



GOVERNMENT OF TAMIL NADU

**BUSINESS MATHEMATICS
AND
STATISTICS**

HIGHER SECONDARY FIRST YEAR

VOLUME - I

Untouchability is Inhuman and a Crime

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HOW TO USE THE BOOK

Career Options

List of Further Studies & Professions.

Learning Objectives:

Learning objectives are brief statements that describe what students will be expected to learn by the end of school year, course, unit, lesson or class period.



Additional information about the concept.



Amazing facts, Rhetorical questions to lead students to Mathematical inquiry

Exercise

Assess students' critical thinking and their understanding



ICT

To enhance digital skills among students

Web links

List of digital resources



To motivate the students to further explore the content digitally and take them in to virtual world

Miscellaneous Problems

Additional problems for the students

Glossary

Tamil translation of Mathematical terms

References

List of related books for further studies of the topic



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CAREER OPTIONS IN BUSINESS MATHEMATICS

Higher Secondary students who have taken commerce with Business mathematics can take up careers in BCA, B.Com., and B.Sc. Statistics. Students who have taken up commerce stream, have a good future in banking and financial institutions.

A lot of students choose to do B.Com with a specialization in computers. Higher Secondary Commerce students planning for further studies can take up careers in professional fields such as Company Secretary, Chartered Accountant (CA), ICAI and so on. Others can take up bachelor's degree in commerce (B.Com), followed by M.Com, Ph.D and M.Phil. There are wide range of career opportunities for B.Com graduates.

After graduation in commerce, one can choose MBA, MA Economics, MA Operational and Research Statistics at Postgraduate level. Apart from these, there are several diploma, certificate and vocational courses which provide entry level jobs in the field of commerce.

Career chart for Higher Secondary students who have taken commerce with Business Mathematics.

Courses	Institutions	Scope for further studies
B.Com., B.B.A., B.B.M., B.C.A., B.Com (Computer), B.A.	<ul style="list-style-type: none"> Government Arts & Science Colleges, Aided Colleges, Self financing Colleges. Shri Ram College of Commerce (SRCC), Delhi Symbiosis Society's College of Arts & Commerce, Pune. St. Joseph's College, Bangalore 	C.A., I.C.W.A, C.S.
B.Sc Statistics	<ul style="list-style-type: none"> Presidency College, Chepauk, Chennai. Madras Christian College, Tambaram Loyola College, Chennai. D.R.B.C.C Hindu College, Pattabiram, Chennai. 	M.Sc., Statistics
B.B.A., LLB, B.A., LLB, B.Com., LL.B. (Five years integrated Course)	<ul style="list-style-type: none"> Government Law College. School of excellence, Affiliated to Dr.Ambethkar Law University 	M.L.
M.A. Economics (Integrated Five Year course) – Admission based on All India Entrance Examination	<ul style="list-style-type: none"> Madras School of Economics, Kotturpuram, Chennai. 	Ph.D.,
B.S.W.	<ul style="list-style-type: none"> School of Social studies, Egmore, Chennai 	M.S.W

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Assessment



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SYLLABUS

1.	MATRICES AND DETERMINANTS	(15 periods)
	<p>Determinants: Recall - Types of matrices - Determinants Minors – Cofactors - Properties of determinants. Inverse of a matrix: Singular matrix – Non singular matrix - Adjoint of a matrix – Inverse of a matrix - Solution of a system of linear equation. Input - Output analysis: Hawkins Simon conditions.</p>	
2.	ALGEBRA	(20 periods)
	<p>Partial fractions: Denominator contains non - repeated Linear factors - Denominator contains Linear factors, repeated for n times - Denominator contains quadratic factor, which cannot be factorized into linear factors. Permutations: Factorial - Fundamental principle of counting – Permutation - Circular permutation. Combinations - Mathematical Induction - Binomial Theorem.</p>	
3.	ANALYTICAL GEOMETRY	(20 periods)
	<p>Locus: Equation of a locus. System of straight lines: Recall – Various forms of straight lines - Angle between two straight lines – Distance of a point from a line - Concurrence of three lines. Pair of straight lines: Combined equation of pair of straight lines - Pair of straight lines passing through the origin - Angle between pair of straight lines passing through the origin - The condition for general second degree equation to represent the pair of straight lines. Circles: The equation of a circle when the centre and radius are given - Equation of a circle when the end points of a diameter are given - General equation of a circle - Parametric form of a circle – Tangents – Length of tangent to the circle. CONICS: Parabola.</p>	
4.	TRIGONOMETRY	(25 periods)
	<p>Trigonometric ratios: Quadrants - Signs of the trigonometric ratios of an angle θ as θ varies from 0° to 360° - Trigonometric ratios of allied angles. Trigonometric ratios of compound angles: Compound angles - Sum and difference formulae of sine, cosine and tangent - Trigonometric ratios of multiple angles. Transformation formulae: Transformation of the products into sum or difference - Transformation of sum or difference into product. Inverse Trigonometric Functions: Properties of Inverse Trigonometric Functions.</p>	
5.	DIFFERENTIAL CALCULUS	(25 periods)
	<p>Functions and their graphs: Some basic concepts – Function - Various types of functions - Graph of a function. Limits and derivatives: Existence of limit - Indeterminate forms and evaluation of limits – Continuous function - Differentiability at a point - Differentiation from first principle. Differentiation techniques: Some standard results - General rules for differentiation - Successive differentiation.</p>	

MATRICES AND DETERMINANTS



Learning Objectives

After studying this chapter, the students will be able to understand

- the definition of matrices and determinants
- the properties of determinants
- the concept of inverse matrix
- the concept of adjoint matrix
- the solving simultaneous linear equations
- the input-output analysis



1.1 Determinants

Introduction

The idea of a determinant was believed to be originated from a Japanese Mathematician Seki Kowa (1683) while systematizing the old Chinese method of solving simultaneous equations whose coefficients were represented by calculating bamboos or sticks. Later the German Mathematician Gottfried Wilhelm Von Leibnitz formally developed determinants. The present vertical notation was given in 1841 by Arthur Cayley. Determinant was invented independently by Cramer whose well known rule for solving simultaneous equations was published in 1750.



Seki Kowa

In class X, we have studied matrices and algebra of matrices. We have also learnt that a system of algebraic equations can be expressed in the form of matrices. We know that the area of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

To minimize the difficulty in remembering this type of expression, Mathematicians developed the idea of representing the expression in determinant form and the above expression can be represented in the form

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus a determinant is a particular type of expression written in a special concise form. Note that the quantities are arranged in the form of a square between two vertical lines. This arrangement is called a determinant.

In this chapter, we study determinants up to order three only with real entries. Also we shall study various properties of determinant (without proof), minors, cofactors, adjoint, inverse of a square matrix and business applications of determinants.

We have studied matrices in the previous class. Let us recall the basic concepts and operations on matrices.

1.1.1 Recall

Matrix

Definition 1.1

A matrix is a rectangular arrangement of numbers in horizontal lines (rows) and vertical lines (columns). Numbers are enclosed in square brackets or a open brackets or pair of double bars. It is denoted by A, B, C, \dots

For example , $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 8 & 6 \end{bmatrix}$

Order of a matrix

If a matrix A has m rows and n columns, then A is called a matrix of order $m \times n$.

For example, If , $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 8 & 6 \end{bmatrix}$ then order of A is 2×3

General form of a Matrix

Matrix of order $m \times n$ is represented as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1i} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2i} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mi} & \cdots & \cdots & a_{mn} \end{bmatrix}$$

It is shortly written as $[a_{ij}]_{m \times n}$ $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ Here a_{ij} is the element in the i^{th} row and j^{th} column of the matrix.

Types of matrices

Row matrix

A matrix having only one row is called a row matrix.

For example $A = [a_{11} \ a_{12} \ a_{13} \ \dots \ a_{1i} \ \dots \ a_{1n}]_{1 \times n}$; $B = [1 \ 2]_{1 \times 2}$

Column matrix

A matrix having only one column is called a column matrix

For examples, $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix}_{m \times 1}$, $B = \begin{bmatrix} 3 \\ -5 \end{bmatrix}_{2 \times 1}$

Zero matrix (or) Null matrix

If all the elements of a matrix are zero, then it is called a zero matrix. It is represented by an English alphabet 'O'

For examples, $O = [0]$, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are all zero matrices.

Square matrix

If the number of rows and number of columns of a matrix are equal then it is called a square matrix.

For examples, $A = \begin{bmatrix} 8 & 2 \\ 4 & 2 \end{bmatrix}$ is a square matrix of order 2

$B = \begin{bmatrix} 3 & -1 & 0 \\ \sqrt{2} & 0 & 5 \\ 2 & \frac{3}{5} & -4 \end{bmatrix}$ is a square matrix of order 3.

Triangular matrix

A square matrix, whose elements above or below the main diagonal (leading diagonal) are all zero is called a triangular matrix.

For examples, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 7 \end{bmatrix}$



NOTE

Zero matrix can be of any order.



NOTE

The sum of the diagonal elements of a square matrix is called the **trace** of matrix.

Diagonal matrix

A square matrix in which all the elements other than the main diagonal elements are zero is called the diagonal matrix.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix of order 3

Scalar matrix

A diagonal matrix with all diagonal elements are equal to K (a scalar) is called a scalar matrix.

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix of order 3.

Unit matrix (or) Identity matrix

A scalar matrix having each diagonal element equal to one (unity) is called a Unit matrix.

For examples, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

Multiplication of a matrix by a scalar

If $A = [a_{ij}]$ is a matrix of any order and if k is a scalar, then the scalar multiplication of A by the scalar k is defined as $kA = [ka_{ij}]$ for all i, j .

In other words to multiply a matrix A by a scalar k , multiply every element of A by k .

For example, if $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & -2 & 8 \end{bmatrix}$ then, $2A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & -4 & 16 \end{bmatrix}$

Negative of a matrix

The negative of a matrix $A = [a_{ij}]_{m \times n}$ is defined as $-A = [-a_{ij}]_{m \times n}$ for all i, j and is obtained by changing the sign of every element.

For example, if $A = \begin{bmatrix} 2 & -5 & 7 \\ 0 & 5 & 6 \end{bmatrix}$ then $-A = \begin{bmatrix} -2 & 5 & -7 \\ 0 & -5 & -6 \end{bmatrix}$

Equality of matrices

Two matrices A and B are said to be equal if

(i) they have the same order and (ii) their corresponding elements are equal.

Addition and subtraction of matrices

Two matrices A and B can be added, provided both the matrices are of the same order. Their sum $A+B$ is obtained by adding the corresponding entries of both the matrices A and B .

Symbolically if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, then $A + B = [a_{ij} + b_{ij}]_{m \times n}$

Similarly $A - B = A + (-B) = [a_{ij}]_{m \times n} + [-b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$

Multiplication of matrices

Multiplication of two matrices is possible only when the number of columns of the first matrix is equal to the number of rows of the second matrix. The product of matrices A and B is obtained by multiplying every row of matrix A with the corresponding elements of every column of matrix B element-wise and add the results.

Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [b_{ij}]$ be a $p \times n$ matrix, then the product AB is a matrix $C = [c_{ij}]$ of order $m \times n$.

Transpose of a matrix

Let $A = [a_{ij}]$ be a matrix of order $m \times n$. The transpose of A , denoted by A^T of order $n \times m$ is obtained by interchanging either rows into columns or columns into rows of A .

For example, if $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 5 & 6 \end{bmatrix}$



NOTE

It is believed that the students might be familiar with the above concepts and our present syllabus continues from the following.

Definition 1.2

To every square matrix A of order n with entries as real or complex numbers, we can associate a number called **determinant** of matrix A and it is denoted by $|A|$ or $\det(A)$ or Δ .

Thus determinant can be formed by the elements of the matrix A .

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$

Example 1.1

Evaluate: $\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix}$

Solution

$$\begin{vmatrix} 2 & 4 \\ -1 & 4 \end{vmatrix} = (2)(4) - (-1)4 \\ = 8 + 4 = 12$$

To evaluate the determinant of order 3 or more, we define minors and cofactors.

1.1.2 Minors:

Let $|A| = |[a_{ij}]|$ be a determinant of order n . The minor of an arbitrary element a_{ij} is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij} .

1.1.3 Cofactors:

The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as

$$A_{ij} = (-1)^{i+j} M_{ij}$$

The minors and cofactors of a_{11}, a_{12}, a_{13} in a third order determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ are as follows

(i) Minor of a_{11} is $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

Cofactor of a_{11} is $A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}$

(ii) Minor of a_{12} is $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}$

Cofactor of a_{12} is $A_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = - (a_{21}a_{33} - a_{31}a_{23})$

(iii) Minor of a_{13} is $M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Cofactor of a_{13} is $A_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{31}a_{22}$

Example 1.2

Find the minor and cofactor of all the elements in the determinant $\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$

Solution

$$\text{Minor of 1} = M_{11} = 3$$

$$\text{Minor of } -2 = M_{12} = 4$$

$$\text{Minor of 4} = M_{21} = -2$$

$$\text{Minor of 3} = M_{22} = 1$$

$$\text{Cofactor of 1} = A_{11} = (-1)^{1+1}M_{11} = 3$$

$$\text{Cofactor of } -2 = A_{12} = (-1)^{1+2}M_{12} = -4$$

$$\text{Cofactor of 4} = A_{21} = (-1)^{2+1}M_{21} = 2$$

$$\text{Cofactor of 3} = A_{22} = (-1)^{2+2}M_{22} = 1$$

Example 1.3

Find the minor and cofactor of each element of the determinant. $\begin{vmatrix} 3 & 1 & 2 \\ 2 & 2 & 5 \\ 4 & 1 & 0 \end{vmatrix}$

Solution

$$\text{Minor of 3 is } M_{11} = \begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = 0 - 5 = -5$$

$$\text{Minor of 1 is } M_{12} = \begin{vmatrix} 2 & 5 \\ 4 & 0 \end{vmatrix} = 0 - 20 = -20$$

$$\text{Minor of 2 is } M_{13} = \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} = 2 - 8 = -6$$

$$\text{Minor of 2 is } M_{21} = \begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 0 - 2 = -2$$

$$\text{Minor of 2 is } M_{22} = \begin{vmatrix} 3 & 2 \\ 4 & 0 \end{vmatrix} = 0 - 8 = -8$$

$$\text{Minor of 5 is } M_{23} = \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = 3 - 4 = -1$$

$$\text{Minor of 4 is } M_{31} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$$

$$\text{Minor of 1 is } M_{32} = \begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} = 15 - 4 = 11$$

$$\text{Minor of 0 is } M_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} = 6 - 2 = 4$$

$$\text{Cofactor of 3 is } A_{11} = (-1)^{1+1}M_{11} = M_{11} = -5$$

$$\text{Cofactor of 1 is } A_{12} = (-1)^{1+2}M_{12} = -M_{12} = 20$$

$$\text{Cofactor of 2 is } A_{13} = (-1)^{1+3} M_{13} = M_{13} = -6$$

$$\text{Cofactor of 2 is } A_{21} = (-1)^{2+1} M_{21} = -M_{21} = 2$$

$$\text{Cofactor of 2 is } A_{22} = (-1)^{2+2} M_{22} = M_{22} = -8$$

$$\text{Cofactor of 5 is } A_{23} = (-1)^{2+3} M_{23} = -M_{23} = 1$$

$$\text{Cofactor of 4 is } A_{31} = (-1)^{3+1} M_{31} = M_{31} = 1$$

$$\text{Cofactor of 1 is } A_{32} = (-1)^{3+2} M_{32} = -M_{32} = -11$$

$$\text{Cofactor of 0 is } A_{33} = (-1)^{3+3} M_{33} = M_{33} = 4$$

For example

$$\text{If } \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad (\text{or}) \quad a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad (\text{expanding along } R_1)$$

$$\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \quad (\text{or}) \quad a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \quad (\text{expanding along } C_1)$$



NOTE

Value of a determinant can be obtained by using any row or column

Example 1.4

$$\text{Evaluate: } \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$$

Solution

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix} &= 1 (\text{Minor of 1}) - 2 (\text{Minor of 2}) + 4 (\text{Minor of 4}) \\ &= 1 \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} -1 & 0 \\ 4 & 0 \end{vmatrix} + 4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} \\ &= 0 - 0 - 52 = -52. \end{aligned}$$

1.1.4 Properties of determinants (without proof)

1. The value of a determinant is unaltered when its rows and columns are interchanged
2. If any two rows (columns) of a determinant are interchanged then the value of the determinant changes only in sign.
3. If the determinant has two identical rows (columns) then the value of the determinant is zero.

4. If all the elements in a row (column) of a determinant are multiplied by constant k , then the value of the determinant is multiplied by k .
5. If any two rows (columns) of a determinant are proportional then the value of the determinant is zero.
6. If each element in a row (column) of a determinant is expressed as the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants of the same order.
7. The value of the determinant is unaltered when a constant multiple of the elements of any row (column) is added to the corresponding elements of a different row (column) in a determinant.

Example 1.5

Show that
$$\begin{vmatrix} x & y & z \\ 2x + 2a & 2y + 2b & 2z + 2c \\ a & b & c \end{vmatrix} = 0$$

Solution

$$\begin{aligned} \begin{vmatrix} x & y & z \\ 2x + 2a & 2y + 2b & 2z + 2c \\ a & b & c \end{vmatrix} &= \begin{vmatrix} x & y & z \\ 2x & 2y & 2z \\ a & b & c \end{vmatrix} + \begin{vmatrix} x & y & z \\ 2a & 2b & 2c \\ a & b & c \end{vmatrix} \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Example 1.6

Evaluate
$$\begin{vmatrix} x & x + 1 \\ x - 1 & x \end{vmatrix}$$

Solution

$$\begin{aligned} \begin{vmatrix} x & x + 1 \\ x - 1 & x \end{vmatrix} &= x^2 - (x - 1)(x + 1) \\ &= x^2 - (x^2 - 1) \\ &= x^2 - x^2 + 1 = 1 \end{aligned}$$

Example 1.7

Solve
$$\begin{vmatrix} x - 1 & x & x - 2 \\ 0 & x - 2 & x - 3 \\ 0 & 0 & x - 3 \end{vmatrix} = 0$$

Solution

$$\begin{vmatrix} x - 1 & x & x - 2 \\ 0 & x - 2 & x - 3 \\ 0 & 0 & x - 3 \end{vmatrix} = 0$$



$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 11 & 13 \end{pmatrix}$$

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 7 & 8 \end{vmatrix} \neq \begin{vmatrix} 4 & 8 \\ 11 & 13 \end{vmatrix}$$



NOTE

The value of the determinant of triangular matrix is equal to the product of the main diagonal elements.

$$\Rightarrow (x-1)(x-2)(x-3) = 0$$

$$x = 1, x = 2, x = 3$$

Example 1.8

$$\text{Evaluate } \begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 3 & 4 \\ 102 & 18 & 36 \\ 17 & 3 & 6 \end{vmatrix} = 6 \begin{vmatrix} 1 & 3 & 4 \\ 17 & 3 & 6 \\ 17 & 3 & 6 \end{vmatrix} \\ = 0 \quad (\text{since } R_2 \equiv R_3)$$

Example 1.9

$$\text{Evaluate } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} \begin{matrix} R_1 \rightarrow R_1 - R_2 \\ R_2 \rightarrow R_2 - R_3 \end{matrix} \\ = \begin{vmatrix} 0 & a-b & (a-b)(a+b) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix} \\ = (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} \\ = (a-b)(b-c)[0-0+\{b+c-(a+b)\}] \\ = (a-b)(b-c)(c-a).$$



Exercise 1.1

- Find the minors and cofactors of all the elements of the following determinants.

$$(i) \begin{vmatrix} 5 & 20 \\ 0 & -1 \end{vmatrix} \quad (ii) \begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

- Evaluate $\begin{vmatrix} 3 & -2 & 4 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

3. Solve: $\begin{vmatrix} 2 & x & 3 \\ 4 & 1 & 6 \\ 1 & 2 & 7 \end{vmatrix} = 0$
4. Find $|AB|$ if $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & -2 \end{bmatrix}$
5. Solve: $\begin{vmatrix} 7 & 4 & 11 \\ -3 & 5 & x \\ -x & 3 & 1 \end{vmatrix} = 0$
6. Evaluate: $\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$
7. Prove that $\begin{vmatrix} \frac{1}{a} & bc & b+c \\ \frac{1}{b} & ca & c+a \\ \frac{1}{c} & ab & a+b \end{vmatrix} = 0$
8. Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

1.2 Inverse of a matrix

1.2.1 Singular matrix :

A Square matrix A is said to be singular, if $|A| = 0$

1.2.2 Non – singular matrix:

A square matrix A is said to be non – singular, if $|A| \neq 0$

Example 1.10

Show that $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a singular matrix.

Solution

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= 4 - 4 = 0$$

$\therefore A$ is a singular matrix.

Example 1.11

Show that $\begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$ is non – singular.

Solution

$$\text{Let } A = \begin{bmatrix} 8 & 2 \\ 4 & 3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 8 & 2 \\ 4 & 3 \end{vmatrix}$$

$$= 24 - 8 = 16$$

$$\neq 0$$

$\therefore A$ is a non-singular matrix



If A and B are non – singular matrices of the same order then AB and BA are also non – singular matrices of the same order.

1.2.3 Adjoint of a Matrix

The adjoint of a square matrix A is defined as the transpose of a cofactor matrix. Adjoint of the matrix A is denoted by $adj A$

i.e., $adj A = [A_{ij}]^T$ where $[A_{ij}]$ is the cofactor matrix of A .

Example 1.12

Find $adj A$ for $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$adj A = [A_{ij}]^T$$

$$= \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Example 1.13

Find adjoint of $A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 0 \\ -4 & 1 & 0 \end{bmatrix}$

Solution

$$A_{ij} = (-1)^{i+j} M_{ij}$$

$$A_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$$

**NOTE**

- (i) $|adj A| = |A|^{n-1}$ n is the order of the matrix A
- (ii) $|kA| = k^n |A|$ n is the order of the matrix A
- (iii) $adj(kA) = k^{n-1} adj A$ n is the order of the matrix A
- (iv) $A(adj A) = (adj A)A = |A| I$
- (v) $Adj I = I$, I is the unit matrix.
- (vi) $adj(AB) = (adj B)(adj A)$
- (vii) $|AB| = |A| |B|$

$$A_{12} = (-1)^{1+2}M_{12} = -\begin{vmatrix} 0 & 0 \\ -4 & 0 \end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3}M_{13} = \begin{vmatrix} 0 & 1 \\ -4 & 1 \end{vmatrix} = 0 - (-4) = 4$$

$$A_{21} = (-1)^{2+1}M_{21} = -\begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = -(0 - (-3)) = -3$$

$$A_{22} = (-1)^{2+2}M_{22} = \begin{vmatrix} 1 & -3 \\ -4 & 0 \end{vmatrix} = 0 - 12 = -12$$

$$A_{23} = (-1)^{2+3}M_{23} = -\begin{vmatrix} 1 & -2 \\ -4 & 1 \end{vmatrix} = -(1 - 8) = 7$$

$$A_{31} = (-1)^{3+1}M_{31} = \begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = 0 - (-3) = 3$$

$$A_{32} = (-1)^{3+2}M_{32} = -\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = 0 - 0 = 0$$

$$A_{33} = (-1)^{3+3}M_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$$

$$[A_{ij}] = \begin{bmatrix} 0 & 0 & 4 \\ -3 & -12 & 7 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\text{Adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 0 & -3 & 3 \\ 0 & -12 & 0 \\ 4 & 7 & 1 \end{bmatrix}$$

1.2.4 Inverse of a matrix

Let A be any non-singular matrix of order n . If there exists a square matrix B of order n such that $AB = BA = I$ then, B is called the inverse of A and is denoted by A^{-1}

$$\text{We have } A (\text{adj } A) = (\text{adj } A)A = |A| I$$

$$\implies A \left(\frac{1}{|A|} \text{adj } A \right) = \left(\frac{1}{|A|} \text{adj } A \right) A = I$$

$$\implies AB = BA = I \text{ where } B = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$



NOTE

(i) If B is the inverse of A then, A is the inverse of B

$$\text{i.e., } B = A^{-1} \Rightarrow A = B^{-1}$$

(ii) $AA^{-1} = A^{-1}A = I$

(iii) The inverse of a matrix, if it exists, is unique.

(iv) Order of inverse of A will be the same as that of order of A .

(v) $I^{-1} = I$, where I is the identity matrix.

(vi) $(AB)^{-1} = B^{-1}A^{-1}$ (vii) $|A^{-1}| = \frac{1}{|A|}$



Let A be a non-singular matrix, then

$$A^2 = I \Leftrightarrow A = A^{-1}$$

Example 1.14

If $A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$ then, find A^{-1} .

Solution

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 4 \\ -3 & 2 \end{vmatrix}$$

$$= 16 \neq 0$$

Since A is a nonsingular matrix, A^{-1} exists

$$\text{Now } \text{adj } A = \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{16} \begin{bmatrix} 2 & -4 \\ 3 & 2 \end{bmatrix}$$

Example 1.15

If $A = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$ then, find A^{-1}

Solution

$$|A| = \begin{vmatrix} -2 & 6 \\ 3 & -9 \end{vmatrix} = 0$$

Since A is a singular matrix, A^{-1} does not exist.

Example 1.16

If $A = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 5 & 3 \end{bmatrix}$ then find A^{-1}

Solution

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 4 & 4 \\ 2 & 5 & 4 \\ 2 & 5 & 3 \end{vmatrix} \\ &= 2 \begin{vmatrix} 5 & 4 \\ 5 & 3 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} \\ &= 2(15 - 20) - 4(6 - 8) + 4(10 - 10) \\ &= 2(-5) - 4(-2) + 4(0) \\ &= -10 + 8 + 0 = -2 \neq 0 \end{aligned}$$

Since A is a nonsingular matrix, A^{-1} exists.

$$A_{11} = -5; A_{21} = 8; A_{31} = -4$$

$$A_{12} = 2; A_{22} = -2; A_{32} = 0$$

$$A_{13} = 0; A_{23} = -2; A_{33} = 2$$

$$[A_{ij}] = \begin{bmatrix} -5 & 2 & 0 \\ 8 & -2 & -2 \\ -4 & 0 & 2 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} -5 & 8 & -4 \\ 2 & -2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{-2} \begin{bmatrix} -5 & 8 & -4 \\ 2 & -2 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{2} & -4 & 2 \\ -1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

Example 1.17

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - kA + I_2 = O$ then, find k and also A^{-1} .

Solution

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^2 = A.A$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$\text{Given } A^2 - kA + I_2 = O$$

$$\Rightarrow \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - k \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 - 2k + 1 & 12 - 3k + 0 \\ 4 - k + 0 & 7 - 2k + 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 - 2k & 12 - 3k \\ 4 - k & 8 - 2k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 4 - k = 0$$

$$\Rightarrow k = 4.$$

$$\begin{aligned} \text{Also } |A| &= \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \\ &= 4 - 3 = 1 \neq 0 \end{aligned}$$

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Example 1.18

If $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$ then, show that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution

$$|A| = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \neq 0$$

$\therefore A^{-1}$ exists.

$$|B| = \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} = 0 + 1 = 1 \neq 0$$

B^{-1} also exists.

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\begin{aligned} |AB| &= \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ &= -1 \neq 0 \end{aligned}$$

$(AB)^{-1}$ exists.

$$\text{adj}(AB) = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{|AB|} \text{adj}(AB)$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \dots (1)$$

$$\text{adj } A = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{-1} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\text{adj } B = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{adj } B$$

$$= \frac{1}{1} \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \quad \dots (2)$$

From (1) and (2), $(AB)^{-1} = B^{-1}A^{-1}$. Hence proved.

Example 1.19

Show that the matrices $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix}$ are inverses of each other.

Solution

$$AB = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix}$$

$$= \frac{35}{35} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\begin{aligned}
 BA &= \begin{bmatrix} \frac{-4}{35} & \frac{11}{35} & \frac{-5}{35} \\ \frac{-1}{35} & \frac{-6}{35} & \frac{25}{35} \\ \frac{6}{35} & \frac{1}{35} & \frac{-10}{35} \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \\
 &= \frac{1}{35} \begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \\
 &= \frac{1}{35} \begin{bmatrix} 35 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 35 \end{bmatrix} \\
 &= \frac{35}{35} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
 \end{aligned}$$

Thus $AB = BA = I$

Therefore A and B are inverses of each other.



Exercise 1.2

- Find the adjoint of the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$
- If $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then verify that $A (\text{adj } A) = |A| I$ and also find A^{-1}
- Find the inverse of each of the following matrices
 - $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$
 - $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$
 - $\begin{bmatrix} -3 & -5 & 4 \\ -2 & 3 & -1 \\ 1 & -4 & -6 \end{bmatrix}$
- If $A = \begin{bmatrix} 2 & 3 \\ 1 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 4 \\ 1 & -2 \end{bmatrix}$, then verify $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- If $A = \begin{bmatrix} 2 & -2 & 2 \\ 2 & 3 & 0 \\ 9 & 1 & 5 \end{bmatrix}$ then, show that $(\text{adj } A)A = O$
- If $A = \begin{bmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ 4 & -4 & 5 \end{bmatrix}$ then, show that the inverse of A is A itself.
- If $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ then, find A .

8. Show that the matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$ are inverses of each other.
9. If $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$ then, verify that $(AB)^{-1} = B^{-1}A^{-1}$
10. Find λ if the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 2 & \lambda & 4 \\ 9 & 7 & 11 \end{bmatrix}$ has no inverse.
11. If $X = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$ and $Y = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & p & q \end{bmatrix}$ then, find p, q if $Y = X^{-1}$

1.2.5 Solution of a system of linear equations

Consider a system of n linear non – homogeneous equations with n unknowns x_1, x_2, \dots, x_n as

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

.....

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

Thus we get the matrix equation $AX = B$.(. 1)

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

From (1) solution is

$$\boxed{X = A^{-1}B}$$

Example 1.20

Solve by using matrix inversion method:

$$2x + 5y = 1$$

$$3x + 2y = 7$$

Solution

The given system can be written as

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e., $AX = B$

$$X = A^{-1}B$$

where $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$\begin{aligned} |A| &= \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} \\ &= -11 \neq 0 \end{aligned}$$

A^{-1} exists.

$$\text{adj } A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{|A|} \text{adj } A \\ &= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \end{bmatrix} \\ &= -\frac{1}{11} \begin{bmatrix} -33 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$x = 3 \text{ and } y = -1.$$

Example 1.21

Solve by using matrix inversion method:

$$3x - 2y + 3z = 8; 2x + y - z = 1; 4x - 3y + 2z = 4$$

Solution

The given system can be written as

$$\begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

i.e., $AX = B$

$$X = A^{-1}B$$

Here $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

$$\begin{aligned} |A| &= \begin{vmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{vmatrix} \\ &= -17 \neq 0 \end{aligned}$$

A^{-1} exists

$$A_{11} = -1 \quad A_{12} = -8 \quad A_{13} = -10$$

$$A_{21} = -5 \quad A_{22} = -6 \quad A_{23} = 1$$

$$A_{31} = -1 \quad A_{32} = 9 \quad A_{33} = 7$$

$$[A_{ij}] = \begin{bmatrix} -1 & -8 & -10 \\ -5 & -6 & 1 \\ -1 & 9 & 7 \end{bmatrix}$$

$$\begin{aligned} \text{adj } A &= [A_{ij}]^T \\ &= \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \end{aligned}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= -\frac{1}{17} \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$= -\frac{1}{17} \begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$x = 1, y = 2$ and $z = 3$.

Example 1.22

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is ₹320. The cost of 2kg onion, 4 kg wheat and 6 kg rice is ₹560. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is ₹380. Find the cost of each item per kg by matrix inversion method.

Solution

Let x, y and z be the cost of onion, wheat and rice per kg respectively.

$$\text{Then, } 4x + 3y + 2z = 320$$

$$2x + 4y + 6z = 560$$

$$6x + 2y + 3z = 380.$$

It can be written as

$$\begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$AX = B$$

$$\text{where } A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 50 \neq 0$$

A^{-1} exist

$$A_{11} = 0 \quad A_{12} = 30 \quad A_{13} = -20$$

$$A_{21} = -5 \quad A_{22} = 0 \quad A_{23} = 10$$

$$A_{31} = 10 \quad A_{32} = -20 \quad A_{33} = 10$$

$$[A_{ij}] = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$$

$$\text{adj } A = [A_{ij}]^T$$

$$= \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{50} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 320 \\ 560 \\ 380 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix} \begin{bmatrix} 32 \\ 56 \\ 38 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 - 280 + 380 \\ 960 + 0 - 760 \\ -640 + 560 + 380 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$$

$x = 20$, $y = 40$ and $z = 60$.

Cost of 1 kg onion is ₹20, cost of 1 kg wheat is ₹40 and cost of 1 kg rice is ₹60 .



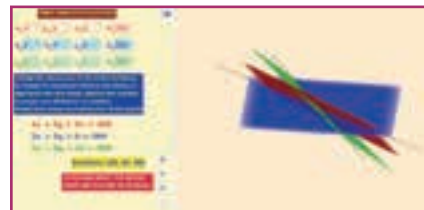
ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

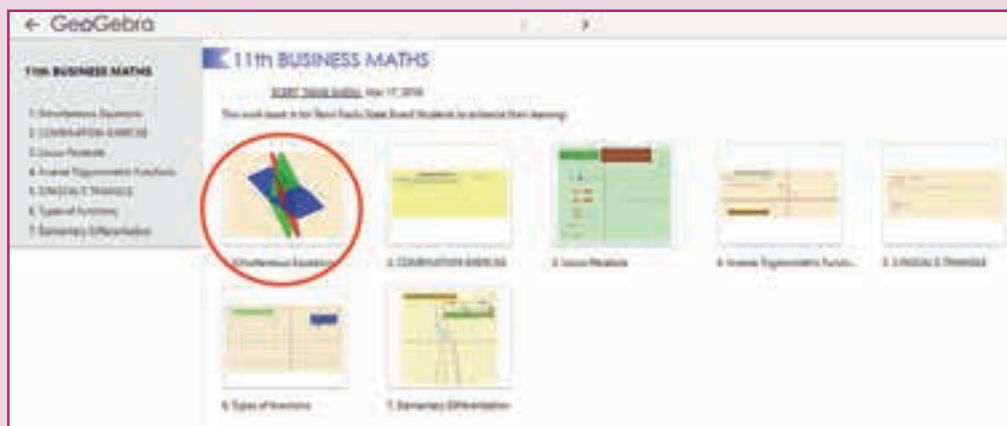
GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Simultaneous Equations”



Step - 2

Type the parameters for three simultaneous equations in the respective boxes. You will get the solutions for the equations. Also, you can see the planes for the equations and the intersection point (Solution) on the right-hand side. Solve yourself and check the answer. If the planes do not intersect then there is no solution.

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code





Exercise 1.3

- Solve by matrix inversion method : $2x + 3y - 5 = 0$; $x - 2y + 1 = 0$
- Solve by matrix inversion method :
 - $3x - y + 2z = 13$; $2x + y - z = 3$; $x + 3y - 5z = -8$
 - $x - y + 2z = 3$; $2x + z = 1$; $3x + 2y + z = 4$.
 - $2x - z = 0$; $5x + y = 4$; $y + 3z = 5$
- A sales person Ravi has the following record of sales for the month of January, February and March 2009 for three products A, B and C He has been paid a commission at fixed rate per unit but at varying rates for products A, B and C

Months	Sales in Units			Commission
	A	B	C	
January	9	10	2	800
February	15	5	4	900
March	6	10	3	850

Find the rate of commission payable on A, B and C per unit sold using matrix inversion method.

- The prices of three commodities A, B and C are ₹ x , ₹ y and ₹ z per unit respectively. P purchases 4 units of C and sells 3 units of A and 5 units of B. Q purchases 3 units of B and sells 2 units of A and 1 unit of C. R purchases 1 unit of A and sells 4 units of B and 6 units of C. In the process P, Q and R earn ₹6,000 , ₹5,000 and ₹13,000 respectively. By using matrix inversion method, find the prices per unit of A, B and C.
- The sum of three numbers is 20. If we multiply the first by 2 and add the second number and subtract the third we get 23. If we multiply the first by 3 and add second and third to it, we get 46. By using matrix inversion method find the numbers.
- Weekly expenditure in an office for three weeks is given as follows. Assuming that the salary in all the three weeks of different categories of staff did not vary, calculate the salary for each type of staff, using matrix inversion method.

week	Number of employees			Total weekly Salary (in rupees)
	A	B	C	
1st week	4	2	3	4900
2 nd week	3	3	2	4500
3 rd week	4	3	4	5800

1.3 Input – output Analysis

Input–Output analysis is a technique which was invented by Prof. Wassily W. Leontief. Input Output analysis is a form of economic analysis based on the interdependencies between economic sectors. The method is most commonly used for estimating the impacts of positive or negative economic shocks and analyzing the ripple effects throughout an economy.

The foundation of Input - Output analysis involves input – output tables. Such tables include a series of rows and columns of data that quantify the supply chain for sectors of the economy. Industries are listed in the heads of each row and each column. The data in each column corresponds to the level of inputs used in that industry's production function. For example the column for auto manufacturing shows the resources required for building automobiles (ie., requirement of steel, aluminum, plastic, electronic etc.,). Input – Output models typically includes separate tables showing the amount of labour required per rupee unit of investment or production.

Consider a simple economic model consisting of two industries A_1 and A_2 where each produces only one type of product. Assume that each industry consumes part of its own output and rest from the other industry for its operation. The industries are thus interdependent. Further assume that whatever is produced that is consumed. That is the total output of each industry must be such as to meet its own demand, the demand of the other industry and the external demand (final demand).

Our aim is to determine the output levels of each of the two industries in order to meet a change in final demand, based on knowledge of the current outputs of the two industries, of course under the assumption that the structure of the economy does not change.

Let a_{ij} be the rupee value of the output of A_i consumed by A_j , $i, j = 1, 2$

Let x_1 and x_2 be the rupee value of the current outputs of A_1 and A_2 respectively.

Let d_1 and d_2 be the rupee value of the final demands for the outputs of A_1 and A_2 respectively.

The assumptions lead us to frame the two equations

$$a_{11}x_1 + a_{12}x_2 + d_1 = x_1; \quad a_{21}x_1 + a_{22}x_2 + d_2 = x_2 \quad \dots (1)$$

Let $b_{ij} = \frac{a_{ij}}{x_j}$, $i, j = 1, 2$

$$b_{11} = \frac{a_{11}}{x_1}; \quad b_{12} = \frac{a_{12}}{x_2}; \quad b_{21} = \frac{a_{21}}{x_1}; \quad b_{22} = \frac{a_{22}}{x_2}$$

The equations (1) take the form

$$b_{11}x_1 + b_{12}x_2 + d_1 = x_1 \qquad b_{21}x_1 + b_{22}x_2 + d_2 = x_2$$

The above equations can be rearranged as

$$(1 - b_{11})x_1 - b_{12}x_2 = d_1$$

$$-b_{21}x_1 + (1 - b_{22})x_2 = d_2$$

The matrix form of the above equations is

$$\begin{bmatrix} 1 - b_{11} & -b_{12} \\ -b_{21} & 1 - b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$(I - B)X = D$$

$$\text{where } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

By solving we get

$$X = (I - B)^{-1}D$$

The matrix B is known as the *Technology matrix*.

1.3.1 The Hawkins – Simon conditions

Hawkins – Simon conditions ensure the viability of the system.

If B is the technology matrix then Hawkins – Simon conditions are

- (i) the main diagonal elements in $I - B$ must be positive and
- (ii) $|I - B|$ must be positive.

Example 1.23

The technology matrix of an economic system of two industries is

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}. \text{ Test whether the system is viable as per Hawkins – Simon conditions.}$$

Solution

$$B = \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.8 & 0.2 \\ 0.9 & 0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{vmatrix}$$

$$= (0.2)(0.3) - (-0.2)(-0.9)$$

$$= 0.06 - 0.18$$

$$= -0.12 < 0$$

Since $|I - B|$ is negative, Hawkins – Simon conditions are not satisfied.

Therefore, the given system is not viable.

Example 1.24

The following inter – industry transactions table was constructed for an economy of the year 2016.

Industry	1	2	Final consumption	Total output
1	500	1,600	400	2,500
2	1,750	1,600	4,650	8,000
Labours	250	4,800	-	-

Construct technology co-efficient matrix showing direct requirements. Does a solution exist for this system.

Solution

$$a_{11} = 500 \quad a_{12} = 1600 \quad x_1 = 2500$$

$$a_{21} = 1750 \quad a_{22} = 1600 \quad x_2 = 8000$$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{500}{2500} = 0.20; \quad b_{12} = \frac{a_{12}}{x_2} = \frac{1600}{8000} = 0.20$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{1750}{2500} = 0.70; \quad b_{22} = \frac{a_{22}}{x_2} = \frac{1600}{8000} = 0.20$$

The technology matrix is $= \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.2 & 0.2 \\ 0.7 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{bmatrix}, \text{ elements of main diagonal are positive}$$

$$\text{Now, } |I - B| = \begin{vmatrix} 0.8 & -0.2 \\ -0.7 & 0.8 \end{vmatrix}$$

$$= (0.8)(0.8) - (-0.7)(-0.2)$$

$$= 0.64 - 0.14$$

$$= 0.50 > 0$$

Since diagonal elements of are positive and $|I - B|$ is positive,

Hawkins – Simon conditions are satisfied

Hence this system has a solution.

Example 1.25

In an economy there are two industries P_1 and P_2 and the following table gives the supply and the demand position in crores of rupees.

Production sector	Consumption sector		Final demand	Gross output
	P_1	P_2		
P_1	10	25	15	50
P_2	20	30	10	60

Determine the outputs when the final demand changes to 35 for P_1 and 42 for P_2 .

Solution

$$a_{11} = 10 \quad a_{12} = 25 \quad x_1 = 50$$

$$a_{21} = 20 \quad a_{22} = 30 \quad x_2 = 60$$

$$b_{11} = \frac{a_{11}}{x_1} = \frac{10}{50} = \frac{1}{5}; \quad b_{12} = \frac{a_{12}}{x_2} = \frac{25}{60} = \frac{5}{12}$$

$$b_{21} = \frac{a_{21}}{x_1} = \frac{20}{50} = \frac{2}{5}; \quad b_{22} = \frac{a_{22}}{x_2} = \frac{30}{60} = \frac{1}{2}$$

$$\begin{aligned} \text{The technology matrix is } B &= \begin{bmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{bmatrix} \\ I - B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{5}{12} \\ \frac{2}{5} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{bmatrix}, \text{ elements of main diagonal are positive} \\ \text{Now, } |I - B| &= \begin{vmatrix} \frac{4}{5} & -\frac{5}{12} \\ -\frac{2}{5} & \frac{1}{2} \end{vmatrix} \\ &= \frac{7}{30} > 0 \end{aligned}$$

Main diagonal elements of $|I - B|$ are positive and $|I - B|$ is positive

\therefore The problem has a solution

$$\begin{aligned} \text{adj}(I - B) &= \begin{bmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} \\ (I - B)^{-1} &= \frac{1}{|I - B|} \text{adj}(I - B) \text{ (Since } |I - B| \neq 0, (I - B)^{-1} \text{ exist)} \\ &= \frac{1}{\frac{7}{30}} \text{adj}(I - B) \\ &= \frac{30}{7} \begin{bmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 15 & 25 \\ 12 & 24 \end{bmatrix} \\ X &= (I - B)^{-1}D, \text{ where } D = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 15 & \frac{12}{2} \\ 12 & 24 \end{bmatrix} \begin{bmatrix} 35 \\ 42 \end{bmatrix} = \begin{bmatrix} 150 \\ 204 \end{bmatrix} \end{aligned}$$

The output of industry P_1 should be ₹150 crores and P_2 should be ₹204 crores.

Example 1.26

An economy produces only coal and steel. These two commodities serve as intermediate inputs in each other's production. 0.4 tonne of steel and 0.7 tonne of coal are needed to produce a tonne of steel. Similarly 0.1 tonne of steel and 0.6 tonne of coal are required to produce a tonne of coal. No capital inputs are needed. Do you think that

the system is viable? 2 and 5 labour days are required to produce a tonne of coal and steel respectively. If economy needs 100 tonnes of coal and 50 tonnes of steel, calculate the gross output of the two commodities and the total labour days required.

Solution

Here the technology matrix is given under

	Steel	Coal	Final demand
Steel	0.4	0.1	50
Coal	0.7	0.6	100
Labour days	5	2	-

The technology matrix is $B = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$

$$I - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{bmatrix}$$

$$|I - B| = \begin{vmatrix} 0.6 & -0.1 \\ -0.7 & 0.4 \end{vmatrix}$$

$$= (0.6)(0.4) - (-0.7)(-0.1)$$

$$= 0.24 - 0.07 = 0.17$$

Since the diagonal elements of $I - B$ are positive and value of $|I - B|$ is positive, the system is viable.

$$\text{adj}(I - B) = \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

$$(I - B)^{-1} = \frac{1}{|I - B|} \text{adj}(I - B)$$

$$= \frac{1}{0.17} \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix}$$

$$X = (I - B)^{-1}D, \text{ where } D = \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$$= \frac{1}{0.17} \begin{bmatrix} 0.4 & 0.1 \\ 0.7 & 0.6 \end{bmatrix} \begin{bmatrix} 50 \\ 100 \end{bmatrix}$$

$$= \frac{1}{0.17} \begin{bmatrix} 30 \\ 95 \end{bmatrix}$$

$$= \begin{bmatrix} 176.5 \\ 558.8 \end{bmatrix}$$

Steel output = 176.5 tonnes

Coal output = 558.8 tonnes

Total labour days required

$$= 5(\text{steel output}) + 2(\text{coal output})$$

$$= 5(176.5) + 2(558.8)$$

$$= 882.5 + 1117.6 = 2000.1$$

$$\simeq 2000 \text{ labour days.}$$



Exercise 1.4

- The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.30 \\ 0.41 & 0.33 \end{bmatrix}$. Test whether the system is viable as per Hawkins Simon conditions.
- The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.6 & 0.9 \\ 0.20 & 0.80 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.
- The technology matrix of an economic system of two industries is $\begin{bmatrix} 0.50 & 0.25 \\ 0.40 & 0.67 \end{bmatrix}$. Test whether the system is viable as per Hawkins-Simon conditions.
- Two commodities A and B are produced such that 0.4 tonne of A and 0.7 tonne of B are required to produce a tonne of A . Similarly 0.1 tonne of A and 0.7 tonne of B are needed to produce a tonne of B . Write down the technology matrix. If 6.8 tonnes of A and 10.2 tonnes of B are required, find the gross production of both of them.
- Suppose the inter-industry flow of the product of two industries are given as under.

