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Convergences en analyse multivoque et unilatérale

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RÉSUMÉS DES CONFÉRENCES

ÉDITÉS PAR
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CONVERGENCES EN ANALYSE MULTIVOQUE ET UNILATERALE

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CIRM, Marseille Luminy

Résumés des Conférences

Convergences en Analyse Multivoque et Unilatérale

Organisateurs: H. Attouch (Montpellier)
M. Théra (Limoges)

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Thèmes abordés

Convergences d'ensembles (Painlevé-Kuratowski, Mosco, Hausdorff et Hausdorff bornée, slice,...). G-convergence des fonctions (épiconvergence, epi-hypo convergence,...). Convergence en graphes des opérateurs. Topologies sur les hyperespaces.

Applications à l'approximation, la perturbation, l'analyse sensitive en :

• optimisation et analyse non-lisse (épi-dérivées des fonctions, proto-différentiation des opérateurs);

• optimisation stochastique (schémas numériques basés sur des méthodes d'approximation) et statistique (loi des grands nombres pour les variables aléatoires semi-continues);

• calcul des variations et mécanique des milieux continus, homogénéisation des matériaux composites, relaxation des problèmes de contrôle,...)

• analyse asymptotique des fonctions et des ensembles (fonctions de récession,...).

• Géométrie des espaces de Banach et problèmes d'optimisation bien posés.

Conférences plénières

Z. Artstein (Rehovot)
E. Balder (Utrecht)
G. Beer (Los Angeles)
A. Cellina (Trieste)
E. De Giorgi (Pisa)
U. Mosco (Rome)
S. M. Robinson (Madison)
R. T. Rockafellar (Seattle)
R. J.-B. Wets (Davis)

Conférenciers et Auteurs

Arstein, Zvi (Rehovot)
Attouch, Hedy (Montpellier)
Auslender, Alfred (Clermont-Ferrand)
Azé, Dominique (Perpignan)
Bahraoui, Mohamed (Montpellier)
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Beer, Gerald (Los Angeles)
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De Giorgi, Ennio (Pise)
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Flam, Sjur (Bergen)
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Vitolo, Paolo (Potenza)
Volle, Michel (Avignon)
Wets, Roger J.-B (Davis)
Zalinescu, Constantin (Cluj)
Zolezzi, Tullio (Gênes)

Présentation scientifique

H. Attouch et **M. Théra**

Introduction. Les convergences de suites d'ensembles et les topologies sur les fermés d'un espace topologique se sont révélées au cours des dernières années comme étant la clé de la compréhension de nombreux problèmes de convergence, approximation, perturbation en analyse non-linéaire, optimisation, statistique... Le congrès du CIRM du 22 au 26 Juin 1992, est consacré à ces questions. Nous présentons dans les pages qui suivent ces nouveaux concepts et quelques unes de leurs applications.

Les notions fondamentales de l'analyse mathématique, telles que continuité, dérivation, intégration, approximation, développements (en série...) reposent sur le choix d'un concept de limite. Dans l'approche classique, c'est la convergence simple des fonctions et des opérateurs qui sous-tend la plupart de ces notions, avec bien sûr des adaptations pour chaque domaine d'utilisation (convergence presque sûre, convergence au sens des distributions, convergence uniforme sur les bornés, sur les compacts...).

Les développements récents en analyse non-linéaire et optimisation ont amené une rupture par rapport à cette vision classique: déjà, l'analyse convexe nous a familiarisé avec l'idée que la convexité d'une fonction se formule naturellement en termes de convexité de son **épigraphe**, et sa semicontinuité inférieure en termes de fermeture de ce dernier. En analyse linéaire, les propriétés des opérateurs se formulent souvent simplement à l'aide de leurs **graphes** (théorème du graphe fermé...). L'analyse des opérateurs à partir de leurs graphes s'est imposée tout naturellement avec la théorie des opérateurs maximaux monotones. Ces exemples nous font sentir le rôle géométrique clé que jouent l'épigraphe pour les fonctions et le graphe pour les opérateurs dans l'analyse des problèmes unilatéraux (minimisation...). Il est naturel de les retrouver au coeur des notions de convergences adaptées à l'analyse variationnelle et l'optimisation.
L'epi-convergence d'une suite de fonctions se définit géométriquement par la convergence de la suite des épigraphes correspondants. La graph-convergence d'une suite d'opérateurs se définit en termes de convergence des graphes correspondants. A la base de ces concepts se trouve donc la notion de **convergence pour une suite d'ensembles**.

L'epi-convergence, la graph-convergence possèdent bien sûr, des versions analytiques qui, contrairement à la convergence simple où la variable est gelée, prennent en compte le comportement des fonctions ou des opérateurs en des points voisins. Ils sont particulièrement utiles lorsque l'on considère des fonctions ou des opérateurs de domaines variables ou lorsque l'on rencontre de fortes variations ou oscillations dans les données d'un problème. Ils ont ainsi permis de traiter de nombreux problèmes de convergences, approximation, perturbation, sensitivité échappant à une analyse utilisant les concepts classiques de convergence basés sur la convergence simple. Ces développements récents se rattachent à un courant mathématique dont nous retraçons brièvement l'historique.

Un bref historique. La théorie des **applications multivoques** (semicontinues, mesurables), et les questions de différentiation (cônes tangents, dérivées généralisées) ou d'intégration (multivoque) qui s'y rattachent, ainsi que que l'étude des **ensembles aléatoires** et la théorie de la décision statistique ont longtemps été les moteurs principaux de la théorie des convergences d'ensembles et des hypertopologies (topologies sur les fermés d'un espace topologique). Ces concepts se sont progressivement dégagés grâce aux travaux de P. Painlevé (1909), L. Vietoris (1923), G. Choquet (1947), K. Kuratowski (1948), G. Bouligand (1950), E. Michael (1951), J. M. G. Fell (1962), G. Matheron (1975), et, plus directement inspirés par l'étude des ensembles aléatoires et la théorie de la décision statistique, R. A. Wijsman (1964), E. Effros (1965), D. Walkup & R. Wets (1967), B. Van Cutsem (1971). Son champ d'applications a été en s'élargissant avec la théorie des jeux C. Berge (1959), l'économie mathématique R. Aumann (1964), W. Hildenbrand (1979), J.-P Aubin (1979), la théorie des points fixes de type Ky-Fan, la viabilité et le contrôle des systèmes dynamiques J.-P. Aubin (1981), B. Cornet (1981), G. Haddad (1981), J.-P. Aubin & A. Cellina (1984), H. Frankowska (1984), l'étude des fonctionnelles intégrales et le contrôle optimal avec R. T. Rockafellar (1975), Ch. Castaing & M. Valadier (1977), Z. Artstein (1972), C. Olech (1976), E. Balder (1984).

Une étape décisive dans le développement de ces concepts a été leur mariage avec l'**analyse variationnelle**. La compréhension des méthodes d'approximation variationnelle (Galerkin, éléments finis) ou de perturbation (en liaison avec la capacité et la théorie du potentiel) a amené U. Mosco en Italie (1969) et J. L. Joly en France (1970) à faire le lien avec les inéquations variationnelles et l'analyse convexe en dimension infinie, et à introduire la **Mosco-convergence**. En liaison avec l'étude de problèmes liés aux hypersurfaces minima, E. De Giorgi dégageait vers 1975 les concepts topologiques généraux de convergence pour des suites de fonctions (non nécessairement convexes), qui permettent de passer à la limite sur les problèmes de minimisation correspondants, à savoir la théorie de **la Γ-convergence**.

Dans cette orientation variationnelle, on trouve un grand nombre de travaux d'**analyse convexe** portant sur les convergences de suites de convexes fermés, de fonctions convexes s.c.i. et de leurs sous-différentiels dans les espaces de Banach. La **bicontinuité de la tranformation de Fenchel** relativement à la Mosco epi-convergence dans les espaces de Banach réflexifs est démontrée par Mosco en 1971, cette propriété fondamentale étant en grande partie à l'origine de l'introduction de ce concept. En 1976, la continuité de l'opération de sous-différentiation $f \rightarrow \partial f$ lorsque l'on munit les fonctions convexes s.c.i. de la Mosco epi-convergence et les opérateurs sous-différentiels de la graph-convergence est mise en évidence par H. Attouch. Ce dernier résultat montre clairement le lien entre epi-convergence des fonctions et graph-convergence des opérateurs. Plus récemment, G. Beer montre l'existence d'une topologie induisant la Mosco convergence (1988), qu'il étend ensuite aux Banach généraux sous le nom de **slice convergence**. Outre la continuité de la polarité, une étude systématique des propriétés de continuité des opérations sur les convexes (intersection, addition vectorielle...) et sur les fonctions convexes (addition, inf-convolution...) est entreprise dans les travaux R. C. Bergstrom & L. Mc Linden (1981), H. Attouch & D. Azé & R. Wets, R. T. Rockafellar & R. Wets, D. Azé & J.-P. Penot, G. Beer & R. Lucchetti, M. Volle, J.-P. Aubin & H. Frankowska, S. Dolecki, H. Rihai...

Des progrès décisifs ont été également accomplis dans le cadre non convexe avec l'introduction de versions "localisées" des métriques de type Hausdorff par H. Attouch & R. Wets avec comme corollaires les notions d'**epi-distance** et de **graph-distance** (1988). Une analyse quantitative des questions de convergence et d'approximation pour les problèmes de minimisation, point-selles... est ainsi rendue possible; citons dans cette orientation les travaux de G. Beer, R. Lucchetti, D. Azé & J. P. Penot, D. Pai & P. Shunmugaraj, H. Attouch & J.-L. Ndoutoume & M. Théra , M. Soueycatt, J. Lahrache.

Les travaux récents de G Beer, R. Lucchetti, Y. Sonntag & C. Zalinescu, A. Lechicki & S. Levi ont permis de classer et de comprendre de façon unifiée les nombreuses convergences d'ensembles et hypertopologies apparues ces dernières années.

Ces travaux théoriques ont été accompagnés de nombreuses **applications**:

a) *Mécanique des milieux continus, calcul des variations et E.D.P.* (problèmes à petits paramètres, **homogénéisation** de milieux composites, couches minces, transition de phase, contrôle et relaxation) avec les travaux en Italie de l'école de Pise animée par E. De Giorgi (G. Buttazzo, G. Dal Maso, P. Marcellini, A. Marino, L. Modica, C. Sbordone, S. Spagnolo), en France de H. Attouch, D. Azé , G. Bouchitte, A. Brillard, A. Damlamian, G. Michaille, F. Murat, C. Picard, P. Suquet, L. Tartar, en Russie de A. V. Marchenko & E. Y. Hruslov et aux USA de B. Kohn.

b) *Optimisation stochastique et théorie de la décision statistique* (approximation numérique, **lois des grands nombres pour des ensembles ou des fonctions s.c.i. aléatoires**, problème variationnel moyen) avec les travaux de R. Wets, G. Salinetti & R.

Wets, R. T. Rockafellar & R. Wets, Z. Artstein, H. Attouch & R. Wets, Ch. Hess, F. Hiai, P. Kall, Ch. Castaing, A. Choukairi, A. Truffert.

c) *Analyse non régulière, optimisation et contrôle* (cône tangent, dérivées généralisées, **epi-dérivées directionnelles** du premier et du deuxième ordre, théorème d'inversion locale pour une application multivoque) avec les travaux de J. P. Aubin, A. Auslender, D. Azé , R. Cominetti, L. Contesse & J.-P. Penot, R. Correa, S. Dolecki, H. Frankowska, D. Klatte, P. Michel & J.-P. Penot, J. L. Ndoutoume, D. Noll, J.M. Borwein & D. Noll, J. P. Penot, R. A. Poliquin, S.M. Robinson, R. T. Rockafellar, A. Seeger, M. Soueycatt, L. Thibault.

d) *Analyse numérique et optimisation* (approximation, stabilité et conditionnement numérique, combinaison d'algorithmes avec des méthodes d'approximation, sensitivité, étude de problèmes non coercifs et fonctions de récession) avec les travaux de A. Auslender, A. Auslender & J.-P. Crouzeix, H. Attouch & R. Wets, S. Flam & R. Wets, J.-B. Hiriart-Urruty, B. Lemaire, A. Moudafi, P.-L. Papini, Y. Sonntag, S. Flam & R. Wets, A. Seeger, P. Tossing, T. Zolezzi, méthodes d'homotopie et de continuation en optimisation paramétrique avec les travaux de J. Guddat, Th. Jongen, H. Attouch & J.-P. Penot & H. Rihai.

e) *Principes variationnels* (Ekeland, Borwein-Preiss), géométrie des Banach, problèmes bien posés (au sens de Hadamard, de Tychonov), propriétés génériques et differentiabilité, avec les travaux de R.R. Phelps, G. Beer, R. Deville & G. Godefroy & V. Zizler, R. Lucchetti, J.M. Borwein, J.M. Borwein & S. Fitzpatrick, P. Kenderov, H. Attouch & H. Riahi, F. De Blasi, P.- L. Papini, J. Mijak, J. P. Revalski, S. Simons.

f) *Théorie de l'approximation* où le rôle fondamental de l'approximation épigraphique (inf-convolution) et ses propriétés de régularisation ont été mis progressivement en évidence dans un cadre de plus en plus général: approximation de **Baire-Wijsman** puis **approximation Moreau-Yosida** introduite par J.-J. Moreau et H. Brezis dans le cas de fonctions convexes et un cadre Hilbertien (1973) suivis par les travaux de R. Wets, H. Attouch, J.-P. Penot, M. Bougeard, J.-B. Hiriart-Urruty. Très récemment, l'approximation-régularisation de **Lasry-Lions** a permis de s'affranchir de toute hypothèses de convexité ou de croissance sur les fonctions et ce, en dimension infinie (H. Attouch & D. Azé , M. Volle, J. Benoist).

g) *Convergences de suites d'opérateurs maximaux monotones* ou d'opérateurs accrétifs et des semi-groupes associés. Initialement introduites par H. Brezis, A. Pazy, M. Crandall, Ph. Benilan, H. Attouch, L. Mc Linden ces convergences ont été utilisées dans l'étude de problèmes d'évolution avec opérateurs dépendant du temps par J.-J. Moreau (problème de rafle en élastoplasticité), H. Attouch & A. Damlamian, et en les combinant avec des perturbations multivoques semicontinues par A. Cellina, B. Cornet, Ch. Castaing, N.S. Papageorgiou. Combinée avec l'approximation Yosida, la graph-convergence de suites d'opérateurs maximaux monotones a permis de définir une notion de somme étendue (variationnelle) H. Attouch & J.-B. Baillon & M. Thera.

h) *Théorie des jeux, problèmes de min-max et économie mathématique* où l'étude des questions de stabilité et approximation pour les problèmes de min-max, de points selles ont amené l'introduction de nouvelles notions de convergences, H. Attouch & R. Wets (epi-hypo convergence pour les problèmes de point selle), G. Greco (Γ-convergence), E. Cavazzuti (équilibre de Nash), J. Morgan et P. Loridan (équilibre de Stackelberg), B. Lemaire (optima de Pareto), B. Cornet (cônes tangents et viabilité), S. Simons, S. Flam.

Outre les ouvrages de référence classiques C. Berge (1959), C. Kuratowski (1958), G. Matheron (1975), Ch. Castaing & M. Valadier (1977), plusieurs ouvrages récents traitent des convergences d'ensembles: Y. Sonntag (1982), H. Attouch (Pitman; 1984), E. Klein & A. C. Thomson (John-Wiley; 1984) J.-P. Aubin & H. Frankowska (Birkhäuser; 1990), R. T. Rockafellar & R. Wets, G. Beer , Y. Sonntag & C. Zalinescu, (à paraître).

