CALCULUS &
ANALYTIC GEOMETRY

B Sc. MATHEMATICS

2011 Admission onwards

IV SEMESTER

CORE COURSE

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

CALICUT UNIVERSITY.P.O., MALAPPURAM, KERALA, INDIA – 673 635

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Layout & Settings
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MODULE I

CHAPTER 1: NATURAL LOGARITHMS

The natural logarithm of a positive number x is the value of the integral $\int_1^x \frac{1}{t}dt$. It is written as $\ln x$. i.e.,,

$$\ln x = \int_1^x \frac{1}{t}dt, \quad x > 0 \qquad \ldots(1)$$

Remarks

1. If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$.
2. For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1.
3. For $x = 1$, $\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0,$ as upper and lower limits equal.
4. The natural logarithm function is not defined for $x \le 0$.

The Derivative of *y* = ln *x*

Using the first part of the Fundamental Theorem of Calculus, for every positive value of x,

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}dt = \frac{1}{x}.$$

If u is a differentiable function of x whose values are positive, so that ln u is defined, then applying the Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

to the function $y = \ln u$ (with $u > 0$) gives

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx}$$

or simply $$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$

Problem Evaluate $\ln(x^2 + 1)$.

Solution

Using Eq.(1), with $u = x^2 + 1$,

$$\frac{d}{dx}\ln(x^2+1) = \frac{1}{x^2+1}\cdot\frac{d}{dx}(x^2+1)$$

$$= \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$$

Properties of Logarithms

For any numbers $a > 0$ and $x > 0$,

1. $\ln ax = \ln a + \ln x$ **(Product Rule)**
2. $\ln \frac{a}{x} = \ln a - \ln x$ **(Quotient Rule)**
3. $\ln \frac{1}{x} = -\ln x$ **(Reciprocal Rule)**
4. $\ln x^n = n \ln x$ **(Power Rule)**

Theorem $\ln ax = \ln a + \ln x$.

Proof We first note that $\ln ax$ and $\ln x$ have the same derivative. Using Corollary to the Mean Value Theorem, then, the functions must differ by a constant, which means that

$$\ln ax = \ln x + C$$

for some C. It remains only to show that C equals $\ln a$.

Equation holds for all positive values of x, so it must in particular hold for $x = 1$. Hence,

$$\ln(a \cdot 1) = \ln 1 + C$$

$$\ln a = 0 + C, \qquad \text{since } \ln 1 = 0$$

$$C = \ln a \cdot$$

Hence, substituting $C = \ln a$,

$$\ln ax = \ln a + \ln x \cdot$$

Theorem $\ln \frac{a}{x} = \ln a - \ln x.$

Proof We use

$$\ln ax = \ln a + \ln x \cdot$$

With a replaced by $1/x$ gives

$$\ln \frac{1}{x} + \ln x = \ln\left(\frac{1}{x} \cdot x\right)$$

$$= \ln 1 = 0,$$

hence

$$\ln \frac{1}{x} = -\ln x \cdot$$

x replaced by $\frac{1}{x}$ gives

$$\ln \frac{a}{x} = \ln\left(a \cdot \frac{1}{x}\right) = \ln a + \ln \frac{1}{x}$$

$$= \ln a - \ln x \cdot$$

Theorem $\ln x^n = n \ln x$ (assuming n rational).

Proof: For all positive value of x,

$$\frac{d}{dx}\ln x^n = \frac{1}{x^n}\frac{d}{dx}(x^n) \text{, using Eq. (1) with } u = x^n$$

$$= \frac{1}{x^n} n x^{n-1} \text{, here is where we need } n \text{ to be rational.}$$

$$= n \cdot \frac{1}{x} = \frac{d}{dx}(n \ln x) \cdot$$

Since ln x^n and $n \ln x$ have the same derivative, by corollary to the Mean Value Theorem,

$$\ln x^n = n \ln x + C$$

for some constant C. Taking $x = 1,$ we obtain $\ln 1 = n \ln 1 + C$ or $C = 0$. Hence the proof.

The Graph and Range of ln *x*

The derivative $\frac{d}{dt}(\ln x) = \frac{1}{x}$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative , so the graph of $\ln x$ is concave down. We can estimate $\ln 2$ by numerical integration to be about 0.69 and, obtain

$$\ln 2^n = n \ln 2 > n\left(\frac{1}{2}\right) = \frac{n}{2}$$

and $$\ln 2^{-n} = -n \ln 2 < -n\left(\frac{1}{2}\right) = -\frac{n}{2}.$$

Hence, it follows that

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to\infty} \ln x = -\infty.$$

The domain of ln x is the set of positive real numbers; the range is the entire real line.

Logarithmic Differentiation

The derivatives of positive functions given by formulas that involve products, quotients, and powers can often be found more quickly if we take the natural logarithm of both sides before differentiating. This enables us to use the properties of natural logarithm to simplify the formulas before differentiating. The process, called **logarithmic differentiation,** is illustrated in the coming examples.

Problem Find $\frac{dy}{dx}$ where $y = (\sin x)^{\cos x} > 0$

Solution Given $y = (\sin x)^{\cos x}$.

Taking logarithms on both sides, we obtain

$$\ln y = \cos x \ln \sin x$$

Now differentiating both sides with respect to x , we obtain

$$\frac{d}{dx}\ln y = \frac{d}{dx}(\cos x \ln \sin x) = \frac{d}{dx}(\cos x)\ln \sin x + \cos x \frac{d}{dx}(\ln \sin x)$$

$$= -\sin x \cdot \ln \sin x + \cos x \cdot \frac{1}{\sin x}\frac{d}{dx}(\sin x)\ .$$

i.e.,
$$\frac{1}{y}\frac{dy}{dx} = \cot x \cos x - \sin x \ln \sin x$$

$\therefore$
$$\frac{dy}{dx} = y\left(\cot x \cos x - \sin x \ln \sin x\right)$$

i.e.,
$$\frac{dy}{dx} = (\sin x)^{\cos x}\left(\cot x \cos x - \sin x \ln \sin x\right).$$

Problem Find $\frac{dy}{dx}$, where $y = \sqrt{\frac{x^2+x+1}{x^2-x+1}}$.

Solution Given
$$y = \left(\frac{x^2+x+1}{x^2-x+1}\right)^{1/2}$$

Taking logarithms on both sides, we get

$$\ln y = \tfrac{1}{2}\,[\ln(x^2 + x + 1)\ \mathrm{J}\ \ln(x^2 - x + 1)].$$

Now differentiating both sides with respect to x , we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{\left(x^2+x+1\right)}\frac{d}{dx}\left(x^2+x+1\right) - \frac{1}{2}\frac{1}{x^2-x+1}\frac{d}{dx}\left(x^2-x+1\right).$$

$$= \frac{1}{2}\frac{1}{\left(x^2+x+1\right)}(2x+1) - \frac{1}{2}\frac{1}{x^2-x+1}(2x-1).$$

$$\frac{dy}{dx} = y\left\{\frac{2x+1}{2\left(x^2+x+1\right)} - \frac{2x-1}{2\left(x^2-x+1\right)}\right\}$$

or
$$\frac{dy}{dx} = \frac{1-x^2}{\left(x^2+x+1\right)^{1/2}\left(x^2-x+1\right)^{3/2}}\ .$$

The Integral $\int$(1/u)du

If u is a nonzero differentiable function,

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Proof When u is a positive differentiable function, Eq. (1) leads to the integral formula

$$\int \frac{1}{u}\,du = \ln u + C\ ,$$

If u is negative, then $-u$ is positive and

$$\int \frac{1}{u}\,du = \int \frac{1}{(-u)}\,d(-u)$$

$$= \ln(-u) + C$$

We can combine the above equations into a single formula by noticing that in each case the expression on the right is

$$\ln|u| + C\,.$$

Proof. $\ln u = \ln|u|$ because $u > 0$;

$\ln(-u) = \ln|u|$ because $u < 0$.

Hence whether u is positive or negative, the integral of $(1/u)du$ is $\ln|u| + C$. This completes the proof.

We recall that

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C\,, \qquad n \neq -1\,.$$

The case of $n = -1$ is given in Eq. (9). Hence,

$$\int u^n\,du = \begin{cases} \dfrac{u^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|u|, & n = -1 \end{cases}$$

Integration Using Logarithms

Integrals of a certain form lead to logarithms. That is,

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

whenever $f(x)$ is a differentiable function that maintains a constant sign on the domain given for it.

Problem Evaluate $\int_0^2 \frac{2x}{x^2-5}\,dx$.

Answer

$$\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u}, \text{ letting } u = x^2 - 5,\ du = 2x\,dx,\ u(0) = -5,\ u(2) = -1 \quad = \ln|u|\Big]_{-5}^{-1}$$

$$= \ln|-1| - \ln|-5| = \ln 1 - \ln 5 = -\ln 5\,.$$

Problem Evaluate $\int_{-\text{®}/2}^{\text{®}/2} \frac{4\cos\text{¯}}{3+2\sin\text{¯}}\,d\text{¯}$.

Solution

$$\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta}d\theta = \int_1^5 \frac{2}{u}du, \text{taking } u = 3+2\sin\theta\ , du = 2\cos\theta\, d\theta\ ,$$

$$u(-\pi/2)=1,\ u(\pi/2)=5.$$

$$= 2\ln|u|\Big]_1^5$$

$$= 2\ln|5| - 2\ln|1| = 2\ln 5.$$

The Integrals of tan x and cot x

Problem Evaluate $\int \tan x\, dx$ and $\int \cot x\, dx$

Answer

(i) $\int \tan x dx = \int \frac{\sin x}{\cos x}dx = \int \frac{-du}{u}$, taking $u = \cos x, \quad du = -\sin x\, dx.$

$$= -\int \frac{du}{u} = -\ln|u| + C \text{ , using Eq. (9)}$$

$$= -\ln|\cos x| + C = \ln\frac{1}{|\cos x|} + C, \quad \text{by Reciprocal Rule}$$

$$= \ln|\sec x| + C.$$

(ii) $\int \cot x dx = \int \frac{\cos x dx}{\sin x} = \int \frac{du}{u}$, taking $u = \sin x,\ du = \cos x\, dx$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

In general, we have

$$\int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C$$

$$\int \cot u\, du = -\ln|\sin u| + C = -\ln|\csc u| + C$$

Problem Evaluate $\int_0^{\pi/6} \tan 2x\, dx.$

Answer

$$\int_0^{\pi/6} \tan 2x dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} \text{, taking } u = 2x,\ dx = du/2,\ u(0) = 0,$$

$$u(\pi/6) = \pi/3$$

$$= \frac{1}{2}\int_0^{\pi/3} \tan u\, du$$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3}$$

$$= \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2.$$

Exercises

In Exercises 1-6 express the logarithms in terms of ln 5 and ln7.

1. $\ln(1/125)$ 2. $\ln 9.8$ 3. $\ln 7\sqrt{7}$ 4. $\ln 1225$ 5. $\ln 0.056$ 6. $(\ln 35 + \ln(1/7)/(\ln 25)$

In Exercises 7-12, Express the logarithms in terms of ln 2 and ln 3.

7) $\ln 0.75$ 8) $\ln(4/9)$ 9) $\ln(1/2)$ 10) $\ln\sqrt[3]{9}$ 11) $\ln 3\sqrt{2}$ 12) $\ln\sqrt{13.5}$

In Exercises 13-15, simplify the expressions using the properties of logarithms

13. $\ln\sin\text{¯} - \ln\left(\frac{\sin\text{¯}}{5}\right)$ 14. $\ln(3x^2 - 9x) + \ln\left(\frac{1}{3x}\right)$ 15. $\frac{1}{2}\ln(4x^4) - \ln 2$

In Exercises 16-25, find the derivatives of y with respect to x, t or ¯ , as appropriate**.**

16. $y = \ln 3x$ 17. $y = \ln(t^2)$ 18. $y = \ln\frac{3}{x}$ 19. $y = \ln(\text{¯} + 1)$

20. $y = \text{¯}\left(\sin(\ln\text{¯}) + \cos(\ln\text{¯})\right)$ 21. $y = \ln\frac{1}{x\sqrt{x+1}}$ 22. $y = \frac{1+\ln t}{1-\ln t}$

23. $y = \ln\left(\sec(\ln\text{¯})\right)$ 24. $y = \ln\left(\frac{(x^2+1)^5}{\sqrt{1-x}}\right)$ 25. $y = \int_{x^2/2}^{x^2} \ln\sqrt{t}\,dt$

In Exercises 26-32, use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

26. $y = \sqrt{(x^2+1)(x-1)^2}$ 27. $y = \sqrt{\frac{1}{t(t+1)}}$

28. $y = (\tan\text{¯})\sqrt{2\text{¯} + 1}$ 29. $y = \frac{1}{t(t+1)(t+2)}$

30. $y = \frac{\text{¯}\sin\text{¯}}{\sqrt{\sec\text{¯}}}$ 31. $y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}}$

32. $y = \sqrt[3]{\frac{x(x-1)(x-2}{(x^2+1)(2x+3)}}$

Evaluate the integrals in Exercises 33-41

33. $\int_{-1}^{0}\frac{3dx}{3x-2}$ 34. $\int\frac{8r\,dr}{4r^2-5}$ 35. $\int_0^{\text{®}/3}\frac{4\sin\text{¯}}{1-4\cos\text{¯}}d\text{¯}$

36. $\int_2^4\frac{dx}{x\ln x}$ 37. $\int_2^{16}\frac{dx}{2x\sqrt{\ln x}}$ 38. $\int\frac{\sec y\tan y}{2+\sec y}dy$

39. $\int_{\text{®}/4}^{\text{®}/2}\cot t\,dt$ 40. $\int_0^{\text{®}/12}6\tan 3x\,dx$ 41. $\int\frac{\sec x\,dx}{\sqrt{\ln(\sec x + \tan x)}}$

Differentiate the following expressions in Exercise 42-47 with respect to x

42. ln $6x$ 43. (ln x)^2 44. ln (tan x + sec x)

45. x^2 ln (x^2) 46. (ln x)^3 47. $x\sec^{-1}x - \ln\left(x + \sqrt{x^2-1}\right)$, $x > 1$.

CHAPTER 2 : THE EXPONENTIAL FUNCTION

In this chapter we discuss the exponential function (it is the inverse of $\ln x$) and explores its properties. Before giving formula definition we consider an example.

The Inverse of ln *x* and the Number *e*

The function $\ln x$, being an increasing function of x with domain $(0,\infty)$ and range $(-\infty,\infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty,\infty)$ and range $(0,\infty)$. The graph of $\ln^{-1} x$ is the graph of $\ln x$ reflected across the line $y = x$. Also,

$$\lim_{x\to-\infty} \ln^{-1} x = \infty \quad \text{and} \quad \lim_{x\to\infty} \ln^{-1} x = 0.$$

The number $\ln^{-1} 1$ is denoted by the letter e.

Definition $e = \ln^{-1} 1$.

Remark e is not a rational number, its value can be computed using the formula

$$e = \lim_{n\to\infty}\left(1 + 1 + \frac{1}{2} + \frac{1}{6} + \ldots + \frac{1}{n!}\right)$$

and is approximately given by

e = 2.7 1828 1828 45 90 45.

Problem Consider a quantity y whose rate of change over time is proportional to the amount of y present. Then $\frac{dy}{dt} \alpha\ y$ and y satisfies the differential equation

$$\frac{dy}{dt} = ky, \qquad \ldots(1)$$

where k is the proportionality constant. By separating variables, we obtain

$$\frac{dy}{y} = k\,dt.$$

Integrating both sides, we get

$\ln y = kt + c$ or $y = e^{kt+c}$, where e^x is the exponential function (it is the inverse of $\ln x$) simply we can write

$$y = Ce^{kt},. \qquad \ldots(2)$$

taking $C = e^c$.

If, in addition to (1), $y = y_0$ when $t = 0$, then (2) gives $y_0 = C \cdot e^0$ or $C = y_0$. Hence the function satisfying the differential equation (1) and $y = y_0$ when $t = 0$ is the exponential function $y = y_0 e^{kt}$.

The Function $y = e^x$

We note that $e^2 = e \cdot e,\quad e^{-2} = \dfrac{1}{e^2},\quad e^{1/2} = \sqrt{e},$ and so on. Since e positive, e^x is positive for any rational number x. Hence e^x has a logarithm and is given by

$$\ln e^x = x \ln e = x \cdot 1 = x.$$

Since $\ln x$ is one - to - one and $\ln(\ln^{-1} x) = x,$ the above equation tells us that, for x rational

$$e^x = \ln^{-1} x\,.$$

The above equation provides a way to extend the definition of e^x to irrational values of x. The function $\ln^{-1} x$ is defined for all real x, so we can use it to assign a value to e^x at every point. The definition follows:

Definition For every real number x, $e^x = \ln^{-1} x.$

Equations Involving ln x and e^x

Since $\ln x$ and e^x are inverses of one another, we have

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$

$$\ln(e^x) = x \qquad (\text{all } x)$$

The above are **inverse equations for e^x and ln x**¸ respectively.

