Building polyhedra and a lot of other related structures using double-sided concave hexagonal origami units.

Francesco De Comité

University of Sciences of Lille

Abstract

Modular origami is a branch of origami where one can build complex structures by interlocking elementary folded units (the modules). We will explore the possibilities of one of those modules, an show it can be used in a wide range of different constructions, sometimes by infringing the rules of pure origami (no glue, no scissors, no graduated ruler).

Introduction

A lot of different units have have been proposed by several origamists. The double-sided concave hexagonal ring solid unit was introduced by Tomoko Fuse in her book « Unit Polyhedron Origami ».

She used it to build several archimedean solids (and one platonician solid : the dodecahedron).In all of those solids, the hexagonal module plays the role of the edge, hence only one size of unit is needed (all vertices in platonician and archimedean polyhedra are equals).

We will show that we can play with colours to exhibit some properties of those solids (hamiltonian circuits) and that we can build a large set of related solids (fullerenes, nanotubes, Johnson solids).

Then we will show that we can use units of different sizes to widen the set of buildable solids (Catalan polyhedra, Penrose tiling-like balls).

Folding a hexagonal module

The table on page 2 shows how to fold an elementary unit. The relative sizes of the rectangle sides is not really important, we will see later that they are to be within certain limits. A standard ration is 1:2, easy to obtain by cutting a standard square sheet of paper.

The module has two flaps and two pockets, that will be used to connect several modules together.

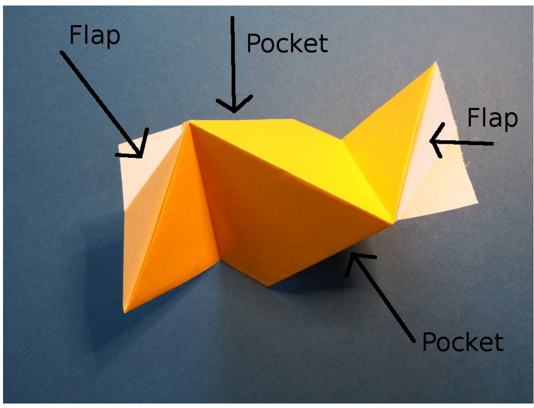
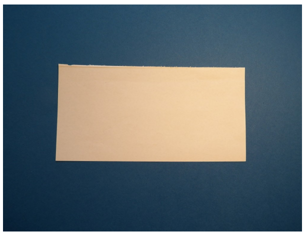
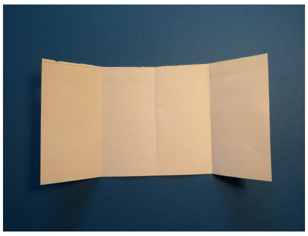
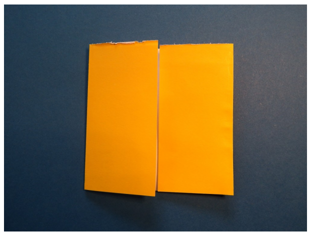
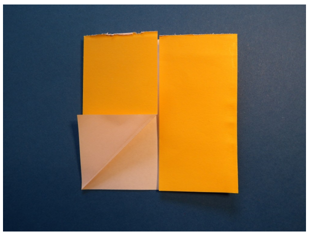
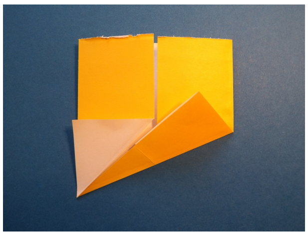
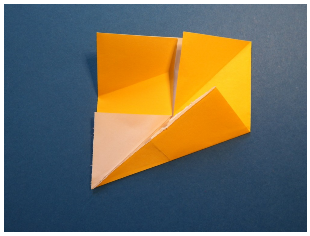
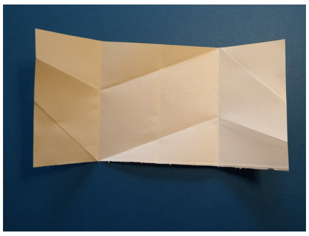
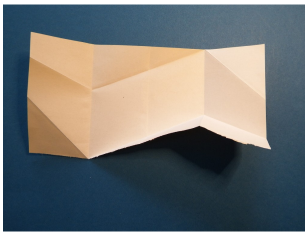
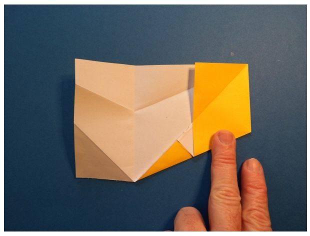
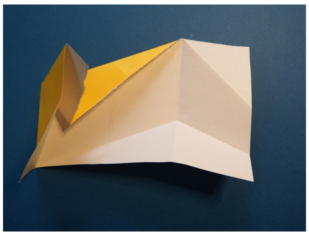
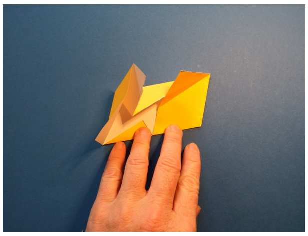
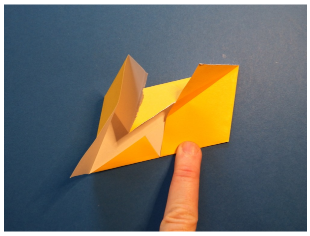
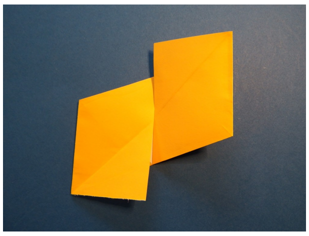
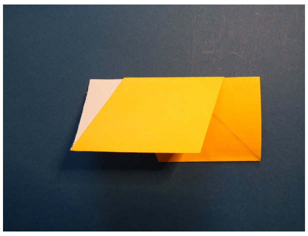
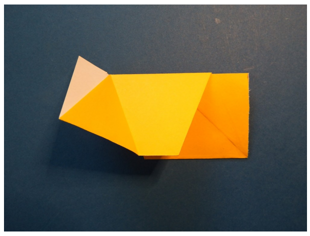
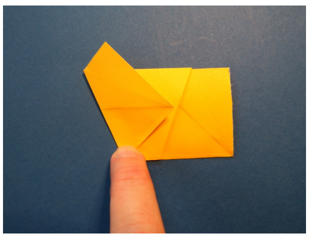
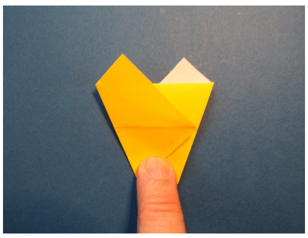
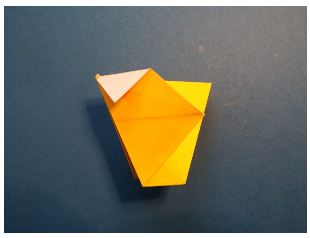


