_s \overset{(5.92)}{=} \int_0^t \mathrm{sign}(X_s)\, dB_s,$$

In other words $Y_. = X_.$ solves (7.41), but one can prove, see [8], p. 302, or (7.144) below, that, in the notation of (7.22), for all $t \geq 0$,

$$F_t^{B.} = F_t^{|X.|} \subsetneq F_t^X = F_t\,.$$

As a matter of fact, one can show that whenever $Y_.$ satisfies (7.41) with σ as in (7.44), then $F_t^B \subsetneq F_t^Y$, for $t > 0$. □

Proof.

• (7.42):

We know from (7.24) that for $f \in C_c^2(\mathbb{R}^d, \mathbb{R})$, under P

$$f(Y_t) - f(Y_0) - \int_0^t Lf(Y_s)\,ds \;\text{ is a } (\mathcal{G}_t)\text{-martingale.} \tag{7.46}$$

Hence, for $0 \leq s_0 < \dots < s_m \leq s < t$, $g_0, \dots, g_m \in b\mathcal{B}(\mathbb{R}^d)$, denoting by P_x the law on $C(\mathbb{R}_+, \mathbb{R}^d)$ of Y under P, we see that:

canonical process

$$\begin{aligned} &E^{P_x}\Big[\Big(f(X_t) - f(X_s) - \int_s^t (Lf)(X_u)\,du\Big)\, g_0(X_{s_0}) \dots g_m(X_{s_m})\Big] \overset{P_x = \text{law of } Y.}{=} \\ &E\Big[\Big(f(Y_t) - f(Y_s) - \int_s^t (Lf)(Y_u)\,du\Big)\, g_0(Y_{s_0}) \dots g_m(Y_{s_m})\Big] \overset{(7.46)}{=} 0\,. \end{aligned}$$

Using Dynkin's lemma, it follows that under P_x,

$$M_t^f \overset{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Lf(X_u)\,du$$

is an $(\mathcal{F}_t)$-martingale for any $f \in C_c^2(\mathbb{R}^d, \mathbb{R})$. Moreover since $Y_0 = x$, P-a.s., we see that $P_x[X_0 = x] = 1$. Hence P_x is a solution of the martingale problem (7.2).

• (7.43):

We will only prove (7.43) in a special case, namely when

(7.47) $n = d$, and $a(x)$ is locally elliptic (i.e. for $U \neq \emptyset$ a bounded open subset of $\mathbb{R}^d$, $\exists c(U) > 0$, such that ${}^t\xi\, a(y)\, \xi \geq c(U)\, |\xi|^2$, for all $\xi \in \mathbb{R}^d$, and y in U).

For a proof in the general case we refer to [11], p. 91.

Note that due to (7.47) and $a(\cdot) = \sigma(\cdot)\, {}^t\sigma(\cdot)$, $\sigma(\cdot)$ is invertible and for $y \in U$, $\xi \in \mathbb{R}^d$ one has

$$|\xi|^2 =^t \xi\, \sigma^{-1}(y)\, a(y)\, {}^t\sigma^{-1}(y)\; \xi \geq c(U) |\, {}^t\sigma^{-1}(y)\, \xi|^2\,, \tag{7.48}$$

so that (using the explicit formula for $\sigma^{-1}(\cdot)$):

$$\sigma^{-1}(\cdot) \text{ is locally bounded measurable.} \tag{7.49}$$

On $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}, P_x)$, we introduce for $t \geq 0$, the σ-algebra $\mathcal{H}_t$ generated by $\mathcal{F}_t$ and the negligible sets of P_x and $\mathcal{G}_t = \mathcal{H}_t^+$, $t \geq 0$ (satisfying the usual conditions). We define

$$(7.50) \qquad M_t^i = X_t^i - X_0^i - \int_0^t b_i(X_s)\,ds, \quad i = 1, \dots, d\,.$$

If we apply (7.2) and stopping we see (since for $f(y) = y^i$, $Lf(y) = b_i(y)$) that the M_t^i are continuous $(\mathcal{G}_t)$-local martingales (see exercise below). Analogously, choosing $f(y) = y^i y^j$, so that $Lf(y) = a_{i,j}(y) + y^i\, b_j(y) + y^j\, b_i(y)$, we see that

$$(7.51) \qquad \begin{array}{l} X_t^i\, X_t^j - X_0^i\, X_0^j - \displaystyle\int_0^t \big(a_{i,j}(X_s) + X_s^i\, b_j(X_s) + X_s^j\, b_i(X_s)\big)\,ds, \\ \text{is a continuous } (\mathcal{G}_t)\text{-local martingale under } P_x\,. \end{array}$$

From Ito's formula we know that P_x-a.s.,

$$(7.52) \qquad \begin{aligned} X_t^i\, X_t^j &= X_0^i\, X_0^j + \int_0^t X_s^i\, dX_s^j + \int_0^t X_s^j\, dX_s^i + \langle X^i, X^j\rangle_t \\ &= X_0^i\, X_0^j + \int_0^t \big(X_s^i\, b_j(X_s) + X_s^j\, b_i(X_s)\big)\,ds + \langle M^i, M^j\rangle_t \\ &\qquad + \text{ continuous local martingale.} \end{aligned}$$

Comparing (7.51) and (7.52), we conclude that P-a.s.,

$$(7.53) \qquad \langle M^i, M^j\rangle_t = \int_0^t a_{i,j}(X_s)\,ds, \quad \text{for } t \geq 0\,.$$

We now define

$$(7.54) \qquad \begin{array}{l} \beta_t = \displaystyle\int_0^t \sigma^{-1}(X_s)\cdot dM_s, \quad t \geq 0 \\ \Big(\text{that is } \beta_t^i = \displaystyle\sum_{j=1}^d \int_0^t \sigma_{i,j}^{-1}(X_s)\,dM_s^j\Big), \end{array}$$

so that

$$(7.55) \qquad \beta_t,\ t \geq 0, \text{ is an } \mathbb{R}^d\text{-valued continuous } (\mathcal{G}_t)\text{-local martingale}$$

and

$$(7.56) \qquad \begin{array}{l} \langle \beta^i, \beta^j\rangle_t = \Big\langle \displaystyle\sum_{k=1}^d \int_0^\cdot \sigma_{i,k}^{-1}(X_s)\,dM_s^k,\ \sum_{\ell=1}^d \int_0^\cdot \sigma_{j,\ell}^{-1}(X_s)\,dM_s^\ell \Big\rangle_t = \\ \displaystyle\sum_{k,\ell=1}^d \int_0^t \sigma_{i,k}^{-1}(X_s)\,\sigma_{j,\ell}^{-1}(X_s)\,d\langle M^k, M^\ell\rangle_s \overset{(7.53)}{=} \\ \displaystyle\sum_{k,\ell=1}^d \int_0^t \sigma_{i,k}^{-1}(X_s)\,a_{k,\ell}(X_s)\,\sigma_{j,\ell}^{-1}(X_s)\,ds = \int_0^t (\sigma^{-1}(X_s)\,a(X_s)\,{}^t\sigma^{-1}(X_s))_{i,j}\,ds \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \delta_{i,j}\,t\,. \end{array}$$

It thus follows from Paul Lévy's characterization, cf. (6.44), (6.45), that

$$(\beta_t)_{t\ge 0} \text{ is } d\text{-dimensional } (\mathcal{G}_t)\text{-Brownian motion under } P_x\,. \tag{7.57}$$