Problem a) $\ln e^5 = 5$

b) $\ln e^{-5} = -5$

c) $\ln \sqrt[3]{e} = \ln e^{1/3} = \dfrac{1}{3}$

d) $\ln e^{\sin x} = \sin x$

e) $e^{\ln 4} = 4$

f) $e^{\ln(x^4+x^2+3)} = x^4 + x^2 + 3$ (this is possible, since $x^4 + x^2 + 3 > 0$)

Problem Evaluate $e^{3\ln 2}$

Answer $e^{3\ln 2} = e^{\ln 2^3} = e^{\ln 8} = 8\,.$

Aliter: $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8\,.$

Problem Find y if $\ln y = 7t + 9\,.$

Answer Exponentiating both sides, we obtain

$$e^{\ln y} = e^{7t+9}$$

$\therefore$ $y = e^{7t+9}$, using Eq.(5)

Problem Find k if $e^{2k} = 10$.

Answer Taking the natural logarithm on both sides, we get

$$\ln e^{2k} = \ln 10$$

$\therefore$ $2k = \ln 10$, using Eq. (6)

$\therefore$ $k = \dfrac{1}{2}\ln 10.$

Laws of Exponents

For all real numbers x, x_1 and x_2, then the following laws of exponents hold:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$
2. $e^{-x} = \dfrac{1}{e^x}$
3. $\dfrac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
4. $\left(e^{x_1}\right)^{x_2} = e^{x_1x_2} = \left(e^{x_2}\right)^{x_1}$

Problem

a) $e^{x+\ln 2} \quad = e^x \cdot e^{\ln 2} = 2e^x$, by law 1

b) $e^{-\ln x} \quad = \dfrac{1}{e^{\ln x}}$,by law 2

$= \dfrac{1}{x}$

c) $\dfrac{e^{2x}}{e} \quad = e^{2x-1}$, by law 3

d) $\left(e^3\right)^x \quad = e^{3x} = \left(e^x\right)^3$, by law 4

Problem Solve the following for the value of y.

(i) $e^{x^2} \cdot e^{2x+1} = e^y$ (ii) $e^{\sqrt{y}} = x^2$ (iii) $e^{3y} = 2 + \cos x$.

Answer

(i) $e^y = e^{x^2} \times e^{2x+1} = e^{x^2+2x+1}$

Now taking logarithms on both sides, we get

$$y = x^2 + 2x + 1 = (x+1)^2 \quad \text{as the solution.}$$

(*ii*) Given $e^{\sqrt{y}} = x^2$.

Taking logarithms on both sides, we get

$\sqrt{y} = 1\text{n}\, x^2$ or $\sqrt{y} = 2\ln x$.

Squaring both sides we get

$$y = [2 \ln x]^2 = 4\, [\ln x]^2.$$

(*iii*) Given $e^{3y} = 2 + \cos x$.

Taking logarithms on both sides, we get

$3y = \ln(2 + \cos x)$, so that

$$y = \frac{1}{3}\ln(2 + \cos x).$$

The Derivative of e^x

The exponential function is differentiable because it is the inverse of a differentiable function whose derivative is never zero. Consider

$$y = e^x .$$

Applying logarithms on both sides, we obtain

$$\ln y = x .$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx} = 1$$

or
$$\frac{dy}{dx} = y$$

Replacing y by e^x, we obtain

$$\frac{d}{dx}e^x = e^x$$

Problem Evaluate $\frac{d}{dx}\left(5e^x\right)$.

Answer

$$\frac{d}{dx}\left(5e^x\right) = 5\frac{d}{dx}e^x$$
$$= 5e^x .$$

The Derivative of e^u

If u is any differentiable function of x, then using the Chain Rule

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} .$$

Problem

a) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x)$, using the above equation. with $u = -x$

$$= e^{-x}(-1) = -e^{-x} ,$$

b) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x)$, using the above equation with $u = \sin x$

$$= e^{\sin x} \cdot \cos x$$

Integral of e^u

$$\int e^u du = e^u + C .$$

Problem Solve the initial value problem

$$e^y\frac{dy}{dx} = 2x , \quad x > \sqrt{3} ; \quad y(2) = 0 .$$

Solution

By separating variables, the given differential equation becomes $e^{y}dy = 2x\,dx$.

Integrating both sides of the differential equation, we obtain

$$e^{y} = x^{2} + C. \qquad \ldots(11)$$

To determine *C* we use the initial condition. Given $y = 0$, when $x = 2$.

Hence $\quad e^{0} = (2)^{2} + C$

or $\quad C = e^{0} - (2)^{2}$

$= 1 - 4 = -3$.

Substituting this values of *C* in (11), we obtain

$$e^{y} = x^{2} - 3 \qquad \ldots(12)$$

To find *y*, we take logarithms on both sides of (12) and get

$$\ln e^{y} = \ln(x^{2} - 3)$$

or $\quad y = \ln(x^{2} - 3) \qquad \ldots(13)$

Clearly $\ln(x^{2} - 3)$ is well defined for $x^{2} - 3 > 0$ and hence the solution is valid for $x > \sqrt{3}$.

Checking of the solution in the original equation .

Now $\quad e^{y}\dfrac{dy}{dx} = e^{y}\dfrac{d}{dx}\ln(x^{2} - 3)$, using Eq.(13)

$= e^{y}\dfrac{2x}{x^{2} - 3}$, as $\dfrac{d}{dx}\ln u = \dfrac{1}{u}\cdot\dfrac{du}{dx}$

$= (x^{2} - 3)\dfrac{2x}{x^{2} - 3}$, using Eq.(12)

$= 2x.$

Hence the solution is checked.

Exercises

Find simpler expressions for the quantities in Exercises1- 6

1. $e^{\ln(x^{2} - y^{2})}$ 2. $e^{-\ln 0.3}$ 3. $e^{\ln ® x - \ln 2}$

4. $\ln(e^{\sec^{-}})$ 5. $\ln(e^{(e^{x})})$ 6. $\ln(e^{2\ln x})$

In Exercises 7-9, solve for *y* in terms of *t* or *x*, as appropriate.

7. $\ln y = -t + 5$ 8. $\ln(1 - 2y) = t$

9. $\ln(y^{2} - 1) - \ln(y + 1) = \ln(\sin x)$

In Exercise10-12, solve for k.

10. a) $e^{5k} = \dfrac{1}{4}$ 11. $80e^{k} = 1$ 12. $e^{(\ln 0.8)k} = 0.8$

In Exercises 13-16, solve for t.

13. a) $e^{-0.01t}$ =1000 14. $e^{kt} = \frac{1}{10}$ 15. $e^{(\ln 2)t} = \frac{1}{2}$

16. $e^{(x^2)}e^{(2x+1)} = e^t$

In Exercises 17-26, find the derivatives of y with respect to x, t, or¯ , as appropriate.

17. $y = e^{2x/3}$ 18. $y = e^{(4\sqrt{x}+x^2)}$ 19. $y =$ $(1+2x)e^{-2x}$

20. $y = (9x^2 - 6x + 2)e^{3x}$ 21. $y = \ln(3¯e^{-¯})$

22. $y = ¯^3 e^{-2¯} \cos 5¯$ 23. $y = \ln(2e^{-t}\sin t)$

24. $y = \ln\left(\frac{\sqrt{¯}}{1+\sqrt{¯}}\right)$ 25. $y = e^{\sin t}(\ln t^2 + 1)$

26. $y = \int_{e^{4\sqrt{x}}}^{e^{2x}} \ln t dt$

In Exercises 27-28, find dy/dx.

27. $\ln xy = e^{x+y}$ 28. $\tan y = e^x + \ln x$

Evaluate the integrals in Exercises 29-39.

29. $\int (2e^x + 3e^{-2x})dx$ 30. $\int_{-\ln 2}^{0} e^{-x}dx$

31. $\int 2e^{(2x-1)}dx$ 32. $\int_0^{\ln 16} e^{x/4}dx$

33. $\int \frac{e^{-\sqrt{r}}}{\sqrt{r}}dr$ 34. $\int t^3 e^{(t^4)}dt$

35. $\int \frac{e^{1/x^2}}{x^3}dx$ 36. $\int_{®/4}^{®/2}\left(1+e^{\cot ¯}\right)\csc^2 ¯ \, d¯$

37. $\int e^{\csc(® + t)}\csc(® + t)\cot(® + t)dt$

38. $\int_0^{\sqrt{\ln ®}} 2xe^{x^2}\cos(e^{x^2})dx$ 39. $\int \frac{dx}{1+e^x}$

Solve the initial value problems in Exercises 40-41.

40. $\frac{dy}{dt}$ $e^{-t}\sec^2(®e^{-t})$, $y(\ln 4) = 2/®$

41. $\frac{d^2y}{dt^2}$ $1 - e^{2t}$, $y(1) = -1$ and $y'(1)$ =0

CHAPTER 3 : a^x and $\log_a x$

The Function a^x

Since $a = e^{\ln a}$ for any positive number a, we can write a^x as $(e^{\ln a})^x = e^{x \ln a}$ and we state this in the following definition.

Definition For any number $a > 0$ and x,

$$a^x = e^{x \ln a}$$

Problem

a) $3^{\sqrt{5}} = e^{\sqrt{5} \ln 3}$

b) $6^{®} = e^{® \ln 6} =$

Table: Laws of exponents
For $a > 0$, and any x and y:

1. $a^x \cdot a^y = a^{x+y}$
2. $a^{-x} = \dfrac{1}{a^x}$
3. $\dfrac{a^x}{a^y} = a^{x-y}$
4. $(a^x)^y = a^{xy} = (a^y)^x$

The Power Rule (Final Form)

For any $x > 0$ and any real number n, we can define $x^n = e^{n \ln x}$. Therefore, the n in the equation $\ln x^n = n \ln x$ no longer needs to be rational- it can be any number as long as $x > 0$:

$$\ln x^n = \ln(e^{n \ln x}) = n \ln x . \ln e \text{, as } \quad \ln e^u = u \text{, for any } u$$

$$= n \ln x .$$

Differentiating x^n with respect to x,

$$\frac{d}{dx} x^n = \frac{d}{dx} e^{n \ln x} \text{, as for } x > 0 \text{, } x^n = e^{n \ln x}$$

$$= e^{n \ln x} . \frac{d}{dx}(n \ln x) \text{, as } \frac{d}{dx} e^u = e^u \cdot \frac{du}{dx}$$

$$= x^n . \frac{n}{x} \text{, as } x^n = e^{n \ln x} \text{, and } \frac{d}{dx} \ln x = \frac{1}{x} .$$

$$= n x^{n-1} \text{, as } \frac{x^n}{x^1} = x^{n-1}$$

Hence, as long as $x > 0$, $\frac{d}{dx}x^n = nx^{n-1}$.

Using the Chain Rule, we can extend the above equation to the **Power Rule's final form:**

If u is a positive differentiable function of x and n is any real number, then u^n is a differentiable function of x and

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}.$$

Problem

a) $\frac{d}{dx}x^{\sqrt{2}} = \sqrt{2}\,x e^{\sqrt{2}-1}$ $\quad (x > 0)$

b) $\frac{d}{dx}(\sin x)^{®} = ®(\sin x)^{®-1}\cos x$ $\quad (\sin x > 0)$

The Derivative of a^x

Differentiating $a^x = e^{x\ln a}$ with respect to x, we obtain

$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}.\frac{d}{dx}(x\ln a)$, taking $u = 1\mathrm{n}\,a$ and using the Chain Rule

$$\frac{d}{dx}e^u = \frac{d}{du}e^u.\frac{du}{dx}$$

$= a^x \ln a$, as $\frac{d}{dx}(x\,1\mathrm{n}\,a) = 1\mathrm{n}\,a\frac{dx}{dx} = 1\mathrm{n}\,a$

That is, if $a > 0$, then

$$\frac{d}{dx}a^x = a^x\ln a.$$

Using the Chain Rule, we can extend the above equation to the following general form.

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u\ln a\frac{du}{dx}.$$

If $a = e$, then $\ln a = \ln e = 1$ and the above equation simplifies to

$$\frac{d}{dx}e^x = e^x.$$

Problem

(a) $\frac{d}{dx}3^x = 3^x\ln 3$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}\ln 3\frac{d}{dx}(-x) = -3^{-x}\ln 3$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}\ln 3\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

The derivative of a^x is positive if $\ln a > 0$, or $a > 1$, and negative if $\ln a < 0$, or $0 < a < 1$. Thus, a^x is an increasing function of x if $a > 1$ and a decreasing function of x if $0 < a < 1$. In each case, a^x is one-to-one. The second derivative

$$\frac{d^2}{dx^2}(a^x) = \frac{d}{dx}\left(\frac{d}{dx}(a^x)\right) = = \frac{d}{dx}(a^x \ln a) = (\ln a)^2 a^x$$

is positive for all x, so the graph of a^x is concave up on every interval of the real line.

OTHER POWER FUNCTIONS

The ability to raise positive numbers to arbitrary real powers makes it possible to define functions like x^x and $x^{\ln x}$ for $x > 0$. We find the derivatives of such functions by rewriting the functions as powers of e.

Problem

Find $\frac{dy}{dx}$ if $y = x^x$, $x > 0$.

Answer

With $a = x$, we can write x^x as $e^{x \ln x}$, a power of e, so that

$$y = x^x = e^{x \ln x}$$

Differentiating both sides with respect to x, we obtain

$$\frac{dy}{dx} = \frac{d}{dx} e^{x \ln x}.$$

$= e^{x \ln x} \frac{d}{dx}(x \ln x)$, using Eq.(2) with $a = e$, $u = x \ln x$ and noting that $\ln e = 1$, or simply using Eq. (9) of the previous chapter

$= x^x \left(x.\frac{1}{x} + \ln x \right)$, applying product rule of differentiation

$= x^x (1 + \ln x)$.

The Integral of a^u

If $a \neq 1$, then $\ln a \neq 0$, so

$$a^u \frac{du}{dx} = \frac{1}{\ln a} \frac{d}{dx}(a^u).$$

Integrating with respect to x, we obtain

$$\int a^u \frac{du}{dx} dx = \int \frac{1}{\ln a} \frac{d}{dx}(a^u) dx = \frac{1}{\ln a} \int \frac{d}{dx}(a^u)\, dx = \frac{1}{\ln a} a^u + C.$$

Writing the first integral in differential form gives

$$\int a^u du = \frac{a^u}{\ln a} + C$$

Problem Evaluate $\int 2^x dx$.

Answer $\int 2^x dx = \frac{2^x}{\ln 2} + C$, *using Eq. (3) with* $a = 2$, $u = x$

Problem Evaluate $\int 2^{\sin x} \cos x\, dx$

Answer

$$\int 2^{\sin x} \cos x dx = \int 2^u du,$$

$$= \frac{2^u}{\ln 2} + C$$

$$= \frac{2^{\sin x}}{\ln 2} + C.$$

Logarithms with Base *a*

We have noted that if a is any positive number other than 1, the function a^x is one- to –one and has a nonzero derivative at every point. It therefore has a differentiable inverse. We call the inverse the **logarithm of** x **with base** a and denote it by $\log_a x$.

Definition For any positive number $a \neq 1$,

$$\log_a x = \text{inverse of } a^x.$$

The graph of $y = \log_a x$ can be obtained by reflecting the graph of $y = a^x$ across the line $y = x$ (Fig.2). Since $\log_a x$ and a^x are inverse of one another, composing them in either order gives the identity function. That is,

$$a^{\log_a x} = x \qquad (x > 0)$$

and $$\log_a (a^x) = x \qquad (\text{all } x)$$

The above are the inverse equations for a^x and $\log_a x$

The Evaluation of $\log_a x$

$$a^{\log_a (x)} = x,$$

Taking the natural logarithm of both sides,

$$\ln a^{\log_a (x)} = \ln x.$$

Using Power Rule,

$$\log_a (x).\ln a = \ln x.$$

Solving for $\log_a x$, we obtain

$$\log_a x = = \frac{\ln x}{\ln a}$$

Problem

$$\log_{10} 3 = \frac{\ln 3}{\ln 10}$$

Properties of base *a* logarithms

For any number $x > 0$ and $y > 0$,

1. Product Rule: $\log_a xy = \log_a x + \log_a y$
2. Quotient Rule: $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. Reciprocal Rule: $\log_a \frac{1}{y} = -\log_a y$
4. Power Rule: $\log_a x^y = y \log_a x$

Proof

For natural logarithms, we have $\ln xy = \ln x + \ln y$

Dividing both sides by $\ln a$, we get

$$\frac{\ln xy}{\ln a} = \frac{\ln x}{\ln a} + \frac{\ln y}{\ln a}$$

i.e., $\log_a xy = \log_a x + \log_a y$.

The Derivative of $\log_a u$

Prove that, if u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}$$

Proof

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a} \frac{d}{dx}(\ln u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}.$$

Problem Evaluate $\frac{d}{dx} \log_{10}(3x+1)$

Solution

Taking $a = 10$ and $u = 3x+1$, Eq.(7) gives

$$\frac{d}{dx} \log_{10}(3x+1) = \frac{1}{\ln 10} \cdot \frac{1}{3x+1} \frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

Integrals Involving $\log_a x$

To evaluate integrals involving base a logarithms, we convert them to natural logarithms.

Problem Evaluate $\int \frac{\log_2 x}{x} dx.$

Solution

$$\int \frac{\log_2 x}{x} dx = \frac{1}{\ln 2}\int \frac{\ln x}{x} dx \text{ , since } \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u du \text{ , taking } u = \ln x,\ du = \frac{dx}{x}$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C$$

$$= \frac{(\ln x)^2}{2\ln 2} + C.$$

Exercises

Find the derivative of y with respect to the given independent variable.