1. Convergences de suites d'ensembles.

1.1 Convergence au sens de Painlevé-Kuratowski.

Soit (X,τ) un espace topologique, que, pour la simplicité de l'exposé, on supposera métrisable. Etant donnée une suite de parties $A_1, A_2, A_3,\ldots$ de l'espace topologique X, les limites inférieures et supérieures (topologiques) de la suite $\{A_n;\ n \in \mathbb{N}\}$ sont définies par les formules

$$\mathrm{Li}\, A_n = \{\, x\in X :\ \exists\, (a_n) \to x \text{ avec pour tout } n \in \mathbb{N},\ a_n \in A_n \,\}$$

$$\mathrm{Ls}\, A_n = \{\, x\in X :\ \exists\ n(1) < n(2) < n(3) < \ldots \text{ et } \forall k\in \mathbb{N}\quad a_k \in A_{n(k)} \text{ avec } (a_k) \to x \,\}.$$

En d'autres termes, Li A_n est l'ensemble formé par toutes les limites possibles de suites $\{a_n;\ n \in \mathbb{N}\}$ avec $a_n \in A_n$ pour tout n, alors que Ls A_n est formé par toutes les valeurs d'adhérence de telles suites. Etant donnée une suite $\{A_n;\ n \in \mathbb{N}\}$, on peut toujours définir ces ensembles Li A_n et Ls A_n, qui peuvent être éventuellement vides, et l'on a l'inclusion

$$\mathrm{Li}\, A_n \subset \mathrm{Ls}\, A_n .$$

Lorsque l'égalité a lieu, on dit que la suite $\{A_n;\ n\in \mathbb{N}\}$ converge ou plus précisément converge au sens de Painlevé-Kuratowski, et l'on note

$$\mathrm{Lim}\, A_n = \mathrm{Li}\, A_n = \mathrm{Ls}\, A_n .$$

La référence à la topologie τ est implicite dans ces formules; lorsque l'on veut la mettre en évidence on écrit $A = \tau\text{-}\,\mathrm{Lim}\, A_n$. Dans les formules suivantes "cl" désigne l'opération de fermeture et $\mathcal{N}^{\#}_{\infty}$ désigne la grille du filtre de Fréchet $\mathcal{N}_{\infty}$ sur $\mathbb{N}$:

$$\mathrm{Li}\, A_n = \bigcap_{N\in \mathcal{N}^{\#}_{\infty}} \mathrm{cl} \bigcup_{k\in N} A_k$$

$$\mathrm{Ls}\, A_n = \bigcap_{N\in \mathcal{N}_{\infty}} \mathrm{cl} \bigcup_{k\in N} A_k .$$

Ces formules sont intéressantes à double titre: elles mettent clairement en évidence que les ensembles τ- LiA_n, τ-LsA_n et donc τ-LimA_n (lorsque la limite existe) sont des ensembles fermés pour la topologie τ. De plus, ces limites restent inchangées lorsque l'on remplace A_n par sa fermeture. C'est la raison pour laquelle, sans diminuer la généralité des résultats, on peut raisonner sur des ensembles fermés. Ces formules permettent d'étendre très simplement les concepts précédents aux cas d'ensembles indexés par un paramètre continu $\{A_\varepsilon ;\ \varepsilon\to 0\}$.

Donnons quelques exemples élémentaires en dimension finie:

a) Une suite de boules $\mathbb{B}(x_n, \rho_n)$ converge vers la boule $\mathbb{B}(x, \rho)$ si et seulement si, $\{x_n;\ n \in \mathbb{N}\}$ converge vers x et $\{\rho_n;\ n \in \mathbb{N}\}$ converge vers ρ. Si ρ_n tend vers l'infini, alors ces boules convergent vers l'espace tout entier, et leurs complémentaires vers l'ensemble vide.

b) Considérons une suite qui prend alternativement deux valeurs, $A_n = A_0$ pour n pair et $A_n = A_1$ pour n impair, où A_0 et A_1 désignent deux fermés distincts. Alors cette suite ne converge pas, on a $\mathrm{Li}\, A_n = A_0 \cap A_1$, $\mathrm{Ls}\, A_n = A_0 \cup A_1$.

c) Dans l'exemple suivant, les ensembles A_n sont des épigraphes: $A_n = \mathrm{epi}\, f_n$ avec

$$f_n(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ -nx & \text{si } 0 \leq x \leq \frac{1}{n} \\ nx-2 & \text{si } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{si } x \geq \frac{2}{n} \end{cases}$$

La suite des ensembles A_n converge, sa limite est encore un épigraphe à savoir ,

$$\mathrm{Lim}\, A_n = A = \mathrm{epi}\, f$$

où f est la fonction qui vaut zéro partout, sauf à l'origine où elle vaut -1. Cet exemple élémentaire d'épi-convergence illustre bien la différence avec la convergence simple, puisque sur cet exemple la suite $\{f_n;\, n \in \mathbb{N}\}$ converge simplement vers la fonction identiquement nulle.

d) Considérons à présent une suite d'opérateurs monotones $\mathcal{A}_n$ de $\mathbb{R}$ dans $\mathbb{R}$ et prenons pour ensembles A_n dans $\mathbb{R}^2$ leurs graphes: $A_n = \mathrm{graph}\, \mathcal{A}_n$ (dans la pratique, on identifie l'opérateur et son graphe qui sont alors désignés par le même symbole).

$$\mathcal{A}_n(x) = \begin{cases} -1 & \text{si } x \leq -\frac{1}{n} \\ nx & \text{si } -\frac{1}{n} \leq x \leq \frac{1}{n} \\ 1 & \text{si } x \geq \frac{1}{n} \end{cases}$$

La suite des ensembles A_n converge et sa limite est l'ensemble $A = \mathrm{graph}\, \mathcal{A}$ où

$$\mathcal{A}(x) = \begin{cases} -1 & \text{si } x \leq 0 \\ [-1,+1] & \text{si } x=0 \\ +1 & \text{si } x \geq 0 \end{cases}$$

Cet exemple illustre l'approximation Yosida de l'opérateur maximal monotone multivoque $\mathcal{A}$: les opérateurs $\mathcal{A}_n$ sont maximaux monotones lipschitziens, et la convergence a bien lieu en graphe, la convergence simple des opérateurs $\mathcal{A}_n$ ne permettant de recupérer que $\mathcal{A}^0$ la section de norme minimale de l'opérateur $\mathcal{A}$.

Lorsque la topologie τ est métrisable, notant d une distance induisant cette topologie, et posant:

$$d(x,A) := \inf_{y \in A} d(x,y),$$

on peut reformuler les limites topologiques de suites d'ensembles:

$$\mathrm{Li}\, A_n = \{ x \in X : \limsup_{n\to\infty} d(x,A_n) = 0 \} = \{ x \in X : \lim_{n\to\infty} d(x,A_n) = 0 \}.$$

$$\mathrm{Ls}\, A_n = \{ x \in X : \liminf_{n\to\infty} d(x,A_n) = 0 \}.$$

1.2 Convergence au sens de Wijsman et Mosco convergence.

Les formules ci-dessus suggèrent naturellement la notion de convergence suivante: la suite $\{A_n; n\in \mathbb{N}\}$ converge au sens de Wijsman vers A, on note $A = \text{W-Lim}A_n$, si pour tout $x\in X$,

$$d(x,A) = \lim_{n\to\infty} d(x,A_n).$$

La convergence au sens de Wijsman entraîne la convergence au sens de Kuratowski-Painlevé. Elle est, en général, strictement plus forte que cette dernière. La convergence au sens de Wijsman permet de topologiser de façon simple la convergence topologique d'ensembles: identifiant un ensemble A à la fonction distance $d(.,A)$, on considère la topologie de la convergence simple des fonctions distances. Si X est un espace métrisable séparable, on obtient ainsi une topologie métrisable séparable sur l'ensemble $\mathcal{F}(X)$ des fermés de X, dont la tribu borélienne est précisément la **tribu d'Effrös** $\mathcal{E}$(Ch. Hess; 1986). Cette tribu joue un rôle clé dans l'étude des application multivoques mesurables, elle est engendrée par les ensembles

$$U^- = \{ F\in \mathcal{F}(X) ; F\cap U \neq \varnothing \}$$

où U décrit l'ensemble des ouverts de X. Une application multivoque mesurable n'est autre qu'une application mesurable à valeurs dans $\mathcal{F}(X)$ muni de la tribu $\mathcal{E}$(R.T. Rockafellar; 1977). La topologie de Wijsman se révèle d'un usage délicat en dimension infinie car elle dépend du choix de la métrique (ou de la norme) induisant la topologie τ. Lorsque l'on se restreint aux convexes fermés d'un espace de Hilbert et que l'on prend pour d la distance associée à la norme, on obtient la Mosco-convergence:

Soit $\{A_n; n \in \mathbb{N}\}$ une suite de fermés d'un espace normé X. La suite $\{A_n; n\in \mathbb{N}\}$ **Mosco-converge** vers A, et l'on notera $A = \text{M-lim}A_n$, si on a les deux propriétés suivantes:

(i) pour tout x dans A il existe une suite x_n convergeant **fortement** vers x telle que x_n appartienne à A_n pour tout $n \in \mathbb{N}$.

(ii) pour toute sous-suite $n(1) < n(2) < n(3) <\ldots$ et $\forall k\in \mathbb{N}$ $y_k\in A_{n(k)}$ avec y_k convergeant **faiblement** vers y, on a $y \in A$.

Théorème (H. Attouch-Y. Sonntag-M. Tsukada). Soit X un espace de Banach réflexif muni d'une norme strictement convexe et Kadec ainsi que la norme duale (la convergence faible et la convergence des normes entraînent la convergence forte), ce qui est toujours possible après renormage. Soit $\{A_n; n \in \mathbb{N}\}$, A une suite de convexes fermés non vides de X.
Les propositions suivantes sont équivalentes:

(i) la suite $\{A_n; n\in \mathbb{N}\}$ Mosco-converge vers A ;

(ii) $\forall x\in X$, $d(x,A) = \lim_{n\to\infty} d(x,A_n)$;

(iii) $\forall x\in X$, $\text{proj}_A x = \lim_{n\to\infty} \text{proj}_{A_n} x$.

où $\text{proj}_A x$ désigne la projection de x sur A.

La convergence au sens de Mosco joue un rôle clé en analyse variationnelle convexe dans les espaces réflexifs. Nous verrons plus loin ses propriétés remarquables relativement à la dualité. Lorsque l'espace X n'est pas réflexif, elle perd une bonne partie de ses propriétés.

1.3 Convergences de type Hausdorff.

$(X, \|.\|)$ désigne un espace normé de boule unité fermée $\mathbb{B}$. Etant donnés $C, D \subset X$, l'excès de l'ensemble C sur D est défini par

$$e(C, D) := \sup_{x \in C} d(x, D)$$

avec la convention $e = 0$ si $C = \emptyset$. La distance de Hausdorff entre C et D est définie par

$$\text{haus}(C, D) := \max \{ e(C, D), e(D, C) \}.$$

On a la formule suivante:

$$\text{haus}(C, D) = \sup_{x \in X} | d(x, C) - d(x, D) | .$$

Cette métrique qui joue un rôle fondamental en analyse est souvent trop forte lorsque l'on manie des ensembles non bornés tels que épigraphes, cônes, graphes...A cet effet, une version "localisée" a été introduite par H. Attouch & R. Wets (1987):

Pour tout $\rho \geq 0$, la **ρ-Hausdorff distance** entre C et D est donnée par :

$$\text{haus}_\rho(C, D) := \max \{ e(C_\rho, D), e(D_\rho, C) \}$$

où C_ρ (resp. D_ρ) est défini par $C_\rho := C \cap \rho \mathbb{B}$ (resp. $D_\rho := D \cap \rho \mathbb{B}$).

Une suite $\{C_n \subset X; n \in \mathbb{N}\}$ de parties de X converge au sens des ρ-Hausdorff distances vers un ensemble C, si pour tout $\rho > 0$,

$$\lim_{n \to \infty} \text{haus}_\rho(C_n, C) = 0 .$$

Cette convergence est associée à une topologie métrisable; lorsque l'on travaille avec les fermés non vides d'un espace normé X, on peut prendre comme métrique D

$$D(A, B) := \sum_{n=1}^{+\infty} 2^{-n} \frac{d_n(A,B)}{1 + d_n(A,B)}$$

où

$$d_n(A,B) := \sup_{\|x\| \leq n} | d(x, A) - d(x, B) | .$$

Cette topologie, associée à la convergence uniforme sur les bornés des fonctions distances est donc intermédiaire entre la topologie de Wijsman (associée à la convergence simple des fonctions distances) et la topologie de Hausdorff (associée à la convergence uniforme des fonctions distances). En dimension finie, la topologie de Wijsman coincide avec la topologie des ρ-Hausdorff distances et ces topologies induisent la convergence au sens de Painlevé-Kuratowski.

1.4 Hypertopologies

L'étude des topologies sur les fermés d'un espace topologique (X,τ) (hypertopologies) induisant les convergences topologiques d'ensembles est un thème central dans toutes les questions évoquées précédemment. Historiquement, ce sont les hypertopologies du type "hit and miss" qui sont apparues les premières. Etant donné $E \subset X$, on introduit

$$E^- = \{ A \in \mathcal{F}(X);\ A \cap E \neq \emptyset \}$$

$$E^+ = \{ A \in \mathcal{F}(X);\ A \subset E \}.$$

(1) La topologie de **Vietoris** sur $\mathcal{F}(X)$ a pour sous-base les parties de la forme V^- et les parties de la forme W^+ où V et W sont des ouverts de (X,τ).

(2) La topologie de **Fell** sur $\mathcal{F}(X)$ a pour sous-base les parties de la forme V^- où V est un ouvert de (X,τ) et les parties de la forme W^+ où W est de complémentaire compact.

(3) Si X est un espace vectoriel normé, la **Mosco** topologie (introduite par G. Beer) sur les fermés faibles de X a pour sous-base les parties de la forme V^- où V est ouvert de X muni de la topologie de la norme et les parties de la forme W^+ où W est de complémentaire faiblement compact.

Cette approche géométrique se marie bien avec la théorie des multiapplications mesurables et l'analyse stochastique. L'approche "moderne" pour définir des hypertopologies est plus fonctionnelle, elle consiste à associer à un ensemble une famille de fonctions et de considérer la topologie initiale qui leurs est associée:
On a vu que la **Wijsman** topologie est la topologie la moins fine (topologie initiale) rendant continue les applications $A \rightarrow d(x,A)$ pour tout x dans X.
La **Slice topologie de Beer** sur les convexes fermés d'un espace normé peut être définie comme topologie initiale associée aux fonctions de saut

$$A \rightarrow D(A,B) := \inf \{ \|x - y\| ;\ x \in A,\ y \in B \}$$

où B parcourt l'ensemble des convexes fermés bornés de X. Elle est associée à la famille de semi-métriques

$$p_F(A,B) = |D(A,F) - D(B,F)|$$

où F appartient à une famille de parties de X (ici les convexes fermés bornés de X). Y. Sonntag et C. Zalinescu obtiennent une classification des hypertopologies en distinguant celles associéees à une famille de semi-métriques du type précédent (le type p) et celles associéees à une famille de semi-métriques du type suivant (le type q)

$$q_F(A,B) = \sup_{x \in F} |d(x,A) - d(x,B)| .$$

Dans ce dernier type on reconnait la métrique de **Hausdorff** en prenant pour F l'espace tout entier, la topologie de **Attouch-Wets** ("bounded Hausdorff topology") lorsque F décrit l'ensemble des parties bornées de X.

2. Epi-convergence de suites de fonctions.

De nombreuses questions liées à la minimisation de fonctions $f: X \to \mathbb{R} \cup \{+\infty\}$ peuvent être abordées de façon géométrique en identifiant la fonction et son épigraphe

$$\text{epi } f \equiv \{(x,\alpha) : x \in X, \alpha \in \mathbb{R}, \alpha \geq f(x)\}.$$

Les convergences épigraphiques (epi-convergences) de suites de fonctions se définissent géométriquement en termes de convergences topologiques de la suite de leurs épigraphes. On a donc les correspondances suivantes:

$$f = \text{epi-lim } f_n \Leftrightarrow \text{epi } f = \text{Lim epi } f_n$$

$$f = \text{W-epi-lim } f_n \Leftrightarrow \text{epi } f = \text{W-Lim epi } f_n$$

$$f = \text{M-epi-lim } f_n \Leftrightarrow \text{epi } f = \text{M-Lim epi } f_n$$

$$f = \text{epi-dist-lim } f_n \Leftrightarrow \text{epi } f = \text{haus}_\rho\text{-Lim epi } f_n \quad \forall \rho > 0$$

On dira alors que la suite de fonctions $\{f_n; n \in \mathbb{N}\}$ respectivement **epi-converge**, Wijsman epi-converge, Mosco epi-converge, epi-distance converge vers f. Une limite topologique d'épigraphes est encore un épigraphe, plus précisément:

$$\text{Ls epi } f_n = \text{epi (epi-liminf } f_n)$$

$$\text{Li epi } f_n = \text{epi (epi-limsup } f_n),$$

avec

$$(\text{epi-liminf } f_n)(x) = \min \{ \liminf f_n(x_n);\ x_n \to x \},$$

$$(\text{epi-limsup } f_n)(x) = \min \{ \limsup f_n(x_n);\ x_n \to x \}.$$

L'epi-convergence de la suite de fonctions $\{f_n; n \in \mathbb{N}\}$ se traduit par l'égalité entre les deux fonctions epi-liminf f_n et epi-limsup f_n. On a donc que $\{f_n; n \in \mathbb{N}\}$ epi-converge vers f au point x, si les deux propriétés suivantes sont satisfaites:

(i) pour tout suite x_n convergeant vers x, $\liminf f_n(x_n) \geq f(x)$
(ii) il existe une suite $\hat{x}_n$ convergeant vers x telle que $f(x) \geq \limsup f_n(\hat{x}_n)$.