Now, from (7.54) we deduce that P_x-a.s., for $t \ge 0$,

$$\int_0^t \sigma(X_s)\cdot d\beta_s \overset{(5.92)}{=} \int_0^t \sigma(X_s)\cdot\sigma^{-1}(X_s)\cdot dM_s = M_t\,, \tag{7.58}$$

and therefore P_x-a.s., for $t \ge 0$,

$$X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot d\beta_s\,. \tag{7.59}$$

This yields the representation (7.43). □

Exercise 7.8. *Show that when P_x is a solution of the martingale problem (7.2) on $(C(\mathbb{R}_+,\mathbb{R}^d),\mathcal{F})$, then for any $f \in C^2(\mathbb{R}^d)$*

$$f(X_t) - f(X_0) = \int_0^t Lf(X_s)\,ds,\ t\ge 0\,,$$

is a $(\mathcal{G}_t)$-local martingale, where $(\mathcal{G}_t)_{t\ge 0}$ is the filtration $\mathcal{G}_t = \mathcal{H}_t^+$ $(= \bigcap_{\varepsilon>0}\mathcal{H}_{t+\varepsilon})$, where $\mathcal{H}_t = \sigma(\mathcal{F}_t,\mathcal{N})$, where $\mathcal{N}$ is the collection of P_x-negligible sets of $\mathcal{F}$ (so that $(C(\mathbb{R}_+,\mathbb{R}^d),\mathcal{F}, (\mathcal{G}_t)_{t\ge 0}, P_x)$ satisfies the usual conditions). □

We further discuss the martingale problem (7.2) and its link with the SDE (7.1). As an application of Theorems 7.1 and 7.6 we have the following

Corollary 7.9. *Assume that $b(\cdot)\colon \mathbb{R}^d\to\mathbb{R}^d$ and $\sigma(\cdot)\colon \mathbb{R}^d \to M_{d\times n}$ satisfy the Lipschitz condition (7.3). Then, for $x\in\mathbb{R}^d$,*

$$\begin{aligned}&\text{there is a unique solution of the martingale problem (7.2) attached to}\\ &L = \tfrac{1}{2}\sum_{i,j=1}^d a_{ij}(\cdot)\,\partial^2_{i,j} + \sum_{i=1}^d b_i(\cdot)\,\partial_i,\ \text{ with } a(\cdot) = \sigma(\cdot)\,{}^t\sigma(\cdot)\end{aligned} \tag{7.60}$$

(one says that the martingale problem attached to L is well-posed).

Proof.

• Existence:

We consider some filtered probability space $(\Omega,\mathcal{G},(\mathcal{G}_t)_{t\ge 0},P)$ endowed with an n-dimensional $(\mathcal{G}_t)$-Brownian motion $(B_t)_{t\ge 0}$. One can for instance pick the canonical space $(C(\mathbb{R}_+,\mathbb{R}^d),\mathcal{F}, (F_t)_{t\ge 0}, W_0)$, and $B_. = X_.$, the canonical process). By (7.4), we know that we can construct a "solution", i.e. a continuous adapted $(Y_t)_{t\ge 0}$, such that P-a.s., for $t\ge 0$,

$$Y_t = x + \int_0^t b(Y_s)\,ds + \int_0^t \sigma(Y_s)\cdot dB_s\,. \tag{7.61}$$

It then follows from (7.42) that

$$\text{the law of } (Y_t)_{t\geq 0} \text{ on } (C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}) \text{ is a solution of the martingale problem attached to } L \text{ and } x\,. \tag{7.62}$$

• Uniqueness:

Assume that P_x is a solution of the martingale problem (7.2) attached to L and x. By (7.43) we know that we can find some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ and $(\beta_t)_{t\geq 0}$, which is an n-dimensional Brownian motion, and a continuous $(\mathcal{G}_t)$-adapted $\mathbb{R}^d$-valued process $(Z_t)_{t\geq 0}$, such that P-a.s., for $t \geq 0$:

$$\begin{aligned} Z_t =&\ x + \int_0^t b(Z_s)\, ds + \int_0^t \sigma(Z_s) \cdot d\beta_s, \ \text{ and} \\ P_x =&\ \text{law of } (Z_t)_{t\geq 0} \text{ on } (C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})\,. \end{aligned} \tag{7.63}$$

From (7.11), (7.21), we see that P-a.s., for all $t \geq 0$,

$$Z_t = X_t^\infty\,, \tag{7.64}$$

where X_t^∞ is, see below (7.19), defined as the P-a.s. uniform limit on compact intervals of X_t^m, $t \geq 0$, where

$$\begin{aligned} X_t^0 \equiv&\ x,\ X_t^1 = x + \int_0^t b(X_s^0)\, ds + \int_0^t \sigma(X_s^0) \cdot d\beta_s, \ \text{ and for } m \geq 1, \\ X_t^{m+1} =&\ x + \int_0^t b(X_s^m)\, ds + \int_0^t \sigma(X_s^m) \cdot d\beta_s, \ \text{ for } m \geq 1\,. \end{aligned} \tag{7.65}$$

By inspection of (7.65) we see that the law of $(X_t^m)_{t\geq 0}$ or of $(X_t^\infty)_{t\geq 0}$ are unchanged if instead of $\beta_.$ one uses the canonical n-dimensional Brownian motion. Combining this observation with (7.64), we see that the law of $Z_.$ (i.e. P_x) is uniquely determined. □

Remark 7.10. A not very satisfactory feature of the above theorem has to do with the fact that the assumptions are made on the coefficients $\sigma(\cdot)$ and $b(\cdot)$, that appear in (7.3), but the conclusion concerns the martingale problem where only $a(\cdot)$ and $b(\cdot)$ are involved.

It is clear that it does not suffice to assume $a(\cdot)$ Lipschitz continuous, in order to find a Lipschitz continuous $\sigma(\cdot)$ such that $\sigma\, {}^t\sigma = a$ (for instance when $d = 1$, $a(x) = |x|$ yields such an example).

However, one can show that when $a(\cdot) : \mathbb{R}^d \to M_{d\times d}$ satisfy a global ellipticity condition, i.e. for some $\varepsilon > 0$,

$${}^t\xi\, a(x)\, \xi \geq \varepsilon\, |\xi|^2, \ \text{ for all } x, \xi \text{ in } \mathbb{R}^d\,, \tag{7.66}$$

and a Lipschitz condition

$$|a(y) - a(z)| \leq K\, |y - z|, \ \text{ for } y, z \in \mathbb{R}^d\,, \tag{7.67}$$

then $a^{1/2}(\cdot)$ satisfies a Lipschitz condition as well., cf. [11], p. 97. Of course $a^{1/2}(\cdot)$ can then play the role of $\sigma(\cdot)$ in the Corollary 7.9.

A similar Lipschitz property of $a^{1/2}(\cdot)$ can be proved when instead of (7.66), (7.67) one assumes that

$$\sup_{x\in\mathbb{R}^d} |a(x)| \le C < \infty, \text{ and} \tag{7.68}$$

$$\begin{aligned} & x \in \mathbb{R}^d \to a(x) \in M_{d\times d} \text{ is a } C^2\text{-function and} \\ & \sup_{1\le i,j\le d}\ \sup_{x\in\mathbb{R}^d} |\partial^2_{i,j}\, a(x)| \le C < \infty\,. \end{aligned} \tag{7.69}$$

In this last case, $a(\cdot)$ need not be uniformly elliptic. As direct application of the Corollary 7.9, one now has:

(7.70) if $b(\cdot)$ satisfies a global Lipschitz condition, and $a(\cdot)$ either satisfies (7.66), (7.67), or (7.68), (7.69), then the martingale problem attached to L is well-posed.