1. $y = \log_3(1 + {}^{-}\ln 3)$
2. $y = \log_{25} e^x - \log_5 \sqrt{x}$
3. $y = \log_3 r \cdot \log_9 r$
4. $y = \log_5 \sqrt{\left(\frac{7x}{3x+2}\right)^{\ln 5}}$
5. $y = \log_7\left(\frac{\sin^{-}\cos^{-}}{e^{-}2^{-}}\right)$
6. $y = \log_2 \frac{x^2 e^2}{2\sqrt{x+1}}$
7. $y = 3\log_8(\log_2 t)$
8. $y = t\log_3(e^{(\sin t)(\ln 3)})$

Use logarithmic differentiation to find the derivative of y with respect to the given independent variable.

9. $y = x^{(x+1)}$
10. $y = t^{\sqrt{t}}$
11. $y = x^{\sin x}$
12. $y = (\ln x)^{\ln x}$

CHAPTER 4: GROWTH AND DECAY

In this chapter, we derive the law of exponential change and describe some of the applications that account for the importance of logarithmic and exponential functions.

The Law of Exponential Change

Consider a quantity y (velocity, temperature, electric current, whatever) that increases or decreases at a rate that at any given time t is proportional to the amount present. If we also know the amount present at time $t=0$, call it y_0, we can find y as a function of t by solving the following initial value problem:

Differential equation: $\frac{dy}{dt} = ky$

Initial condition: $y = y_0$ when $t = 0$

If y is positive and increasing, then k is positive and we use the first equation to say that the **rate of growth** is proportional to what has already been accumulated. If y is positive and decreasing, then k is negative and we use the second equation to say that the **rate of decay** is proportional to the amount still left.

Clearly the constant function $y = 0$ is a solution of the differential equation in Eq. (1). Now to find the nonzero solutions, we proceed as follows:

By separating variables, the differential equation in Eq.(1a) gives

$$\frac{dy}{y} = k\,dt\,.$$

Integrating both sides, we obtain

$$\ln|y| = kt + C\,.$$

By exponentiating, we obtain

$$|y| = e^{kt+C}$$

i.e., $|y| = e^C \cdot e^{kt}$, since $e^{a+b} = e^a \cdot e^b$

i.e., $y = \pm e^C e^{kt}$, noting that if $|y| = r$, then $y = \pm r$.

i.e., $y = Ae^{kt}$, as A is a more convenient than $\pm e^C$.

To find the right value of A that satisfies the initial value problem, we solve for A when $y = y_0$ and $t = 0$:

$$y_0 = Ae^{k\cdot 0} = A\,.$$

Hence the solution of the initial value problem is

$$y = y_0 e^{kt}\,.$$

The law of Exponential Change says that the above equation gives a **growth** when $k > 0$ and **decay** when $k > 0$

The number k is the **rate constant** of the equation.

Population Growth

Consider the number of individuals in a population of people. It is a discontinuous function of time because it takes on discrete values. However, as soon as the number of individuals becomes large enough, it can safely be described with a continuous or even differentiable function.

If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t the birth rate is proportional to the number $y(t)$ of individuals present. If, further, we neglect departures, arrivals and deaths, the growth rate dy/dt will be the same as the birth rate ky. In other words, $dy/dt = ky$, so that $y = y_0 e^{kt}$. In real life all kinds of growth, may have limitations imposed by the surrounding environment, but we ignore them.

Problem One model for the way diseases spread assumes the rate dy/dt at which the number of infected people changes is proportional to the number y. The more infected people there are, the faster the disease will spread. The fewer there are, the slower it will spread.

Suppose that in the course of any given year the number of cases of a disease is reduced by 20%. If there are 10,000 cases today, how many years will it take to reduce the number to 1000?

Answer

We use the equation $y = y_0 e^{kt}$. There are three things to find:

1. the value of y_0,
2. the value of k,
3. the value of t that makes y =1000.

Determination of the value of y_0. We are free to count time beginning anywhere we want. If we count from today, then y =10,000 when $t=0$, so y_0 =10,000. Our equation is now

$$y = 10000e^{kt}. \qquad \ldots(3)$$

Determination of the value of k. When $t=1$ year, the number of cases will be 80% of its present value, or 8000. Hence,

8000 = 10,000 $e^{k(1)}$, using Eq. (3) with $t=1$ and y =8000

$$\therefore \qquad e^k = 0.8$$

Taking logarithms on both sides, we obtain

$$\ln\left(e^k\right) = \ln 0.8.$$

Hence

$$\therefore \qquad k = \ln 0.8.$$

Using Eq.(3), at any given time t,

$$y = 10{,}000\, e^{(\ln 0.8)t}.$$

Determination of the value of t that makes y=1000. We set y equal to 1000 in Eq.(4) and solve for t:

$$1000 = 10{,}000\, e^{(\ln 0.8)t}$$

$$e^{(\ln 0.8)t} = 0.1$$

Taking logarithms on both sides, we obtain

$$(\ln 0.8)t = \ln 0.1$$

$$\therefore \qquad t = \frac{\ln 0.1}{\ln 0.8} \approx 10.32\,.$$

It will take a little more than 10 years to reduce the number of cases to 1000.

Continuously Compounded Interest

If you invest an amount A_0 of money at a fixed annual interest rate r (expressed as a decimal) and if interest is added to your account k times a year, it turns out that the amount of money you will have at the end of t years is

$$A_t = A_0\left(1+\frac{r}{k}\right)^{kt}.$$

The interest might be added ("compounded") monthly $(k=12)$, weekly $(k=52)$, daily $(k=365)$, or even more frequently, say by the hour or by the minute. But there is still a limit to how much you will earn that way, and the limit is

$$\lim_{k\to\infty} A_t = \lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}$$

$$= A_0 e^{rt}, \text{ as } \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} = e^{x}.$$

The resulting formula for the amount of money in your account after t years is

$$A(t) = A_0 e^{rt}\,.$$

Interest paid according to this formula is said to be **compounded continuously.** The number r is called the **continuous interest rate.**

Problem Suppose you deposit Rs.62100 in a bank account that pays 6% compounded continuously. How much money will you have 8 years later? If bank pays 6% interest quarterly how much money will you have 8 years later? Compare the two compounding.

Answer

With $A_0 = 62100,$ r =0.06 and t =8:

$A(8)$ = 62100 $e^{(0.06)(8)}$ = 62100 $e^{0.48}$ = 100358, approximately.

If the bank pays 6% interest quarterly, we have to put $k=4$ in Eq. (5) and

$A(8) = 62100\left(1+\frac{0.06}{4}\right)^{4\times 8} = 100001$, approximately. Thus the effect of continuous compounding, as compared with quarterly compounding, has been an addition of Rs.357.

Radioactivity

When an atom emits some of its mass as radiation, the remainder of the atom re-forms to make an atom of some new element. This process of radiation and change is called **radioactive decay**, and an element whose atoms go spontaneously through this process is called **radioactive**. Thus, radioactive carbon-14 decays into nitrogen. Also, radium, through a number of intervening radioactive steps, decays into lead.

Experiments have shown that at any given time the rate at which a radioactive element decays (as measured by the number of nuclei that change per unit time) is approximately proportional to the number of nuclei present. Thus the decay of a radioactive element is described by the equation $dy/dt = -ky$, $k > 0$. If y_0 is the number of radioactive nuclei present at time zero, the number still present at any later time t will be

$$y = y_0 e^{-kt}, \; k > 0$$

Problem The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay. Show that the half life is a constant that does not depend on the number of radioactive nuclei initially present in the sample, but only on the radioactive substance.

Answer

Let y_0 be the number of radioactive nuclei initially present in the sample. Then the number y present at any later time t will be $y = y_0 e^{-kt}$. We search the value of t at which the number of radioactive nuclei present equals half the original number:

$$y_0 e^{-kt} = \frac{1}{2} y_0$$

$$e^{-kt} = \frac{1}{2}$$

$$-kt = \ln\frac{1}{2} = -\ln 2$$, using Reciprocal Rule for logarithms

$$\therefore \qquad t = \frac{\ln 2}{k}.$$

This value of t is the half-life of the element. It depends only on the value of k for a radioactive element, not on y_0 the number of radioactive nuclei present. Thus,

$$\text{Half-life} = \frac{\ln 2}{k},$$

where k depends only on the radio active substance.

Problem The number of radioactive Polonium-200 atoms remaining after t days in a sample that starts with y_0 atoms is given by the Polonium decay equation

$$y = y_0 e^{-5\times10^{-3} t} .$$

Find the Polonium-200 half –life.

Answer

Comparing Polonium decay equation with Eq. (7), we have $k = 5\times10^{-3}$.

$$\text{Half-life} = \frac{\ln 2}{k}, \text{ using Eq. (8)}$$

$$= \frac{\ln 2}{5\times10^{-3}}$$

$$\approx 139 \text{ days}$$

Problem Using Carbon-14 dating, find the age of a sample in which 10% of the radioactive nuclei originally present have decayed. (The half life of Carbon-14 is 5700 years)

Answer

We note that 10% of the radio active nuclei originally present have decayed is equivalent to say that 90% of the radioactive nuclei is still present.

We use the decay equation $y = y_0 e^{-kt}$. There are two things to find:

1. the value of k,
2. the value of t when $y_0 e^{-kt} = 0.9 y_0$, or e^{-kt} =0.9

Determination of *the value of* k. We use the half-life equation (8), to get

$$k = \frac{\ln 2}{\text{half - life}} = \frac{\ln 2}{5700} .$$

Hence the decay equation becomes $y = y_0 \overline{e} (\ln 2/5700) t$

Determination of the value of t that makes $e^{-(\ln 2/5700)t} = 0.9$

Taking logarithm of both sides,

$$-\frac{\ln 2}{5700} t = \ln 0.9$$

$$\therefore \qquad t = -\frac{5700 \ln 0.9}{\ln 2} \approx 866$$

Hence the sample is about 866 years old.

CHAPTER.5 L'HOSPITAL'S RULE

L` Hospital rule for forms of type 0/0

Theorem Suppose that $\lim_{x\to u} f(x) = \lim_{x\to u} g(x) = 0$. If $\lim_{x\to u} \frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense (that is, if this limit is a finite number or $-\infty$ or $+\infty$), then

$$\lim_{x\to u} \frac{f(x)}{g(x)} = \lim_{x\to u} \frac{f'(x)}{g'(x)}.$$

Here u may stand for any of the symbols $a, a^-, a^+, -\infty, or\ +\infty$.

Problem Find $\lim_{x\to 0} \frac{e^x - 1}{x}$.

Answer Here both the numerator and denominator have limit 0. Therefore limit has 0/0 form.

y $$\lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1}$$, applying l'Hôpital's Rule

$$= \frac{e^0}{1} = 1.$$

Problem Use l'Hôpital's rule to show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Answer Here limits of both the numerator and denominator is 0. Therefore $\lim_{x\to 0} \frac{\sin x}{x}$ is in the 0/0 form. Now

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1}$$, using l'Hôpital's Rule and noting that derivative of sin x is cos x and that of x is 1.

$$= \frac{\lim_{x\to 0} \cos x}{\lim_{x\to 0} 1},$$ using quotient rule for limits

$$= \frac{1}{1} = 1.$$

Problem Find $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$.

Answer Here both the numerator and denominator have limit 0. Therefore limit has 0/0 form.

y $$\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \lim_{x\to 0} \frac{n(1+x)^{n-1}}{1}$$, applying l'Hôpital's Rule

$$= \frac{n(1+0)^{n-1}}{1} = n.$$

Successive Application of l'Hôpital's Rule

Problem Evaluate $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

Answer Here the limit is in $0/0$ form.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}\overset{\mathcal{L}}{=}\lim_{x\to 0}\frac{1-\cos x}{3x^2}, \text{ again in } 0/0 \text{ form}$$

$$\overset{\mathcal{L}}{=}\lim_{x\to 0}\frac{\sin x}{6x}, \text{ again in } 0/0 \text{ form}$$

$$\overset{\mathcal{L}}{=}\lim_{x\to 0}\frac{\cos x}{6}, \text{ now limit can be evaluated}$$

$$=\frac{1}{6}.$$

Problem

Find $\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}$.

Answer $\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}\overset{\mathcal{L}}{=}\lim_{x\to 0}\frac{\sin x}{2x+3}\overset{\mathcal{L}}{=}\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2}$ **This is wrong,** as the first application of l'Hôpital's Rule was correct; the second was not, since at that stage the limit did not have the 0/0 form. Here is what we should have done:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2+3x}\overset{\mathcal{L}}{=}\lim_{x\to 0}\frac{\sin x}{2x+3}=0$$ **This is right.**

Problem Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$

Answer The given is in the $\frac{0}{0}$ form.

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}$$ Not $\frac{0}{0}$

$$=\frac{0}{1}=0$$

If we continue to differentiate in an attempt to apply L'Hopital's rule once more, we get

$$\lim_{x\to 0}\frac{1-\cos x}{x+x^2}=\lim_{x\to 0}\frac{\sin x}{1+2x}=\lim_{x\to 0}\frac{\cos x}{2}=\frac{1}{2},$$

which is wrong.

Problem Find $\displaystyle\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}$.

Answer Here the given limit can be written as

$$\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}=\lim_{x\to\infty}\frac{\log\left(1+\frac{1}{x}\right)}{\log\left(1-\frac{1}{x}\right)}$$

and the limit is in $0/0$ form.

Also,
$$\lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}=\lim_{x\to 0}\frac{\log(x+1)-\log x}{\log(x-1)-\log x}.$$

Now we are ready to apply l'Hôpital's Rule:

$$\therefore \qquad \lim_{x\to\infty}\frac{\log\left(\frac{x+1}{x}\right)}{\log\left(\frac{x-1}{x}\right)}\overset{\mathcal{L}}{=}\lim_{x\to\infty}\frac{\frac{1}{x+1}-\frac{1}{x}}{\frac{1}{x-1}-\frac{1}{x}}$$

$$=\lim_{x\to\infty}\frac{\frac{-1}{x(x+1)}}{\frac{1}{x(x-1)}} \quad \text{(by algebraic manipulations)}$$

$$=\lim_{x\to\infty}-\frac{x-1}{x+1}\ =\lim_{x\to\infty}\frac{1-x}{x+1}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x}-1}{1+\frac{1}{x}}=\frac{0-1}{1+0}=-1.$$

Exercises

Evaluate the following limits.

1. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{\tan^{-1}x}$ **2.** $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$

3. $\displaystyle\lim_{x\to 1}\frac{1+\cos\text{®}\,x}{(1-x)^2}$ **4.** $\displaystyle\lim_{x\to 0}\frac{e^x-b^x}{x}$

5. $\displaystyle\lim_{x\to 1}\frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$ **6.** $\displaystyle\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$

7. $\lim\limits_{x\to 0}\dfrac{x^2}{1-\cos x}$ **8.** $\lim\limits_{x\to 0}\left[\dfrac{x-\sin x}{x^3}+\dfrac{\sin 3x}{x}\right]$

9. $\lim\limits_{x\to 0}\dfrac{xe^x-\log(1+x)}{x^2}$ **10.** $\lim\limits_{x\to 0}\dfrac{e^x+\sin x-1}{\log(1+x)}$

11. $\lim\limits_{x\to 0}\dfrac{e^{\sin x}-1-x}{x^2}$ **12.** $\lim\limits_{x\to 0}\dfrac{\tan x-x}{x-\sin x}$

13. $\lim\limits_{x\to 0}\dfrac{\sin\log(1+x)}{\log(1+x)}$ **14.** $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

15. $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}+2\sin x+x^3-4x}{10x^3}$

16. $\lim\limits_{x\to 0}\dfrac{\log(1-x^2)}{\log\cos x}$ **17.** $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$

18. $\lim\limits_{x\to 0}\dfrac{1+x\cos x-\cosh x-\log(1+x)}{\tan x-x}$

19. $\lim\limits_{x\to 0}\dfrac{1-2\cos x+\cos^2 x}{x^4}$ **20.** $\lim\limits_{x\to 0}\dfrac{\log(1+kx^2)}{1-\cos x}$

21. $\lim\limits_{x\to 0}\left(e^x-e^{-x}-\dfrac{2x}{x^2\sin x}\right)$ **22.** $\lim\limits_{x\to 3}\dfrac{x^2-9}{x^2-x-6}$

L` Hospital rule for forms of type ã/ã

Theorem Suppose that $\lim\limits_{x\to u}|f(x)|=\lim\limits_{x\to u}|g(x)|=\infty$. If $\lim\limits_{x\to u}\dfrac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense (that is, if this limit is a finite number or –∞ or +∞), then

$$\lim_{x\to u}\frac{f(x)}{g(x)}=\lim_{x\to u}\frac{f'(x)}{g'(x)}.$$

Here u may stand for any of the symbols $a, a^-, a^+, -\infty, or\ +\infty$.

Problem Find $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$

Answer

Both x and e^x tend to ∞ as $x\to\infty$. Hence limit is in ∞/∞ form.

$$\lim_{x\to\infty}\frac{x}{e^x}=\lim_{x\to\infty}\frac{1}{e^x}\text{, applying l'Hôpital's Rule}$$

$$= 0.$$

ProblemEvaluate $\lim\limits_{x\to\infty} \frac{x^n}{e^x}$, where n is natural number.