Illustration 1: Flaps and pockets

(see Illustration 1: Flaps and pockets) . Table 1: Joining modules on page 3 shows how to link two or more modules together into a closed ring. Of course, you can link more than three modules : it's not rare to link five or six modules together without problems.

Even if pure origami prohibits the use of glue, a little bit of stick glue can help avoiding the structure you build to fall apart when a lot of modules are already connected: it let the user concentrate on the geometry of his/her work.

What can we build with such units ? Plasticity constraints make it difficult to build platonician solids, except the dodecahedron. But quite all the archimedean solids can be build (to be verified...). An illustrated catalogue of possibilities is detailed at chapter Objects you can build on page 3. Construction is often easier when the modules are made longer, by increasing the ratio length:width of the initial rectangle.

		
Begin with a rectangle	Fold it twice in two	Close it
		
Fold the corner at 45°	Fold the bottom at 22,5°	Idem on the other side
		
Unfold everything	Raise the lower right crease...	...as in this picture
		
Same on the other side	Mark the creases	Mark the creases
		
Until you obtain this	Fold in the middle	Fold the left loose end
		
Like that	Same on the right	Fold the two upper triangles

18 steps for folding a module

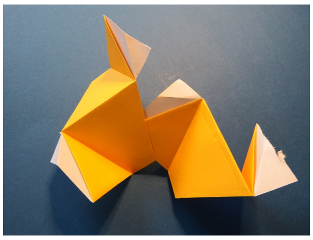
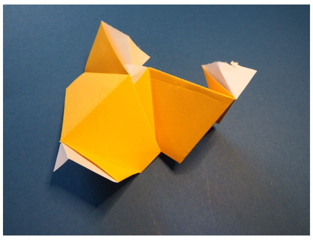
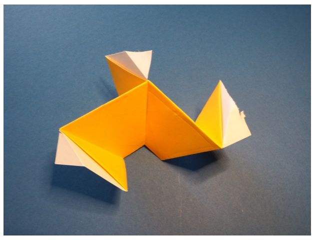
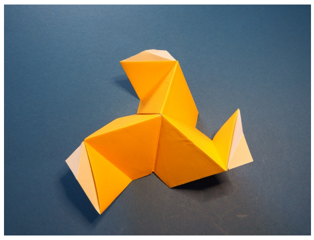
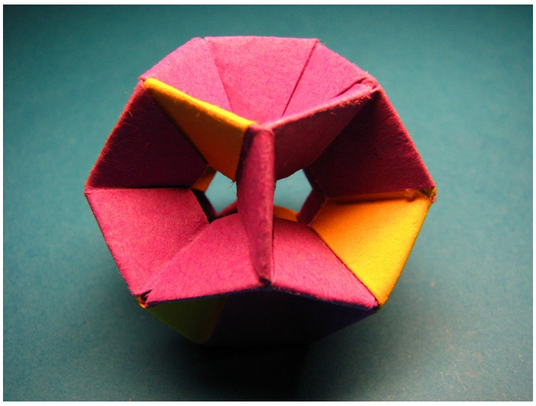
			
Insert a flap in a pocket	Insert the little triangle in the other part of the pocket.	Mark the creases firmly.	Continue with another module and close the ring.

Table 1: Joining modules

Objects you can build

Platonician solids

As said above, physical constraints make it difficult, if not impossible, to build any of the regular solids except the dodecahedron.



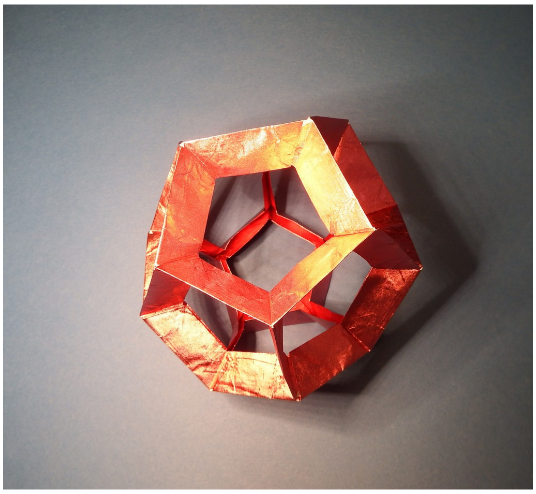
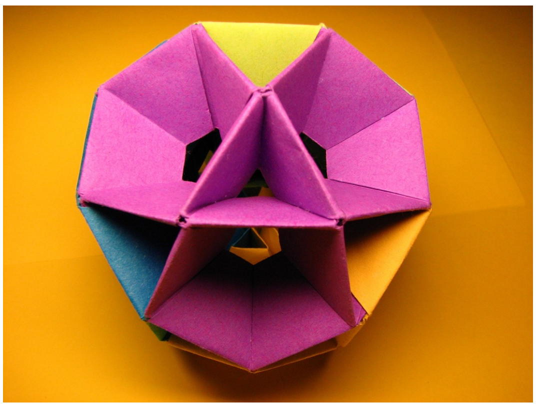


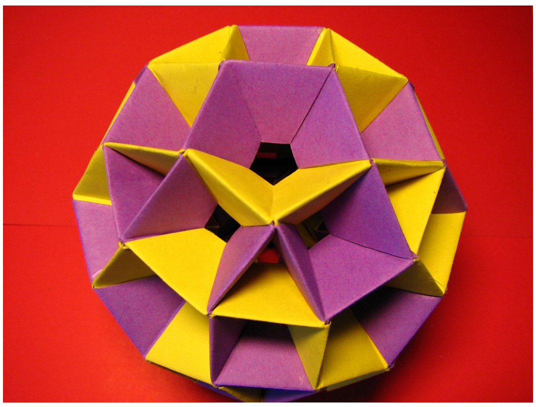
Illustration 2 and 3 show two versions of the dodecahedron, with different sizes for the edges.