Production sector	Consumption sector		Domestic demand	Total output
	X	Y		
X	30	40	50	120
Y	20	10	30	60

Determine the technology matrix and test Hawkin's -Simon conditions for the viability of the system. If the domestic demand changes to 80 and 40 units respectively, what should be the gross output of each sector in order to meet the new demands.

6. You are given the following transaction matrix for a two sector economy.

Sector	Sales		Final demand	Gross output
	1	2		
1	4	3	13	20
2	5	4	3	12

- (i) Write the technology matrix
- (ii) Determine the output when the final demand for the output sector 1 alone increases to 23 units.
7. Suppose the inter-industry flow of the product of two sectors X and Y are given as under.

Production sector	Consumption Sector		Domestic demand	Gross output
	X	Y		
X	15	10	10	35
Y	20	30	15	65

Find the gross output when the domestic demand changes to 12 for X and 18 for Y.



Exercise 1.5



Choose the correct answer



1. The value of x if $\begin{vmatrix} 0 & 1 & 0 \\ x & 2 & x \\ 1 & 3 & x \end{vmatrix} = 0$ is
- (a) 0, -1 (b) 0, 1 (c) -1, 1 (d) -1, -1
2. The value of $\begin{vmatrix} 2x+y & x & y \\ 2y+z & y & z \\ 2z+x & z & x \end{vmatrix}$ is
- (a) xyz (b) $x + y + z$ (c) $2x + 2y + 2z$ (d) 0
3. The cofactor of -7 in the determinant $\begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ is
- (a) -18 (b) 18 (c) -7 (d) 7

4. If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}$ then $\begin{vmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \end{vmatrix}$ is
- (a) Δ (b) $-\Delta$ (c) 3Δ (d) -3Δ
5. The value of the determinant $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}^2$ is
- (a) abc (b) 0 (c) $a^2 b^2 c^2$ (d) $-abc$
6. If A is square matrix of order 3 then $|kA|$ is
- (a) $k|A|$ (b) $-k|A|$ (c) $k^3|A|$ (d) $-k^3|A|$
7. $adj(AB)$ is equal to
- (a) $adjA adjB$ (b) $adjA^T adjB^T$ (c) $adjB adjA$ (d) $adjB^T adjA^T$
8. The inverse matrix of $\begin{pmatrix} \frac{4}{5} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{2} \end{pmatrix}$ is
- (a) $\frac{7}{30} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ (b) $\frac{7}{30} \begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{1}{5} \end{pmatrix}$ (c) $\frac{30}{7} \begin{pmatrix} \frac{1}{2} & \frac{5}{12} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}$ (d) $\frac{30}{7} \begin{pmatrix} \frac{1}{2} & \frac{-5}{12} \\ \frac{-2}{5} & \frac{4}{5} \end{pmatrix}$
9. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$ then A^{-1} is
- (a) $\frac{1}{ad - bc} \begin{pmatrix} d & b \\ -c & a \end{pmatrix}$ (b) $\frac{1}{ad - bc} \begin{pmatrix} d & b \\ c & a \end{pmatrix}$
- (c) $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ (d) $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ c & a \end{pmatrix}$
10. The number of Hawkins-Simon conditions for the viability of an input-output analysis is
- (a) 1 (b) 3 (c) 4 (d) 2
11. The inventor of input - output analysis is
- (a) Sir Francis Galton (b) Fisher
- (c) Prof. Wassily W. Leontief (d) Arthur Caylay

12. Which of the following matrix has no inverse

(a) $\begin{pmatrix} -1 & 1 \\ 1 & -4 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix}$

(d) $\begin{pmatrix} \sin a & \sin a \\ -\cos a & \cos a \end{pmatrix}$

13. The inverse matrix of $\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$ is

(a) $\begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}$

(b) $\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & -1 \\ -5 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix}$

14. If $A = \begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}$ then $A(adjA)$ is

(a) $\begin{pmatrix} -4 & -2 \\ -1 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & -2 \\ -1 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

15. If A and B non-singular matrix then, which of the following is incorrect?

(a) $A^2 = I$ implies $A^{-1} = A$

(b) $I^{-1} = I$

(c) If $AX = B$ then $X = B^{-1}A$

(d) If A is square matrix of order 3 then $|adj A| = |A|^2$

16. The value of $\begin{vmatrix} 5 & 5 & 5 \\ 4x & 4y & 4z \\ -3x & -3y & -3z \end{vmatrix}$ is

(a) 5

(b) 4

(c) 0

(d) 3

17. If A is an invertible matrix of order 2 then $\det(A^{-1})$ be equal to

(a) $\det(A)$

(b) $\frac{1}{\det(A)}$

(c) 1

(d) 0

18. If A is 3×3 matrix and $|A| = 4$ then $|A^{-1}|$ is equal to

(a) $\frac{1}{4}$

(b) $\frac{1}{16}$

(c) 2

(d) 4

19. If A is a square matrix of order 3 and $|A| = 3$ then $|adjA|$ is equal to

(a) 81

(b) 27

(c) 3

(d) 9

20. The value of $\begin{vmatrix} x & x^2 - yz & 1 \\ y & y^2 - zx & 1 \\ x & z^2 - xy & 1 \end{vmatrix}$ is
 (a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $-xyz$
21. If $A = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$, then $|2A|$ is equal to
 (a) $4 \cos 2\theta$ (b) 4 (c) 2 (d) 1
22. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is cofactor of a_{ij} , then value of Δ is given by
 (a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
 (c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$
23. If $\begin{vmatrix} x & 2 \\ 8 & 5 \end{vmatrix} = 0$ then the value of x is
 (a) $\frac{-5}{6}$ (b) $\frac{5}{6}$ (c) $\frac{-16}{5}$ (d) $\frac{16}{5}$
24. If $\begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} = -5$ then the value of $\begin{vmatrix} 20 & 15 \\ 15 & 5 \end{vmatrix}$ is
 (a) -5 (b) -125 (c) -25 (d) 0
25. If any three rows or columns of a determinant are identical then the value of the determinant is
 (a) 0 (b) 2 (c) 1 (d) 3

Miscellaneous Problems

1. Solve: $\begin{vmatrix} x & 2 & -1 \\ 2 & 5 & x \\ -1 & 2 & x \end{vmatrix} = 0$
2. Evaluate $\begin{vmatrix} 10041 & 10042 & 10043 \\ 10045 & 10046 & 10047 \\ 10049 & 10050 & 10051 \end{vmatrix}$
3. Without actual expansion show that the value of the determinant $\begin{vmatrix} 5 & 5^2 & 5^3 \\ 5^2 & 5^3 & 5^4 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is zero.
4. Show that $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$.

5. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 4 & 7 \\ 1 & -1 & 1 \end{bmatrix}$ verify that $A(\text{adj}A) = (\text{adj}A)A = |A|I_3$.
6. If $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ then, find the Inverse of A.
7. If $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ show that $A^2 - 4A + 5I_2 = O$ and also find A^{-1}
8. Solve by using matrix inversion method : $x - y + z = 2, 2x - y = 0, 2y - z = 1$.
9. The cost of 2 Kg of Wheat and 1 Kg of Sugar is ₹70. The cost of 1 Kg of Wheat and 1 Kg of Rice is ₹70. The cost of 3 Kg of Wheat, 2 Kg of Sugar and 1 Kg of rice is ₹170. Find the cost of per kg each item using matrix inversion method.
10. The data are about an economy of two industries A and B. The values are in crores of rupees.

Producer	User		Final demand	Total output
	A	B		
A	50	75	75	200
B	100	50	50	200

Find the output when the final demand changes to 300 for A and 600 for B

Summary

- Determinant is a number associated to a square matrix.
- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then its $|A| = a_{11}a_{22} - a_{21}a_{12}$
- Minor of an arbitrary element a_{ij} of the determinant of the matrix is the determinant obtained by deleting the i^{th} row and j^{th} column in which the element a_{ij} stands. The minor of a_{ij} is denoted by M_{ij} .
- The cofactor is a signed minor. The cofactor of a_{ij} is denoted by A_{ij} and is defined as $A_{ij} = (-1)^{i+j}M_{ij}$.
- Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then
 $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$ or $a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

- $\text{adj } A = [A_{ij}]^T$ where $[A_{ij}]$ is the cofactor matrix of the given matrix.
- $|\text{adj } A| = |A|^{n-1}$, where n is the order of the matrix A .
- $A(\text{adj } A) = (\text{adj } A)(A) = |A| I$.
- $\text{adj } I = I$, I is the unit Matrix.
- $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$.
- A square matrix A is said to be singular, if $|A| = 0$.
- A square matrix A is said to be non-singular if $|A| \neq 0$
- Let A be any square matrix of order n and I be the identity matrix of order n . If there exists a square matrix B of order n such that $AB = BA = I$ then, B is called the inverse of A and is denoted by A^{-1}
- Inverse of A (if it exists) is $A^{-1} = \frac{1}{|A|} \text{adj } A$.
- Hawkins - Simon conditions ensure the viability of the system. If B is the technology matrix then Hawkins-Simon conditions are
 - (i) the main diagonal elements in $I - B$ must be positive and
 - (ii) $|I - B|$ must be positive.

GLOSSARY

Adjoint Matrix	சேர்ப்பு அணி
Analysis	பகுப்பாய்வு
Cofactor	இணைக் காரணி
Determinant	அணிக்கோவை
Digonal Matrix	மூலை விட்ட அணி
Input	உள்ளீடு
Inverse Matrix	நேர்மாறு அணி
Minors	சிற்றணிக் கோவைகள்
Non-Singular Matrix	பூச்சியமற்றக் கோவை அணி
Output	வெளியீடு
Scalar	திசையிலி
Singular Matrix	பூச்சியக் கோவை அணி
Transpose of a Matrix	நிரை நிரல் மாற்று அணி
Triangular Matrix	முக்கோண அணி



Learning Objectives

After studying this chapter, the students will be able to understand

- definition of partial fractions
- the various techniques to resolve into partial fraction
- different cases of permutation such as word formation
- different cases of combination such as word formation
- difference between permutation and combination
- the concept and principle of mathematical induction
- Binomial expansion techniques



Introduction

Algebra is a major component of mathematics that is used to unify mathematics concepts. Algebra is built on experiences with numbers and operations along with geometry and data analysis. The word “algebra” is derived from the Arabic word “al-Jabr”. The Arabic mathematician Al-Khwarizmi has traditionally been known as the “Father of algebra”. Algebra is used to find perimeter, area, volume of any plane figure and solid.



Al-Khwarizmi

2.1 Partial Fractions

Rational Expression

An expression of the form $\frac{p(x)}{q(x)} = \frac{a_0x^n + a_1x^{n-1} + \dots + a_n}{b_0x^m + b_1x^{m-1} + \dots + b_m}$, $[q(x) \neq 0]$ is called a rational algebraic expression.

If the degree of the numerator $p(x)$ is less than that of the denominator $q(x)$ then $\frac{p(x)}{q(x)}$ is called a proper fraction. If the degree of $p(x)$ is greater than or equal to that of $q(x)$ then $\frac{p(x)}{q(x)}$ is called an improper fraction.

An improper fraction can be expressed as the sum of an integral function and a proper fraction.

$$\text{That is, } \frac{p(x)}{q(x)} = f(x) + \frac{r(x)}{q(x)}$$

$$\text{For example, } \frac{x^2 + x + 1}{x^2 + 2x + 1} = 1 - \frac{x}{x^2 + 2x + 1}$$

Partial Fractions

We can express the sum or difference of two fractions $\frac{3}{x-1}$ and $\frac{2}{x-2}$ as a single fraction.

$$\text{i.e., } \frac{3}{x-1} + \frac{2}{x-2} = \frac{5x-8}{(x-1)(x-2)}$$

$$\frac{5x-8}{(x-1)(x-2)} = \frac{3}{x-1} + \frac{2}{x-2}$$

Process of writing a single fraction as a sum or difference of two or more simpler fractions is called splitting up into *partial fractions*.

Generally if $p(x)$ and $q(x)$ are two rational integral algebraic functions of x and the fraction $\frac{p(x)}{q(x)}$ be expressed as the algebraic sum (or difference) of simpler fractions according to certain specified rules, then the fraction $\frac{p(x)}{q(x)}$ is said to be resolved into partial fractions.

2.1.1 Denominator contains non-repeated Linear factors

In the rational fraction $\frac{p(x)}{q(x)}$, if $q(x)$ is the product of non-repeated linear factors of the form $(ax+b)(cx+d)$, then $\frac{p(x)}{q(x)}$ can be resolved into partial fraction of the form $\frac{A}{ax+b} + \frac{B}{cx+d}$, where A and B are constants to be determined.

For example $\frac{3x+7}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$, the values of A and B are to be determined.

Example 2.1

$$\text{Find the values of } A \text{ and } B \text{ if } \frac{1}{(x^2-1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Solution

$$\text{Let } \frac{1}{(x^2-1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

Multiplying both sides by $(x-1)(x+1)$, we get

$$1 = A(x+1) + B(x-1) \quad \dots (1)$$

Put $x=1$ in (1) we get, $1 = A(2)$

$$\therefore A = \frac{1}{2}$$

Put $x=-1$ in (1) we get, $1 = A(0)+B(-2)$

$$\therefore B = -\frac{1}{2}$$

Example 2.2

Resolve into partial fractions : $\frac{7x-1}{x^2-5x+6}$

Solution

Write the denominator into the product of linear factors.

Here $x^2 - 5x + 6 = (x-2)(x-3)$

$$\text{Let } \frac{7x-1}{x^2-5x+6} = \frac{A}{(x-2)} + \frac{B}{(x-3)} \quad \dots\dots\dots(1)$$

Multiplying both the sides of (1) by $(x-2)(x-3)$, we get

$$7x-1 = A(x-3) + B(x-2) \quad \dots\dots\dots(2)$$

Put $x = 3$ in (2), we get

$$21 - 1 = B(1)$$

$$\Rightarrow B = 20$$

Put $x = 2$ in (2), we get

$$14 - 1 = A(-1)$$

$$\Rightarrow A = -13$$

Substituting the values of A and B in (1), we get

$$\frac{7x-1}{x^2-5x+6} = \frac{-13}{(x-2)} + \frac{20}{(x-3)}$$



$$\begin{aligned}\frac{7x-1}{x^2-5x+6} &= \frac{1}{(x-2)} \left[\frac{(7x-1)}{(x-3)} \right]_{x=2} + \frac{1}{(x-3)} \left[\frac{(7x-1)}{(x-2)} \right]_{x=3} \\ &= \frac{-13}{(x-2)} + \frac{20}{(x-3)}\end{aligned}$$

Example 2.3

Resolve into partial fraction : $\frac{x+4}{(x^2-4)(x+1)}$

Solution

Write the denominator into the product of linear factors

Here $(x^2-4)(x+1) = (x-2)(x+2)(x+1)$

$$\therefore \frac{x+4}{(x^2-4)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+2)} + \frac{C}{(x+1)} \quad \dots(1)$$

Multiplying both the sides by $(x-2)(x+2)(x+1)$, we get

$$x+4 = A(x+2)(x+1) + B(x-2)(x+1) + C(x-2)(x+2) \quad \dots (2)$$

Put $x = -2$ in (2), we get

$$-2+4 = A(0) + B(-4)(-1) + C(0)$$

$$\therefore B = \frac{1}{2}$$

Put $x = 2$ in (2), we get

$$2+4 = A(4)(3) + B(0) + C(0)$$

$$\therefore A = \frac{1}{2}$$

Put $x = -1$ in (2), we get

$$-1+4 = A(0) + B(0) + C(-3)(1)$$

$$\therefore C = -1$$

Substituting the values of A, B and C in (1), we get

$$\frac{x+4}{(x^2-4)(x+1)} = \frac{1}{2(x-2)} + \frac{1}{2(x+2)} - \frac{1}{x+1}$$

2.1.2 Denominator contains Linear factors, repeated n times

In the rational fraction $\frac{p(x)}{q(x)}$, if $q(x)$ is of the form $(ax+b)^n$ [the linear factor is the product of $(ax+b)$, n times], then $\frac{p(x)}{q(x)}$ can be resolved into partial fraction of the form.

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots + \frac{A_n}{(ax+b)^n}$$

For example,
$$\frac{9x+7}{(x+4)(x+1)^2} = \frac{A}{(x+4)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

Example 2.4

Find the values of A, B and C if
$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

Solution

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x = A(x+1)^2 + B(x-1)(x+1) + C(x-1) \quad \dots (1)$$

Put $x = 1$ in (1)

$$1 = A(1+1)^2 + B(0) + C(0)$$

$$\therefore A = \frac{1}{4}$$

Put $x = -1$ in (1)

$$-1 = A(0) + B(0) + C(-1-1)$$

$$\therefore C = \frac{1}{2}$$

Equating the constant term on both sides of (1), we get

$$A - B - C = 0$$

$$\Rightarrow B = A - C$$

$$\therefore B = -\frac{1}{4}$$

Example 2.5

Resolve into partial fraction $\frac{x+1}{(x-2)^2(x+3)}$

Solution

$$\text{Let } \frac{x+1}{(x-2)^2(x+3)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x+3)} \quad \dots (1)$$

Multiplying both the sides by $(x-2)^2(x+3)$, we get

$$x+1 = A(x-2)(x+3) + B(x+3) + C(x-2)^2 \quad \dots (2)$$

Put $x = 2$ in (2), we have

$$2+1 = A(0) + B(5) + C(0)$$

$$\therefore B = \frac{3}{5}$$

Put $x = -3$ in (2), we have

$$-3+1 = A(0) + B(0) + C(-3-2)^2$$

$$\therefore C = \frac{-2}{25}$$

Equating the coefficient of x^2 on both the sides of (2), we get

$$0 = A + C$$

$$A = -C$$

$$\therefore A = \frac{2}{25}$$

Substituting the values of A, B and C in (1),

$$\frac{x+1}{(x-2)^2(x+3)} = \frac{2}{25(x-2)} + \frac{3}{5(x-2)^2} - \frac{2}{25(x+3)}$$

Example 2.6

Resolve into partial fraction $\frac{9}{(x-1)(x+2)^2}$.

Solution

$$\frac{9}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \quad \dots (1)$$

Multiplying both sides by $(x-1)(x+2)^2$

$$\therefore 9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \quad \dots (2)$$

Put $x = -2$ in (2)

$$9 = A(0) + B(0) + C(-2 - 1)$$

$$\therefore C = -3$$

Put $x = 1$ in (2)

$$9 = A(3)^2 + B(0) + C(0)$$

$$\therefore A = 1$$

Equating the coefficient of x^2 on both the sides of (2), we get

$$0 = A + B$$

$$B = -A$$

$$\therefore B = -1$$

Substituting the values of A, B and C in (1), we get

$$\frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

2.1.3 Denominator contains quadratic factor, which cannot be factorized into linear factors

In the rational fraction $\frac{p(x)}{q(x)}$, if $q(x)$ is the quadratic factor $ax^2 + bx + c$ and cannot be factorized into linear factors, then $\frac{p(x)}{q(x)}$ can be resolved into partial fraction of the form $\frac{Ax + B}{ax^2 + bx + c}$.

Example 2.7

Resolve into partial fractions: $\frac{2x+1}{(x-1)(x^2+1)}$

Solution

Here $x^2 + 1$ cannot be factorized into linear factors.

$$\text{Let } \frac{2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+c}{x^2+1} \quad \dots (1)$$

Multiplying both sides by $(x-1)(x^2+1)$

$$2x+1 = A(x^2+1) + (Bx+C)(x-1) \quad \dots (2)$$

Put $x = 1$ in (2)

$$2+1 = A(1+1)+0$$

$$\therefore A = \frac{3}{2}$$

Put $x = 0$ in (2)

$$0+1 = A(0+1)+(0+C)(-1)$$

$$1 = A - C$$

$$\therefore C = \frac{1}{2}$$

Equating the coefficient of x^2 on both the sides of (2), we get

$$A+B = 0$$

$$B = -A$$

$$\therefore B = -\frac{3}{2}$$

Substituting the values of A, B and C in (1), we get

$$\begin{aligned} \frac{2x+1}{(x-1)(x^2+1)} &= \frac{\frac{3}{2}}{x-1} + \frac{-\frac{3}{2}x + \frac{1}{2}}{x^2+1} \\ &= \frac{3}{2(x-1)} - \frac{3x-1}{2(x^2+1)} \end{aligned}$$



Exercise 2.1

Resolve into partial fractions for the following:

1. $\frac{3x+7}{x^2-3x+2}$
2. $\frac{4x+1}{(x-2)(x+1)}$
3. $\frac{1}{(x-1)(x+2)^2}$
4. $\frac{1}{x^2-1}$
5. $\frac{x-2}{(x+2)(x-1)^2}$
6. $\frac{2x^2-5x-7}{(x-2)^3}$
7. $\frac{x^2-6x+2}{x^2(x+2)}$
8. $\frac{x^2-3}{(x+2)(x^2+1)}$
9. $\frac{x+2}{(x-1)(x+3)^2}$
10. $\frac{1}{(x^2+4)(x+1)}$

2.2 Permutations

2.2.1 Factorial

For any natural number n , n factorial is the product of the first n natural numbers and is denoted by $n!$ or $\lfloor n \rfloor$.

$$\text{For any natural number } n, n! = n(n-1)(n-2) \dots 3 \times 2 \times 1$$

For examples,

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$3! = 3 \times 2 \times 1 = 6$$

$$2! = 2 \times 1 = 2$$

$$1! = 1$$



NOTE

$$0! = 1$$

For any natural number n

$$n! = n(n-1)(n-2) \dots 3 \times 2 \times 1$$

$$= n(n-1)!$$

$$= n(n-1)(n-2)!$$

In particular,

$$8! = 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)$$

$$= 8 \times 7!$$

Example 2.8

Evaluate the following:

(i) $\frac{7!}{6!}$ (ii) $\frac{8!}{5!}$ (iii) $\frac{9!}{6!3!}$

Solution

(i) $\frac{7!}{6!} = \frac{7 \times 6!}{6!} = 7.$

(ii) $\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 336.$

(iii) $\frac{9!}{6!3!} = \frac{9 \times 8 \times 7 \times 6!}{6! \times 3!} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84.$

Example 2.9

Rewrite $7!$ in terms of $5!$

Solution

$$7! = 7 \times 6!$$

$$= 7 \times 6 \times 5! = 42 \times 5!$$

Example 2.10

Find n , if $\frac{1}{9!} + \frac{1}{10!} = \frac{n}{11!}$

Solution

$$\begin{aligned}\frac{1}{9!} + \frac{1}{10!} &= \frac{n}{11!} \\ \frac{1}{9!} + \frac{1}{10 \times 9!} &= \frac{n}{11!} \\ \frac{1}{9!} \left[1 + \frac{1}{10} \right] &= \frac{n}{11!} \\ \frac{1}{9!} \times \frac{11}{10} &= \frac{n}{11!} \\ n &= \frac{11! \times 11}{9! \times 10} \\ &= \frac{11! \times 11}{10!} \\ &= \frac{11 \times 10! \times 11}{10!} \\ &= 121.\end{aligned}$$

2.2.2 Fundamental principle of counting**Multiplication principle of counting:**

Consider the following situation in an auditorium which has three entrance doors and two exit doors. Our objective is to find the number of ways a person can enter the hall and then come out.

Assume that, P_1, P_2 and P_3 are the three entrance doors and S_1 and S_2 are the two exit doors. A person can enter the hall through any one of the doors P_1, P_2 or P_3 in 3 ways. After entering the hall, the person can come out through any of the two exit doors S_1 or S_2 in 2 ways.

Hence the total number of ways of entering the hall and coming out is

$$3 \times 2 = 6 \text{ ways.}$$

These possibilities are explained in the given flow chart (Fig. 2.1).

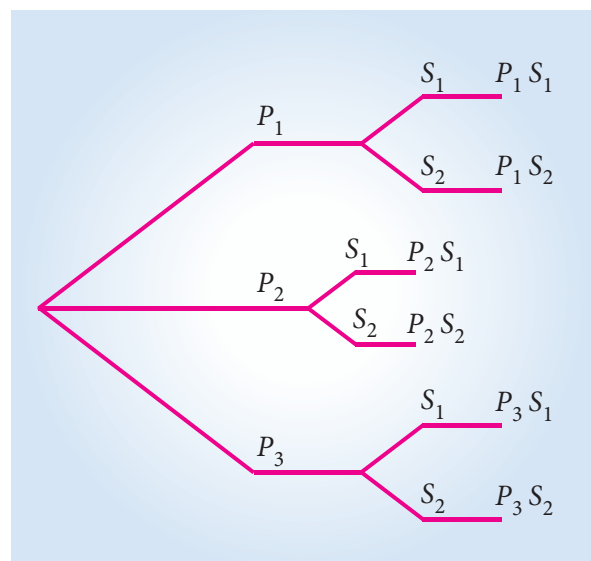


Fig. 2.1

The above problem can be solved by applying multiplication principle of counting.

Definition 2.1

There are two jobs. Of which one job can be completed in m ways, when it has completed in any one of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in $m \times n$ ways. This is called **multiplication principle of counting**.

Example 2.11

Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word “NOTE”, where the repetition of the letters is not allowed.

Solution

Let us allot four vacant places (boxes) to represent the four letters. N,O,T,E of the given word NOTE.



Fig : 2.2

The first place can be filled by any one of the 4 letters in 4 ways. Since repetition is not allowed, the number of ways of filling the second vacant place by any of the remaining 3 letters in 3 ways. In the similar way the third box can be filled in 2 ways and that of last box can be filled in 1 way. Hence by multiplication principle of counting, total number of words formed is $4 \times 3 \times 2 \times 1 = 24$ words.

Example 2.12

If each objective type questions having 4 choices, then find the total number of ways of answering the 4 questions.

Solution

Since each question can be answered in 4 ways, the total number of ways of answering 4 questions = $4 \times 4 \times 4 \times 4 = 256$ ways.

Example 2.13

How many 3 digits numbers can be formed if the repetition of digits is not allowed?

Solution

Let us allot 3 boxes to represent the digits of the 3 digit number

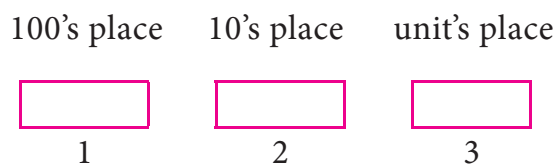


Fig : 2.3

Three digit number never begins with “0”. \therefore 100's place can be filled with any one of the digits from 1 to 9 in 9 ways.

Since repetition is not allowed, 10's place can be filled by the remaining 9 numbers (including 0) in 9 ways and the unit place can be filled by the remaining 8 numbers in 8 ways. Then by multiplication principle of counting, number of 3 digit numbers = $9 \times 9 \times 8 = 648$.

2.2.3 Addition principle of counting

Let us consider the following situation in a class, there are 10 boys and 8 girls. The class teacher wants to select either a boy or a girl to represent the class in a function. Our objective is to find the number of ways that the teacher can select the student. Here the teacher has to perform either of the following two jobs.

One student can be selected in 10 ways among 10 boys.

One student can be selected in 8 ways among 8 girls

Hence the jobs can be performed in $10 + 8 = 18$ ways

This problem can be solved by applying Addition principle of counting.

Definition 2.2

If there are two jobs, each of which can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m+n)$ ways. This is called **addition principle of counting**.

Example 2.14

There are 6 books on commerce and 5 books on accountancy in a book shop. In how many ways can a student purchase either a book on commerce or a book on accountancy?

Solution

Out of 6 commerce books, a book can be purchased in 6 ways.

Out of 5 accountancy books, a book can be purchased in 5 ways.

Then by addition principle counting the total number of ways of purchasing any one book is $5+6=11$ ways.



1. Find x if $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$.
2. Evaluate $\frac{n!}{r!(n-r)!}$ when $n = 5$ and $r = 2$.
3. If $(n+2)! = 60[(n-1)!]$, find n
4. How many five digits telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67 with no digit appears more than once?
5. How many numbers lesser than 1000 can be formed using the digits 5, 6, 7, 8 and 9 if no digit is repeated?

2.2.4 Permutation

Suppose we have a fruit salad with combination of “APPLES, GRAPES & BANANAS”. We don’t care what order the fruits are in. They could also be “bananas, grapes and apples” or “grapes, apples and bananas”. Here the order of mixing is not important. Any how, we will have the same fruit salad.

But, consider a number lock with the number code 395. If we make more than three attempts wrongly, then it locked permanently. In that situation we do care about the order 395. The lock will not work if we input 3-5-9 or 9-5-3. The number lock will work if it is exactly 3-9-5. We have so many such situations in our practical life. So we should know the order of arrangement, called **Permutation**.

Definition 2.3

The number of arrangements that can be made out of n things taking r at a time is called the **number of permutation** of n things taking r at a time.

For example, the number of three digit numbers formed using the digits 1, 2, 3 taking all at a time is 6

6 three digit numbers are 123,132,231,213,312,321

Notations: If n and r are two positive integers such that $0 \leq r \leq n$, then the number of all permutations of n distinct things taken r at a time is denoted by

$$P(n, r) \text{ (or) } nP_r$$

We have the following theorem *without proof*.

Theorem :
$$nP_r = \frac{n!}{(n-r)!}$$

Example 2.15

Evaluate : $5P_3$ and $P(8, 5)$

Solution

$$5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 60.$$

$$\begin{aligned} P(8, 5) &= 8P_5 \\ &= \frac{8!}{(8-5)!} \\ &= \frac{8!}{3!} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720 \end{aligned}$$

Results

- (i) $0! = 1$
- (ii) $nP_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
- (iii) $nP_1 = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$
- (iv) $nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$
- (v) $nP_r = n(n-1)(n-2) \dots [n-(r-1)]$

Example 2.16

Evaluate: (i) $8P_3$ (ii) $5P_4$

Solution

(i) $8P_3 = 8 \times 7 \times 6 = 336.$

(ii) $5P_4 = 5 \times 4 \times 3 \times 2 = 120.$

Example 2.17

In how many ways 7 pictures can be hung from 5 picture nails on a wall ?

Solution

The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is nothing but the number of permutation of 7 things taken 5 at a time $= {}^7P_5 = \frac{7!}{(7-5)!}$
 $= 2520$ ways.

Example 2.18

Find how many four letter words can be formed from the letters of the word “LOGARITHMS” (words are with or without meanings)

Solution

There are 10 letters in the word LOGARITHMS

Therefore $n = 10$

Since we have to find four letter words, $r = 4$

Hence the required number of four letter words $= {}^n P_r = {}^{10} P_4$
 $= 10 \times 9 \times 8 \times 7$
 $= 5040.$

Example 2.19

If ${}^n P_r = 360$, find n and r .

Solution

$$\begin{aligned} {}^n P_r &= 360 = 36 \times 10 \\ &= 3 \times 3 \times 4 \times 5 \times 2 \\ &= 6 \times 5 \times 4 \times 3 = {}^6 P_4 \end{aligned}$$

Therefore $n = 6$ and $r = 4$.

Permutation of repeated things:

The number of permutation of n different things taken r at a time, when repetition is allowed is n^r .

Example 2.20

Using 9 digits from 1,2,3,.....9, taking 3 digits at a time, how many 3 digits numbers can be formed when repetition is allowed?

Solution

Here, $n = 9$ and $r = 3$

$$\begin{aligned} \therefore \text{Number of 3 digit numbers} &= n^r \\ &= 9^3 = 729 \text{ numbers} \end{aligned}$$

Permutations when all the objects are not distinct:

The number of permutations of n things taken all at a time, of which p things are of one kind and q things are of another kind, such that $p + q = n$ is $\frac{n!}{p!q!}$.

In general, the number of permutation of n objects taken all at a time, p_1 are of one kind, p_2 are of another kind, p_3 are of third kind, p_k are of k^{th} kind such that $p_1 + p_2 + \dots + p_k = n$ is $\frac{n!}{p_1!p_2!\dots p_k!}$.

Example 2.21

How many distinct words can be formed using all the letters of the following words.
(i) MISSISSIPPI (ii) MATHEMATICS.

Solution

(i) There are 11 letters in the word MISSISSIPPI

In this word M occurs once

I occurs 4 times

S occurs 4 times

P occurs twice.

$$\text{Therefore required number of permutation} = \frac{11!}{4!4!2!}$$

(ii) In the word MATHEMATICS

There are 11 letters

Here M occurs twice

T occurs twice

A occurs twice

and the rest are all different. Therefore number of permutations = $\frac{11!}{2!2!2!}$

2.2.5 Circular permutation

In the last section, we have studied permutation of n different things taken all together is $n!$. Each permutation is a different arrangement of n things in a row or on a straight line. These are called Linear permutation. Now we consider the permutation of n things along a circle, called **circular permutation**.

Consider the four letters A, B, C, D and the number of row arrangement of these 4 letters can be done in $4!$ Ways. Of those $4!$ arrangements, the arrangements ABCD, BCDA, CDAB, DABC are one and the same when they are arranged along a circle.

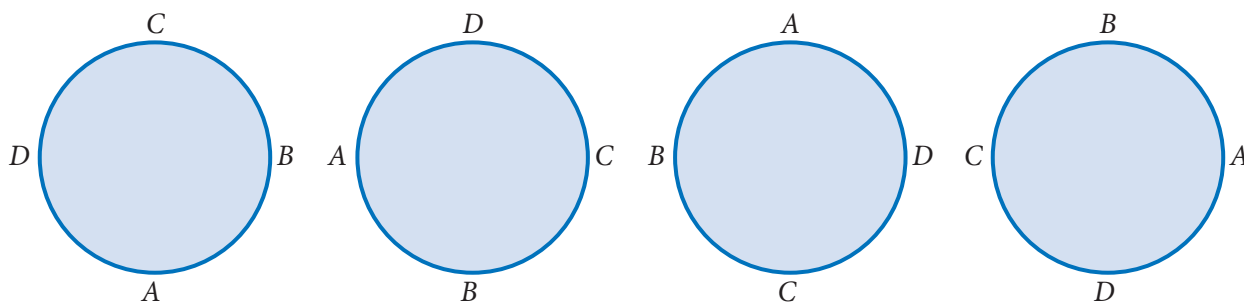


Fig. 2.4

So, the number of permutation of '4' things along a circle is $\frac{4!}{4} = 3!$

In general, circular permutation of n different things taken all at the time = $(n-1)!$



NOTE

If clock wise and anti-clockwise circular permutations are considered to be same (identical), the number of circular permutation of n objects taken all at a time is $\frac{(n-1)!}{2}$

Example 2.22

In how many ways 8 students can be arranged

(i) in a line (ii) along a circle

Solution

- (i) Number of permutations of 8 students along a line = $8P_8 = 8!$
- (ii) When the students are arranged along a circle, then the number of permutation is $(8-1)! = 7!$

Example 2.23

In how many ways 10 identical keys can be arranged in a ring?

Solution

Since keys are identical, both clock wise and anti-clockwise circular permutation are same.

The number of permutation is

$$\frac{(n-1)!}{2} = \frac{(10-1)!}{2} = \frac{9!}{2}$$

Example 2.24

Find the rank of the word 'RANK' in dictionary.

Solutions

Here maximum number of the word in dictionary formed by using the letters of the word 'RANK' is $4!$

The letters of the word RANK in alphabetical order are A, K, N, R

Number of words starting with A = $3! = 6$

Number of words starting with K = $3! = 6$

Number of words starting with N = $3! = 6$

Number of words starting with RAK = $1! = 1$

Number of words starting with RANK = $0! = 1$

\therefore Rank of the word RANK is $6 + 6 + 6 + 1 + 1 = 20$



Rank of word in dictionary

The **rank** of a word in a dictionary is to arrange the words in alphabetical order. In this dictionary the word may or may not be meaningful.


Exercise 2.3

1. If $nP_4 = 12(nP_2)$, find n .
2. In how many ways 5 boys and 3 girls can be seated in a row, so that no two girls are together?
3. How many 6- digit telephone numbers can be constructed with the digits 0,1,2,3,4,5,6,7,8,9 if each numbers starts with 35 and no digit appear more than once?
4. Find the number of arrangements that can be made out of the letters of the word "ASSASSINATION"
5. (a) In how many ways can 8 identical beads be strung on a necklace?
(b) In how many ways can 8 boys form a ring?
6. Find the rank of the word 'CHAT' in dictionary.

2.3 Combinations

A combination is a selection of items from a collection such that the order of selection does not matter. That is the act of combining the elements irrespective of their order. Let us suppose that in an interview two office assistants are to be selected out of 4 persons namely A, B, C and D. The selection may be of AB, AC, AD, BC, BD, CD (we cannot write BA, since AB and BA are of the same selection). Number of ways of selecting 2 persons out of 4 persons is 6. It is represented as $4C_2 = \frac{4 \times 3}{1 \times 2} = 6$.

The process of different selection without repetition is called Combination.

Definition 2.4

Combination is the selection of n things taken r at a time without repetition. Out of n things, r things can be selected in nC_r ways.

$$nC_r = \frac{n!}{r!(n-r)!}, 0 \leq r \leq n$$

Here $n \neq 0$ but r may be 0.

For example, out of 5 balls 3 balls can be selected in $5C_3$ ways.

$$5C_3 = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \left(\frac{5 \times 4 \times 3}{1 \times 2 \times 3} \right) = 10 \text{ ways.}$$

Example 2.25Find 8C_2 **Solution**

$${}^8C_2 = \frac{8 \times 7}{2 \times 1} = 28$$



Permutations are for lists (order matters) and combinations are for groups (order doesn't matter)

Properties

- (i) ${}^nC_0 = {}^nC_n = 1.$
- (ii) ${}^nC_1 = n.$
- (iii) ${}^nC_2 = \frac{n(n-1)}{2!}$
- (iv) ${}^nC_x = {}^nC_y,$
then either $x = y$ or $x + y = n.$
- (v) ${}^nC_r = {}^nC_{n-r}.$
- (vi) ${}^nC_r + {}^nC_{r-1} = (n+1)C_r.$
- (vii) ${}^nC_r = \frac{{}^nP_r}{r!}$

Example 2.26If ${}^nC_4 = {}^nC_6$, find ${}^{12}C_n$.**Solution**If ${}^nC_x = {}^nC_y$, then $x + y = n.$ Here ${}^nC_4 = {}^nC_6$

$$\therefore n = 4 + 6 = 10$$

$$\begin{aligned} {}^{12}C_n &= {}^{12}C_{10} \\ &= {}^{12}C_2 \\ &= \frac{12 \times 11}{1 \times 2} = 66 \end{aligned}$$

Example 2.27If ${}^nP_r = 720$; ${}^nC_r = 120$, find r

Solution

$$\begin{aligned} \text{We know that } nC_r &= \frac{nP_r}{r!} \\ 120 &= \frac{720}{r!} \\ r! &= \frac{720}{120} = 6 = 3! \\ \Rightarrow r &= 3. \end{aligned}$$

Example 2.28

If $15C_{3r} = 15C_{r+3}$, find r

Solution

$$15C_{3r} = 15C_{r+3}$$

Then by the property, $nC_x = nC_y \Rightarrow x + y = n$, we have

$$3r + r + 3 = 15$$

$$\Rightarrow r = 3.$$

Example 2.29

From a class of 32 students, 4 students are to be chosen for competition. In how many ways can this be done?

Solution

$$\begin{aligned} \text{The number of combination} &= 32C_4 \\ &= \frac{32!}{4!(32-4)!} \\ &= \frac{32!}{4!(28)!} \end{aligned}$$

Example 2.30

A question paper has two parts namely Part A and Part B. Each part contains 10 questions. If the student has to choose 8 from part A and 5 from part B, in how many ways can he choose the questions?

Solution

In part A, out of 10 questions 8 can be selected in $10C_8$ ways. In part B out of 10 questions 5 can be selected in $10C_5$. Therefore by multiplication principle the total number of selection is

$$\begin{aligned}
 10C_8 \times 10C_5 &= 10C_2 \times 10C_5 \\
 &= \frac{10 \times 9}{2 \times 1} \times \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \\
 &= 11340.
 \end{aligned}$$

Example 2.31

A Cricket team of 11 players is to be formed from 16 players including 4 bowlers and 2 wicket-keepers. In how many different ways can a team be formed so that the team contains at least 3 bowlers and at least one wicket-keeper?

Solution

A Cricket team of 11 players can be formed in the following ways:

- (i) 3 bowlers, 1 wicket keeper and 7 other players can be selected in

$$4C_3 \times 2C_1 \times 10C_7 \text{ ways}$$

$$\begin{aligned}
 4C_3 \times 2C_1 \times 10C_7 &= 4C_1 \times 2C_1 \times 10C_3 \\
 &= 960 \text{ ways}
 \end{aligned}$$

- (ii) 3 bowlers, 2 wicket keepers and 6 other players can be selected in

$$4C_1 \times 2C_2 \times 10C_6 \text{ ways}$$

$$\begin{aligned}
 4C_1 \times 2C_2 \times 10C_6 &= 4C_1 \times 2C_2 \times 10C_4 \\
 &= 840 \text{ ways}
 \end{aligned}$$

- (iii) 4 bowlers and 1 wicket keeper and 6 other players $4C_4 \times 2C_1 \times 10C_6$

$$4C_4 \times 2C_1 \times 10C_4 = 420 \text{ ways}$$

4 bowlers, 2 wicket keepers and 5 other players can be selected in

$$4C_4 \times 2C_2 \times 10C_5 \text{ ways}$$

$$4C_4 \times 2C_2 \times 10C_5 = 252 \text{ ways}$$

By addition principle of counting.

$$\begin{aligned}
 \text{Total number of ways} &= 960 + 840 + 420 + 252 \\
 &= 2472
 \end{aligned}$$

Example 2.32

If $4(nC_2) = (n+2)C_3$, find n

Solution

$$4(nC_2) = (n+2)C_3$$

$$4 \frac{n(n-1)}{1 \times 2} = \frac{(n+2)(n+1)(n)}{1 \times 2 \times 3}$$

$$12(n-1) = (n+2)(n+1)$$

$$12(n-1) = (n^2 + 3n + 2)$$

$$n^2 - 9n + 14 = 0$$

$$(n-2)(n-7) = 0 \Rightarrow n = 2, n = 7$$

Example 2.33

If $(n+2)C_n = 45$, find n

Solution

$$(n+2)C_n = 45$$

$$(n+2)C_{n+2-n} = 45$$

$$(n+2)C_2 = 45$$

$$\frac{(n+2)(n+1)}{2} = 45$$

$$n^2 + 3n - 88 = 0$$

$$(n+11)(n-8) = 0$$

$$n = -11, 8$$

$n = -11$ is not possible

$$\therefore n = 8$$



ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

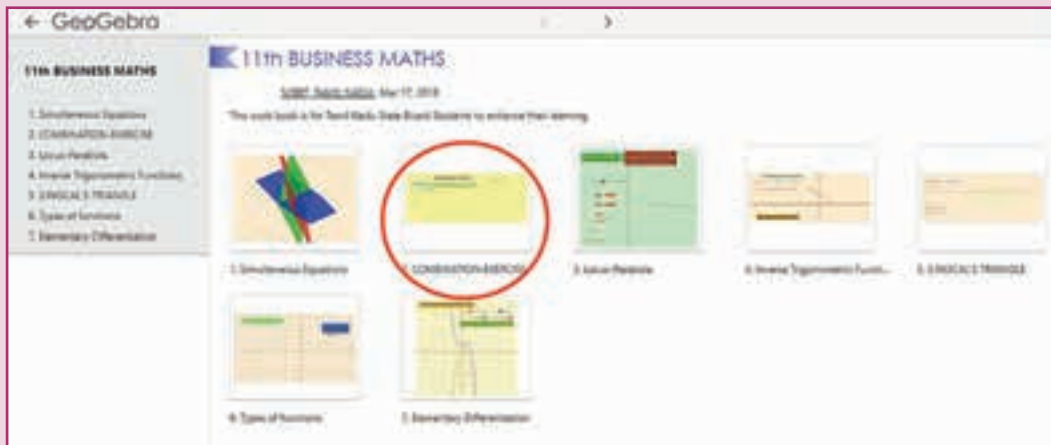
GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Combination Exercise”



Step - 2

Combination practice exercise will open. You can generate as many questions you like by clicking “New question” Solve the problem yourself and check your answer by clicking on the “Show solution” box. Also click “Short method” and follow this method.

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code



 Exercise 2.4

1. If $nP_r = 1680$ and $nC_r = 70$, find n and r .
2. Verify that $8C_4 + 8C_3 = 9C_4$.
3. How many chords can be drawn through 21 points on a circle?
4. How many triangles can be formed by joining the vertices of a hexagon?
5. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?
6. If four dice are rolled, find the number of possible outcomes in which atleast one die shows 2.
7. There are 18 guests at a dinner party. They have to sit 9 guests on either side of a long table, three particular persons decide to sit on one particular side and two others on the other side. In how many ways can the guests to be seated?
8. If a polygon has 44 diagonals, find the number of its sides.
9. In how many ways can a cricket team of 11 players be chosen out of a batch of 15 players?
 - (i) There is no restriction on the selection.
 - (ii) A particular player is always chosen.
 - (iii) A particular player is never chosen.
10. A Committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done when
 - (i) atleast two ladies are included
 - (ii) atmost two ladies are included

2.4 Mathematical induction

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n , where n is a positive integer.

Mathematical Induction is used in the branches of Algebra, Geometry and Analysis where it turns out to be necessary to prove some truths of propositions.

The principle of mathematical induction:

Let $P(n)$ be a given statement for $n \in N$.

- (i) **Initial step:** Let the statement is true for $n = 1$ i.e., $P(1)$ is true and
- (ii) **Inductive step:** If the statement is true for $n = k$ (where k is a particular but arbitrary natural number) then the statement is true for $n = k + 1$. i.e., truth of $P(k)$ implies the truth of $P(k+1)$. Then $P(n)$ is true for all natural numbers n .