$\square$

Girsanov transformations and applications to martingale problems

We will now bring into play certain exponential martingales, and use them as a tool to solve various martingales problems. An important role is played by a theorem due in various forms to Cameron-Martin (1944), Girsanov (1960), Maruyama (1954, 1955).

Setting:

$(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ is a filtered probability space satisfying the usual conditions, cf. (4.5), (4.6).

$(M_t)_{t\ge 0}$, is a continuous local martingale such that

$$Z_t = \exp\Big\{M_t - \frac{1}{2}\,\langle M\rangle_t\Big\},\ t \ge 0, \text{ is a martingale.} \tag{7.71}$$

As we know from Novikov's criterion, cf. (6.72), (6.74), this is for instance the case when $E[\exp\{\frac{1}{2}\,\langle M\rangle_t\}] < \infty$, for each $t \ge 0$.

We pick a fixed $T > 0$, and since $E[Z_T] = 1$ $(= E[Z_0])$, we introduce the new probability Q on $(\Omega, \mathcal{G})$ defined by

$$Q \stackrel{\text{def}}{=} Z_T\, P\,. \tag{7.72}$$

Of course when $A \in \mathcal{G}_t$ with $t \le T$, one has

$$\begin{aligned} & Q(A) = E[1_A\, Z_T] = E[1_A\, Z_t], \text{ so that} \\ & \frac{dQ}{dP}\Big|_{\mathcal{G}_t} = Z_t,\ 0 \le t \le T\,, \end{aligned} \tag{7.73}$$

in other words Z_t, for $t \le T$, represents the density of the restriction of Q to $\mathcal{G}_t$ with respect to the restriction of P to $\mathcal{G}_t$. Note that P and Q have the same null sets.

Theorem 7.11. *(Cameron-Martin, Girsanov, Maruyama)*

If $(N_t)_{0\le t\le T}$ is a continuous local martingale under P, the "Girsanov transform of N" defined as

$$\widetilde{N}_t = N_t - \langle N, M\rangle_t,\ 0 \le t \le T,\ \text{ is a continuous local martingale under } Q\,. \tag{7.74}$$

Moreover, if N^1, N^2 are continuous local martingales under P, then P-a.s. (or equivalently Q-a.s.),

$$\langle \widetilde{N}^1, \widetilde{N}^2\rangle_t^Q = \langle N^1, N^2\rangle_t^P,\ \text{ for } 0 \le t \le T\,. \tag{7.75}$$

Proof. Without loss of generality we assume $N_0 = 0$. We first claim that:

$$\widetilde{N}_t\, Z_t,\ 0 \le t \le T,\ \text{ is a local martingale under } P\,. \tag{7.76}$$

Indeed, it follows by Ito's formula (6.22) and by (6.28) that P-a.s., for $t \ge 0$,

$$\begin{aligned}
\widetilde{N}_t\, Z_t &= \int_0^t Z_s\, d\widetilde{N}_s + \int_0^t \widetilde{N}_s\, dZ_s + \langle \widetilde{N}, Z\rangle_t, \text{ and} \\
Z_t &= 1 + \int_0^t Z_s\, dM_s, \text{ so } \langle \widetilde{N}, Z\rangle_t = \langle N - \langle N, M\rangle, Z\rangle_t = \langle N, Z\rangle_t \\
&\qquad\qquad\qquad \overset{(5.90)}{=} \int_0^t Z_s\, d\,\langle N, M\rangle_s\,.
\end{aligned}$$

As a result,

$$\begin{aligned}
\widetilde{N}_t\, Z_t &= \int_0^t Z_s\, dN_s - \int_0^t Z_s\, d\langle N, M\rangle_s + \int_0^t \widetilde{N}_s\, dZ_s + \int_0^t Z_s\, d\langle N, M\rangle_s \\
&= \int_0^t Z_s\, dN_s + \int_0^t \widetilde{N}_s\, dZ_s, \ \text{ and (7.76) follows.}
\end{aligned} \tag{7.77}$$

We will now see that $\widetilde{N}_t$, $0 \le t \le T$, is a local martingale under Q. We define:

$$\begin{aligned}
T_n &= \inf\{u \ge 0;\ |\widetilde{N}_u| \ge n\}\ \ (\uparrow \infty,\ \text{as } n \to \infty)\,, \\
S_m &= \inf\{u \ge 0;\ |M_u| \ge m \text{ or } \langle M\rangle_u \ge m\}\ (\uparrow \infty,\ \text{as } m \to \infty)\,.
\end{aligned} \tag{7.78}$$

The calculation (7.77) applied to $M_{t\wedge S_m}$, $Z_{t\wedge S_m}$, $N_{t\wedge T_n\wedge S_m}$, $\widetilde{N}_{t\wedge T_n\wedge S_m}$ shows that:

$$\begin{aligned}
&\widetilde{N}_{t\wedge T_n\wedge S_m}\, Z_{t\wedge S_m} \text{ is a continuous local martingale, which} \\
&\text{is bounded, and hence a martingale.}
\end{aligned} \tag{7.79}$$

As a result, we see that for $0 \le s \le t \le T$, and $A \in \mathcal{G}_s$, we have:

$$\begin{aligned}
&E^P\big[\underbrace{1_A\, \widetilde{N}_{t\wedge T_n\wedge S_m}}_{\mathcal{G}_t\text{-meas.}}\ \underbrace{Z_{T\wedge S_m}}_{\text{martingale}}\ \big] = E^P\big[\ \underbrace{1_A}_{\mathcal{G}_s\text{-meas.}}\ \ \underbrace{\widetilde{N}_{t\wedge T_n\wedge S_m}\, Z_{T\wedge S_m}}_{\text{martingale}}\big] = \\
&E^P\big[\underbrace{1_A\, \widetilde{N}_{s\wedge T_n\wedge S_m}}_{\mathcal{G}_s\text{-meas.}}\ \underbrace{Z_{s\wedge S_m}}_{\text{martingale}}\ \big] = E^P[1_A\, \widetilde{N}_{s\wedge T_n\wedge S_m}\, Z_{T\wedge S_m}]\,.
\end{aligned} \tag{7.80}$$

Letting $m \to \infty$, and observing that $Z_{T \wedge S_m} \underset{m\to\infty}{\overset{L^1(P)}{\longrightarrow}} Z_T$ by uniform integrability and a.s. convergence, and keeping in mind that $|\widetilde{N}_{\cdot \wedge T_n}| \le n$, we find that

$$\begin{array}{rcl} E^P[1_A\, \widetilde{N}_{t\wedge T_n}\, Z_T] & = & E^P[1_A\, \widetilde{N}_{s\wedge T_n}\, Z_T] \\ \| & & \| \\ E^Q[1_A\, \widetilde{N}_{t\wedge T_n}] & = & E^Q[1_A\, \widetilde{N}_{s\wedge T_n}] \end{array} \tag{7.81}$$

Since $T_n \uparrow \infty$, P and Q-a.s., this proves that

$$\widetilde{N}_t,\ 0 \le t \le T, \text{ is a continuous local martingale under } Q\,. \tag{7.82}$$

We then turn to the proof of (7.75). We introduce

$$I_t \overset{\text{def}}{=} N_t^2 - \langle N \rangle_t = 2 \int_0^t N_s\, dN_s$$

using Ito's formula (6.22) and the notation $\langle\ \rangle = \langle\ \rangle^P$. By (7.82) we find that

$$\begin{array}{l} \widetilde{I}_t \overset{(7.74)}{=} I_t - \langle I, M \rangle_t = N_t^2 - \langle N \rangle_t - 2 \displaystyle\int_0^t N_s\, d\langle N, M\rangle_s \\ \text{is a continuous local martingale under } Q\,. \end{array} \tag{7.83}$$