Answer

Here both the numerator and denominator tend to ∞ as $x \to \infty$. Hence limit is in ∞ / ∞ form.

$$\lim_{x\to\infty} \frac{x^n}{e^x} \overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{n\,x^{n-1}}{e^x}$$

$$\overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x}$$

$$\overset{\mathcal{L}}{=} \cdots$$

$$\overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{e^x}$$

$$\overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{e^x}$$

$$= \lim_{x\to\infty} \frac{n!}{e^x} = 0$$

Problem Show that if a is any positive real number,

$$\lim_{x\to\infty} \frac{x^a}{e^x} = 0.$$

Answer

Here both the numerator and denominator tend to ∞ as $x \to \infty$. Hence limit is in ∞ / ∞ form.

Suppose as a special case that a = 2.1. Then three applications of l'Hôpital's Rule give

$$\lim_{x\to\infty} \frac{x^{2.1}}{e^x} \overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{2.1x^{1.1}}{e^x} \overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{(2.1)(1.1)x^{0.1}}{e^x}$$

$$\overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{(2.1)(1.1)(0.1)}{x^{0.9}e^x} = 0.$$

A similar argument works for any $a > 0$.

Problem Show that if a is any positive real number,

$$\lim_{x\to\infty} \frac{\ln x}{x^a} = 0.$$

Answer

Here both the numerator and denominator tend to ∞ as $x \to \infty$. Hence limit is in ∞ / ∞ form.

$$\therefore \lim_{x\to\infty} \frac{\ln x}{x^a} \overset{\mathcal{L}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{a\,x^{a-1}} = \lim_{x\to\infty} \frac{1}{a\,x^a} = \frac{\lim\limits_{x\to\infty} 1}{\lim\limits_{x\to\infty} a\,x^a} = 0.$$

Problem Show that $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\cot x} = 0.$

Answer

Here both the numerator and denominator tend to ∞ as $x \to \infty$. Hence limit is in ∞ / ∞ form.

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} \overset{\mathcal{L}}{=} \lim_{x\to 0^+} \frac{1/x}{-\,cosec^2 x}$$

This is still indeterminate (∞ / ∞ form) as it stands, but rather than apply l'Hôpital's Rule again (which only makes things worse), we rewrite:

$$\frac{1/x}{-\,cosec^2 x} = -\frac{\sin^2 x}{x} = -\sin x \frac{\sin x}{x}$$

Thus

$$\lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+}\left(-\sin x \frac{\sin x}{x}\right)$$

$$= -\lim_{x\to 0^+} \sin x \lim_{x\to 0^+} \frac{\sin x}{x} = 0 \cdot 1 = 0.$$

The Indeterminate Products and Differences:

Indeterminate forms 0.ã, ã J ã

Problem Evaluate $\lim\limits_{x\to 0}$ tan x log x

Answer

Write $\lim\limits_{x\to 0}$ tan x log x (which is in $0 \cdot \infty$ form) as:

$\lim\limits_{x\to 0}$ tan x log x $= \lim\limits_{x\to 0} \dfrac{\ln x}{\cot x}$ (now in ∞/∞ form)

$= 0$, by Example 5 in the previous section.

Problem Evaluate $\lim\limits_{x\to 1^+}\left(\dfrac{x}{x-1} - \dfrac{1}{\ln x}\right)$

Answer

$\dfrac{x}{x-1}$ and $\dfrac{1}{\ln x}$ tend to ∞ as $x \to 1$. So the limit is an $\infty - \infty$ form.

Before applying L`Hospital's Rule we rewrite:

$$\lim_{x\to 1^+}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1^+} \frac{x\ln x - x + 1}{(x-1)\ln x} \quad (0/0 \text{ form})$$

Now apply L`Hospital's Rule:

$$\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1^+}\frac{x\ln x-x+1}{(x-1)\ln x}\overset{L}{=}\lim_{x\to 1^+}\frac{x\cdot 1/x+\ln x-1}{(x-1)(1/x)+\ln x}$$

$$=\lim_{x\to 1^+}\frac{x\ln x}{x-1+x\ln x}\overset{L}{=}\lim_{x\to 1^+}\frac{1+\ln x}{2+\ln x}=\frac{1}{2}.$$

Exercises

1. $\lim_{x\to \pi/2}(\sec x-\tan x)$ **2.** $\lim_{x\to 0}\left(\frac{1}{x}-\cot x\right)$

3. $\lim_{x\to a}(a-x)\tan\left(\frac{\pi x}{2a}\right)$ **4.** $\lim_{x\to 1}(\log x\,\log(1-x))$

5. $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

The Indeterminate Powers: Indeterminate forms 0^0, ∞^0, 1^∞

Three indeterminate forms of exponential type are 0^0, ∞^0 and 1^∞. Here the trick is to consider not the original expression, but rather its logarithm. Usually l`Hospital's Rule will apply to the logarithm.

Problem Evaluate $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}$.

Answer

The limit takes the indeterminate form 1^∞.

Let $y=(\sin x)^{\tan x}$,

so taking logarithims, we obtain

$$\log y=\tan x\log\sin x=\frac{\log\sin x}{\cot x}.$$

Applying l`Hospital's Rule for 0/0 forms,

$$\lim_{x\to\frac{\pi}{2}}\log y=\lim_{x\to\frac{\pi}{2}}\frac{\log\sin x}{\cot x}\overset{L}{=}\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\sin x}\cdot\cos x}{-\operatorname{cosec}^2 x}$$

$$=\lim_{x\to\frac{\pi}{2}}-\sin x\cos x=-\sin\tfrac{\pi}{2}\cos\tfrac{\pi}{2}=0.$$

Now $y=e^{\ln y}$, and since the exponential function $f(x)=e^x$ is continuous,

$$\lim_{x\to\frac{\pi}{2}}y=\lim_{x\to\frac{\pi}{2}}\exp(\log y)=\exp\left(\lim_{x\to\frac{\pi}{2}}\log y\right)=\exp 0=1.$$

i.e., $$\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=1.$$

Problem Prove that $\lim\limits_{x\to\pi/2^-}(\tan x)^{\cos x}=1.$

Answer

The limit takes the indeterminate form ∞^0.

Let $y=(\tan x)^{\cos x}$, so that

$$\ln y=\cos x\ln\tan x=\frac{\ln\tan x}{\sec x}.$$

By l`Hospital's Rule,

$$\lim_{x\to\pi/2^-}\ln y=\lim_{x\to\pi/2^-}\frac{\ln\tan x}{\sec x}\overset{\mathcal{L}}{=}\lim_{x\to\pi/2^-}\frac{\dfrac{1}{\tan x}\cdot\sec^2 x}{\sec x\tan x}$$

$$=\lim_{x\to\pi/2^-}\frac{\sec x}{\tan^2 x}=\lim_{x\to\pi/2^-}\frac{\cos x}{\sin^2 x}=0$$

Now $y=e^{\ln y}$, and since the exponential function $f(x)=e^x$ is continuous,

$$\lim_{x\to\pi/2^-}y=\lim_{x\to\pi/2^-}\exp(\ln y)=\exp\left(\lim_{x\to\pi/2^-}\ln y\right)=\exp 0=e^0=1.$$

Problem Show that $\lim\limits_{x\to0^+}(1+x)^{\frac{1}{x}}=e.$

Answer

The limit leads to the indeterminate form 1^∞.

Let $y=(1+x)^{\frac{1}{x}}$, so that $\log y=\dfrac{1}{x}\log(1+x).$

$\therefore \quad \lim\limits_{x\to0^+}\ln y=\lim\limits_{x\to0^+}\dfrac{1}{x}\log(1+x) \qquad\qquad$ (0/0 form)

$$\overset{\mathcal{L}}{=}\lim_{x\to0^+}\frac{\dfrac{1}{1+x}}{1}=\frac{1}{1}=1.$$

$\therefore \quad \lim\limits_{x\to0^+}(1+x)^{\frac{1}{x}}=\lim\limits_{x\to0^+}y=\lim\limits_{x\to0^+}e^{\ln y}=e^1=e.$

Exercises

Evaluate the following limits:

1. $\lim\limits_{x\to0}\left[(1+x)^{\frac{1}{x}}-e^{\frac{1}{x}}\right]$ **2.** $\lim\limits_{x\to\frac{\pi}{2}}\left(\dfrac{\pi}{2}-x\right)^{\tan x}$ **3.** $\lim\limits_{x\to\infty}2^x\tan\left(\dfrac{a}{2^x}\right)$

4. $\lim\limits_{x\to\frac{\pi}{2}}(\cos x)^{\frac{\pi}{2}-x}$ **5.** $\lim\limits_{x\to0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x}}$ **6.** $\lim\limits_{x\to0}(1+\sin x)^{\cot x}$

7. $\lim\limits_{x\to 0}\left(3(1+x)^{1/2}-\left(\dfrac{1+\frac{x}{3}}{x^2}\right)\right)$ **8.** $\lim\limits_{x\to 0}\dfrac{e^x+\sin x-1}{\log(1+x)}$ **9.** $\lim\limits_{x\to 0}\left(e^{2x}+x\right)^{\frac{1}{x}}$

10. $\lim\limits_{x\to a}(x-a)^{x-a}$ **11.** $\lim\limits_{x\to n}(x-n)\cot \circledR\, x$ **12.** $\lim\limits_{x\to\infty}\left(1+x^2\right)^{e^{-x}}$

14. $\lim\limits_{x\to\frac{\circledR}{4}}(\tan x)^{\tan 2x}$ **15.** $\lim\limits_{x\to 0}\left(a^x+\dfrac{b^x}{2}\right)^{\frac{1}{x}}$ **16.** $\lim\limits_{x\to 0}\left(\dfrac{x}{e^x}-1\right)^{\frac{1}{x}}$

17. $\lim\limits_{x\to 1}\left(1-x^2\right)^{\frac{1}{\log(1-x)}}$ **18.** $\lim\limits_{x\to 0}(\cos x)^{\cot 2x}$ **19.** $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^x$

20. $\lim\limits_{x\to 0}\left(\sin 2x+2\sin^2 x-2\dfrac{\sin x}{\cos x}-\cos^2 x\right)$ **21.** $\lim\limits_{x\to 0}\log_{\tan x}\tan 2x$

22. $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{\frac{1}{x}}$ **23.** $\lim\limits_{x\to 0}(\cot x)^{\frac{1}{\log x}}$ **24.** $\lim\limits_{x\to a}\left(2-\dfrac{x}{a}\right)^{\tan\frac{\circledR\, x}{2a}}$

CHAPTER. 6 HYPERBOLIC FUNCTIONS

Hyperbolic Functions

Hyperbolic cosine of x: $\cosh x = \frac{e^x + e^{-x}}{2}$

Hyperbolic sine of x: $\sinh x = \frac{e^x - e^{-x}}{2}$

Remark: $\cosh x + \sinh x = e^x$.

Definition Using the above Definition we can define four other hyperbolic functions and are listed below:

Hyperbolic tangent: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Hyperbolic secant: $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

Hyperbolic cosecant: $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Identities in hyperbolic functions

- $\cosh(-x) = \cosh x.$
- $\sinh(-x) = -\sinh x.$
- $\cosh 0 = 1.$
- $\sinh 0 = 0.$
- $\cosh^2 x - \sinh^2 x = 1.$
- $\tanh^2 x + \operatorname{sech}^2 x = 1.$
- $\coth^2 x - \operatorname{csch}^2 x = 1.$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$
- $\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y.$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$
- $\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y.$
- $\cosh 2x = \cosh^2 x + \sinh^2 x = 1 + 2\sinh^2 x = 2\cosh^2 x - 1.$
- $\sinh 2x = 2\sinh x \cosh x.$
- $\sinh x = 2\sinh\frac{x}{2}\cosh\frac{x}{2}.$
- $\cosh^2 x = \frac{\cosh 2x + 1}{2}.$
- $\sinh^2 x = \frac{\cosh 2x - 1}{2}.$
- $\cosh 3x = 4\cosh^3 x - 3\cosh x.$

õ $\sin\text{h}\, 3x = 3\sinh x + 4\sinh^3 x.$

õ $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

õ $\tanh(x-y) = \dfrac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

õ $\sinh 2x = \dfrac{2\tanh x}{1-\tanh^2 x}.$

õ $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}.$

õ $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}.$

õ $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}.$

Problem Given $\sinh x = -\dfrac{3}{4}$. Find the other five hyperbolic functions.

Using $\cosh^2 x - \sinh^2 x = 1,$

$$\cosh x = \sqrt{1+\sinh^2 x} = \sqrt{1+\left(-3/4\right)^2} = \frac{5}{4}.$$

Also,

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{-3/4}{5/4} = -\frac{3}{5}; \qquad \coth x = \frac{1}{\tanh x} = -\frac{5}{3}.$$

$$\sec\text{h}x = \frac{1}{\cosh x} = \frac{1}{5/4} = \frac{4}{5}; \qquad \text{and} \qquad \cos\text{ech}x = \frac{1}{\sinh x} = -\frac{4}{3}.$$

Derivatives of Hyperbolic Functions

- $\dfrac{d}{dx}(\sinh x) = \cosh x.$
- $\dfrac{d}{dx}(\cosh x) = \sinh x.$
- $\dfrac{d}{dx}(\tanh x) = \text{sech}^2 x.$
- $\dfrac{d}{dx}(\coth x) = -\text{csch}^2 x.$
- $\dfrac{d}{dx}(\text{sech}\, x) = -\text{sech}\, x \tanh x.$
- $\dfrac{d}{dx}(\text{csch}\, x) = -\text{csch}\, x \coth x.$

Problem Find $\frac{dy}{dx}$, where $y = x\sinh x - \cosh x$.

Solution
$$\frac{dy}{dx} = \frac{d}{dx}(x\sinh x - \cosh x)$$
$$= \frac{d}{dx}(x\sinh x) - \frac{d}{dx}\cosh x$$
$$= x\frac{d}{dx}(\sinh x) + \frac{d}{dx}(x)\times\sinh x - \sinh x$$, applying product rule of differentiation
$$= x\cosh x + \sinh x - \sinh x$$
$$= x\cosh x.$$

Problem Evaluate $\frac{d}{dx}\left(\tanh\sqrt{1+x^2}\right)$.

Solution Take $u = \sqrt{1+x^2}$. Then, using formula 3 above,
$$\frac{d}{dx}\left(\tanh\sqrt{1+x^2}\right) = \operatorname{sech}^2\sqrt{1+x^2}\cdot\frac{d}{dx}\left(\sqrt{1+x^2}\right)$$
$$= \frac{x}{\sqrt{1+x^2}}\operatorname{sech}^2\sqrt{1+x^2}.$$

Formulae for Integral of Hyperbolic Functions

- $\int \sinh u\, du = \cosh u + C.$
- $\int \cosh u\, du = \sinh u + C.$
- $\int \operatorname{sech}^2 u\, du = \tanh u + C.$
- $\int \operatorname{csch}^2 u\ du = -\coth u + C.$
- $\int \operatorname{sech} u \tanh u\ du = -\operatorname{sech} u + C.$
- $\int \operatorname{csch} u \coth u\ du = -\operatorname{csch} u + C.$

Problem Evaluate $\int_0^{\ln 2} 4e^x \sinh x\, dx.$

Solution
$$\int_0^{\ln 2} 4e^x\sinh x\, dx = \int_0^{\ln 2} 4e^x\frac{e^x - e^{-x}}{2}\, dx = \int_0^{\ln 2}(2e^{2x} - 2)dx$$
$$= [e^{2x} - 2x]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1-0)$$
$$= e^{\ln 4} - 2\ln 2 - 1$$
$$= 4 - 2\ln 2 - 1$$
$$\approx 1.6137$$

The Inverse Hyperbolic Functions

$\sinh^{-1} x$ is the **inverse hyperbolic sine of x.**

Identities for inverse hyperbolic functions

- $\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}.$
- $\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}.$
- $\coth^{-1} x = \tanh^{-1} \frac{1}{x}.$

Relation between inverse hyperbolic functions and natural logarithm

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right),\ x \ge 1$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right),\ -\infty < x < \infty$$

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x},\ |x| < 1$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), 0 < x \le 1$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right),\ x \ne 0$$

$$\coth^{-1} x = \frac{1}{2} \ln \frac{x+1}{x-1},\ |x| > 1$$

Derivatives of inverse hyperbolic functions

1 Derivative of $\sinh^{-1}x$

Let $y = \sinh^{-1}x$. Then $x = \sinh y$.

Differentiating both sides with respect to x, we get

$$1 = \cosh y. \frac{dy}{dx}$$

Therefore $$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1 + \sinh^2 y}} = \frac{1}{\sqrt{1 + x^2}},\ \text{for real } x.$$

i.e. $$\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{1 + x^2}},\ \text{for real } x.$$

In a similar manner, we have the following derivatives.

2. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}$

3. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1 - x^2}$, for $|x| < 1$.

4. $\dfrac{d}{dx}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2}$, for $|x| > 1$.