Lorsque l'on veut mettre en évidence la topologie τ prise sur X on note $f = \tau\text{-epi-lim } f_n$.
On peut noter que ces conditions entraînent que f est semicontinue inférieurement. L'epiconvergence n'est pas comparable en général avec la convergence simple et fait intervenir les valeurs des fonctions au voisinage du point considéré. Dans le cas de suites monotones, les deux notions coïncident modulo des opérations de fermeture:

Si la suite $\{f_n; n \in \mathbb{N}\}$ est croissante, alors epi-lim $f_n = \sup_n (\text{cl } f_n)$
Si la suite $\{f_n ; n \in \mathbb{N}\}$ est décroissante, alors epi-lim $f_n = \text{cl } \inf_n (f_n)$.

L'importance de l'epi-convergence tient au fait que cette notion est minimale parmi les notions permettant le passage à la limite sur des suites de problèmes de minimisation. Son utilisation combinée avec les méthodes de compacité est mise en évidence dans l'énoncé suivant où l'on désigne par $\varepsilon\text{-argmin } f = \{ x \in X;\ f(x) \leq \inf f + \varepsilon \}$:

Théorème (E. De Giorgi; 1977). Soit $\{f, f_n: X \to \mathbb{R} \cup \{+\infty\}: n=1,2,\ldots\}$ une suite de fonctions à valeurs réelles étendues. On suppose qu'il existe une suite $(\hat{x}_n;\, n\in \mathbb{N})$ τ-relativement compacte pour une topologie τ sur X et une suite de réels positifs $\{\varepsilon_n;\, n\in \mathbb{N}\}$ tendant vers zéro tels que $\hat{x}_n \in \varepsilon_n$-argmin f_n pour tout $n\in \mathbb{N}$. Alors, l'epi-convergence pour la topologie τ de la suite $\{f_n;\, n\in \mathbb{N}\}$ vers f entraîne :

(i) $\min f = \lim_{n \to \infty} \inf f_n$

(ii) $\mathrm{Ls}\,(\varepsilon_n\text{-argmin } f_n) \subset \text{argmin } f$.

Dans les applications, cette propriété variationnelle est complétée par le résultat de stabilité:

Si $f = \tau\text{-epi-lim } f_n$ et $g: X \to \mathbb{R}$ est τ-continue, alors $f + g = \tau\text{-epi-lim } (f_n + g)$.

La topologie τ intervient donc naturellement en liaison avec les propriétés de coercivité, d'inf-compacité de la suite f_n. Le fait de travailler avec des fonctions pouvant éventuellement prendre des valeurs infinies permet de formuler un résultat abstrait de compacité pour l'epi-convergence:

Théorème Soit (X, τ) un espace topologique métrisable séparable. Alors, de toute suite de fonctions $\{f_n: X \to \overline{\mathbb{R}}\, ;\ n=1,2,\ldots\}$ on peut extraire une sous suite τ-epi-convergente.

L'**approximation epigraphique ou inf-convolution** joue un rôle central en théorie de l'epi-convergence. Etant donné une fonction $f: X \to \mathbb{R} \cup \{+\infty\}$ où $(X, \|.\|)$ est un espace normé, pour toute valeur strictement positive du paramètre λ, on définit

$$f_\lambda (x) := \inf_{u\in X} \{ f(u) + \frac{1}{2\lambda} \|x-u\|^2 \}$$

l'**approximation Moreau-Yosida** d'indice λ de f, et

$$f_{[\lambda]} (x) := \inf_{u\in X} \{ f(u) + \frac{1}{\lambda} \|x-u\| \}$$

l'**approximation Baire-Wijsman** d'indice λ de f. Ces fonctions sont respectivement localement lipschitziennes et lipschitziennes partout définies sur l'espace tout entier, et jouent le rôle des fonctions distances lorsque les ensembles considérés sont des épigraphes. De nombreuses propriétés d'epi-convergence se reformulent donc en termes de convergence simple des approximées épigraphiques. Par exemple, la Mosco epi-convergence d'une suite de fonctions convexes s.c.i . sur un espace de Hilbert est équivalente à la convergence simple de toutes les approximées Moreau-Yosida:

$$f = \text{M-epi-lim } f^n \Leftrightarrow \forall \lambda>0\ \ \forall x\in X\ \ f_\lambda (x) = \lim_{n \to \infty} (f^n)_\lambda (x).$$

Un résultat du même type pour les approximées Baire-Wijsman a éte obtenu par D. Aze.

La suite $\{ f_\lambda ; \lambda \to 0\}$ converge simplement et de façon monotone croissante vers f lorsque l'on suppose f s.c.i.. En fait, cette convergence a lieu au sens de l'epi-distance, ce qui donne un contrôle métrique de cette approximation.

3. Epi-convergence de fonctions convexes et dualité.

Soit $(X, \|.\|)$ un espace normé, $\Gamma(X)$ désigne l'ensemble des fonctions convexes semicontinues inférieurement propres sur X, et dualement $\Gamma^*(X^*)$ désigne l'ensemble des fonctions convexes semicontinues inférieurement pour la topologie $\sigma(X^*,X)$, propres définies sur X^*.
Etant donné $f \in \Gamma(X)$, sa *conjuguée* $f^* \in \Gamma^*(X^*)$ est définie par la formule

$$f^*(y) := \sup \{ \langle x,y \rangle - f(x) \ : \ x \in X \}.$$

où $\langle x,y \rangle$ désigne le couplage entre $x \in X$ et $y \in X^*$.
Etant donné $h \in \Gamma^*(X^*)$, on adopte la convention habituelle qui consiste à définir h^* seulement sur X plutôt que sur tout X^{**}, de sorte que $h^* \in \Gamma(X)$. La transformation de Fenchel $f \to f^*$ est alors une involution bijective entre $\Gamma(X)$ et $\Gamma^*(X^*)$. On dit que $y_0 \in X^*$ appartient au sous-différentiel de $f \in \Gamma(X)$ au point $x_0 \in X$, et l'on écrit $y_0 \in \partial f(x_0)$, si pour tout $x \in X$, on a $f(x) \geq f(x_0) + \langle x - x_0 , y_0 \rangle$. En termes de fonctions convexes conjuguées, on a que $y_0 \in \partial f(x_0)$ si et seulement si l'égalité de Fenchel est satisfaite : $f(x_0) + f^*(y_0) = \langle x_0,y_0 \rangle$. Le *sous-differentiel* de f est l'opérateur dont le graphe dans $X \times X^*$ est donné par la formule:

$$\partial f := \{(x,y) \in X \times X^* : \ y \in \partial f(x)\}$$

$$= \{(x,y) \in X \times X^* : \ f(x) + f^*(y) = \langle x,y \rangle\}.$$

Rappelons la formulation analytique de la Mosco epi-convergence: Une suite$\{f_n;\ n=1,2,...\}$ de fonctions de $\Gamma(X)$ Mosco epi-converge vers f si en tout point $x \in X$ les deux propriétés suivantes sont satisfaites:

(i) pour toute suite x_n convergeant **faiblement** vers x, $\liminf f_n(x_n) \geq f(x)$
(ii) il existe une suite $\hat{x}_n$ convergeant **fortement** vers x telle que $f(x) \geq \limsup f_n(\hat{x}_n)$

Théorème (U. Mosco; 1971) Soit X un espace de Banach réflexif. Alors la transformation de Fenchel $f \to f^*$ est bicontinue pour la Mosco epi-convergence. En d'autres termes

$$f = \text{M-epi-lim}\, f_n \iff f^* = \text{M-epi-lim}\, (f_n)^* .$$

Ce résultat, qui explique le rôle important joué par la Mosco-epi-convergence en analyse convexe, tombe en défaut lorsque l'espace n'est plus réflexif. Pour pallier les défauts de la Mosco convergence dans un cadre non réflexif, G. Beer a introduit récemment la slice convergence:

Soit X un espace normé général et $\{f_n;\ n \in \mathbb{N}\}$ une suite de fonctions de $\Gamma(X)$. On dira que f_n **slice epi-converge** vers f et l'on notera $f = \tau_s\text{-lim}\ f_n$ si les deux propriétés suivantes sont satisfaites:

$$\text{(ii)} \begin{cases} \forall\, x_0 \in X \ \ \exists \langle x_n \rangle \to x_0 \text{ tel que } f(x_0) = \lim\limits_{n \to \infty} f_n(x_n) \\ \forall\, y_0 \in X^* \ \ \exists \langle y_n \rangle \to y_0 \text{ tel que } f^*(y_0) = \lim\limits_{n \to \infty} f_n^*(y_n) \end{cases}$$

La slice epi-convergence coincide avec la Mosco epi-convergence lorsque l'espace X est réflexif et la transformation de Fenchel $f \rightarrow f^*$ est bicontinue pour la slice epi-convergence dans les espaces normés généraux (Beer; 1991).

Le résultat suivant fait le lien entre la Mosco epi-convergence d'une suite de fonctions convexes s.c.i . et la graph-convergence de la suite de leurs sous-différentiels.

Théorème (H. Attouch; 1977). Pour toute suite $\{f, f_n: X \rightarrow \mathbb{R} \cup \{+\infty\};\ n \in \mathbb{N}\}$ de fonctions convexes s.c.i . propres, où X désigne un espace de Banach réflexif, les propriétés suivantes sont équivalentes:

(i) $f = M\text{-}\underset{n\to\infty}{\text{epi}}\text{-}\lim f_n$

(ii) $\text{gph}\ \partial f = s{\times}s \lim_{n\to\infty} \text{gph}\ \partial f_n$ plus la *"condition de normalisation (NC)"*:

La condition de normalisation provient du fait que f est déterminée par ∂f à une constante additive près et s'énonce ainsi:

$$(NC)\quad \begin{cases} \exists (x_0, x_0^*) \in \text{gph}\ \partial f,\ \exists (x_{0n}, x_{0n}^*) \in \text{gph}\ \partial f_n \text{ pour tout } n \in \mathbb{N} \text{ tel que} \\ x_{0n} \overset{s}{\rightarrow} x_0,\ x_{0n}^* \overset{s}{\rightarrow} x_0^* \text{ et } f_n(x_{0n}) \rightarrow f(x_0). \end{cases}$$

Ce résultat a été étendu au cas d'espaces de Banach généraux en remplaçant la Mosco epi-convergence par la slice epi-convergence (H. Attouch & G. Beer; 1991).

Les opérations de polarité et de sous-différentiation jouissent également de propriétés de continuité remarquable vis à vis des topogies associées aux ρ–Hausdorff distances:

Théorème (G. Beer; 1990). Soit X un espace normé arbitraire. La transformation de Fenchel $f \rightarrow f^*$ est bicontinue pour la topologie de l'epidistance. En d'autres termes

$$f = \text{epi-dist-lim}\ f_n \Leftrightarrow f^* = \text{epi-dist-lim}\ (f_n)^* .$$

Théorème (Attouch-Ndoutoume-Théra; 1991). Soit $\{f, f_n : X \rightarrow \mathbb{R} \cup \{+\infty\} ;\ n \in \mathbb{N}\}$ une suite de fonctions convexes s.c.i. propres, où X désigne un espace de Banach général. Alors, l'implication (i) ⇒ (ii) a lieu:

(i) $f = \text{epi-dist}\lim f_n$;

(ii) $\partial f = \text{gph-dist}\lim \partial f_n$ plus la condition de normalisation (NC).

$$(NC)\quad \begin{cases} \exists (x_0, x_0^*) \in \text{gph}\ \partial f,\ \exists (x_{0n}, x_{0n}^*) \in \text{gph}\ \partial f_n \text{ pout tout } n \in \mathbb{N} \text{ tel que} \\ x_{0n} \overset{s}{\rightarrow} x_0,\ x_{0n}^* \overset{s}{\rightarrow} x_0^* \text{ et } f_n(x_{0n}) \rightarrow f(x_0). \end{cases}$$

Si de plus X est super-réflexif, alors (i) et (ii) sont équivalents.

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Programme des Conférences

Lundi 22 juin

Président de séance: R. J.-B. Wets (Davis)

9h-10h G.A. Beer (Los Angeles)
10h-10h30 S. Levi (Milan)

Pause Café

Président de séance: D. Klatte (Zürich)

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Déjeuner

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16h15-16h45 S. Gautier (Pau)-**L. Thibault** (Pau)
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Mardi 23 juin

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Président de séance: S. M. Robinson (Madison)

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11h15-11h45 H. Attouch (Montpellier)-**J.-B. Baillon** (Lyon 1)-M. Théra (Limoges)

Pause

Président de séance: L. Thibault (Pau)

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Déjeuner

14h **Excursion dans les calanques de Cassis**

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Jeudi 25 juin

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10h-10h30 A. Auslender (Clermont-Ferrand)

Pause Café

Président de séance: Th. Jongen (Aachen)

10h45-11h15 M. A. Bahraoui-**B. Lemaire** (Montpellier)
11h15-11h45 A. Seeger (Avignon)
11h45-12h15 D. Klatte (Zurich)

Déjeuner

Président de séance: G. Bouchitté (Toulon)

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Pause Café

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Pause

Président de séance: W. Oettli (Mannheim)

17h30-18h S. J. Flam (Bergen)
18h-18h30 J.-E. Martinez-Legaz (Barcelone)
18h30-19h **J.-P. Crouzeix**-R. Kebbour (Clermont-Ferrand)

Vendredi 26 juin

Président de séance: A. Cellina (Trieste)

9h-10h	E. Balder (Utrecht)
10h-10h30	M. Valadier (Montpellier)

Pause Café

Président de séance: A. Auslender (Clermont-Ferrand)

10h45-11h15	**R. Correa**-A. Jofré (Santiago)-L. Thibault (Pau)
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Déjeuner

Président de séance: P.-L. Papini (Bologne)

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Fin du colloque

Table des matières /Contents

Differentiating set-valued maps

Zvi Artstein

The Weizmann Institute Rehovot 76100, Israel

Abstract

Derivatives of compact set-valued maps will be defined,with the goal of developing a calculus suitable to express evolutions of sets, and eventually differential equations for set evolution. The motivation arises in controlled differential inclusions. The talk will compare the suggested derivative to the available notions of derivatives for multifunctions

Approximation and Regularization of Arbitrary Functions in Hilbert Spaces by the Lasry-Lions Method

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and

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Abstract

The Lasry-Lions's regularization method is extended to arbitrary functions. In particular, to any proper lower semicontinuous function $f : X \to I\!R \cup \{+\infty\}$ defined on a Hilbert space X and which is quadratically minorized (i.e. $f(x) \geq -c(1 + \|x\|^2)$ for some $c \geq 0$), is associated a sequence of differentiable functions with Lipschitz continuous derivatives which approximate f from below. Some variants of the method are considered including the case of non quadratic kernels.

Approximation methods play an important role in nonlinear analysis. A number of problems in variational analysis and in optimization theory give rise to nonsmooth functions with possibly infinite values defined on finite or infinite dimensional spaces. By using a regularization procedure based on the infimal convolution or epigraphical sum (see [1]), these problems can be attacked with the help of classical analysis tools. Let us mention in this direction the pioneering works of K. Yosida [6], H. Brézis [2], J.-J. Moreau [5]. These authors deal with convex lower semicontinuous functions in Hilbert spaces and with the corresponding subdifferential operators which are maximal monotone. The regularized function is proved to be $C^{1,1}$ (continuously differentia ble with Lipschitz continuous gradient). Some direct extensions have been recently obtained in [1] for more general kernels than the square of the norm. A difficult problem is *to extend these results to the non convex case.* A decisive step in this direction has been done recently by J.-M. Lasry and P.-L. Lions in [4]. They were motivated by the study of the Hamilton-Jacobi equations and worked with bounded uniformly continuous functions. In [3] Theorem 2.6, boundedness and uniform continuity assumptions are removed: the approximation/regularization result is obtained assuming that the absolute value of the function is quadratically majorized. Our main results state that, given any quadratically minorized function f defined on a Hilbert space X with values in $\mathbb{R} \cup \{+\infty\}$, the function $(f_\lambda)^\mu$ defined by the formula

$$(f_\lambda)^\mu(x) := Sup_{y\in X}\, Inf_{u\in X} \left\{ f(u) + \frac{\|u - y\|^2}{2\lambda} - \frac{\|y - x\|^2}{2\mu} \right\}$$

is $C^{1,1}$ whenever $0 < \mu < \lambda$ and approaches f from below as the parameters λ and μ go to 0. Observe that our growth assumption on f allows to treat the case of an indicator function. Clearly, by exchanging the order of the inf-sup operations, one obtains a corresponding approximation from above. The paper is organized with respect to increasing generality. We consider successively the convex, then the convex up to a square case, and finally the general case. A natural question that arises concerns the flexibility of the method. The case of non quadratic kernels is also considered. These results open new perspectives in nonsmooth analysis and ask for further developments: one may think defining generalized derivatives by relying on these approximation techniques. Regularization of sets can be considered too by using their indicator functions.

