CROSS PRODUCTS AND THE PERMUTATION TENSOR

In class we have studied that the vector product between two vectors **A** and **B** is written as:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

and has a magnitude equal to |A| |B| sin θ, and a direction determined by application of the right hand rule.

If we write each vector in component form, and take the term by term vector product, we obtain for the resulting vector **C:**

$$\mathbf{C} = (A_yB_z - A_zB_y)\mathbf{i} + (-A_xB_z + A_zB_x)\mathbf{j} + (A_xB_y - A_yB_x)\mathbf{k} \quad (1)$$

In class we discussed the patterns observed in (1). First, each component is the difference of products. Second, each product in a component of the cross product represents a permutation of the components of the vectors **A** and **B**. Finally, we notice that the **i** component of the cross product involves no *x* terms; similarly the **j** and **k** components of the cross product involve no *y* or *z* terms.

We will learn to write cross products in summation notation, however, in order to do that we need a mathematical structure that will allow us to reproduce the patterns we see in eq. (1).

The mathematical formalism that allows us to write cross products and curls in summation notation is the **Levi-Civita Permutation Tensor.** In three dimensions, the Levi-Civita permutation tensor (called henceforth the permutation tensor) is written as ε_{ijk} and has the properties:

$$\varepsilon_{ijk} = \begin{cases} 0 \text{ if any two indices are equal} \\ +1 \text{ if all indices are different and are cyclic} \\ -1 \text{ if all indices are different and are anti-cyclic} \end{cases}$$

Cyclic permutations are 123, 231 and 312; anti-cyclic permutations are 132, 213 and 321.

Let's spend a little time investigating the properties of the permutation tensor before moving on to writing cross products.

Question: How many different ways can we write the permutation tensor (in 3 space)?

The answer is 27. We have 3 choices for the first index, 3 for the second, and 3 for the third; 3x3x3 = 27.

What are these 27 different ways of writing ε_{ijk}? It gets a little tedious, but it is instructive to write out all these permutations:

The twenty-seven possible permutations are:

ε_{111}	ε_{211}	ε_{311}
ε_{112}	ε_{212}	ε_{312}
ε_{113}	ε_{213}	ε_{313}
ε_{121}	ε_{221}	ε_{321}
ε_{122}	ε_{222}	ε_{322}
ε_{123}	ε_{223}	ε_{323}
ε_{131}	ε_{231}	ε_{331}
ε_{132}	ε_{232}	ε_{332}
ε_{133}	ε_{233}	ε_{333}

We can use the definition of the permutation tensor given above to realize that only six of these terms are non-zero. These are $\varepsilon_{123,}$ $\varepsilon_{132,}$ $\varepsilon_{213,}$ ε_{231}, ε_{312} and ε_{321}. You should be able to determine which of these equal +1 or -1.

Let's look at equation (1) in a little more detail to see how we can write cross products in summation notation. In this write-up we will denote *x, y, z* components by the indices *1, 2, 3* respectively.

As eq. (1) shows, the *i* component of **C** can be written as $A_2B_3 - A_3B_2$; the *j* component can be written as $A_3B_1 - A_1B_{3;}$ and the *k* component as $A_1B_2 - A_2B_1$. The complete vector **C** is of course the sum of all these components.

Let's consider now the expression:

$$\varepsilon_{ijk} A_j B_k \qquad (2)$$

First, notice that there are two repeated indices, *j* and *k*; this means that we will have to sum over *j* and *k*. The *i* index is not repeated and so is not summed over in this expression. Of course, *i* can be any integer between 1-3.

Let's write explicitly the sum represented by (2). Remember that we will sum over *j* and *k*; *i* is a dummy index that will run from 1 to 3. Remember also though that once we choose a value of *i,* the values of *j* and *k* must be chosen so that no indices are the same (or else the the value of that particular term will be zero). Thus, we can write out the expression in (2) explicitly:

$$\varepsilon_{123} A_2B_3 + \varepsilon_{132} A_3B_2 + \varepsilon_{231} A_3B_1 + \varepsilon_{213} A_1B_3 + \varepsilon_{312} A_1B_2 + \varepsilon_{321} A_2B_1 \quad (3)$$

Make sure you review the expression in (3) carefully. Notice that these terms contain the only non-zero terms of the permutation tensor. This pattern of components in (3) should look very familiar by now.