Archimedean solids

Here again, the only limit are the physical rules of the real world. One thing that makes modular origami interesting is how you can feel in your hands the physical constraints, when you try to assemble modules which don't want to meet each other !

I never studied rigorously which solid one can assemble or not. Illustrations 4 to 7 demonstrate that





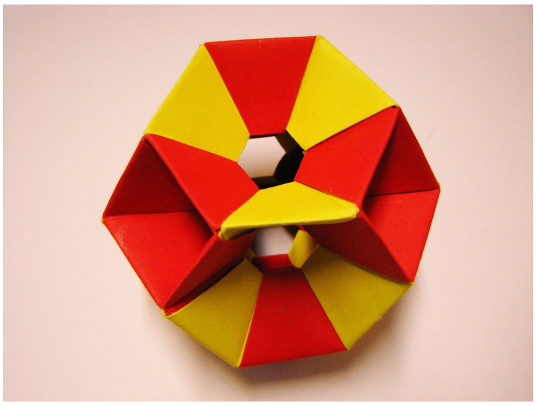


Illustration 6: Truncated octahedron

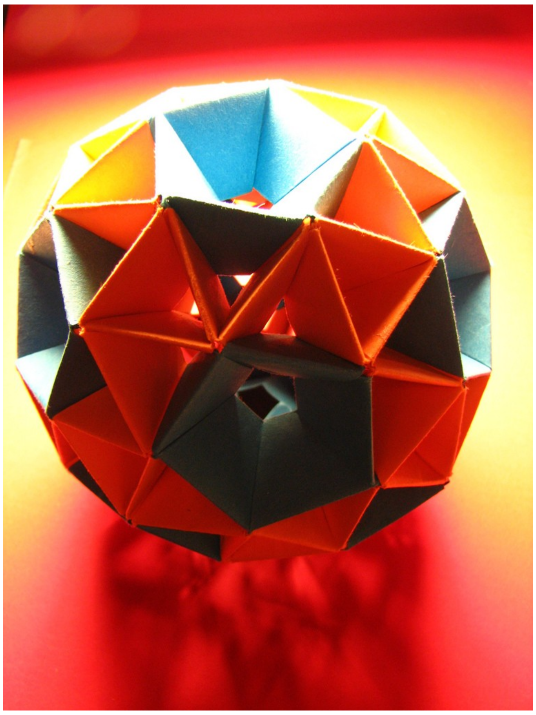


Illustration 7: Snub dodecahedron

at least 4 of them are possible. Illustration 5 shows an interesting feature : the purple edges are following a hamiltonian path on the surface of the rhombicosidodecahedron. Even if we know that every archimedean solid has a hamiltonian path, it is not easy to find. But once you got one, building the corresponding modular solid is quite easy, and give you an illustrative mathematical model at no cost.

Fullerenes

Since fullerenes are based on regular polygons (pentagons and hexagons), they are also candidates for a modular origami construction. If nanotubes are easy to build, a new physical reality imposes its law : gravity. Big models begin to be heavy, and can overcome the strength of the paper. A solution is either to make thinner edges, or to make smaller models. The second solution should be better, since the weight of an edge varies like the third power of its size. Another limiting factor is human : patience. But planning to build huge models can be the occasion to define a team work !

Illustration 8: Fullerene C240

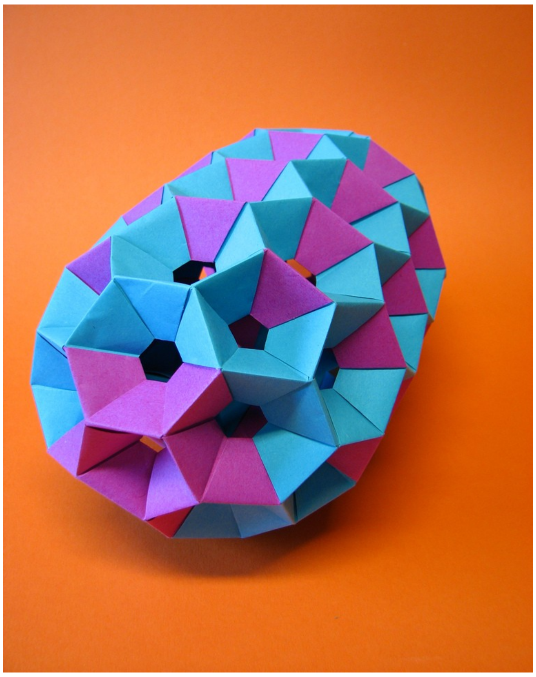


Illustration 9: A short nanotube

In general, it's not a good idea to build separately big parts of the model and join them together at the end : big parts are heavy, and the flaps are fragile. Trying to assemble big parts often results in damaging the flaps and pockets one try to assemble all at the same time. By experience, it's better to have one construction, where one add modules one by one. This implies to have clearly in mind the geometry of the object. Using different colours might help.