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Sum of Maximal Monotone Operators revisited: The variational sum

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Abstract

The sum of (nonlinear) maximal monotone operators is reconsidered from the Yosida approximation and graph-convergence point of view. This leads to a new conncept, called *variational sum*, which coincides with the classical (pointwise) sum when the classical sum is maximal monotone. In the case of subdifferentials of convex functions, the variational sum is equal to the subdifferential of the epigraphical sum (inf-convolution) of the functions. A general feature of the variational sum is to involve not only the values of the two operators at the given point but also their values at nearby points.

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CONVERGENCE OF STATIONARY SEQUENCES FOR VARIATIONAL INEQUALITIES WITH MAXIMAL MONOTONE OPERATORS

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Abstract. Let T be a maximal monotone operator defined on $\mathbb{R}^N$. In this paper we consider the associated variational inequality : $0 \in T(x^*)$ and stationary sequences $\{x_k^*\}$ for this operator, i.e., satisfying $T(x_k^*) \to 0$. The aim of this paper is to give sufficient conditions ensuring that these sequences converge to the solution set $T^{-1}(0)$ especially when they are unbounded. For this we generalize and improve the directionally local boundedness Theorem of Rockafellar to maximal monotone operators T defined on $\mathbb{R}^N$.

A survey of Young measure theory and some applications

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Abstract

A survey will be given of Young measure theory and some of its applications. As for the theory, it has been discovered in the past decade that Young measure theory forms an extension of the classical weak convergence theory for probability measures on a topological space [Ba1,Ba2,Va], with a useful reinforcement in the form of K-convergence [Ba3,Ba4,Ba5]. As for the applications, I shall limit myself mainly to discussing Fatou-type lemmas [Ba1,Ba6,BaHe] and results à la Visintin [Ba7,Ca,Ba5,Ba8].

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Wijsman Convergence of Closed and Convex Sets: an Overview

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Abstract

A natural way to define convergence for a sequence (or net) of closed sets in a metric space is to insist that the associated sequence (or net) of distance functionals convergences pointwise to the distance functional of the limit set. This point of view stems from the seminal paper of R. Wijsman on convergence and convex duality, and has since been developed by numerous authors. In this talk, we give an overview of the major results involving this mode of convergence, indicating connections with convex analysis, Banach space geometry, and measurable multifunctions.

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Approximation and regularization of arbitrary sets in finite dimension

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Abstract

In this paper, we define the regularization of a set in R^n by using two kernels. More precisely, let (K_1, K_2) be a pair of two kernels (nonempty open subsets of R^n); then, for a nonempty closed subset $X \subset R^n$, we associate its regularizing family $(R_{\lambda,\mu}(X))_{0<\mu<\lambda}$ defined by:

$$R_{\lambda,\mu}(X) \;=\; {}^c({}^c(X + \lambda K_1) + \mu K_2),$$

where $0 < \mu < \lambda$ be two real numbers and ${}^cA$ denotes the complement of A in R^n. When $K_1 = K_2$ is the open unit ball, we can easily build the regularizing family. When $K_1 = -K_2$ is a quadratic kernel and when X is the epigraph of a lower semicontinuous function, we retrieve the Lasry-Lions regularization (see [2]). Results are obtained in four directions:

1. general properties about $R_{\lambda,\mu}(X)$ which are issued of its definition ;
2. the convergence of the regularizing family when $(\lambda, \mu) \longrightarrow 0$;
3. the smoothness of the set $R_{\lambda,\mu}(X)$;
4. the asymptotic behaviour of Clarke's normal cone (see [1]) of the regularizing family.

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ASYMPTOTIC CONES, CLOSED IMAGES AND A THEOREM OF DIEUDONNE

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Asymptotic cones were introduced to describe the behaviour of convex sets "at infinity". If A is a convex, closed, nonempty subset of a real separated topological vector space E, then the asymptotic cone A_∞ of A is classically defined as

$$(1) \qquad A_\infty := \bigcap_{\lambda>0} \lambda(A-a),$$

when $a \in A$ is fixed arbitrarily. The definition is independent of the chosen $a \in A$. Dieudonné [3] has proved the following result :

Theorem. Let A, B be convex, closed, nonempty subsets of E. Let A be locally compact and $A_\infty \cap B_\infty = \{0\}$. Then $A-B$ is closed.

Dedieu [2] has generalized the definition of asymptotic cone for arbitrary subsets $A \subseteq E$ as follows :

$$(2) \qquad A_\infty := \bigcap_{\varepsilon>0} \overline{(0,\varepsilon).A}\,.$$

He extended Dieudonné's result to the case where A and B, instead of being convex, are raditating in $a \in A$ and $b \in B$ respectively. Dedieu's proof follows rather closely the proof of Dieudonné. This is possible, because if A is radiating in $a \in A$, then (1) still holds, as in the convex case.

Groinner [4] has carried over Dieudonné's result to convex multivalued mappings $T : E \rightrightarrows F$, where F is another topological vector space. Using Dieudonné's result he gave conditions for $T(A)$ to be closed, where A is a closed convex subset of E.

Here we want to prove the following result, which generalizes Dedieu's and Groinner's results :

Theorem. Let E, F be real topological vector spaces. Let $T : E \rightrightarrows F$ be a multivalued mapping. Let graph T be closed. Let $A \subseteq E$ be closed, locally compact and radiating in $a \in A$. Let $\mathfrak{M} := \{t \in E \setminus t \in A_\infty, (t,0) \in (\text{graph } T)_\infty\}$. If $\mathfrak{M} = \{0\}$, then T(A) is closed.

Here A_∞ is to be undertood in the sense of (2). If we set $T(X) := X-B$ we obtain Dedieu's result under weak assumptions. It should be noted that the requirement $\mathfrak{M} = \{0\}$ above can be replaced by the requirement that $\mathfrak{M}$ is a linear subspace and

$$A+\mathfrak{M} \subseteq A, \ \text{graph } T + (\mathfrak{M} \times \{0\}) \subseteq \text{graph} T.$$

It is interesting to note that already Debreu in [1,p. 23] gave without proof the following result : If A, B are arbitrary closed subsets of $\mathbb{R}^n$ with $A_\infty \cap B_\infty = \{0\}$, then A-B is closed.

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EQUICOERCIVITE DE PROBLEMES VARIATIONNELS LE ROLE DES FONCTIONS DE RECESSION

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en collaboration avec P.SUQUET (LMA Marseille)[1]

Soit X un espace de Banach et τ une topologie rendant compactes les boules fermées de X. Etant données $F^n, F : X \to]-\infty,+\infty]$ des fonctionnelles convexes propres telles que :

$$\tau\text{-}\lim_{n\to\infty}{}_e\ F^n = F$$

et une forme linéaire τ - continue $L : X \to \mathbb{R}$, on se propose de caractériser l'ensemble des scalaires λ pour lesquels la suite $F^n_\lambda := F^n - \lambda L(.)$ est équi-coercive dans X (ce qui assurera la convergence des infima associés). Lorsque cet ensemble J contient 0, il est montré que $J =]\lambda^-, \lambda^+[$ où :

$$\lambda^\pm = \text{Min}\ \{ F_\infty(u) ; u \in X, L(u) = \pm 1 \}$$

F_∞ étant la fonction de récession de F.

Diverses applications sont données en théorie de la plasticité, capillarité ..etc..

En liaison avec l'approximation des coefficients $\lambda^\pm$ à l'aide des problèmes approchés $\lambda_n^\pm = \inf \{ F^n_\infty(u) ; L(u) = \pm 1 \}$, on s'intéresse ensuite à la convergence des fonctions de récession F^n_∞ vers F_∞. En général l' égalité $\tau\text{-}\lim_{n\to\infty}{}_e\ F^n_\infty = F_\infty$ est fausse. Ele est vraie cependant dans le cas $X \subset Y$ avec Y Banach réflexif et injection compacte sous une hypothèse supplémentaire de compatibilité des domaines des fonctionnelles conjuguées sur Y* :

$$w\text{-}\operatorname*{Ls}_{n\to\infty}\ \text{dom}\,(F^n)^* \subset \overline{\text{dom}\,F^*}$$

En application au problème des charges limites en plasticité, on obtient ainsi via l'inégalité de Deny-Lions un résultat d'homogénéisation en milieu incompressible.

[1] à paraitre dans Séminaire EDP Collège de France 1991, Pitman

About proximal determinations of Huber's estimators

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Abstract

In recent years, *robustness* is a problem that has been given much attention in the statistical literature on the linear regression model. One reason of this interest is an increasing sensitivity of applied statisticians to potential deficiencies of least squares methods under the occurrence of outliers. Several robust alternatives have been considered. Among them, we find the class of M-estimators introduced by Huber [3]. *Huber's M-estimator*, proved to be the M-solution to a contamined Gaussian model, is of particular interest. It consists in solving the following problem:

$$\text{Find } \hat{\beta} \in \text{argmin } \varphi(\beta) = \sum_{i=1}^{n} \rho\left[(X\beta - y)_i\right] \qquad (P)$$

where X is a given $n \times m$ matrix, $y \in^n$ and ρ is the cost function, depending on a given

tuning constant c, defined by

$$\rho(r) = \begin{cases} \frac{1}{2}r^2 & \text{if} \quad |r| \le c \\ c|r| - \frac{1}{2}c^2 & \text{otherwise} \end{cases}$$

Several iterative resolution procedures have already been investigated to find the *Huber M-estimator.*

Here, according to the closed connection between Huber's cost function ρ and the Moreau-Yosida c-approximate of the ℓ_1 cost function observed in [1], we propose to solve the optimization problem (P) by a proximal approach combined with duality whose ability in solving efficiently some closely related problems, namely location problems, has been proved [5][6].

In a first part of this paper we reformulate the Huber problem as a linearly constrained optimization problem. Then, thanks to duality (in the sense of Fenchel), we derive optimality conditions and we give several applications of these conditions (description of the entire set of optimal solutions, asymptotical results,..). We also show these optimality conditions can be solved by the *Partial Inverse Method* developed by Spingarn [7]. The algorithm, which can be viewed as a decomposition technique, gives very simple updating rules and permits parallel computations, what is of interest for large size data. As instance of the well known basic proximal algorithm, the procedure is known to be very stable and always globally convergent.

In a second part, we give a new equivalent formulation of the Huber problem that leads to a second duality scheme with decomposition properties in terms of "small residuals" and outliers. We show that this new formulation is also convenient to deal with duality and can be solved by the partial inverse method.

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A Eulogy on Nonconvexity

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Abstract

There is some interest in problems of the Calculus of Variations and Optimization theory without the usual Convexity conditions.

This talk is mainly devoted in exploring and debating the reasons in favor of this approach to the problem.

Some recent existence results for problems involving the gradient will be discussed.

Convergence Issues in Penalty Methods. Linear Programming for Instance.

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Abstract

Since the contributions of Kachiyan (1979) and Karmarkar (1984) on polynomial algorithms for linear programming, intensive research has been directed towards interior point methods for linear as well as nonlinear programming. A number of polynomial algorithms for LP and QP have emerged from this research, some of which have shown to be "competitive" with respect to more standard algorithms such as the simplex method. Recent advances have clarified the closed relationship between interior point methods and the classical (and somewhat forgotten) penalty methods.

In this lecture we present the analysis of trajectories associated to an exponential penalty function: the existence and convergence of these trajectories towards a "center" point of the optimal set, the exponentially fast rate of convergence towards this center, together with a fairly complete duality theory are all established under the sole and very weak assumption of boundedness of the solution set.

The analysis is presented in the simplest setting of LP, but some results of nonlinear nature are exposed as well. The computational efficiency of an algorithm based on exponential penalties, predicted by the fast convergence rate, have been confirmed by (limited) computational experience.

On the other hand, the (unexpected) good behavior of trajectories raise interesting questions concerning the limits to which one can push a satisfactory sensitivity analysis of nonlinear parametric programs in the absence of the usual strong second order hypothesis, linear independence of active gradients, etc.; and suggest that significant extensions are still possible in this area.

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Suprema of Wijsman topologies in normed spaces

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Absrtact

Our work starts from a theorem of G. Beer which states that, on the hyperspace of closed and convex sets of a normed linear space X, the supremum of the Wijsman topologies ranging over the equivalent norms coincide with the Slice topology. In this context, we proved that, on the hyperspace of closed sets, it suffices to weaken a little the Slice topology to find a similar statement. That is to say, on the closed sets of X, the supremum of the Wijsman topologies made as above coincide with the Hit-and-Miss topology generated by the V^- with V open set of X, and the $(B^c)^{++}$ with B closed convex bounded and symmetric set of X. Here symmetric means that B is the translated of a set symmetric at the origin of X. This new topology always coincide with the Slice topology on the closed and convex sets, while on the closed sets this is true if and only if X is finite dimensional. Moreover, since a result of Borwein and Fitzpatrick states that, when X is reflexive, the supremum of the Wijsman is in fact a maximum, we examined the situation on the closed sets and found that in this case the supremum is realized if and only if X is finite dimensional.

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A splitting property of the upper bounded-Hausdorff convergence

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Abstract

In the Hyperspace of a metric space (X, d), a caracterization of the upper bounded-Hausdorff convergence is provided. Precisely, we prove that a net $(A_i)_{i\in I}$ of closed subsets of X is bH^+-convergent to a *closed subset* A *if and only if for every* $i \in I$ there exist closed subsets B_i,C_i of X such that:

1) $A_i = B_i \cup C_i$;

2) $B_i \xrightarrow{H^+} A$;

3) $\forall x \in X$: $\lim_{i\in I} d(x, C_i) = +\infty$ (or, equivalently, $C_i \xrightarrow{w^+} \emptyset$).

As an application, we obtain that if every A_i is connected and the set A is bounded, then the upper bounded-Hausdorff convergence of the A_i's to A implies their upper Hausdorff convergence.

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Subdifferential Monotonicity as Characterization of Convex Functions *

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Abstract

It is known and easy to prove (Clarke (1983)) that the monotonicity of the Clarke subdifferential of a locally Lipschitz real valued function is equivalent to the convexity of this function. In order to prove the same result for a lower semicontinuous function $f : E \to \mathbf{R} \cup \{+\infty\}$ we have considered in Correa, Jofré and Thibault (1990) the Moreau-Yosida proximal approximation

$$f_\lambda(x) := \inf_{y \in E}[f(y) + \frac{1}{2\lambda}\|x - y\|^2] \tag{1}$$

*This work was partially supported by Fondo Nacional de Ciencia y Tecnología, FONDECYT

since (under some general conditions) f_λ is locally Lipschitz and

$$f(x) = \sup_{\lambda>0} f_\lambda(x).$$

Thus our procedure consisted in deriving the monotonicity of ∂f_λ from that of ∂f. This method required the reflexivity of the space E because it depended heavily on the fact that the above infimum is attained whenever the Frechet subdifferential $\partial^F f(x)$ of f at x is nonempty. This has been obtained by supposing (thanks to the reflexivity of E) that the norm of E is Kadec and by showing that (when $\partial^F f(x) \neq \phi$) there exists some minimizing sequence of (1) weakly converging to some point z and whose norms converge to $\|z\|$, which implies the strong convergence of the sequence. The same result was proved before by R.A. Poliquin (1988) for $E = \mathbf{R}^n$ with the help of his notion of quadratic conjugate function.

In this paper, by a completely different approach, we avoid the use of the Moreau-Yosida approximation f_λ in order to get the result for any Banach space E. In fact we prove that the monotonicity of any classical subdifferential ∂f of a lower semicontinuous function $f : E \to \mathbf{R} \cup \{+\infty\}$ defined on a Banach space E is equivalent to the convexity of this function f.

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Index multiplicatifs de convexité/concavité et applications

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Résumé

L'étude de la concavité/convexité d'un produit de fonctions séparables est faite à partir de la notion d'index de convexité/concavité dérivés de l'index de convexité introduite par Debreu-Koopmans et Crouzeix-Lindberg. Des applications sont données en économie et pour des fonctions potentielles introduites récemment en programmation linéaire.

Title to be announced

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Stability of subdifferentials of non convex functionals

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Abstract

We present some recent variational principles and we apply them to show various stability results of subdifferentials and superdifferentials of non convex functionals in infinite dimensions. These results are applied to the geometry of Banach spaces and to the study of Hamilton-Jacobi equations or of fully non linear second order partial differential equations in infinite dimensions.