Using the properties of the permutation tensor described above, we can rewrite and rearrange(3) as:

$$(A_2B_3 - A_3B_2) + (A_3B_1 - A_1B_3) +(A_1B_2 - A_2B_1)$$

Notice that each parenthesis consists of one of the components of the cross product vector **C** from eq. (1). Notice further that the terms in the first parenthesis correspond to i =1; the terms in the second parenthesis correspond to i = 2; and the final parenthesis corresponds to i =3.

Thus, we can express the cross product **C = A x B** in summation notation as:

$$C_i = \varepsilon_{ijk} A_jB_k \qquad (4)$$

In other words, the i^{th} component of the vector C is given by the expression above, and the complete vector **C** results from summing all its components.

A simple proof:

Let's use this description of the cross product to prove a simple vector result, and also to get practice in the use of summation notation in deriving and proving vector identities.

We know that the cross product of two vectors is perpendicular to each of the vectors; that is, we expect **C** ⊥ **A** and also **C** ⊥ **B.** In terms of vector multiplication, this means that **A·C** = 0 = **B·C.** But we already know that in summation notation, the dot product between two vectors can be written as A_iC_i, since in summation notation you sum over repeated indices, and the product $A_iC_i = A_1C_1 + A_2C_2 + A_3C_3 =$ **A·C**.

So, if we wish to prove that **A·C** = 0 if **C=AxB** using only summation notation, let's begin with the expression in (4) and realize that:

$$A_iC_i = A_i (\varepsilon_{ijk} A_jB_k) \qquad (5)$$

Since we are now working only with scalar quantities, we can reorder the multiplications on the right hand side any way we wish, and we can rewrite (5) as:

$$A_iC_i = B_k (\varepsilon_{ijk} AiAj) \qquad (6)$$

Examine the term in parentheses in (6). This term is simply the cross product of **AxA**. However, since the cross product of any vector with itself is zero (since the magnitude is proportional to sinθ), the expression A_iC_i is zero, and we have proven that the cross product is perpendicular to each of the original vectors.

USE OF SUMMATION NOTATION TO PROVE VECTOR IDENTITIES

THE "BAC-CAB" RULE

Let us consider the triple vector product:

$$\mathbf{G} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \qquad (1)$$

You can write the cross products out term by term, but this becomes lengthy and messy. Using summation notation provides an elegant, terse and quick means of proving these identities.

Here, we will show that:

$$\mathbf{G} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \quad (2)$$

the so-called "BAC-CAB" rule.

First, we set **D = B x C**, and we can write the i^{th} component of **D** as:

$$D_i = \varepsilon_{ijk} B_j C_k \qquad (3)$$

Then, we can write **G = A x D**. We can write this cross product in summation notation as:

$$G_m = \varepsilon_{mni} A_n D_i \qquad (4)$$

It is important to understand why these subscripts are chosen as they are in eq. (4). We cannot use the same set of subscripts "ijk" again in the permutation tensor in (4); these subscripts were used in (3). While the exact choice of subscript is often arbitrary, we should use a different set of subscripts in writing the new cross product.

Notice, however, that we *do* use the subscript "i" for the D term. This is because **D** is the cross product of **B** and **C,** and we must use (3) as the expression for D in equation (4).

Notice also the pattern of subscripts in (3) and (4). The first subscript in each Levi-Civita tensor refers to a component of the vector resulting from the cross product; in other words, the "i" in (3) means we are computing the "ith" component of **D**; the "m" in (4) means we are computing the "mth" component of **G.** The second subscript refers to a component of the first vector in the cross product, and the final subscript labels a component of the second vector in the cross product.