Example 2.34

Using mathematical induction method, Prove that

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}, n \in N.$$

Solution

Let the given statement $P(n)$ be defined as

$$1+2+3+\dots+n = \frac{n(n+1)}{2} \text{ for } n \in N.$$

Step 1: Put $n = 1$

$$\text{LHS} \quad P(1) = 1$$

$$\text{RHS} \quad P(1) = \frac{1(1+1)}{2} = 1$$

$$\text{LHS} = \text{RHS for } n = 1$$

\therefore $P(1)$ is true

Step 2: Let us assume that the statement is true for $n = k$.

i.e., $P(k)$ is true

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \text{ is true}$$

Step 3: To prove that $P(k + 1)$ is true

$$\begin{aligned} P(k + 1) &= 1+2+3+\dots+k + (k + 1) \\ &= P(k) + k + 1 \end{aligned}$$

$$\begin{aligned}
 &= \frac{k(k+1)}{2} + k + 1 \\
 &= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}
 \end{aligned}$$

$\therefore P(k+1)$ is true

Thus if $P(k)$ is true, then $P(k+1)$ is also true.

$\therefore P(n)$ is true for all $n \in N$

$$\text{Hence } 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}, n \in N$$

Example 2.35

By the principle of Mathematical Induction, prove that $1 + 3 + 5 \dots + (2n-1) = n^2$, for all $n \in N$.

Solution

Let $P(n)$ denote the statement

$$1 + 3 + 5 \dots + (2n-1) = n^2$$

$$\text{Put } n = 1$$

$$\text{LHS} = 1$$

$$\text{RHS} = 1^2$$

$$= 1$$

$$\text{LHS} = \text{RHS}$$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$$\text{i.e., } 1 + 3 + 5 \dots + (2k-1) = k^2$$

To prove: $P(k+1)$ is true.

$$\begin{aligned}
 \therefore 1 + 3 + 5 \dots + (2k-1) + (2k+1) &= P(k) + (2k+1) \\
 &= k^2 + 2k + 1 \\
 &= (k+1)^2
 \end{aligned}$$

$P(k + 1)$ is true whenever $P(k)$ is true.

$\therefore P(n)$ is true for all $n \in N$.

Example 2.36

By Mathematical Induction, prove that $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$, for all $n \in N$.

Solution

Let $P(n)$ denote the statement:

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Put $n = 1$

$$\therefore \text{LHS} = 1^2$$

$$= 1$$

$$\text{RHS} = \frac{1(1+1)(2+1)}{6}$$

$$= 1$$

$$\text{LHS} = \text{RHS}$$

$\therefore P(1)$ is true.

Assume that $P(k)$ is true.

$$P(k) = 1^2 + 2^2 + 3^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now,

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2 &= p(k) + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

$\therefore P(k + 1)$ is true whenever $P(k)$ is true

$\therefore P(n)$ is true for all $n \in N$.

Example 2.37

Show by the principle of mathematical induction that $2^{3n}-1$ is a divisible by 7, for all $n \in N$.

Solution

Let the given statement $P(n)$ be defined as $P(n) = 2^{3n}-1$

Step 1: Put $n = 1$

$$\therefore P(1) = 2^3 - 1$$

$$= 7 \text{ is divisible by } 7$$

i.e., $P(1)$ is true.

Step 2: Let us assume that the statement is true for $n = k$ i.e., $P(k)$ is true.

$$\text{We assume } 2^{3k} - 1 \text{ is divisible by } 7 \Rightarrow 2^{3k} - 1 = 7m$$

Step 3: To prove that $P(k + 1)$ is true

$$P(k + 1) = 2^{3(k+1)} - 1$$

$$= 2^{(3k+3)} - 1$$

$$= 2^{3k} \cdot 2^3 - 1$$

$$= 2^{3k} \cdot 8 - 1$$

$$= 2^{3k} \cdot (7 + 1) - 1$$

$$= 2^{3k} \cdot 7 + 2^{3k} - 1$$

$$= 2^{3k} \cdot 7 + 7m = 7(2^{3k} + m)$$

which is divisible by 7

$P(k + 1)$ is true whenever $p(k)$ is true

$\therefore P(n)$ is true for all $n \in N$.

Hence the proof.

Example 2.38

By the principle of mathematical induction prove that $n^2 + n$ is an even number, for all $n \in N$.

Solution

Let $P(n)$ denote the statement that, “ $n^2 + n$ is an even number”.

$$\text{Put } n = 1$$

$$\therefore 1^2 + 1 = 1 + 1 = 2, \text{ an even number.}$$

Let us assume that $P(k)$ is true.

$$\therefore k^2 + k \text{ is an even number is true}$$

$$\therefore \text{take } k^2 + k = 2m \quad \dots(1)$$

To prove $P(k + 1)$ is true

$$\begin{aligned} \therefore (k + 1)^2 + (k + 1) &= k^2 + 2k + 1 + k + 1 \\ &= k^2 + k + 2k + 2 \\ &= 2m + 2(k + 1) \text{ by (1)} \\ &= 2(m + k + 1) \text{ (a multiple of 2)} \end{aligned}$$

$$\therefore (k + 1)^2 + (k + 1) \text{ is an even number}$$

$$\therefore P(k + 1) \text{ is true whenever } P(k) \text{ is true.}$$

$P(n)$ is true for $n \in N$.



By the principle of mathematical induction, prove the following

1. $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$ for all $n \in N$.
2. $1.2 + 2.3 + 3.4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}$, for all $n \in N$.
3. $4 + 8 + 12 + \dots + 4n = 2n(n+1)$, for all $n \in N$.
4. $1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n - 1)}{2}$, for all $n \in N$.
5. $3^{2n} - 1$ is a divisible by 8, for all $n \in N$.
6. $a^n - b^n$ is divisible by $a - b$, for all $n \in N$.
7. $5^{2n} - 1$ is divisible by 24, for all $n \in N$.

8. $n(n+1)(n+2)$ is divisible by 6, for all $n \in N$.
9. $2^n > n$, for all $n \in N$.

2.5 Binomial theorem

An algebraic expression of sum or the difference of two terms is called a binomial. For example $(x+y)$, $(5a-2b)$, $(x+\frac{1}{y})$, $(p+\frac{5}{p})$, $(\frac{7}{4}+\frac{1}{y^2})$ etc., are binomials. The Binomial theorem or Binomial Expression is a result of expanding the powers of binomials. It is possible to expand $(x+y)^n$ into a sum involving terms of the form $ax^b y^c$, exponents b and c are non-negative integers with $b+c=n$, the coefficient 'a' of each term is a positive integer called binomial coefficient.

Expansion of $(x+a)^2$ was given by Greek mathematician Euclid on 4th century and there is an evidence that the binomial theorem for cubes i.e., expansion of $(x+a)^3$ was known by 6th century in India. The term Binomial coefficient was first introduced by Michael Stifle in 1544. Blaise Pascal (19 June 1623 to 19 August 1662) a French Mathematician, Physicist, inventor, writer and catholic theologian. In his Treatise on Arithmetical triangle of 1653 described the convenient tabular presentation for Binomial coefficient now called Pascal's triangle. Sir Issac Newton generalized the Binomial theorem and made it valid for any rational exponent.

Now we study the **Binomial theorem for $(x+a)^n$**

Theorem (without proof)

If x and 'a' are real numbers, then for all $n \in N$

$$(x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_r x^{n-r} a^r + \dots \\ + nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n = \sum_{r=0}^n nC_r x^{n-r} a^r$$



NOTE

When $n=0$, $(x+a)^0 = 1$

When $n=1$, $(x+a) = 1C_0 x + 1C_1 a = x+a$

When $n=2$, $(x+a)^2 = 2C_0 x^2 + 2C_1 x a + 2C_2 a^2 = x^2 + 2xa + a^2$

When $n=3$, $(x+a)^3 = 3C_0 x^3 + 3C_1 x^2 a + 3C_2 x a^2 + 3C_3 a^3 \\ = x^3 + 3x^2 a + 3x a^2 + a^3$ and so on.

Given below is the Pascal's Triangle showing the co-efficient of various terms in Binomial expansion of $(x + a)^n$

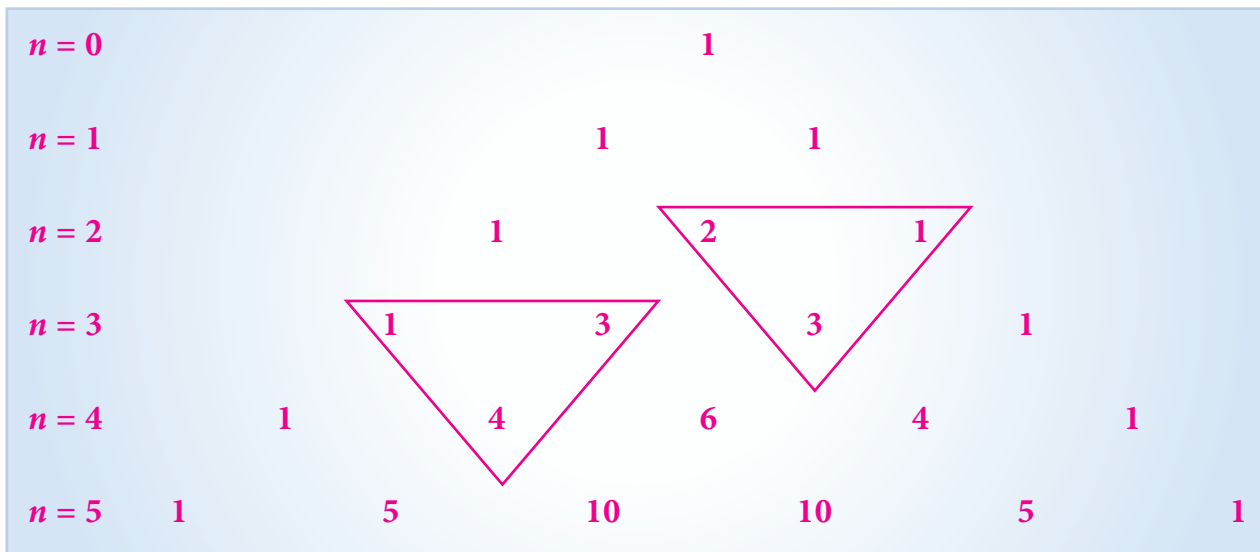


Fig. 2.5



NOTE

- (i) Number of terms in the expansion of $(x + a)^n$ is $n+1$
- (ii) Sum of the indices of x and a in each term in the expansion is n
- (iii) $nC_0, nC_1, nC_2, nC_3, \dots, nC_r, \dots, nC_n$ are also represented as $C_0, C_1, C_2, C_3, \dots, C_r, \dots, C_n$, are called Binomial co-efficients.
- (iv) Since $nC_r = nC_{n-r}$, for $r = 0, 1, 2, \dots, n$ in the expansion of $(x + a)^n$, binomial co-efficients of terms equidistant from the beginning and from the end of the expansion are equal.
- (v) Sum of the co-efficients in the expansion of $(1+x)^n$ is equal of 2^n
- (vi) In the expansion of $(1+x)^n$, the sum of the co-efficients of odd terms = the sum of the co-efficients of the even terms = 2^{n-1}
- (vii) General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$

Middle term of $(x + a)^n$

Case (i) If n is even, then the number of terms $n + 1$ is odd, then there is only one middle term, given by $t_{\frac{n}{2}+1}$

Case (ii) If n is odd, then the number of terms $(n+1)$ is even.. Therefore, we have two middle terms given by $t_{\frac{n+1}{2}}$ and $t_{\frac{n+3}{2}}$

Sometimes we need a **particular term** in the expansion of $(x + a)^n$. For this we first write the general term t_{r+1} . The value of r can be obtained by taking the term t_{r+1} as the required term. To find the **term independent of x** (term without x), equate the power of x in t_{r+1} to zero, we get the value of ' r '. By substituting the value of r in t_{r+1} , we get the term independent of x .



NOTE

$$(x + a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_r x^{n-r} a^r + \dots + a^n \quad \dots(1)$$

(i) To find $(x-a)^n$, replace by $-a$

$$\begin{aligned} (x - a)^n &= nC_0 x^n a^0 + nC_1 x^{n-1} (-a) + nC_2 x^{n-2} (-a)^2 + \dots + nC_r x^{n-r} (-a)^r + \dots + (-a)^n \\ &= nC_0 x^n a^0 - nC_1 x^{n-1} (a) + nC_2 x^{n-2} (a)^2 + \dots + (-1)^r nC_r x^{n-r} (a)^r + \dots + (-1)^n (a)^n \\ &\quad \dots(2) \end{aligned}$$

Note that we have the signs alternatively.

(ii) If we put $a = 1$ in (1), we get

$$(1 + x)^n = 1 + nC_1 x + nC_2 x^2 + \dots + nC_r x^r + \dots + nC_n x^n \quad \dots(3)$$

(iii) If we replace x by $-x$ in (3), we get

$$(1 - x)^n = 1 - nC_1 x + nC_2 x^2 + \dots + nC_r (-1)^r x^r + \dots + nC_n (-1)^n x^n$$

Example 2.39

Expand $(2x + 3y)^5$ using binomial theorem.

Solution

$$\begin{aligned} (x + a)^n &= nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_r x^{n-r} a^r + \dots \\ &\quad + nC_{n-1} x^1 a^{n-1} + nC_n x^0 a^n \\ (2x + 3y)^5 &= 5C_0 (2x)^5 + 5C_1 (2x)^4 (3y) + 5C_2 (2x)^3 (3y)^2 + 5C_3 (2x)^2 (3y)^3 + \\ &\quad 5C_4 (2x)(3y)^4 + 5C_5 (3y)^5 \\ &= 32x^5 + 5(16)x^4(3y) + \frac{5 \times 4}{2 \times 1} (8x^3)(9y^2) + \frac{5 \times 4 \times 3}{3 \times 2 \times 1} 4x^2(27y^3) + \\ &\quad 5(2x)81y^4 + 243y^5 \\ &= 32x^5 + 240x^4y + 720x^3y^2 + 1080x^2y^3 + 810xy^4 + 243y^5 \end{aligned}$$

Example 2.40

Using binomial theorem, expand $\left(x^2 + \frac{1}{x^2}\right)^4$

Solution

$$\begin{aligned}\left(x^2 + \frac{1}{x^2}\right)^4 &= (x^2)^4 + 4C_1(x^2)^3 \frac{1}{x^2} + 4C_2(x^2)^2 \left(\frac{1}{x^2}\right)^2 + 4C_3(x^2) \left(\frac{1}{x^2}\right)^3 + 4C_4 \left(\frac{1}{x^2}\right)^4 \\ &= x^8 + 4x^4 + 6 + \frac{4}{x^2} + \frac{4}{x^4} + \frac{1}{x^8}\end{aligned}$$

Example 2.41

Using binomial theorem, evaluate $(101)^5$

Solution

$$\begin{aligned}(101)^5 &= (100 + 1)^5 \\ &= (100)^5 + 5C_1(100)^4 + 5C_2(100)^3 + 5C_3(100)^2 + 5C_4(100) + 5C_5 \\ &= 10000000000 + 5(100000000) + 10(1000000) + 10(10000) + 5(100) + 1 \\ &= 10000000000 + 500000000 + 10000000 + 100000 + 500 + 1 \\ &= 10510100501\end{aligned}$$

Example 2.42

Find the 5th term in the expansion of $\left(x - \frac{3}{x^2}\right)^{10}$

Solution

General term in the expansion of $(x + a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$ -----(1)

To find the 5th term of $\left(x - \frac{3}{x^2}\right)^{10}$

For this take $r = 4$

$$\begin{aligned}\text{Then, } t_{4+1} = t_5 &= 10C_4(x)^6 \left(-\frac{3}{x^2}\right)^4 \quad (\text{Here } n = 10, x = x, a = -\frac{3}{x^2}) \\ &= 10C_4(x)^6 \frac{3^4}{x^8} \\ &= \frac{17010}{x^2}\end{aligned}$$

Example 2.43

Find the middle term in the expansion of $\left(x^2 - \frac{2}{x}\right)^{10}$

Solution

Compare $\left(x^2 - \frac{2}{x}\right)^{10}$ with $(x + a)^n$

$$n = 10, \quad x = x^2, \quad a = -\frac{2}{x}$$

Since $n = 10$, we have 11 terms (odd)

\therefore 6th term is the middle term.

The general term is $t_{r+1} = nC_r x^{n-r} a^r$ (1)

To get t_6 , put $r = 5$

$$\begin{aligned} t_{5+1} = t_6 &= 10C_5 (x^2)^5 \left(-\frac{2}{x}\right)^5 \\ &= 10C_5 x^{10} \frac{(-2)^5}{x^5} = -8064x^5 \end{aligned}$$

Example 2.44

Find the middle term in the expansion of $\left(\frac{x}{3} + 9y\right)^9$

Solution

Compare $\left(\frac{x}{3} + 9y\right)^9$ with $(x + a)^n$

Since $n = 9$, we have 10 terms (even)

\therefore There are two middle terms namely $\frac{t_{n+1}}{2}, \frac{t_{n+3}}{2}$ i.e., $\frac{t_{9+1}}{2}, \frac{t_{9+3}}{2}$

General term in the expansion of $(x + a)^n$ is

$$t_{r+1} = nC_r x^{n-r} a^r \quad (1)$$

Here t_5 and t_6 are middle terms.

$$\begin{aligned} \text{put } r = 4 \text{ in (1), } t_{4+1} = t_5 &= 9C_4 \left(\frac{x}{3}\right)^5 \cdot (9y)^4 \\ &= 9C_4 \frac{x^5}{3^5} \cdot 9^4 y^4 \\ &= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \cdot \frac{x^5}{3^5} \cdot 9^4 y^4 \\ &= 3402 x^5 y^4 \end{aligned}$$

$$\begin{aligned} \text{put } r = 5 \text{ in (1), } t_{5+1} = t_6 &= 9C_5 \left(\frac{x}{3}\right)^4 \cdot (9y)^5 \\ &= 9C_5 \frac{x^4}{3^4} \cdot 9^5 y^5 \\ &= 91854 x^4 y^5 \end{aligned}$$

Example 2.45

Find the Coefficient of x^{10} in the binomial expansion of $\left(2x^2 - \frac{3}{x}\right)^{11}$

Solution

General term of $(x + a)^n$ is $t_{r+1} = {}^nC_r x^{n-r} a^r$ (1)

Compare $\left(2x^2 - \frac{3}{x}\right)^{11}$ with $(x + a)^n$

$$\begin{aligned} t_{r+1} &= {}^{11}C_r (2x^2)^{11-r} \left(\frac{-3}{x}\right)^r \\ &= {}^{11}C_r 2^{11-r} x^{2(11-r)} (-3)^r \left(\frac{1}{x}\right)^r \\ &= {}^{11}C_r 2^{11-r} (-1)^r \cdot 3^r x^{22-3r} \end{aligned}$$

To find the co-efficient of x^{10} , take $22-3r = 10$

$$r = 4$$

Put $r = 4$ in (1), $t_5 = {}^{11}C_4 2^{11-4} 3^4 x^{10} = {}^{11}C_4 \cdot 2^7 3^4 x^{10}$

\therefore Co-efficient of x^{10} is ${}^{11}C_4 2^7 3^4$.

Example 2.46

Find the term independent of x in the expansion of $\left(2x + \frac{1}{2x^2}\right)^9$.

Solution

General term in the expansion of $(x + a)^n$ is $t_{r+1} = {}^nC_r x^{n-r} a^r$

Compare $\left(2x + \frac{1}{2x^2}\right)^9$ with $(x + a)^n$

$$\begin{aligned} \therefore t_{r+1} &= {}^9C_r (2x)^{9-r} \left(\frac{1}{2x^2}\right)^r \\ &= {}^9C_r 2^{9-r} x^{9-r} \frac{1}{2^r} x^{-2r} \\ t_{r+1} &= {}^9C_r \frac{2^{9-r}}{2^r} \cdot x^{9-3r} \quad \dots (1) \end{aligned}$$

To get the term independent of x , equating the power of x as 0

$$9 - 3r = 0$$

$$\therefore r = 3$$

Put $r = 3$ in (1)

$$t_{3+1} = {}^9C_3 \frac{2^{9-3}}{2^3} \cdot x^0 = {}^9C_3 \frac{2^6}{2^3} = 1792$$



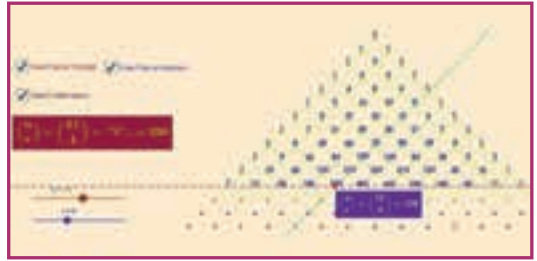
ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

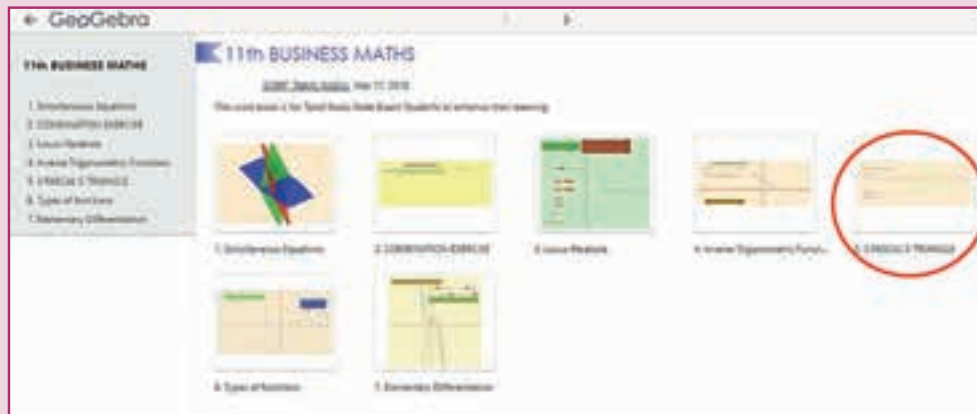
GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Pascal’s Triangle”



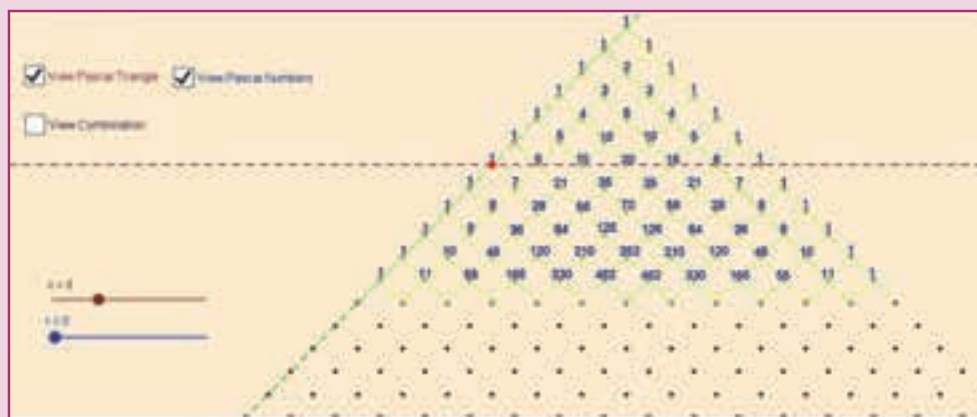
Step - 2

Pascal’s triangle page will open. Click on the check boxes “View Pascal Triangle” and “View Pascal’s Numbers” to see the values. Now you can move the sliders n and r to move the red point to respective ${}^n C_r$. Now click on view Combination to see the ${}^n C_r$ calculation.

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code





Exercise 2.6

- Expand the following by using binomial theorem
 - $(2a - 3b)^4$
 - $\left(x + \frac{1}{y}\right)^7$
 - $\left(x + \frac{1}{x^2}\right)^6$
- Evaluate the following using binomial theorem:
 - $(101)^4$
 - $(999)^5$
- Find the 5th term in the expansion of $(x-2y)^{13}$.
- Find the middle terms in the expansion of
 - $\left(x + \frac{1}{x}\right)^{11}$
 - $\left(3x + \frac{x^2}{2}\right)^8$
 - $\left(2x^2 - \frac{3}{x^3}\right)^{10}$
- Find the term independent of x in the expansion of
 - $\left(x^2 - \frac{2}{3x}\right)^9$
 - $\left(x - \frac{2}{x^2}\right)^{15}$
 - $\left(2x^2 + \frac{1}{x}\right)^{12}$
- Prove that the term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \dots (2n-1) 2^n}{n!}$.
- Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \dots (2n-1) 2^n x^n}{n!}$



Exercise 2.7



Choose the correct answer



- If $nC_3 = nC_2$ then the value of nC_4 is
 - 2
 - 3
 - 4
 - 5
- The value of n , when $nP_2 = 20$ is
 - 3
 - 6
 - 5
 - 4
- The number of ways selecting 4 players out of 5 is
 - 4!
 - 20
 - 25
 - 5

4. If $nP_r = 720(nC_r)$, then r is equal to
 (a) 4 (b) 5 (c) 6 (d) 7
5. The possible out comes when a coin is tossed five times
 (a) 2^5 (b) 5^2 (c) 10 (d) $\frac{5}{2}$
6. The number of diagonals in a polygon of n sides is equal to
 (a) nC_2 (b) $nC_2 - 2$ (c) $nC_2 - n$ (d) $nC_2 - 1$
7. The greatest positive integer which divide $n(n+1)(n+2)(n+3)$ for all $n \in N$ is
 (a) 2 (b) 6 (c) 20 (d) 24
8. If n is a positive integer, then the number of terms in the expansion of $(x+a)^n$ is
 (a) n (b) $n+1$ (c) $n-1$ (d) $2n$
9. For all $n > 0$, $nC_1 + nC_2 + nC_3 + \dots + nC_n$ is equal to
 (a) 2^n (b) $2^n - 1$ (c) n^2 (d) $n^2 - 1$
10. The term containing x^3 in the expansion of $(x-2y)^7$ is
 (a) 3rd (b) 4th (c) 5th (d) 6th
11. The middle term in the expansion of $(x + \frac{1}{x})^{10}$ is
 (a) $10C_4(\frac{1}{x})$ (b) $10C_5$ (c) $10C_6$ (d) $10C_7x^4$
12. The constant term in the expansion of $(x + \frac{2}{x})^6$ is
 (a) 156 (b) 165 (c) 162 (d) 160
13. The last term in the expansion of $(3 + \sqrt{2})^8$ is
 (a) 81 (b) 16 (c) $8\sqrt{2}$ (d) $27\sqrt{3}$
14. If $\frac{kx}{(x+4)(2x-1)} = \frac{4}{x+4} + \frac{1}{2x-1}$ then k is equal to
 (a) 9 (b) 11 (c) 5 (d) 7
15. The number of 3 letter words that can be formed from the letters of the word number when the repetition is allowed are
 (a) 206 (b) 133 (c) 216 (d) 300

16. The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is
(a) 18 (b) 12 (c) 9 (d) 6
17. There are 10 true or false questions in an examination. Then these questions can be answered in
(a) 240 ways (b) 120 ways (c) 1024 ways (d) 100 ways
18. The value of $(5C_0 + 5C_1) + (5C_1 + 5C_2) + (5C_2 + 5C_3) + (5C_3 + 5C_4) + (5C_4 + 5C_5)$ is
(a) $2^6 - 2$ (b) $2^5 - 1$ (c) 2^8 (d) 2^7
19. The total number of 9 digit number which have all different digit is
(a) 10! (b) 9! (c) $9 \times 9!$ (d) $10 \times 10!$
20. The number of ways to arrange the letters of the word "CHEESE"
(a) 120 (b) 240 (c) 720 (d) 6
21. Thirteen guests has participated in a dinner. The number of handshakes happened in the dinner is
(a) 715 (b) 78 (c) 286 (d) 13
22. Number of words with or without meaning that can be formed using letters of the word "EQUATION", with no repetition of letters is
(a) 7! (b) 3! (c) 8! (d) 5!
23. Sum of Binomial co-efficient in a particular expansion is 256, then number of terms in the expansion is
(a) 8 (b) 7 (c) 6 (d) 9
24. The number of permutation of n different things taken r at a time, when the repetition is allowed is
(a) r^n (b) n^r (c) $\frac{n!}{(n-r)!}$ (d) $\frac{n!}{(n+r)!}$
25. Sum of the binomial coefficients is
(a) 2^n (b) n^2 (c) $2n$ (d) $n+17$

Miscellaneous Problems

1. Resolve into Partial Fractions : $\frac{5x+7}{(x-1)(x+3)}$
2. Resolve into Partial Fractions: $\frac{x-4}{x^2-3x+2}$
3. Decompose into Partial Fractions: $\frac{6x^2-14x-27}{(x+2)(x-3)^2}$
4. Decompose into Partial Fractions: $\frac{5x^2-8x+5}{(x-2)(x^2-x+1)}$
5. Evaluate the following expression.
 - (i) $\frac{7!}{6!}$ (ii) $\frac{8!}{5!}$ (iii) $\frac{9!}{6!3!}$
6. How many code symbols can be formed using 5 out of 6 letters A, B, C, D, E, F so that the letters a) cannot be repeated b) can be repeated c) cannot be repeated but must begin with E d) cannot be repeated but end with CAB .
7. From 20 raffle tickets in a hat, four tickets are to be selected in order. The holder of the first ticket wins a car, the second a motor cycle, the third a bicycle and the fourth a skateboard. In how many different ways can these prizes be awarded?
8. In how many different ways, 2 Mathematics, 2 Economics and 2 History books can be selected from 9 Mathematics, 8 Economics and 7 History books?
9. Let there be 3 red, 2 yellow and 2 green signal flags. How many different signals are possible if we wish to make signals by arranging all of them vertically on a staff?
10. Find the Co-efficient of x^{11} in the expansion of $\left(x + \frac{2}{x^2}\right)^{17}$

Summary

- For any natural number n , n factorial is the product of the first n natural numbers and is denoted by $n!$ or \underline{n} .
- For any natural number n $n! = n(n-1)(n-2)\dots 3 \times 2 \times 1$
- $0! = 1$
- If there are two jobs, each of which can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m+n)$ ways.
- The number of arrangements that can be made out of n things taking r at a time is called the number of permutation of n things taking r at a time.

- ${}^n P_r = \frac{n!}{(n-r)!}$
- ${}^n P_0 = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
- ${}^n P_1 = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$
- ${}^n P_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$
- ${}^n P_r = n(n-1)(n-2) \dots [n-(r-1)]$
- The number of permutation of n different things taken r at a time, when repetition is allowed is n^r .
- The number of permutations of n things taken all at a time, of which p things are of one kind and q things are of another kind, such that $p+q = n$ is $\frac{n!}{p! q!}$
- Circular permutation of n different things taken all at the time = $(n-1)!$
- The number of circular permutation of n identical objects taken all at a time is $\frac{(n-1)!}{2}$
- Combination is the selection of n things taken r at a time without repetition.
- Out of n things, r things can be selected in ${}^n C_r$ ways.
- ${}^n C_r = \frac{n!}{r!(n-r)!}$, $0 \leq r \leq n$
- ${}^n C_n = 1$.
- ${}^n C_1 = n$.
- ${}^n C_2 = \frac{n(n-1)}{2!}$
- ${}^n C_x = {}^n C_y$, then either $x = y$ or $x + y = n$.
- ${}^n C_r = {}^n C_{n-r}$.
- ${}^n C_r + {}^n C_{r-1} = (n+1)C_r$
- $(x+a)^n = {}^n C_0 x^n a^0 + {}^n C_1 x^{n-1} a^1 + {}^n C_2 x^{n-2} a^2 + \dots + {}^n C_r x^{n-r} a^r + {}^n C_{n-1} x^1 a^{n-1} + {}^n C_n x^0 a^n = \sum_{r=0}^n {}^n C_r x^{n-r} a^r$
- Number of terms in the expansion of $(x+a)^n$ is $n+1$
- Sum of the indices of x and a in each term in the expansion of $(x+a)^n$ is n

- $nC_0, nC_1, nC_2, nC_3, \dots, nC_r, \dots, nC_n$ are also represented as $C_0, C_1, C_2, C_3, \dots, C_r, \dots, C_n$, called Binomial co-efficients.
- Since $nC_r = nC_{n-r}$, for $r = 0, 1, 2, \dots, n$ in the expansion of $(x+a)^n$, Binomial co-efficients which are equidistant from either end are equal
- $nC_0 = nC_n, nC_1 = nC_{n-1}, nC_2 = nC_{n-2}$
- Sum of the co-efficients is equal of 2^n
- Sum of the co-efficients of odd terms = sum of the co-efficients of even terms = 2^{n-1}
- General term in the expansion of $(x+a)^n$ is $t_{r+1} = nC_r x^{n-r} a^r$

GLOSSARY

Binomial	ஈருறுப்பு
Circular Permutation	வட்ட வரிசை மாற்றம்
Coefficient	கெழு
Combination	சேர்வு
Factorial	காரணியப் பெருக்கம்
Independent term	சாரா உறுப்பு
Linear factor	ஒரு படிக்காரணி
Mathematical Induction	கணிதத் தொகுத்தறிதல்
Middle term	மைய உறுப்பு
Multiplication Principle of counting	எண்ணுதலின் பெருக்கல் கொள்கை
Partial fraction	பகுதிப் பின்னங்கள்
Pascal's Triangle	பாஸ்கலின் முக்கோணம்
Permutations	வரிசை மாற்றங்கள்
Principle of counting	எண்ணுதலின் கொள்கை
Rational Expression	விகிதமுறு கோவை

ANALYTICAL GEOMETRY



Learning Objectives

After studying this chapter, the students will be able to understand

- the locus
- the angle between two lines.
- the concept of concurrent lines.
- the pair of straight lines
- the general equation and parametric equation of a circle.
- the centre and radius of the general equation of a circle.
- the equation of a circle when the extremities of a diameter are given.
- the equation of a tangent to the circle at a given point.
- identification of conics.
- the standard equation of a parabola, its focus, directrix, latus rectum and commercial applications



Introduction

The word “Geometry” is derived from the word “geo” meaning “earth” and “metron” meaning “measuring”. The need of measuring land is the origin of geometry.

“Geometry” is the study of points, lines, curves, surface etc., and their properties. The importance of analytical geometry is that it establishes a correspondence between geometric curves and algebraic equations.

A systematic study of geometry by the use of algebra was first carried out by celebrated French Philosopher and mathematician Rene Descartes (1596-1650), in his book ‘La Geometry’, published in 1637. The resulting combination of analysis and geometry is referred now as analytical geometry. He is known as the father of Analytical geometry



Rene Descartes
(1596-1650)

Analytical geometry is extremely useful in the aircraft industry, especially when dealing with the shape of an airplane fuselage.

3.1 Locus

Definition 3.1

The path traced by a moving point under some specified geometrical condition is called its **locus**.

3.1.1 Equation of a locus

Any relation in x and y which is satisfied by every point on the locus is called the equation of the locus.

For example,

- (i) The locus of a point $P(x_1, y_1)$ whose distance from a fixed point $C(h, k)$ is constant, is a circle. The fixed point 'C' is called the centre.

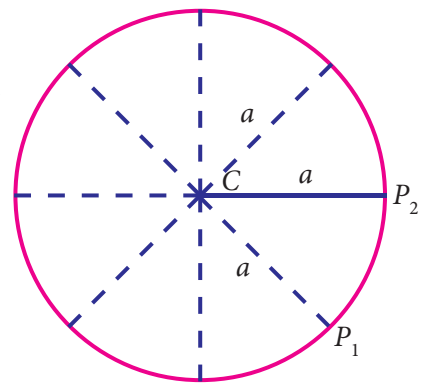


Fig. 3.1

- (ii) The locus of a point whose distances from two points A and B are equal is the perpendicular bisector of the line segment AB .

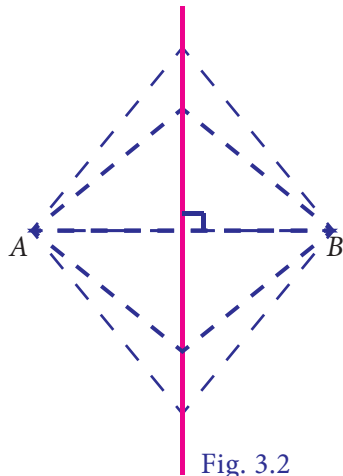


Fig. 3.2

DO YOU KNOW?

Straight line is the locus of a point which moves in the same direction.

Example 3.1

A point in the plane moves so that its distance from the origin is thrice its distance from the y -axis. Find its locus.

Solution

Let $P(x_1, y_1)$ be any point on the locus and A be the foot of the perpendicular from $P(x_1, y_1)$ to the y -axis.

$$\text{Given that } OP = 3 AP$$

$$OP^2 = 9AP^2$$

$$(x_1 - 0)^2 + (y_1 - 0)^2 = 9x_1^2$$

$$x_1^2 + y_1^2 = 9x_1^2$$

$$8x_1^2 - y_1^2 = 0$$

∴ The locus of $P(x_1, y_1)$ is

$$8x^2 - y^2 = 0$$

Example 3.2

Find the locus of the point which is equidistant from $(2, -3)$ and $(3, -4)$.

Solution

Let $A(2, -3)$ and $B(3, -4)$ be the given points

Let $P(x_1, y_1)$ be any point on the locus.

Given that $PA = PB$.

$$PA^2 = PB^2$$

$$(x_1 - 2)^2 + (y_1 + 3)^2 = (x_1 - 3)^2 + (y_1 + 4)^2$$

$$x_1^2 - 4x_1 + 4 + y_1^2 + 6y_1 + 9 = x_1^2 - 6x_1 + 9 + y_1^2 + 8y_1 + 16$$

i.e., $2x_1 - 2y_1 - 12 = 0$

i.e., $x_1 - y_1 - 6 = 0$

The locus of $P(x_1, y_1)$ is $x - y - 6 = 0$.

Example 3.3

Find the locus of a point, so that the join of $(-5, 1)$ and $(3, 2)$ subtends a right angle at the moving point.

Solution

Let $A(-5, 1)$ and $B(3, 2)$ be the given points

Let $P(x_1, y_1)$ be any point on the locus.

Given that $\angle APB = 90^\circ$.

Triangle APB is a right angle triangle.

$$BA^2 = PA^2 + PB^2$$

$$(3+5)^2 + (2-1)^2 = (x_1 + 5)^2 + (y_1 - 1)^2 + (x_1 - 3)^2 + (y_1 - 2)^2$$

$$65 = x_1^2 + 10x_1 + 25 + y_1^2 - 2y_1 + 1 + x_1^2 - 6x_1 + 9 + y_1^2 - 4y_1 + 4$$

$$\text{i.e., } 2x_1^2 + 2y_1^2 + 4x_1 - 6y_1 + 39 - 65 = 0$$

$$\text{i.e., } x_1^2 + y_1^2 + 2x_1 - 3y_1 - 13 = 0$$

The locus of $P(x_1, y_1)$ is $x^2 + y^2 - 2x - 3y - 13 = 0$



Exercise 3.1

1. Find the locus of a point which is equidistant from $(1, 3)$ and x axis.
2. A point moves so that it is always at a distance of 4 units from the point $(3, -2)$
3. If the distance of a point from the points $(2,1)$ and $(1,2)$ are in the ratio $2 : 1$, then find the locus of the point.
4. Find a point on x axis which is equidistant from the points $(7, -6)$ and $(3, 4)$.
5. If $A(-1, 1)$ and $B(2, 3)$ are two fixed points, then find the locus of a point P so that the area of triangle $APB = 8$ sq.units.

3.2 System of straight lines

3.2.1 Recall

In lower classes, we studied the basic concept of coordinate geometry, like distance formula, section - formula, area of triangle and slope of a straight lines etc.

We also studied various form of equations of lines in X std. Let us recall the equations of straight lines. Which will help us for better understanding the new concept and definitions of XI std co-ordinate geometry.

Various forms of straight lines:

(i) Slope-intercept form

Equation of straight line having slope m and y -intercept ' c ' is $y = mx + c$

(ii) Point- slope form

Equation of Straight line passing through the given point $P(x_1, y_1)$ and having a slope m is

$$y - y_1 = m(x - x_1)$$

(iii) Two Point form

Equation of a straight line joining the given points $A(x_1, y_1), B(x_2, y_2)$ is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}.$$

In determinant form, equation of straight line joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(iv) Intercept form

Equation of a straight line whose x and y intercepts are a and b , is $\frac{x}{a} + \frac{y}{b} = 1$.

(v) General form

Equation of straight line in general form is $ax + by + c = 0$ where a , b and c are constants and a , b are not simultaneously zero.

3.2.2 Angle between two straight lines

Let l_1 and l_2 be two straight lines represented by the equations $l_1: y = m_1x + c_1$ and $l_2: y = m_2x + c_2$ intersecting at P .

If θ_1 and θ_2 are two angles made by l_1 and l_2 with x -axis then slope of the lines are $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$.

From fig 3.3, if θ is angle between the lines l_1 & l_2 then

$$\theta = \theta_1 - \theta_2$$

$$\begin{aligned} \therefore \tan \theta &= |\tan(\theta_1 - \theta_2)| \\ &= \left| \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} \right| \end{aligned}$$

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

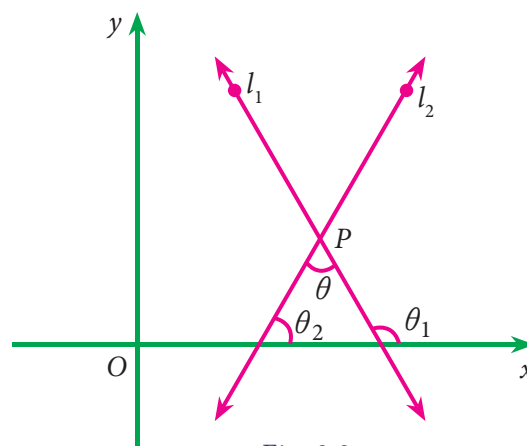


Fig. 3.3



NOTE

- (i) If $\frac{m_1 - m_2}{1 + m_1 m_2}$ is positive, then θ , the angle between l_1 and l_2 is acute and if it is negative, then, θ is the obtuse.
- (ii) We know that two straight lines are parallel if and only if their slopes are equal.
- (iii) We know that two lines are perpendicular if and only if the product of their slopes is -1 . (Here the slopes m_1 and m_2 are finite.)



The straight lines x -axis and y -axis are perpendicular to each other. But, the condition $m_1 m_2 = -1$ is not true because the slope of the x -axis is zero and the slope of the y -axis is not defined.

Example 3.4

Find the acute angle between the lines $2x - y + 3 = 0$ and $x + y + 2 = 0$

Solution

Let m_1 and m_2 be the slopes of $2x - y + 3 = 0$ and $x + y + 2 = 0$

Now $m_1 = 2$, $m_2 = -1$

Let θ be the angle between the given lines

$$\begin{aligned}\tan \theta &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \\ &= \left| \frac{2 + 1}{1 + 2(-1)} \right| = 3 \\ \theta &= \tan^{-1}(3)\end{aligned}$$

3.2.3 Distance of a point from a line

(i) The length of the perpendicular from a point $P(l, m)$ to the line $ax + by + c = 0$ is $d = \left| \frac{al + bm + c}{\sqrt{a^2 + b^2}} \right|$

(ii) The length of the perpendicular from the origin $(0,0)$ to the line $ax + by + c = 0$ is $d = \left| \frac{c}{\sqrt{a^2 + b^2}} \right|$

Example 3.5

Show that perpendicular distances of the line $x - y + 5 = 0$ from origin and from the point $P(2, 2)$ are equal.

Solution

Given line is $x - y + 5 = 0$

$$\begin{aligned} \text{Perpendicular distance of the given line from } P(2, 2) &= \left| \frac{2 - 2 + 5}{\sqrt{1^2 + 1^2}} \right| \\ &= \left| \frac{5}{\sqrt{2}} \right| \end{aligned}$$

$$\text{Distance of } (0,0) \text{ from the given line} = \left| \frac{5}{\sqrt{1^2 + 1^2}} \right| = \left| \frac{5}{\sqrt{2}} \right|$$

The given line is equidistance from origin and $(2, 2)$

Example 3.6

If the angle between the two lines is $\frac{\pi}{4}$ and slope of one of the lines is 3, then find the slope of the other line.

Solution

We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\text{Given } m_1 = 3 \text{ and } \theta = \frac{\pi}{4}$$

$$\therefore \tan \frac{\pi}{4} = \left| \frac{3 - m_2}{1 + 3m_2} \right|$$

$$1 = \left(\frac{3 - m_2}{1 + 3m_2} \right)$$

$$1 + 3m_2 = 3 - m_2$$

$$\Rightarrow m_2 = \frac{1}{2}$$

Hence the slope of the other line is $\frac{1}{2}$.

3.2.4 Concurrence of three lines

If two lines l_1 and l_2 meet at a common point P , then that P is called point of intersection of l_1 and l_2 . This point of intersection is obtained by solving the equations of l_1 and l_2 .

If three or more straight lines will have a point in common then they are said to be concurrent.

The lines passing through the common point are called concurrent lines and the common point is called concurrent point.

Conditions for three given straight lines to be concurrent

$$\text{Let } a_1x + b_1y + c_1 = 0 \quad \rightarrow (1)$$

$$a_2x + b_2y + c_2 = 0 \quad \rightarrow (2)$$

$$a_3x + b_3y + c_3 = 0 \quad \rightarrow (3)$$

be the equations of three straight lines, then the condition that these lines to be concurrent is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Example 3.7

Show that the given lines $3x - 4y - 13 = 0$, $8x - 11y = 33$ and $2x - 3y - 7 = 0$ are concurrent and find the concurrent point.

Solution

$$\text{Given lines } 3x - 4y - 13 = 0 \quad \dots (1)$$

$$8x - 11y = 33 \quad \dots (2)$$

$$2x - 3y - 7 = 0 \quad \dots (3)$$

$$\text{Condition for concurrent lines is } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\begin{aligned} \text{i.e., } \begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & -7 \end{vmatrix} &= 3(77-99) + 4(-56+66) - 13(-24+22) \\ &= -66 + 40 + 26 = 0 \end{aligned}$$

\Rightarrow Given lines are concurrent.