5. $\dfrac{d}{dx}\left(\text{sech}^{-1} x\right) = -\dfrac{1}{x\sqrt{1-x^2}}$, $0 < x < 1.$

6. $\dfrac{d}{dx}\left(\text{cosech}^{-1} x\right) = -\dfrac{1}{|x|\sqrt{1-x^2}}$, $x \neq 0.$

Problem Find the derivatives of the following functions with respect to x:

(i) $\cosh^{-1}(x^2)$ (ii) $\sinh^{-1}(\tan x)$

Solution

(i) Let $y = \cosh^{-1}(x^2)$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cosh^{-1} x^2\right) = \frac{d}{du}\left(\cosh^{-1} u\right)\frac{du}{dx} \text{with } u = x^2$$

$$= \frac{1}{\sqrt{\left(x^2\right)^2 - 1}} 2x = \frac{2x}{\sqrt{x^4 - 1}} .$$

(ii) Let $y = \sinh^{-1}(\tan x)$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sinh^{-1}(\tan x)\right) = \frac{d}{du}\left(\sinh^{-1} u\right)\frac{du}{dx}$$

$$= \frac{1}{\sqrt{1+(\tan x)^2}}\sec^2 x$$

$$= \frac{\sec^2 x}{\sec x} = \sec x .$$

Integrals leading to inverse hyperbolic functions

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C,\ a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C,\ u > a > 0$

3. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{u}{a}\right) + C & \text{if } u^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{u}{a}\right) + C & \text{if } u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = -\frac{1}{a}\text{sech}^{-1}\left(\frac{u}{a}\right) + C,\quad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\text{csch}^{-1}\left|\frac{u}{a}\right| + C,\quad u \neq 0$

Problem Evaluate the definite integral $\int_0^1 \frac{2dx}{\sqrt{3+4x^2}}$

Answer

$$\int_0^1 \frac{2dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1$$

$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

Exercises

Each of Exercises 1- 2 gives a value of $\sinh x$ or $\cosh x$. Use the definitions and the identity $\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1. $\sinh x = \frac{4}{3}$ 2. $\cosh x = \frac{13}{5},\ x > 0$

Rewrite the expressions in Exercises 3-5 in terms of exponentials and simplify as much as you can.

3. $\sinh(21\text{n}\, x)$ 4. $\cosh 3x - \sinh 3x$

5. $1\text{n}\,(\cosh x + \sinh x) + 1\text{n}(\cosh x - \sinh x)$

In Exercises 6-17, find the derivative of y with respect to the appropriate variable.

6. $y = \frac{1}{2}\sinh(2x+1)$ 7. $y = t^2 \tanh\frac{1}{t}$ 8. $y = 1\text{n}(\cosh z)$

9. $y - \text{csch}^{¯}(1 - 1\text{n}\,\text{csch}^{¯})$ 10. $y = 1\text{n}\sinh ¬ - \frac{1}{2}\coth^2 ¬$

11. $y = (4x^2 - 1)\,\text{csch}(1\text{n}\,2x)$ 12. $y = \cosh^{-1} 2\sqrt{x+1}$

13. $y = (¯^2 + 2¯)\tanh^{-1}(¯ + 1)$ 14. $y = (1 - t^2)\coth^{-1} t$

15. $y = 1\text{n}x + \sqrt{1 - x^2}\,\text{sech}^{-1}x$ 16. $y = \text{csch}^{-1} 2^{¯}$

17. $y = \cosh^{-1}(\sec x),\ 0 < x < ® / 2$

In Exercises 18-19, verify the following integration formulae:

18. $\int x\,\text{sech}^{-1}x\,dx = \frac{x^2}{2}\text{sech}^{-1}x - \frac{1}{2}\sqrt{1-x^2} + C$

19. $\int \tanh^{-1}x\,dx = x\tanh^{-1}x + \frac{1}{2}1\text{n}(1 - x^2) + C$

In Exercises 20- 24, evaluate the indefinite integrals:

20. $\int \sinh\frac{x}{5}dx$ 21. $\int 4\cosh(3x - 1\text{n}\,2)dx$ 22. $\int \coth\frac{¯}{\sqrt{3}}d¯$

23. $\int \text{csch}^2(5 - x)dx$ 24. $\int \frac{\text{csch}(1\text{n}\ t)\coth(1\text{n}\ t)dt}{t}$

In Exercises 25- 29, evaluate the definite integrals:

25. $\int_0^{1n2} \tanh 2x dx$ 26. $\int_0^{1n2} 4e^{-} \sinh^{-} d^{-}$

27. $\int_0^{®/2} 2\sinh(\sin^{-})\cos^{-} d^{-}$ 28. $\int_1^4 \frac{8\cosh\sqrt{x}}{\sqrt{x}} dx$

29. $\int_0^{1n\,10} 4\sinh^2\left(\frac{x}{2}\right) dx$

Express the numbers in Exercises 30-32 in terms of natural logarithms:

30. $\cosh^{-1}(5/3)$ 31. $\coth^{-1}(5/4)$ 32. $\operatorname{csch}^{-1}(-1/\sqrt{3})$

In Exercises 33-36, evaluate the integrals in terms of (a) inverse hyperbolic functions, (b) natural logarithms.

33. $\int_0^{1/3} \frac{6dx}{\sqrt{1+9x^2}}$ 34. $\int_0^{1/2} \frac{dx}{1-x^2}$

35. $\int_1^2 \frac{dx}{x\sqrt{4+x^2}}$ 36. $\int_1^e \frac{dx}{x\sqrt{1+(1nx)^2}}$

MODULE II

CHAPTER 7: SEQUENCES

Sequences

Definition (*Sequence*) If to each positive integer n there is assigned a (real or complex) number u_n, then these numbers $u_1, u_2, \ldots, u_n, \ldots$ are said to form an **infinite sequence** or, briefly, a **sequence**, and the numbers u_n are called the terms of the sequence. A sequence whose terms are real numbers is called *real sequence.* We discuss real sequences only.

Definition An **infinite sequence** (or **sequence**) of numbers is a function whose domain is the set of integers greater than or equal to some integer n_0. Usually n_0 is 1 and the domain of the sequence is the set of positive integers and in that case sequences are the functions from the set of positive integers.

- Based on the above definition an example of a sequence is $u(n) = \dfrac{n+1}{n}$.

The number $u(n)$ is the nth term of the sequence, or the term with index n. If $u(n) = \dfrac{n+1}{n}$, we have

First term	Second term	Third term		nth term
$u(1) = 2$	$u(2) = \dfrac{3}{2}$	$u(3) = \dfrac{4}{3},$	$\ldots$	$u(n) = \dfrac{n+1}{n}$

When we use the subscript notation u_n for $u(n)$, the sequence is written

$$u_1 = 2, \qquad u_2 = \frac{3}{2} \qquad u_3 = \frac{4}{3}, \qquad \ldots \qquad u_n = \frac{n+1}{n}$$

Some other examples of sequences are

$$u(n) = \sqrt{n}, \qquad u(n) = (-1)^{n+1}\frac{1}{n}, \qquad u(n) = \frac{n-1}{n}.$$

We refer to the sequence whose nth term is u_n with the notation $\{u_n\}$

- If $b \in$, the set of real numbers, the sequence $B = (b, b, b, \ldots)$, all of whose terms equal b, is called the **constant sequence** b. Thus the constant sequence 1 is the sequence $(1, 1, 1, \ldots)$, all of whose terms equal 1, and the constant sequence 0 is the sequence $(0, 0, 0, \ldots)$.
- If $a \in$, then the sequence $A = \{a^n\}$ is the sequence $a, a^2, a^3, \ldots, a^n, \ldots$. In particular, if $a = \frac{1}{2}$, then we obtain the sequence

$$\left\{\frac{1}{2^n}\right\} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right\}.$$

Definition A sequence $\{u_n\}$ is said to **converge** or to **be convergent** if there is a number l with the following property : For every £ > 0 (i.e., £ is a positive real number that may be very small, but not zero) we can find a positive integer N such that

$$n > N \;\Rightarrow\; |u_n - l| < £.$$

l is called the **limit** of the sequence. Then we write

$$\lim_{n\to\infty} u_n = l$$

or simply

$$u_n \to l \quad \text{as} \quad n \to \infty$$

and we say that the sequence **converges** to l or **has the limit** l. If no such number l exists, we say that $\{u_n\}$ **diverges.**

Problem Show that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Answer

Here $u_n = \frac{1}{n}$ and $l = 0$. Let $£ > 0$ be given. We must show that there exists an integer N such that

$$n > N \;\Rightarrow\; |u_n - l| < £$$

i.e., to show that there exists a positive integer N such that

$$n > N \;\Rightarrow\; \left|\frac{1}{n} - 0\right| < £ .$$

The implication in (1) will hold if $\frac{1}{n} < £$ or $n > \frac{1}{£}$. If N is any integer greater than $\frac{1}{£}$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$.

Problem Show that $\lim_{n\to\infty} k = k$ (where k is a constant).

Answer

Let $£ > 0$ be given. We must show that there exists a positive integer N such that

$$n > N \;\Rightarrow\; |k - k| < £.$$

Since $k - k = 0$, we can use any positive integer for N and the implication will hold. This proves that $\lim_{n\to\infty} k = k$ for any constant k.

Problem The sequence $(0, 2, 0, 2, \ldots, 0, 2, \ldots)$ does *not* converge to 0.

Answer Here

$$u_n = \begin{cases} 0, & \text{when } n \text{ is odd} \\ 2, & \text{when } n \text{ is even} \end{cases}$$

If we choose $£ = 1$, then, for any positive integer N, one can always select an even number $n > N$, for which the corresponding value $u_n = 2$ and for which $|u_n - 0| = |2 - 0| = 2 > 1$. Thus, the number 0 is *not* the limit of the given sequence (u_n).

Recursive Definitions

So far, we have calculated each u_n directly from the value of n. But, some sequences are defined **recursively** by giving

1. The value(s) of the initial term (or terms), and
2. A rule, called a **recursion formula,** for calculating any later term from terms that precede it.

Problem The statements $u_1 = 1$ and $u_n = u_{n-1} + 1$ define the sequence $1, 2, 3, \ldots, n, \ldots$ of positive integers. With $u_1 = 1,$ we have $u_2 = u_1 + 1 = 2,$ $u_3 = u_2 + 1 = 3,$ and so on.

SUBSEQUENCES

If the terms of one sequence appear in another sequence in their given order, we call the first sequence a **subsequence** of the second.

Problem Some subsequences of $X = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$ are

$$\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{n+2}, \cdots\right\}, \quad \left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2n-1}, \cdots\right\}, \qquad \text{and } \left\{\frac{1}{2!}, \frac{1}{4!}, \cdots, \frac{1}{(2n)!}, \cdots\right\}.$$

But $Y = \left\{\frac{1}{2}, \frac{1}{1}, \frac{1}{3}, \cdots\right\}$ is not a subsequence of X, because the terms of Y **do not appear** in X **in the given order.**

Definition A **tail** of a sequence is a subsequence that consists of all terms of the sequence from some index N on. In other words, a tail is one of the sets $\{u_n \mid n \geq N\}$.

Definition If $X = \{u_1, u_2, \ldots, u_n, \ldots\}$ is a sequence of real numbers and if m is a given natural number, then the m-tail of X is the sequence

$X_m = \{u_{m+1}, u_{m+2}, \ldots\}$ and its nth term is u_{m+n}.

For example, the 3-tail of the sequence

$X = \{2, 4, 6, 8, 10, \ldots, 2n, \ldots\}$,

is the sequence

$X_3 = \{8, 10, 12, \ldots, 2n+6, \ldots\}$.

Remark Another way to say that $u_n \to L$ is to say that for every £ $> 0,$ the £ - open interval $(L - £, L + £)$ about L contains a tail of the sequence.

Bounded Nondecreasing Sequences

Definition A sequence $\{u_n\}$ with the property that $u_n \leq u_{n+1}$ for all n is called a **nondecreasing sequence.** Some examples of nondecreasing sequences are

i) The sequence $1, 2, 3, \ldots, n, \ldots$ of natural numbers

ii) The constant sequence {3}

Definition A sequence $\{u_n\}$ is **bounded from above** if there exists a number M such that $u_n \le M$ for all n. The number M is an **upper bound** for $\{u_n\}$. If M is an upper bound for $\{u_n\}$ and no number less than M is an upper bound for $\{u_n\}$, then M is the **least upper bound** for $\{u_n\}$.

Theorem 1 A non-decreasing sequence that is bounded from above always has a least upper bound.

- The sequence $1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots$ is bounded from above with an upper bound 1. 1 is the least upper bound as no number less than 1 is an upper bound. Also note that any real number greater than or equal to 1 is also an upper bound.
- The sequence $1, 2, 3, \ldots, n, \ldots$ has no upper bound.

Theorem 2 (The Nondecreasing sequence theorem)

A nondecreasing sequence of real numbers converges if and only if it is bounded from above. If a nondecreasing sequence converges, it converges to its least upper bound.

Exercises

Each of Exercises 1-7 gives a formula for the nth term u_n of a sequence $\{u_n\}$. Find the values of $u_1, u_2, u_3,$ and u_4.

1. $u_n = \dfrac{1}{n!}$ 2. $u_n = 2 + (-1)^n$ 3. $u_n = \dfrac{2^n - 1}{2^n}$

4. $u_n = 1 + (-1)^n$, 5. $u_n = \dfrac{(-1)^n}{n}$, 6. $u_n = \dfrac{1}{n(n+1)}$

7. $u_n = \dfrac{1}{n^2 + 2}$.

In Exercises 8-11 the first few terms of a sequence $\{u_n\}$ are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the nth term u_n.

8. $5, 7, 9, 11, \ldots,$ 9. $\dfrac{1}{2}, -\dfrac{1}{4}, \dfrac{1}{8}, -\dfrac{1}{16}, \cdots,$

10. $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \cdots,$ 11. $1, 4, 9, 16, \ldots$

Each of Exercises 12-18 gives the first term or two of a sequence along with a recursion formula for the remaining terms. Write out the first ten terms of the sequence.

12. $u_1 = 1,$ $u_{n+1} = \dfrac{u_n}{n+1}$

13. $u_1 = -2,$ $u_{n+1} = \dfrac{nu_n}{n+1}$

14. $u_1 = 2,\qquad u_2 = -1,\qquad u_{n+2} = \dfrac{u_{n+1}}{u_n}$

15. $u_1 = 1,\qquad u_{n+1} = 3u_n + 1,$

16. $v_1 = 2,\qquad v_{n+1} = \frac{1}{2}(y_n + \frac{2}{y_n})$

17. $u_1 = 1,\qquad u_2 = 2,\qquad u_{n+2} = \dfrac{(u_{n+1} + u_n)}{(u_{n+1} - u_n)}$

18. $u_1 = 3,\qquad u_2 = 5,\qquad u_{n+2} = u_n + u_{n+1}$

In Exercises 19-23, find a formula for the *n*th term of the sequence.

19. The sequence $-1, 1, -1, 1, -1, \ldots$

20. The sequence $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

21. The sequence $-3, -2, -1, 0, 1, \ldots$

22. The sequence 2, 6, 10, 14, 18, …

23. The sequence 0,1,1,2,2,3,3,4,…

CHAPTER.8 THEOREMS FOR CALCULATING LIMITS OF SEQUENCES

Definition If $X = \{u_n\}$ and $Y = \{v_n\}$ are sequences of real numbers, then we define their **sum** to be the sequence $X + Y = \{u_n + v_n\}$, their **difference** to be the sequence $X - Y = \{u_n + v_n\}$, and their **product** to be the sequence $X \cdot Y = \{u_n v_n\}$. If $c \in$ we define the **multiple** of X by

c to be the sequence $cX = \{cu_n\}$. Finally, if $Z = \{w_n\}$ is a sequence of real numbers with $w_n \neq 0$ for all $n \in$, then we define the **quotient** of X and Z to be the sequence $\frac{X}{Z} = \left\{ \frac{u_n}{w_n} \right\}$.

Theorem 3 Let $\{u_n\}$ and $\{v_n\}$ be sequences of real numbers. The following rules hold if $\lim_{n\to\infty} u_n = A$ and $\lim_{n\to\infty} v_n = B$ where A and B be real numbers.

1. *Sum Rule* : $\lim_{n\to\infty}(u_n + v_n) = A + B$
2. *Difference Rule* : $\lim_{n\to\infty}(u_n - v_n) = A - B$
3. *Product Rule* : $\lim_{n\to\infty}(u_n \cdot v_n) = A \cdot B$
4. *Constant Multiple Rule* : $\lim_{n\to\infty}(k \cdot v_n) = k \cdot B$ (Any number k)
5. *Quotient Rule* : $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{A}{B}$ if $B \neq 0$

Problem Show that $\lim_{n\to\infty} \frac{2n+1}{n} = 2$

Answer

Since $\frac{2n+1}{n} = 2 + \frac{1}{n}$, we have

$$\lim_{n\to\infty} \frac{2n+1}{n} = \lim_{n\to\infty}\left(2 + \frac{1}{n}\right) = \lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{1}{n} = 2 + 0 = 2.$$

Theorem 4 The Sandwich Theorem for Sequences

Let $\{u_n\}, \{v_n\}$, and $\{w_n\}$ be sequences of real numbers. If $u_n \leq v_n \leq w_n$ holds for all n beyond some index N, and if $\lim_{x\to 0} u_n = \lim_{x\to 0} w_n = L$, then $\lim_{n\to\infty} v_n = L$ also.