On the Convergence of Efficient Sets

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Abstract

Let E be a normed space partially ordered by a convex cone C. The set of efficient points and of weakly efficient points of A are defined by

$$MinA = \{a \in A : a \in a' + C \; for\ some\ a' \in A\ implies\ a' \in a + C\}$$

$$WMinA = \{a \in A : there\ is\ no\ a' \in A\ with\ a \in a' + intC\}$$

The question studied here is the dependence of $MinA$ and $WMinA$ on perturbations of A. The main results existing on this topic [1][2][4], deal with the Kuratovski-Painlevé limits. For instance, one can prove that if $\lim_{n\to\infty} A_n = A$ then $\limsup_{n\to\infty} WMinA_n \subset WMinA$. When E is finite dimentional and A is closed this result can be read as

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: $\lim_{n\to\infty} d(An, x) = d(A, x)$, for all $x \in E$, implies $\liminf_{n\to\infty} d(WMinA_n, x) \geq d(WMinA, x)$ for all $x \in E$. Examples show that the same conclusion is not always valid in an infinite dimentional setting. Here we present two particular cases in which the result holds, either supposing that E is an Hilbert space with C a polyhedral cone, or with the condition that $\cup\, WMinA_n$ is relatively compact. Another interesting topic is to obtain an opposite inequality in the form $\limsup_{n\to\infty} d(MinA_n, x) \leq d(MinA, x)$ for all $x \in E$. For this type of conclusion we present sufficient conditions ensuring at the same time that Min A is nonempty.

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ON VARIATIONAL STABILITY IN COMPETITIVE ECONOMIES

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ABSTRACT. We explore the variational stability of supply, demand and equilibria in perfectly competitive economies. The appropriate, and in fact, minimal limit notion is furnished by the concept of Kuratowski-Painlevé convergence together with the derived concepts of epi and hypo convergence. When technologies and preferences converge in such manners we show, subject to compactness assumptions, that equilibria of approximate economies cluster to those of the limiting economy.

Key words: Epi-convergence, hypo-convergence, slice topology, Hotelling's lemma, Shepard's lemma, excess demand, indirect utility, expenditure function, competitive equilibria.

1 INTRODUCTION.

Microeconomic theory deals with the behavior of firms and consumers interacting via markets. Typically one wants to understand (or predict) equilibrium demand and supply of these agents. The dominating paradigm is then constrained optimization: each firm (or consumer) is assumed to maximize its achievable profit (resp. utility) subject to constraints. As a rule, only one half of such optimization problems is visible for an outside observer: For a given, fixed price-vector p the objective of a profit-maximizing firm is a known linear function $\langle p,\cdot\rangle$. By contrast, its *technology*, that is, the set Y of all possible production plans y is often hard to describe. This situation is precisely reversed for a representative consumer: On one hand, an adequate description of his feasible set is readily given by the budget constraint $\langle p,x\rangle \leq b$, b denoting here his budget. On the other hand, his *utility function* u mapping the commodity space E into $[-\infty,\infty)$, is usually unknown, or at least, not made directly explicit.

This state of affairs prompts at least two questions. <u>First</u>, one regarding a *dual approach* (Diewert, 1982): Observing the *optimal choice correspondences*

$$p \rightarrow X(p,b)\ , \quad p \rightarrow Y(p),$$

portraying demand and supply of an individual consumer and a single firm is it possible to reconstruct the underlying preference $u: E \rightarrow [-\infty,\infty)$ and the technology Y ?
<u>Second</u>, in terms of *sensitivity analysis* (also named comparative statics), knowing the two optimal value mappings, namely the *expenditure function*

$$p \ \rightarrow \ e(p) := \langle p, x\rangle,\ x \in X(p,b)$$

and the *profit function*,

$$p \rightarrow \pi(p) := \langle p, y\rangle,\ y \in Y(p)$$

may one predict changes $\Delta X(p,b)$, $\Delta Y(p)$ that result from price perturbations Δp? Not surprisingly, good answers to these questions have been granted by the application of convex analysis. Some of these results are recalled in Prop. 3.1-2 below.
However, since economic models suffer from inaccuracies of various sorts, such answers are usually inexact. In fact, to provide estimates of competitive supply and demand is not well founded unless the correct, yet unknown economic model is stable in some appropriate sense. This observation motivates the present note to inquire: *What kind of convergence* $\mathbf{E}^n \to \mathbf{E}$ *between economies* $\mathbf{E}^n, \mathbf{E}$ *ensures that competitive equlibria of the approximate economies* $\mathbf{E}^n$, n=1,2,..., *cluster to those of the limit economy* $\mathbf{E}$?

At the level of technologies the convenient and natural notion $Y^n \to Y$ is that *Kuratowski-Painlevé convergence*. For preferences it turns out that the most suitable convergence mode $u^n \to u$ is that of *hypo-convergence*. All definitions are recalled in Section 2. As customary in microeconomics we focus first on *one* firm (Sect. 3) and *one* consumer (Sect. 4) the objective being to explore stability of individual demand and supply. Thereafter the results obtained for single agents are synthesized into a general equilibrium analysis (Sect. 5). To facilitate the exposition we shall assume, for the most part, that the commodity space E is finite-dimensional Euclidean. Sect. 6 points out generalizations.

The novelties of this paper are mostly in terms of applications; it opens up for variational analysis in competitive economic contexts, a branch so far not much developed.

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Viability for Constrained Stochastic Differential Equations

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Abstract

The existence of viable solutions of constrained stochastic differential equations (with Brownian motion) is established under a stochastic tangential condition. The approach is based on an appropriate notion of approximate solutions.

Multivalued strong law of large numbers in the Wijsman topology and the slice topology.

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In recent years, multivalued strong laws of large numbers (SLLN), especially for unbounded random sets, proved to be useful in applications to stochastic optimization, statistics and related fields. Results of such type were first provided by Artstein and Hart [ArH] for random sets (r. s.) with closed values in finite dimensional spaces. Later, Hiaï [Hi] and Hess [He3, 4] independently proved similar results for random sets with values in an infinite dimensional Banach space. These multivalued SLLN were proved assuming that the set of all closed subsets of the Banach space X, denoted by $\mathcal{C}(X)$, is endowed with the Mosco topology (which reduces to the Kuratowski-Painlevé topology for finite dimensional spaces).

Although these SLLN provide useful probabilistic convergence properties, one may ask for other results involving more explicitelly the norm of the space X. A natural and well-known topology for this purpose is the Wijsman topology, denoted by $\mathcal{T}_W$. It is the topology of pointwise convergence of distance functions. So, a natural question is : in a general (separable) Banach space, does the SLLN for random sets with values in $\mathcal{C}(X)$ hold when this set is endowed with $\mathcal{T}_W$? An affirmative answer to this question is provided by theorem 1 (A) below.

On the other hand, more recently, G. Beer [Be1, 2] introduced the "slice topology" on $\mathcal{C}(X)$, which is a natural extension of the Mosco topology. Indeed, Beer showed that, in a general Banach space, the slice topology, denoted by $\mathcal{T}_s$, is stronger than the Mosco topology, and that both topologies coincide if and only if X is reflexive. Further, on $\mathcal{C}(X)$ the slice topology is stronger than $\mathcal{T}_W$. Consequently, one may also ask if the multivalued SLLN holds in the slice topology. A positive answer to that question is furnished by theorem 1 (B).

Theorem 1 - A) Consider a separable Banach space X, an integrable r. s. Γ with values in $\mathcal{C}(X)$ and a sequence $(\Gamma_n)_{n\geq 1}$ of pairwise independent r. s. having the same distribution as Γ.

Then, there exists a negligible subset N such that, for any $\omega \in \Omega\backslash N$, one has

$$\overline{\text{co}}\, I(\Gamma, \mathcal{A}) = \mathcal{T}\text{-}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \Gamma_i(\omega) \tag{1}$$

where $\mathcal{T} = \mathcal{T}_W$, that is, for any $x \in X$,

$$d(x, \overline{\text{co}}\, I(\Gamma, \mathcal{A})) = \lim_{n\to\infty} d(x, \frac{1}{n}\sum_{i=1}^{n} \Gamma_i(\omega)).$$

B) Moreover, if X* is strongly separable then (1) holds with $\mathcal{T} = \mathcal{T}_s$.

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Théorème 2 (Application to integrands) - Consider an integrable convex lower semi-continuous normal integrand f defined on $\Omega \times X$ with values in $\bar{\mathbf{R}}$ and a sequence $(f_n)_{n\geq 1}$ of lsc normal integrands, pairwise independent with the same distribution as f. Also define the <u>mean functional</u> of f, namely the function F given by

$$F(x) := \int_\Omega f(\omega, x)\, dx \qquad\qquad x \in X.$$

Then for almost all $\omega \in \Omega$, one has

$$(2) \qquad F^{**} = \mathcal{T}_S\text{-}\lim_{n\to\infty}{}_e \frac{1}{n}\sum_{i=1}^{n} f_i(\omega, .).$$

where F^{**} is the biconjugate of F. In (2) '$\lim_e$' means 'epigraphical limit'. More precisely, for almost all ω, the sequence of epigraphs

$$\text{epi}\left(\frac{1}{n}\sum_{i=1}^{n} f_i(\omega, .)\right) \qquad\qquad n \geq 1$$

$\mathcal{T}_S$-converges to epi F^{**}.

See [He5, 6] for the proofs and further discussions. The proof of theorem 1 (A) heavily relies upon specific properties on the distribution of random sets ([ArH] and [He1, 2]). Also note that results similar to theorem 2, but for other topologies on $\mathcal{C}(X)$, has been obtained by Attouch and Wets [AtW].

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SUBDIFFERENTIAL CALCULUS WITHOUT "QUALIFICATION" CONDITIONS

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Abstract. Calculus with subdifferentials of convex functions is important for dealing with variational problems ; if f is constructed from some other convex functions f_j , the problem is to compute (exactly) ∂f in terms of the ∂f_j's. When the functions involved are extended-valued, some "qualification" conditions are usually required to get the desired formulas. For example, various conditions ensure that $\partial(f_1+f_2) = \partial f_1 + \partial f_2$, but these conditions are not always satisfied in variational problems.The aim of the present work is to derive **a new set of calculus rules** for subdifferentials of convex functions, **without any qualification condition,** substituting the information $(\partial_\varepsilon f_j(x), \varepsilon \in]0,\bar{\varepsilon}])$ for $\partial f_j(x)$. The threshold $\bar{\varepsilon} > 0$ can be taken as small as desired, the "zero" of your computer for example. Again the role of the ε - subdifferential $\partial_\varepsilon f$ as a "smoothing effect" or "viscosity enlargement" of ∂f is emphasized. We give hereunder some snapshots of the calculus rules obtained.

(1) **Sum**. Suppose that f_1 and f_2 are proper lower semicontinuous convex functions ; then (with the bar denoting the closure in the weak * topology)

$$\partial(f_1+f_2)(x) = \bigcap_{0<\varepsilon\leq\bar{\varepsilon}} [\, \overline{\partial_\varepsilon f_1(x) + \partial_\varepsilon f_2(x)}\,] \qquad (\bar{\varepsilon} > 0 \text{ arbitrary})$$

$$\left(= \lim_{\varepsilon \downarrow 0} [\, \overline{\partial_\varepsilon f_1(x) + \partial_\varepsilon f_2(x)}\,] \right).$$

(2) **Inf-convolution.** If f_1 and f_2 are convex lower semicontinuous,

$$\partial(f_1 \,\square\, f_2)(x) = \bigcap_{0<\varepsilon\leq\bar{\varepsilon}} \; \bigcup_{y\in E_\varepsilon(x)} \left[\partial_\varepsilon f_1(y) \cap \partial_\varepsilon f_2(x-y)\right],$$

where $E_\varepsilon(x) = \left\{y : f_1(y) + f_2(x-y) < (f_1 \,\square\, f_2)(x) + \varepsilon\right\}$.

(3) **Maximum.** If f_1 and f_2 are convex lower semicontinuous, if $f_1(x) = f_2(x)$,

$$\partial[\mathrm{Max}(f_1,f_2)](x) = \bigcap_{0<\varepsilon\leq\bar{\varepsilon}} \overline{\mathrm{co}}\left[\partial_\varepsilon f_1(x) \cup \partial_\varepsilon f_2(x)\right]$$ (A. BRONDSTED, 1972).

Once some key-formulas are derived (like the one on the sum of functions), formulas concerning further operations are deduced by using the usual tricks and transformations in Convex Analysis.

The cases of marginal functions and differences of functions have been more thoroughly studied in AVIGNON (A. SEEGER, M. VOLLE, R. MOUSSAOUI) with applications in the area of Calculus of Variations.

Lipschitz Approximation of Lattice Valued Functions

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ABSTRACT

Let E be an order-complete Banach lattice with norm $\|\cdot\|$ and positive cone E_+. We adjoin to E a greatest element $+\infty$ and we set $E^\bullet = E \cup \{+\infty\}$, $E_+^\bullet = E_+ \cup \{+\infty\}$. We introduce a new concept of lower semi-continuity for the functions ψ from a separable metric space (X, d) into $E^\bullet$: a function $\psi : X \longrightarrow E^\bullet$ is *inf-continuous* at $x \in X$ if for every $e \in E$ with $e \leq \psi(x)$ and for every neighbourhood V of 0 in E, there exists a neighbourhood U of x such that $\inf \psi(U) \in e + V + E_+^\bullet$. This notion is stronger than the usual one (cf. [2]). Both coincide when E_+ has a non empty interior. We can now state the result of Lipschitz approximation.

THEOREM. *Let $\psi : X \longrightarrow E^\bullet$ an inf-continuous proper function on X. Assume that there exist $a, b \in E_+$ and $x_0 \in X$ such that $\psi(x) \geq -a - d(x, x_0)b$, $\forall x \in X$. Then there exist $h \in E_+$ with $\|h\| \leq 1$ and $k_0 \in \mathbf{N}^*$ such that for all $k \geq k_0$, the functions $\psi^k : X \longrightarrow E, x \longmapsto \inf_{y \in X} \{\psi(y) + kd(x, y)h\}$ verify the following properties:*

(1) $|\psi^k(x) - \psi^k(y)| \leq kd(x, y)h, \quad \forall (x, y) \in X \times X$.
(2) $\|\psi^k(x) - \psi^k(y)\| \leq kd(x, y), \quad \forall (x, y) \in X \times X$.
(3) $\psi(x) = \sup_k \psi^k(x), \quad \forall x \in X$.
(4) $\psi(x) = \lim_k \psi^k(x), \quad \forall x \in X$ *such that* $\psi(x) \in E$.

We point out that the element h is not unique and that it depends on ψ. Also, if ψ is such that there exists a sequence $(\psi_n)_n$ of functions verifiyng (1) to (4) then ψ is inf-continuous.

We use this approximation result to characterize an epi-like convergence of lattice valued functions: let $(\psi_n)_n$ a sequence of functions from X into $E^\bullet$ and define $\mathrm{li}_e\, \psi_n = \sup_p \underline{\lim}_n \inf \psi_n(B(x, 1/p))$, $\mathrm{ls}_e\, \psi_n = \sup_p \overline{\lim}_n \inf \psi_n(B(x, 1/p))$. If

ψ, $(\psi_n)_n$ are finite inf-continuous functions minored by $-a-d(x,x_0)b$ and such that $\psi_n \leq \psi$, then one has $\mathrm{li_e}\,\psi_n(x) = \sup_k \underline{\lim}_n \psi_n^k(x)$, $\mathrm{ls_e}\,\psi_n(x) = \sup_k \overline{\lim}_n \psi_n^k(x)$. It seems that the "good" definition of convergence is the following: a sequence $(\psi_n)_n$ *$*$-epi-converges uniformly* on X to ψ_∞ if from every subsequence of $(\psi_n)_n$ one can extract another subsequence $(\psi_{n_p})_p$ such that $\mathrm{li_e}\,\psi_{n_p} = \mathrm{ls_e}\,\psi_{n_p} = \psi_\infty$. This notion have many variational properties such that $\inf \psi_\infty \geq \overline{\lim}_p[\inf \psi_{n_p}]$ for a subsequence $(\psi_{n_p})_p$. Actually, the only relation between uniform $*$-epi-convergence and the convergence of the epigraphs is the following inclusion: $\mathrm{Epi}(\mathrm{ls_e}\,\psi_n) \subset \mathrm{Li}(\mathrm{Epi}\,\psi_n)$.

We also extend to lattice valued integrands the previous approximation result and prove a strong law of large numbers for this integrands.

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On B-subgradients and applications

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Abstract

The set of b-subgradients of a function f, $\partial_b f$, is a notion which generalizes the derivative, in the sense that, it is reduced to a singleton when the function f is differentiable. This subgradient set and the related normal cone N_b are always smaller than the Clarke generalized gradient and its correspondent normal cone. Recently, an important number of calculus rules (Michel & Penot, Treiman), optimality conditions in mathematical programming and optimal control, Proximal normal and b-subgradient formulae (Jofre & Thibault, Treiman) and Frechet (ϵ, r)-normal formula (Jofre & Thibault) have been proved using these concepts. In this talk, we show two consequences of the proximal b-normal formulae. Firstly, we give estimates to b-subgradient of the optimal value function $p(u)$

$$p(u) = \inf\{f(x) : g(x) + u \in -K, x \in C\}$$

associated to the parameterized nonsmooth optimization problem

$$\begin{aligned}&\text{minimize } f(x) \text{ subject to}\\&g(x) + u \in -K\\&x \in C\end{aligned}$$

where $f : E \to \mathbf{R}$ and $g : E \to \mathbf{R}^p$ are locally Lipschitzian functions, C is a nonempty closed subset of a Banach space E, and K is a convex cone in $\mathbf{R}^p$. Our proof follows the ideas of Rockafellar and Clarke's works with a major difference, the set-valued map $\partial_b f(\cdot)$ is not upper semicontinuous even if f is locally lipschitzian.