Penrose-like tiling

My first idea was to use hexagonal units to build planar Penrose tilings, which can be made with edges all of the same size. But, while making my first experiment, I realized that the structure I was building is not at all planar. This is due to the fact that both parts of an edge are not of the same size.

But every structure that has a five-fold symmetry can be put on the faces of a dodecahedron to produce a spatial symmetric structure. I was then able to recycle my Penrose tiling into a quite complex three-dimensional object.

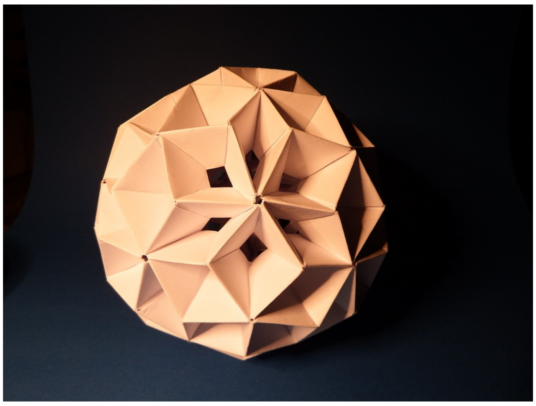


Illustration 10: A simple Penrose-like structure. The five-fold motif is clearly visible.

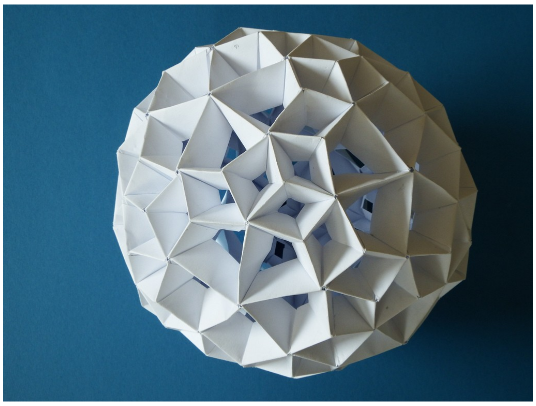


Illustration 11: A more complicated Penrose-like tiling : can you find the motif ?

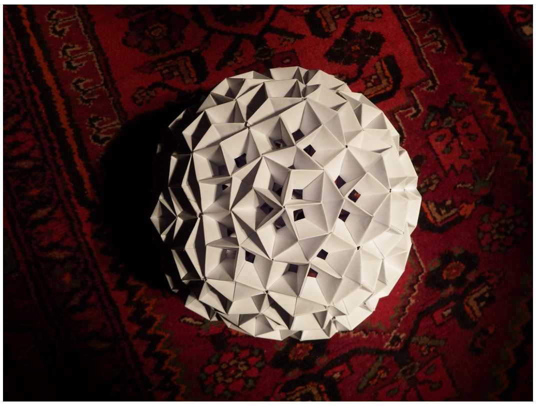


Illustration 13: A big Penrose-like tiling

The model in Illustration 12 is at the limit of heaviness problem : while it poses no problem during the construction (apart from patience), it is beginning to collapse under its own weight.