As a result

$$\begin{aligned} \widetilde{N}_t^2 - \langle N \rangle_t &\overset{(7.74)}{=} N_t^2 - 2N_t \langle N, M\rangle_t + \langle N, M\rangle_t^2 - \langle N \rangle_t \\ &\overset{(7.83)}{=} \widetilde{I}_t + \langle N, M\rangle_t^2 + 2 \int_0^t N_s\, d\langle N, M\rangle_s - 2N_t \langle N, M\rangle_t\,. \end{aligned} \tag{7.84}$$

It follows from Ito's formula that:

$$\begin{array}{l} J_t = 2N_t \langle N, M\rangle_t - 2 \displaystyle\int_0^t N_s\, d\langle N, M\rangle_s = 2 \int_0^t \langle N, M\rangle_s\, dN_s = \\ \text{continuous local martingale under } P\,, \end{array}$$

and

$$\widetilde{J}_t \overset{(7.74)}{=} J_t - 2 \int_0^t \langle N, M\rangle_s\, d\langle N, M\rangle_s = J_t - \langle N, M\rangle_t^2\,. \tag{7.85}$$

We can then come back to (7.84) to conclude that

$$\widetilde{N}_t^2 - \langle N \rangle_t = \widetilde{I}_t - \widetilde{J}_t \overset{(7.74)}{=} \text{continuous local martingale under } Q\,. \tag{7.86}$$

This shows that

$$\langle \widetilde{N} \rangle_t^Q = \langle N \rangle_t^P, \ \ 0 \le t \le T\,, \tag{7.87}$$

and the claim (7.75) now follows by polarization. $\square$

We will now apply the above theorem in order to construct the solution of certain martingale problems.

Theorem 7.12. *Assume that $b(\cdot), c(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ and $a(\cdot) : \mathbb{R}^d \to M^+_{d\times d}$ (i.e. the set of $d \times d$ non-negative matrices) are measurable locally bounded functions, with b, a, ${}^t cac$ bounded. Then, there is a bijective correspondence between the solutions of the martingale problem attached to*

$$L = \tfrac{1}{2} \sum_{i,j=1}^d a_{ij}\, \partial^2_{i,j} + \sum_{i=1}^d b_i\, \partial_i, \quad x \in \mathbb{R}^d\,, \tag{7.88}$$

and

$$\widetilde{L} = \tfrac{1}{2} \sum_{i,j=1}^d a_{ij}\, \partial^2_{i,j} + \sum_{i=1}^d (b_i + (ac)_i)\, \partial_i, \quad x \in \mathbb{R}^d\,. \tag{7.89}$$

This bijective correspondence is the following. To each P solution of the martingale problem attached to (L, x), one associates the law $\widetilde{P}$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$, which is specified by the fact that

$$\frac{d\widetilde{P}}{dP}\Big|_{\mathcal{F}_t} = \exp\Big\{\int_0^t c(X_s) \underset{\substack{\uparrow \\ \mathbb{R}^d\text{-scalar product}}}{\cdot} d\overline{X}_s - \tfrac{1}{2} \int_0^t {}^t cac(X_s)\, ds\Big\}, \ \text{ for } t \ge 0\,, \tag{7.90}$$

where $\overline{X}_t = X_t - \int_0^t b(X_s)\, ds$, $t \ge 0$.

Proof. Note that under P, the process $\overline{X}$ is a continuous local martingale and

$$\langle \overline{X}^i, \overline{X}^j \rangle_t \overset{(7.53)}{=} \int_0^t a_{ij}(X_s) ds, \ \text{ for } t \ge 0\,. \tag{7.91}$$

Hence, $\int_0^t c(X_s) \cdot d\overline{X}_s$ is a continuous local martingale and P-a.s.:

$$\Big\langle \int_0^\cdot c(X_s) \cdot d\overline{X}_s \Big\rangle_t = \int_0^t {}^t cac(X_s)\, ds, \ \text{for } t \ge 0\,. \tag{7.92}$$

As a result, the expression in (7.90) is the stochastic exponential of $\int_0^\cdot c(X_s) \cdot d\overline{X}_s$. By Novikov's criterion (6.72) or by (6.29), we know that:

$$Z_t = \exp\Big\{\int_0^t c(X_s) \cdot d\overline{X}_s - \tfrac{1}{2} \int_0^t {}^t cac(X_s)\, ds\Big\}, \ t \ge 0, \tag{7.93}$$

is a continuous martingale.

We will now use the following

Lemma 7.13.

$$\text{There is a unique probability } Q \text{ on } (C(\mathbb{R}_+,\mathbb{R}^d),\mathcal{F}) \text{ such that } \frac{dQ}{dP}\Big|_{\mathcal{F}_t} = Z_t, \text{ for each } t \ge 0\,. \tag{7.94}$$

Proof. Uniqueness follows from Dynkin's lemma.

We now prove the existence. Consider for $n \ge 2$ the laws π_n on

$$\begin{aligned}&\textstyle\sum_n \overset{\text{def}}{=} C([0,1],\mathbb{R}^d) \times \overbrace{C_0([0,1],\mathbb{R}^d) \times \dots \times C_0([0,1],\mathbb{R}^d)}^{(n-1)\text{ copies}}, \text{ of}\\ &((X_\cdot)_{0\le\cdot\le1},\, (X_{1+\cdot}-X_1)_{0\le\cdot\le1}, \dots, (X_{n-1+\cdot}-X_{n-1})_{0\le\cdot\le1}) \text{ under } Z_n\,P\end{aligned}$$

(here $C_0([0,1],\mathbb{R}^d) \overset{\text{def}}{=} \{w \in C([0,1],\mathbb{R}^d);\ w(0)=0\}$).

The laws $\pi_n, n \ge 2$, are consistent, i.e. the image of π_{n+1} on Σ_n under the "projection" from $\Sigma_{n+1} \to \sum_n$, which drops the last component is equal to π_n, for all $n \ge 2$, thanks to the martingale property of Z_t, $t \ge 0$, under P.

By Kolmogorov's extension theorem (see for instance [12], p. 129), there is a (unique) probability π on $\Sigma = C([0,1],\mathbb{R}^d) \times \prod_1^\infty C_0([0,1],\mathbb{R}^d)$, such that for each $n \ge 2$, the image of π under the natural "projection" $\Sigma \to \Sigma_n$ is π_n. If one now defines the map φ: $\Sigma \to C(\mathbb{R}_+,\mathbb{R}^d)$ via

$$\begin{aligned}&\varphi((w_1,w_2,\dots)) = w \text{ such that}\\ &w(t) = w_1(t), \text{ for } 0 \le t \le 1\,,\\ &w(t) = w_1(1) + w_2(t-1), \text{ for } 1 \le t \le 2\,,\\ &w(t) = w_1(1) + w_2(1) + \dots + w_n(1) + w_{n+1}(t-n), \text{ for } n \le t \le n+1\,,\end{aligned} \tag{7.95}$$

then, the image Q of π under φ satisfies (by definition of π_n):

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_n} = Z_n, \text{ for any } n \ge 2\,. \tag{7.96}$$

The martingale property (7.93) now implies the property (7.94). □

We will now see that the unique Q constructed by the above lemma satisfies:

$$Q \text{ solves the martingale problem } \widetilde{L}, x\,. \tag{7.97}$$

To this end, we first note by (7.74) that under $Z_n\,P$

$$\begin{aligned}\overline{X}^i_t - \langle \overline{X}^i, \int_0^\cdot c(X_s)\cdot d\overline{X}_s\rangle_t &= X^i_t - \int_0^t b_i(X_s)\,ds - \sum_{j=1}^d \int_0^t c_j(X_s)\,d\langle \overline{X}^j, \overline{X}^i\rangle_s\\ &\overset{(7.91)}{=} X^i_t - \int_0^t \big(b_i(X_s) + (ac)_i(X_s)\big)\,ds,\ 0 \le t \le n\,,\end{aligned}$$

is a local martingale.