Remark An immediate consequence of Theorem 4 is that, if $|v_n| \leq w_n$ and $w_n \to 0$, then $v_n \to 0$ because $-w_n \leq v_n \leq w_n$. We use this fact in the coming examples.

Problem Show that $\left\{ \frac{1}{2^n} \right\}$ converges to 0.

Answer

$\frac{1}{2^n} \to 0$ because $\frac{1}{2^n} \leq \frac{1}{n}$ and since $\lim_{n\to\infty} \frac{1}{n} = 0.$

Theorem 5: The Continuous Function Theorem for Sequences

Let $\{u_n\}$ be a sequence of real numbers. If $u_n \to L$ and if f is a function that is continuous at L and defined at all u_n, then $f(u_n) \to f(L)$.

Using l'Hôpital's Rule

Theorem 6 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{u_n\}$ is a sequence of real numbers such that $u_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \quad \Rightarrow \quad \lim_{n\to\infty} u_n = L.$$

***Example* 39** Find $\lim_{n\to\infty} \frac{2^n}{5n}$.

Solution $\lim_{n\to\infty} \frac{2^n}{5n}$ is in the $\frac{\infty}{\infty}$ form. Hence

$\lim_{n\to\infty} \frac{2^n}{5n} = \lim_{n\to\infty} \frac{2^n \cdot \ln 2}{5}$, applying L'Hôpital's rule, and noting that the derivative of 2^n with respect to n is $2^n \ln 2$.

$= \infty$.

Problem Does the sequence whose *n*th term is

$$u_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} u_n$.

Solution The limit leads to the indeterminate form 1^∞. We can apply l'Hôpital's rule if we first change it to the form $\infty \cdot 0$ by taking the natural logarithm of u_n.

$$\ln u_n = \ln\left(\frac{n+1}{n-1}\right)^n$$

$$= n \ln\left(\frac{n+1}{n-1}\right).$$

$$\lim_{n\to\infty} \ln u_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad (\infty \cdot 0 \text{ form})$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{\frac{1}{n}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{n\to\infty} \frac{-\frac{2}{(n^2-1)}}{-\frac{1}{n^2}}, \text{ applying l'Hôpital's rule}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = \lim_{n\to\infty} \frac{2}{1-\frac{1}{n^2}} = 2.$$

Since $\ln u_n \to 2$ as $n \to \infty$, and $f(x) = e^x$ is continuous everywhere, Theorem 5 tells us that

$$u_n = e^{\ln u_n} \to e^2.$$

That is, the sequence $\{u_n\}$ converges to e^2.

Exercises

Which of the sequences $\{u_n\}$ in Exercises 1-31 converge, and which diverge? Find the limit of each convergent sequence.

1. $u_n = \dfrac{n+(-1)^n}{n}$ 2. $u_n = \dfrac{2n+1}{1-3\sqrt{n}}$ 3. $u_n = \dfrac{n+3}{n^2+5n+6}$ 4. $u_n = \dfrac{n}{n+1}$

5. $u_n = \dfrac{(-1)^n n}{n+1}$ 6. $u_n = \dfrac{n^2}{n+1}$ 7. $u_n = \dfrac{2n^2+3}{n^2+1}.$ 8. $u_n = \dfrac{1-n^3}{70-4n^2}$

9. $u_n = (-1)^n\left(1-\dfrac{1}{n}\right)$ 10. $u_n = \left(2-\dfrac{1}{2^n}\right)\left(3+\dfrac{1}{2^n}\right)$ 11. $u_n = \left(-\dfrac{1}{2}\right)^n$

12. $u_n = \dfrac{1}{(0.9)^n}$ 13. $u_n = n\text{®}\cos(n\text{®})$ 14. $u_n = \dfrac{\sin^2 n}{2^n}$

15. $u_n = \dfrac{3^n}{n^3}$ 16. $u_n = \dfrac{\ln n}{\ln 2n}$ 17. $u_n = (0.03)^{\frac{1}{n}}$

18. $u_n = \left(1-\dfrac{1}{n}\right)^n$ 19. $u_n = \sqrt[n]{n^2}$ 20. $u_n = (n+4)^{\frac{1}{(n+4)}}$

21. $u_n = \ln n - \ln(n+1)$ 22. $u_n = \sqrt[n]{3^{2n+1}}$ 23. $u_n = \dfrac{(-4)^n}{n!}$

24. $u_n = \dfrac{n!}{2^n \cdot 3^n}$ 25. $u_n = \ln\left(1+\dfrac{1}{n}\right)^n$ 26. $u_n = \left(\dfrac{n}{n+1}\right)^n$

27. $u_n = \left(1-\dfrac{1}{n^2}\right)^n$ 28. $u_n = \dfrac{\left(\frac{10}{11}\right)^n}{\left(\frac{9}{10}\right)^n + \left(\frac{11}{12}\right)^n}$ 29. $u_n = \sinh(\ln n)$

30. $u_n = n\left(1-\cos\dfrac{1}{n}\right)$ 31. $u_n = \dfrac{1}{\sqrt{n}}\tan^{-1} n$ 32. $u_n = \sqrt[n]{n^2+n}$

33. $u_n = \dfrac{(\ln n)^5}{\sqrt{n}}$ 34. $u_n = \dfrac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$ 35. $u_n = \int_1^n \dfrac{1}{x^p}dx,\ p>1$

36. Give an example of two divergent sequences X, Y such that their product XY converges.
37. Show that if X and Y are sequences such that X and $X+Y$ are convergent, then Y is convergent.
38. Show that if X and Y are sequences such that X converges to $x \neq 0$ and XY converges, then Y converges.
39. Show that the sequence $\{2^n\}$ is not convergent.
40. Show that the sequence $\{(-1)^n n^2\}$ is not convergent.

In Exercises 41-44, find the limits of the following sequences:

41. $\left\{\left(2+\dfrac{1}{n}\right)^2\right\}$ 42. $\left\{\dfrac{(-1)^n}{n+2}\right\}$ 43. $\left\{\dfrac{\sqrt{n}-1}{\sqrt{n}+1}\right\}$ 44. $\left\{\dfrac{n+1}{n\sqrt{n}}\right\}$

CHAPTER.9 SERIES

Definition If $u_1, u_2, u_3, \ldots, u_n, \ldots$ be a sequence of real numbers, then

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is called an **infinite series** or, briefly, **series**. u_n is the ***n*th term** of the series. An infinite series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

is denoted by

$$\sum_{n=1}^{\infty} u_n$$

The sum

$$s_n = u_1 + u_2 + \ldots + u_n$$

(i.e., the sum of the first n terms of the series) is called the n^{th} **partial sum** of the series

The sequence

$$s_1, s_2, s_3, \ldots, s_n, \ldots$$

where s_n is the n^{th} partial sum of the series $\sum_{n=1}^{\infty} u_n$, is called the **sequence of** n^{th} **partial sum.**

Problem Find the n^{th} partial sum of the series $\sum_{n=1}^{\infty} u_n$, where $u_n = (-1)^n$.

Answer

The given series is $\sum_{n=1}^{\infty} u_n = -1 + 1 - 1 + 1 - 1 + \ldots + (-1)^n + \ldots$

The n^{th} partial sum is given by

$$s_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Convergence, Divergence and Oscillation of a series

Consider the infinite series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots + u_n + \ldots \qquad \ldots (1)$$

and let the sum of the first n terms be

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n$$

Then, is the n^{th} partial sum of the series (1). The sequence

$$s_1, s_2, s_3, \ldots, s_n, \ldots \qquad \ldots (2)$$

is the sequence of n^{th} partial sums of the series (1)

As $n \to \infty$ three possibilities arise:

(i) The sequence given by (2) converges to a finite number *l*; in this case the series $\sum_{n=1}^{\infty} u_n$ is said to be **convergent** and has the **sum** *l*.

i.e., $\sum_{n=1}^{\infty} u_n = l$. (i.e., the series is **summable** with sum *l*).

(ii) The sequence (2) doesn't converge but tends to $+\infty$ or $-\infty$ as $n \to \infty$; in this case the series $\sum_{n=1}^{\infty} u_n$ is said to be **divergent** and has no sum. (i.e., the series is **not summable**)

(iii) If the both the cases (i) and (ii) above do not occur, then the series $\sum_{n=1}^{\infty} u_n$ is said to be **oscillatory** or **non-convergent**. (In this case also the series is not summable).

Problem Show that the series $1+\frac{1}{2}+\frac{1}{2^2}+\ldots$ converges and also find its sum.

Solution

Let $u_n = \frac{1}{2^{n-1}}$

Then the n^{th} partial sum is given by

$$s_n = 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} = \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 2-\frac{1}{2^{n-1}}$$

Since $\frac{1}{2^{n-1}} \to 0$ as $n \to \infty$, $s_n \to 2$ as $n \to \infty$.

Since (s_n), the sequence of n^{th} partial sums, converges to 2, the given series also converges and the sum of the series is 2

Problem Show that the **geometric series**

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$$

converges if $|r| < 1$ and diverges if $|r| \ge 1$.

Proof

***Case* 1** $|r| < 1$

The n^{th} partial sum of the series is given by

$$s_n = a + ar + ar^2 + \ldots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \qquad \ldots (3)$$

We note that when $|r| < 1, r^n \to 0$ as $|r| < 1, r^n \to 0$, Hence,

$$\lim_{n\to\infty} \frac{ar^n}{1-r} = \frac{a}{1-r} \lim_{n\to\infty} r^n = a \cdot 0 = 0 \quad \text{for} \quad |r| < 1$$

Hence from (3), we obtain

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

i.e., the sequence (s_n) of n^{th} partial sums converges to $\frac{a}{1-r}$.

Hence the given series also converges to $\frac{a}{1-r}$ for $|r|<1$. In otherwords, $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r},\ |r|<1$.

***Case* 2** When $r=1$, we have

$$s_n = 1+1+\ldots+1 = n.$$

Hence as $n\to\infty, s_n\to\infty$.

So the sequence of n^{th} partial sums diverges and hence the given series also diverges.

***Case* 3** When $|r|>1 |r^n| \to\infty$ as $n\to\infty$,

$$s_n = a + ar + \ldots + ar^n$$

$$= \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1} \to \infty \text{ as } n\to\infty .$$

So in this case the sequence of n^{th} partial sums diverges. Hence the geometric series

$$a + ar + ar^2 + \ldots$$

converges if $|r|<1$ and diverges if $|r|>1$.

If $|r|<1$, the geometric series $a + ar + ar^2 + \ldots + ar^{n-1} + \ldots$ converges to $\frac{a}{1-r}$ and if $|r|\geq 1$, the series diverges.

Problem Show that the series $\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots$ converges. Find its sum.

Solution The given is a geometric series with $a=\frac{1}{9}$ and $r=\frac{1}{3}$. Here $|r|<1$. Hence the series is convergent and its sum is given by

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots = \frac{\frac{1}{9}}{1-\left(\frac{1}{3}\right)} = \frac{1}{6}.$$

Problem Discuss the convergence of the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$$

Also find it sum.

Answer

Here $u_n = \dfrac{1}{n(n+1)}$. By partial fraction, $\dfrac{1}{n(n+1)} = \dfrac{A}{n} + \dfrac{B}{n+1}$, which gives, $1 = A(n+1) + Bn$. Putting $n = -1$, we have $B = -1$ and putting $n = 0, A = 1$.

$$\because \quad u_n = \frac{1}{n} - \frac{1}{n+1}$$

The n^{th} partial sum of the series is given by

$$\begin{aligned} s_n &= u_1 + u_2 + \ldots + u_n \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Hence $\quad \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(1 - \dfrac{1}{n+1}\right) = 1 - 0 = 1$

Since the sequence of n^{th} partial sums converges to 1, the series also converges to 1. Hence we can write

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n(n+1)} + \ldots = 1.$$

Theorem 1 If the series $u_1 + u_2 + \ldots + u_n + \ldots = \sum\limits_{n=1}^{\infty} u_n$

converges then $\lim\limits_{n\to\infty} u_n = 0$. i.e., the n^{th} term of a convergent series must tend to zero as $n \to \infty$.

Proof Let s_n denote the n^{th} partial sum of the series. Then we note that

$$\lim_{n\to\infty} s_{n-1} = \lim_{n\to\infty} s_n.$$

Since $s_n = u_1 + u_2 + \ldots + u_n$ and $s_{n-1} = u_1 + u_2 + \ldots + u_{n-1}$

we have $\quad s_n = s_{n-1} + u_n$

or $\quad u_n = s_n - s_{n-1}$

Hence

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_n = 0.$$

Hence the proof.

DIVERGENT SERIES

Geometric series with $|r| \geq 1$ are not the only series to diverge. We discuss some other divergent series.

Problem Show that the series

$$\sum_{m=1}^{\infty} \frac{n+1}{n} = \frac{2}{1} + \frac{3}{2} + \frac{4}{3} + \cdots + \frac{n+1}{n} \cdots$$

diverges.

Solution

The given senes diverges because the partial sums eventually outgrow every preassigned number. Each term is greater than 1, so the sum of *n* terms is greater than *n*.

Simple Test for Divergence (nth Term Test)

Theorem 2 (n^{th} Term Test) A necessary condition for the convergence of a series

$$\sum_{m=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

is that

$$\lim_{n\to\infty} u_n = 0$$

i.e, if the series $\sum_{n=1}^{\infty} u_n$ converges, then $\lim_{n\to\infty} u_n = 0$.

Attention! The condition in Theorem 2 is only necessary for convergence, ***but*** not sufficient. As an example, the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

satisfies the condition $\lim_{n\to\infty} x_n = 0$ but it diverges.

Divergence Test

In view of Theorem 2 we have the following:

If $\lim_{n\to\infty} u_n \neq 0$, the series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + ... + u_n + ...$$

diverges.

In view of n^{th} Term Test, $\sum_{n=1}^{\infty} n^2$ is not convergent as $\lim_{n\to\infty} u_n = \lim_{n\to\infty} n^2 \neq 0$.

Problem Discuss the convergence of the series

$$\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} + \sqrt{\frac{3}{4}} + \cdots$$

Answer

Let $u_n = \sqrt{\frac{n}{n+1}}$ Then

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = \sqrt{\frac{1}{1+0}} = 1 \neq 0$$

Since $\lim_{n\to\infty} u_n \neq 0$, the given series cannot converge.

Theorem 3

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule*: $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule*: $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule*: $\sum ka_n = k\sum a_n = kA$

Problem Show that the series $\sum_{n=1}^{\infty} \frac{4}{2^{n-1}}$ converges.

Answer

$\sum_{n=1}^{\infty} \frac{4}{2^{n-1}} = 4\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$, by constant multiple Rule as the above is a geometric series with $a = 1, r = \frac{1}{2}$. Hence the given series converges.

Adding or Deleting Terms

We can always add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$ and $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_{k-1} + \sum_{n=1}^{\infty} a_n$

Conversely, if $\sum_{n=k}^{\infty} a_n$ converges for any $k > 1$, then $\sum_{n=1}^{\infty} a_n$ converges. Thus,

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=4}^{\infty} \frac{1}{5^n}\right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}$$

Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in the formula for a_n by $n-h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = a_1 + a_2 + a_3 + \dots .$$

To lower the starting value of the index h units, replace the n in the formula for a_n by $n+h$:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1-h}^{\infty} a_{n+h} = a_1 + a_2 + a_3 + \dots .$$

Remark The partial sums remain the same no matter what indexing we choose. We usually give preference to indexings that lead to simple expressions.

Exercises

In Exercises 1-3, find a formula for the nth partial sum of each series and use it to find the series sum if the series converges.

1. $\frac{9}{100} + \frac{9}{100^2} + \frac{9}{100^3} + \cdots + \frac{9}{100^n} + \cdots$

2. $1 - 2 + 4 - 8 + \cdots + (-1)^{n-1} 2^{n-1} + \cdots$

3. $\frac{5}{1\cdot 2} + \frac{5}{2\cdot 3} + \frac{5}{3\cdot 4} + \cdots + \frac{5}{n(n+1)} + \cdots$

In Exercises 4 - 7, write out the first few terms of each series to show how the series starts. Then find the sum of the series.

4. $\sum_{n=2}^{\infty} \frac{1}{4^n}$

5. $\sum_{n=0}^{\infty} (-1)^n \frac{5}{4^n}$

6. $\sum_{n=0}^{\infty} \left(\frac{5}{2^n} - \frac{1}{3^n} \right)$

7. $\sum_{n=0}^{\infty} \left(\frac{2^{n+1}}{5^n} \right)$

Use partial fractions to find the sum of each series in Exercises 8-11.

8. $\sum_{n=1}^{\infty} \frac{6}{(2n-1)(2n+1)}$

9. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

10. $\sum_{n=1}^{\infty} \left(\frac{1}{2^{\frac{1}{n}}} - \frac{1}{2^{\frac{1}{(n+1)}}} \right)$

11. $\sum_{n=1}^{\infty} (\tan^{-1}(n) - \tan^{-1}(n+1))$

Which series in Exercises 12-20 converge, and which diverge? Give reasons for your answers. If a series converges, find its sum.