Secondly, we give a result on the b-normal cones, in the vein of Cornet-Rockafellar's theorem (1989), which coincide with a separation theorem when the sets are convex.

Semi-Infinite Optimization: Structure and Stability of the Feasible Set

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Abstract

This research was done in collaboration with F. Twilt and G.-W. Weber. The problem of the minimization of a function $f : \mathbb{R}^n \to \mathbb{R}$ under finitely many equality constraints and perhaps infinitely many inequality constraints gives rise to a structural analysis of the feasible set $M[H,G] = \{x \in \mathbb{R}^n : H(x) = 0,\ G(x,y) \geq 0,\ y \in Y\}$ with compact Y in $\mathbb{R}^r$. An extension of the well-known *Mangasarian-Fromowitz* constraint qualification (EMFCQ) is introduced. The main result for compact M[H,G] is the equivalence of the topological stability of the feasible set M[H,G] and the validity of EMFCQ. As a byproduct, we obtain under EMFCQ that the feasible set admits local linearizations and also that M[H,G] depends continuously of the pair (H,G). Moreover, EMFCQ is shown to be satisfied generically.

Metric regularity and stability in semi–infinite optimization

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Abstract

In the paper we study parametric nonlinear optimization problems of the type SIP(t):

$$\begin{aligned}&f(x,t) \longrightarrow \min_x\\ &\text{subject to}\\ &h_i(x,t) = 0\ ,\ i \in I\ ,\\ &g(j,x,t) \geq 0\ ,\ j \in K\ ,\end{aligned}$$

where t is regarded as a parameter varying over some metric space T, $I := \{1,\dots,p\}$, K is a compact subset of $\mathbf{R}^s$, and $h_i : \mathbf{R}^s \times T \longrightarrow \mathbf{R}$ and $g : \mathbf{R}^s \times \mathbf{R}^n \times T \longrightarrow \mathbf{R}$ are continuous functions being C^1 w.r.t. to x resp. (j,x).
Given $t^o \in T$ and some solution x^o to SIP(t^o) of interest (e.g., a local minimizer or a stationary solution in the Kuhn-Tucker sense), we are looking for conditions ensuring regularity of the constraint system near (x^o,t^o) and local stability of the solution. As a main regularity requirement on the constraints, we use the Extended Mangasarian Fromovitz Constraint Qualification (EMFCQ), cf. Jongen, Twilt and Weber [3]. First we show that EMFCQ at some feasible point z is necessary and sufficient for metric regularity

of the constraint system at z, cf. Henrion and Klatte [2]. This fact increases the number of equivalent characterizations of EMFCQ: In Henrion [1], equivalence between EMFCQ and some local epigraph representability of the constraint set was observed, and in [3], there is shown that the feasible set of a semi-infinite program is (topologically) stable in Jongen's sense if and only if EMFCQ holds on the whole set.

Then, assuming EMFCQ at a strict local minimizer x^o of the initial semi-infinite problem (i.e., at t^o), the continuity of some portion of the local minimizing set mapping at t^o immediately follows, and, under more structure, even "upper Hoelder" continuity holds, cf. Klatte [4]. Using a certain extension of the Gauvin–Robinson result on local boundedness of the Lagrange multiplier set mapping under MFCQ to our situation, one may show the upper semicontinuity of some portion of the (Kuhn–Tucker) stationary solution set mapping.

Finally, we present strong stability results under the well-known reduction ansatz. This approach leads – even under restrictive smoothness and regularity conditions on the original data – to a $C^{1,1}$ optimization problem with finitely many constraints, and one may apply the stability properties known in this case, cf. Klatte [4].

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Convergence of diagonally stationary sequences in Convex Optimization

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Let f be a closed proper convex function defined on a real normed vector space X. Denoting by $\partial f(x)$ the subdifferential of f at x and by $d_*(0, C)$ the distance, in the dual space X^* of X, from 0 to the subset C of X^*, we say that a sequence $\{x_n\}$ in X is **stationary** for f if it satisfies

$$\lim_{n \to +\infty} d_*(0, \partial f(x_n)) = 0$$

that is, for each $n \in \mathbb{N}, x_n$ is determined along with a subgradient $x_n^* \in \partial f(x_n)$ such that $x_n^* \to 0$ strongly in X^*. Some infinite constructive processes for minimizing f generate such a sequence. Actually, in most situations (infinite dimension of X, constraints in f), such a process is not applied to f itself but to a **fixed** approximation of it.

The idea of diagonal processes, as introduced in [Boy 74], is to combine a basic process with a sequence $\{f^n\}$ of (closed proper convex) approximations of f **changing** the approximation at each step of the process. For stationary sequence generating processes this leads to generating what we call a **diagonally stationary sequence** (DSS) for $\{f^n\}$, that is, a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} d_*(0, \partial f^n(x_n)) = 0$$

A natural question arises: under what conditions on $\{f^n\}$ with respect to f is the considered basic process stable in the sense that the diagonal process inherits the convergence properties of the basic one ? Variational convergence theory has revealed powerfull tools to study this question in the context of particular basic processes [Lem 88,Mou 89, Tos 90,A-L 91,Lem 91]. The purpose of the present work is to proceed on that way for general DSS. We first consider the case of **bounded** DSS with different kinds of convergence of f^n to f. For **unbounded** DSS we introduce the notion of **equi well asymptotical behaviour** (of the sequence $\{f^n\}$) which extends the notion of **well asymptotical behaviour** introduced in [A-C 89] for a single convex function. If X is a (general) Banach space, this notion appears to be equivalent to a notion of **equi well posedness** closely related to the one introduced in [Zol 78], equivalence yet established in [Lem 92] for a single convex function between well asymptotical behaviour and well posedness in the (extended to nonuniqueness) sense of Thikhonov.

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SOME NEW RESULTS ON WIJSMAN TOPOLOGIES

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Following the paper [BLLN], new results are appearing (see [Be] and [CLP] that point towards the centrality of Wijsman topologies within the theory of set convergences.

It is therefore of interest to describe the conditions under which two equivalent metrics generate the same Wijsman convergence in the hyperspace.

$c_0(X)$ will denote the hyperspace of nonempty closed subsets of the metrizable space X.

We recall the following results from [BLLN] :

- the Vietoris topology on $c(X)$ is the supremum of the Wijsman topologies over all equivalent metrics on X ;

- the upper Hausdorff topology generated by a metric d on X is the supremum of the Wijsman topologies over all metrics that are uniformly equivalent to d.

and from [Be] :

- *the slice topology* on the closed convex subsets of a normed space is the supremum of the Wijsman topologies over all metrics generated by equivalent norms.

In the forthcoming paper [CLP] the following result is proved :

- Kuratowski convergence on $c(X)$ is the infimum - in the lattice of convergences - of the Wijsman topologies over all equivalent metrics on X.

As for equality of two Wijsman topologies we have the

Theorem [CLZ] : Let d and r equivalent metrics on X ; then the Wijsman topology generated by r is stronger on $c_0(X)$ than that generated by d if and only if for every x in X and $0 < e < a$ there exists $x_1,\dots,x_n$ in X and $0 < d_i < s_i$ $(i = 1,\dots,n)$ with

$$B^d_e(x) \subset \bigcup_{i=1}^{n} B^r_{d_i}(x_i) \subset \bigcup_{i=1}^{n} B^r_{s_i}(x_i) \subset B^d_a(x)$$

where $B^d_e(x)$ is the d-open ball of center x and radius e.

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As a corollary we obtain that on the hyperspace of a normed space two Wijsman topologies generated by equivalent non homothetic norms agree if and only if the space is finite dimensional.

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WEAK TWO-LEVEL OPTIMIZATION PROBLEMS AND TYKHONOV REGULARIZATION

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Let U and V be two finite dimensional euclidian spaces, X a non empty subset of U, K a multifunction from U to the nonempty subsets of V, f_1 and f_2 two functionals defined on XxV and valued in $\mathbb{R}$. We consider the following two-level optimization problem (with explicit constraints in the lower level problem) corresponding to a weak Steckelberg problem in which the set of solutions to the lower level is not a singleton :

Find $\bar{x}\in X$ such that: $\sup_{y\in M_2(\bar{x})} f_1(\bar{x},y) = \min_{x\in X}\sup_{y\in M_2(x)} f_1(x,y)$

(S) where $M_2(x)$ is the set of optimal solutions to the lower level problem

$P(x)$: $\inf_{y\in K(x)} f_2(x,y)$

When the problem (S) fail to have a solution, in order to obtain an approximation of (S), ε-regularizations have been considered in preceeding papers ([LO-MO.1],[LO.-MO.2],[LO.-MO.3],[LO.-MO.4],[LI.-MO.],[MA.-MO.], ...). Here, in order to trasform the problem (S) in a "better" one, different new regularizations are presented and more particularly a regularization of a Tykhonov type, which allows to substitute an ill-posed two-level problem by a quasi well-posed and approximating well-posed problem ([MO.]. Approximation results for the regularized problems and connections between the different approaches are given.

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Hypertopologies: old and new approaches

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Abstract

Two recent papers [BL1,SZ] deal with the description of the topologies on the subspace CL(X) of the closed sets of a metrizable space X. They are presented as the weakest topologies making continuous certain families of geometric functionals defined on CL(X). Not only this allows to unify the subject, but it also suggests the applications of general methods to get results in optimization, in presence of perturbations described by means of moving sets, as epigraphs, constrained sets etc. [BL2,LSS]. A new way of describing hypertopologies also outside the metric case is investigated in [LP].

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Bifurcation for variational inequalities with constraints on the derivatives

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Abstract

There are several problems which can be framed in the following abstract scheme : to study, for a given real smooth function f on a manifold V, the equations

$$\operatorname{grad}_V f(u) = 0 \qquad u \in V; \tag{1}$$

$$\mathcal{U}' = -\operatorname{grad}_V f(\mathcal{U}) \qquad \mathcal{U} : I \to V \ (I \text{ is an interval}). \tag{2}$$

For the nonsmooth case, several important theories have been developed with various aims by Brézis, Clarke, Rockafellar, Temam, among others.

The theory of subdifferential and evolution equations for convex functions (and suitable perturbations of theirs), V being a convex set, provides a classical extension of (1) and (2), which allows to study problems of differential equations and inequalities of elliptic and parabolic type.

Nevertheless, several problems related to the above mentioned ones lead to consider some classes of nonconvex constraints V.

A typical example is provided by the study of Van Karman's Inequalities, which are related to the problem of the well-clamped plate, in presence of unilateral constraints on the displacement or on the curvature. To be more precise, given a bounded open subset

Ω of $\mathbf{R}^2$, a C^2 function $F_o : \bar{\Omega} \to \mathbf{R}$ and two functions (the obstacles) $\varphi_1, \varphi_2 : \Omega \to \mathbf{R}$ with $\varphi_1 \leq 0 \leq \varphi_2$, one considers the problem

$$\text{(V.K.I.)} \qquad \begin{cases} \Delta^2 h + [u,u] = 0; \\ \int_\Omega (\Delta u \Delta(v-u) - [\lambda F_o + h, u](v-u))\, dx dy \geq 0 \quad \forall v \in \mathbf{K}; \\ \lambda \in \mathbf{R}, h \in W_o^{2,2}(\Omega), u \in \mathbf{K}, u \neq 0 \end{cases}$$

(u represents the displacement of the plate and h the Airy stress function), having defined $[u,v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}$ and

$$\mathbf{K} = \mathbf{K}_1 = \{u \in W_o^{2,2}(\Omega) \mid \varphi_1 \leq u \leq \varphi_2\}$$

or

$$\mathbf{K} = \mathbf{K}_2 = \{u \in W_o^{2,2}(\Omega) \mid \varphi_1 \leq \text{eigenvalues of } H_u \leq \varphi_2\};$$

(H_u is the Hessian matrix of u).

To study the previous problem, we may introduce suitable definitions of "subdifferential" and "lower critical point" (extending the meaning of equation (1)). Now, introducing the functional $f : W_o^{2,2}(\Omega) \to \mathbf{R}$ defined by

$$f(u) = \int_\Omega \left(\frac{1}{2}(\Delta u)^2 + \frac{1}{4}(\left[\Delta^{-2}[u,u], u\right] u\right) dx dy$$

and $\mathbf{M}_\rho$ defined by

$$\mathbf{M}_\rho = \{u \in W_o^{2,2}(\Omega) \mid \int_\Omega [F_o, u] u \, dx dy = \rho\}$$

the problem (V.K.I.) is reduced to finding the lower critical points of f on the constraint $\mathbf{V} = \mathbf{K} \cap \mathbf{M}_\rho$, as the parameter ρ varies.

The problem with assigned ρ has been studied for $\mathbf{K} = \mathbf{K}_1$, and some results are exposed in [2,4].

We wish to consider here the bifurcation problem for (V.K.I.), namely the relationship between "branches of solutions" (λ, h, u) of (V.K.I.), with ρ close to zero, and the nontrivial solutions of the corresponding "0-asymptotic" problem :

$$\text{(V.K.I.)}_\omega \qquad \begin{cases} \int_\Omega (\Delta u \Delta(v-u) - [\lambda F_o, u](v-u))\, dx dy \geq 0 \quad \forall v \in \mathbf{K}_\omega; \\ \lambda \in \mathbf{R}, u \in \mathbf{K}_\omega, u \neq 0 \end{cases}$$

where $\mathbf{K}_\omega = \overline{\bigcup_{t>0} t\mathbf{K}}$.

Notice that this problem is twofold:

I) If (λ, u) solves (V.K.I.)$_\omega$, does the "eigenvalue" λ give rise to bifurcating branches of solutions of (V.K.I.) (and *viceversa*)?

II) Do eigenvalues of $(\mathrm{V.K.I.})_\omega$ exist?

It is worth noticing that for the study of (V.K.I.), in the case of assigned ρ and in the bifurcation case as well, the "curves of steepest descent" for f on $\mathbf{K} \cap \mathbf{M}_\rho$ (which extend the meaning of equation (2)) are used.

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DUAL REPRESENTATION OF COOPERATIVE GAMES

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Abstract.

One can associate to a cooperative game, defined by a set of players $N = \{1,2,\dots,n\}$ and the characteristic function $v : 2^N \to \mathbb{R}$ (assumed to be non decreasing with respect to inclusion and to satisfy $b(\emptyset) = 0$), the function $\pi : R^n_+ \to \mathbb{R}$ given by $\pi(w) = \max_S \{v(S) - \sum_{i\in S} w_i\}$ for all $w = (w_1,\dots,w_n) \in \mathbb{R}^n_+$. Interpreting the components of w as wages for the players, π associates to each possible set of wages the net profit an employer can get hiring some set of players according to these wages and then obtaining the value they generate by acting as a coalition. Clearly, π is a nonincreasing polyhedral convex function such that $\inf_w \pi(w) = 0$ and, at any $w \in R^n_+$, the subdifferential $\partial\pi(w)$ intersects the set $\{0,-1\}^n$. The function π provides exactly the same information as the characteristic function v, since one has $v(S) = \inf_w \{\pi(w) + \sum_{i\in S} w_i\}$; it is worth noticing that, for this equality to hold, no assumption on v is needed. The interpretation of this expression is the following : the value of a coalition is the total amount of agents (i.e., the players and the employer) are sure to obtain whatever the wages may be. If v is superadditive, the function π satisfies the condition $\pi(w) \geq \inf_{w_{N\setminus S}} \pi(w_S,w_{N\setminus S}) + \inf_{w_S} \pi(w_S,w_{N\setminus S})$ for all $w \in \mathbb{R}^n_+$ and $S \subset N$, and conversely. For arbitrary games, the core coincides with the set of vectors $x \in \mathbb{R}^n_+$ such that $\pi(x) = 0$ and $(-1,\dots,-1) \in \partial\pi(x)$.

ERGODIC THEORY AND THE CALCULUS OF VARIATION

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Let $(\Sigma, \mathcal{T}, P)$ be a probability space, Ω a bounded regular open set in $\mathbb{R}^d$, $M(\mathbb{R}^d)$ the set of non negative regular Borel measure on $\mathbb{R}^d$ equipped with its Borel field $\mathcal{B}(\mathbb{R}^d)$. $M(\Omega)$ is the set of non negative bounded regular Borel measure on Ω.

A Random Borel measure is any map from Σ into $M(\mathbb{R}^d)$ which is $(\mathcal{T}, \mathcal{M})$ measurable, $\mathcal{M}$ denoting the trace of the product tribe of $\mathbb{R} \cup \{+\infty\}^{\mathcal{B}(\mathbb{R}^d)}$.