If we now define $T_m = \inf\{u \ge 0;\, |X_u - \int_0^u (b+ac)(X_s)\,ds| \ge m\}$, so that $T_m \uparrow \infty$, as $m \to \infty$ ($ac = a^{1/2}(a^{1/2}c)$ is bounded, since a and ${}^t cac$ are bounded),

$$\begin{aligned} &X_{t\wedge T_m} - \int_0^{t\wedge T_m} (b+ac)(X_s)\,ds,\ 0 \le t \le n,\ \text{is a martingale} \\ &\text{under } Z_n\,P \text{ (and also under } Q)\,. \end{aligned} \tag{7.98}$$

As a result, we see that for each m, the $\mathbb{R}^d$-valued

$$M_t = X_t - \int_0^t (b+ac)(X_s)\,ds \text{ is a local martingale under } Q\,. \tag{7.99}$$

By (7.75) and (7.91) we see that Q-a.s.,

$$\langle M^i, M^j\rangle_t = \int_0^t a_{ij}(X_s)\,ds,\ t \ge 0\,. \tag{7.100}$$

The application of Ito's formula yields that for any f in $C^2(\mathbb{R}^d)$, Q-a.s., for $t \ge 0$:

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^d \int_0^t \partial_i f(X_s)\,dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial^2_{ij} f(X_s)\,d\langle X^i, X^j\rangle_s \\ &\overset{(7.99)}{\underset{(7.100)}{=}} f(X_0) + \int_0^t \nabla f(X_s)\cdot(b+ac)(X_s)\,ds + \int_0^t \nabla f(X_s)\cdot dM_s \\ &\qquad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial^2_{ij} f(X_s)\,a_{ij}(X_s)\,ds \\ &= f(X_0) + \int_0^t \widetilde{L} f(X_s)\,ds + \text{ continuous local martingale under } Q\,. \end{aligned} \tag{7.101}$$

This completes the proof of (7.97).

Since Q coincides with $\widetilde{P}$ in (7.90), we see that for any $x \in \mathbb{R}^d$

$$\begin{aligned} &P \to \widetilde{P} \text{ sends solutions of the martingale problem } (L,x) \text{ into solutions} \\ &\text{of the martingale problem } (\widetilde{L},x)\,. \end{aligned} \tag{7.102}$$

There now remains to see that the correspondence is bijective. We first show that

$$\text{the correspondence } P \to \widetilde{P} \text{ is injective.} \tag{7.103}$$

Assume that $P_1 \to \widetilde{P}$ and $P_2 \to \widetilde{P}$. Then for $t \ge 0$, $P_1 \sim P_2 \sim \widetilde{P}$ on $\mathcal{F}_t$, and the integral $\int_0^s c(X_u)\,d\overline{X}_u$, $0 \le s \le t$, is identical $P_1, P_2, \widetilde{P}$-a.s. (this feature goes back to (4.50), (4.57), (4.59) and the explicit approximating sequences used to construct stochastic integrals, see exercise 2) below). Now the equalities $\widetilde{P} = Z_t\, P_1 = Z_t\, P_2$ on $\mathcal{F}_t$, imply that $P_1 = P_2$ on $\mathcal{F}_t$, for $t \ge 0$, and (7.103) follows, by Dynkin's lemma.

(7.104) The correspondence $P \to \widetilde{P}$ is surjective.

Indeed, consider Q a solution of the martingale problem $(\widetilde{L}, x)$. The above shows that there is a unique P on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ such that for any $t \ge 0$

$$\begin{aligned}(7.105)\quad \frac{dP}{dQ}\Big|_{\mathcal{F}_t} &= \exp\Big\{ - \int_0^t c(X_s)\cdot d\Big(X_s - \int_0^s (b+ac)(X_u)\, du\Big) - \tfrac{1}{2} \int_0^t {}^t cac(X_u)\, du\Big\} \\ &\stackrel{\text{def}}{=} \widetilde{Z}_t\,,\end{aligned}$$

and P is a solution of the martingale problem (L, x) (this is an application of (7.102) with Q playing the role of P).

Note that for $t \ge 0$,

$$\widetilde{Z}_t = \exp\Big\{ - \int_0^t c(X_s)\cdot d\overline{X}_s + \tfrac{1}{2} \int_0^t ({}^t cac)(X_u) du\Big\} = Z_t^{-1}\,,$$

so that for $t \ge 0$,

$$\frac{d\widetilde{P}}{dP}\Big|_{\mathcal{F}_t} = Z_t = \frac{dQ}{dP}\Big|_{\mathcal{F}_t}$$

and hence $\widetilde{P} = Q$, thus completing the proof of (7.104). □

Example:

We know by (7.60) that the martingale problem attached to $L = \frac{1}{2}\,\Delta$ is well-posed (this is the case $b = 0$, $\sigma =$ Identity matrix $(d \times d)$). The solution to the martingale problem attached to (L, x) is W_x, the "Wiener measure starting from x".

Consider now $b(\cdot)$: $\mathbb{R}^d \to \mathbb{R}^d$, bounded measurable.

When $\widetilde{L} = \frac{1}{2}\,\Delta + b\cdot\nabla$, we see by (7.90) that the martingale problem attached to $(\widetilde{L}, x)$ has a unique solution, which is a probability $\widetilde{W}_x$ on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ such that

$$(7.106)\qquad \frac{d\widetilde{W}_x}{dW_x}\Big|_{\mathcal{F}_t} = \exp\Big\{ \int_0^t b(X_s)\cdot dX_s + \tfrac{1}{2} \int_0^t |b(X_s)|^2\, ds\Big\},\ \text{ for any } t \ge 0\,.$$

Note that $\widetilde{W}_x$-a.s.,

$$(7.107)\qquad X_t = x + \int_0^t b(X_s)\, ds + \beta_t,\ \text{ for } t \ge 0\,,$$

where $(\beta_t)_{t\ge 0}$, is an $(\mathcal{F}_t)$-Brownian motion under $\widetilde{W}_x$ (thanks to (7.75) and Paul Lévy's characterization Theorem 6.10).

In particular this yields a (weak) solution to the stochastic differential equation attached to $b(\cdot)$ (which is only bounded measurable!) and $\sigma(\cdot) = Id$.

Exercise 7.14.

1) Show that all solutions to (7.107) have the same law (hint: use Theorem 7.12).

2) Consider P a solution of the martingale problem (L, x), where L is as in (7.88) and $\widetilde{P}$ such that (7.90) holds (with the same assumptions on $a(\cdot), b(\cdot), c(\cdot)$).

a) Show that when H is a bounded progressively measurable process, $\int_0^t H_s \, d\overline{X}_s$ is well-defined regardless of whether one uses that under P, $\overline{X}_.$ is a continuous martingale or that under $\widetilde{P}$, $\overline{X}_.$ is a continuous semimartingale (hint: we use the approximating sequences from (4.57), with $A_s = s$, and (4.59), alternatively use (5.82), (7.75)).

b) Show that $\int_0^t c(X_s) \cdot d\overline{X}_s$ is well-defined regardless of whether one works with P or $\widetilde{P}$ to interpret the stochastic integral.