12. $\sum_{n=0}^{\infty} \left(\sqrt{2}\right)^n$

13. $\sum_{n=0}^{\infty} (-1)^{n+1} n$

14. $\sum_{n=0}^{\infty} \frac{\cos n®}{5^n}$

15. $\sum_{n=1}^{\infty} \ln \frac{1}{n}$ 16. $\sum_{n=0}^{\infty} \frac{1}{x^n}, |x| > 1$ 17. $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

18. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ 19. $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$ 20. $\sum_{n=0}^{\infty} \frac{e^{nx}}{®^{ne}}$

In each of the geometric series in Exercises 21-22, write out the first few terms of the series to find *a* and *r*, and find the sum of the series. Then express the inequality $|r| < 1$ in terms of *x* and find the values of *x* for which the inequality holds and the series converges.

21. $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ 22. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2} \left(\frac{1}{3 + \sin x}\right)^n$

In Exercises 23-25, find the values of *x* for which the given geometric series converges. Also, find the sum of the series (as a function of *x*) for those values of *x*.

23. $\sum_{n=0}^{\infty} (-1)^n x^{-2n}$ 24. $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (x-3)^n$ 25. $\sum_{n=0}^{\infty} (\ln x)^n$.

NONDECREASING PARTIAL SUMS

Theorem 4 A series $\sum_{n=1}^{\infty} u_n$ of nonnegative terms converges if and only if its partial sums are bounded from above.

***Problem* (*The Harmonic Series*)**

The series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is called the **harmonic series**. It diverges because there is no upper bound for its partial sums. To see why, group the terms of the series in the following way:

$$1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{> \frac{2}{4} = \frac{1}{2}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{> \frac{4}{8} = \frac{1}{2}} + \underbrace{\left(\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16}\right)}_{> \frac{8}{16} = \frac{1}{2}} + \cdots$$

The sum of the first two terms is 1.5. The sum of the next two terms is $\frac{1}{3} + \frac{1}{4}$,which is greater than $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. The sum of the next four terms is $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$, which is greater than $\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$. The sum of the next eight terms is $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$, which is greater than $\frac{8}{16} = \frac{1}{2}$. The sum of the next 16 terms is greater than $\frac{16}{32} = \frac{1}{2}$, and so on. In general, the sum of 2^n terms ending with $\frac{1}{2^{n+1}}$ is greater than $\frac{2^n}{2^{n+1}} = \frac{1}{2}$. Hence the sequence of partial sums is not bounded from above: For, if $n = 2^k$ the partial sum s_n is greater than $\frac{k}{2}$. Hence, by Theorem 4, the harmonic series diverges.

The Integral Test and *p*-series

Theorem 5 The Integral Test

Let $\{u_n\}$ be a sequence of positive terms. Suppose that $u_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} u_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

Problem Using the Integral Test, show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \qquad (1)$$

(p a real constant) converges if $p > 1$ and diverges if $p \leq 1$.

Solution

***Case* 1** If $p > 1$ then $f(x) = \dfrac{1}{x^p}$ is a positive decreasing function of x for $x > 1$. Now,

$$\int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \lim_{b \to \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b$$

$$= \frac{1}{1-p} \lim_{b \to \infty} \left(\frac{1}{b^{p-1}} - 1 \right) = \frac{1}{1-p}(0-1) \text{ , since } b^{p-1} \to \infty \text{ as } b \to \infty$$

because $p - 1 > 0$.

$$= \frac{1}{p-1}.$$

Hence $\int_1^{\infty} \frac{1}{x^p} dx$ converges and hence, by the Integral Test, the given series converges.

***Case* 2** If $p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{1-p} \lim_{b \to \infty} (b^{1-p} - 1) = \infty \text{ as } \lim_{b \to \infty} b^{1-p} = \infty \text{ for } 1 - p > 0$$

Hence, by the Integral Test,

The series diverges for $p < 1$.

If $p = 1$, we have the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,$$

which is known by Example 22, to be divergent,

Hence we conclude that the series converges for $p > 1$but diverges when $p \leq 1$.

Remark We not that Theorem says, in particular, that

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \qquad \text{diverges.}$$

Exercises

Which of the series in Exercises 1-15 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} e^{-n}$ 2. $\sum_{n=1}^{\infty} \frac{5}{n+1}$ 3. $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}}$

4. $\sum_{n=1}^{\infty} \frac{-8}{n}$ 5. $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ 6. $\sum_{n=1}^{\infty} \frac{5^n}{4^n+3}$

7. $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ 8. $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ 9. $\sum_{n=1}^{\infty} n \tan\frac{1}{n}$

10. $\sum_{n=1}^{\infty} \frac{1}{(\ln 3)^n}$ 11. $\sum_{n=1}^{\infty} \frac{1}{n(1+\ln^2 n)}$ 12. $\sum_{n=1}^{\infty} n \tan\frac{1}{n}$

13. $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ 14. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ 15. $\sum_{n=1}^{\infty} \operatorname{sech}^2 n$

COMPARISON TESTS FOR SERIES OF NON-NEGATIVE TERMS

Theorem 6 (Direct Comparison Test) Let $\sum u_n$ and $\sum v_n$ be two series with non-negative terms such that $u_n \le v_n$ for all $n > N$, for some integer N. Then

(a) if $\sum v_n$ is convergent, then $\sum u_n$ is also convergent.

(b) if $\sum u_n$ is divergent, then $\sum v_n$ is also divergent.

Theorem 7 Limit Comparison Test

Suppose that $u_n > 0$ and $v_n > 0$ for all $n \ge N$ (*where N is* an integer).

1. If $\lim_{n\to\infty} \frac{u_n}{v_n} = c > 0$, then $\sum u_n$ and $\sum v_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$ and $\sum v_n$ converges, then $\sum u_n$ converges.

3. If $\lim_{n\to\infty} \frac{u_n}{v_n} = \infty$ and $\sum v_n$ diverges, then $\sum u_n$ diverges.

Problem Test the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n^3}$$

Solution

Let $$u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n^3}$$

Then

$$u_n = \frac{\left(\sqrt{n+1}-\sqrt{n-1}\right)\left(\sqrt{n+1}+\sqrt{n-1}\right)}{n^3\left(\sqrt{n+1}-\sqrt{n-1}\right)}$$

$$= \frac{(n+1)-(n-1)}{n^3 \cdot \sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)} = \frac{2}{n^3\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

Take $\quad v_n = \dfrac{1}{n^{\frac{7}{2}}}$.

Then $\quad \lim\limits_{n\to\infty}\dfrac{u_n}{v_n} = 2\cdot\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}} = 2\times\dfrac{1}{2} = 1 \neq 0$

$\therefore$ The given series $\sum u_n$ converges as $\sum v_n$, being a harmonic series with $p=\frac{7}{2}>1$, converges.

Problem Test for convergence or divergence the series

$$\frac{2}{1^h}+\frac{3}{2^h}+\frac{4}{3^h}+\frac{5}{4^h}+\cdots$$

Solution

Let n^{th} term be u_n. Then

$$u_n = \frac{n+1}{n^h} = \frac{n\left(1+\frac{1}{n}\right)}{n\left(n^{h-1}\right)}$$

Let $v_n = \dfrac{1}{n^{h-1}}$ so that $\sum v_n = \sum \dfrac{1}{n^{h-1}}$, a harmonic series with $p=h-1$, which is convergent for $h-1>1$ and divergent for $h-1\le 1$.

Now $\lim\limits_{n\to\infty}\left(\dfrac{u_n}{v_n}\right) = \lim\limits_{n\to\infty}\dfrac{\left(1+\frac{1}{n}\right)}{\left(n^{h-1}\right)}\cdot\dfrac{n^{h-1}}{1} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right) = 1$,

a finite non zero number. Hence, by comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

Since the series $\sum\limits_{1}^{\infty} v_n$ is convergent for $h-1>1$ is also convergent for $h-1\le 1$ i.e., for $h>2$.

The series $\sum\limits_{1}^{\infty} v_n$ is divergent for $h-1\le 1$, hence $\sum u_n$ is also divergent for $h-1\le 1$ i.e., for $h\le 2$.

D'Alembert's Ratio-Test for Convergence

Theorem 7 D'Alembert's Ratio-Test

If $\sum_{n=1}^{\infty} u_n$ is a series with positive terms, and if $\lim_{x\to\infty} \frac{u_{n+1}}{u_n} = 1$, then

(i) $\sum_{n=1}^{\infty} u_n$ is convergent when $l < 1$,

(ii) $\sum_{n=1}^{\infty} u_n$ is divergent when $l > 1$

(iii) the test is inconclusive when $l = 1$. i.e., the series may converge or diverge when $l = 1$.

Problem Test the convergence of the series $\sum_{n=0}^{\infty} \frac{n^3+1}{5^n+1}$

Answer

Let $u_n = \frac{n^3+1}{5^n+1}$, then $u_{n+1} = \frac{(n+1)^3+1}{5^{n+1}+1}$

and $$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^3+1}{5^{n+1}+1} \times \frac{5^n+1}{n^3+1} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^3 + \frac{1}{n^3}}{1+\frac{1}{n^3}} \cdot \frac{1+\frac{1}{5^n}}{5+\frac{1}{5^n}}$$

$$= \frac{1+0}{1+0} \frac{1+0}{5+0} = \frac{1}{5} < 1$$

So by D'Alembert's ratio test the given series converges.

Problem Test the convergence of $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

Answer

Take $u_n = \frac{n^n}{n!}$ Then $u_{n+1} = \frac{(n+1)^{n+1}}{(n+1)!}$

and $\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)$.

$$\therefore \quad \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \left\{\lim\left(1+\frac{1}{n}\right)^n\right\} = e > 1$$

and hence by D'Alemberts' ratio test, is divergent.

The n^{th} Root test

Theorem 9 (Cauchy's n^{th} Root Test) If $\sum u_n$ is a series with non-negative terms such that $\lim\limits_{n\to\infty} \sqrt[n]{u_n} = l$ then the series $\sum u_n$

(i) converges if $l < 1$,

(ii) diverges if $l > 1$ or l is infinite,

(iii) the test is in conclusive if $l = 1$.

Problem Investigate the behaviour (convergence or divergence) of $\sum u_n$, if $u_n = \left(\sqrt[n]{n} - 1\right)^n$

Answer

Let $u_n = \left(n^{\frac{1}{n}} - 1\right)^n$. Then $u_n^{\frac{1}{n}} = n^{\frac{1}{n}} - 1$

and $\lim\limits_{n\to\infty}\left(a_n^{\frac{1}{n}}\right) = \lim\limits_{n\to\infty}\left(n^{\frac{1}{n}} - 1\right) = 1 - 1 = 0 < 1$

Hence, by Root Test, $\sum u_n$ is convergent.

Problem Show that the series $\sum \frac{[(n+1)r]^n}{n^{n+1}}$ is convergent if $r < 1$ and divergent if $r \geq 1$.

Answer

Taking $u_n = \frac{[(n+1)r]^n}{n^{n+1}}$, we have

$$(u_n)^{\frac{1}{n}} = \frac{(n+1)r}{n^{\frac{n+1}{n}}} = \frac{n+1}{n^{1+\frac{1}{n}}} \cdot r = \frac{1 + \frac{1}{n}}{n^{\frac{1}{n}}} r$$

Since $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$, the above implies $\lim\limits_{n\to\infty}(u_n)^{\frac{1}{n}} = r$.

Therefore $\sum u_n$ converges if $r < 1$ and diverges if $r > 1$.

If $r = 1$, $u_n = \frac{(n+1)^n}{n^{n+1}} = \left(\frac{n+1}{n}\right)^n \cdot \frac{1}{n} = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{n}$

Let $\sum v_n = \sum \frac{1}{n}$ Then $\sum v_n$ is a harmonic series with $p = 1$ and hence is divergent. Also,

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e = 1, \text{ a finite non zero value.}$$

Hence, by Comparison Test, $\sum u_n$ diverges. Thus the series converges when $r < 1$ and diverges when $r \geq 1$.

Exercises

Which of the series in Exercises 1-13 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} n^2 e^{-n}$ 2. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$ 3. $\sum_{n=1}^{\infty} \left(\frac{n-2}{2}\right)^n$

4. $\sum_{n=1}^{\infty} \frac{(-2)^n}{3^n}$ 5. $\sum_{n=1}^{\infty} \left(1-\frac{1}{3n}\right)^n$ 6. $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

7. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\frac{1}{n^2}\right)^n$ 8. $\sum_{n=1}^{\infty} \frac{n\ln n}{2^n}$ 9. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$

10. $\sum_{n=1}^{\infty} \frac{n2^n(n+1)!}{3^n n!}$ 11. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

12. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^{\left(\frac{n}{2}\right)}}$ 13. $\sum_{n=2}^{\infty} \frac{3^n}{n^3 2^n}$

Which of the series $\sum_{n=1}^{\infty} a_n$ defined by the formulas in Exercises 14-19 converge, and which diverge? Give reasons for your answers.

14. $a_1 = 1, a_{n+1} = \frac{1+\tan^{-1} n}{n} a_n$ 15. $a_1 = 3, a_{n+1} = \frac{n}{n+1} a_n$

16. $a_1 = 5, a_{n+1} = \frac{\sqrt[n]{n}}{2} a_n$ 17. $a_1 = \frac{1}{2}, a_{n+1} = \frac{n+\ln n}{n+10} a_n$

18. $a_1 = \frac{1}{2}, a_{n+1} = (a_n)^{n+1}$ 19. $a_n = \frac{(3n)!}{n!(n+1)!(n+2)!}$

Which of the series in Exercises 20-22 converge and which diverge? Give reasons for your answers.

20. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{(n^2)}}$ 21. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

22. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{[2\cdot 4\cdots(2n)](3^n+1)}$

CHAPTER.10 ALTERNATING SERIES

Alternating series and Leibniz test

Definition (*Alternating series*) A series in which the terms are alternatively positive and negative is an **alternating series**.

Problem Each of the three series

$$1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{8}+\cdots+\tfrac{(-1)^{n-1}}{2^{n-1}}+\cdots;$$

$$-1+2-3+4-\cdots+(-1)^n n+\cdots;$$

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\cdots+\tfrac{(-1)^{n+1}}{n}+\ldots$$

is an alternating series. The third series, called the **alternating harmonic series,** is convergent. This is described in Problem 2.

Notation An alternating series may be written as $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ where each u_n is positive and the first term is positive. If the first term in the series is negative, then we write the series as $\sum_{n=1}^{\infty}(-1)^n u_n$.

Theorem (Leibniz Test for testing the nature of alternating series)

Suppose $\{u_n\}$ is a sequence of positive numbers such that

(a) $u_1 \ge u_2 \ge \cdots \ge u_n \ge u_{n+1} \ge \cdots$ and

(b) $\lim_{n\to\infty} u_n = 0$,

Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ is convergent

Proof

If n is an even integer, say $n=2m$ then the sum of the first n terms is

$$s_{2m} = (u_1-u_2)+(u_3-u_4)+\cdots+(u_{2m-1}-u_{2m})$$

$$= u_1-(u_2-u_3)-(u_4-u_5)-\cdots(u_{2m-2}-u_{2m-1})-u_{2m}$$

The first equality shows that s_{2m} is the sum of m nonnegative terms, since, by assumption (a), each term in parentheses is positive or zero. Hence $s_{2m+2} \ge s_{2m}$, and the sequence $\{s_{2m}\}$ is nondecreasing. The second equality shows that $s_{2m} \le u_1$. Since $\{s_{2m}\}$ is nondecreasing and bounded from above, by non decreasing Sequence Theorem (Theorem 2 of Chapter "*Sequence*"), it has a limit, say

$$\lim_{m\to\infty} s_{2m} = L \qquad \ldots (1)$$

If n is an odd integer, say $n = 2m+1$ then the sum of the first n terms is $s_{2m+1} = s_{2m} + u_{2m+1}$. Since, by assumption (b), $\lim_{n\to\infty} u_n = 0$.

$$\lim_{m\to\infty} u_{2m+1} = 0$$

and, as $m \to \infty$,

$$s_{2m+1} = s_{2m} + u_{2m+1} \to L + 0 = L \qquad \ldots (2)$$

Combining the results of (1) and (2) gives $\lim_{n\to\infty} s_n = L$. As the sequence of n^{th} partial sums of the given series converges, the given series converges.

Problem Show that the **alternating Harmonic series**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \text{ is convergent.}$$

Answer

The given series is an alternating series and $u_1 = 1, u_2 = \frac{1}{2}, \ u_3 = \frac{1}{3}, \ u_4 = \frac{1}{4}, \ldots$ with $u_1 > u_2 > u_3 > u_4 \ldots$.

In general, $u_n = \frac{1}{n}$ and $u_n > u_{n+1}$ for all n, since $n < n+1 \Rightarrow \frac{1}{n} > \frac{1}{n+1}$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$. Hence all the conditions of Leibniz Test are satisfied by the given alternating series and so it is convergent.

Problem Test the convergence of

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\log n}$$

Answer

The given series is an alternating series; also

$$u_2 = \frac{1}{\log 2}, \ u_3 = \frac{1}{\log 3}, \ u_4 = \frac{1}{\log 4}.$$

with $\qquad u_2 > u_3 > u_4 > u_5 > \cdots$

In general $\qquad u_n = \frac{1}{\log n}$ with $u_n > u_{n+1}$

Also $\qquad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(\frac{1}{\log n} \right) = \frac{1}{\lim_{n\to\infty} \log n} = 0.$

Hence all the conditions of Leibniz's Test are satisfied by the given alternating series and so it is convergent.