To get the point of concurrency solve the equations (1) and (3)

$$\text{Equation (1)} \times 2 \quad \Rightarrow \quad 6x - 8y = 26$$

$$\text{Equation (3)} \times 3 \quad \Rightarrow \quad 6x - 9y = 21$$

$$\begin{array}{r} \hline y = 5 \\ \hline \end{array}$$

When $y=5$ from (2) $8x = 88$

$$x = 11$$

Point of concurrency is $(11, 5)$

Example 3.8

If the lines $3x - 5y - 11 = 0$, $5x + 3y - 7 = 0$ and $x + ky = 0$ are concurrent, find the value of k .

Solution

Given the lines are concurrent.

$$\text{Therefore } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

$$\begin{vmatrix} 3 & -5 & -11 \\ 5 & 3 & -7 \\ 1 & k & 0 \end{vmatrix} = 0$$

$$1(35+33) - k(-21+55) = 0$$

$$\Rightarrow 34k = 68. \quad \therefore k = 2.$$

Example 3.9

A private company appointed a clerk in the year 2012, his salary was fixed as ₹20,000. In 2017 his salary raised to ₹25,000.

- (i) Express the above information as a linear function in x and y where y represent the salary of the clerk and x -represent the year
- (ii) What will be his salary in 2020?

Solution

Let y represent the salary (in Rs) and x represent the year

year (x)	salary (y)
2012(x_1)	20,000(y_1)
2017(x_2)	25,000(y_2)
2020	?

The equation of straight line expressing the given information as a linear equation in x and y is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

$$\frac{y - 20,000}{25,000 - 20,000} = \frac{x - 2012}{2017 - 2012}$$

$$\frac{y - 20,000}{5000} = \frac{x - 2012}{5}$$

$$y = 1000x - 2012000 + 20,000$$

$$y = 1000x - 19,92,000$$

In 2020 his salary will be

$$y = 1000(2020) - 19,92,000$$

$$y = ₹28000$$



Exercise 3.2

- Find the angle between the lines whose slopes are $\frac{1}{2}$ and 3
- Find the distance of the point (4,1) from the line $3x - 4y + 12 = 0$
- Show that the straight lines $x + y - 4 = 0$, $3x + 2 = 0$ and $3x - 3y + 16 = 0$ are concurrent.
- Find the value of 'a' for which the straight lines $3x + 4y = 13$; $2x - 7y = -1$ and $ax - y - 14 = 0$ are concurrent.
- A manufacturer produces 80 TV sets at a cost of ₹2,20,000 and 125 TV sets at a cost of ₹2,87,500. Assuming the cost curve to be linear, find the linear expression of the given information. Also estimate the cost of 95 TV sets.

3.3 Pair of straight lines

3.3.1 Combined equation of the pair of straight lines

Let us consider the two individual equations of straight lines

$$l_1x + m_1y + n_1 = 0 \text{ and } l_2x + m_2y + n_2 = 0$$

Then their combined equation is

$$(l_1x + m_1y + n_1) (l_2x + m_2y + n_2) = 0$$

$$l_1 l_2 x^2 + (l_1 m_2 + l_2 m_1)xy + m_1 m_2 y^2 + (l_1 n_2 + l_2 n_1)x + (m_1 n_2 + m_2 n_1)y + n_1 n_2 = 0$$

Hence the general equation of pair of straight lines can be taken as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

where a, b, c, f, g and h are all constants.

3.3.2 Pair of straight lines passing through the origin

The homogeneous equation $ax^2 + 2hxy + by^2 = 0$... (1)

of second degree in x and y represents a pair of straight lines passing through the origin.

Let $y = m_1 x$ and $y = m_2 x$ be two straight lines passing through the origin.

Then their combined equation is

$$\begin{aligned} (y - m_1 x)(y - m_2 x) &= 0 \\ \Rightarrow m_1 m_2 x^2 - (m_1 + m_2)xy + y^2 &= 0 \quad \dots (2) \end{aligned}$$

(1) and (2) represent the same pair of straight lines

$$\begin{aligned} \therefore \frac{a}{m_1 m_2} &= \frac{2h}{-(m_1 + m_2)} = \frac{b}{1} \\ \Rightarrow m_1 m_2 &= \frac{a}{b} \text{ and } m_1 + m_2 = -\frac{2h}{b}. \end{aligned}$$

i.e., product of the slopes = $\frac{a}{b}$ and sum of the slopes = $-\frac{2h}{b}$

3.3.3 Angle between pair of straight lines passing through the origin

The equation of the pair of straight lines passing through the origin is

$$ax^2 + 2hxy + by^2 = 0$$

Let m_1 and m_2 be the slopes of above lines.

Here $m_1 + m_2 = -\frac{2h}{b}$ and $m_1 m_2 = \frac{a}{b}$.

Let θ be the angle between the pair of straight lines.

Then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$= \left| \frac{\pm 2\sqrt{h^2 - ab}}{a + b} \right|$$

Let us take ' θ ' as acute angle

$$\therefore \theta = \tan^{-1} \left[\left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \right]$$



NOTE

(i) If θ is the angle between the pair of straight lines

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ then } \theta = \tan^{-1} \left[\left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \right]$$

(ii) If the straight lines are parallel, then $h^2 = ab$.

(iii) If the straight lines are perpendicular, then $a + b = 0$

$$\text{i.e., coefficient of } x^2 + \text{coefficient of } y^2 = 0$$

3.3.4 The condition for general second degree equation to represent the pair of straight lines

The condition for a general second degree equation in x, y namely

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$



NOTE

The condition in determinant form is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Example 3.10

Find the combined equation of the given straight lines whose separate equations are $2x + y - 1 = 0$ and $x + 2y - 5 = 0$.

Solution

The combined equation of the given straight lines is

$$(2x + y - 1)(x + 2y - 5) = 0$$

$$\text{i.e., } 2x^2 + xy - x + 4xy + 2y^2 - 2y - 10x - 5y + 5 = 0$$

$$\text{i.e., } 2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$$

Example 3.11

Show that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Also find the angle between them.

Solution

Compare the equation

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0 \text{ with}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get}$$

$$a = 2, b = 3, h = \frac{5}{2}, g = 3, f = \frac{7}{2} \text{ and } c = 4$$

Condition for the given equation to represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 24 + \frac{105}{2} - \frac{49}{2} - 27 - 25 = 0$$

Hence the given equation represents a pair of straight lines.

Let θ be the angle between the lines.

$$\text{Then } \theta = \tan^{-1} \left[\left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| \right] = \tan^{-1} \left[\frac{2\sqrt{\frac{25}{4} - 6}}{5} \right]$$

$$\therefore \theta = \tan^{-1} \left(\frac{1}{5} \right)$$

Example 3.12

The slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other, show that $8h^2 = 9ab$.

Solution

Let m_1 and m_2 be the slopes of the pair of straight lines

$$ax^2 + 2hxy + by^2 = 0$$

$$\therefore m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

It is given that one slope is twice the other, so let $m_2 = 2m_1$

$$\therefore m_1 + 2m_1 = -\frac{2h}{b} \text{ and } m_1 \cdot 2m_1 = \frac{a}{b}$$

$$\begin{aligned} \therefore \quad m_1 &= -\frac{2h}{3b} \text{ and } 2m_1^2 = \frac{a}{b} \\ \Rightarrow \quad 2\left(-\frac{2h}{3b}\right)^2 &= \frac{a}{b} \\ \Rightarrow \quad \frac{8h^2}{9b^2} &= \frac{a}{b} \\ \Rightarrow \quad 8h^2 &= 9ab \end{aligned}$$

Example 3.13

Show that the equation $2x^2 + 7xy + 3y^2 + 5x + 5y + 2 = 0$ represent two straight lines and find their separate equations.

Solution

Compare the equation $2x^2 + 7xy + 3y^2 + 5x + 5y + 2 = 0$ with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get,}$$

$$a = 2, b = 3, h = \frac{7}{2}, g = \frac{5}{2}, f = \frac{5}{2}, c = 2$$

$$\begin{aligned} \text{Now} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} &= \begin{vmatrix} a & \frac{7}{2} & \frac{5}{2} \\ \frac{7}{2} & 3 & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} & 2 \end{vmatrix} \\ &= 2\left(6 - \frac{25}{4}\right) - \frac{7}{2}\left(9 - \frac{25}{4}\right) + \frac{5}{2}\left(\frac{35}{4} - \frac{15}{2}\right) \\ &= 2\left(-\frac{1}{4}\right) - \frac{7}{2} \cdot \frac{3}{4} + \frac{5}{2} \cdot \frac{5}{4} \\ &= -\frac{1}{2} - \frac{21}{8} + \frac{25}{8} = 0 \end{aligned}$$

Hence the given equation represents a pair of straight lines.

Now consider,

$$\begin{aligned} 2x^2 + 7xy + 3y^2 &= 2x^2 + 6xy + xy + 3y^2 \\ &= 2x(x + 3y) + y(x + 3y) \\ &= (x + 3y)(2x + y) \end{aligned}$$

$$\text{Let } 2x^2 + 7xy + 3y^2 + 5x + 5y + 2 = (x + 3y + l)(2x + y + m)$$

$$\text{Comparing the coefficient of } x, \quad 2l + m = 5 \quad (1)$$

$$\text{Comparing the coefficient of } y, \quad l + 3m = 5 \quad (2)$$

$$\text{Solving (1) and (2), we get } m = 1 \text{ and } l = 2$$

\therefore The separate equations are $x + 3y + 2 = 0$ and $2x + y + 1 = 0$.

Example 3.14

Show that the pair of straight lines $4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$ represents a pair of parallel straight lines and find their separate equations.

Solution

The given equation is $4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$

Here $a = 4$, $b = 9$ and $h = -6$

$$h^2 - ab = 36 - 36 = 0$$

Hence the given equation represents a pair of parallel straight lines.

$$\text{Now } 4x^2 - 12xy + 9y^2 = (2x - 3y)^2$$

Consider $4x^2 - 12xy + 9y^2 + 18x - 27y + 8 = 0$

$$\Rightarrow (2x - 3y)^2 + 9(2x - 3y) + 8 = 0$$

Put $2x - 3y = z$

$$z^2 + 9z + 8 = 0$$

$$(z+1)(z+8) = 0$$

$$z+1 = 0 \qquad z+8 = 0$$

$$2x-3y+1 = 0 \qquad 2x-3y+8 = 0$$

Hence the separate equations are

$$2x-3y+1 = 0 \quad \text{and} \quad 2x-3y+8 = 0$$

Example 3.15

Find the angle between the straight lines $x^2 + 4xy + y^2 = 0$

Solution

The given equation is $x^2 + 4xy + y^2 = 0$

Here $a = 1$, $b = 1$ and $h = 2$.

If θ is the angle between the given straight lines, then

$$\begin{aligned}\theta &= \tan^{-1} \left\| \frac{2\sqrt{h^2 - ab}}{a + b} \right\| \\ &= \tan^{-1} \left\| \frac{2\sqrt{4 - 1}}{2} \right\| \\ &= \tan^{-1}(\sqrt{3}) \\ \theta &= \frac{\pi}{3}\end{aligned}$$

Example 3.16

For what value of k does $2x^2 + 5xy + 2y^2 + 15x + 18y + k = 0$ represent a pair of straight lines.

Solution

Here $a = 2$, $b = 2$, $h = \frac{5}{2}$, $g = \frac{15}{2}$, $f = 9$, $c = k$.

The given line represents a pair of straight lines if,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\text{i.e., } 4k + \frac{675}{2} - 162 - \frac{225}{2} - \frac{25}{4}k = 0$$

$$\Rightarrow 16k + 1350 - 648 - 450 - 25k = 0$$

$$\Rightarrow 9k = 252 \quad \therefore k = 28$$



Exercise 3.3

1. If the equation $ax^2 + 5xy - 6y^2 + 12x + 5y + c = 0$ represents a pair of perpendicular straight lines, find a and c .
2. Show that the equation $12x^2 - 10xy + 2y^2 + 14x - 5y + 2 = 0$ represents a pair of straight lines and also find the separate equations of the straight lines.
3. Show that the pair of straight lines $4x^2 + 12xy + 9y^2 - 6x - 9y + 2 = 0$ represents two parallel straight lines and also find the separate equations of the straight lines.
4. Find the angle between the pair of straight lines $3x^2 - 5xy - 2y^2 + 17x + y + 10 = 0$

3.4 Circles

Definition 3.2

A **circle** is the locus of a point which moves in such a way that its distance from a fixed point is always constant. The fixed point is called the **centre** of the circle and the constant distance is the **radius** of the circle.

3.4.1 The equation of a circle when the centre and radius are given.

Let $C(h, k)$ be the centre and ' r ' be the radius of the circle

Let $P(x, y)$ be any point on the circle

$$CP = r$$

$$CP^2 = r^2$$

$(x-h)^2 + (y-k)^2 = r^2$ is the equation of the circle.

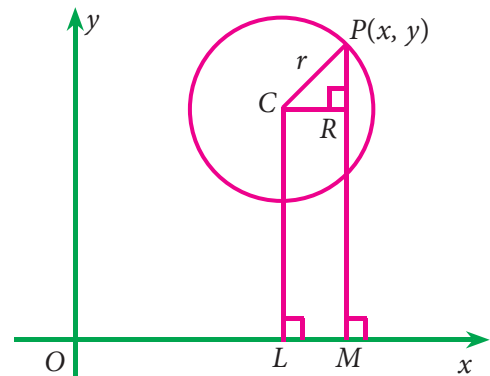


Fig. 3.4

In particular, if the centre is at the origin, the equation of circle is $x^2 + y^2 = r^2$

Example 3.17

Find the equation of the circle with centre at $(3, -1)$ and radius is 4 units.

Solution

Equation of circle is $(x-h)^2 + (y-k)^2 = r^2$

Here $(h, k) = (3, -1)$ and $r = 4$

Equation of circle is

$$(x-3)^2 + (y+1)^2 = 16$$

$$x^2 - 6x + 9 + y^2 + 2y + 1 = 16$$

$$x^2 + y^2 - 6x + 2y - 6 = 0$$

Example 3.18

Find the equation of the circle with centre at origin and radius is 3 units.

Solution

Equation of circle is $x^2 + y^2 = r^2$

Here $r = 3$

i.e equation of circle is $x^2 + y^2 = 9$

3.4.2 Equation of a circle when the end points of a diameter are given

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the end points of a diameter of a circle and

$P(x, y)$ be any point on the circle.

We know that angle in the semi circle is 90°

$$\therefore \angle APB = 90^\circ$$

$$\therefore (\text{Slope of AP}) (\text{Slope of BP}) = -1$$

$$\left(\frac{y - y_1}{x - x_1} \right) \times \left(\frac{y - y_2}{x - x_2} \right) = -1$$

$$(y - y_1)(y - y_2) = -(x - x_1)(x - x_2)$$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ is the required equation of circle.

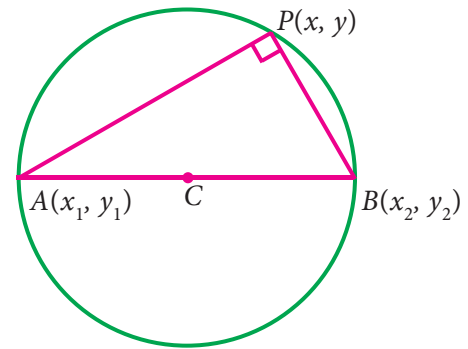


Fig. 3.5

Example 3.19

Find the equation of the circle when the end points of the diameter are (2, 4) and (3, -2).

Solution

Equation of a circle when the end points of the diameter are given is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Here $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (3, -2)$

$$\therefore (x - 2)(x - 3) + (y - 4)(y + 2) = 0$$

$$x^2 + y^2 - 5x - 2y - 2 = 0$$

3.4.3 General equation of a circle

The general equation of circle is $x^2 + y^2 + 2gx + 2fy + c = 0$ where g , f and c are constants.

$$\text{i.e. } x^2 + y^2 + 2gx + 2fy = -c$$

$$x^2 + 2gx + g^2 - g^2 + y^2 + 2fy + f^2 - f^2 = -c$$

$$(x+g)^2 - g^2 + (y+f)^2 - f^2 = -c$$

$$(x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

$$[x - (-g)]^2 + [y - (-f)]^2 = [\sqrt{g^2 + f^2 - c}]^2$$

Comparing this with the circle $(x-h)^2 + (y-k)^2 = r^2$

We get, centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$



NOTE

The general second degree equation $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a circle if

- (i) $a = b$ i.e., co-efficient of $x^2 =$ co-efficient of y^2 .
- (ii) $h = 0$ i.e., no xy term.

Example 3.20

Find the centre and radius of the circle $x^2 + y^2 - 8x + 6y - 24 = 0$

Solution

Equation of circle is $x^2 + y^2 - 8x + 6y - 24 = 0$

Here $g = -4$, $f = 3$ and $c = -24$

Centre = $C(-g, -f) = C(4, -3)$ and

$$\begin{aligned} \text{Radius: } r &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{16 + 9 + 24} = 7 \text{ unit.} \end{aligned}$$

Example 3.21

For what values of a and b does the equation

$(a-2)x^2 + by^2 + (b-2)xy + 4x + 4y - 1 = 0$ represents a circle? Write down the resulting equation of the circle.

Solution

The given equation is $(a - 2)x^2 + by^2 + (b - 2)xy + 4x + 4y - 1 = 0$

As per conditions noted above,

$$(i) \text{ coefficient of } xy = 0 \Rightarrow b - 2 = 0$$

$$\therefore b = 2$$

$$(ii) \text{ coefficient of } x^2 = \text{Coefficient of } y^2 \Rightarrow a - 2 = b$$

$$a - 2 = 2 \Rightarrow a = 4$$

$$\therefore \text{Resulting equation of circle is } 2x^2 + 2y^2 + 4x + 4y - 1 = 0$$

Example 3.22

If the equation of a circle $x^2 + y^2 + ax + by = 0$ passing through the points (1, 2) and (1, 1), find the values of a and b

Solution

The circle $x^2 + y^2 + ax + by = 0$ passing through (1, 2) and (1, 1)

$$\therefore \text{ We have } 1 + 4 + a + 2b = 0 \text{ and } 1 + 1 + a + b = 0$$

$$\Rightarrow a + 2b = -5 \quad (1)$$

$$\text{and } a + b = -2 \quad (2)$$

solving (1) and (2), we get $a = 1, b = -3$.

Example 3.23

If the centre of the circle $x^2 + y^2 + 2x - 6y + 1 = 0$ lies on a straight line $ax + 2y + 2 = 0$, then find the value of 'a'

Solution

Centre C(-1, 3)

It lies on $ax + 2y + 2 = 0$

$$-a + 6 + 2 = 0$$

$$a = 8$$

Example 3.24

Show that the point $(7, -5)$ lies on the circle $x^2 + y^2 - 6x + 4y - 12 = 0$ and find the coordinates of the other end of the diameter through this point .

Solution

Let $A(7, -5)$

Equation of circle is $x^2 + y^2 - 6x + 4y - 12 = 0$

Substitute $(7, -5)$ for (x, y) , we get

$$\begin{aligned} x^2 + y^2 - 6x + 4y - 12 &= 7^2 + (-5)^2 - 6(7) + 4(-5) - 12 \\ &= 49 + 25 - 42 - 20 - 12 = 0 \end{aligned}$$

$\therefore (7, -5)$ lies on the circle

Here $g = -3$ and $f = 2$

\therefore Centre = $C(3, -2)$

Let the other end of the diameter be $B(x, y)$

Midpoint of $AB = \left(\frac{x+7}{2}, \frac{y-5}{2} \right) = C(3, -2)$

$$\begin{array}{ccc} \frac{x+7}{2} = 3 & | & \frac{y-5}{2} = -2 \\ x = -1 & & y = 1 \end{array}$$

Other end of the diameter is $(-1, 1)$.

Example 3.25

Find the equation of the circle passing through the points $(0,0)$, $(1, 2)$ and $(2,0)$.

Solution

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$

The circle passes through the point $(0, 0)$

$$c = 0 \quad \dots (1)$$

The circle passes through the point $(1, 2)$

$$1^2 + 2^2 + 2g(1) + 2f(2) + c = 0$$

$$2g + 4f + c = -5 \quad \dots (2)$$

The circle passes through the point (2, 0)

$$2^2 + 0 + 2g(2) + 2f(0) + c = 0$$

$$4g + c = -4 \quad \dots (3)$$

Solving (1), (2) and (3), we get

$$g = -1, \quad f = -\frac{3}{4} \quad \text{and} \quad c = 0$$

\therefore The equation of the circle is

$$x^2 + y^2 + 2(-1)x + 2\left(-\frac{3}{4}\right)y + 0 = 0$$

$$\text{ie.,} \quad 2x^2 + 2y^2 - 4x - 3y = 0$$

3.4.4 Parametric form of a circle

Consider a circle with radius r and centre at the origin. Let $P(x, y)$ be any point on the circle. Assume that OP makes an angle θ with the positive direction of x -axis.

Draw PM perpendicular to x -axis.

$$\text{From the figure,} \quad \cos \theta = \frac{x}{r} \Rightarrow x = r \cos \theta$$

$$\sin \theta = \frac{y}{r} \Rightarrow y = r \sin \theta$$

The equations $x = r \cos \theta$, $y = r \sin \theta$ are called the parametric equations of the circle $x^2 + y^2 = r^2$. Here ' θ ' is called the parameter and $0 \leq \theta \leq 2\pi$.

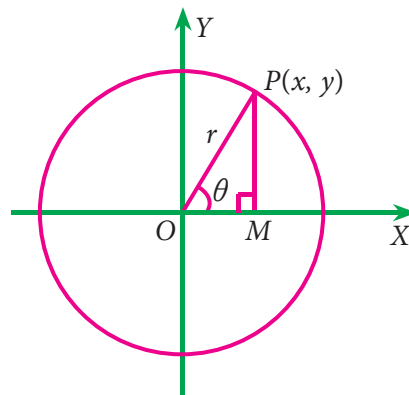


Fig. 3.6

Example 3.26

Find the parametric equations of the circle $x^2 + y^2 = 25$

Solution

$$\text{Here } r^2 = 25 \Rightarrow r = 5$$

$$\text{Parametric equations are } x = r \cos \theta, \quad y = r \sin \theta$$

$$\Rightarrow x = 5 \cos \theta, \quad y = 5 \sin \theta, \quad 0 \leq \theta \leq 2\pi$$


Exercise 3.4

- Find the equation of the following circles having
 - the centre (3,5) and radius 5 units
 - the centre (0,0) and radius 2 units
- Find the centre and radius of the circle
 - $x^2 + y^2 = 16$
 - $x^2 + y^2 - 22x - 4y + 25 = 0$
 - $5x^2 + 5y^2 + 4x - 8y - 16 = 0$
 - $(x + 2)(x - 5) + (y - 2)(y - 1) = 0$
- Find the equation of the circle whose centre is (-3, -2) and having circumference 16π
- Find the equation of the circle whose centre is (2,3) and which passes through (1,4)
- Find the equation of the circle passing through the points (0, 1), (4, 3) and (1, -1).
- Find the equation of the circle on the line joining the points (1,0), (0,1) and having its centre on the line $x + y = 1$
- If the lines $x + y = 6$ and $x + 2y = 4$ are diameters of the circle, and the circle passes through the point (2, 6) then find its equation.
- Find the equation of the circle having (4, 7) and (-2, 5) as the extremities of a diameter.
- Find the Cartesian equation of the circle whose parametric equations are $x = 3\cos\theta$, $y = 3\sin\theta$, $0 \leq \theta \leq 2\pi$.

3.4.5 Tangents

The equation of the tangent to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ at } (x_1, y_1) \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0 .$$

Corollary:

The equation of the tangent at (x_1, y_1) to the circle

$$x^2 + y^2 = a^2 \text{ is } xx_1 + yy_1 = a^2$$

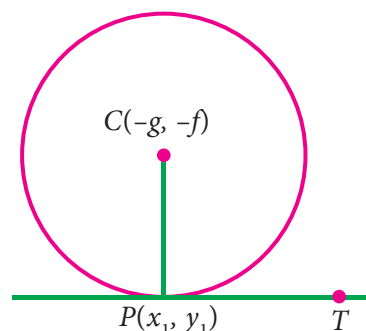


Fig. 3.7



NOTE

To get the equation of the tangent at (x_1, y_1) to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots (1)$$

replace x^2 by xx_1 , y^2 by yy_1 , x by $\frac{x+x_1}{2}$ and y by $\frac{y+y_1}{2}$ in equation (1).

Example 3.27

Find the equation of tangent at the point $(-2, 5)$ on the circle

$$x^2 + y^2 + 3x - 8y + 17 = 0.$$

Solution

The equation of the tangent at (x_1, y_1) to the given circle $x^2 + y^2 + 3x - 8y + 17 = 0$ is $xx_1 + yy_1 + 3 \times \frac{1}{2}(x + x_1) - 8 \times \frac{1}{2}(y + y_1) + 17 = 0$

Here $(x_1, y_1) = (-2, 5)$

$$-2x + 5y + \frac{3}{2}(x - 2) - 4(y + 5) + 17 = 0$$

$$-2x + 5y + \frac{3}{2}x - 3 - 4y - 20 + 17 = 0$$

$$-4x + 10y + 3x - 6 - 8y - 40 + 34 = 0$$

$x - 2y + 12 = 0$ is the required equation.

Length of the tangent to the circle

Length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point $P(x_1, y_1)$

is $PT = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

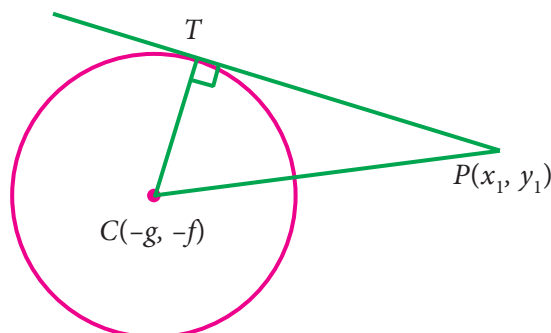


Fig. 3.8



NOTE

- (i) If the point P is on the circle then $PT^2 = 0$
- (ii) If the point P is outside the circle then $PT^2 > 0$
- (iii) If the point P is inside the circle then $PT^2 < 0$

Example 3.28

Find the length of the tangent from the point (2, 3) to the circle

$$x^2 + y^2 + 8x + 4y + 8 = 0$$

Solution

The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$$\begin{aligned} \text{Length of the tangent} &= \sqrt{x_1^2 + y_1^2 + 8x_1 + 4y_1 + 8} \\ &= \sqrt{2^2 + 3^2 + 8(2) + 4(3) + 8} \quad [\text{Here } (x_1, y_1) = (2, 3)] \\ &= \sqrt{49} \end{aligned}$$

$$\text{Length of the tangent} = 7 \text{ units}$$

Example 3.29

Determine whether the points $P(0,1)$, $Q(5,9)$, $R(-2, 3)$ and $S(2, 2)$ lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x + 4y - 8 = 0$

Solution

The equation of the circle is $x^2 + y^2 - 4x + 4y - 8 = 0$

$$PT^2 = x_1^2 + y_1^2 - 4x_1 + 4y_1 - 8$$

$$\text{At } P(0, 1) \quad PT^2 = 0 + 1 + 0 + 4 - 8 = -3 < 0$$

$$\text{At } Q(5, 9) \quad QT^2 = 25 + 81 - 20 + 36 - 8 = 114 > 0$$

$$\text{At } R(-2, 3) \quad RT^2 = 4 + 9 + 8 + 12 - 8 = 25 > 0$$

$$\text{At } S(2, 2) \quad ST^2 = 4 + 4 - 8 + 8 - 8 = 0$$

\therefore The point P lies inside the circle. The points Q and R lie outside the circle and the point S lies on the circle.

Result

Condition for the straight line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$

Example 3.30

Find the value of k so that the line $3x + 4y - k = 0$ is a tangent to $x^2 + y^2 - 64 = 0$

Solution

The given equations are $x^2 + y^2 - 64 = 0$ and $3x + 4y - k = 0$

The condition for the tangency is $c^2 = a^2(1 + m^2)$

Here $a^2 = 64$, $m = \frac{-3}{4}$ and $c = \frac{k}{4}$

$$c^2 = a^2(1 + m^2) \Rightarrow \frac{k^2}{16} = 64\left(1 + \frac{9}{16}\right)$$

$$k^2 = 64 \times 25$$

$$k = \pm 40$$

**Exercise 3.5**

1. Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 4y - 8 = 0$ at $(-2, -2)$
2. Determine whether the points $P(1, 0)$, $Q(2, 1)$ and $R(2, 3)$ lie outside the circle, on the circle or inside the circle $x^2 + y^2 - 4x - 6y + 9 = 0$
3. Find the length of the tangent from $(1, 2)$ to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$
4. Find the value of P if the line $3x + 4y - P = 0$ is a tangent to the circle $x^2 + y^2 = 16$

3.5 Conics**Definition 3.3**

If a point moves in a plane such that its distance from a fixed point bears a constant ratio to its perpendicular distance from a fixed straight line, then the path described by the moving point is called a **conic**.

In figure, the fixed point F is called focus, the fixed straight line l is called directrix and P is the moving point such that $\frac{FP}{PM} = e$, a constant. Here the locus of P is called a conic and the constant 'e' is called the eccentricity of the conic.

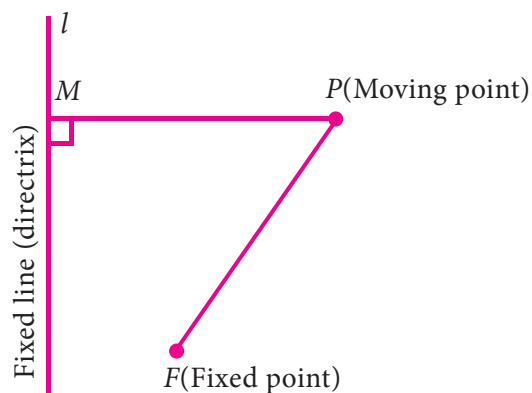


Fig. 3.9

Based on the value of eccentricity we can classify the conics namely,

- If $e = 1$, then, the conic is called a parabola
- If $e < 1$, then, the conic is called an ellipse
- If $e > 1$, then, the conic is called a hyperbola.

The general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

- a pair of straight lines if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$
- a circle if $a = b$ and $h = 0$

If the above two conditions are not satisfied, then $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents,

- a parabola if $h^2 - ab = 0$
- an ellipse if $h^2 - ab < 0$
- a hyperbola if $h^2 - ab > 0$

In this chapter, we study about parabola only.

3.5.1 Parabola

Definition 3.4

The locus of a point whose distance from a fixed point is equal to its distance from a fixed line is called a **parabola**.

$y^2 = 4ax$ is the standard equation of the parabola. It is perpendicular

3.5.2 Definitions regarding a parabola: $y^2 = 4ax$

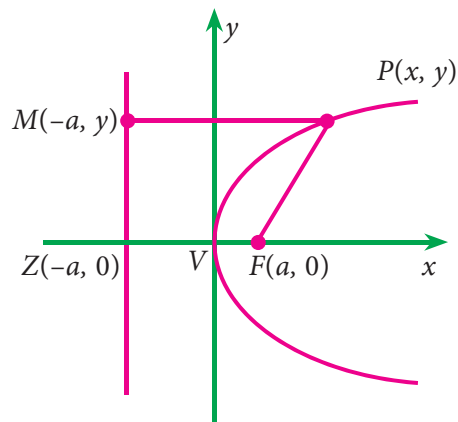


Fig. 3.10

Focus	The fixed point used to draw the parabola is called the focus (F). Here, the focus is $F(a, 0)$.
Directrix	The fixed line used to draw a parabola is called the directrix of the parabola. Here, the equation of the directrix is $x = -a$.
Axis	The axis of the parabola is the axis of symmetry. The curve $y^2 = 4ax$ is symmetrical about x-axis and hence x-axis or $y = 0$ is the axis of the parabola $y^2 = 4ax$. Note that the axis of the parabola passes through the focus and perpendicular to the directrix.
Vertex	The point of intersection of the parabola with its axis is called its vertex. Here, the vertex is $V(0, 0)$.
Focal distance	The distance between a point on the parabola and its focus is called a focal distance
Focal chord	A chord which passes through the focus of the parabola is called the focal chord of the parabola.
Latus rectum	It is a focal chord perpendicular to the axis of the parabola. Here, the equation of the latus rectum is $x = a$. Length of the latus rectum is $4a$.

3.5.3 Other standard parabolas :

1. Open leftward : $y^2 = -4ax [a > 0]$

If $x > 0$, then y become imaginary. i.e., the curve exist for $x \leq 0$.

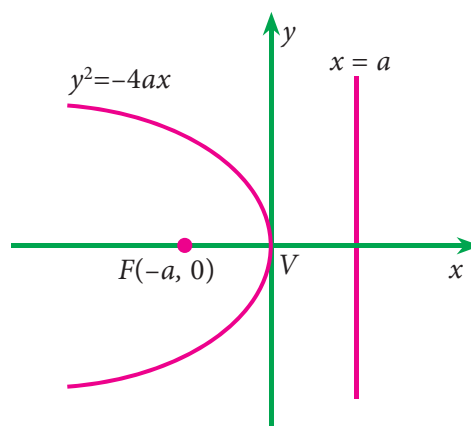


Fig. 3.11

2. Open upward : $x^2 = 4ay [a > 0]$

If $y < 0$, then x becomes imaginary. i.e., the curve exist for $y \geq 0$.

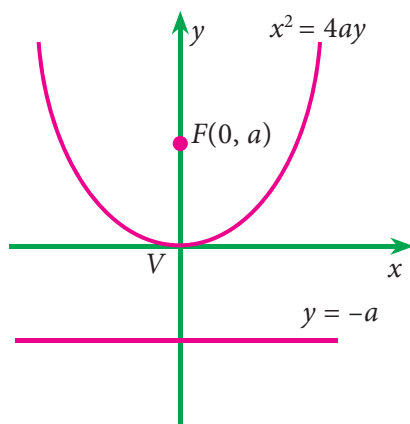


Fig. 3.12

3. Open downward : $x^2 = -4ay$ [$a > 0$]

If $y > 0$, then x becomes imaginary. i.e., the curve exist for $y \leq 0$.

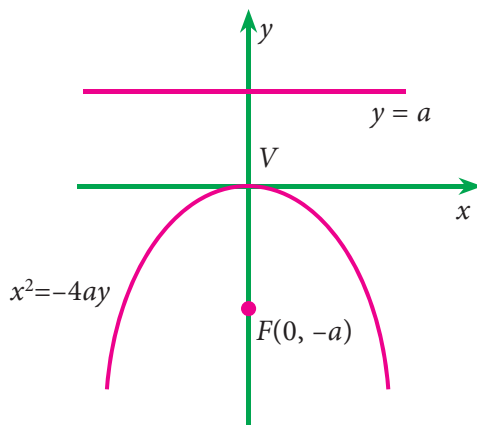


Fig. 3.13

Equations	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Vertex	$V(0, 0)$	$V(0, 0)$	$V(0, 0)$	$V(0, 0)$
Focus	$F(a, 0)$	$F(-a, 0)$	$F(0, a)$	$F(0, -a)$
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Length of Latus rectum	$4a$	$4a$	$4a$	$4a$
Equation of Latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$

The process of shifting the origin or translation of axes.

Consider the xoy system. Draw a line parallel to x -axis (say X axis) and draw a line parallel to y -axis (say Y axis). Let $P(x, y)$ be a point with respect to xoy system and $P(X, Y)$ be the same point with respect to $XO'Y$ system.

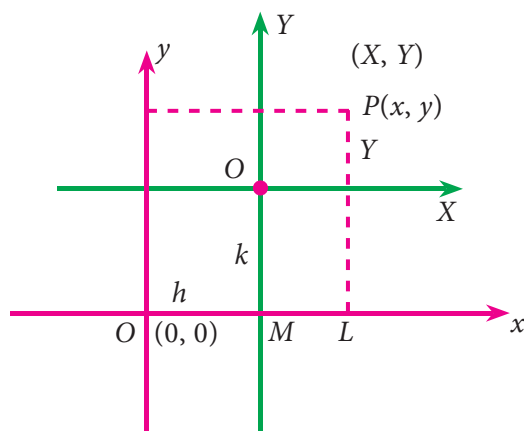


Fig. 3.14

Let the co-ordinates of O' with respect to xoy system be (h, k)

The co-ordinate of P with respect to xoy system:

$$\begin{aligned} OL &= OM + ML \\ &= h + X \end{aligned}$$

$$\text{i.e) } x = X + h$$

$$\text{similarly } y = Y + k$$

\therefore The new co-ordinates of P with respect to $XO'Y$ system.

$$X = x - h \text{ and } Y = y - k$$

3.5.4 General form of the standard equation of a parabola, which is open rightward (i.e., the vertex other than origin) :

Consider a parabola with vertex V whose co-ordinates with respect to XOY system is $(0, 0)$ and with respect to xoy system is (h, k) . Since it is open rightward, the equation of the parabola w.r.t. XOY system is $Y^2 = 4aX$.

By shifting the origin $X = x - h$ and $Y = y - k$, the equation of the parabola with respect to old xoy system is $(y - k)^2 = 4a(x - h)$.

This is the general form of the standard equation of the parabola, which is open rightward.

Similarly the other general forms are

$$(y - k)^2 = -4a(x - h) \text{ (open leftwards)}$$

$$(x - h)^2 = 4a(y - k) \text{ (open upwards)}$$

$$(x - h)^2 = -4a(y - k) \text{ (open downwards)}$$

Example 3.31

Find the equation of the parabola whose focus is (1,3) and whose directrix is $x - y + 2 = 0$.

Solution

F is (1, 3) and directrix is $x - y + 2 = 0$

$$x - y + 2 = 0$$

let $P(x, y)$ be any point on the parabola.

$$\text{For parabola } \frac{FP}{PM} = 1$$

$$FP = PM$$

$$\begin{aligned} FP^2 &= (x - 1)^2 + (y - 3)^2 \\ &= x^2 - 2x + 1 + y^2 - 6y + 9 \end{aligned}$$

$$= x^2 + y^2 - 2x - 6y + 10$$

$$PM = \pm \frac{(x - y + 2)}{\sqrt{1 + 1}}$$

$$= \pm \frac{(x - y + 2)}{\sqrt{2}}$$

$$PM^2 = \frac{(x - y + 2)^2}{2} = \frac{x^2 + y^2 + 4 - 2xy - 4y + 4x}{2}$$

$$FP^2 = PM^2$$

$$2x^2 + 2y^2 - 4x - 12y + 20 = x^2 + y^2 - 2xy - 4y + 4x + 4$$

The required equation of the parabola is

$$\Rightarrow x^2 + y^2 + 2xy - 8x - 8y + 16 = 0$$

Example 3.32

Find the focus, the vertex, the equation of the directrix, the axis and the length of the latus rectum of the parabola $y^2 = -12x$

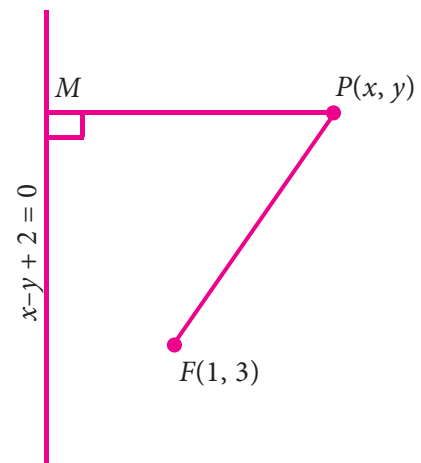


Fig. 3.15

Solution

The given equation is of the form $y^2 = -4ax$

where $4a = 12$, $a = 3$.

The parabola is open left, its focus is

$$F(-a, 0), F(-3, 0)$$

Its vertex is $V(h, k) = V(0, 0)$

The equation of the directrix is $x = a$

$$\text{i.e., } x = 3$$

Its axis is x – axis whose equation is $y = 0$.

Length of the latus rectum = $4a = 12$

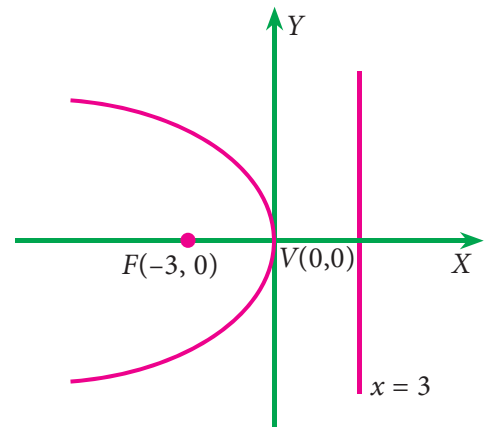


Fig. 3.16

Example 3.33

Show that the demand function $x = 10p - 20 - p^2$ is a parabola and price is maximum at its vertex.

Solution

$$\begin{aligned} x &= 10p - 20 - p^2 \\ &= -p^2 - 20 + 10p + 5 - 5 \end{aligned}$$

$$\begin{aligned} x - 5 &= -p^2 + 10p - 25 \\ &= -(p^2 - 10p + 25) \end{aligned}$$

$$= -(p - 5)^2$$

Put $X = x - 5$ and $P = p - 5$

$$X = -P^2 \quad \text{i.e. } P^2 = -X$$

It is a parabola open downward.

\therefore At the vertex $p = 5$ when $x = 5$. i.e., price is maximum at the vertex.

Example 3.34

Find the axis, vertex, focus, equation of directrix and length of latus rectum for the parabola $x^2 + 6x - 4y + 21 = 0$

Solution

$$4y = x^2 + 6x + 21$$

$$4y = (x^2 + 6x + 9) + 12$$

$$4y - 12 = (x + 3)^2$$

$$(x + 3)^2 = 4(y - 3)$$

$$X = x + 3, Y = y - 3$$

$$x = X - 3 \text{ and } y = Y + 3$$

$$X^2 = 4Y$$

Comparing with $X^2 = 4aY$, $4a = 4$

$$a = 1$$

	Referred to (X, Y)	Referred to (x, y) $x = X - 3, y = Y + 3$
Axis $y = 0$	$Y = 0$	$y = 3$
Vertex $V(0,0)$	$V(0,0)$	$V(-3,3)$
Focus $F(0, a)$	$F(0, 1)$	$F(-3, 4)$
Equation of directrix ($y = -a$)	$Y = -1$	$y = 2$
Length of Latus rectum ($4a$)	$4(1) = 4$	4

Example 3.35

The supply of a commodity is related to the price by the relation $x = \sqrt{5P - 15}$. Show that the supply curve is a parabola.

Solution

The supply price relation is given by $x^2 = 5p - 15$

$$= 5(p - 3)$$

$$\Rightarrow x^2 = 4aP \text{ Where } X = x \text{ and } P = p - 3$$

\therefore the supply curve is a parabola whose vertex is $(X = 0, P = 0)$

i.e., The supply curve is a parabola whose vertex is $(0, 3)$



ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Locus-Parabola”



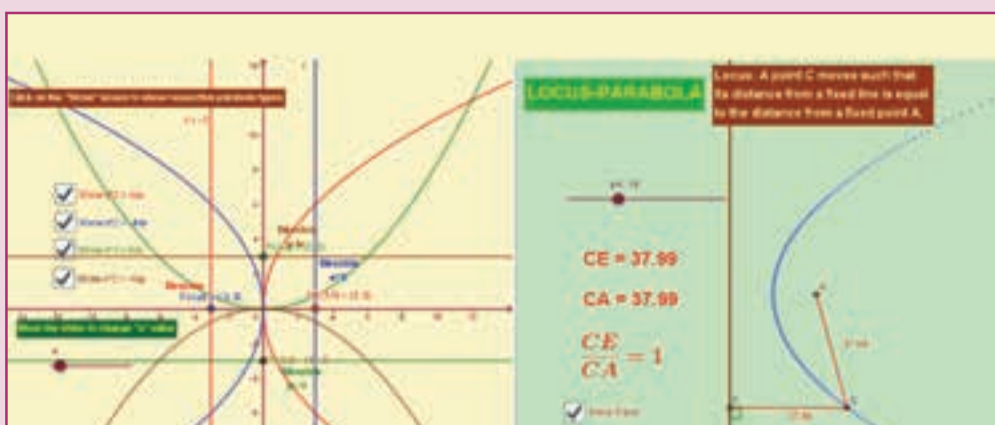
Step - 2

Locus-Parabola page will open. Click on the check boxes on the Left-hand side to see the respective parabola types with Directrix and Focus. On right-hand side Locus for parabola is given. You can play/pause for the path of the locus and observe the condition for the locus.

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code





Exercise 3.6

- Find the equation of the parabola whose focus is the point $F(-1, -2)$ and the directrix is the line $4x - 3y + 2 = 0$.
- The parabola $y^2 = kx$ passes through the point $(4, -2)$. Find its latus rectum and focus.
- Find the vertex, focus, axis, directrix and the length of latus rectum of the parabola $y^2 - 8y - 8x + 24 = 0$
- Find the co-ordinates of the focus, vertex, equation of the directrix, axis and the length of latus rectum of the parabola
 - $y^2 = 20x$
 - $x^2 = 8y$
 - $x^2 = -16y$
- The average variable cost of a monthly output of x tonnes of a firm producing a valuable metal is ₹ $\frac{1}{5}x^2 - 6x + 100$. Show that the average variable cost curve is a parabola. Also find the output and the average cost at the vertex of the parabola.
- The profit ₹ y accumulated in thousand in x months is given by $y = -x^2 + 10x - 15$. Find the best time to end the project.