3) When $b = 1$, in (7.106), show that although the restrictions of W_x and $\widetilde{W}_x$ to $\mathcal{F}_t$ are equivalent, for each $t \geq 0$, one has $W_x \perp \widetilde{W}_x$ (i.e. there is $A \in \mathcal{F}$ with $W_x(A) = 1 = \widetilde{W}_x(A^c)$). □

Explosions of solutions of stochastic differential equations: an application of Girsanov transformations.

We consider $c(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ a locally Lipschitz function:

(7.108) $$\forall M > 0, \exists K_M > 0, \text{ such that } |c(x) - c(y)| \leq K_M |x - y|, \text{ for } |x|, |y| \leq M\,.$$

If we now consider $(X_t)_{t \geq 0}$, the canonical process on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F}, (F_t)_{t \geq 0}, W_x)$, i.e. the canonical Brownian motion starting from x, we know from (6.27) that

(7.109) $$Z_t = \exp\Big\{ \int_0^t c(X_s) \cdot dX_s - \frac{1}{2} \int_0^t |c(X_s)|^2 \, ds \Big\}, \ t \geq 0\,,$$

is a continuous local martingale.

However when $c(\cdot)$ "grows too fast at infinity", it need not be a martingale. The key quantity, cf. (6.84), is

(7.110) $$e_t = 1 - E_x[Z_t] \geq 0\,,$$

since we know, from (6.84), that $e_t = 0$ implies that $Z_s, 0 \leq s \leq t$, is a martingale.

We will now provide an interpretation of $e_t, t \geq 0$, in terms of the possible "explosions" of the SDE:

(7.111) $$\begin{cases} dY_t = c(Y_t) \, dt + dB_t\,, \\ \ \ Y_0 = x\,. \end{cases}$$

For this purpose we choose a sequence of bounded, Lipschitz functions $c_N(\cdot)$ on $\mathbb{R}^d$, such that

(7.112) $$c_N(\cdot) = c(\cdot) \text{ on } \overline{B}_N = \{z \in \mathbb{R}^d; \, |z| \leq N\}\,.$$

Using the canonical Brownian motion $X_{\cdot}$ as "driving noise", we have (7.3), (7.4) a unique solution $(Y_t^N)_{t \geq 0}$ of

$$Y_t^N = X_t + \int_0^t c_N(Y_s^N)\, ds \tag{7.113}$$

(since $\sigma(\cdot) = Id$, $Y_{\cdot}^N$ is even $(\mathcal{F}_t)$-adapted and actually a continuous function of $X_{\cdot}$, when $C(\mathbb{R}_+, \mathbb{R}^d)$ is endowed with the topology of uniform convergence on compact time intervals, cf. (7.21)).

We then define the $(\mathcal{F}_t)$-stopping time:

$$T_N = \inf\{u \geq 0: \ |Y_u^N| \geq N\}\,. \tag{7.114}$$

Lemma 7.15. $(N, k \geq 0)$

$$\textit{For all } \ t \geq 0, \ Y_{t \wedge T_N}^{N+k} = Y_{t \wedge T_N}^N\,. \tag{7.115}$$

Proof. Define $T_N^k = \inf\{u \geq 0; \ |Y_u^{N+k}| \geq N\}$. As below (7.5) we find that

$$Y_{t \wedge T_N \wedge T_N^k}^{N+k} - Y_{t \wedge T_N \wedge T_N^k}^N \overset{(7.111)}{=} \int_0^{t \wedge T_N \wedge T_N^k} \big(c_{N+k}(Y_s^{N+k}) - c_N(Y_s^N)\big)\, ds$$

and since $|Y_s^{N+k}|, |Y_s^N| \leq N$ for $0 < s \leq T_N \wedge T_N^k$, and $c_{N+k}(\cdot) = c_N(\cdot) = c(\cdot)$ on $\overline{B}_N$, we see that for $t_0 \geq 0$,

$$\sup_{t \leq t_0} |Y_{t \wedge T_N \wedge T_N^k}^{N+k} - Y_{t \wedge T_N \wedge T_N^k}^N| \overset{(7.108)}{\leq} K_N \int_0^{t_0} |Y_{s \wedge T_N \wedge T_N^k}^{N+k} - Y_{s \wedge T_N \wedge T_N^k}^N|\, ds\,. \tag{7.116}$$

By Gronwall's lemma, cf. (7.9), we see that

$$\text{for all } t_0 \geq 0, \ \sup_{t \leq t_0} |Y_{t \wedge T_N \wedge T_N^k}^{N+k} - Y_{t \wedge T_N \wedge T_N^k}^N| = 0\,. \tag{7.117}$$

This last equality immediately implies that $T_N = T_N^k$, for all $k \geq 0$, and the claim (7.115) follows. □

By (7.115), we see that for $N' \geq N$, Y^N and $Y^{N'}$ coincide up to time T_N, and in particular,

$$T_N \text{ is a non-decreasing sequence of } (\mathcal{F}_t)\text{-stopping times.} \tag{7.118}$$

We can now define the **explosion time** of the SDE

$$Y_t = X_t + \int_0^t c(Y_s)\, ds, \ t \geq 0\,,$$

as the $(\mathcal{F}_t)$-stopping time:

$$T = \lim_{N \to \infty} T_N \in (0, \infty]\,. \tag{7.119}$$

The relation between the explosion time and e_t in (7.110) comes in the following:

Theorem 7.16.

$$\text{(7.120)} \qquad \textit{For } t \geq 0,\ e_t = W_x[T \leq t]\,.$$

Proof. The case $t = 0$ is obvious and we thus assume $t > 0$.

By (7.106), (7.107), we know that if we define the probability

$$\text{(7.121)} \qquad Q_N = \exp\Big\{ \int_0^t c_N(X_s) \cdot dX_s - \frac{1}{2} \int_0^t |c_N(X_s)|^2\, ds \Big\}\, W_x\,,$$

under Q_N,

$$\text{(7.122)} \qquad X_s = x + \beta_s + \int_0^s c_N(X_u)\, du, \text{ for } 0 \leq s \leq t\,,$$

where $(\beta_s)_{0 \leq s \leq t}$ is a d-dimensional Brownian motion, and the law (on $C([0,t], \mathbb{R}^d)$) of $(X_s)_{0 \leq s \leq t}$ under Q_N is that of the restriction to time $[0,t]$ of the unique solution of the martingale problem attached to $L = \frac{1}{2}\,\Delta + c_N \cdot \nabla$, and x. By (7.42) the law of $(Y_s^N)_{0 \leq s \leq t}$, under W_x, with $Y_{\cdot}^N$ as in (7.113), coincides with the law of $(X_s)_{0 \leq s \leq t}$ under Q_N. As a result, we find that setting

$$S_N = T_{B_N} \big(\stackrel{\text{def}}{=} \inf\{s \geq 0;\ |X_s| \geq N\} \big)\,,$$

we have

$$\text{(7.123)} \qquad \begin{aligned} &W_x[T_N > t] = Q_N[S_N > t] = \\ &E_x\Big[S_N > t,\ \exp\Big\{ \int_0^t c_N(X_s) \cdot dX_s - \frac{1}{2} \int_0^t |c_N(X_s)|^2\, ds \Big\}\Big]\,. \end{aligned}$$