Illustration 12: A four-fold motif on a cubic symmetry

Modules of different sizes

The geometry and measures of the module

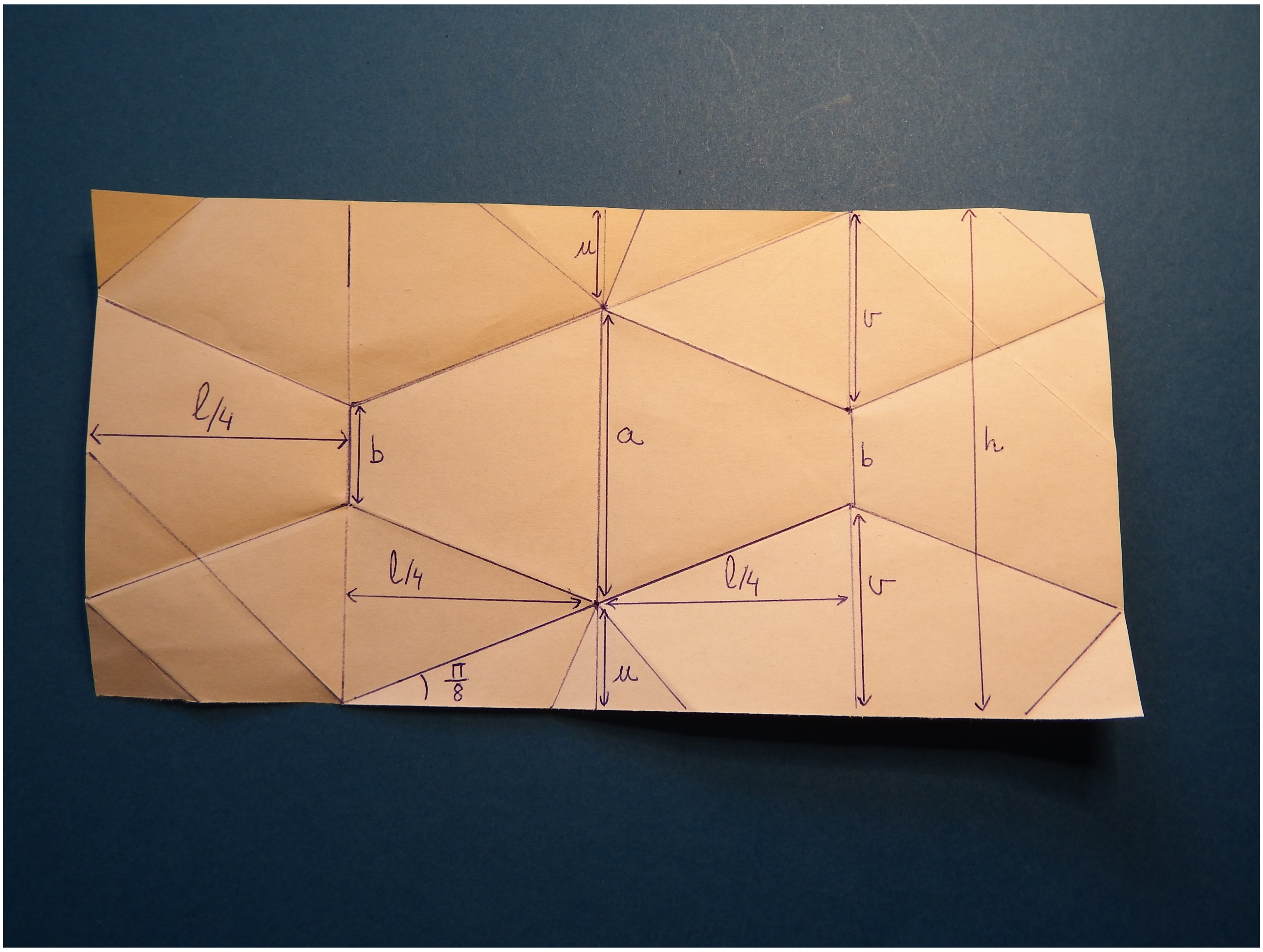


Illustration 14: Naming the creases and their measures and angles

The illustration above shows an unfolded unit. When folded, the unit has three important measures : a, b, and l/4. When we work with equal units, we don't matter too much about those quantities, except for controlling the thickness of the edges (namely l/4). If we want to design and work with edges of different sizes, we have to control those values. Since we want the units to join perfectly, we must maintain the same value of l/4, for each size of module: hence the length of modules will not change. The only thing we can modify then is h, the height of the rectangle. How a and b are related to h ? Some elementary trigonometry is necessary :

$$a = h - 2\mathrm{u}$$

$$\tan\left(\frac{\pi}{8}\right) = \frac{u}{\frac{L}{4}} = \frac{4u}{L}$$

$$u = \frac{L}{4}\tan\left(\frac{\pi}{8}\right) = (\sqrt{2}-1)\frac{L}{4}$$

$$a = h - \frac{L}{2}(\sqrt{2}-1)$$

Following the same logic, it is easy to express b as a function of h and L :

$$b = h - L(\sqrt{2}-1)$$

This last formula gives the minimal value of h, since b must be non-negative.