Exercise 3.7



Choose the correct answer

- If m_1 and m_2 are the slopes of the pair of lines given by $ax^2 + 2hxy + by^2 = 0$, then the value of $m_1 + m_2$ is
 - $\frac{2h}{b}$
 - $-\frac{2h}{b}$
 - $\frac{2h}{a}$
 - $-\frac{2h}{a}$
- The angle between the pair of straight lines $x^2 - 7xy + 4y^2 = 0$ is
 - $\tan^{-1}\left(\frac{1}{3}\right)$
 - $\tan^{-1}\left(\frac{1}{2}\right)$
 - $\tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$
 - $\tan^{-1}\left(\frac{5}{\sqrt{33}}\right)$
- If the lines $2x - 3y - 5 = 0$ and $3x - 4y - 7 = 0$ are the diameters of a circle, then its centre is
 - $(-1, 1)$
 - $(1, 1)$
 - $(1, -1)$
 - $(-1, -1)$
- The x -intercept of the straight line $3x + 2y - 1 = 0$ is
 - 3
 - 2
 - $\frac{1}{3}$
 - $\frac{1}{2}$

5. The slope of the line $7x + 5y - 8 = 0$ is
 (a) $\frac{7}{5}$ (b) $-\frac{7}{5}$ (c) $\frac{5}{7}$ (d) $-\frac{5}{7}$
6. The locus of the point P which moves such that P is at equidistance from their coordinate axes is
 (a) $y = \frac{1}{x}$ (b) $y = -x$ (c) $y = x$ (d) $y = \frac{-1}{x}$
7. The locus of the point P which moves such that P is always at equidistance from the line $x + 2y + 7 = 0$ is
 (a) $x + 2y + 2 = 0$ (b) $x - 2y + 1 = 0$ (c) $2x - y + 2 = 0$ (d) $3x + y + 1 = 0$
8. If $kx^2 + 3xy - 2y^2 = 0$ represent a pair of lines which are perpendicular then k is equal to
 (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 2 (d) -2
9. $(1, -2)$ is the centre of the circle $x^2 + y^2 + ax + by - 4 = 0$, then its radius
 (a) 3 (b) 2 (c) 4 (d) 1
10. The length of the tangent from $(4,5)$ to the circle $x^2 + y^2 = 16$ is
 (a) 4 (b) 5 (c) 16 (d) 25
11. The focus of the parabola $x^2 = 16y$ is
 (a) $(4, 0)$ (b) $(-4, 0)$ (c) $(0, 4)$ (d) $(0, -4)$
12. Length of the latus rectum of the parabola $y^2 = -25x$.
 (a) 25 (b) -5 (c) 5 (d) -25
13. The centre of the circle $x^2 + y^2 - 2x + 2y - 9 = 0$ is
 (a) $(1, 1)$ (b) $(-1, -1)$ (c) $(-1, 1)$ (d) $(1, -1)$
14. The equation of the circle with centre on the x axis and passing through the origin is
 (a) $x^2 - 2ax + y^2 = 0$ (b) $y^2 - 2ay + x^2 = 0$
 (c) $x^2 + y^2 = a^2$ (d) $x^2 - 2ay + y^2 = 0$

15. If the centre of the circle is $(-a, -b)$ and radius is $\sqrt{a^2 - b^2}$, then the equation of circle is
- (a) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$ (b) $x^2 + y^2 + 2ax + 2by - 2b^2 = 0$
(c) $x^2 + y^2 - 2ax - 2by - 2b^2 = 0$ (d) $x^2 + y^2 - 2ax - 2by + 2b^2 = 0$
16. Combined equation of co-ordinate axes is
- (a) $x^2 - y^2 = 0$ (b) $x^2 + y^2 = 0$ (c) $xy = c$ (d) $xy = 0$
17. $ax^2 + 4xy + 2y^2 = 0$ represents a pair of parallel lines then 'a' is
- (a) 2 (b) -2 (c) 4 (d) -4
18. In the equation of the circle $x^2 + y^2 = 16$ then y intercept is (are)
- (a) 4 (b) 16 (c) ± 4 (d) ± 16
19. If the perimeter of the circle is 8π units and centre is (2,2) then the equation of the circle is
- (a) $(x - 2)^2 + (y - 2)^2 = 4$ (b) $(x - 2)^2 + (y - 2)^2 = 16$
(c) $(x - 4)^2 + (y - 4)^2 = 2$ (d) $x^2 + y^2 = 4$
20. The equation of the circle with centre (3, -4) and touches the x - axis is
- (a) $(x - 3)^2 + (y - 4)^2 = 4$ (b) $(x - 3)^2 + (y + 4)^2 = 16$
(c) $(x - 3)^2 + (y - 4)^2 = 16$ (d) $x^2 + y^2 = 16$
21. If the circle touches x axis, y axis and the line $x = 6$ then the length of the diameter of the circle is
- (a) 6 (b) 3 (c) 12 (d) 4
22. The eccentricity of the parabola is
- (a) 3 (b) 2 (c) 0 (d) 1
23. The double ordinate passing through the focus is
- (a) focal chord (b) latus rectum (c) directrix (d) axis

24. The distance between directrix and focus of a parabola $y^2 = 4ax$ is
 (a) a (b) $2a$ (c) $4a$ (d) $3a$
25. The equation of directrix of the parabola $y^2 = -x$ is
 (a) $4x + 1 = 0$ (b) $4x - 1 = 0$ (c) $x - 4 = 0$ (d) $x + 4 = 0$

Miscellaneous Problems

1. A point P moves so that P and the points $(2, 2)$ and $(1, 5)$ are always collinear. Find the locus of P .
2. As the number of units produced increases from 500 to 1000 and the total cost of production increases from ₹ 6000 to ₹9000. Find the relationship between the cost (y) and the number of units produced (x) if the relationship is linear.
3. Prove that the lines $4x + 3y = 10$, $3x - 4y = -5$ and $5x + y = 7$ are concurrent.
4. Find the value of p for which the straight lines $8px + (2 - 3p)y + 1 = 0$ and $px + py - 7 = 0$ are perpendicular to each other.
5. If the slope of one of the straight lines $ax^2 + 2hxy + by^2 = 0$ is thrice that of the other, then show that $3h^2 = 4ab$.
6. Find the values of a and b if the equation $(a - 1)x^2 + by^2 + (b - 8)xy + 4x + 4y - 1 = 0$ represents a circle.
7. Find whether the points $(-1, -2)$, $(1, 0)$ and $(-3, -4)$ lie above, below or on the line $3x + 2y + 7 = 0$.
8. If $(4, 1)$ is one extremity of a diameter of the circle $x^2 + y^2 - 2x + 6y - 15 = 0$, find the other extremity.
9. Find the equation of the parabola which is symmetrical about x-axis and passing through $(-2, -3)$.
10. Find the axis, vertex, focus, equation of directrix length of latus rectum of the parabola $(y - 2)^2 = 4(x - 1)$

Summary



- Angle between the two intersecting lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is $\theta = \tan^{-1} \left| \frac{m_1 - m_2}{1 + m_1m_2} \right|$
- Condition for the three straight lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, and $a_3x + b_3y + c_3 = 0$ to be concurrent is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$
- The condition for a general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
- Pair of straight lines passing through the origin is $ax^2 + 2hxy + by^2 = 0$.
- Let $ax^2 + 2hxy + by^2 = 0$ be the straight lines passing through the origin then the product of the slopes is $\frac{a}{b}$ and sum of the slopes is $\frac{-2h}{b}$.
- If θ is the angle between the pair of straight lines
- $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then $\theta = \tan^{-1} \left[\left| \frac{\pm 2\sqrt{h^2 - ab}}{a + b} \right| \right]$.
- The equation of a circle with centre (h, k) and the radius r is $(x - h)^2 + (y - k)^2 = r^2$
- The equation of a circle with centre at origin and the radius r is $x^2 + y^2 = r^2$.
- The equation of the circle with (x_1, y_1) and (x_2, y_2) as end points of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$.
- The parametric equations of the circle $x^2 + y^2 = r^2$ are $x = r \cos \theta$, $y = r \sin \theta$
 $0 \leq \theta \leq 2\pi$
- The equation of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$
- The equation of the tangent at (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$.
- Length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from a point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$.
- Condition for the straight line $y = mx + c$ to be a tangent to the circle $x^2 + y^2 = a^2$ is $c^2 = a^2(1 + m^2)$.
- The standard equation of the parabola is $y^2 = 4ax$

GLOSSARY

Centre	மையம்
Chord	நாண்
Circle	வட்டம்
Concentric Lines	ஒரு புள்ளி வழிக் கோடு
Conics	கூம்பு வெட்டிகள்
Diameter	விட்டம்
Directrix	இயக்குவரை
Equation	சமன்பாடு
Focal distance	குவியத்தூரம்
Foci	குவியம்
Latus rectum	செவ்வகலம்
Locus of the tag n	தொடுகோட்டின் நீளம்
Locus	நியமப்பாதை அல்லது இயங்குவரை
Origin	ஆதி
Pair of straight lines	இரட்டை நேர்கோடு
Parabola	பரவளையம்
Parallel lines	இணை கோடு
Parameter	துணையலகு
Perpendicular lines	செங்குத்து கோடு
Point of concurrency	ஒருங்கிணைவுப் புள்ளி
Point of intersection	வெட்டும் புள்ளி
Radii	ஆரம்
Straight line	நேர்கோடு
Tag n	தொடுகோடு
Vertex	முனை

TRIGONOMETRY



Learning Objectives

After studying this chapter, the students will be able to understand

- trigonometric ratio of angles
- addition formulae, multiple and sub-multiple angles
- transformation of sums into products and vice versa
- basic concepts of inverse trigonometric functions
- properties of inverse trigonometric functions



Introduction

The word trigonometry is derived from the Greek word 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, trigonometry is the study of relationships between the sides and angles of a triangle. Around second century A.D. George Rheticus was the first to define the trigonometric functions in terms of right angles. The study of trigonometry was first started in India. The ancient Indian Mathematician, Aryabhatta, Brahmagupta, Bhaskara I and Bhaskara II obtained important results. Bhaskara I gave formulae to find the values of sine functions for angles more than 90 degrees. The earliest applications of trigonometry were in the fields of navigation, surveying and astronomy.



Brahmagupta



Currently, trigonometry is used in many areas such as electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analyzing a musical tone.

Recall

1. $\sin\theta = \frac{\text{side opposite to angle } \theta}{\text{Hypotenuse}}$
2. $\cos\theta = \frac{\text{Adjacent side to angle } \theta}{\text{Hypotenuse}}$
3. $\tan\theta = \frac{\text{side opposite to angle } \theta}{\text{Adjacent side to angle } \theta}$
4. $\cot\theta = \frac{\text{Adjacent side to angle } \theta}{\text{side opposite to angle } \theta}$
5. $\sec\theta = \frac{\text{Hypotenuse}}{\text{Adjacent side to angle } \theta}$
6. $\text{cosec}\theta = \frac{\text{Hypotenuse}}{\text{side opposite to angle } \theta}$

Relations between trigonometric ratios

1. $\sin\theta = \frac{1}{\text{cosec}\theta}$ or $\text{cosec}\theta = \frac{1}{\sin\theta}$
2. $\cos\theta = \frac{1}{\sec\theta}$ or $\sec\theta = \frac{1}{\cos\theta}$
3. $\tan\theta = \frac{\sin\theta}{\cos\theta}$ or $\cot\theta = \frac{\cos\theta}{\sin\theta}$
4. $\tan\theta = \frac{1}{\cot\theta}$ or $\cot\theta = \frac{1}{\tan\theta}$

Trigonometric Identities

1. $\sin^2\theta + \cos^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \text{cosec}^2\theta.$

Angle

Angle is a measure of rotation of a given ray about its initial point. The ray 'OA' is called the initial side and the final position of the ray 'OB' after rotation is called the terminal side of the angle. The point 'O' of rotation is called the vertex.

If the direction of rotation is anticlockwise, then angle is said to be positive and if the direction of rotation is clockwise, then angle is said to be negative.

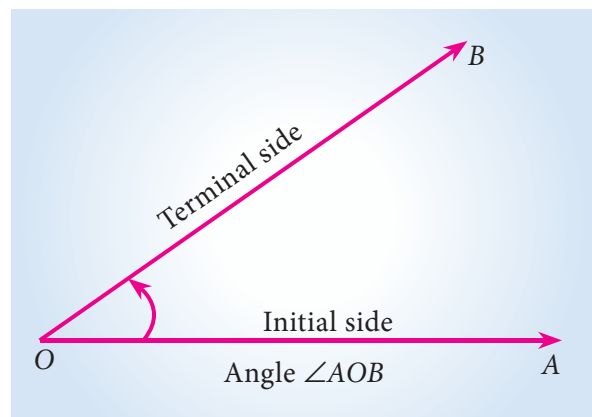


Fig. 4.1

Measurement of an angle

Two types of units of measurement of an angle which are most commonly used namely degree measure and radian measure.

Degree measure

If a rotation from the initial position to the terminal position $\left(\frac{1}{360}\right)^{th}$ of the revolution, the angle is said to have a measure of one degree and written as 1° . A degree is divided into minutes and minute is divided into seconds.

$$\begin{aligned}\text{One degree} &= 60 \text{ minutes } 60' \\ \text{One minute} &= 60 \text{ seconds } (60'')\end{aligned}$$

Radian measure

The angle subtended at the centre of the circle by an arc equal to the length of the radius of the circle is called a radian, and it is denoted by 1^c

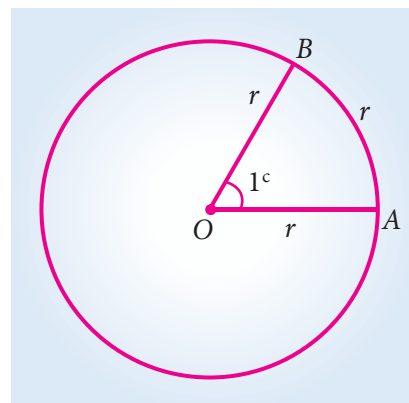


Fig. 4.2



NOTE

The number of radians in an angle subtended by an arc of a circle at the centre of a circle is

$$= \frac{\text{length of the arc}}{\text{radius}}$$

i.e., $\theta = \frac{s}{r}$, where θ is the angle subtended at the centre of a circle of radius r by an arc of length s .

Relation between degrees and radians

We know that the circumference of a circle of radius 1 unit is 2π . One complete revolution of the radius of unit circle subtends 2π radians. A circle subtends at the centre an angle whose degree measure is 360° .

$$2\pi \text{ radians} = 360^\circ$$

$$\pi \text{ radians} = 180^\circ$$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

Example 4.1

Convert (i) 160° into radians (ii) $\frac{4\pi}{5}$ radians into degree (iii) $\frac{1}{4}$ radians into degree

Solution

$$i) 160^\circ = 160 \times \frac{\pi}{180} \text{ radians} = \frac{8}{9} \pi$$

$$ii) \frac{4\pi}{5} \text{ radians} = \frac{4}{5} \pi \times \frac{180^\circ}{\pi} = 144^\circ$$

$$iii) \frac{1}{4} \text{ radian} = \frac{1}{4} \frac{180}{\pi} = \frac{1}{4} \times 180 \times \frac{7}{22} = 14^\circ 19' 5''$$

4.1 Trigonometric ratios**4.1.1 Quadrants:**

Let $X'OX$ and $Y'OY$ be two lines at right angles to each other as in the figure. We call $X'OX$ and $Y'OY$ as X axis and Y axis respectively. Clearly these axes divided the entire plane into four equal parts called "Quadrants".

The parts XOY , YOX' , $X'OY'$ and $Y'OX$ are known as the first, second, third and the fourth quadrant respectively.

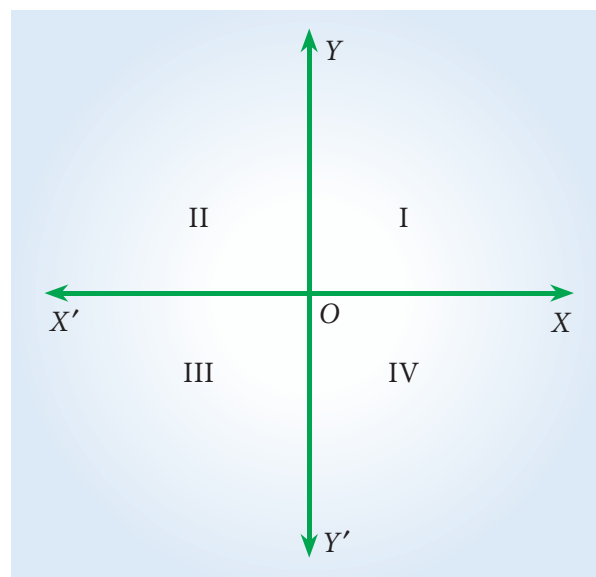


Fig. 4.3

Example 4.2

Find the quadrants in which the terminal sides of the following angles lie.

- (i) -70° (ii) -320° (iii) 1325°

Solution (i)

1. (i) The terminal side of -70° lies in IV quadrant.

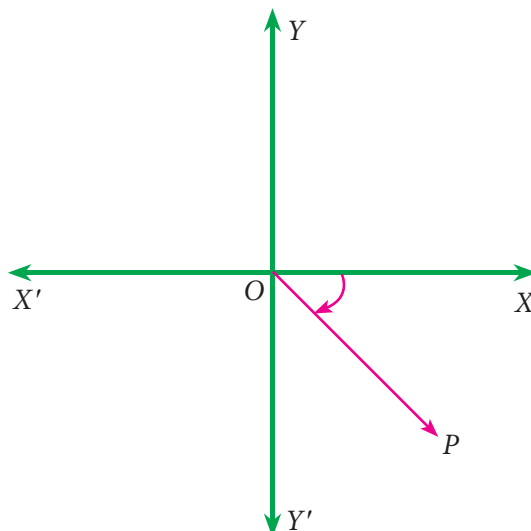


Fig. 4.4

(ii) The terminal side -320° of lies in I quadrant.

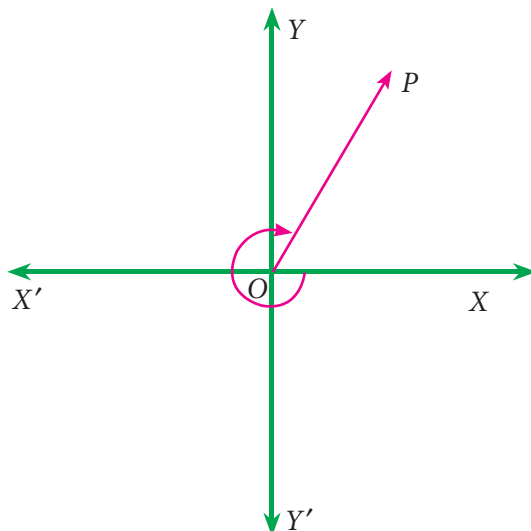


Fig. 4.5

(iii) $1325^\circ = (3 \times 360) + 180^\circ + 65^\circ$ the terminal side lies in III quadrant.

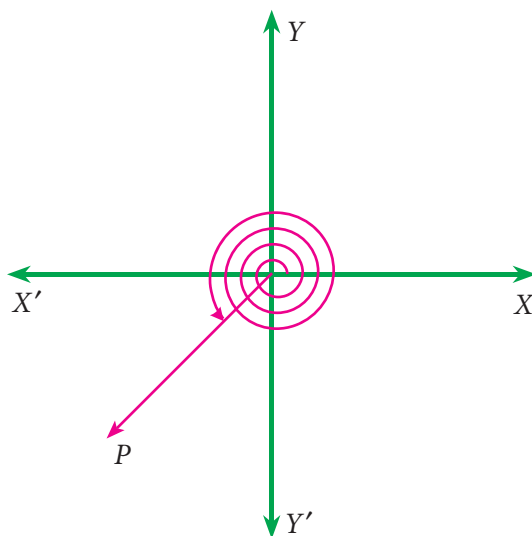
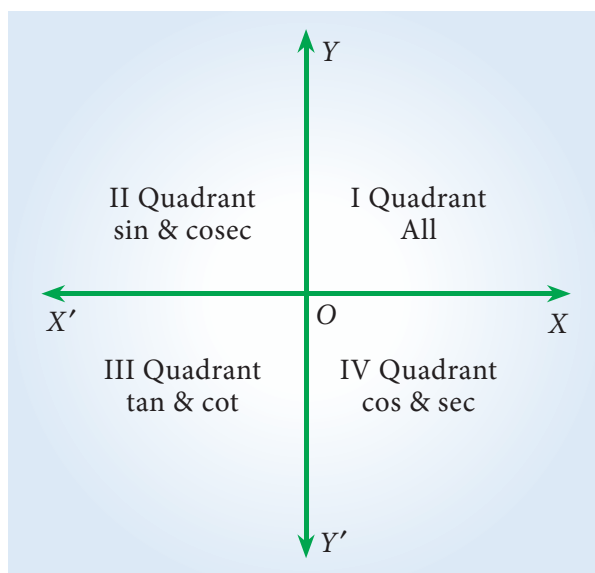


Fig. 4.6

4.1.2 Signs of the trigonometric ratios of an angle θ as it varies from 0° to 360°

In the first quadrant both x and y are positive. So all trigonometric ratios are positive. In the second quadrant ($90^\circ < \theta < 180^\circ$) x is negative and y is positive. So trigonometric ratios $\sin \theta$ and $\operatorname{cosec} \theta$ are positive. In the third quadrant ($180^\circ < \theta < 270^\circ$) both x and y are negative. So trigonometric ratios $\tan \theta$ and $\cot \theta$ are positive. In the fourth quadrant ($270^\circ < \theta < 360^\circ$) x is positive and y is negative. So trigonometric ratios $\cos \theta$ and $\sec \theta$ are positive.

A function $f(x)$ is said to be odd function if $f(-x) = -f(x)$. $\sin \theta$, $\tan \theta$, $\cot \theta$ and $\operatorname{cosec} \theta$ are all odd function. A function $f(x)$ is said to be even function if $f(-x) = f(x)$. $\cos \theta$ and $\sec \theta$ are even function.



ASTC : All Sin Tan Cos

Fig. 4.7

4.1.3 Trigonometric ratios of allied angles:

Two angles are said to be allied angles when their sum or difference is either zero or a multiple of 90° . The angles $-\theta$, $90^\circ \pm \theta$, $180^\circ \pm \theta$, $360^\circ \pm \theta$ etc., are angles allied to the angle θ . Using trigonometric ratios of the allied angles we can find the trigonometric ratios of any angle.

Angle/ Function	$-\theta$	$90^\circ - \theta$ or $\frac{\pi}{2} - \theta$	$90^\circ + \theta$ or $\frac{\pi}{2} + \theta$	$180^\circ - \theta$ or $\pi - \theta$	$180^\circ + \theta$ or $\pi + \theta$	$270^\circ - \theta$ or $\frac{3\pi}{2} - \theta$	$270^\circ + \theta$ or $\frac{3\pi}{2} + \theta$	$360^\circ - \theta$ or $2\pi - \theta$	$360^\circ + \theta$ or $2\pi + \theta$
Sine	$-\sin \theta$	$\cos \theta$	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$
Cosine	$\cos \theta$	$\sin \theta$	$-\sin \theta$	$-\cos \theta$	$-\cos \theta$	$-\sin \theta$	$\sin \theta$	$\cos \theta$	$\cos \theta$
tangent	$-\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$	$\cot \theta$	$-\cot \theta$	$-\tan \theta$	$\tan \theta$
cotangent	$-\cot \theta$	$\tan \theta$	$-\tan \theta$	$-\cot \theta$	$\cot \theta$	$\tan \theta$	$-\tan \theta$	$-\cot \theta$	$\cot \theta$
secant	$\sec \theta$	$\operatorname{cosec} \theta$	$-\operatorname{cosec} \theta$	$-\sec \theta$	$-\sec \theta$	$-\operatorname{cosec} \theta$	$\operatorname{cosec} \theta$	$\sec \theta$	$\sec \theta$
cosecant	$-\operatorname{cosec} \theta$	$\sec \theta$	$\sec \theta$	$\operatorname{cosec} \theta$	$-\operatorname{cosec} \theta$	$-\sec \theta$	$-\sec \theta$	$-\operatorname{cosec} \theta$	$\operatorname{cosec} \theta$

Table : 4.1

Example 4.3

Find the values of each of the following trigonometric ratios.

(i) $\sin 150^\circ$ (ii) $\cos(-210^\circ)$ (iii) $\operatorname{cosec} 390^\circ$ (iv) $\tan(-1215^\circ)$

(v) $\sec 1485^\circ$

Solution

$$(i) \quad \sin 150^\circ = \sin(1 \times 90^\circ + 60^\circ)$$

Since 150° lies in the second quadrant, we have

$$\begin{aligned} \sin 150^\circ &= \sin(1 \times 90^\circ + 60^\circ) \\ &= \cos 60^\circ = \frac{1}{2} \end{aligned}$$

$$(ii) \quad \text{we have } \cos(-210^\circ) = \cos 210^\circ$$

Since 210° lies in the third quadrant, we have

$$\cos 210^\circ = \cos(180^\circ + 30^\circ) = -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$(iii) \quad \operatorname{cosec} 390^\circ = \operatorname{cosec}(360^\circ + 30^\circ) = \operatorname{cosec} 30^\circ = 2$$

$$\begin{aligned} (iv) \quad \tan(-1215^\circ) &= -\tan(1215^\circ) = -\tan(3 \times 360^\circ + 135^\circ) \\ &= -\tan 135^\circ = -\tan(90^\circ + 45^\circ) = -(-\cot 45^\circ) = 1 \end{aligned}$$

$$(v) \quad \sec 1485^\circ = \sec(4 \times 360^\circ + 45^\circ) = \sec 45^\circ = \sqrt{2}$$

Example 4.4

Prove that $\sin 600^\circ \cos 390^\circ + \cos 480^\circ \sin 150^\circ = -1$

Solution

$$\sin 600^\circ = \sin(360^\circ + 240^\circ) = \sin 240^\circ = \sin(180^\circ + 60^\circ) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos 390^\circ = \cos(360^\circ + 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\cos 480^\circ = \cos(360^\circ + 120^\circ) = \cos 120^\circ$$

$$= \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$$

$$\sin 150^\circ = \sin(180^\circ - 30^\circ) = \sin 30^\circ = \frac{1}{2}$$

Now $\sin 600^\circ \cos 390^\circ + \cos 480^\circ \sin 150^\circ$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = \frac{-3-1}{4} = -1$$

Example 4.5

Prove that $\frac{\sin(-\theta)\tan(90^\circ - \theta)\sec(180^\circ - \theta)}{\sin(180 + \theta)\cot(360 - \theta)\operatorname{cosec}(90^\circ - \theta)} = 1$

Solution

$$\begin{aligned} \text{L.H.S} &= \frac{\sin(-\theta)\tan(90^\circ - \theta)\sec(180^\circ - \theta)}{\sin(180 + \theta)\cot(360 - \theta)\operatorname{cosec}(90^\circ - \theta)} \\ &= \frac{(-\sin\theta)\cot\theta(-\sec\theta)}{(-\sin\theta)(-\cot\theta)\sec\theta} = 1 \end{aligned}$$

**Exercise 4.1**

- Convert the following degree measure into radian measure
 - 60°
 - 150°
 - 240°
 - -320°
- Find the degree measure corresponding to the following radian measure.
 - $\frac{\pi}{8}$
 - $\frac{9\pi}{5}$
 - -3
 - $\frac{11\pi}{18}$
- Determine the quadrants in which the following degree lie.
 - 380°
 - -140°
 - 1195°
- Find the values of each of the following trigonometric ratios.
 - $\sin 300^\circ$
 - $\cos(-210^\circ)$
 - $\sec 390^\circ$
 - $\tan(-855^\circ)$
 - $\operatorname{cosec} 1125^\circ$
- Prove that:
 - $\tan(-225^\circ)\cot(-405^\circ) - \tan(-765^\circ)\cot(675^\circ) = 0$
 - $2\sin^2\frac{\pi}{6} + \operatorname{cosec}^2\frac{7\pi}{6} \cos^2\frac{\pi}{3} = \frac{3}{2}$
 - $\sec\left(\frac{3\pi}{2} - \theta\right)\sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right)\tan\left(\theta - \frac{5\pi}{2}\right) = -1$
- If A, B, C, D are angles of a cyclic quadrilateral, prove that :

$$\cos A + \cos B + \cos C + \cos D = 0$$
- Prove that :
 - $\frac{\sin(180^\circ - \theta)\cos(90^\circ + \theta)\tan(270 - \theta)\cot(360 - \theta)}{\sin(360 - \theta)\cos(360 + \theta)\sin(270 - \theta)\operatorname{cosec}(-\theta)} = -1$
 - $\sin\theta \cdot \cos\theta \left\{ \sin\left(\frac{\pi}{2} - \theta\right) \cdot \operatorname{cosec}\theta + \cos\left(\frac{\pi}{2} - \theta\right) \cdot \sec\theta \right\} = 1$

8. Prove that : $\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ = -1$
9. Prove that: (i) $\tan(\pi + x) \cot(x - \pi) - \cos(2\pi - x) \cos(2\pi + x) = \sin^2 x$
(ii) $\frac{\sin(180^\circ + A) \cos(90^\circ - A) \tan(270^\circ - A)}{\sin(540^\circ - A) \cos(360^\circ + A) \operatorname{cosec}(270^\circ + A)} = -\sin A \cos^2 A$
10. If $\sin \theta = \frac{3}{5}$, $\tan \varphi = \frac{1}{2}$ and $\frac{\pi}{2} < \theta < \pi < \varphi < \frac{3\pi}{2}$, then find the value of $8 \tan \theta - \sqrt{5} \sec \varphi$

4.2 Trigonometric ratios of compound angles

4.2.1 Compound angles

When we add or subtract angles, the result is called a compound angle. i.e., the algebraic sum of two or more angles are called compound angles

For example, If A, B, C are three angles then $A \pm B, A + B + C, A - B + C$ etc., are compound angles.

4.2.2 Sum and difference formulae of sine, cosine and tangent

- (i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$
(ii) $\sin(A - B) = \sin A \cos B - \cos A \sin B$
(iii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$
(iv) $\cos(A - B) = \cos A \cos B + \sin A \sin B$
(v) $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
(vi) $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

Example 4.6

If $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, $\frac{3\pi}{2} < A, B < 2\pi$, find the value of

- (i) $\sin(A - B)$ (ii) $\cos(A + B)$.

Solution

Since $\frac{3\pi}{2} < A, B < 2\pi$, both A and B lie in the fourth quadrant,

$\therefore \sin A$ and $\sin B$ are negative.

Given $\cos A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$,

$$\begin{aligned}\text{Therefore, } \sin A &= -\sqrt{1 - \cos^2 A} \\ &= -\sqrt{1 - \frac{16}{25}} \\ &= -\sqrt{\frac{25 - 16}{25}} \\ &= -\frac{3}{5}\end{aligned}$$

$$\begin{aligned}\sin B &= -\sqrt{1 - \cos^2 B} \\ &= -\sqrt{1 - \frac{144}{169}} \\ &= -\sqrt{\frac{169 - 144}{169}} \\ &= -\frac{5}{13}\end{aligned}$$

$$\begin{aligned}\text{(i) } \cos(A + B) &= \cos A \cos B - \sin A \sin B \\ &= \frac{4}{5} \times \frac{12}{13} - \left(\frac{-3}{5}\right) \times \left(\frac{-5}{13}\right) \\ &= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}\end{aligned}$$

$$\begin{aligned}\text{(ii) } \sin(A - B) &= \sin A \cos B - \cos A \sin B \\ &= \left(\frac{-3}{5}\right) \left(\frac{12}{13}\right) - \left(\frac{4}{5}\right) \left(\frac{-5}{13}\right) \\ &= \frac{-36}{65} + \frac{20}{65} = \frac{-16}{65}\end{aligned}$$

Example 4.7

Find the values of each of the following trigonometric ratios.

(i) $\sin 15^\circ$ (ii) $\cos(-105^\circ)$ (iii) $\tan 75^\circ$ (iv) $-\sec 165^\circ$

Solution

$$\begin{aligned}\text{(i) } \sin 15^\circ &= \sin(45^\circ - 30^\circ) \\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} = \frac{\sqrt{3} - 1}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\text{(ii) } \cos(-105^\circ) &= \cos 105^\circ \\ &= \cos(60^\circ + 45^\circ) \\ &= \cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ \\ &= \frac{1}{2} \times \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{1 - \sqrt{3}}{2\sqrt{2}}\end{aligned}$$

$$\text{(iii) } \tan 75^\circ = \tan(45^\circ + 30^\circ)$$

$$\begin{aligned}
 &= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\
 &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \cos 165^\circ &= \cos(180^\circ - 15^\circ) \\
 &= -\cos 15^\circ \\
 &= -\cos(60^\circ - 45^\circ) \\
 &= -(\cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ) \\
 &= -\left(\frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) \\
 &= -\left(\frac{1 + \sqrt{3}}{2\sqrt{2}}\right) \\
 \therefore -\sec 165^\circ &= \frac{2\sqrt{2}}{1 + \sqrt{3}} \\
 &= \frac{2\sqrt{2}}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}} \\
 &= \sqrt{2}(\sqrt{3} - 1)
 \end{aligned}$$

Example 4.8

If $\tan A = m \tan B$, prove that $\frac{\sin(A+B)}{\sin(A-B)} = \frac{m+1}{m-1}$

Solution

Given $\tan A = m \tan B$

$$\begin{aligned}
 \frac{\sin A}{\cos A} &= m \frac{\sin B}{\cos B} \\
 \frac{\sin A \cos B}{\cos A \sin B} &= m
 \end{aligned}$$



Componendo and dividendo rule:

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

Applying componendo and dividendo rule, we get

$$\begin{aligned}
 \frac{\sin A \cos B + \cos A \sin B}{\sin A \cos B - \cos A \sin B} &= \frac{m+1}{m-1} \\
 \frac{\sin(A+B)}{\sin(A-B)} &= \frac{m+1}{m-1}, \text{ which completes the proof.}
 \end{aligned}$$

4.2.3 Trigonometric ratios of multiple angles

Identities involving $\sin 2A$, $\cos 2A$, $\tan 2A$, $\sin 3A$ etc., are called multiple angle identities.

$$(i) \quad \sin 2A = \sin(A + A) = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A.$$

$$(ii) \quad \cos 2A = \cos(A + A) = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A$$

$$\begin{aligned} (iii) \quad \sin 3A &= \sin(2A+A) \\ &= \sin 2A \cos A + \cos 2A \sin A \\ &= (2 \sin A \cos A) \cos A + (1 - 2 \sin^2 A) \sin A \\ &= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A \\ &= 2 \sin A(1 - \sin^2 A) + \sin A - 2 \sin^3 A \\ &= 3 \sin A - 4 \sin^3 A \end{aligned}$$

Thus we have the following multiple angles formulae

$$1. \quad (i) \quad \sin 2A = 2 \sin A \cos A$$

$$(ii) \quad \sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$$

$$2. \quad (i) \quad \cos 2A = \cos^2 A - \sin^2 A$$

$$(ii) \quad \cos 2A = 2 \cos^2 A - 1$$

$$(iii) \quad \cos 2A = 1 - 2 \sin^2 A$$

$$(iv) \quad \cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

$$3. \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$4. \quad \sin 3A = 3 \sin A - 4 \sin^3 A$$

$$5. \quad \cos 3A = 4 \cos^3 A - 3 \cos A$$

$$6. \quad \tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$



NOTE

$$(i) \quad \cos^2 A = \frac{1 + \cos 2A}{2} \quad (\text{or}) \quad \cos A = \pm \sqrt{\frac{1 + \cos 2A}{2}}$$

$$(ii) \quad \sin^2 A = \frac{1 - \cos 2A}{2} \quad (\text{or}) \quad \sin A = \pm \sqrt{\frac{1 - \cos 2A}{2}}$$

Example 4.9

Show that $\frac{\sin 2\theta}{1 + \cos 2\theta} = \tan \theta$

Solution

$$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2 \sin \theta \cos \theta}{2 \cos^2 \theta} = \frac{\sin \theta}{\cos \theta} = \tan \theta$$

Example 4.10

Using multiple angle identity, find $\tan 60^\circ$

Solution

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Put $A = 30^\circ$ in the above identity, we get

$$\begin{aligned} \tan 60^\circ &= \frac{2 \tan 30^\circ}{1 - \tan^2 30^\circ} \\ &= \frac{2 \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}} = \frac{\frac{2}{\sqrt{3}}}{\frac{2}{3}} = \frac{2}{\sqrt{3}} \times \frac{3}{2} \\ &= \sqrt{3} \end{aligned}$$

Example 4.11

If $\tan A = \frac{1}{7}$ and $\tan B = \frac{1}{3}$, show that $\cos 2A = \sin 4B$

Solution

$$\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A} = \frac{1 - \frac{1}{49}}{1 + \frac{1}{49}} = \frac{48}{49} \times \frac{49}{50} = \frac{24}{25} \dots (1)$$

$$\begin{aligned} \text{Now } \sin 4B &= 2 \sin 2B \cos 2B \\ &= 2 \frac{2 \tan B}{1 + \tan^2 B} \times \frac{1 - \tan^2 B}{1 + \tan^2 B} = \frac{24}{25} \dots (2) \end{aligned}$$

From (1) and (2) we get,

$$\cos 2A = \sin 4B.$$

Example 4.12

If $\tan A = \frac{1 - \cos B}{\sin B}$ then prove that $\tan 2A = \tan B$

Solution

$$\text{Consider } \frac{1 - \cos B}{\sin B} = \frac{2 \sin^2 \frac{B}{2}}{2 \sin \frac{B}{2} \cos \frac{B}{2}} = \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} = \tan \frac{B}{2}$$

$$\tan A = \frac{1 - \cos B}{\sin B}$$

$$= \tan \frac{B}{2}$$

$$A = \frac{B}{2}$$

$$2A = B$$

$$\therefore \tan 2A = \tan B$$

Example 4.13

If $\tan \alpha = \frac{1}{3}$ and $\tan \beta = \frac{1}{7}$ then prove that $(2\alpha + \beta) = \frac{\pi}{4}$.

Solution

$$\tan(2\alpha + \beta) = \frac{\tan 2\alpha + \tan \beta}{1 - \tan 2\alpha \cdot \tan \beta}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{2 \times \frac{1}{3}}{1 - \frac{1}{9}} = \frac{3}{4}$$

$$\tan(2\alpha + \beta) = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}} = \frac{\frac{25}{28}}{\frac{25}{28}}$$

$$= 1 = \tan \frac{\pi}{4}$$

$$2\alpha + \beta = \frac{\pi}{4}$$

**Exercise 4.2**

- Find the values of the following : (i) cosec15° (ii) sin(-105°) (iii) cot75°
- Find the values of the following
 - sin76° cos16° - cos76° sin16°
 - sin $\frac{\pi}{4}$ cos $\frac{\pi}{12}$ + cos $\frac{\pi}{4}$ sin $\frac{\pi}{12}$
 - cos70° cos10° - sin70° sin10°
 - cos² 15° - sin² 15°
- If $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ and $\cos B = \frac{-12}{13}$, $\pi < B < \frac{3\pi}{2}$ find the values of the following :
 - cos(A+B)
 - sin(A-B)
 - tan(A-B)

4. If $\cos A = \frac{13}{4}$ and $\cos B = \frac{1}{7}$ where A, B are acute angles prove that $A - B = \frac{\pi}{3}$
5. Prove that $2\tan 80^\circ = \tan 85^\circ - \tan 5^\circ$
6. If $\cot \alpha = \frac{1}{2}$, $\sec \beta = \frac{-5}{3}$, where $\pi < \alpha < \frac{3\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$, find the value of $\tan(\alpha + \beta)$. State the quadrant in which $\alpha + \beta$ terminates.
7. If $A + B = 45^\circ$, prove that $(1 + \tan A)(1 + \tan B) = 2$ and hence deduce the value of $\tan 22\frac{1}{2}^\circ$
8. Prove that (i) $\sin(A + 60^\circ) + \sin(A - 60^\circ) = \sin A$
(ii) $\tan 4A \tan 3A \tan A + \tan 3A + \tan A - \tan 4A = 0$
9. (i) If $\tan \theta = 3$ find $\tan 3\theta$
(ii) If $\sin A = \frac{12}{13}$, find $\sin 3A$
10. If $\sin A = \frac{3}{5}$, find the values of $\cos 3A$ and $\tan 3A$
11. Prove that $\frac{\sin(B - C)}{\cos B \cos C} + \frac{\sin(C - A)}{\cos C \cos A} + \frac{\sin(A - B)}{\cos A \cos B} = 0$.
12. If $\tan A - \tan B = x$ and $\cot B - \cot A = y$ prove that $\cot(A - B) = \frac{1}{x} + \frac{1}{y}$.
13. If $\sin \alpha + \sin \beta = a$ and $\cos \alpha + \cos \beta = b$, then prove that $\cos(\alpha - \beta) = \frac{a^2 + b^2 - 2}{2}$.
14. Find the value of $\tan \frac{\pi}{8}$.
15. If $\tan \alpha = \frac{1}{7}$, $\sin \beta = \frac{1}{\sqrt{10}}$ Prove that $\alpha + 2\beta = \frac{\pi}{4}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.

4.3 Transformation formulae

The two sets of transformation formulae are described below.

4.3.1 Transformation of the products into sum or difference

$$\sin A \cos B + \cos A \sin B = \sin(A + B) \quad \dots (1)$$

$$\sin A \cos B - \cos A \sin B = \sin(A - B) \quad \dots (2)$$

$$\cos A \cos B - \sin A \sin B = \cos(A + B) \quad \dots (3)$$

$$\cos A \cos B + \sin A \sin B = \cos(A - B) \quad \dots (4)$$

Adding (1) and (2), we get

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

Subtracting (1) from (2), we get

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

Adding (3) and (4), we get

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

Subtracting (3) from (4), we get

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

Thus, we have the following formulae;

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

Also $\sin^2 A - \sin^2 B = \sin(A + B) \sin(A - B)$

$$\cos^2 A - \sin^2 B = \cos(A + B) \cos(A - B)$$

4.3.2 Transformation of sum or difference into product

$$\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2 \cos\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{C-D}{2}\right)$$

Example 4.14

Express the following as sum or difference

- (i) $2 \sin 2\theta \cos \theta$ (ii) $2 \cos 3\theta \cos \theta$ (iii) $2 \sin 4\theta \sin 2\theta$
 (iv) $\cos 7\theta \cos 5\theta$ (v) $\cos \frac{3A}{2} \cos \frac{5A}{2}$ (vi) $\cos 9\theta \sin 6\theta$
 (vii) $2 \cos 13A \sin 15A$

Solution

- (i) $2 \sin 2\theta \cos \theta = \sin(2\theta + \theta) + \sin(2\theta - \theta)$
 $= \sin 3\theta + \sin \theta$
- (ii) $2 \cos 3\theta \cos \theta = \cos(3\theta + \theta) + \cos(3\theta - \theta)$
 $= \cos 4\theta + \cos 2\theta$
- (iii) $2 \sin 4\theta \sin 2\theta = \cos(4\theta - 2\theta) - \cos(4\theta + 2\theta)$
 $= \cos 2\theta - \cos 6\theta$
- (iv) $\cos 7\theta \cos 5\theta = \frac{1}{2}[\cos(7\theta + 5\theta) + \cos(7\theta - 5\theta)]$
 $= \frac{1}{2}[\cos 12\theta + \cos 2\theta]$
- (v) $\cos \frac{3A}{2} \cos \frac{5A}{2} = \frac{1}{2} \cos\left(\frac{3A}{2} + \frac{5A}{2}\right) + \cos\left(\frac{3A}{2} - \frac{5A}{2}\right)$
 $= \frac{1}{2} \left[\cos \frac{8A}{2} + \cos\left(\frac{-2A}{2}\right) \right]$
 $= \frac{1}{2} [\cos 4A + \cos(-A)]$
 $= \frac{1}{2} [\cos 4A + \cos A]$
- (vi) $\cos 9\theta \sin 6\theta = \frac{1}{2} [\sin(9\theta + 6\theta) - \sin(9\theta - 6\theta)]$
 $= \frac{1}{2} [\sin(15\theta) - \sin 3\theta]$
- (vii) $2 \cos 13A \sin 15A = \sin(13A + 15A) - \sin(13A - 15A)$
 $= \sin 28A + \sin 2A$

Example 4.15

Show that $\sin 20^\circ \sin 40^\circ \sin 80^\circ = \frac{\sqrt{3}}{8}$

Solution

$$\begin{aligned}
 \text{LHS} &= \sin 20^\circ \sin 40^\circ \sin 80^\circ \\
 &= \sin 20^\circ \sin(60^\circ - 20^\circ) \sin(60^\circ + 20^\circ) \\
 &= \sin 20^\circ [\sin^2 60^\circ - \sin^2 20^\circ] \\
 &= \sin 20^\circ \left[\frac{3}{4} - \sin^2 20^\circ \right] \\
 &= \sin 20^\circ \left[\frac{3 - 4 \sin^2 20^\circ}{4} \right] \\
 &= \frac{3 \sin 20^\circ - 4 \sin^3 20^\circ}{4} \\
 &= \frac{\sin 60^\circ}{4} = \frac{\sqrt{3}/2}{4} = \frac{\sqrt{3}}{8}.
 \end{aligned}$$

Example 4.16

Show that $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ = \frac{3}{16}$

Solution

$$\begin{aligned}
 \text{L.H.S} &= \sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ \\
 &= \sin 60^\circ \sin 20^\circ \sin(60^\circ - 20^\circ) \sin(60^\circ + 20^\circ) \\
 &= \frac{\sqrt{3}}{2} \sin 20^\circ (\sin^2 60^\circ - \sin^2 20^\circ) \\
 &= \frac{\sqrt{3}}{2} \sin 20^\circ \left(\frac{3}{4} - \sin^2 20^\circ \right) \\
 &= \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{4} \right) (3 \sin 20^\circ - 4 \sin^3 20^\circ) \\
 &= \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{4} \right) \sin 60^\circ = \left(\frac{\sqrt{3}}{2} \right) \left(\frac{1}{4} \right) \left(\frac{\sqrt{3}}{2} \right) = \frac{3}{16} = \text{R.H.S.}
 \end{aligned}$$

Example 4.17

Prove that $\cos^2 A + \cos^2(A + 120^\circ) + \cos^2(A - 120^\circ) = \frac{3}{2}$

Solution

$$\begin{aligned}
 &\cos^2 A + \cos^2(A + 120^\circ) + \cos^2(A - 120^\circ) \\
 &= \cos^2 A + \cos^2(A + 120^\circ) + (1 - \sin^2(A - 120^\circ)) \quad (\text{Since } \cos^2 A = 1 - \sin^2 A) \\
 &= \cos^2 A + 1 + (\cos^2(A + 120^\circ)) - \sin^2(A - 120^\circ) \\
 &= \cos^2 A + 1 + \cos[(A + 120^\circ) + (A - 120^\circ)] \cos[(A + 120^\circ) - (A - 120^\circ)]
 \end{aligned}$$

$$\begin{aligned}
&= \cos^2 A + 1 + \cos 2A \cos 240^\circ \quad (\text{Since } \cos^2 A - \sin^2 B = \cos(A+B)\cos(A-B) \text{)} \\
&= \cos^2 A + 1 + (2\cos^2 A - 1)\cos(180^\circ + 60^\circ) \quad (\text{since } 2\cos^2 A = 1 + \cos 2A) \\
&= \cos^2 A + 1 + (2\cos^2 A - 1)(-\cos 60^\circ) \\
&= \cos^2 A + 1 + (2\cos^2 A - 1)\left(-\frac{1}{2}\right) \\
&= \cos^2 A + 1 - \cos^2 A + \frac{1}{2} = \frac{3}{2}
\end{aligned}$$

Example 4.18

Convert the following into the product of trigonometric functions.