Note that W_x-a.s. on $\{S_N > t\}$,

$$\int_0^t c_N(X_s) \cdot dX_s = \int_0^{t \wedge S_N} c_N(X_s) \cdot dX_s \underset{(7.112)}{\overset{(5.60)}{=}} \int_0^{t \wedge S_N} c(X_s) \cdot dX_s = \int_0^t c(X_s) \cdot dX_s\,,$$

and therefore:

$$\text{(7.124)} \qquad W_x[T_N > t] = E_x[S_N > t, Z_t]\,.$$

Now W_x-a.s. $S_N \uparrow \infty$, and thus letting $N \to \infty$, we find

$$W_x[T > t] = \lim_n\ W_x[T_N > t] = E_x[Z_t]\,,$$

and in view of (7.110) the claim (7.120) follows. $\square$

Complement: An example of SDE with no strong solution, which is weakly well-posed

We resume here the discussion (cf. Remark 7.7) of the stochastic differential equation on $\mathbb{R}$:

$$Y_t = \int_0^t \text{sign}(Y_s)\, dB_s\,, \tag{7.125}$$

where B is a one-dimensional Brownian motion and, cf. (7.44),

$$\begin{aligned} \text{sign}(x) &= 1, \quad \text{when } x \ge 0\,, \\ &= -1, \ \text{ when } x < 0\,. \end{aligned}$$

We have explained below (7.45) that we can always find a solution (in a weak sense) of this equation: if we have on some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ satisfying the usual conditions a $(\mathcal{G}_t)$-Brownian motion $Y_.$, we define

$$B_t = \int_0^t \text{sign}(Y_s)\, dY_s,\ t \ge 0\,. \tag{7.126}$$

By Lévy's characterization, (6.44), (6.45), we know that

$$(B_t)_{t\ge 0} \ \text{ is a } (\mathcal{G}_t)\text{-Brownian motion.} \tag{7.127}$$

In addition, P-a.s.,

$$Y_t = \int_0^t \text{sign}^2(Y_s)\, dY_s \underset{(7.126)}{\overset{(5.92)}{=}} \int_0^t \text{sign}(Y_s)\, dB_s,\ \text{for } t \ge 0\,.$$

In other words, with (Y, B) as above, we have a solution of the SDE (7.125), in the weak sense. We will now explain that the above structure is "typical" and necessarily, for all $t > 0$, $F_t^B \subsetneq F_t^Y$, in the notation of (7.22).

By a **weak solution** of (7.125), we mean an $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ satisfying the usual conditions, endowed with a $(\mathcal{G}_t)$-Brownian motion $(B_t)_{t\ge 0}$, and a continuous adapted process $(Y_t)_{t\ge 0}$, such that (7.125) holds.

Theorem 7.17. *Given a weak solution of (7.125), then $Y_.$ is a $(\mathcal{G}_t)$-Brownian motion and P-a.s.,*

$$B_t = \int_0^t \text{sign}(Y_s)\, dY_s,\ \textit{for}\ t \ge 0\,. \tag{7.128}$$

Moreover, one has the identity:

$$|Y_t| = B_t + L_t,\ \ \textit{for } t \ge 0,\ \ \text{“}\textbf{Tanaka's formula}\text{”}, \tag{7.129}$$

where $(L_t)_{t\ge 0}$, the "local time of Y at 0", is a continuous, adapted, non-decreasing process, with $L_0 = 0$, characterized by the fact that P-a.s.,

$$L_t = \int_0^t 1\{Y_s = 0\}\, dL_s, \text{ for all } t \ge 0\,. \tag{7.130}$$

In addition, P-a.s.,

$$L_t = \sup_{s\le t}\,(B_s)^- = -\inf_{s\le t} B_s, \ \text{ for } t \ge 0 \quad (\text{where } x^- = \max(-x,0)), \tag{7.131}$$

and for all $t \ge 0$,

$$F_t^{|Y|} = F_t^B\,. \tag{7.132}$$

Proof.

• (7.128):

From (7.125) and P. Lévy's characterization, it follows that $Y_{\cdot}$ is a $(\mathcal{G}_t)$-Brownian motion and

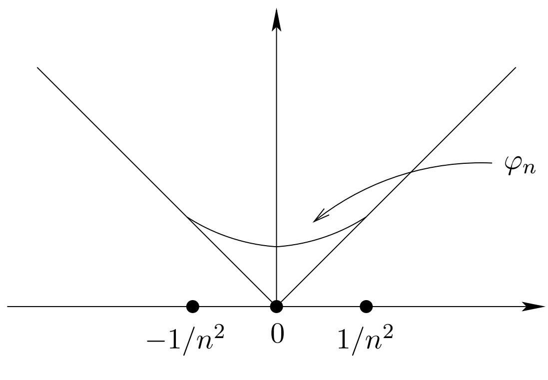
$$\int_0^t \text{sign}(Y_s)\, dY_s \overset{(5.92)}{=} \int_0^t \text{sign}^2(Y_s)\, dB_s = B_t,\ P\text{-a.s., for all } t \ge 0,$$

whence our claim.

• (7.129):

We consider a decreasing sequence of C^2, symmetric, convex functions φ_n on $\mathbb{R}$, such that

$$\begin{aligned} &\varphi_n(x) = |x| \text{ on } \Big(-\frac{1}{n^2}, \frac{1}{n^2}\Big)^c, \\ &\varphi_n'(x) \uparrow 1, \text{ for } x > 0,\ \varphi_n'(x) \downarrow -1,\ x < 0\,. \end{aligned} \tag{7.133}$$



Note that $\varphi_n(x) \downarrow |x|$ for all $x \in \mathbb{R}$, and in fact $0 \le \varphi_n(x) - |x| \le \frac{1}{n^2}$, so, $\varphi_n(\cdot)$ converges uniformly to $|\cdot|$ on $\mathbb{R}$. Also, by convexity and (7.133),

$$-1 \le \varphi_n'(x) \le 1, \text{ for } x \in \mathbb{R},\ n \ge 1\,. \tag{7.134}$$

Applying Ito's formula we find that:

$$\varphi_n(Y_t) = \varphi_n(Y_0) + \int_0^t \varphi'_n(Y_s)\,dY_s + \frac{1}{2}\int_0^t \varphi''_n(Y_s)\,ds\,. \tag{7.135}$$

We already know that $|\varphi_n(Y_t) - |Y_t|| \le \frac{1}{n^2}$, and $|\varphi_n(Y_0) - |Y_0|| \le \frac{1}{n^2}$, and in addition, by (7.128) and Doob's inequality,

$$\begin{aligned} &E\Big[\sup_{0\le s\le t}\Big|\int_0^t \varphi'_n(Y_s)\,dY_s - B_t\Big|^2\Big] \overset{(4.76)}{\le} 4E\Big[\int_0^t \big(\varphi'_n(Y_s) - \mathrm{sign}(Y_s)\big)^2\,ds\Big] \le \\ &c\int_0^t P\Big[|Y_s| \le \frac{1}{n^2}\Big]\,ds \underset{\text{B.M.}}{\overset{Y_\cdot \text{ is a}}{\le}} \\ &c\int_0^t \int_{-\frac{1}{n^2}}^{\frac{1}{n^2}} \frac{ds\,du}{\sqrt{2\pi s}} \le \frac{c'}{n^2}\,\sqrt{t} \leftarrow \text{ convergent series in } n\,. \end{aligned} \tag{7.136}$$

It now follows from (7.136) and (7.135) that

(7.137) P-a.s., $\int_0^t \varphi'_n(Y_s)\,dY_s$ converges to B_t, uniformly on bounded time intervals, and $L^n_t \overset{\text{def}}{=} \int_0^t \frac{1}{2}\,\varphi''_n(Y_s)\,ds$ converges uniformly to a continuous non-decreasing process L_t, on bounded time intervals.