Given $(T_z)_{z \in S}$ a group of P-preserving transformation on Σ where S is any subgroup $k\mathbb{Z}^d$ of $(\mathbb{Z}^d, +)$, and μ a Random Borel measure satisfying, for every bounded set A in $\mathcal{B}(\mathbb{R}^d)$, the two following properties

$$\mu(\omega)(A+z) = \mu(T_{-z}\omega)(A)$$

$$\omega \rightarrow \mu(\omega)(A) \text{ belongs to } L^1(\Sigma, \mathcal{T}, P),$$

we define the sequence μ_n from Σ into $M(\Omega)$ by

$$\mu_n(\omega)(A) := \varepsilon_n^d \, \mu(\omega)\left(\frac{1}{\varepsilon_n} A\right)$$

where ε_n tends to 0^+.

Using Ergodic Theory and more precisely, an additive ergodic theorem due to Nguyen Xuan Xanh and H. Zessin, we prove the following result

THEOREM 1.

(i) If a.s., $\{ \mu_n(\omega) ; n \in N \}$ is tight then a.s., $\mu_n(\omega)$ converges for the narrow topology towards $\theta(\omega)\, dx$ where

$$\theta(\omega) := \frac{1}{k^d} E^{\mathcal{F}} \mu(.)([0,k[^d),$$

$E^{\mathcal{F}}$ denoting the conditional expectation operator with respect to the tribe $\mathcal{F} := \{E \in \mathcal{T}, T_{-z}E = E \;\; \forall z \in S\}$;

(ii) if $(T_z)_{z\in S}$ is Ergodic, that is to say if $\mathfrak{F}$ contains only sets of $\mathcal{T}$ with probability 0 or 1 then a.s., $\mu_n(\omega)$ converges for the narrow topology towards θ dx where

$$\theta := \frac{1}{k^d} \, E\,\mu(.)([0,k[^d) .$$

In the case where $\mu(\omega) = f(\omega,.)\,dx$, $f(\omega,.)$ belonging to $L^p_{loc}(\mathbb{R}^d)$, $1\leq p\leq +\infty$, we get under suitable measurability hypothesis, the following stronger result.

THEOREM 2. Setting $f_n(\omega,x) := f(\omega,\frac{x}{\varepsilon_n})$, we have

(i) in the case $1 < p \leq +\infty$, a.s. $f_n(\omega,.)$ converges towards $\frac{1}{k^d} E^{\mathfrak{F}} \int_{]0,k[^d} f(.,x)dx$ in $L^p(\Omega)$ weak (* weak if $p = +\infty$);

(ii) in the case $p = 1$, when $(T_z)_{z\in S}$ is Ergodic, a.s. $f_n(\omega,.)$ converges towards $\frac{1}{k^d} E \int_{]0,k[^d} f(.,x)dx$ in $L^1(\Omega)$ weak .

These two results have been used to solve stochastic homogenization problems, in particular to construct suitable random test functions and random Radon measure in nonconvex stochastic homogenization and in a stochastic version of Darcy's Law .

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VARIATIONAL TOPOLOGICAL PROPERTIES OF THE SPACE OF DIRICHLET FORMS

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Abstract.

We study compactness and closure properties of families of iocal and nonlocal Dirichlet forms with respect to suitable variational topologies.

Generalized second-order derivatives of convex functions in locally convex topological vector spaces

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Abstract

Generalized second-order derivatives introduced by T. Rockafellar in the finite dimensional setting are extended to convex functions defined on locally convex topological spaces. The main result which is obtained is the exhibition of a particular generalized Hessian when the function admits a generalized second derivative

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Graphical convergence and generalized second derivatives for nonsmooth functions

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Abstract

Second order epi derivatives for convex and nonsmooth functions, as introduced by R.T. Rockafellar around 1987, have emerged in an important role in nonsmooth analysis and optimization. Here we present some results obtained recently by D. Noll [3], [4], and J. Borwein and D. Noll [1]. In particular, we address the following questions:

- Give a refined analysis of the interrelation between second epi derivatives and the usual pointwise second derivatives.
- Give a geometric characterization of second order epi differentiability. A nonsmooth version of the Theorem of Meusnier.
- In the convex case, give a link between second epi derivatives and approximate second derivatives in the sense of J.B. Hiriart-Urruty. How to define the Dupin indicatrix using second epi derivatives?
- The infinite dimensional dilemma! What is the right notion of convergence for the second order difference quotients of non-convex functions in Hilbert space?

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Approximation by convex functions

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Abstract

Consider the following problem: let X be a space of real functions, endowed with a norm; for $f \in X$, consider the function(s) which best approximate f from the class of convex functions. Also, if C denotes the subset of X containing all continuous functions, let us consider the same problem by using as approximants the elements of the set $A \cap C$. The sets A and $A \cap C$ are cones; in case the norm if X is "good" and A (or $A \cap C$) is closed, then some obvious results spring from the general theory of approximation, while simple characterizatiuons of best approximations are possible. But in different situations, some problems can arise, for example concerning convergence of sequences which best approximate f with approximating norms; in this context, many papers appeared recently. Here we try to survey some results of this type.

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Convergent and divergent concepts from convergence theory and their uses in optimization theory

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Abstract

We endeavour to give a general account of the impact of convergence notions for optimization theory. Whether these tools can be considered as a convergent array of concepts or as a divergent and diverse bundle of tools is still a matter of taste or opinion. At least it cannot be denied that the number of topics in optimization theory for which notions of convergence become instrumental is increasing and covers a large extend of subjects: approximation of functions and of sets, optimality conditions, sensitivity analysis, well-posedness... A number of recent or less recent contributions of the author to this stream are reviewed. The appearance of "alien terms" in optimality conditions is pointed out as it corresponds to a well known phenomenon in homogenization theory.

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Calculus of epi-derivatives and proto-derivatives

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Abstract

A calculus of first- and second-order epi-derivatives is presented for amenable and fully amenable functions (all functions are considered on a finite dimensional space). An amenable function is the composition of a lower semicontinuous proper convex function with a smooth mapping and a natural constraint qualification. A fully amenable function is an amenable function where the convex function is piecewise linear-quadratic and the smooth function is of class C^2. The calculus rules have direct applications in optimization because most problems encountered in practice can be reformulated using fully amenable functions; in addition, first- and second-order optimality conditions can be derived using epi-derivatives. Finally, calculus rules for the proto-derivatives of subgradient mappings of fully amenable functions are presented. These calculus rules for proto-derivatives have applications in the sensitivity analysis of optimal solution mappings in parametric optimization.

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Solving Nonsmooth equations via Generalized Jacobians and Iteration Functions

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Abstract

Systems of nonlinear equations defined by nondifferentiable, locally Lipschitzian functions arise from many different areas including partial differential equations, nonlinear complementarity, variational inequality, nonlinear programming and maximal monotone operator problems [2]. The generalized Jacobians of the functions defining these nonsmooth equations are set-valued. Superlinearly convergent Newton's methods for solving nonsmooth equations are constructed via generalized Jacobians [3] [4]. A characterization of superlinear convergence under the condition of semismoothness extends the classic results Dennis-Moré for smooth equations [2]. Broyden-type methods are also developed for solving nonsmooth equations [1]. Global convergent methods are established via iteration functions.

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Necessary and Sufficient Conditions for the Existence of Densely Defined Selections of Multivalued Mappings

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Abstract

Let $F: X \to Y$ be a multivalued mapping with non-empty images acting from the topological space X into the topological space Y (i.e. for each $x \in X$ the value $F(x)$ is a non-empty subset of Y). We give here sufficient (and necessary) conditions for the existence of an upper semicontinuous and non-empty-compact-valued (usco) mapping $G: X_1 \to Y$, where X_1 is a dense G_δ-subset of X and G is a selection of F. In contrast to what is widely accepted under "*a selection of F on X_1*" we understand here a set-valued mapping $G: X_1 \to Y$ such that $\emptyset \neq G(x) \subset F(x)$ for every $x \in X_1$. In some particular cases the selection G coincides with the restriction of F on X_1. *If the range space Y is completely metrizable*, then G can be considered to be single-valued.

More precisely, we call F *lower demicontinuous* (ldc) in X if for every open V in Y, the interior of the closure of the set $F^{-1}(V) := \{x \in X : F(x) \bigcap V \neq \emptyset\}$ is dense in the closure of $F^{-1}(V)$, i.e. $\mathrm{IntCl}(F^{-1}(V))$ is dense in $\mathrm{Cl}(F^{-1}(V))$. The class of ldc mappings contains for example all lower semicontinuous mappings and all minimal usco mappings. Several results are proved asserting that a given ldc mapping F admits an usco selection defined on a dense subset of X. A typical theorem reads as follows:

Let $F: X \to Y$ be an open ldc mapping with closed graph, where X is a Baire space and Y contains a dense Čech complete subspace Y_1. Then there exist a dense G_δ-subset X_1 of X and an usco mapping $G: X_1 \to Y_1$ which is a selection of F. If, in addition, Y_1 is completely metrizable, then G can be considered to be single-valued.

In fact, the existence of such kind of selections for every mapping from the above class is a characterization of the fact that the space Y contains a dense Čech complete subspace or a dense completely metrizable subspace.

The above result generalizes some of the results in the recent paper of E.Michael [M].

We describe also situations in which the mapping F itself (not merely a selection of it) is usco at the points of a dense G_δ-subset of X. Here a typical result says that, if $F: X \to Y$ has a closed graph, X is a Baire space, Y is Čech complete and for every open $V \subset Y$ the interior of the set $\{x \in X \, olon \, F(x) \subset V\}$ is dense in $F^{-1}(V)$, then there exists a dense G_δ-subset X_1 of X at the points of which F is usco. If, in addition, Y is completely metrizable then F is single-valued at the points of X_1.

These results are used to obtain some new as well as known results like:

-to get a new version of the classical Lavrentieff theorem concerning the extension of a densely defined homeomorphism to a subset containing a dense G_δ-subset of the domain space;

-to prove that a given convex and continuous function is differentiable at the points of some dense G_δ-subset of its domain ([Ph]);

-to obtain generic results about well-posed optimization problems ([LP], [ČK1], [ČK2], [ČKR]);

-to derive results about generic non-multivaluedness of metric projections and antiprojections ([Zh]);

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TOPOLOGICAL DEGREE FOR MAXIMAL MONOTONE OPERATORS

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Abstract

The generalized degree theory is a replacement for the Brouwer and Leray-Schauder degrees. The purpose of this work is two-fold. One goal is to show how Browder's degree, given for operators of class (S_+), see [4], can be extended to the case of maximal monotone operators by relying on generalized Yosida approximations. Particular attention is paid to the normalization and invariance under homotopies for the topological degree we define. This allows us to extend some recent results of Attouch, Penot and Riahi [2] about the continuation method for solutions of parametrized monotone nonlinear equations by withdrawing the compactness assumptions. It is also possible that our definition, by relying on subdifferentials, could be used to establish definitions of topological degree for real convex functions and convex-concave *saddle bifunctions*.

1- Let be given a real reflexive Banach space X with the topological dual space X^*, such that X and X^* are locally uniformly convex. This implies that the duality mapping J of X is a homeomorphism between X and X^*.

In the sequel we will identify an operator $A: X \rightrightarrows X^*$ *with its graph in* $X\times X^*$. An operator A is said to be monotone if for any $(x_i, y_i) \in A$, one has $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$. A is <u>maximal monotone</u> if it is maximal in the family of monotone operators in $X\times X^*$, ordered by inclusion.

The Yosida approximation is given by $A_\lambda x = (A^{-1} + \lambda J^{-1})^{-1}$. An extension, called <u>generalized Yosida approximation</u>, we'll use is given by $A^\lambda = A_\lambda + \lambda J$. A^λ is of class (S_+).

<u>**Theorem 1.**</u> If a family of maximal monotone operators $\{A_t \subset X\times X^*)\,;\, t\in T\}$ is graph-continuous (given $t_i \to t$ in T one has $\operatorname{Limsup}_{i\in I} A_{t_i} \subset A_t \subset \operatorname{Liminf}_{i\in I} A_{t_i}$) then for any open bounded subset $\Omega \subset X$, the family $\{(A_t)^\lambda : \bar{\Omega} \to X^*\,;\, t\in T, \lambda>0\}$ is of class (S_+).

2- Now, we are in a position to define the topological degree for a maximal monotone operator A over Ω at zero by the formula :

$$\deg(A,\Omega,0) = \lim_{\lambda \downarrow 0} d\ (A^{\lambda},\Omega,0). \qquad (1)$$

where d denotes the Browder's degree for operators of class (S_+), we refer to [4] for more details. We verify that in the definition above, the degree function deg is independent of $\lambda>0$ for λ sufficiently small. The following Theorem summarizes familiar properties of degree theory for maximal monotone mappings.

Theorem 2. Let Ω be an open bounded subset, $A \subset X\text{x}X^*$ be maximal monotone. Then we have : (i) $\deg(J,\Omega,0) = 1$, provided $0 \in \Omega$;

(ii) $\deg(A,\Omega,0) = 0$ whenever $0 \notin A(\bar{\Omega})$;

(iii) if the homotopy of maximal monotone operators $\{A_t \subset X\text{x}X^*;); t \in T\}$ is graph continuous and satisfies $0 \notin \cup\{A_t(\partial\Omega) ; t \in T\}$, then $\deg(A_t, \Omega, 0)$ is independent of t in T ;

(iv) if Ω_1 and Ω_2 are two disjoint subsets of Ω such that $0 \notin A(\bar{\Omega}\backslash\Omega_1 \cup \Omega_2)$, then

$$\deg(A, \Omega, 0) = \deg(A, \Omega_1, 0) + \deg(A, \Omega_2, 0).$$

(v) On the family of maximal monotone operators, there exists one and only one degree function, with the invariance under graph-continuous homotopies.

As a consequence of this theorem we shall sharpen and extend [2] results for Hilbert to reflexive spaces, as well as ruling out the compactness conditions.

Corollary 3. Suppose that assumptions of the preceding theorem (iii) hold and the following condition is satisfied : for some $t_0 \in T$ one has $\deg(A_{t_0}, \Omega, 0) \neq 0$.

Then for each t in T one has $A_t^{-1}(0)$ is nonempty and contained in Ω.

3- In this brief section we apply the results of section 2 to convex functions and convex- concave saddle bifunctions.

A- Since the subdifferential $\partial\phi$ of a proper lsc function ϕ is a maximal monotone operator, see [1, 5] one can define the degree at 0 relatively to an open bounded subset of X by :

$$\deg(\phi, \Omega, 0) = \deg(\partial\phi, \Omega, 0). \qquad \textbf{(2)}$$

Proposition 4. a) For a proper convex lsc function ϕ, $\deg(\phi, \Omega, 0) \neq 0$ and $M(\phi) \cap \partial\Omega = \emptyset$ implies that $\emptyset \neq M(\phi) \subset \Omega$, where $M(\phi)$ denotes the minimum set of the function ϕ.

b) Let $\{\phi_t ; t\in T\}$ be a Mosco-epicontinuous family of proper convex lsc functions such that $M(\phi_t)\cap\partial\Omega=\emptyset \quad \forall\, t\in T$ and $\deg(\phi_{t_0}, \Omega, 0)\neq 0$ for some $t_0\in T$.

Then for each t in T one has $\emptyset\neq M(\phi_t)\subset\Omega$.

B- For a closed convex-concave saddle bifunction $F : X\text{x}Y \to \overline{\mathbb{R}}$ which is convex lsc (resp. concave usc) with respect to the variable x in X (resp. y in Y), the operator $A = \partial_1 F\text{x}(-\partial_2 F)$, where $\partial_1 F$ and $\partial_2 F$ denote the partial subdifferentials of the convex functions F(.,y) and −F(x,.), see [6], is maximal monotone .

The degree of F at (x,y) is defined as follows $\deg(F,\Omega,(x,y)) = \deg(A,\Omega,(x,y))$.
With the above definition one can state the analogue of Proposition 4 for bifunctions :

<u>Proposition 5</u>. a) $\deg(F,\Omega,0)\neq 0$ implies that $S(F)=\{\text{saddle points of F in XxY}\}\cap\Omega\neq\emptyset$.
b) Let $\{F_t ; t\in T\}$ be a Mosco-epi/hypocontinuous homotopy such that $\deg(F_{t_0},\Omega,0)\neq 0$ for some $t_0\in T$ and $S(F_t)\cap\partial\Omega=\emptyset$ for each t in T. Then for every $t\in T$ one has $\emptyset\neq S(F_t)\subset\Omega$.

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Some Topological Min-Max Theorems via an Alternative Principle for Multifunctions

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Abstract

Let X, Y be two non-empty sets and let f be a real function defined on $X \times Y$. We are interested in the classical problem of finding suitable conditions under which the equality

$$(1) \qquad \sup_{y \in Y} \inf_{x \in X} f(x,y) = \inf_{x \in X} \sup_{y \in Y} f(x,y)$$

does hold.