Then if we want two different units of respective length a_1 and a_2 , such that $a_2 = k\,a_1$ and a_1 is known, we must set h_2 to :

$$h_2 = k\,h_1 - \frac{L}{2}(\sqrt{(2)} - 1)(k - 1)$$

Units of different sizes : application

The Catalan solids are the dual of the archimedean solids. All their faces are equal, but the length of their edges are different. As for the archimedean polyhedra, it is possible to build a large part of the Catalan solids set.

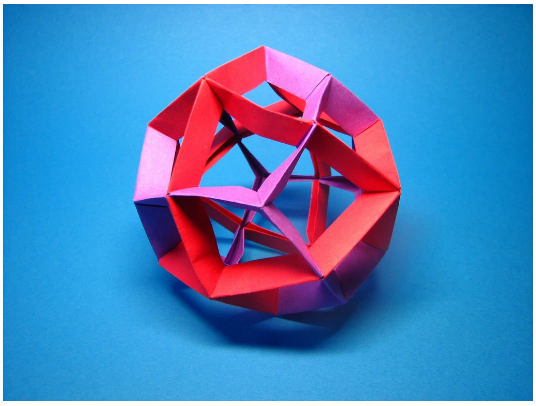


Illustration 15: Kite hexecontahedron

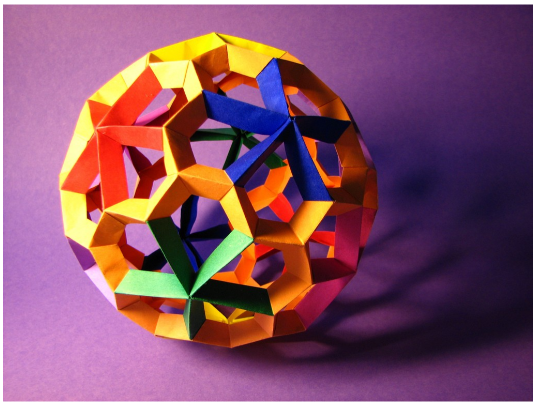


Illustration 16: Pentagonal hexecontahedron

Illustration 15: Kite hexecontahedron And Illustration 16: Pentagonal hexecontahedron show two examples of modular structures build on this principle. The exact dimensions of the edges can be found in any good geometry book, or on the internet, and is left to the reader.

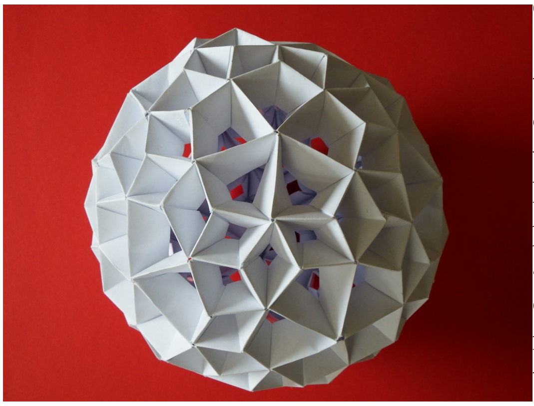


Illustration 17: Pentagonal hexecontahedron variation

One can also decide to build those polyhedra not by showing their edges, but different paths from one vertice to the “center” of the face (this center can be defined in different ways). This leads to still another rich variation, which is illustrated on Illustration 17: Pentagonal hexecontahedron variation, based on the geometry of the pentagonal hexecontahedron. Note that this variation is also appliable to the archimedean solids, and since one can choose its own definition of the “center” of a face, it gives raise to an infinity of variations, which might be worth trying virtually before to begin folding.

Conclusion

The hexagonal concave unit is an easy to fold easy to assemble modular origami unit, which allows to build sophisticated geometric structures, from standard polyhedra to more exotic ones. With a little patience, one can obtain real-world objects, lead tangible experiments at the fraction of the cost and work implied by standard model making, like 3D printing, for example. It can be used in classrooms, since the need of a lot of modules can make the folding a group activity, and I think it's valuable for children to be able to conduct an experiment from the very beginning (imagine the structure), then the construction itself (fold the paper) and the final result (assemble the units).