Moreover, we have P-a.s., for all $t \ge 0$,

$$|Y_t| = B_t + L_t \;\; \text{(i.e. (7.129) holds).} \tag{7.138}$$

By (7.137) we also see that P-a.s.,

$$\int_0^t \psi(s)\,dL^n_s \underset{n\to\infty}{\longrightarrow} \int_0^t \psi(s)\,dL_s, \text{ for all } t \ge 0, \text{ and continuous bounded } \psi(\cdot) \text{ on } [0,\infty)\,.$$

Applying this fact to $\psi(s) = h(Y_s)$, where h is a continuous bounded function on $\mathbb{R}$, vanishing on some open interval containing 0, yields

$$\begin{aligned} \int_0^t h(Y_s)\,dL_s = \lim_n \int_0^t h(Y_s)\,dL^n_s = \lim_n \int_0^t h(Y_s)\,\frac{1}{2}\,\varphi''_n(Y_s)\,ds \\ \overset{(7.133)}{=} 0\,. \end{aligned}$$

It now follows that

$$P\text{-a.s., } L_t = \int_0^t 1\{Y_s = 0\}\,dL_s, \text{ for all } t \ge 0\,, \tag{7.139}$$

i.e. (7.130) holds. In addition, note that $\varphi''_n(\cdot)$ is symmetric so $L^n_t = \int_0^t \frac{1}{2}\,\varphi''_n(|Y_s|)\,ds$, and (7.137) together with (7.138) imply that

$$F^L_t \subseteq F^{|Y|}_t, \text{ and } F^B_t \subseteq F^{|Y|}_t, \text{ for all } t \geq 0\,. \tag{7.140}$$

There only remains to prove that L necessarily satisfies (7.131), then the claim $F^B_t \supseteq F^{|Y|}_t$ will follow from (7.129), (7.131). For this purpose we will use a deterministic lemma:

Lemma 7.18. *(Skorohod)*

Let $b(\cdot)$ a continuous real-valued function on $[0,\infty)$ such that $b(0) \geq 0$. There exists a unique pair of continuous functions $z(\cdot)$ and $\ell(\cdot)$ on $[0,\infty)$ such that

$$\begin{cases} \text{i)} & z(\cdot) = b(\cdot) + \ell(\cdot)\,, \\ \text{ii)} & z(\cdot) \geq 0\,, \\ \text{iii)} & \ell(\cdot) \textit{ is non-decreasing, } \ell(0) = 0, \textit{ and } d\ell(s) \textit{ is supported by} \\ & \{s \geq 0;\, z(s) = 0\}\,. \end{cases} \tag{7.141}$$

The function $\ell(\cdot)$ is moreover given by

$$\begin{aligned} &\ell(t) = \sup_{s \leq t}\, (b(s))^- \;\; \textit{(where } x^- = (-x) \vee 0) \\ &\textit{(in addition when } b(0) = 0,\ \ell(t) = -\inf_{s \leq t} b(s)\textit{).} \end{aligned} \tag{7.142}$$

Proof. Note first that $z(t) \stackrel{\text{def}}{=} b(t) + \sup_{s \leq t}\,(b(s))^-$, $\ell(t) \stackrel{\text{def}}{=} \sup_{s \leq t}\,(b(s))^-$, $t \geq 0$, satisfy (7.141), i), ii), iii), for iii) note that t such that $z(t) > 0$ does not belong to the support of $d\ell$ (note also that the right-hand side of (7.142) equals $\max(-\inf_{s \leq t} b(s), 0)$).

To prove uniqueness, we consider an other pair $z'(\cdot)$, $\ell'(\cdot)$ of continuous functions on $[0,\infty)$ satisfying (7.141). In particular $z - z' = \ell - \ell'$ is continuous with bounded variation and vanishes at time 0. So,

$$\begin{aligned} 0 \leq (z(t) - z'(t))^2 &= 2 \int_0^t (z(s) - z'(s))\, d(\ell(s) - \ell'(s)) \\ &\stackrel{(7.141)\, iii)}{=} -2 \int_0^t z'(s)\, d\ell(s) - 2 \int_0^t z(s)\, d\ell'(s) \stackrel{(7.141)\, ii), iii)}{\leq} 0\,. \end{aligned}$$

Hence, $z(t) = z'(t)$ and $\ell(t) = \ell'(t)$, for all $t \geq 0$. □

Skorohod's lemma immediately yields (7.131), and as noted above, this concludes the proof of the theorem. □

Corollary 7.19. *For any weak solution of (7.125),*

$$\textit{the law of } (Y,B) \textit{ on } C(\mathbb{R}_+,\mathbb{R})^2 \textit{ is uniquely determined} \tag{7.143}$$

and

$$\textit{for any } t > 0,\ F^B_t = F^{|Y|}_t \subsetneq F^Y_t\,. \tag{7.144}$$

Proof.

• (7.143):

This follows from the fact $Y_{\textstyle .}$ is a $(\mathcal{G}_t)$-Brownian motion and (7.128).

• (7.144):

We note that for $t \geq 0$:

$$\text{sign}(Y_t) \text{ is equidistributed on } \{-1,1\}, \text{ and independent of } |Y_{\textstyle .}|\,. \tag{7.145}$$

Indeed, since $-Y$ has same distribution as Y, we see that for any B in $\mathcal{F}$ (the canonical σ-algebra on $C(\mathbb{R}_+,\mathbb{R})$), one has:

$$\begin{aligned} P[\text{sign}(Y_t) = 1,\, |Y| \in B] = P[\text{sign}(-Y_t) = 1, |-Y| \in B] \underset{P\text{-a.s.}}{\overset{Y_t \neq 0}{=}} \\ P[\text{sign}(Y_t) = -1,\, |Y| \in B] = \frac{1}{2}\, P[|Y| \in B]\,, \end{aligned}$$

and (7.145) follows.

By (7.132), the claim (7.144) now readily follows (if $F_t^{|Y|} = F_t^Y$ was true, we would have $P[\text{sign}(Y_t) = 1 \,|\, F_t^{|Y|}] \overset{P\text{-a.s.}}{=} 1\{\text{sign}(Y_t) = 1\} \overset{(7.145)}{=} \frac{1}{2}$, a contradiction). □

Remark 7.20.

1) We have thus shown that (7.125) is weakly well-posed in the sense that there are weak solutions for (7.125) and the law of (Y,B) for any such weak solution is uniquely determined.

However, there are no strong solutions of (7.125) due to (7.144). Note for instance that if we choose B to be the canonical Brownian motion (i.e. $B_t = X_t$, $t \geq 0$, and $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P) = (C(\mathbb{R}_+,\mathbb{R}), \mathcal{F},\ (F_t)_{t\geq 0}, W_0))$, we cannot find an adapted stochastic process $Y_{\textstyle .}$ such that (7.125) holds.

2) We have in fact shown in (7.129) that given any $(\mathcal{G}_t)$-Brownian motion Y, one has the identity, P-a.s., for all $t \geq 0$:

$$|Y_t| = \int_0^t \text{sign}(Y_s)\, dY_s + L_t \tag{7.146}$$

where $L_0 = 0$, and L_t is a continuous, non-decreasing stochastic process such that $L_t = \int_0^t 1\{Y_s = 0\}\, dL_s$.

This is the so-called Tanaka Formula for the local time of Brownian motion. We refer to Chapter 3 §6 of [8] for more on this matter. □

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