- (i) $\sin 9A + \sin 7A$ (ii) $\sin 7\theta - \sin 4\theta$ (iii) $\cos 8A + \cos 12A$
(iv) $\cos 4\alpha - \cos 8\alpha$ (v) $\cos 20^\circ - \cos 30^\circ$ (vi) $\cos 75^\circ + \cos 45^\circ$
(vii) $\cos 55^\circ + \sin 55^\circ$

Solution

$$\begin{aligned}
\text{(i)} \quad \sin 9A + \sin 7A &= 2 \sin\left(\frac{9A+7A}{2}\right) \cos\left(\frac{9A-7A}{2}\right) \\
&= 2 \sin\left(\frac{16A}{2}\right) \cos\left(\frac{2A}{2}\right) \\
&= 2 \sin 8A \cos A \\
\text{(ii)} \quad \sin 7\theta - \sin 4\theta &= 2 \cos\left(\frac{7\theta+4\theta}{2}\right) \sin\left(\frac{7\theta-4\theta}{2}\right) \\
&= 2 \cos\left(\frac{11\theta}{2}\right) \sin\left(\frac{3\theta}{2}\right) \\
\text{(iii)} \quad \cos 8A + \cos 12A &= 2 \cos\left(\frac{8A+12A}{2}\right) \cos\left(\frac{8A-12A}{2}\right) \\
&= 2 \cos\left(\frac{20A}{2}\right) \cos\left(\frac{-4A}{2}\right) \\
&= 2 \cos 10A \cos(-2A) \\
&= 2 \cos 10A \cos 2A \\
\text{(iv)} \quad \cos 4\alpha - \cos 8\alpha &= -2 \sin\left(\frac{4\alpha+8\alpha}{2}\right) \sin\left(\frac{4\alpha-8\alpha}{2}\right) \\
&= 2 \sin\left(\frac{12\alpha}{2}\right) \sin\left(\frac{4\alpha}{2}\right) \\
&= 2 \sin 6\alpha \sin 2\alpha \\
\text{(v)} \quad \cos 20^\circ - \cos 30^\circ &= -2 \sin\left(\frac{20^\circ+30^\circ}{2}\right) \sin\left(\frac{20^\circ-30^\circ}{2}\right) \\
&= 2 \sin\left(\frac{50^\circ}{2}\right) \sin\left(\frac{10^\circ}{2}\right) \\
&= 2 \sin 25^\circ \sin 5^\circ
\end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad \cos 75^\circ + \cos 45^\circ &= 2 \cos\left(\frac{75^\circ + 45^\circ}{2}\right) \cos\left(\frac{75^\circ - 45^\circ}{2}\right) \\
 &= 2 \cos\left(\frac{120^\circ}{2}\right) \cos\left(\frac{30^\circ}{2}\right) \\
 &= 2 \cos 60^\circ \cos 15^\circ
 \end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \quad \cos 55^\circ + \sin 55^\circ &= \cos 55^\circ + \cos(90^\circ - 55^\circ) \\
 &= \cos 55^\circ + \cos 35^\circ \\
 &= 2 \cos\left(\frac{55^\circ + 35^\circ}{2}\right) \cos\left(\frac{55^\circ - 35^\circ}{2}\right) \\
 &= 2 \cos 45^\circ \cos 10^\circ = 2\left(\frac{1}{\sqrt{2}}\right) \cdot \cos 10^\circ \\
 &= \sqrt{2} \cos 10^\circ
 \end{aligned}$$

Example 4.19

If three angles A , B and C are in arithmetic progression, Prove that

$$\cot B = \frac{\sin A - \sin C}{\cos C - \cos A}$$

Solution

$$\begin{aligned}
 \frac{\sin A - \sin C}{\cos C - \cos A} &= \frac{2 \sin\left(\frac{A-C}{2}\right) \cos\left(\frac{A+C}{2}\right)}{2 \sin\left(\frac{A+C}{2}\right) \sin\left(\frac{A-C}{2}\right)} \\
 &= \frac{\cos\left(\frac{A+C}{2}\right)}{\sin\left(\frac{A+C}{2}\right)} = \cot\left(\frac{A+C}{2}\right) \\
 &= \cot B \quad \left(\text{since } A, B, C \text{ are in A.P., } B = \frac{A+C}{2}\right)
 \end{aligned}$$

Example 4.20

Prove that $\frac{\sin 5x - 2 \sin 3x + \sin x}{\cos 5x - \cos x} = \tan x$

Solution

$$\begin{aligned}
 \frac{\sin 5x - 2 \sin 3x + \sin x}{\cos 5x - \cos x} &= \frac{\sin 5x + \sin x - 2 \sin 3x}{\cos 5x - \cos x} \\
 &= \frac{2 \sin 3x \cos 2x - 2 \sin 3x}{\cos 5x - \cos x} \\
 &= \frac{2 \sin 3x (\cos 2x - 1)}{-2 \sin 3x \sin 2x} = \frac{1 - \cos 2x}{\sin 2x} \\
 &= \frac{2 \sin^2 x}{2 \sin x \cos x} = \tan x
 \end{aligned}$$

Example 4.21

Find $\sin 105^\circ + \cos 105^\circ$

Solution

$$\begin{aligned}\sin 105^\circ + \cos 105^\circ &= \sin(90^\circ + 15^\circ) + \cos 105^\circ \\ &= \cos 15^\circ + \cos 105^\circ \\ &= \cos 105^\circ + \cos 15^\circ = 2 \cos\left(\frac{105^\circ + 15^\circ}{2}\right) \cos\left(\frac{105^\circ - 15^\circ}{2}\right) \\ &= 2 \cos\left(\frac{120^\circ}{2}\right) \cos\left(\frac{90^\circ}{2}\right) \\ &= 2 \cos 60^\circ \cos 45^\circ = \frac{1}{\sqrt{2}}\end{aligned}$$

Example 4.22

Prove that $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2\left(\frac{\alpha - \beta}{2}\right)$

Solution

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cdot \cos\left(\frac{\alpha - \beta}{2}\right) \quad \dots\dots\dots(1)$$

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cdot \cos\left(\frac{\alpha - \beta}{2}\right) \quad \dots\dots\dots(2)$$

Squaring and adding (1) and (2)

$$\begin{aligned}(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 &= 4 \cos^2\left(\frac{\alpha + \beta}{2}\right) \cos^2\left(\frac{\alpha - \beta}{2}\right) + 4 \sin^2\left(\frac{\alpha + \beta}{2}\right) \cos^2\left(\frac{\alpha - \beta}{2}\right) \\ &= 4 \cos^2\left(\frac{\alpha - \beta}{2}\right) \left[\cos^2\left(\frac{\alpha + \beta}{2}\right) + \sin^2\left(\frac{\alpha + \beta}{2}\right) \right] \\ &= 4 \cos^2\left(\frac{\alpha - \beta}{2}\right)\end{aligned}$$


Exercise 4.3

1. Express each of the following as the sum or difference of sine or cosine:

(i) $\sin \frac{A}{8} \sin \frac{3A}{8}$

(ii) $\cos(60^\circ + A) \sin(120^\circ + A)$

(iii) $\cos \frac{7A}{3} \sin \frac{5A}{3}$

(iv) $\cos 7\theta \sin 3\theta$

2. Express each of the following as the product of sine and cosine

(i) $\sin A + \sin 2A$

(ii) $\cos 2A + \cos 4A$

(iii) $\sin 6\theta - \sin 2\theta$

(iv) $\cos 2\theta - \cos \theta$

3. Prove that

(i) $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$ (ii) $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \sqrt{3}$

4. Prove that

(i) $(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \sin^2 \left(\frac{\alpha - \beta}{2} \right)$

(ii) $\sin A \sin (60^\circ + A) \sin (60^\circ - A) = \frac{1}{4} \sin 3A$

5. Prove that

(i) $\sin(A - B) \sin C + \sin(B - C) \sin A + \sin(C - A) \sin B = 0$

(ii) $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} = 0$

6. Prove that

(i) $\frac{\cos 2A - \cos 3A}{\sin 2A - \sin 3A} = \tan \frac{A}{2}$

(ii) $\frac{\cos 7A + \cos 5A}{\sin 7A - \sin 5A} = \cot A$

7. Prove that $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ = \frac{1}{16}$

8. Evaluate

(i) $\cos 20^\circ + \cos 100^\circ + \cos 140^\circ$

(ii) $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ$

9. If $\cos A + \cos B = \frac{1}{2}$ and $\sin A + \sin B = \frac{1}{4}$, prove that $\tan \left(\frac{A+B}{2} \right) = \frac{1}{2}$

10. If $\sin(y+z-x), \sin(z+x-y), \sin(x+y-z)$ are in A.P, then prove that $\tan x, \tan y$ and $\tan z$ are in A.P

11. If $\operatorname{cosec} A + \sec A = \operatorname{cosec} B + \sec B$ prove that $\cot \left(\frac{A+B}{2} \right) = \tan A \tan B$

4.4 Inverse Trigonometric Functions

4.4.1 Inverse Trigonometric Functions

The quantities such as $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc., are known as inverse trigonometric functions.

i.e., if $\sin \theta = x$, then $\theta = \sin^{-1}x$. The domains and ranges (Principal value branches) of trigonometric functions are given below

Function	Domain x	Range y
$\sin^{-1}x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$R = (-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\operatorname{cosec}^{-1}x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq 0$
$\sec^{-1}x$	$R - (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], y \neq \frac{\pi}{2}$
$\cot^{-1}x$	R	$(0, \pi)$

4.4.2 Properties of Inverse Trigonometric Functions

Property (1)

$$(i) \quad \sin^{-1}(\sin x) = x$$

$$(ii) \quad \cos^{-1}(\cos x) = x$$

$$(iii) \quad \tan^{-1}(\tan x) = x$$

$$(iv) \quad \cot^{-1}(\cot x) = x$$

$$(v) \quad \sec^{-1}(\sec x) = x$$

$$(vi) \quad \operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$$

Property (2)

$$(i) \quad \sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}(x)$$

$$(ii) \quad \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}(x)$$

$$(iii) \quad \tan^{-1}\left(\frac{1}{x}\right) = \cot^{-1}(x)$$

$$(iv) \quad \operatorname{cosec}^{-1}\left(\frac{1}{x}\right) = \sin^{-1}(x)$$

$$(v) \quad \sec^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(x)$$

$$(vi) \quad \cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(x)$$

Property (3)

- | | |
|--|--|
| (i) $\sin^{-1}(-x) = -\sin^{-1}(x)$ | (ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ |
| (iii) $\tan^{-1}(-x) = -\tan^{-1}(x)$ | (iv) $\cot^{-1}(-x) = -\cot^{-1}(x)$ |
| (v) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$ | (vi) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$ |

Property (4)

- | | |
|---|--|
| (i) $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ | (ii) $\tan^{-1}(x) + \cot^{-1}(x) = \frac{\pi}{2}$ |
| (iii) $\sec^{-1}(x) + \operatorname{cosec}^{-1}(x) = \frac{\pi}{2}$ | |

Property (5)

If $xy < 1$, then

- | |
|---|
| (i) $\tan^{-1}(x) + \tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ |
| (ii) $\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ |

Property (6)

$$\sin^{-1}(x) + \sin^{-1}(y) = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})$$

Example 4.23

Find the principal value of (i) $\sin^{-1}(1/2)$ (ii) $\tan^{-1}(-\sqrt{3})$

Solution

(i) Let $\sin^{-1}(1/2) = y$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\therefore \sin y = (1/2) = \sin\left(\frac{\pi}{6}\right)$$

$$y = \frac{\pi}{6}$$

The principal value of $\sin^{-1}(1/2)$ is $\frac{\pi}{6}$

(ii) Let $\tan^{-1}(-\sqrt{3}) = y$ where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\therefore \tan y = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$$

$$y = -\frac{\pi}{3}$$

The principal value of $\tan^{-1}(-\sqrt{3})$ is $-\frac{\pi}{3}$

Example 4.24

Evaluate the following (i) $\cos\left(\sin^{-1}\frac{5}{13}\right)$ (ii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$

Solution

$$(i) \text{ Let } \left(\sin^{-1}\frac{5}{13}\right) = \theta \quad \dots (1)$$

$$\sin \theta = \frac{5}{13}$$

$$\begin{aligned} \cos \theta &= \sqrt{1 - \sin^2 \theta} \\ &= \sqrt{1 - \frac{25}{169}} \\ &= \frac{12}{13} \quad \dots (2) \end{aligned}$$

From (1) and (2), we get

$$\text{Now } \cos\left(\sin^{-1}\frac{5}{13}\right) = \cos \theta = \frac{12}{13}$$

$$(ii) \text{ Let } \left(\cos^{-1}\frac{8}{17}\right) = \theta$$

$$\cos \theta = \frac{8}{17}$$

$$\begin{aligned} \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ &= \sqrt{1 - \frac{64}{289}} \\ &= \frac{15}{17} \quad \dots (2) \end{aligned}$$

From (1) and (2), we get

$$\begin{aligned} \text{Now } \tan\left(\cos^{-1}\frac{8}{17}\right) &= \tan \theta = \frac{\sin \theta}{\cos \theta} \\ &= \frac{\frac{15}{17}}{\frac{8}{17}} = \frac{15}{8} \end{aligned}$$

Example 4.25

$$\text{Prove that (i) } \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$$

$$(ii) \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{27}{11}\right)$$

Solution

$$(i) \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left[\frac{\frac{1}{7} + \frac{1}{13}}{1 - \left(\frac{1}{7}\right)\left(\frac{1}{13}\right)}\right]$$

$$= \tan^{-1}\left(\frac{20}{90}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

(ii) Let $\cos^{-1}\left(\frac{4}{5}\right) = \theta$. Then $\cos \theta = \frac{4}{5} \Rightarrow \sin \theta = \frac{3}{5}$

$$\therefore \tan \theta = \frac{3}{4} \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\therefore \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\begin{aligned} \text{Now } \cos^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{3}{5}\right) &= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) \\ &= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \times \frac{3}{5}}\right) \\ &= \tan^{-1}\left(\frac{27}{11}\right) \end{aligned}$$

Example 4.26

Find the value of $\tan\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{8}\right)\right]$

Solution

$$\text{Let } \tan^{-1}\left(\frac{1}{8}\right) = \theta$$

$$\text{Then } \tan \theta = \frac{1}{8}$$

$$\begin{aligned} \therefore \tan\left[\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{8}\right)\right] &= \tan\left(\frac{\pi}{4} - \theta\right) \\ &= \frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \\ &= \frac{1 - \frac{1}{8}}{1 + \frac{1}{8}} = \frac{7}{9} \end{aligned}$$

Example 4.27

Solve $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

Solution

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}\left[\frac{\frac{x-1}{x-2} + \frac{x+1}{x+2}}{1 - \left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)}\right] = \tan^{-1}\left(\frac{2x^2 - 4}{-3}\right)$$

Given that

$$\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$$

$$\begin{aligned} \therefore \quad \tan^{-1}\left(\frac{2x^2-4}{-3}\right) &= \frac{\pi}{4} \\ \frac{2x^2-4}{-3} &= \tan \frac{\pi}{4} = 1 \\ 2x^2 - 4 &= -3 \\ \Rightarrow \quad 2x^2 - 1 &= 0 \\ \Rightarrow \quad x &= \pm \frac{1}{\sqrt{2}} \end{aligned}$$

Example 4.28

Simplify: $\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right)$

Solution

We know that

$$\begin{aligned} \sin^{-1}(x) + \sin^{-1}(y) &= \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right] \\ \therefore \quad \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2}{3}\right) &= \sin^{-1}\left[\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right] \\ &= \sin^{-1}\left[\frac{1}{3}\sqrt{\frac{5}{9}} + \frac{2}{3}\sqrt{\frac{8}{9}}\right] \\ &= \sin^{-1}\left(\frac{\sqrt{5} + 4\sqrt{2}}{9}\right) \end{aligned}$$

Example 4.29

Solve $\tan^{-1}(x+2) + \tan^{-1}(2-x) = \tan^{-1}\left(\frac{2}{3}\right)$

Solution

$$\begin{aligned} \tan^{-1}\left[\frac{(x+2)+(2-x)}{1-(x+2)(2-x)}\right] &= \tan^{-1}\left(\frac{2}{3}\right) \\ \Rightarrow \quad \tan^{-1}\left(\frac{4}{1-(4-x^2)}\right) &= \tan^{-1}\left(\frac{2}{3}\right) \\ \Rightarrow \quad 2x^2 - 6 &= 12 \\ \Rightarrow \quad x^2 &= 9 \\ \therefore \quad x &= \pm 3 \end{aligned}$$

Example 4.30

If $\tan(x+y) = 42$ and $x = \tan^{-1}(2)$, then find y

Solution

$$\tan(x+y) = 42$$

$$x + y = \tan^{-1}(42)$$

$$\tan^{-1}(2) + y = \tan^{-1}(42)$$

$$y = \tan^{-1}(42) - \tan^{-1}(2)$$

$$= \tan^{-1}\left[\frac{42-2}{1+(42 \times 2)}\right]$$

$$= \tan^{-1}\left(\frac{40}{85}\right)$$

$$y = \tan^{-1}\left[\frac{8}{17}\right]$$

**Exercise 4.4**

- Find the principal value of the following
 - $\sin^{-1}\left(-\frac{1}{2}\right)$
 - $\tan^{-1}(-1)$
 - $\operatorname{cosec}^{-1}(2)$
 - $\sec^{-1}(-\sqrt{2})$
- Prove that
 - $2 \tan^{-1}(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$
 - $\tan^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$
- Show that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{2}{11}\right) = \tan^{-1}\left(\frac{3}{4}\right)$
- Solve $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$
- Solve $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{4}{7}\right)$
- Evaluate (i) $\cos\left[\tan^{-1}\left(\frac{3}{4}\right)\right]$ (ii) $\sin\left[\frac{1}{2} \cos^{-1}\left(\frac{4}{5}\right)\right]$
- Evaluate: $\cos\left(\sin^{-1}\left(\frac{4}{5}\right) + \sin^{-1}\left(\frac{12}{13}\right)\right)$
- Prove that $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$
- Show that $\sin^{-1}\left(-\frac{3}{5}\right) - \sin^{-1}\left(-\frac{8}{7}\right) = \cos^{-1}\frac{84}{85}$
- Express $\tan^{-1}\left[\frac{\cos x}{1-\sin x}\right]$, $-\frac{\pi}{2} < x < \frac{3\pi}{2}$, in the simplest form.



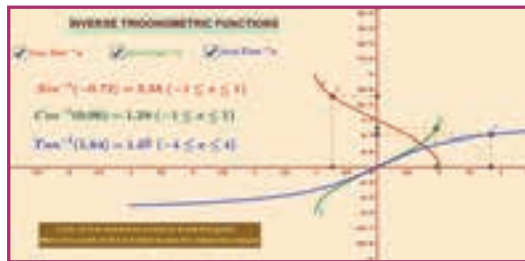
ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

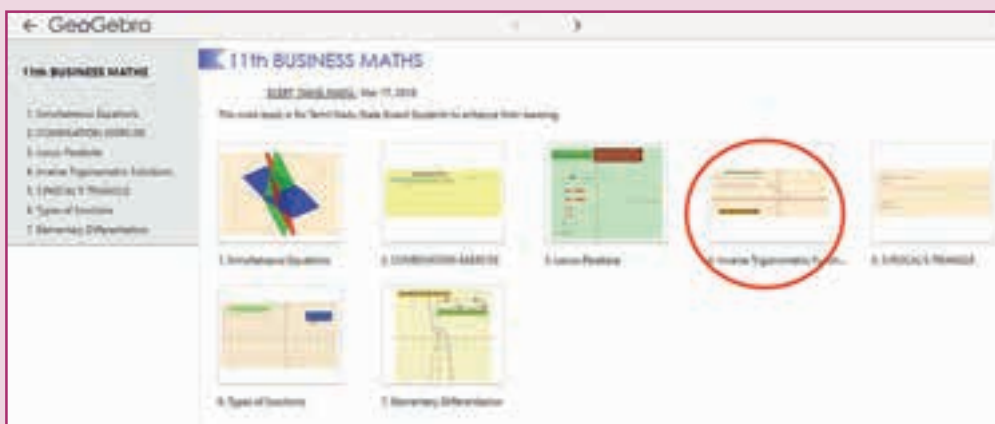
GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Inverse Trigonometric Functions”



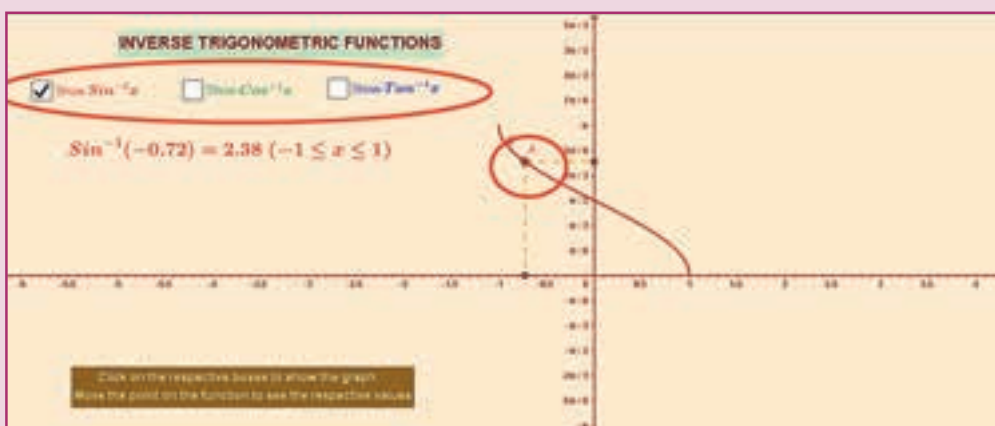
Step - 2

Inverse Trigonometric Functions page will open. Click on the check boxes on the Left-hand side to see the respective Inverse trigonometric functions. You can move and observe the point on the curve to see the x-value and y-value along the axes. Sine and cosine inverse are up to the max limit. But Tangent inverse is having customised limit.

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code



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Exercise 4.5



Choose the correct answer

- The degree measure of $\frac{\pi}{8}$ is
 (a) $20^{\circ}60'$ (b) $22^{\circ}30'$ (c) $22^{\circ}60'$ (d) $20^{\circ}30'$
- The radian measure of $37^{\circ}30'$ is
 (a) $\frac{5\pi}{24}$ (b) $\frac{3\pi}{24}$ (c) $\frac{7\pi}{24}$ (d) $\frac{9\pi}{24}$
- If $\tan \theta = \frac{1}{\sqrt{5}}$ and θ lies in the first quadrant then $\cos \theta$ is
 (a) $\frac{1}{\sqrt{6}}$ (b) $\frac{-1}{\sqrt{6}}$ (c) $\frac{\sqrt{5}}{\sqrt{6}}$ (d) $\frac{-\sqrt{5}}{\sqrt{6}}$
- The value of $\sin 15^{\circ}$ is
 (a) $\frac{\sqrt{3}+1}{2\sqrt{2}}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}}{\sqrt{2}}$ (d) $\frac{\sqrt{3}}{2\sqrt{2}}$
- The value of $\sin(-420^{\circ})$ is
 (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$
- The value of $\cos(-480^{\circ})$ is
 (a) $\sqrt{3}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{-1}{2}$
- The value of $\sin 28^{\circ} \cos 17^{\circ} + \cos 28^{\circ} \sin 17^{\circ}$ is
 (a) $\frac{1}{\sqrt{2}}$ (b) 1 (c) $\frac{-1}{\sqrt{2}}$ (d) 0
- The value of $\sin 15^{\circ} \cos 15^{\circ}$ is
 (a) 1 (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{4}$
- The value of $\sec A \sin(270^{\circ} + A)$ is
 (a) -1 (b) $\cos^2 A$ (c) $\sec^2 A$ (d) 1
- If $\sin A + \cos A = 1$ then $\sin 2A$ is equal to
 (a) 1 (b) 2 (c) 0 (d) $\frac{1}{2}$

11. The value of $\cos^2 45^\circ - \sin^2 45^\circ$ is
 (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) 0 (d) $\frac{1}{\sqrt{2}}$
12. The value of $1 - 2 \sin^2 45^\circ$ is
 (a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 0
13. The value of $4 \cos^3 40^\circ - 3 \cos 40^\circ$ is
 (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$
14. The value of $\frac{2 \tan 30^\circ}{1 + \tan^2 30^\circ}$ is
 (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$
15. If $\sin A = \frac{1}{2}$ then $4 \cos^3 A - 3 \cos A$ is
 (a) 1 (b) 0 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$
16. The value of $\frac{3 \tan 10^\circ - \tan^3 10^\circ}{1 - 3 \tan^2 10^\circ}$ is
 (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{2}}$
17. The value of $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$ is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$
18. $\sec^{-1} \frac{2}{3} + \operatorname{cosec}^{-1} \frac{2}{3} =$
 (a) $\frac{-\pi}{2}$ (b) $\frac{\pi}{2}$ (c) π (d) $-\pi$
19. If α and β be between 0 and $\frac{\pi}{2}$ and if $\cos(\alpha + \beta) = \frac{12}{13}$ and $\sin(\alpha - \beta) = \frac{3}{5}$ then $\sin 2\alpha$ is
 (a) $\frac{16}{15}$ (b) 0 (c) $\frac{56}{65}$ (d) $\frac{64}{65}$
20. If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$ then $\tan(2A + B)$ is equal to
 (a) 1 (b) 2 (c) 3 (d) 4
21. $\tan\left(\frac{\pi}{4} - x\right)$ is
 (a) $\left(\frac{1 + \tan x}{1 - \tan x}\right)$ (b) $\left(\frac{1 - \tan x}{1 + \tan x}\right)$ (c) $1 - \tan x$ (d) $1 + \tan x$

22. $\sin\left(\cos^{-1}\frac{3}{5}\right)$ is
 (a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) $\frac{4}{5}$ (d) $\frac{5}{4}$
23. The value of $\frac{1}{\operatorname{cosec}(-45^\circ)}$ is
 (a) $\frac{-1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) $-\sqrt{2}$
24. If $p \sec 50^\circ = \tan 50^\circ$ then p is
 (a) $\cos 50^\circ$ (b) $\sin 50^\circ$ (c) $\tan 50^\circ$ (d) $\sec 50^\circ$
25. $\left(\frac{\cos x}{\operatorname{cosec} x}\right) - \sqrt{1 - \sin^2 x} \sqrt{1 - \cos^2 x}$ is
 (a) $\cos^2 x - \sin^2 x$ (b) $\sin^2 x - \cos^2 x$ (c) 1 (d) 0

Miscellaneous Problems

1. Prove that $\frac{\cos 4x + \cos 3x + \cos 2x}{\sin 4x + \sin 3x + \sin 2x} = \cot 3x$
2. Prove that $\sqrt{3} \operatorname{cosec} 20^\circ - \sin 20^\circ = 4$
3. Prove that $\cot 4x(\sin 5x + \sin 3x) = \cot x(\sin 5x - \sin 3x)$.
4. If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$ then find the value of $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$
5. Prove that $2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{3\pi}{4} = 10$
6. Find the value of (i) $\sin 75^\circ$ (ii) $\tan 15^\circ$
7. If $\sin A = \frac{1}{3}$, $\sin B = \frac{1}{4}$ then find the value of $\sin(A + B)$, where A and B are acute angles.
8. Show that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$
9. If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$ where $\alpha + \beta$ and $\alpha - \beta$ are acute, then find $\tan 2\alpha$
10. Express $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$, $0 < x < \pi$ in the simplest form.

Summary



- If in a circle of radius r , an arc of length l subtends an angle of θ radians, then $l = r \theta$
- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\cot^2 x + 1 = \operatorname{cosec}^2 x$
- $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
- $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$
- $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$
- $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
- $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$
- $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$
- $\sin 2\alpha = 2 \sin \alpha \cos \alpha$
- $\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1$
- $\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$
- $\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$
- $\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$
- $\tan 3\alpha = \frac{3 \tan \alpha - \tan^3 \alpha}{1 - 3 \tan^2 \alpha}$
- $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$
- $\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$
- $\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$
- $\sin x \cos y = \frac{1}{2} [\sin(x+y) + \sin(x-y)]$
- $\cos x \sin y = \frac{1}{2} [\sin(x+y) - \sin(x-y)]$
- $\cos x \cos y = \frac{1}{2} [\cos(x+y) + \cos(x-y)]$
- $\sin x \sin y = \frac{1}{2} [\cos(x-y) - \cos(x+y)]$

GLOSSARY	
Allied angles	துணைக் கோணங்கள்
Angle	கோணம்
Componendo and dividendo.	கூட்டல் மற்றும் கழித்தல் விகித சமம்
Compound angle	கூட்டுக் கோணம்
Degree measure	பாகை அளவை
Inverse function	நேர்மாறு சார்பு
Length of the arc.	வில்லின் நீளம்
Multiple angle	மடங்கு கோணம்
Quadrants	கால் பகுதிகள்
Radian measure	ரேடியன் அளவு / ஆரையன் அளவு
Transformation formulae	உருமாற்று சூத்திரங்கள்
Trigonometric identities	திரிகோணமிதி முற்றொருமைகள்
Trigonometry Ratios	திரிகோணமிதி விகிதங்கள்

DIFFERENTIAL CALCULUS



Learning Objectives

After studying this chapter, the students will be able to understand

- the concept of limit and continuity
- formula of limit and definition of continuity
- basic concept of derivative
- how to find derivative of a function by first principle
- derivative of a function by certain direct formula and its respective application
- how to find higher order derivatives



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Introduction

Calculus is a Latin word which means that *Pebble or a small stone* used for calculation. The word calculation is also derived from the same Latin word. Calculus is primary mathematical tool for dealing with change. The concept of derivative is the basic tool in science of calculus. Calculus is essentially concerned with the rate of change of dependent variable with respect to an independent variable. Sir Issac Newton (1642 – 1727 CE) and the German mathematician G.W. Leibnitz (1646 – 1716 CE) invented and developed the subject independently and almost simultaneously.



Isaac Newton

In this chapter we will study about functions and their graphs, limits, derivatives and differentiation technique.

5.1 Functions and their graphs

Some basic concepts

5.1.1 Quantity

Anything which can be performed on basic mathematical operations like addition, subtraction, multiplication and division is called a quantity.



G.W. Leibnitz

5.1.2 Constant

A quantity which retains the same value throughout a mathematical investigation is called a constant.

Basically constant quantities are of two types

(i) **Absolute constants** are those which do not change their values in any mathematical investigation. In other words, they are fixed for ever.

Examples: $3, \sqrt{3}, \pi, \dots$

(ii) **Arbitrary constants** are those which retain the same value throughout a problem, but we may assign different values to get different solutions. The arbitrary constants are usually denoted by the letters a, b, c, \dots

Example: In an equation $y=mx+4$, m is called arbitrary constant.

5.1.3 Variable

A variable is a quantity which can assume different values in a particular problem. Variables are generally denoted by the letters x, y, z, \dots

Example: In an equation of the straight line $\frac{x}{a} + \frac{y}{b} = 1$,

x and y are variables because they assumes the co-ordinates of a moving point in a straight line and thus changes its value from point to point. a and b are intercept values on the axes which are arbitrary.

There are two kinds of variables

- (i) A variable is said to be an **independent variable** when it can have any arbitrary value.
- (ii) A variable is said to be a **dependent variable** when its value depend on the value assumed by some other variable.

Example: In the equation $y = 5x^2 - 2x + 3$,

“ x ” is the independent variable,

“ y ” is the dependent variable and

“3” is the constant.

5.1.4 Intervals

The real numbers can be represented geometrically as points on a number line called real line. The symbol R denotes either the real number system or the real line. A subset of the real line is an interval. It contains atleast two numbers and all the real numbers lying between them.

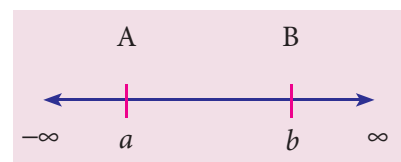


Fig. 5.1

(i) Open interval

The set $\{x : a < x < b\}$ is an open interval, denoted by (a, b) .

In this interval, the boundary points a and b are not included.

For example, in an open interval $(4, 7)$, 4 and 7 are not elements of this interval.

But, 4.001 and 6.99 are elements of this interval.

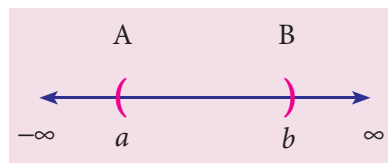


Fig. 5.2

(ii) Closed interval

The set $\{x : a \leq x \leq b\}$ is a closed interval and is denoted by $[a, b]$.

In the interval $[a, b]$, the boundary points a and b are included.

For example, in an interval $[4, 7]$, 4 and 7 are also elements of this interval.

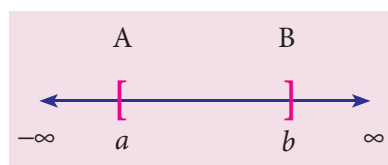


Fig. 5.3

Also we can make a mention about semi closed or semi open intervals.

$(a, b] = \{x : a < x \leq b\}$ is called left open interval

and $[a, b) = \{x : a \leq x < b\}$ is called right open interval.

In all the above cases, $b - a = h$ is called the length of the interval.

5.1.5 Neighbourhood of a point

Let 'a' be any real number. Let $\epsilon > 0$ be arbitrarily small real number.

Then $(a - \epsilon, a + \epsilon)$ is called an ϵ - neighbourhood of the point 'a' and denoted by $N_{a, \epsilon}$

For examples,

$$N_{5, \frac{1}{4}} = \left(5 - \frac{1}{4}, 5 + \frac{1}{4}\right)$$

$$= \left\{x : \frac{19}{4} < x < \frac{21}{4}\right\}$$

$$N_{2, \frac{1}{7}} = \left(2 - \frac{1}{7}, 2 + \frac{1}{7}\right) = \left\{x : \frac{13}{7} < x < \frac{15}{7}\right\}$$

5.1.6 Function

Let X and Y be two non-empty sets of real numbers. If there exists a rule f which associates to every element $x \in X$, a unique element $y \in Y$, then such a rule f is called a function (or mapping) from the set X to the set Y . We write $f: X \rightarrow Y$.



The function notation
 $y = f(x)$
was first used by
Leonhard Euler in 1734 - 1735

The set X is called the domain of f , Y is called the co-domain of f and the range of f is defined as $f(X) = \{f(x) / x \in X\}$. Clearly $f(X) \subseteq Y$.

A function of x is generally denoted by the symbol $f(x)$, and read as “function of x ” or “ f of x ”.

5.1.7 Classification of functions

Functions can be classified into two groups.

(i) Algebraic functions

Algebraic functions are algebraic expressions using a finite number of terms, involving only the algebraic operations addition, subtraction, multiplication, division and raising to a fractional power.

Examples : $y = 3x^4 - x^3 + 5x^2 - 7$, $y = \frac{2x^3 + 7x - 3}{x^3 + x^2 + 1}$, $y = \sqrt{3x^2 + 6x - 1}$ are algebraic functions.

(a) $y = 3x^4 - x^3 + 5x^2 - 7$ is a polynomial or rational integral function.

(b) $y = \frac{2x^3 + 7x - 3}{x^3 + x^2 + 1}$ is a rational function.

(c) $y = \sqrt{3x^2 + 6x - 1}$ is an irrational function.

(ii) Transcendental functions

A function which is not algebraic is called transcendental function.

Examples: $\sin x$, $\sin^{-1} x$, e^x , $\log_a x$ are transcendental functions.

(a) $\sin x$, $\tan 2x$, ... are trigonometric functions.

(b) $\sin^{-1} x$, $\cos^{-1} x$, ... are inverse trigonometric functions.

(c) e^x , 2^x , x^x , ... are exponential functions.

(d) $\log_a x$, $\log_e (\sin x)$, ... are logarithmic functions.

5.1.8 Even and odd functions

A function $f(x)$ is said to be an even function of x , if $f(-x) = f(x)$.

A function $f(x)$ is said to be an odd function of x , if $f(-x) = -f(x)$.

Examples: $f(x) = x^2$ and $f(x) = \cos x$
are even functions.

$f(x) = x^3$ and $f(x) = \sin x$
are odd functions.



NOTE

$f(x) = x^3 + 5$ is neither even nor odd function

5.1.9 Explicit and implicit functions

A function in which the dependent variable is expressed explicitly in terms of some independent variables is known as **explicit function**.

Examples: $y = x^2 + 3$ and $y = e^x + e^{-x}$ are explicit functions of x .

If two variables x and y are connected by the relation or function $f(x, y) = 0$ and none of the variable is directly expressed in terms of the other, then the function is called an **implicit function**.

Example: $x^3 + y^3 - xy = 0$ is an implicit function.

5.1.10 Constant function

If k is a fixed real number then the function $f(x)$ given by $f(x) = k$ for all $x \in R$ is called a constant function.

Examples: $y = 3, f(x) = -5$ are constant functions.



Graph of a constant function
 $y = f(x)$
is a straight line parallel to
 x -axis

5.1.11 Identity function

A function that associates each real number to itself is called the identity function and is denoted by I .

i.e. A function defined on R by $f(x) = x$ for all $x \in R$ is an identity function.

Example: The set of ordered pairs $\{(1, 1), (2, 2), (3, 3)\}$ defined by $f : A \rightarrow A$ where $A = \{1, 2, 3\}$ is an identity function.



Graph of an identity function on R is a straight line passing through the origin and makes an angle of 45° with positive direction of x -axis

5.1.12 Modulus function

A function $f(x)$ defined by $f(x) = |x|$, where $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ is called the modulus function.

It is also called an absolute value function.

Remark

Domain is R and range set is $[0, \infty)$



NOTE

$$|5| = 5, \quad |-5| = -(-5) = 5$$

5.1.13 Signum function

A function $f(x)$ is defined by $f(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is called the Signum function.

Remark

Domain is R and range set is $\{-1, 0, 1\}$

5.1.14 Step function

(i) Greatest integer function

The function whose value at any real number x is the greatest integer less than or equal to x is called the greatest integer function. It is denoted by $\lfloor x \rfloor$.

i.e. $f: R \rightarrow R$ defined by $f(x) = \lfloor x \rfloor$ is called the greatest integer function.



NOTE

$$\begin{aligned} \lfloor 2.5 \rfloor &= 2, \quad \lfloor -2.1 \rfloor = -3, \\ \lfloor 0.74 \rfloor &= 0, \quad \lfloor -0.3 \rfloor = -1, \\ \lfloor 4 \rfloor &= 4. \end{aligned}$$

(ii) Least integer function

The function whose value at any real number x is the smallest integer greater than or equal to x is called the least integer function. It is denoted by $\lceil x \rceil$.

i.e. $f: R \rightarrow R$ defined by $f(x) = \lceil x \rceil$ is called the least integer function



NOTE

$$\begin{aligned} \lceil 4.7 \rceil &= 5, \quad \lceil -7.2 \rceil = -7, \\ \lceil 5 \rceil &= 5, \quad \lceil 0.75 \rceil = 1. \end{aligned}$$

Remark

Function's domain is R and range is Z [set of integers]

5.1.15 Rational function

A function $f(x)$ is defined by the rule $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ is called rational function.

Example: $f(x) = \frac{x^2 + 1}{x - 3}$, $x \neq 3$ is a rational function.

5.1.16 Polynomial function

For the real numbers $a_0, a_1, a_2, \dots, a_n$; $a_0 \neq 0$ and n is a non-negative integer, a function $f(x)$ given by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$ is called as a polynomial function of degree n .

Example: $f(x) = 2x^3 + 3x^2 + 2x - 7$ is a polynomial function of degree 3.

5.1.17 Linear function

For the real numbers a and b with $a \neq 0$, a function $f(x) = ax + b$ is called a linear function.

Example: $y = 2x + 3$ is a linear function.

5.1.18 Quadratic function

For the real numbers a, b and c with $a \neq 0$, a function $f(x) = ax^2 + bx + c$ is called a quadratic function.

Example: $f(x) = 3x^2 + 2x - 7$ is a quadratic function.

5.1.19 Exponential function

A function $f(x) = a^x$, $a \neq 1$ and $a > 0$, for all $x \in R$ is called an exponential function

Remark Domain is R and range is $(0, \infty)$ and $(0, 1)$ is a point on the graph

Examples: e^{2x} , e^{x^2+1} and 2^x are exponential functions.

5.1.20 Logarithmic function

For $x > 0$, $a > 0$ and $a \neq 1$, a function $f(x)$ defined by $f(x) = \log_a x$ is called the logarithmic function.

Remark Domain is $(0, \infty)$, range is R and $(1, 0)$ is a point on the graph

Example: $f(x) = \log_e(x+2)$, $f(x) = \log_e(\sin x)$ are logarithmic functions.

5.1.21 Sum, difference, product and quotient of two functions

If $f(x)$ and $g(x)$ are two functions having same domain and co-domain, then

- (i) $(f \pm g)(x) = f(x) \pm g(x)$
- (ii) $(fg)(x) = f(x)g(x)$

$$(iii) \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$$

$$(iv) (cf)(x) = cf(x), \quad c \text{ is a constant.}$$

5.1.22 Graph of a function

The graph of a function is the set of points $(x, f(x))$ where x belongs to the domain of the function and $f(x)$ is the value of the function at x .

To draw the graph of a function, we find a sufficient number of ordered pairs $(x, f(x))$ belonging to the function and join them by a smooth curve.



NOTE

It is enough to draw a diagram for graph of a function in white sheet itself.

Example 5.1

Find the domain for which the functions $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal.

Solution

$$\text{Given that } f(x) = g(x)$$

$$\Rightarrow 2x^2 - 1 = 1 - 3x$$

$$2x^2 + 3x - 2 = 0$$

$$(x+2)(2x-1) = 0$$

$$\therefore x = -2, x = \frac{1}{2}$$

$$\text{Domain is } \left\{-2, \frac{1}{2}\right\}$$

Example 5.2

$f = \{(1, 1), (2, 3)\}$ be a function described by the formula $f(x) = ax + b$. Determine a and b ?

Solution

$$\text{Given that } f(x) = ax + b, f(1) = 1 \text{ and } f(2) = 3$$

$$\therefore a + b = 1 \text{ and } 2a + b = 3$$

$$\Rightarrow a = 2 \text{ and } b = -1$$

Example 5.3

If $f(x) = x + \frac{1}{x}$, then show that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$

Solution

$$f(x) = x + \frac{1}{x}, \quad f(x^3) = x^3 + \frac{1}{x^3} \quad \text{and} \quad f\left(\frac{1}{x}\right) = f(x)$$

$$\begin{aligned} \text{Now,} \quad LHS &= [f(x)]^3 \\ &= \left[x + \frac{1}{x}\right]^3 = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) \\ &= f(x^3) + 3f(x) = f(x^3) + 3f\left(\frac{1}{x}\right) \\ &= RHS \end{aligned}$$

Example 5.4

Let f be defined by $f(x) = x - 5$ and g be defined by

$$g(x) = \begin{cases} \frac{x^2 - 25}{x + 5} & \text{if } x \neq -5 \\ \lambda & \text{if } x = -5 \end{cases} \quad \text{Find } \lambda \text{ such that } f(x) = g(x) \text{ for all } x \in R$$

Solution

Given that $f(x) = g(x)$ for all $x \in R$

$$\therefore f(-5) = g(-5)$$

$$-5 - 5 = \lambda$$

$$\therefore \lambda = -10$$

Example 5.5

If $f(x) = 2^x$, show that $f(x) \cdot f(y) = f(x + y)$

Solution

$$f(x) = 2^x \quad (\text{Given})$$

$$\begin{aligned} \therefore f(x + y) &= 2^{x+y} \\ &= 2^x \cdot 2^y \\ &= f(x) \cdot f(y) \end{aligned}$$

Example 5.6

If $f(x) = \frac{x-1}{x+1}$, $0 < x < 1$, then show that $f[f(x)] = -\frac{1}{x}$

Solution

$$f(x) = \frac{x-1}{x+1} \quad (\text{Given})$$

$$\begin{aligned} \therefore f[f(x)] &= \frac{f(x)-1}{f(x)+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} \\ &= \frac{x-1-x-1}{x-1+x+1} = -\frac{2}{2x} = -\frac{1}{x} \end{aligned}$$

Example 5.7

If $f(x) = \log \frac{1+x}{1-x}$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$.

Solution

$$f(x) = \log \frac{1+x}{1-x} \quad (\text{Given})$$

$$\begin{aligned} \therefore f\left(\frac{2x}{1+x^2}\right) &= \log \frac{1 + \frac{2x}{1+x^2}}{1 - \frac{2x}{1+x^2}} \\ &= \log \frac{1+x^2+2x}{1+x^2-2x} = \log \frac{(1+x)^2}{(1-x)^2} \\ &= 2 \log \frac{1+x}{1-x} = 2f(x). \end{aligned}$$

Example 5.8

If $f(x) = x$ and $g(x) = |x|$, then find

(i) $(f+g)(x)$ (ii) $(f-g)(x)$ (iii) $(fg)(x)$

Solution

$$\begin{aligned} \text{(i)} \quad (f+g)(x) &= f(x) + g(x) = x + |x| \\ &= \begin{cases} x+x & \text{if } x \geq 0 \\ x-x & \text{if } x < 0 \end{cases} = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (f-g)(x) &= f(x) - g(x) = x - |x| \\ &= \begin{cases} x-x & \text{if } x \geq 0 \\ x-(-x) & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \geq 0 \\ 2x & \text{if } x < 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad (fg)(x) &= f(x)g(x) = x|x| \\ &= \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases} \end{aligned}$$

Example 5.9

A group of students wish to charter a bus for an educational tour which can accommodate atmost 50 students. The bus company requires at least 35 students for that trip. The bus company charges ₹ 200 per student up to the strength of 45. For more than 45 students, company charges each student ₹ 200 less $\frac{1}{5}$ times the number more than 45. Consider the number of students who participates the tour as a function, find the total cost and its domain.