In this talk we intend to discuss some of our recent results on this subject. Here is a sample:

THEOREM - *Let X, Y be topological spaces, with Y connected and admitting a continuous bijection onto $[0,1]$. Assume that, for each $\lambda > \sup_{y \in Y} \inf_{x \in X} f(x,y)$, $x_0 \in X$, $y_0 \in Y$, the sets*

$$\{x \in X : f(x, y_0) \leq \lambda\}$$

and

$$\{y \in Y : f(x_0, y) > \lambda\}$$

are connected. In addition, assume that at least one of the following three sets of conditions is satisfied:

(h_1) *$f(x,\cdot)$ is upper semicontinuous in Y for each $x \in X$, and $f(\cdot, y)$ is lower semicontinuous in X for each $y \in Y$;*

(h_2) *Y is compact, and f is upper semicontinuous in $X \times Y$;*
(h_3) *X is compact, and f is lower semicontinuous in $X \times Y$.*
Under such hypotheses, equality (1) does hold.

Just one remark. Namely, the condition that Y must admit some continuous bijection onto [0,1] is, of course, very restrictive. This is, in practice, the necessary price one has to pay for the generality of the other assumptions. In this connection, consider that, almost every known topological mini-max theorem, ensuring the validity of (1), contains at least the following assumptions: the space X is compact and, for each $\lambda > \sup_{y\in Y}\inf_{x\in X} f(x,y)$ and each non-empty finite set $A \subseteq Y$, the set

$$\bigcap_{y\in A}\{x \in X : f(x,y) \leq \lambda\}$$

is connected (see [1]-[8]).

The fact that the above theorem can badly fail even when Y is only a "little" out of the considered class, is clearly shown by the following

EXAMPLE - Take:

$$X = Y = \{(t,u) \in \mathbf{R}^2 : t^2 + u^2 = 1\}$$

and, for each $(t,u),(v,z) \in X$,

$$f(t,u,v,z,) = tv + uz.$$

In this case, of course, we have

$$-1 = \sup_{y\in Y}\inf_{x\in X} f(x,y) < \inf_{x\in X}\sup_{y\in Y} f(x,y) = 1$$

while, except that on Y, each other assumption of the theorem is satisfied.

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Graph Convergence in Mathematical Programming

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Abstract

Methods using graph convergence, particularly the proto-derivatives introduced by R.T. Rockafellar [3], seem to be the most natural tools available for obtaining certain optimality conditions in mathematical programming (see, *e.g.*, [4]). On the other hand, once the optimality conditions are written, the analysis of solutions to those conditions generally proceeds using very different tools, often some variety of implicit-function theorem.

For such analysis the formalism of *normal maps* [2] leads to equations involving single-valued, though generally nonsmooth, functions. For example, the theorem of [2] together with the functional form first introduced by Kojima [1] permits one to conduct a complete local analysis in very much the same way as one would do for a system of nonlinear equations. Noteworthy in this analysis is the complete absence of any graph convergence methodology.

In this lecture we will investigate connections between the graph convergence methodology for obtaining optimality conditions and the analytical methods for exploiting the mathematical structure of those solutions once they are obtained. We will describe the normal-map approach and ask whether any natural relationships exist between the normal map derived from the optimality conditions and the graph-convergence techniques that lead to those conditions.

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Proto-differentiability of solution mappings in optimization

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Abstract

The most commonly used model for sensitivity analysis of solution mappings in parameterized problems of optimization is the generalized equation model of Robinson, which corresponds closely to optimality conditions in the sense of variational inequalities. Much of the work on generalized equations has centered on obtaining conditions under which the solution is unique and depends Lipschitz continuously on the parameters, but one-sided differentiability of the single-valued mapping obtained under such circumstances has been studied as well. In contrast, proto-differentiability properties can be developed even when the solution mapping is multi-valued. One-sided differentiability is then simply the case where the mapping does happen to be single-valued and locally Lipschitz, which in finite-dimensional spaces can be characterized through Mordukhovich's criterion on coderivatives. Furthermore, proto-differentiability also is an effective tool in broader models in sensitivity analysis formulated in terms of subgradient mappings instead of generalized equations. In that setting it relates to the calculation of perturbations through the solution of auxiliary problems of optimization involving second-order epi-derivatives of functions in the original problem.

APPROXIMATE EULER-LAGRANGE INCLUSION, APPROXIMATE TRANSVERSALITY CONDITION, AND SENSITIVITY ANALYSIS OF CONVEX PARAMETRIC PROBLEMS OF CALCULUS OF VARIATIONS.

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ABSTRACT. We study the first-order behaviour of the optimal value function associated to a convex parametric problem of calculus of variations. An important feature of this paper is that we do not assume the existence of optimal trajectories for the unperturbed problem. The concepts of approximate Euler-Lagrange inclusion and approximate transversality condition are key ingredients in the writing of our sensitivity results.

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Key words : Euler-Lagrange inclusion, transversality condition, approximate subdifferential, sensitivity analysis.

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Swimming below icebergs, and the maximal monotonicity of subdifferentials

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Abstract

Let E be a Banach space, $\phi : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous and Pr_2 be the projection map from $E \times \mathbb{R}$ to $\mathbb{R}$. If B is a nonempty subset of $E \times \mathbb{R}$ such that $(x, \lambda) \in B \implies \lambda \leq \phi(x)$ and $\sup \mathrm{Pr}_2(B) > \phi(E)$ then we write

$$[B \downarrow \phi] := \sup_{(x,\lambda)\in B, \phi(y)<\lambda} \frac{\lambda - \phi(y)}{||x - y||}$$

If B is a rock and ϕ is the bottom of an iceberg, $[B \downarrow \phi]$ tells you at what slope you have to swim down starting from an arbitrary point on the rock and still be sure that you will not hist the iceberg. There are significant situations in which $0 < [B \downarrow \phi] < \infty$. We can deduce from this a number of results on the existence of subtangents to ϕ satisfying various conditions, with sharp lower bounds on the slopes of the subtangents. One of them improbes a recent result of Beer on separating subtangents, and another improves a recent result of Beer on separating subtangents, and another improves a recent result of Attouch and Beer on the approximation of conjugate functions. Our results also lead to generalizations of Rockafellar's fundamental result that $\partial\phi$ is *maximal monotone* , that is to say,

$$if; q \in E,\ a \in E'\ and\ \forall(z, b) \in \partial\phi,\ < q - z, a - b > \geq 0\ then\ (q, a) \in \partial\phi.$$

For instance, we can prove that : if Q is a nonempty weakly compact convex subset of E, $a \in E'$ and,

$$\forall(z, b) \in \partial\phi,\ \exists q \in Q \text{such that} < q - z, a - b > \geq 0$$

then

$$\exists q \in Q \text{ such that } (q, a) \in \partial\phi.$$

Dually, we can prove that : if A is a nonempty weak* compact convex subset of $E', q \in E$ and,

$$\forall (z, b) \in \partial\phi, \exists q \in Q \text{such that} < q - z, a - b > \geq 0$$

then

$$\exists a \in Q \text{ such that } (q, a) \in \partial\phi.$$

Since some of these result are quite technical, in the talk we will simply show how the techniques can be used to give a very short proof that $\partial\phi$ is maximal monotone.

Set convergences: a survey and a classification

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Abstract

Let (E,d) be a metric space and $\mathfrak{F}$ be the class of nonempty and closed subsets of E. By a *set convergence* we mean a procedure that associates an element $A \in \mathfrak{F}$ to a net $(A_i)_{i\in I} \subset \mathfrak{F}$. The number of these convergences continues to increase: there are more than 35 (and, of course, the same number of epigraphical convergences for nets of functions $f_i:E \to \overline{\mathrm{IR}}$ and graphical convergences for nets of operators $\Phi_i:E \to E$). This proliferation is justified by the theory and by the applications, but the theory becomes more and more complicated and the perplexity of the potential user is comprehensible ... The difficulty comes from the disparate aspect of definitions that does not make easy a global view of convergences: having in mind all the results concerning the relative fineness between them causes already problems. A solution consists to modify the original definitions of convergences using only few mathematical notions that can be easily compared (inequalities, inclusions for example).

We proposed the following method in [SZ]: each convergence corresponds to a semi-metric space defined by a couple formed by a family $\mathfrak{X} \subset \mathfrak{F}$ (the *class* of the convergence) and a function $f:\mathfrak{F} \times \mathfrak{X} \to \mathrm{IR}$ or $g:E \times \mathfrak{F} \to \mathrm{IR}$ (the *type* of the convergence). The semi-metric space is defined by the semi-metrics

$$(f_X)_{X\in\mathfrak{X}}, \text{ where } f_X(A,B) = |f(A,X) - f(B,X)| \text{ for } A,B \in \mathfrak{F},$$

or

$$(g_X)_{X\in\mathfrak{X}}, \text{ where } g_X(A,B) = \sup_{x\in X}|g(x,A) - g(x,B)| \text{ for } A,B \in \mathfrak{F}.$$

It is also possible to take several families of such semi-metrics. The natural choices $f(A,X) = d(A,X)$ (the type p) and $g(x,A) = d(x,A)$ (the type q) give the possibility to *redefine* the most part of known convergences. One can use results of G. Beer (for type p), of B. Cornet (for type q) and of the authors. The convergences can be written simply:

$$\mathfrak{X}(f)\text{-}\lim(A_i) = A \text{ iff } \lim f(A_i,X) = f(A,X) \quad \forall\ X \in \mathfrak{X},$$

$$\mathfrak{X}(g)\text{-}\lim(A_i) = A \text{ iff } \lim[\sup_{x\in X}|g(x,A_i) - g(x,A)|] = 0 \ \forall\ X \in \mathfrak{X}.$$

The classification of convergences becomes now very simple and is based on the inequality $p_X \le q_X$ and on evident inclusions of classes (compact sets $\subset$ closed bounded sets $\subset \mathfrak{F}$, etc ...).

Remark that there are many convergences of type p, but very few, till now, of type q: Hausdorff, Attouch-Wets and Wijsman.

Of course, one can define other types taking for f and g other canonical functions associated to (E,d): for example the Chebyshev radius, the Hausdorff excess function, etc...

G. Beer and R. Lucchetti developed the study of types corresponding to the Hausdorff excess function, giving so a good framework for fine and very fine convergences.

The papers of [SZ] and [BL] (and some others), which represent the base for our talk, give a clear and coherent vision of the majority of set convergences. We hope that this presentation will facilitate the work of ès-convergences students and the task of users (optimization, multivalued functions, probability, etc...) in choosing the adequate convergence(s) for their own study.

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A new Topology for the Solid Sets of a Topological Space

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Abstract

In the paper [LTW] a new hypertopology τ_r is defined with the aim of identifying classes of sets where narrow convergence of a sequence of probability measures implies uniform convergence (see also [BT, LSW] and [SW] for applications).

The result can be expressed in this way: in any τ_r-compact class $\mathcal{A}$ of sets which is contained in the family of the continuity sets of a probability P, narrow (i.e. pointwise) convergence of P^n to P implies uniform convergence of P^n to P on the class $\mathcal{A}$. Also, it turns out that the condition of A being a P-continuity set (i.e $P(\partial A) = 0$) is necessary as well as sufficient for the function $P : (CL(X), \tau_r) \longrightarrow [0,1]$ being continuous at A. The paper in vestigates the topological properties of τ_r on the subset of the sets that are the closure of their interior. In particular, it is shown that the hyperspace is metrizable if and only if the space X is separable and locally compact.

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Strong convergence implied by weak in L^1

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Abstract

The Visintin's theorem and its extension by E.J. Balder have been extended to infinite dimensional Banach spaces by several authors. The best conclusion is obtained when some strong compactness is assumed. But the weak topology is also considered. Some of the results can be proved without Young measures. Then truncations are essential parts of proofs and are related to some extensions of the tightness condition.

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Remarks about the Infimum of Hausdorff Metric Topologies

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Abstract

We consider a metrisable topological space X and denote by D the set of all compatible metrics on X. For every d in D, the upper and lower Hausdorff topologies - H(d,+) and H(d,-), respectively - are defined on the collection C(X) of all closed subsets of X. (See [1]). Let P be the infimum of the topologies of the form H(d,+), where d runs over D, and M the infimum of the topologies of the form H(d,-). We show that M coincides with the lower Vietoris topology, if and only if X separable (compare with [3], cor. 6). We also show that (for locally compact X) P coincides with the co-compact topology (see [2]) if and only if X is compact.

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THE USE OF MONOTONE NORMS IN CONVEX ANALYSIS

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Abstract. We introduce some new binary operations for convex sets and convex functions. These operations provide a general framework for dealing with the calculus of epigraphs, polar sets, and Fenchel conjugates.

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Key words : monotone norms, convex duality, epigraphical analysis.

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Convergence of Integral Functionals

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Abstract

Let $(T, \mathcal{A}, \mu)$ be a measure space with σ-additive, nonnegative measure μ, and $\mathcal{X}$ a Banach subspace of $\mathcal{M}(T; \mathbb{R}^n)$, the space of measurable functions from T to $\mathbb{R}^n$. One refers to a bivariate function $f : T \times \mathcal{X} \to \overline{\mathbb{R}}$ as an *integrand* if for all $x \in \mathcal{M}(T; \mathbb{R}^n)$, the function $t \mapsto f(t, x(t))$ is $\mathcal{A}$-measurable. We are interested in *integral functionals* of the type:

$$F : \mathcal{X} \to \overline{\mathbb{R}}, \qquad F(x) := \int_T f(t, x(t))\,\mu(dt).$$

The need for approximation techniques to deal with integral functionals that arise in the calculus of variations, stochastic optimization and other variational problems (pde), provided at least initially, the major motivation for the development of the theory of *epi-convergence* and its extensions (epi/hypo-convergence, Γ-convergence, etc.). This lecture will try to take stock of the progress made during the last decade or so in dealing with the convergence of integral functionals.

More specifically, let $\{f^\nu : T \times \mathbb{R}^n \to \overline{\mathbb{R}}, \nu \in \mathbb{N}\}$ be a sequence of integrands, $\{\mu^\nu : \mathcal{A} \to \mathbb{R}_+, \nu \in \mathbb{N}\}$ a sequence of measures, and F^ν the associated sequence of integral functionals

$$F^\nu(x) := \int_T f^\nu(t, x(t))\,\mu^\nu(dt), \qquad \nu \in \mathbb{N}.$$

Assuming that the f^ν and μ^ν approximate f and μ in some sense, what can be said about the convergence of the integral functionals F^ν to F? In particular, when can we guarantee the epi-convergence of the F^ν to F?

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ON THE CONVERGENCE OF THE GENERALIZED GRADIENTS

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EXTENDED ABSTRACT for the conference " Convergences en Analyse Multivoque et Unilatérale " ; Marseille-Luminy , June 1992.

We consider a real Banach space E , and sequences of locally Lipschitz functions

$$f \, , f_n \; : \; E \to \; (-\infty,+\infty) \, , \; n = 1,2,3,\ldots \, .$$

We address the problem, under which conditions on the convergence of f_n towards f, it is possible to obtain some form of convergence of the sequence of their Clarke's generalized gradients

$$\partial f_n \; \to \; \partial f \; .$$

As an application (and a motivation), we mention the stability of the Clarke multiplier theorem [1] , which is connected to the stability under perturbations of sensitivity estimates for constrained optimization problems under (in)equality constraints.

If each term f_n of the sequence is a proper, convex, lower semicontinuous function, the problem above has been thoroughly investigated. Necessary and sufficient conditions, linking epi - convergence of f_n toward f with the Kuratowski convergence of the graphs of the generalized gradients are known : see [2] , [3], [4] , [5] . The bounded Hausdorff convergence of f_n is also related to the convergence of the subgradients, see [6].

If E is a finite - dimensional space , sufficient conditions are known yielding

$$(1) \qquad \limsup \partial f_n(y) \subset \partial f(x), \text{ as } n \to +\infty \text{ and } y \to x,$$

for every $x \in E$, again under epi - convergence of f_n to f , see [7].

If E is a Hilbert space , and each f_n is (p,q) -convex, then results linking epi-convergence of the functions with various Kuratowski convergences of their lower semigradients are proved in [8].

If E is infinite - dimensional and fulfills some regularity conditions, it is possible to extend (1) under epi-convergence of the sequence f_n . No convexity is required, and an equi - lower - semidifferentiability condition is imposed (as in [7]) on the sequence f_n . The proof makes use of convergence results for lower semigradients of lower semicontinuous functions.

This allows us to pass to the limit in the Clarke's multipliers rule in the infinite - dimensional setting, when data perturbations are present. In this way we obtain sufficient conditions for the stable behavior of the multipliers, under epi - convergence conditions.

Further applications to the behavior of the generalized gradients of integral functionals are also possible using the above result.

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