We can now substitute the expression for D_i from (3) into (4) and obtain:

$$G_m = \varepsilon_{mni} A_n (\varepsilon_{ijk} B_j C_k) \qquad (5)$$

Since all the terms in (5) are scalars, we can rearrange terms and write:

$$G_m = \varepsilon_{mni} \varepsilon_{ijk} A_n B_j C_k \qquad (6)$$

We realize that $\varepsilon_{mni} = \varepsilon_{nim} = \varepsilon_{imn}$, so we can rewrite (6) as:

$$G_m = \varepsilon_{imn} \varepsilon_{ijk} A_n B_j C_k \qquad (7)$$

Recognize that in (7) we have a product of two permutation tensors, and each has the same index ("i") in the same location. This allows us to use the "ε–δ" relationship:

$$\varepsilon_{imn} \varepsilon_{ijk} = \delta_{jm} \delta_{kn} - \delta_{km}\delta_{jn} \qquad (8)$$

Using the relationship in (8) to expand the product of permutation tensors in (7) yields:

$$G_m = (\delta_{jm} \delta_{kn} - \delta_{km}\delta_{jn}) A_n B_j C_k$$

$$= \delta_{jm} \delta_{kn} A_n B_j C_k - \delta_{km}\delta_{jn} A_n B_j C_k \qquad (9)$$

Let's consider each term on the RHS of (9). In order that the first term be non-zero, we have the conditions that j = m **and** k =n. The second term can be non-zero if and only if k=m **and** j =n. Making these substations in (9) gives:

$$G_m = A_n B_m C_n - A_n B_n C_m \qquad (10)$$

Since we are dealing with scalar quantities, we can switch order of multiplication as we please, allowing us to write (10) in the very recognizable form:

$$G_m = B_m (A_n C_n) - C_m (A_n B_n) \qquad (11)$$

Note that the terms in parentheses are **A·C** and **A·B**. So we can readily observe that (11) is the component form of the vector identity:

$$\mathbf{G = B (A{\cdot}C) - C(A{\cdot}B)} \qquad (12)$$

QED

TRIPLE VECTOR PRODUCT

Proof of $\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{F}$

The proof of this identity follows the same path of the proof of the "BAC-CAB" identity.

First, we set $\mathbf{A} = \nabla \times \mathbf{F}$, so that $\mathbf{G} = \nabla \times \mathbf{A}$. We can the write the "ith" component of **A** as:

$$A_i = \varepsilon_{ijk} (\partial/\partial x_j \; F_k) \qquad (1)$$

and $\mathbf{G} = \nabla \times \mathbf{A}$ becomes:

$$G_m = \varepsilon_{mni} (\partial/\partial x_n \; A_i) = \varepsilon_{mni} \, \partial/\partial x_n (\varepsilon_{ijk} \, \partial/\partial x_j \; F_k)$$

$$= \varepsilon_{mni} \, \varepsilon_{ijk} \, \partial/\partial x_n \; (\partial/\partial x_j \; F_k) \qquad (2)$$

As before, we can permute the first Levi-Civita symbol so that (2) becomes:

$$\varepsilon_{imn} \, \varepsilon_{ijk} \, \partial/\partial x_n \; (\partial/\partial x_j \; F_k) \qquad (3)$$

We can now apply the "$\varepsilon - \delta$" relationship to (3) to obtain:

$$G_m = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) (\partial/\partial x_n \; (\partial/\partial x_j \; F_k)) \qquad (4)$$

Equation (4) admits two terms; in the first term on the RHS of (4), we see that:

$$j = m$$
$$k = n$$

In the second term on the RHS:

$$j = n$$
$$k = m$$

Making these substitutions into (4) we get:

$$G_m = \partial/\partial x_n \; (\partial/\partial x_m \, F_n) - \partial/\partial x_n \; (\partial/\partial x_n \; F_m) \qquad (5)$$

We are almost done, believe it or not. We just have to recognize what these terms in (5) represent. Let's look at the first term in the RHS of (5). We can rearrange the order of differentiation (always valid for continuously differentiable functions) and obtain:

$$\partial/\partial x_m \ (\partial/\partial x_n F_n) \qquad (6)$$

However, it should be apparent that the term in parentheses, $(\partial/\partial x_n F_n)$ is merely $\nabla \cdot \mathbf{F}$, and $\partial/\partial x_m$ of $\nabla \cdot \mathbf{F}$ is the "mth" component of the gradient of the scalar function $\nabla \cdot \mathbf{F}$.

We can write the second term on the RHS of (5) as $(\partial/\partial x_n \ \partial/\partial x_n)F_m$. This is just the "mth" component of $\nabla^2\mathbf{F}$, and putting all this together yields the desired identity:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F}$$