Solution

Let 'x' be the number of students who participate the tour.

Then $35 \leq x \leq 50$ and x is a positive integer.

Formula : *Total cost = (cost per student) × (number of students)*

(i) If the number of students are between 35 and 45, then

the cost per student is ₹ 200

∴ The total cost is $y = 200x$.

(ii) If the number of students are between 46 and 50, then

the cost per student is ₹ $\left\{200 - \frac{1}{5}(x - 45)\right\} = 209 - \frac{x}{5}$

∴ The total cost is $y = \left(209 - \frac{x}{5}\right)x = 209x - \frac{x^2}{5}$

∴ $y = \begin{cases} 200x & ; 35 \leq x \leq 45 \\ 209x - \frac{x^2}{5} & ; 46 \leq x \leq 50 \end{cases}$ and

the domain is $\{35, 36, 37, \dots, 50\}$.

Example 5.10

Draw the graph of the function $f(x) = |x|$

Solution

$$\begin{aligned} y &= f(x) = |x| \\ &= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \end{aligned}$$

Choose suitable values for x and determine y .

Thus we get the following table.

x	-2	-1	0	1	2
y	2	1	0	1	2

Table : 5.1

Plot the points $(-2, 2)$, $(-1, 1)$, $(0, 0)$, $(1, 1)$, $(2, 2)$ and draw a smooth curve.

The graph is as shown in the figure

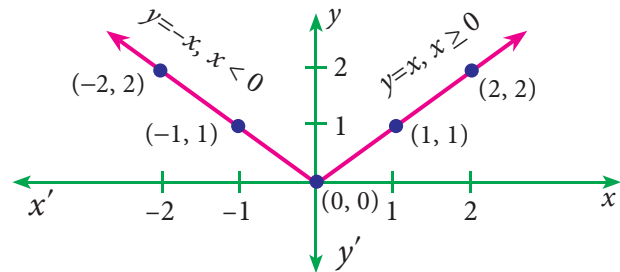


Fig. 5.4

Example 5.11

Draw the graph of the function $f(x) = x^2 - 5$

Solution

Let $y = f(x) = x^2 - 5$

Choose suitable values for x and determine y .

Thus we get the following table.

x	-3	-2	-1	0	1	2	3
y	4	-1	-4	-5	-4	-1	4

Table : 5.2

Plot the points $(-3, 4)$, $(-2, -1)$, $(-1, -4)$, $(0, -5)$, $(1, -4)$, $(2, -1)$, $(3, 4)$ and draw a smooth curve.

The graph is as shown in the figure

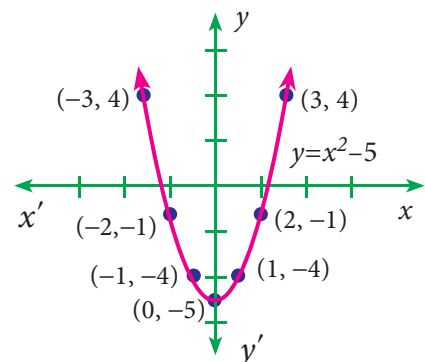


Fig 5.5

Example 5.12

Draw the graph of $f(x) = a^x$, $a \neq 1$ and $a > 0$

Solution

We know that, domain set is R , range set is $(0, \infty)$ and the curve passing through the point $(0, 1)$

Case (i) when $a > 1$

$$y = f(x) = a^x = \begin{cases} < 1 & \text{if } x < 0 \\ = 1 & \text{if } x = 0 \\ > 1 & \text{if } x > 0 \end{cases}$$

We noticed that as x increases, y is also increases and $y > 0$.

\therefore The graph is as shown in the figure.

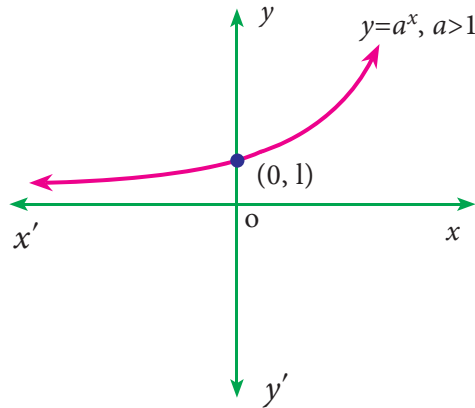


Fig 5.6

Case (ii) when $0 < a < 1$

$$y = f(x) = a^x = \begin{cases} > 1 & \text{if } x < 0 \\ = 1 & \text{if } x = 0 \\ < 1 & \text{if } x > 0 \end{cases}$$

We noticed that as x increases, y decreases and $y > 0$.

\therefore The graph is as shown in the figure.

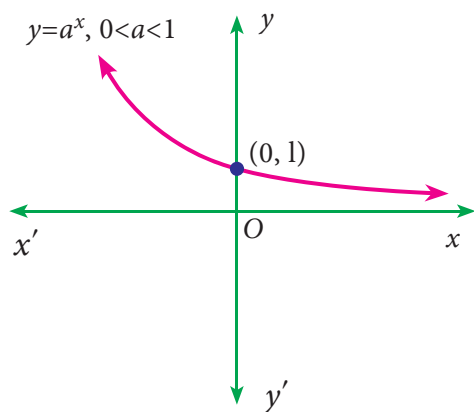


Fig 5.7



NOTE

- (i) The value of e lies between 2 and 3. i.e. $2 < e < 3$
- (ii) The graph of $f(x) = e^x$ is identical to that of $f(x) = a^x$ if $a > 1$ and the graph of $f(x) = e^{-x}$ is identical to that of $f(x) = a^x$ if $0 < a < 1$

Example 5.13

Draw the graph of $f(x) = \log_a x$; $x > 0$, $a > 0$ and $a \neq 1$.

Solution

We know that, domain set is $(0, \infty)$, range set is \mathbb{R} and the curve passing the point $(1, 0)$

Case (i) when $a > 1$ and $x > 0$

$$f(x) = \log_a x = \begin{cases} < 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ > 0 & \text{if } x > 1 \end{cases}$$

We noticed that as x increases, $f(x)$ is also increases.

\therefore The graph is as shown in the figure.

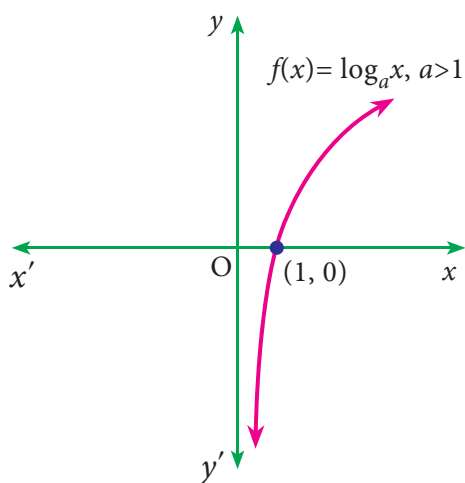


Fig 5.8

Case (ii) when $0 < a < 1$ and $x > 0$

$$f(x) = \log_a x = \begin{cases} > 0 & \text{if } 0 < x < 1 \\ = 0 & \text{if } x = 1 \\ < 0 & \text{if } x > 1 \end{cases}$$

We noticed that x increases, the value of $f(x)$ decreases.

\therefore The graph is as shown in the figure.

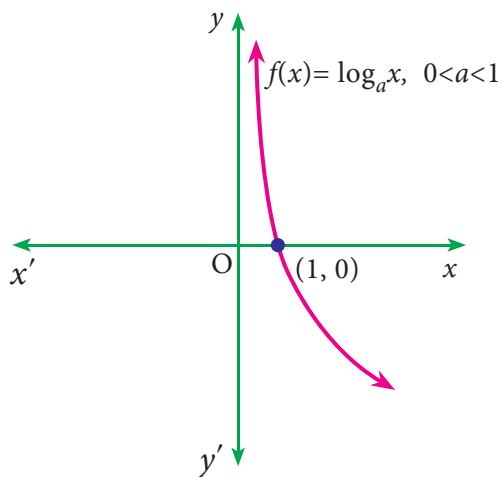


Fig 5.9



ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Types of Functions”



Step - 2

Types of Functions page will open. Click on the check boxes of the functions on the Right-hand side to see the respective functions. You can move the slider and change the function. Observe each function.

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code





Exercise 5.1

- Determine whether the following functions are odd or even?
 - $f(x) = \left(\frac{a^x - 1}{a^x + 1} \right)$
 - $f(x) = \log(x^2 + \sqrt{x^2 + 1})$
 - $f(x) = \sin x + \cos x$
 - $f(x) = x^2 - |x|$
 - $f(x) = x + x^2$
- Let f be defined by $f(x) = x^3 - kx^2 + 2x$, $x \in R$. Find k , if ' f ' is an odd function.
- If $f(x) = x^3 - \frac{1}{x^3}$, then show that $f(x) + f\left(\frac{1}{x}\right) = 0$
- If $f(x) = \frac{x+1}{x-1}$, then prove that $f(f(x)) = x$
- For $f(x) = \frac{x-1}{3x+1}$, write the expressions of $f\left(\frac{1}{x}\right)$ and $\frac{1}{f(x)}$
- If $f(x) = e^x$ and $g(x) = \log_e x$, then find
 - $(f+g)(1)$
 - $(fg)(1)$
 - $(3f)(1)$
 - $(5g)(1)$
- Draw the graph of the following functions :
 - $f(x) = 16 - x^2$
 - $f(x) = |x - 2|$
 - $f(x) = x|x|$
 - $f(x) = e^{2x}$
 - $f(x) = e^{-2x}$
 - $f(x) = \frac{|x|}{x}$

5.2 Limits and derivatives

In this section, we will discuss about limits, continuity of a function, differentiability and differentiation from first principle. Limits are used when we have to find the value of a function near to some value.

Definition 5.1

Let f be a real valued function of x . Let l and a be two fixed numbers. If $f(x)$ approaches the value l as x approaches to a , then we say that l is the limit of the function $f(x)$ as x tends to a , and written as

$$\lim_{x \rightarrow a} f(x) = l$$

Notations: (i) $L[f(x)]_{x=a} = \lim_{x \rightarrow a^-} f(x)$ is known as left hand limit of $f(x)$ at $x=a$.

(ii) $R[f(x)]_{x=a} = \lim_{x \rightarrow a^+} f(x)$ is known as right hand limit of $f(x)$ at $x=a$.

5.2.1 Existence of limit

$\lim_{x \rightarrow a} f(x)$ exist if and only if both $L[f(x)]_{x=a}$ and $R[f(x)]_{x=a}$ exist and are equal.

i.e., $\lim_{x \rightarrow a} f(x)$ exist if and only if $L[f(x)]_{x=a} = R[f(x)]_{x=a}$.



NOTE

$L[f(x)]_{x=a}$ and $R[f(x)]_{x=a}$ are shortly written as $L[f(a)]$ and $R[f(a)]$

5.2.2 Algorithm of left hand limit : $L[f(x)]_{x=a}$

- (i) Write $\lim_{x \rightarrow a^-} f(x)$.
- (ii) Put $x = a - h$, $h > 0$ and replace $x \rightarrow a^-$ by $h \rightarrow 0$.
- (iii) Obtain $\lim_{h \rightarrow 0} f(a-h)$.
- (iv) The value obtained in step (iii) is called as the value of left hand limit of the function $f(x)$ at $x=a$.

5.2.3 Algorithm of right hand limit : $R[f(x)]_{x=a}$

- (i) Write $\lim_{x \rightarrow a^+} f(x)$.
- (ii) Put $x = a+h$, $h > 0$ and replace $x \rightarrow a^+$ by $h \rightarrow 0$.
- (iii) Obtain $\lim_{h \rightarrow 0} f(a+h)$.
- (iv) The value obtained in step (iii) is called as the value of right hand limit of the function $f(x)$ at $x=a$.

Example 5.14

Evaluate the left hand and right hand limits of the function

$$f(x) = \begin{cases} \frac{|x-3|}{x-3} & \text{if } x \neq 3 \\ 0 & \text{if } x = 3 \end{cases} \text{ at } x = 3.$$

Solution

$$\begin{aligned} L[f(x)]_{x=3} &= \lim_{x \rightarrow 3^-} f(x) \\ &= \lim_{h \rightarrow 0} f(3-h), \quad x = 3-h \\ &= \lim_{h \rightarrow 0} \frac{|(3-h)-3|}{(3-h)-3} \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{|-h|}{-h} \\
 &= \lim_{h \rightarrow 0} \frac{h}{-h} \\
 &= \lim_{h \rightarrow 0} -1 \\
 &= -1
 \end{aligned}$$

$$\begin{aligned}
 R[f(x)]_{x=3} &= \lim_{x \rightarrow 3^+} f(x) \\
 &= \lim_{h \rightarrow 0} f(3+h) \\
 &= \lim_{h \rightarrow 0} \frac{|(3+h)-3|}{(3+h)-3} \\
 &= \lim_{h \rightarrow 0} \frac{|h|}{h} \\
 &= \lim_{h \rightarrow 0} 1 \\
 &= 1
 \end{aligned}$$



NOTE

Here,
 $L[f(3)] \neq R[f(3)]$
 $\therefore \lim_{x \rightarrow 3} f(x)$ does not exist.

Example 5.15

Verify the existence of the function $f(x) = \begin{cases} 5x-4 & \text{if } 0 < x \leq 1 \\ 4x^3-3 & \text{if } 1 < x < 2 \end{cases}$ at $x = 1$.

Solution

$$\begin{aligned}
 L[f(x)]_{x=1} &= \lim_{x \rightarrow 1^-} f(x) \\
 &= \lim_{h \rightarrow 0} f(1-h), x = 1-h \\
 &= \lim_{h \rightarrow 0} [5(1-h)-4] \\
 &= \lim_{h \rightarrow 0} (1-5h) = 1 \\
 R[f(x)]_{x=1} &= \lim_{x \rightarrow 1^+} f(x) \\
 &= \lim_{h \rightarrow 0} f(1+h), x = 1+h \\
 &= \lim_{h \rightarrow 0} [4(1+h)^3 - 3(1+h)] \\
 &= 4(1)^3 - 3(1)
 \end{aligned}$$

Clearly, $L[f(1)] = R[f(1)]$

$\therefore \lim_{x \rightarrow 1} f(x)$ exists and equal to 1.



NOTE

Let a be a point and $f(x)$ be a function, then the following stages may happen

- (i) $\lim_{x \rightarrow a} f(x)$ exists but $f(a)$ does not exist.
- (ii) The value of $f(a)$ exists but $\lim_{x \rightarrow a} f(x)$ does not exist.
- (iii) Both $\lim_{x \rightarrow a} f(x)$ and $f(a)$ exist but are unequal.
- (iv) Both $\lim_{x \rightarrow a} f(x)$ and $f(a)$ exist and are equal.

5.2.4 Some results of limits

If $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exists, then

- (i) $\lim_{x \rightarrow a} (f \pm g)(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- (ii) $\lim_{x \rightarrow a} (fg)(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
- (iii) $\lim_{x \rightarrow a} \left(\frac{f}{g} \right)(x) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$, whenever $\lim_{x \rightarrow a} g(x) \neq 0$
- (iv) $\lim_{x \rightarrow a} k f(x) = k \lim_{x \rightarrow a} f(x)$, where k is a constant.

5.2.5 Indeterminate forms and evaluation of limits

Let $f(x)$ and $g(x)$ are two functions in which the limits exist.

If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, then $\frac{f(x)}{g(x)}$ takes $\frac{0}{0}$ form which is meaningless but does not imply that $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ is meaningless.

In many cases, this limit exists and have a finite value. Finding the solution of such limits are called evaluation of the indeterminate form.

The indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0 , ∞^0 and 1^∞ .

Among all these indeterminate forms, $\frac{0}{0}$ is the fundamental one.

5.2.6 Methods of evaluation of algebraic limits

- (i) Direct substitution
- (ii) Factorization
- (iii) Rationalisation
- (iv) By using some standard limits

5.2.7 Some standard limits

- (i) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$ if $n \in \mathbb{Q}$
- (ii) $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$, θ is in radian.
- (iii) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$ if $a > 0$
- (iv) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$
- (v) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
- (vi) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$
- (vii) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

Example 5.16

Evaluate: $\lim_{x \rightarrow 1} (3x^2 + 4x - 5)$.

Solution

$$\begin{aligned} \lim_{x \rightarrow 1} (3x^2 + 4x - 5) \\ = 3(1)^2 + 4(1) - 5 = 2 \end{aligned}$$

Example 5.17

Evaluate: $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 6}{x + 2}$.

Solution

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{x^2 - 4x + 6}{x + 2} &= \frac{\lim_{x \rightarrow 2} (x^2 - 4x + 6)}{\lim_{x \rightarrow 2} (x + 2)} \\ &= \frac{(2)^2 - 4(2) + 6}{2 + 2} = \frac{1}{2} \end{aligned}$$

Example 5.18

Evaluate : $\lim_{x \rightarrow \frac{\pi}{4}} \frac{5 \sin 2x - 2 \cos 2x}{3 \cos 2x + 2 \sin 2x}$.

Solution

$$\lim_{x \rightarrow \frac{\pi}{4}} \frac{5 \sin 2x - 2 \cos 2x}{3 \cos 2x + 2 \sin 2x} = \frac{\lim_{x \rightarrow \frac{\pi}{4}} (5 \sin 2x - 2 \cos 2x)}{\lim_{x \rightarrow \frac{\pi}{4}} (3 \cos 2x + 2 \sin 2x)}$$

$$\begin{aligned}
 &= \frac{5 \sin \frac{\pi}{2} - 2 \cos \frac{\pi}{2}}{3 \cos \frac{\pi}{2} + 2 \sin \frac{\pi}{2}} \\
 &= \frac{5(1) - 2(0)}{3(0) + 2(1)} = \frac{5}{2}.
 \end{aligned}$$

Example 5.19

Evaluate: $\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}$.

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} &\text{ is of the type } \frac{0}{0} \\
 \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)} \\
 &= \lim_{x \rightarrow 1} (x^2 + x + 1) \\
 &= (1)^2 + 1 + 1 = 3
 \end{aligned}$$

Example 5.20

Evaluate: $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$.

Solution

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} &= \lim_{x \rightarrow 0} \frac{(\sqrt{2+x} - \sqrt{2})(\sqrt{2+x} + \sqrt{2})}{x(\sqrt{2+x} + \sqrt{2})} \\
 &= \lim_{x \rightarrow 0} \frac{2+x-2}{x(\sqrt{2+x} + \sqrt{2})} \\
 &= \lim_{x \rightarrow 0} \frac{1}{(\sqrt{2+x} + \sqrt{2})} \\
 &= \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}.
 \end{aligned}$$

Example 5.21

Evaluate: $\lim_{x \rightarrow a} \frac{x^{\frac{3}{5}} - a^{\frac{3}{5}}}{\frac{1}{x^5} - \frac{1}{a^5}}$.

Solution

$$\begin{aligned}
 \lim_{x \rightarrow a} \frac{x^{\frac{3}{5}} - a^{\frac{3}{5}}}{\frac{1}{x^5} - \frac{1}{a^5}} &= \lim_{x \rightarrow a} \frac{\frac{x^{\frac{3}{5}} - a^{\frac{3}{5}}}{x - a}}{\frac{\frac{1}{x^5} - \frac{1}{a^5}}{x - a}} \\
 &= \frac{\frac{3}{5}(a)^{-\frac{2}{5}}}{\frac{1}{5}(a)^{-\frac{4}{5}}} = 3a^{-\frac{2}{5} + \frac{4}{5}} = 3a^{\frac{2}{5}}.
 \end{aligned}$$

Example 5.22

Evaluate: $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x}$.

Solution

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} &= \lim_{x \rightarrow 0} \left\{ \frac{3x \times \frac{\sin 3x}{3x}}{5x \times \frac{\sin 5x}{5x}} \right\} \\ &= \frac{3 \lim_{x \rightarrow 0} \frac{\sin 3x}{3x}}{5 \lim_{x \rightarrow 0} \frac{\sin 5x}{5x}} \\ &= \frac{3 \left(\frac{1}{1} \right)}{5} = \frac{3}{5} \end{aligned}$$

Example 5.23

Evaluate: $\lim_{x \rightarrow \infty} \frac{6 - 5x^2}{4x + 15x^2}$.

Solution

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{6 - 5x^2}{4x + 15x^2} &= \lim_{x \rightarrow \infty} \frac{\frac{6}{x^2} - 5}{\frac{4}{x} + 15} \\ &= \frac{0 - 5}{0 + 15} = -\frac{1}{3}. \end{aligned}$$

Example 5.24

Evaluate: $\lim_{x \rightarrow \infty} x \tan\left(\frac{1}{x}\right)$.

Solution

Let $y = \frac{1}{x}$ and $y \rightarrow 0$ as $x \rightarrow \infty$

$$\lim_{x \rightarrow \infty} x \tan\left(\frac{1}{x}\right) = \lim_{y \rightarrow 0} \frac{\tan y}{y} = 1.$$

Example 5.25

Evaluate: $\lim_{n \rightarrow \infty} \frac{\sum n^2}{n^3}$.

Solution

$$\lim_{n \rightarrow \infty} \frac{\sum n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{n(n+1)(2n+1)}{6n^3}$$

$$\begin{aligned}
&= \lim_{n \rightarrow \infty} \frac{1}{6} \left\{ \left(\frac{n}{n} \right) \left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right) \right\} \\
&= \frac{1}{6} \lim_{n \rightarrow \infty} \left\{ \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) \right\} \\
&= \frac{1}{6} (1)(2) \\
&= \frac{1}{3}.
\end{aligned}$$

Example 5.26

Show that $\lim_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x} = 1$.

Solution

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x} &= \lim_{x \rightarrow 0} \left\{ \frac{\log(1+x^3)}{x^3} \times \frac{x^3}{\sin^3 x} \right\} \\
&= \lim_{x \rightarrow 0} \frac{\log(1+x^3)}{x^3} \times \frac{1}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^3} \\
&= 1 \times \frac{1}{1} = 1
\end{aligned}$$



Exercise 5.2

1. Evaluate the following

(i) $\lim_{x \rightarrow 2} \frac{x^3 + 2}{x + 1}$

(ii) $\lim_{x \rightarrow \infty} \frac{2x + 5}{x^2 + 3x + 9}$

(iii) $\lim_{x \rightarrow \infty} \frac{\sum n}{n^2}$

(iv) $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{5x}$

(v) $\lim_{x \rightarrow a} \frac{x^{\frac{5}{8}} - a^{\frac{5}{8}}}{x^{\frac{2}{3}} - a^{\frac{2}{3}}}$

(vi) $\lim_{x \rightarrow 0} \frac{\sin^2 3x}{x^2}$

2. If $\lim_{x \rightarrow a} \frac{x^9 + a^9}{x + a} = \lim_{x \rightarrow 3} (x + 6)$, find the values of a .

3. If $\lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 448$, then find the least positive integer n .

4. If $f(x) = \frac{x^7 - 128}{x^5 - 32}$, then find $\lim_{x \rightarrow 2} f(x)$

5. Let $f(x) = \frac{ax + b}{x + 1}$, If $\lim_{x \rightarrow 0} f(x) = 2$ and $\lim_{x \rightarrow \infty} f(x) = 1$, then show that $f(-2) = 0$.

Derivative

Before going into the topic, let us first discuss about the continuity of a function. A function $f(x)$ is continuous at $x = a$ if its graph has no break at $x = a$

If there is any break at the point $x = a$ then we say that the function is not continuous at the point $x = a$.

If a function is continuous at all the point in an interval, then it is said to be a continuous in that interval.

5.2.8 Continuous function

A function $f(x)$ is continuous at $x = a$ if

- (i) $f(a)$ exists
- (ii) $\lim_{x \rightarrow a} f(x)$ exists
- (iii) $\lim_{x \rightarrow a} f(x) = f(a)$

Observation

In the above statement, if atleast any one condition is not satisfied at a point $x = a$ by the function $f(x)$, then it is said to be a discontinuous function at $x = a$

5.2.9 Some properties of continuous functions

Let $f(x)$ and $g(x)$ are two real valued continuous functions at $x = a$, then

- (i) $f(x) \pm g(x)$ is continuous at $x = a$.
- (ii) $f(x) g(x)$ is continuous at $x = a$.
- (iii) $kf(x)$ is continuous at $x = a$, k be a real number.
- (iv) $\frac{1}{f(x)}$ is continuous at $x = a$, if $f(a) \neq 0$.
- (v) $\frac{f(x)}{g(x)}$ is continuous at $x = a$, if $g(a) \neq 0$.
- (vi) $|f(x)|$ is continuous at $x = a$.

Example 5.27

Show that $f(x) = \begin{cases} 5x - 4, & \text{if } 0 < x \leq 1 \\ 4x^3 - 3x, & \text{if } 1 < x < 2 \end{cases}$ is continuous at $x = 1$

Solution

$$\begin{aligned} L[f(x)]_{x=1} &= \lim_{x \rightarrow 1^-} f(x) \\ &= \lim_{h \rightarrow 0} f(1-h), \quad x = 1-h \\ &= \lim_{h \rightarrow 0} [5(1-h) - 4] \\ &= 5(1) - 4 = 1 \end{aligned}$$

$$\begin{aligned} R[f(x)]_{x=1} &= \lim_{x \rightarrow 1^+} f(x) \\ &= \lim_{h \rightarrow 0} f(1+h), \quad x = 1+h \\ &= \lim_{h \rightarrow 0} [4(1+h)^3 - 3(1-h)] \\ &= 4(1)^3 - 3(1) \\ &= 4 - 3 = 1 \end{aligned}$$

Now, $f(1) = 5(1) - 4 = 5 - 4 = 1$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1),$$

$f(x)$ is continuous at $x = 1$.

Example 5.28

Verify the continuity of the function $f(x)$ given by

$$f(x) = \begin{cases} 2 - x & \text{if } x < 2 \\ 2 + x & \text{if } x \geq 2 \end{cases} \text{ at } x = 2.$$

Solution

$$\begin{aligned} L[f(x)]_{x=2} &= \lim_{x \rightarrow 2^-} f(x) \\ &= \lim_{h \rightarrow 0} f(2-h), \quad x = 2-h \\ &= \lim_{h \rightarrow 0} \{2 - (2-h)\} \\ &= \lim_{h \rightarrow 0} h = 0 \end{aligned}$$

$$\begin{aligned}
 R[f(x)]_{x=2} &= \lim_{x \rightarrow 2^+} f(x) \\
 &= \lim_{h \rightarrow 0} f(2+h), x = 2+h \\
 &= \lim_{h \rightarrow 0} \{2 + (2+h)\} \\
 &= \lim_{h \rightarrow 0} (4+h) = 4
 \end{aligned}$$

Now, $f(2) = 2+2 = 4.$

Here $\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$

Hence $f(x)$ is not continuous at $x = 2.$



If $f(x) = \begin{cases} \alpha(x) & \text{if } x < a \\ k & \text{if } x = a \\ \beta(x) & \text{if } x > a \end{cases}$ then

$$L[f(x)]_{x=a} = \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a} \alpha(x)$$

$$R[f(x)]_{x=a} = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} \beta(x)$$

It is applicable only when the function has different definition on both sides of the given point $x = a.$

Observations

- (i) A constant function is everywhere continuous.
- (ii) The identity function is everywhere continuous.
- (iii) A polynomial function is everywhere continuous.
- (iv) The modulus function is everywhere continuous.
- (v) The exponential function a^x , $a > 0$ is everywhere continuous.
- (vi) The logarithmic function is continuous in its domain.
- (vii) Every rational function is continuous at every point in its domain.



Exercise 5.3

1. Examine the following functions for continuity at indicated points

$$(a) f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ 0, & \text{if } x = 2 \end{cases} \text{ at } x = 2$$

$$(b) f(x) = \begin{cases} \frac{x^2-9}{x-3}, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases} \text{ at } x = 3$$

2. Show that $f(x) = |x|$ is continuous at $x = 0$

5.2.10 Differentiability at a point

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let

$c \in (a, b)$. $f(x)$ is said to be differentiable or derivable at $x = c$ if and only if $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists finitely.

This limit is called the derivative (or) differential co-efficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ (or) $Df(c)$ (or) $\left[\frac{d}{dx}(f(x)) \right]_{x=c}$.

$$\text{Thus } f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

5.2.11 Left hand derivative and right hand derivative

- (i) $\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c}$ or $\lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h}$ is called the left hand derivative of $f(x)$ at $x = c$ and it denoted by $f'(c^-)$ or $L[f'(c)]$.
- (ii) $\lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c}$ or $\lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$ is called the right hand derivative of $f(x)$ at $x = c$ and it denoted by $f'(c^+)$ or $R[f'(c)]$.

Result

$$f(x) \text{ is differentiable at } x = c \iff L[f'(c)] = R[f'(c)]$$

Remarks:

- (i) If $L[f'(c)] \neq R[f'(c)]$, then we say that $f(x)$ is not differentiable at $x=c$.
- (ii) $f(x)$ is differentiable at $x = c \implies f(x)$ is continuous at $x = c$.



A function may be continuous at a point but may not be differentiable at that point.

Example 5.29

Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

Solution

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\begin{aligned} L[f'(0)] &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}, \\ &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{0-h-0}, \quad x = 0-h \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} \\
&= \lim_{h \rightarrow 0} \frac{|-h| - |0|}{-h} \\
&= \lim_{h \rightarrow 0} \frac{|-h|}{-h} \\
&= \lim_{h \rightarrow 0} \frac{h}{-h} \\
&= \lim_{h \rightarrow 0} (-1) \\
&= -1
\end{aligned}$$

$$\begin{aligned}
R[f'(0)] &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\
&= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{0+h-0}, \quad x = 0+h \\
&= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\
&= \lim_{h \rightarrow 0} \frac{|h| - |0|}{h} \\
&= \lim_{h \rightarrow 0} \frac{|h|}{h} \\
&= \lim_{h \rightarrow 0} \frac{h}{h} \\
&= \lim_{h \rightarrow 0} 1 \\
&= 1
\end{aligned}$$

Here, $L[f'(0)] \neq R[f'(0)]$

$\therefore f(x)$ is not differentiable at $x=0$

Example 5.30

Show that $f(x) = x^2$ differentiable at $x=1$ and find $f'(1)$.

Solution

$$f(x) = x^2$$

$$\begin{aligned}
L[f'(1)] &= \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{1-h-1} \\
&= \lim_{h \rightarrow 0} \frac{(1-h)^2 - 1}{-h} \\
&= \lim_{h \rightarrow 0} \frac{1 + h^2 - 2h - 1}{-h} \\
&= \lim_{h \rightarrow 0} (-h + 2) \\
&= 2
\end{aligned}$$

$$\begin{aligned}
R[f'(1)] &= \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} \\
&= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{1+h-1} \\
&= \lim_{h \rightarrow 0} \frac{(1+h)^2 - 1}{h} \\
&= \lim_{h \rightarrow 0} \frac{1 + h^2 + 2h - 1}{h} \\
&= \lim_{h \rightarrow 0} (h + 2) \\
&= 2
\end{aligned}$$

Here, $L[f'(1)] = R[f'(1)]$

$\therefore f(x)$ is differentiable at $x = 1$

and $f'(1) = 2$

5.2.12 Differentiation from first principle

The process of finding the derivative of a function by using the definition that

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ is called the differentiation from first principle. It is

convenient to write $f'(x)$ as $\frac{dy}{dx}$.

5.2.13 Derivatives of some standard functions using first principle

1. For $x \in \mathbb{R}$, $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof:

Let $f(x) = x^n$

then $f(x+h) = (x+h)^n$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^n - (x)^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^n - (x)^n}{(x+h) - x} \\ &= \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x}, \text{ where } z = x+h \text{ and } z \rightarrow x \text{ as } h \rightarrow 0 \\ &= nx^{n-1} \quad \left[\because \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right] \end{aligned}$$

i.e., $\frac{d}{dx}(x^n) = nx^{n-1}$

2. $\frac{d}{dx}(e^x) = e^x$

proof:

Let $f(x) = e^x$

then $f(x+h) = e^{x+h}$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^x e^h - e^x}{h} \\ &= \lim_{h \rightarrow 0} e^x \left[\frac{e^h - 1}{h} \right] \end{aligned}$$

$$\begin{aligned}
 &= e^x \lim_{h \rightarrow 0} \left[\frac{e^h - 1}{h} \right] \\
 &= e^x \times 1 \qquad \left[\because \lim_{h \rightarrow 0} \left[\frac{e^h - 1}{h} \right] = 1 \right] \\
 &= e^x
 \end{aligned}$$

i.e., $\frac{d}{dx}(e^x) = e^x$.

3. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$

Proof:

Let $f(x) = \log_e x$

Then, $f(x+h) = \log_e(x+h)$

$$\begin{aligned}
 \frac{d}{dx}(f(x)) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\log_e(x+h) - \log_e x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\log_e \left(1 + \frac{h}{x}\right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\log_e \left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \cdot \frac{1}{x} \\
 &= \frac{1}{x} \lim_{h \rightarrow 0} \frac{\log_e \left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \\
 &= \frac{1}{x} \times 1 \qquad \left[\because \lim_{x \rightarrow 0} \frac{\log_e(1+x)}{x} = 1 \right] \\
 &= \frac{1}{x}
 \end{aligned}$$

Example 5.31

Find $\frac{d}{dx}(x^3)$ from first principle.

Solution

Let $f(x) = x^3$ then

$$f(x+h) = (x+h)^3$$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2) \\ &= 3x^2 + 3x(0) + (0)^2 \\ &= 3x^2 \end{aligned}$$

$$\frac{d}{dx}(x^3) = 3x^2$$

Example 5.32

Find $\frac{d}{dx}(e^{3x})$ from first principle.

Solution

Let $f(x) = e^{3x}$

then $f(x+h) = e^{3(x+h)}$

$$\begin{aligned} \frac{d}{dx}(f(x)) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{3(x+h)} - e^{3x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{3x} \cdot e^{3h} - e^{3x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{e^{3x}(e^{3h} - 1)}{h} \end{aligned}$$

$$\begin{aligned}
 &= 3e^{3x} \lim_{h \rightarrow 0} \frac{e^{3h} - 1}{3h} \\
 &= 3e^{3x} \times 1 \qquad \qquad \qquad [\because \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1] \\
 &= 3e^{3x}
 \end{aligned}$$



Exercise 5.4

1. Find the derivative of the following functions from first principle.

- (i) x^2 (ii) e^{-x} (iii) $\log(x+1)$

5.3 Differentiation techniques

In this section we will discuss about different techniques to obtain the derivatives of the given functions.

5.3.1 Some standard results [formulae]

1. $\frac{d}{dx}(x^n) = nx^{n-1}$
2. $\frac{d}{dx}(x) = 1$
3. $\frac{d}{dx}(k) = 0$, k is a constant
4. $\frac{d}{dx}(kx) = k$, k is a constant
5. $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$
6. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(e^{ax}) = ae^{ax}$
9. $\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}$
10. $\frac{d}{dx}(e^{-x^2}) = -2x e^{-x^2}$
11. $\frac{d}{dx}(a^x) = a^x \log_e a$

12. $\frac{d}{dx}(\log_e x) = \frac{1}{x}$
13. $\frac{d}{dx} \log_e(x+a) = \frac{1}{x+a}$
14. $\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{(ax+b)}$
15. $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$
16. $\frac{d}{dx}(\sin x) = \cos x$
17. $\frac{d}{dx}(\cos x) = -\sin x$
18. $\frac{d}{dx}(\tan x) = \sec^2 x$
19. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
20. $\frac{d}{dx}(\sec x) = \sec x \tan x$
21. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

5.3.2 General rules for differentiation

(i) Addition rule

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

(ii) Subtraction rule (or) Difference rule

$$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

(iii) Product rule

$$\frac{d}{dx}[f(x) \cdot g(x)] = \frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)]$$

(or) If $u = f(x)$ and $v = g(x)$ then, $\frac{d}{dx}(uv) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

(iv) Quotient rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(or) If $u = f(x)$ and $v = g(x)$ then, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$

(v) Scalar product

$$\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)], \text{ where } c \text{ is a constant.}$$

(vi) Chain rule:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\text{(or) If } y = f(t) \text{ and } t = g(x) \text{ then, } \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Here we discuss about the differentiation for the explicit functions using standard results and general rules for differentiation.

Example 5.33

Differentiate the following functions with respect to x .

(i) $x^{\frac{3}{2}}$

(ii) $7e^x$

(iii) $\frac{1-3x}{1+3x}$

(iv) $x^2 \sin x$

(v) $\sin^3 x$

(vi) $\sqrt{x^2 + x + 1}$

Solution

(i) $\frac{d}{dx}(x^{\frac{3}{2}}) = \frac{3}{2}x^{\frac{3}{2}-1}$

$$= \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

(ii) $\frac{d}{dx}(7e^x) = 7 \frac{d}{dx}(e^x) = 7e^x$

(iii) Differentiating $y = \frac{1-3x}{1+3x}$ with respect to x

$$\frac{dy}{dx} = \frac{(1+3x) \frac{d}{dx}(1-3x) - (1-3x) \frac{d}{dx}(1+3x)}{(1+3x)^2}$$

$$= \frac{(1+3x)(-3) - (1-3x)(3)}{(1+3x)^2} = \frac{-6}{(1+3x)^2}$$

(iv) Differentiating $y = x^2 \sin x$ with respect to x

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$

$$= x^2 \cos x + 2x \sin x$$

$$= x(x \cos x + 2 \sin x)$$

$$(v) \quad y = \sin^3 x \quad (\text{or}) \quad (\sin x)^3$$

$$\text{Let } u = \sin x \text{ then } y = u^3$$

Differentiating $u = \sin x$ with respect to x

$$\text{We get, } \frac{du}{dx} = \cos x$$

Differentiating $y = u^3$ with respect to u

$$\text{We get, } \frac{dy}{du} = 3u^2 = 3 \sin^2 x$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3 \sin^2 x \cdot \cos x$$

$$(vi) \quad y = \sqrt{x^2 + x + 1}$$

$$\text{Let } u = x^2 + x + 1 \text{ then } y = \sqrt{u}$$

Differentiating $u = x^2 + x + 1$ with respect to x

$$\text{We get, } \frac{du}{dx} = 2x + 1$$

Differentiating $y = \sqrt{u}$ with respect to u .

$$\text{We get, } \frac{dy}{du} = \frac{1}{2\sqrt{u}}$$

$$= \frac{1}{2\sqrt{x^2 + x + 1}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{x^2 + x + 1}} \cdot (2x + 1)$$

$$= \frac{2x + 1}{2\sqrt{x^2 + x + 1}}$$



$$★ \quad \frac{d}{dx}(\sin(\log x)) = \frac{d}{d(\log x)}(\sin(\log x)) \frac{d}{dx}(\log x)$$

$$★ \quad \frac{d}{dx}(x^2 + 3x + 1)^5 = \frac{d}{d(x^2 + 3x + 1)}(x^2 + 3x + 1)^5 \frac{d}{dx}(x^2 + 3x + 1)$$

$$★ \quad \frac{d}{dx}(e^{(x^3+3)}) = \frac{d}{d(x^3+3)}(e^{(x^3+3)}) \frac{d}{dx}(x^3+3)$$

Example 5.34

If $f(x) = x^n$ and $f'(1) = 5$, then find the value of n .

Solution

$$f(x) = x^n$$

$$\therefore f'(x) = nx^{n-1}$$

$$f'(1) = n(1)^{n-1}$$

$$f'(1) = n$$

$$f'(1) = 5 \text{ (given)}$$

$$\Rightarrow n = 5$$

Example 5.35

If $y = \frac{1}{u^2}$ and $u = x^2 - 9$, find $\frac{dy}{dx}$.

Solution

$$y = \frac{1}{u^2} = u^{-2}$$

$$\frac{dy}{du} = -\frac{2}{u^3}$$

$$\frac{dy}{du} = -\frac{2}{(x^2 - 9)^3} \quad (\because u = x^2 - 9)$$

$$u = x^2 - 9$$

$$\frac{du}{dx} = 2x$$

$$\begin{aligned} \text{Now, } \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= -\frac{2}{(x^2 - 9)^3} \cdot 2x \\ &= -\frac{4x}{(x^2 - 9)^3} \end{aligned}$$



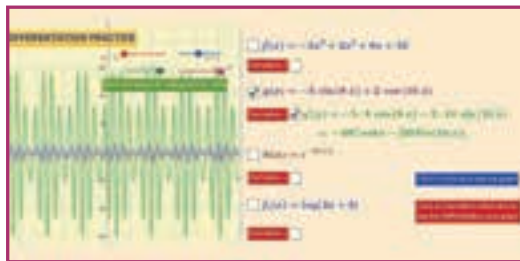
ICT Corner

Expected final outcomes

Step - 1

Open the Browser and type the URL given (or) Scan the QR Code.

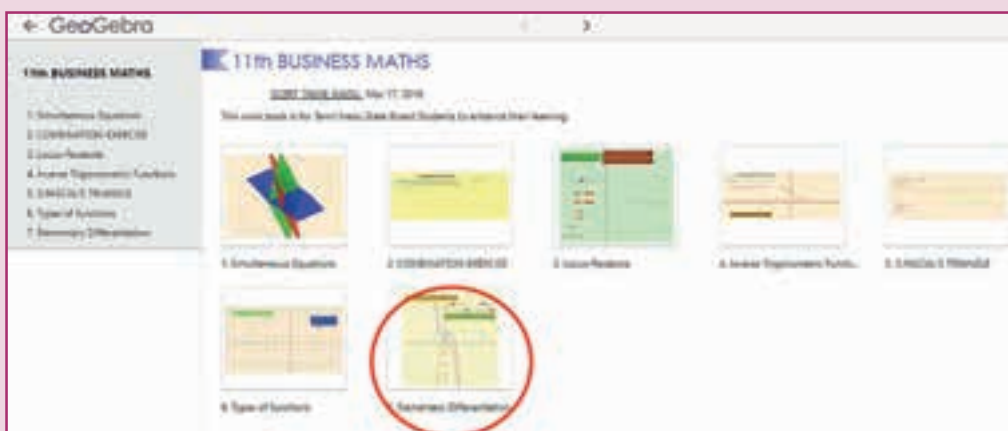
GeoGebra Work book called “11th BUSINESS MATHS” will appear. In this several work sheets for Business Maths are given, Open the worksheet named “Elementary Differentiation”



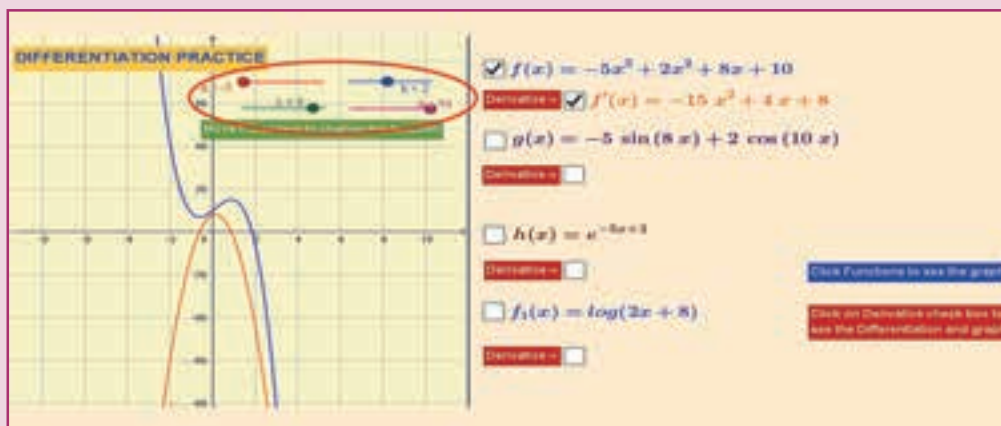
Step - 2

Elementary Differentiation and Functions page will open. Click on the function check boxes on the Right-hand side to see the respective graph. You can move the sliders a, b, c and d to change the co-efficient and work out the differentiation. Then click the Derivatives check box to see the answer and respective graphs

Step 1



Step 2



Browse in the link

11th Business Maths: <https://ggbm.at/qKj9gSTG> (or) scan the QR Code



B184_11_BM_EM



Exercise 5.5

1. Differentiate the following with respect to x .

(i) $3x^4 - 2x^3 + x + 8$

(ii) $\frac{5}{x^4} - \frac{2}{x^3} + \frac{5}{x}$

(iii) $\sqrt{x} + \frac{1}{\sqrt[3]{x}} + e^x$

(iv) $\frac{3 + 2x - x^2}{x}$

(v) $x^3 e^x$

(vi) $(x^2 - 3x + 2)(x + 1)$

(vii) $x^4 - 3 \sin x + \cos x$

(viii) $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

2. Differentiate the following with respect to x .

(i) $\frac{e^x}{1+x}$

(ii) $\frac{x^2 + x + 1}{x^2 - x + 1}$

(iii) $\frac{e^x}{1 + e^x}$

3. Differentiate the following with respect to x .

(i) $x \sin x$

(ii) $e^x \sin x$

(iii) $e^x(x + \log x)$

(iv) $\sin x \cos x$

(v) $x^3 e^x$

4. Differentiate the following with respect to x .

(i) $\sin^2 x$

(ii) $\cos^2 x$

(iii) $\cos^3 x$

(iv) $\sqrt{1+x^2}$

(v) $(ax^2 + bx + c)^n$

(vi) $\sin(x^2)$

(vii) $\frac{1}{\sqrt{1+x^2}}$

5.3.3 Derivative of implicit functions

For the implicit function $f(x, y) = 0$, differentiate each term with respect to x treating y as a function of x and then collect the terms of $\frac{dy}{dx}$ together on left hand side and remaining terms on the right hand side and then find $\frac{dy}{dx}$.

Example 5.36

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ then find $\frac{dy}{dx}$.

Solution

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$



NOTE

If the function $f(x, y) = 0$ is an implicit function, then $\frac{dy}{dx}$ contains both x and y .



Differentiating both side with respect to x ,

$$2ax + 2h\left(x\frac{dy}{dx} + y\right) + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$(2ax + 2hy + 2g) + (2hx + 2by + 2f)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{(ax + hy + g)}{(hx + by + f)}.$$

Example 5.37

If $x^3 + y^3 = 3axy$, then find $\frac{dy}{dx}$.

Solution

$$x^3 + y^3 = 3axy$$

Differentiating both sides with respect to x ,

$$\frac{d}{dx}(x^3 + y^3) = \frac{d}{dx}(3axy)$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

$$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

$$(y^2 - ax)\frac{dy}{dx} = (ay - x^2)$$

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

Example 5.38

Find $\frac{dy}{dx}$ at (1,1) to the curve $2x^2 + 3xy + 5y^2 = 10$

Solution

$$2x^2 + 3xy + 5y^2 = 10$$

Differentiating both sides with respect to x ,

$$\frac{d}{dx}[2x^2 + 3xy + 5y^2] = \frac{d}{dx}[10]$$

$$4x + 3x\frac{dy}{dx} + 3y + 10y\frac{dy}{dx} = 0$$

$$(3x+10y)\frac{dy}{dx} = -3y -4x$$

$$\frac{dy}{dx} = -\frac{(3y+4x)}{(3x+10y)}$$

Now, $\frac{dy}{dx}$ at (1, 1) = $-\frac{3+4}{3+10}$
 $= -\frac{7}{13}$

Example 5.39

If $\sin y = x \sin (a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution

$$\sin y = x \sin (a+y)$$

$$x = \frac{\sin y}{\sin(a+y)}$$

Differentiating with respect to y ,

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y \cdot \cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$= \frac{\sin a}{\sin^2(a+y)}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$



Exercise 5.6

- Find $\frac{dy}{dx}$ for the following functions
 - $xy = \tan(xy)$
 - $x^2 - xy + y^2 = 7$
 - $x^3 + y^3 + 3axy = 1$
- If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, then prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$
- If $4x + 3y = \log(4x - 3y)$, then find $\frac{dy}{dx}$

5.3.4 Logarithmic differentiation

Some times, the function whose derivative is required involves products, quotients, and powers. For such cases, differentiation can be carried out more conveniently if we take logarithms and simplify before differentiation.

Example 5.40

Differentiate the following with respect to x .

(i) x^x (ii) $(\log x)^{\cos x}$

Solution

(i) Let $y = x^x$

Taking logarithm on both sides

$$\log y = x \log x$$

Differentiating with respect to x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\frac{dy}{dx} = y[1 + \log x]$$

$$\frac{dy}{dx} = x^x[1 + \log x]$$

(ii) Let $y = (\log x)^{\cos x}$

Taking logarithm on both sides

$$\therefore \log y = \cos x \log(\log x)$$

Differentiating with respect to x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \frac{1}{\log x} \cdot \frac{1}{x} + [\log(\log x)](-\sin x)$$

$$\frac{dy}{dx} = y \left[\frac{\cos x}{x \log x} - \sin x \log(\log x) \right]$$

$$= (\log x)^{\cos x} \left[\frac{\cos x}{x \log x} - \sin x \log(\log x) \right]$$

Example 5.41

If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$

Solution

$$x^y = e^{x-y}$$

Taking logarithm on both sides,

$$y \log x = (x-y)$$

$$y(1+\log x) = x$$

$$y = \frac{x}{1+\log x}$$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{(1+\log x)(1) - x\left(0 + \frac{1}{x}\right)}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

Example 5.42

Differentiate: $\sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

Solution

Let $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}} = \left[\frac{(x-3)(x^2+4)}{3x^2+4x+5}\right]^{\frac{1}{2}}$

Taking logarithm on both sides,

$$\log y = \frac{1}{2}[\log(x-3) + \log(x^2+4) - \log(3x^2+4x+5)]$$

$$\left[\because \log ab = \log a + \log b \text{ and } \log \frac{a}{b} = \log a - \log b\right]$$

Differentiating with respect to x ,

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} \left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5} \right]$$

$$\frac{dy}{dx} = \frac{1}{2} \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}} \left[\frac{1}{x-3} + \frac{2x}{x^2+4} - \frac{6x+4}{3x^2+4x+5} \right]$$



Exercise 5.7

1. Differentiate the following with respect to x ,

(i) $x^{\sin x}$

(ii) $(\sin x)^x$

(iii) $(\sin x)^{\tan x}$

(iv) $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x^2+x+1)}}$

2. If $x^m \cdot y^n = (x+y)^{m+n}$, then show that $\frac{dy}{dx} = \frac{y}{x}$

5.3.5 Differentiation of parametric functions

If the variables x and y are functions of another variable namely t , then the functions are called a parametric functions. The variable t is called the parameter of the function.

If $x = f(t)$ and $y = g(t)$, then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

5.3.6 Differentiation of a function with respect to another function

Let $u = f(x)$ and $v = g(x)$ be two functions of x . The derivative of $f(x)$ with respect to $g(x)$ is given by the formula,

$$\frac{d(f(x))}{d(g(x))} = \frac{du/dx}{dv/dx}$$

Example 5.43

Find $\frac{dy}{dx}$ if (i) $x = at^2$, $y = 2at$

(ii) $x = a \cos \theta$, $y = a \sin \theta$

Solution

$$(i) \quad \begin{array}{l} x = at^2 \\ \frac{dx}{dt} = 2at \end{array} \quad \left| \quad \begin{array}{l} y = 2at \\ \frac{dy}{dt} = 2a \end{array} \right.$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \\ &= \frac{2a}{2at} \\ &= \frac{1}{t} \end{aligned}$$

(ii) $x = a \cos \theta$, $y = a \sin \theta$

$$\frac{dx}{d\theta} = -a \sin \theta \quad \left| \quad \frac{dy}{d\theta} = a \cos \theta \right.$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{a \cos \theta}{-a \sin \theta} \\ &= -\cot \theta \end{aligned}$$

Example 5.44

If $x = a\theta$ and $y = \frac{a}{\theta}$, then prove that $\frac{dy}{dx} + \frac{y}{x} = 0$

Solution

$$\begin{array}{l|l} x = a\theta & y = \frac{a}{\theta} \\ \frac{dx}{d\theta} = a & \frac{dy}{d\theta} = \frac{-a}{\theta^2} \end{array}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \\ &= \frac{\left(\frac{-a}{\theta^2}\right)}{a} \\ &= -\frac{1}{\theta^2} \\ &= -\frac{y}{x} \end{aligned}$$

i.e. $\frac{dy}{dx} + \frac{y}{x} = 0$

Aliter :

Take $xy = a\theta \cdot \frac{a}{\theta}$

$$xy = a^2$$

Differentiating with respect to x ,

$$x \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} + \frac{y}{x} = 0$$

Example 5.45

Differentiate $\frac{x^2}{1+x^2}$ with respect to x^2

Solution

$$\text{Let } u = \frac{x^2}{1+x^2}$$

$$\begin{aligned} \frac{du}{dx} &= \frac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} \\ &= \frac{2x}{(1+x^2)^2} \end{aligned}$$

$$v = x^2$$

$$\frac{dv}{dx} = 2x$$

$$\begin{aligned} \frac{du}{dv} &= \frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)} \\ &= \frac{\left[\frac{2x}{(1+x^2)^2}\right]}{2x} \\ &= \frac{1}{(1+x^2)^2} \end{aligned}$$

**Exercise 5.8**

- Find $\frac{dy}{dx}$ of the following functions
 - $x = ct, y = \frac{c}{t}$
 - $x = \log t, y = \sin t$
 - $x = a \cos^3 \theta, y = a \sin^3 \theta$
 - $x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$
- Differentiate $\sin^3 x$ with respect to $\cos^3 x$
- Differentiate $\sin^2 x$ with respect to x^2

5.3.7 Successive differentiation

The process of differentiating the same function again and again is called successive differentiation.

- The derivative of y with respect to x is called the first order derivative and is denoted by $\frac{dy}{dx}$ (or) y_1 (or) $f'(x)$
- If $f'(x)$ is differentiable, then the derivative of $f'(x)$ with respect to x is called the second order derivative and is denoted by $\frac{d^2y}{dx^2}$ (or) y_2 (or) $f''(x)$
- Further $\frac{d^n y}{dx^n}$ (or) y_n (or) $f^{(n)}(x)$ is called n^{th} order derivative of the function $y = f(x)$

Remarks

(i) If $y = f(x)$, then $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$.

(ii) If $x = f(t)$ and $y = g(t)$, then $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \frac{dt}{dx} = \frac{d}{dt} \left\{ \frac{g'(t)}{f'(t)} \right\} \cdot \frac{dt}{dx}$

Example 5.46

Find the second order derivative of the following functions with respect to x ,

(i) $3 \cos x + 4 \sin x$

(ii) $x = at^2, y = 2at$

(iii) $x \sin x$

Solution

(i) Let $y = 3 \cos x + 4 \sin x$

$$y_1 = -3 \sin x + 4 \cos x$$

$$y_2 = -3 \cos x - 4 \sin x$$

$$y_2 = -3 (\cos x + 4 \sin x)$$

$$y_2 = -y \text{ (or) } y_2 + y = 0$$

(ii) $x = at^2, y = 2at$

$$\frac{dx}{dt} = 2at \quad \Bigg| \quad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt} \right)}{\left(\frac{dx}{dt} \right)}$$

$$= \frac{1}{t}$$

Now,

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{dy}{dx} \right) \\ &= \frac{d}{dt} \left(\frac{dy}{dx} \right) \frac{dt}{dx} \\ &= \frac{d}{dt} \left(\frac{1}{t} \right) \cdot \frac{1}{2at} \\ &= -\frac{1}{t^2} \cdot \frac{1}{2at} \\ &= -\frac{1}{2at^3} \end{aligned}$$

$$(iii) \quad y = x \sin x$$

$$\frac{dy}{dx} = x \cos x + \sin x$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= -x \sin x + \cos x + \cos x \\ &= 2 \cos x - x \sin x \end{aligned}$$

Example 5.47

If $y = A \sin x + B \cos x$, then prove that $y_2 + y = 0$

Solution

$$y = A \sin x + B \cos x$$

$$y_1 = A \cos x - B \sin x$$

$$y_2 = -A \sin x - B \cos x$$

$$y_2 = -y$$

$$y_2 + y = 0$$



Exercise 5.9

- Find y_2 for the following functions
 - $y = e^{3x+2}$
 - $y = \log x + a^x$
 - $x = a \cos \theta, y = a \sin \theta$
- If $y = 500e^{7x} + 600e^{-7x}$, then show that $y_2 - 49y = 0$
- If $y = 2 + \log x$, then show that $xy_2 + y_1 = 0$
- If $y = a \cos mx + b \sin mx$, then show that $y_2 + m^2y = 0$
- If $y = (x + \sqrt{1+x^2})^m$, then show that $(1+x^2)y_2 + xy_1 - m^2y = 0$
- If $y = \sin(\log x)$, then show that $x^2y_2 + xy_1 + y = 0$.



Exercise 5.10



Choose the correct answer

- If $f(x) = x^2 - x + 1$ then $f(x+1)$ is
 (a) x^2 (b) x (c) 1 (d) $x^2 + x + 1$
- Let $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$ then $f(5)$ is
 (a) -1 (b) 2 (c) 5 (d) 7
- For $f(x) = \begin{cases} x^2 - 4x & \text{if } x \geq 2 \\ x + 2 & \text{if } x < 2 \end{cases}$ then, $f(0)$ is
 (a) 2 (b) 5 (c) -1 (d) 0
- If $f(x) = \frac{1-x}{1+x}$ then $f(-x)$ is equal to
 (a) $-f(x)$ (b) $\frac{1}{f(x)}$ (c) $-\frac{1}{f(x)}$ (d) $f(x)$
- The graph of the line $y = 3$ is
 (a) Parallel to x -axis (b) Parallel to y -axis
 (c) Passing through the origin (d) Perpendicular to x -axis
- The graph of $y = 2x^2$ is passing through
 (a) (0,0) (b) (2,1) (c) (2,0) (d) (0,2)
- The graph of $y = e^x$ intersect the y -axis at
 (a) (0, 0) (b) (1, 0) (c) (0, 1) (d) (1, 1)
- The minimum value of the function $f(x) = |x|$ is
 (a) 0 (b) -1 (c) +1 (d) $-\infty$
- Which one of the following functions has the property $f(x) = f\left(\frac{1}{x}\right)$
 (a) $f(x) = \frac{x^2 - 1}{x}$ (b) $f(x) = \frac{1 - x^2}{x}$ (c) $f(x) = x$ (d) $f(x) = \frac{x^2 + 1}{x}$
- If $f(x) = 2^x$ and $g(x) = \frac{1}{2^x}$ then $(fg)(x)$ is
 (a) 1 (b) 0 (c) 4^x (d) $\frac{1}{4^x}$

11. Which of the following function is neither even nor odd?
 (a) $f(x) = x^3 + 5$ (b) $f(x) = x^5$ (c) $f(x) = x^{10}$ (d) $f(x) = x^2$
12. $f(x) = -5$, for all $x \in R$ is a
 (a) an identity function (b) modulus function
 (c) exponential function (d) constant function
13. The range of $f(x) = |x|$, for all $x \in R$ is
 (a) $(0, \infty)$ (b) $[0, \infty)$ (c) $(-\infty, \infty)$ (d) $[1, \infty)$
14. The graph of $f(x) = e^x$ is identical to that of
 (a) $f(x) = a^x, a > 1$ (b) $f(x) = a^x, a < 1$
 (c) $f(x) = a^x, 0 < a < 1$ (d) $y = ax + b, a \neq 0$
15. If $f(x) = x^2$ and $g(x) = 2x+1$ then $(fg)(0)$ is
 (a) 0 (b) 2 (c) 1 (d) 4
16. $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} =$
 (a) 1 (b) ∞ (c) $-\infty$ (d) θ
17. $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} =$
 (a) e (b) nx^{n-1} (c) 1 (d) 0
18. For what value of x , $f(x) = \frac{x+2}{x-1}$ is not continuous?
 (a) -2 (b) 1 (c) 2 (d) -1
19. A function $f(x)$ is continuous at $x = a$ if $\lim_{x \rightarrow a} f(x)$ is equal to
 (a) $f(-a)$ (b) $f\left(\frac{1}{a}\right)$ (c) $2f(a)$ (d) $f(a)$
20. $\frac{d}{dx}\left(\frac{1}{x}\right)$ is equal to
 (a) $-\frac{1}{x^2}$ (b) $-\frac{1}{x}$ (c) $\log x$ (d) $\frac{1}{x^2}$

21. $\frac{d}{dx}(5e^x - 2 \log x)$ is equal to
 (a) $5e^x - \frac{2}{x}$ (b) $5e^x - 2x$ (c) $5e^x - \frac{1}{x}$ (d) $2 \log x$
22. If $y = x$ and $z = \frac{1}{x}$ then $\frac{dy}{dz} =$
 (a) x^2 (b) 1 (c) $-x^2$ (d) $-\frac{1}{x^2}$
23. If $y = e^{2x}$ then $\frac{d^2y}{dx^2}$ at $x = 0$ is
 (a) 4 (b) 9 (c) 2 (d) 0
24. If $y = \log x$ then $y_2 =$
 (a) $\frac{1}{x}$ (b) $-\frac{1}{x^2}$ (c) $-\frac{2}{x^2}$ (d) e^2
25. $\frac{d}{dx}(a^x) =$
 (a) $\frac{1}{x \log_e a}$ (b) a^a (c) $x \log_e a$ (d) $a^x \log_e a$

Miscellaneous Problems

1. If $f(x) = \frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that $f(f(x)) = \frac{2x+1}{2x+3}$, provided $x \neq -\frac{3}{2}$
2. Draw the graph of $y = 9 - x^2$
3. If $f(x) = \begin{cases} \frac{x-|x|}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$ then show that $\lim_{x \rightarrow 0} f(x)$ does not exist.
4. Evaluate: $\lim_{x \rightarrow 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$
5. Show that the function $f(x) = 2x - |x|$ is continuous at $x = 0$
6. Verify the continuity and differentiability of $f(x) = \begin{cases} 1-x & \text{if } x < 1 \\ (1-x)(2-x) & \text{if } 1 \leq x \leq 2 \\ 3-x & \text{if } x > 2 \end{cases}$ at $x = 1$ and $x = 2$
7. If $x^y = y^x$, then prove that $\frac{dy}{dx} = \frac{y}{x} \left(\frac{x \log y - y}{y \log x - x} \right)$
8. If $xy^2 = 1$, then prove that $2 \frac{dy}{dx} + y^3 = 0$.
9. If $y = \tan x$, then prove that $y_2 - 2yy_1 = 0$.
10. If $y = 2 \sin x + 3 \cos x$, then show that $y_2 + y = 0$.

Summary



- Let A and B be two non empty sets, then a function f from A to B , associates every element of A to an unique element of B .
- $\lim_{x \rightarrow a} f(x)$ exists $\Leftrightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$.
- For the function $f(x)$ and a real number a , $\lim_{x \rightarrow a} f(x)$ and $f(a)$ may not be the same.
- A function $f(x)$ is a continuous function at $x = a$ if only if $\lim_{x \rightarrow a} f(x) = f(a)$.
- A function $f(x)$ is continuous, if it is continuous at every point of its domain.
- If $f(x)$ and $g(x)$ are continuous on their common domain, then $f \pm g$, $f \cdot g$, kf (k is a constant) are continuous and if $g \neq 0$ then $\frac{f}{g}$ is also continuous.
- A function $f(x)$ is differentiable at $x = c$ if and only if $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists finitely and denoted by $f'(c)$.
- $L[f'(c)] = \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h}$ and $R[f'(c)] = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$
- A function $f(x)$ is said to be differentiable at $x = c$ if and only if $L[f'(c)] = R[f'(c)]$.
- Every differentiable function is continuous but, the converse is not necessarily true.
- If $y = f(x)$ then $\frac{d}{dx} \left(\frac{dy}{dx} \right)$ is called second order derivative of y with respect to x .
- If $x = f(t)$ and $y = g(t)$ then $\frac{d^2 y}{dx^2} = \frac{d}{dx} \left\{ \frac{g'(t)}{f'(t)} \right\}$ (or) $\frac{d^2 y}{dx^2} = \frac{d}{dt} \left\{ \frac{g'(t)}{f'(t)} \right\} \cdot \frac{dt}{dx}$

GLOSSARY	
Absolute constants	முழுமையான மாறிலிகள்
Algebraic functions	இயற்கணித சார்புகள்
Arbitrary constants	தன்னிச்சை மாறிலிகள்
Chain rule	சங்கிலி விதி
Closed interval	மூடிய இடைவெளி
Constant	மாறிலி
Continuous	தொடர்ச்சி
Dependent variable	சார்ந்த மாறி
Derivative	வகையீடு
Domain	சார்பகம் / அரங்கம்
Explicit	வெளிபடு
Exponential	அடுக்கு
Function	சார்பு
Identity	சமனி
Implicit	உட்படு
Independent variable	சாரா மாறி
Interval	இடைவெளி
Left limit	இடக்கை எல்லை
Limit	எல்லை
Logarithmic	மடக்கை
Modulus	மட்டு
Neighbourhood	அண்மையகம்
Open interval	திறந்த இடைவெளி
Parametric functions	துணையலகு சார்புகள்
Range	வீச்சகம்
Right limit	வலக்கை எல்லை
Signum function	குறிச் சார்பு
Successive differentiation	தொடர் வகையிடல்
Transcendental functions	விஞ்சிய சார்புகள்
Variable	மாறி

ANSWERS

1. MATRICES AND DETERMINANTS

Exercise 1.1

$$1.(i) M_{11} = -1 \quad M_{12} = 0 \quad M_{21} = 20 \quad M_{22} = 5$$

$$A_{11} = -1 \quad A_{12} = 0 \quad A_{21} = -20 \quad A_{22} = 5$$

$$(ii) M_{11} = -12 \quad M_{12} = 2 \quad M_{13} = 23 \quad M_{21} = -16 \quad M_{22} = -4 \quad M_{23} = 14$$

$$M_{31} = -4 \quad M_{32} = -6 \quad M_{33} = 11$$

$$A_{11} = -12 \quad A_{12} = -2 \quad A_{13} = 23 \quad A_{21} = 16 \quad A_{22} = -4 \quad A_{23} = -14$$

$$A_{31} = -4 \quad A_{32} = 6 \quad A_{33} = 11$$

$$2. 20 \qquad 3. \frac{1}{2} \qquad 4. -30 \qquad 5. \frac{13}{2}, 2 \qquad 6. 0$$

Exercise 1.2

$$1. \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} \quad 2. \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad 3.(i) \frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} \quad (ii) \frac{1}{10} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

$$(iii) \frac{1}{10} \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix} \quad (iv) \frac{1}{151} \begin{bmatrix} -22 & -46 & -7 \\ -13 & 14 & -11 \\ 5 & -17 & -19 \end{bmatrix}$$

$$7. \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix} \quad 10. \lambda = \frac{7}{4} \quad 11. p = 2, q = -3$$

Exercise 1.3

$$1. x = 1, y = 1 \qquad 2.(i) x = 3, y = -2, z = 1$$

$$(ii) x = -1, y = 2, z = 3 \quad (iii) x = 1, y = -1, z = 2 \quad 3. 17.95, 43.08, 103.85$$

$$4. 3000, 1000, 2000 \qquad 5. 13, 2, 5 \qquad 6. 700, 600, 300$$

Exercise 1.4

1. It is viable 2. It is not viable 3. It is viable
4. $A = 250.36$ tonnes $B = 62.44$ tonnes 5. 181.62, 84.32
6. 34.16, 17.31, 7. 42 and 78

Exercise 1.5

1	2	3	4	5	6	7	8	9	10	11	12	13
(b)	(d)	(b)	(b)	(c)	(c)	(c)	(c)	(c)	(d)	(c)	(b)	(a)
14	15	16	17	18	19	20	21	22	23	24	25	
(c)	(c)	(c)	(b)	(a)	(d)	(b)	(b)	(d)	(d)	(b)	(a)	

Miscellaneous Problems

1. $x = 3$. $x = -1$ 2. 0 6. $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$
7. $\frac{1}{5} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ 8. $x = 1, y = 2, z = 3$ 9. $x = 20, y = 30, z = 50$
10. ₹1200 Crores, ₹1600 Crores

2. ALGEBRA

Exercise 2.1

1. $\frac{13}{x-2} - \frac{10}{x-1}$ 2. $\frac{3}{x-2} + \frac{1}{x+1}$
3. $\frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$ 4. $\frac{1}{2(x-1)} - \frac{1}{2(x+1)}$
5. $\frac{-4}{9(x+2)} + \frac{4}{9(x-1)} - \frac{1}{3(x-1)^2}$ 6. $\frac{2}{x-2} + \frac{3}{(x-2)^2} - \frac{9}{(x-2)^2}$
7. $\frac{-7}{2x} + \frac{1}{x^2} + \frac{9}{2(x+2)}$ 8. $\frac{1}{5(x+2)} + \frac{4(x-2)}{5(x^2+1)}$
9. $\frac{3}{16(x-1)} - \frac{3}{16(x+3)} + \frac{1}{4(x+3)^2}$ 10. $\frac{1-x}{5(x^2+4)} + \frac{1}{5(x+1)}$

Exercise 2.2

1. 64 2. 10 3. 3 4. 336 5. 85

Exercise 2.3

1. 6 2. 14400 3. 1680 4. $\frac{13!}{4!3!2!2!}$
5. (a) $\frac{7!}{2}$ (b) 7! 6. Rank of the word CHAT = 9.

Exercise 2.4

1. 4
2. $8C_4 + 8C_3 = 9C_4 = 126$
3. 210 cards
4. 20
5. 25200
6. 15
7. $\frac{13!}{7!6!} \times 9! \times 9!$
8. 11
- 9.(i) 1365
- (ii) 1001
- (iii) 364
- 10.(i) 186
- (ii) 186

Exercise 2.6

- 1.(i) $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$
- (ii) $x^7 + \frac{7x^6}{y} + \frac{21x^5}{y^2} + \frac{35x^4}{y^3} + \frac{35x^3}{y^4} + \frac{21x^2}{y^5} + \frac{7x}{y^6} + \frac{1}{y^7}$
- (iii) $x^6 + 6x^3 + 15 + \frac{20}{x^3} + \frac{15}{x^6} + \frac{6}{x^9} + \frac{1}{x^{12}}$
- 2.(i) 10, 40, 60, 401
- (ii) 995009990004999
3. $11440 x^9 y^4$
- 4.(i) $11C_5 x, \frac{11C_5}{x}$
- (ii) $\frac{8C_4(81)}{16} x^{12}$
- (iii) $\frac{-10C_5(6)^5}{x^5}$
- 5.(i) $9C_6 \frac{2^6}{3^6}$
- (ii) $-32(15C_5)$
- (iii) 7920.

Exercise 2.7

1	2	3	4	5	6	7	8	9	10	11	12	13
(d)	(c)	(d)	(c)	(a)	(c)	(d)	(b)	(b)	(c)	(b)	(d)	(b)
14	15	16	17	18	19	20	21	22	23	24	25	
(a)	(c)	(a)	(c)	(a)	(c)	(a)	(b)	(c)	(d)	(b)	(a)	

Miscellaneous Problems

1. $\frac{3}{x-1} + \frac{2}{x+3}$
2. $\frac{3}{x-1} - \frac{2}{x-2}$
3. $\frac{1}{x+2} + \frac{5}{x-3} - \frac{3}{(x-3)^2}$
4. $\frac{3}{x-2} + \frac{2x-1}{x^2-x+1}$
- 5.(i) 7
- (ii) 336
- (iii) 84
- 6.(a) 720
- (b) 7776
- (c) 120
- (d) 6
7. 74 ways
8. 85 ways
9. 210
10. 544

3. ANALYTICAL GEOMETRY

Exercise 3.1

1. $x^2 - 2x - 6y + 10 = 0$
2. $x^2 + y^2 - 6x + 4y - 3 = 0$
3. $3x^2 + 3y^2 - 4x - 14y + 15 = 0$
4. $\left(\frac{15}{2}, 0\right)$
5. $2x - 3y + 21 = 0$

Exercise 3.2

1. Angle between the line is 45°
2. 4 units
4. $a = 5$
5. $y = 1500x + 100000$, cost of 95 TV sets ₹ 2,42,500

Exercise 3.3

1. $a = 6, c = 6$
2. $2x - y + 2 = 0$ and $6x - 2y + 1 = 0$
3. $2x + 3y - 1 = 0$ and $2x + 3y - 2 = 0$
4. $\theta = \tan^{-1}(7)$

Exercise 3.4

- 1.(i) $x^2 + y^2 - 6x - 10y + 9 = 0$
- (ii) $x^2 + y^2 - 4 = 0$
- 2.(i) $C(0, 0), r = 4$
- (ii) $C(11, 2), r = 10$
- (iii) $C\left(\frac{-2}{5}, \frac{4}{5}\right), r = 2$
- (iv) $C\left(\frac{3}{2}, \frac{3}{2}\right), r = \frac{5}{\sqrt{2}}$
3. $x^2 + y^2 + 6x + 4y - 3 = 0$
4. $x^2 + y^2 - 4x - 6y + 13 = 0$
5. $x^2 + y^2 - 5x - 2y + 1 = 0$
6. $x^2 + y^2 - x - y = 0$
7. $x^2 + y^2 - 16x + 4y - 32 = 0$
8. $x^2 + y^2 - 2x - 12y + 27 = 0$
9. $x^2 + y^2 = 9$

Exercise 3.5

1. $x + 2 = 0$
2. R lies inside, P lies outside and Q lies on the circle.
3. $\sqrt{20}$ units
4. $p = \pm 20$.

Exercise 3.6

1. $9x^2 + 16y^2 + 24xy + 34x + 112y + 121 = 0$
2. $k = \frac{1}{4}$, Length of latus rectum = 1, Focus $F\left(\frac{1}{4}, 0\right)$

3.

Axis	Vertex	Focus	Equation of directrix	Length of latus rectum
$y = 4$	$V(1, 4)$	$F(3, 4)$	$x + 1 = 0$	4 units

4.

Problem	Axis	Vertex	Focus	Equation of directrix	Length of latusrectum
(a) $y^2 = 20x$	$y = 0$	$V(0, 0)$	$F(5, 0)$	$x = -5$	20 units
(b) $x^2 = 8y$	$x = 0$	$V(0, 0)$	$F(0, 2)$	$y = -2$	8 units
(c) $x^2 = -16y$	$x = 0$	$V(0, 0)$	$F(0, -4)$	$y = 4$	16 units

5. $(x - 15)^2 = 5(y - 55)$. The output and the average cost at the vertex are 15 kg. and ₹ 55.6. when $x = 5$ months

Exercise 3.7

1	2	3	4	5	6	7	8	9	10	11	12	13
(b)	(c)	(c)	(c)	(b)	(c)	(a)	(c)	(a)	(b)	(c)	(a)	(d)
14	15	16	17	18	19	20	21	22	23	24	25	
(a)	(a)	(d)	(a)	(c)	(b)	(b)	(a)	(d)	(b)	(b)	(b)	

Miscellaneous problems

- $2x + y - 7 = 0$
- $y = 6x + 3000$
- $p = 1, p = 2$
- $a = 9, b = 8$
- $(-1, -2)$ is on the line, $(1, 0)$ is above the line and $(-3, -4)$ is below the line
- $(-2, -7)$
- $y^2 = -\frac{9}{2}x$
- Axis : $y = 2$, Vertex : $(1, 2)$, Focus : $(2, 2)$,
Equation of directrix : $x = 0$ and Length of latus rectum : 4 units

4. TRIGONOMETRY

Exercise 4.1

- (i) $\frac{\pi}{3}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{4\pi}{3}$ (iv) $\frac{-16\pi}{9}$

- 2.(i) $22^\circ 30'$ (ii) $405^\circ 30'$ (iii) $-171^\circ 48'$ (iv) 110°
 3.(i) 1st quadrant (ii) 3rd quadrant (iii) 2nd quadrant.
 4.(i) $\frac{-\sqrt{3}}{2}$ (ii) $\frac{-1}{2}$ (iii) $\frac{2}{\sqrt{3}}$ (iv) 1
 (v) $\sqrt{2}$ 10. $\frac{-7}{2}$

Exercise 4.2

- 1.(i) $\frac{2\sqrt{2}}{\sqrt{3}-1}$ (ii) $-\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$ (iii) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$
 2.(i) $\sin 92^\circ$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\cos 80^\circ$ (iv) $\frac{\sqrt{3}}{2}$
 3.(i) $\frac{-33}{65}$ (ii) $\frac{-16}{65}$ (iii) $\frac{16}{33}$
 6. $\frac{2}{11}$, $\alpha + \beta$ lies in 1st quadrant. 7. $\pm\sqrt{2} - 1$
 9.(i) $\frac{9}{13}$ (ii) $-\frac{828}{2197}$ 10. $-\frac{44}{125}, -\frac{117}{44}$ 14. $\sqrt{2} - 1$

Exercise 4.3

- 1.(i) $\frac{1}{2}\left(\cos\frac{A}{4} - \cos\frac{A}{2}\right)$ (ii) $\frac{1}{2}\left(-\sin 2A + \frac{\sqrt{3}}{2}\right)$
 (iii) $\frac{1}{2}\left(\sin 4A - \sin\frac{2A}{3}\right)$ (iv) $\frac{1}{2}(\cos 7\theta + \cos \theta)$
 2.(i) $2\sin\frac{3A}{2}\cos\frac{A}{2}$ (ii) $2\cos 3A \cos A$
 (iii) $2\sin 2\theta \cos 4\theta$ (iv) $2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}$

Exercise 4.4

- 1.(i) $\frac{-\pi}{6}$ (ii) $\frac{-\pi}{4}$ (iii) $\frac{\pi}{6}$ (iv) $\frac{\pi}{4}$
 4. $x = \frac{1}{6}$ 5. $x = \frac{1}{2}$
 6.(i) $\frac{4}{5}$ (ii) $\frac{1}{\sqrt{10}}$ 7. $\frac{-27}{65}$ 10. $\frac{\pi}{4} + \frac{x}{2}$

Exercise 4.5

1	2	3	4	5	6	7	8	9	10	11	12	13
(b)	(a)	(c)	(b)	(b)	(d)	(a)	(d)	(a)	(c)	(c)	(d)	(b)
14	15	16	17	18	19	20	21	22	23	24	25	
(c)	(b)	(a)	(c)	(b)	(c)	(c)	(b)	(c)	(a)	(b)	(d)	

Miscellaneous Problems

4. $\frac{3}{\sqrt{10}}$ and $\frac{-1}{\sqrt{10}}$

6.(i) $\frac{\sqrt{6} + \sqrt{2}}{4}$ (ii) $2 - \sqrt{3}$

7. $\frac{\sqrt{15} + 2\sqrt{2}}{12}$ 9. $\frac{56}{33}$

10. $\frac{\pi}{4} - x$

5. DIFFERENTIAL CALCULUS

Exercise 5.1

1.(i) Odd function (ii) Even function

(iii) Neither even nor odd function (iv) Even function

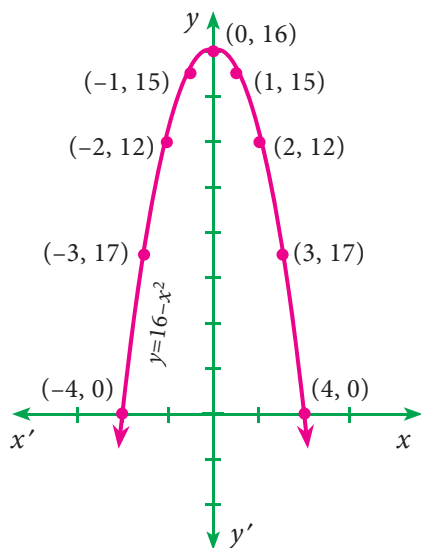
(v) Neither even nor odd function

2. $k = 0$

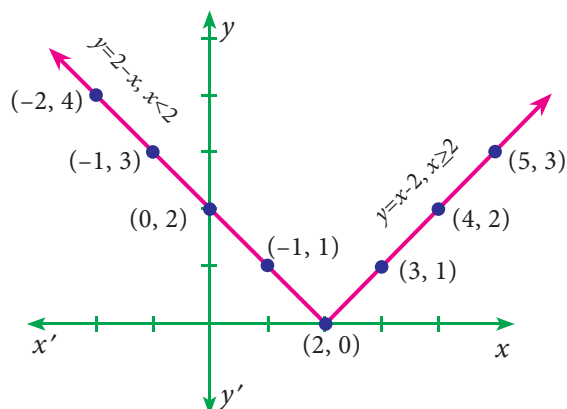
5. $\frac{1-x}{3+x}, \frac{3x+1}{x-1}$

6.(i) e (ii) 0(iii) $3e$ (iv) 0

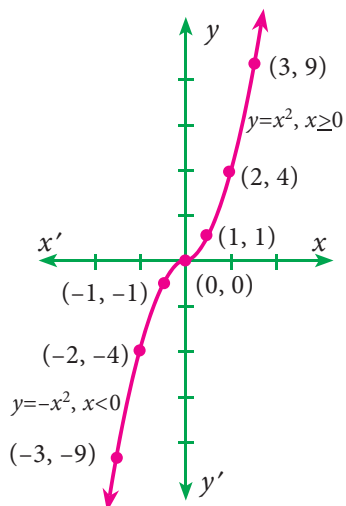
7.(i)



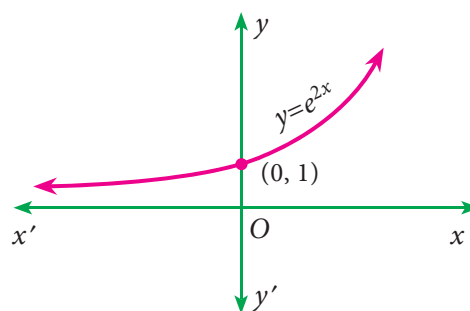
(ii)

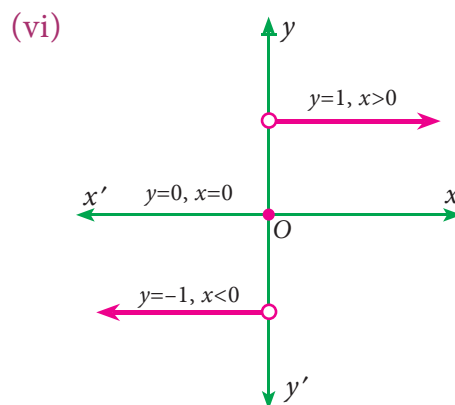
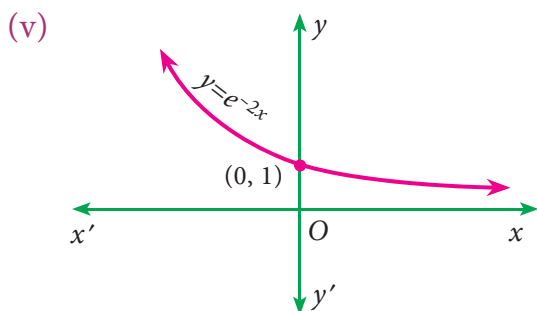


(iii)



(iv)





Exercise 5.2

- 1.(i) $\frac{10}{3}$ (ii) 0 (iii) $\frac{1}{2}$ (iv) 1
 (v) $\frac{15}{16}a^{-\frac{1}{24}}$ (vi) 9
 2. $a = \pm 1$ 3. $n = 7$ 4. $\frac{28}{5}$ 5. $f(-2) = 0$

Exercise 5.3

- 1.(a) Not a continuous function at $x = 2$ (b) continuous function at $x = 3$

Exercise 5.4

- (i) $2x$ (ii) $-e^{-x}$ (iii) $\frac{1}{x+1}$

Exercise 5.5

- 1.(i) $12x^3 - 6x^2 + 1$ (ii) $\frac{-20}{x^5} + \frac{6}{x^4} - \frac{5}{x^2}$ (iii) $\frac{1}{2\sqrt{x}} - \frac{1}{3\sqrt[3]{x^4}} + e^x$
 (iv) $-\frac{3+x^2}{x^2}$ (v) $x^2 e^x (x+3)$ (vi) $3x^2 - 4x - 1$
 (vii) $4x^3 - 3\cos x - \sin x$ (viii) $1 - \frac{1}{x^2}$
 2.(i) $\frac{xe^x}{(1+x)^2}$ (ii) $\frac{2(x^2+1)}{(x^2-x+1)^2}$ (iii) $\frac{e^x}{(1+e^x)^2}$
 3.(i) $x \cos x + \sin x$ (ii) $e^x (\sin x + \cos x)$
 (iii) $e^x \left(1 + \frac{1}{x} + x + \log x\right)$ (iv) $\cos 2x$ (v) $e^x x^2 (x+3)$
 4.(i) $\sin 2x$ (ii) $-\sin 2x$ (iii) $\frac{-3}{2} \cos x \sin 2x$ (iv) $\frac{x}{\sqrt{1+x^2}}$
 (v) $n(ax^2 + bx + c)^{n-1} (2ax + b)$ (vi) $2x(\cos(x^2))$ (vii) $\frac{-x}{(1+x^2)\sqrt{1+x^2}}$

Exercise 5.6

1.(i) $-\frac{y}{x}$ (ii) $\frac{y-2x}{2y-x}$ (iii) $-\frac{(x^2+ay)}{(y^2+ay)}$ 3. $\frac{4}{3}\left(\frac{1-4x+3y}{1+4x-3y}\right)$

Exercise 5.7

1.(i) $x^{\sin x} \left[\frac{\sin x}{x} + \cos x \log x \right]$ (ii) $(\sin x)^x [x \cot x + \log(\sin x)]$
 (iii) $(\sin x)^{\tan x} [1 + \sec^2 x \log(\sin x)]$
 (iv) $\frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x^2+x+1)}} \left\{ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{2x+1}{x^2+x+1} \right\}$

Exercise 5.8

1.(i) $-\frac{1}{t^2}$ (ii) $t \cos t$ (iii) $-\tan \theta$ (iv) $\cot \frac{\theta}{2}$
 2. $-\tan x$ 3. $\frac{\sin 2x}{2x}$

Exercise 5.9

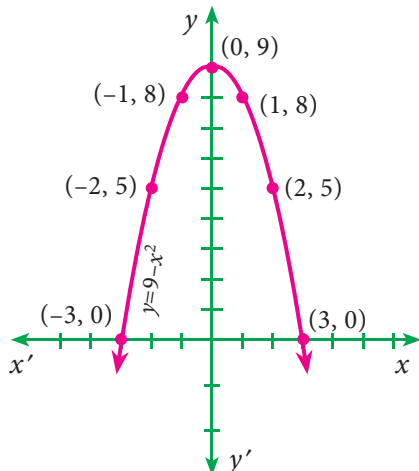
1.(i) $9y$ (ii) $-\frac{1}{x^2} + a^x (\log a)^2$ (iii) $-\frac{1}{a} \operatorname{cosec}^3 \theta$

Exercise: 5.10

1	2	3	4	5	6	7	8	9	10	11	12	13
(d)	(c)	(a)	(b)	(a)	(a)	(c)	(a)	(d)	(a)	(a)	(d)	(b)
14	15	16	17	18	19	20	21	22	23	24	25	
(a)	(a)	(a)	(c)	(b)	(d)	(a)	(a)	(c)	(a)	(b)	(d)	

Miscellaneous Problems

2.



4. $-\frac{1}{10}$

6.(i) continuous at $x = 1$; differentiable at $x = 1$

(ii) continuous at $x = 2$; not differentiable at $x = 2$

GLOSSARY	
Absolute constants	முழுமையான மாறிலிகள்
Adjoint Matrix	சேர்ப்பு அணி
Algebraic functions	இயற்கணித சார்புகள்
Allied angle	துணைக் கோணங்கள்
Analysis	பகுப்பாய்வு
Angle	கோணம்
Arbitrary constants	தன்னிச்சை மாறிலிகள்
Binomial	ஈருறுப்பு
Centre	மையம்
Chain rule	சங்கிலி விதி
Chord	நாண்
Circle	வட்டம்
Circular Permutation	வட்ட வரிசை மாற்றம்
Closed interval	மூடிய இடைவெளி
Coefficient	கெழு
Cofactor	இணைக் காரணி
Combination	சேர்வு
Componendo and dividendo.	கூட்டல் மற்றும் கழித்தல் விகித சமம்
Compound angle	கூட்டுக் கோணம்
Concurrent Line	ஒரு புள்ளி வழிக் கோடு
Conics	கம்பு வெட்டிகள்
Constant	மாறிலி
Continuous	தொடர்ச்சி
Degree measure	பாகை அளவை
Dependent variable	சார்ந்த மாறி
Derivative	வகையீடு
Determinant	அணிக்கோவை
Diameter	விட்டம்
Digonal Matrix	மூலை விட்ட அணி
Directrix	இயக்குவரை
Domain	சார்பகம் / அரங்கம்
Equation	சமன்பாடு
Explicit	வெளிபடு
Exponential	அடுக்கு
Factorial	காரணியப் பெருக்கம்
Focal distance	குவியத்தூரம்
Focus	குவியம்
Function	சார்பு
Identity	சமனி (அல்லது) முற்றொருமை
Implicit	உட்படு
Independent term	சாரா உறுப்பு
Independent variable	சாரா மாறி
Input	உள்ளீடு
Interval	இடைவெளி
Inverse function	நேர்மாறு சார்பு
Inverse Matrix	நேர்மாறு அணி
Latus rectum	செவ்வகலம்
Left limit	இடக்கை எல்லை
Length of the arc.	வில்லின் நீளம்

Length of the tangent	தொடுகோட்டின் நீளம்
Limit	எல்லை
Linear factor	ஒரு படிக்காரணி
Locus	நியமப்பாதை அல்லது இயங்குவரை
Logarithmic	மடக்கை
Mathematical Induction	கணிதத் தொகுத்தறிதல்
Middle term	மைய உறுப்பு
Minors	சிற்றணிக் கோவைகள்
Modulus	மட்டு
Multiple angle	மடங்கு கோணம்
Multiplication Principle of counting	எண்ணுதலின் பெருக்கல் கொள்கை
Neighborhood	அண்மையகம்
Non-Singular Matrix	பூச்சியமற்றக் கோவை அணி
Open interval	திறந்த இடைவெளி
Origin	ஆதி
Output	வெளியீடு
Pair of straight line	இரட்டை நேர்கோடு
Parabola	பரவளையம்
Parallel line	இணை கோடு
Parameter	துணையலகு
Parametric functions	துணையலகு சார்புகள்
Partial fraction	பகுதிப் பின்னங்கள்
Pascal's Triangle	பாஸ்கலின் முக்கோணம்
Permutations	வரிசை மாற்றங்கள்
Perpendicular line	செங்குத்து கோடு
Point of concurrency	ஒருங்கிணைவுப் புள்ளி
Point of intersection	வெட்டும் புள்ளி
Principle of counting	எண்ணுதலின் கொள்கை
Quadrants	கால் பகுதிகள்
Radian measure	ரேடியன் அளவு / ஆரையன் அளவு
Radius	ஆரம்
Range	வீச்சகம் / வீச்சு
Rational Expression	விகித முறு கோவை
Right limit	வலக்கை எல்லை
Scalar	திசையிலி
Signum function	குறிச் சார்பு
Singular Matrix	பூச்சியக் கோவை அணி
Straight line	நேர்க் கோடு
Successive differentiation	தொடர் வகையிடல்
Tangent	தொடுகோடு
Transcendental functions	விஞ்சிய சார்புகள்
Transformation formulae	உருமாற்று சூத்திரங்கள்
Transpose of a Matrix	நிரை நிரல் மாற்று அணி
Triangular Matrix	முக்கோண அணி
Trigonometric identities	திரிகோணமிதி முற்றொருமைகள்
Trigonometry Ratios	திரிகோணமிதி விகிதங்கள்
Variable	மாறி
Vertex	முனை

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