Problem Discuss the convergence of the series

$$\frac{2}{1^2}+\frac{3}{3^2}+\frac{4}{5^2}+\frac{5}{7^2}+\cdots$$

Answer

The given series is an alternating series. The terms of the series are

$$u_1=\frac{2}{1^2},\, u_2=\frac{3}{3^2},\, u_3=\frac{4}{5^2},\, u_4=\frac{5}{7^2},\cdots$$

with $\quad u_1>u_2>u_3>\ldots$

In general $u_n=\dfrac{n+1}{(2n-1)^2}$ with $u_n>u_{n+1}$, since $\dfrac{n+1}{(2n-1)^2}>\dfrac{n+2}{(2n+1)^2}$.

Hence the terms are in the decreasing order.

Also $\quad \lim\limits_{n\to\infty}u_n=\lim\limits_{n\to\infty}\left[\dfrac{n+1}{(2n-1)^2}\right]$

$$=\lim_{n\to\infty}\left[\frac{n}{n^2\left(2-\frac{1}{n}\right)}\right]=\lim_{n\to\infty}\left[\frac{n}{n\left(2-\frac{1}{n}\right)}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\cdot\lim_{n\to\infty}\frac{n}{\left(2-\frac{1}{n}\right)}=0\times\frac{1}{2}=0$$

Hence all the conditions of Leibniz's Test are satisfied by the given alternating series and so it is convergent.

Problem Test the convergence of

$$1-\frac{1}{2^2\sqrt{2}}+\frac{1}{3^2\sqrt{3}}-\frac{1}{4^2\sqrt{4}}+\cdots$$

Answer

The given series an alternating series and the terms are given by

$$u_1=1,\, u_2=\frac{1}{2^2\sqrt{2}},\, u_3=\frac{1}{3^2\sqrt{3}},\, u_4=\frac{1}{4^2\sqrt{4}},\cdots$$

Now $\quad u_n=\dfrac{1}{n^2\sqrt{n}}$

$$n+1>n\Rightarrow\sqrt{n+1}>\sqrt{n}$$

$$\Rightarrow(n+1)^2\sqrt{n+1}>n^2\sqrt{n}$$

$$\Rightarrow \frac{1}{(n+1)^2\sqrt{n+1}} < \frac{1}{n^2\sqrt{n}}$$

i.e., $u_{n+1} < u_n$ or $u_n > u_{n+1}$.

Hence the terms are in the decreasing order.

Hence by the Leibniz's Test, the given series is convergent.

Problem Examine the convergence of the series:

(i) $\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots \quad (p > 0)$

(ii) $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \cdots \quad (0 < x < 1)$

Answer

(i) Here the terms are alternatively positive and negative. Writing the series in the form $\sum(-1)^{n-1}u_n$, where $u_n = 1/n^p$ we get

$$\frac{u_{n+1}}{u_n} = \frac{n^p}{(n+1)^p} = \frac{1}{(1+1/n)^p} < 1, \text{ since } p > 0.$$

Thus $u_{n+1} < u_n$

Also $\lim_{n\to\infty}(u_n) = \lim_{n\to\infty}\left(\frac{1}{n^p}\right) = 0$, since $p > 0$.

Hence all the conditions of Leibniz test are satisfied and the series converges.

(ii) The terms of the series are alternatively positive and negative.

$$u_n = \frac{x^n}{1+x^n} \text{ and } u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

$$\therefore \quad u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= x^n\left[\frac{1}{1+x^n} - \frac{x}{1+x^{n+1}}\right] = \frac{x^n(1-x)}{(1+x^n)1+x^{n+1}}$$

Since x is positive and less than 1, $u_n - u_{n+1} > 0$

$\therefore \quad u_{n+1} < u_n$ for all n.

Also

$\lim(u_n) = \lim\left(\frac{x^n}{1+x^n}\right) = 0$, (since with $x < 1 x^n \to 0$ as $n \to \infty$)

Hence all the conditions of Leibniz test are satisfied and the series is convergent.

Theorem: The Alternating Series Estimation Theorem

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the conditions of Leibniz Test, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1},

i.e., $|s_n - L| \leq u_{n+1}$

Problem Estimate the interval in which the limit of the converging series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{(-1)^{n+1}}{n}+\cdots$ lies.

Answer

The given is the alternating harmonic series, which, by Problem 1, is convergent. Let the series converges to the real number L.

Then $\sum(-1)^{n+1}\frac{1}{n} = L$ and, by the Theorem, for any $n \in N$.

$$\left|\left[1-\frac{1}{2}+\cdots+\frac{(-1)^{n+1}}{n}\right]-L\right| \leq \frac{1}{n+1}, \text{ since } u_n = \frac{1}{n+1}.$$

Putting n = 9,(i.e., if we truncate the series after the ninth term), we get

$|.7456 - L| \leq \frac{1}{10}$ i.e., $.6456 \leq L \leq .8456$.

Exercises

Test the convergence of the following series.

1. $\frac{1}{3}-\frac{1}{6}+\frac{1}{9}-\frac{1}{12}+\cdots$ 2. $\frac{2}{6}-\frac{4}{11}+\frac{6}{16}-\frac{8}{21}+\cdots$

3. $1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\cdots$ 4. $\frac{1}{2^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots$

5. $\sum\frac{(-1)^n n}{2n-1}$

Answers

1. convergent 2. oscillatory 3. convergent.

4. convergent. 5. oscillatory

Absolute and conditional convergence of series

Definition If the series

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

be such that the series

$$|u_1| + |u_2| + |u_3| + \cdots + |u_n| + \cdots$$

is convergent, then the series is said to be **absolutely convergent**.

***Problem* 11** The series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\cdots,$$

is absolutely convergent, since the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots,$$

being a harmonic series with $p=2$ is convergent,

Problem The geometric series

$$1-\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

converges absolutely because the corresponding series of absolute values

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

converges. The alternating harmonic series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ does not converge absolutely as the corresponding series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ of absolute values is the (divergent) harmonic series.

Definition (*Conditionally convergent series*)

If $\sum_{n=1}^{\infty}|u_n|=|u_1|+|u_2|+\cdots+|u_n|+\cdots$ is divergent but $\sum_{n=1}^{\infty}u_n$ is convergent, then $\sum_{n=1}^{\infty}u_n$ is said to be **conditionally convergent**.

Problem Show that the series $1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ is conditionally convergent.

Answer

The series is convergent (by Leibniz's Test as seen in an earlier Problem). Now the series formed by the absolute values of the terms is

$$\sum|u_n|=1-\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum\frac{1}{n}$$

and, being a harmonic series with $p=1$, is divergent. Hence the given series is conditionally convergent.

Theorem (**The Absolute Convergence Test**) An absolutely convergent series is necessarily convergent. i.e, if $\sum_{n=1}^{\infty}|u_n|$ converges $\sum_{n=1}^{\infty}u_n$ then converges.

Proof For each n

$$-|u_n|\le u_n\le|u_n| \quad \text{so} \quad 0\le u_n+|u_n|\le 2|u_n|$$

If $\sum_{n=1}^{\infty} u_n$ converges, then $\sum_{n=1}^{\infty} 2|u_n|$ converges, and by the Direct Comparison Test, the nonnegative series $\sum_{n=1}^{\infty}(u_n + |u_n|)$ converges. The equality $u_n = (u_n + |u_n|) - |u_n|$ now let us express $\sum_{n=1}^{\infty} u_n$ as the difference of two convergent series:

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty}(u_n + |u_n| - |u_n|) = \sum_{n=1}^{\infty}(u_n + |u_n|) - \sum_{n=1}^{\infty}|u_n|$$

Therefore $\sum_{n=1}^{\infty} u_n$ converges.

Remark The converse of the above theorem is not true.

For Problem, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ converges, but the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is divergent.

Problem Test whether the series

$$1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \cdots$$

is absolutely convergent or not? Does the series Converge?

Answer

The series of absolute terms is

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \cdots$$

and, being a harmonic series with $p = 2 > 1$, the series is convergent.

$\therefore$ The given series is absolutely convergent and hence, in view of the theorem, the given series is convergent.

Problem (***Alternating p-series***)If p is a positive constant, the sequence is a decreasing sequence of positive numbers with limit zero. Therefore by Leibniz's Test the alternating p-series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n=1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \quad p > 0$$

converges.

If $p > 1$, the series converges absolutely. If $0 < p \le 1$ the series converges conditionally. In particular,

$1 - \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} - \frac{1}{4^{\frac{3}{2}}} + \cdots$ is a conditionally convergent series while $1 - \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} - \frac{1}{4^{\frac{3}{2}}} + \cdots$ is an absolutely convergent series.

Rearrangements of Series

A **rearrangement of a series** $\sum u_n$ is a series $\sum v_n$ whose terms are the same as those of $\sum u_n$ but occur in different order.

The series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots$$

is rearrangement of the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots$$

We shall see in this section that rearrangement of an absolutely convergent series has no effect on its sum, but that rearrangement of a conditionally convergent series can have drastic effect.

Theorem (Dirichlets' theorem) (The Rearrangement Theorem for Absolutely Convergent Series) Any series obtained from an absolutely convergent series by a rearrangement of terms converges absolutely and has the same sum as the original series .i.e., if $\sum_{n=1}^{\infty} u_n$ converges absolutely, and $v_1, v_2, ..., v_n, ...$ is any arrangement of the sequence $\{u_n\}$, then $\sum v_n$ converges absolutely and $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} u_n$.

Problem As we saw in an earlier Problem, the series

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots+(-1)^{n-1}\frac{1}{n^2}+\cdots$$

converges absolutely. A possible rearrangement of the terms of the series might start with a positive term, then two negative terms, then three positive terms, then four negative terms, and so on: After k terms of one sign, take $k+1$ terms of the opposite sign. The first ten terms of such a series look like this:

$$1-\frac{1}{4}-\frac{1}{16}+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}-\frac{1}{36}-\frac{1}{64}-\frac{1}{100}-\frac{1}{144}+\cdots$$

The Rearrangement Theorem says that both series converge to the same value. In this Problem, if we had the second series to begin with, we would probably be glad to exchange it for the first, if we knew that we could. We can do even better: The sum of either series is also equal to

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}-\sum_{n=1}^{\infty}\frac{1}{(2n)^2}.$$

Caution: For a conditionally convergent series Dirichlets' theorem doesn't hold as the following theorem and Problem illustrates:

Theorem (Riemann's Theorem) The terms of any conditionally convergent series can be rearranged to give either a conditionally convergent series having as sum an arbitrary preassigned number, or a divergent series or an oscillatory series.

Problem We have seen in an earlier Problem that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges conditionally. Let the series converges to L. We note that $L \neq 0$ As limit of the series is its sum, we have

$$L = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \qquad \ldots (1)$$

and so, certainly

$$\frac{1}{2}L = 0 + \frac{1}{2} - 0 - \frac{1}{4} + 0 + \frac{1}{6} - 0 - \frac{1}{8} + \cdots \qquad \ldots (2)$$

Adding (1) and (2), we obtain

$$\frac{3}{2}L = (1+0) + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{3} - 0\right) + \left(-\frac{1}{4} - \frac{1}{4}\right) + \left(\frac{1}{5} + 0\right)$$

$$+ \left(-\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{7} + 0\right) + \cdots$$

i.e., $$\frac{3}{2}L = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots \qquad \ldots (3)$$

The series on the right of (3) is a rearrangement of the series on the right of (1), but they converge to different sums.

Exercises

Which of the alternating series in Exercises 1-5 converge, and which diverge? Give reasons for your answers.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{\frac{3}{2}}}$
2. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{10^n}{n^{10}}$
3. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\ln n}{n}$
4. $\sum_{n=1}^{\infty} (-1)^{n+1} \ln\left(1 + \frac{1}{n}\right)$
5. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{3\sqrt{n}+1}{\sqrt{n}+1}$

Which of the series in Exercises 6-22 converge absolutely, which converge, and which diverge? Give reasons for your answers.

6. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(0.1)^n}{n}$
7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$
8. $\sum_{n=1}^{\infty} (-1)^n \frac{n!}{2^n}$
9. $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n}{n^2}$
10. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(n^3)}$
11. $\sum_{n=1}^{\infty} \frac{\mathrm{c}(-2)^{n+1}}{n+5^n}$
12. $\sum_{n=1}^{\infty} (-1)^{n+1} (\sqrt[n]{10})$
13. $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$
14. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$
15. $\sum_{n=1}^{\infty} (-5)^{-n}$
16. $\sum_{n=2}^{\infty} (-1)^n \left(\frac{\ln n}{\ln n^2}\right)^n$
17. $\sum_{n=1}^{\infty} \frac{\cos n®}{n}$
18. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n!)^2}{(2n)!}$
19. $\sum_{n=1}^{\infty} (-1)^n \frac{(n!)^2 3^n}{(2n+1)!}$
20. $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+n} - n)$
21. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}}$
22. $\sum_{n=1}^{\infty} (-1)^n \operatorname{csc h} n$

In Exercises 23-24, estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the entire series.

23. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$
24. $\frac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n, \; 0 < t < 1$

MODULE III

CHAPTER.11 POWER SERIES

In mathematics and science we often write functions as infinite polynomials, such as

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots,\ x^n + \cdots, |x| < 1$

We call these polynomials power series because they are defined as infinite series of powers of some variable, in our case x. Like polynomials, power series can be added, subtracted, multiplied, differentiated, and integrated to give new power series.

Definition A **power series about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \qquad \text{. . . (1)}$$

A **power series about** $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \qquad \text{. . . (2)}$$

in which the **center** a and the **coefficients** $c_0, c_1, c_2, ..., c_n ...$ are constants.

Remarks

- The power series (1) always converges at $x = 0$ and the limit at that point is its constant term Similarly, the power series (2) converges at the center a and the sum of the series is c_0.
- A power series may converge for some or all values of x or may not converge for some or all values of x, except at the center.

Problem Taking all the coefficients to be 1 in Eq. (1) gives the **geometric power series**

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

This is the geometric series with first term 1 and common ratio x. It converges to $\frac{1}{1-x}$ for $|x| < 1$. We express this fact by writing

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots, \quad -1 < x < 1 \qquad \text{. . . (3)}$$

Up to now, we have used Eq. (3) as a formula for the sum of the series on the right. We now change the focus: We think of the partial sums of the series on the right as polynomials $P_n(x)$ that approximate the function on the left. For values of x near zero, we need take only a few terms of the series to get a good approximation. As we move toward $x = 1$ or -1, we must take more terms. Fig.1 shows the graphs of $f(x) = \frac{1}{1-x}$, and the approximating polynomials $y_n = P_n(x)$ for $n = 0, 1, 2,$ and 8.

Problem The power series

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots \qquad \text{. . . (4)}$$

matches Eq. (2) with $a=2,\ c_0=1,\ c_1=-\frac{1}{2},\ c_2=\frac{1}{4},\cdots,c_n=\left(-\frac{1}{2}\right)^n$. This is a geometric series with first term 1 and ratio $r=-\frac{x-2}{2}$. The series converges for $\left|\frac{x-2}{2}\right|<1$ or $-1<\frac{x-2}{2}<1$ or $-2<x-2<2$ or $0<x<4$. The sum is $\frac{1}{1-r}=\frac{1}{1+\frac{x-2}{2}}=\frac{2}{x}$,

so $$\frac{2}{x}=1-\frac{(x-2)}{2}+\frac{(x-2)^2}{4}-\cdots+\left(-\frac{1}{2}\right)^n(x-2)^n+\cdots,\ 0<x<4$$

Series (4) generates useful polynomial approximations of $f(x)=\frac{2}{x}$ for values of x near 2:

$$P_0(x)=1$$

$$P_1(x)=1-\frac{1}{2}(x-2)=2-\frac{x}{2}$$

$$P_2(x)=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2=3-\frac{3x}{2}+\frac{x^2}{4}$$

and so on (Fig. 2).

How to Test a Power Series for Convergence

Step 1: Use the Ratio Test (or nth Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$|x-a|<R$ or $a-R<x<a+R$

Step 2: If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint, as in Problems (a) and (b) above. (Use Comparison Test, the Integral Test, or the Alternating Series Test.)

Step 3: If the interval of absolute convergence is $a-R<x<a+R$, the series diverges for $|x-a|>R$ (it does not even converge conditionally), because the nth term does not approach zero for those values of x.

THE RADIUS AND INTERVAL OF CONVERGENCE

Theorem 1 (The Convergence Theorem for Power Series)

If $\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+\cdots$ converges for $x=c\neq 0$ then it converges absolutely for all $|x|<|c|$. If the series diverges for $x=d$ then it diverges for all $|x|=|d|$.

Possible Behavior of $\sum c_n(x-a)^n$

A power series $\sum c_n(x-a)^n$ behaves in one of the following three ways.

1. There is a positive number R such that the series diverges for $|x-a|>R$ but converges absolutely for $|x-a|<R$. The series may or may not converge at either of the endpoints $x=a-R$ and $x=a+R$.
2. The series converges absolutely for every x. In this case $(R=\infty)$.
3. The series converges at $x=a$ and diverges elsewhere. In this case $(R=0)$.

In case 1, the set of points at which the series converges is a finite interval, called the **interval of convergence.** We know from the Problems that the interval can be open, half-open, or closed, depending on the series. But no matter which kind of interval it is, R is called the **radius of convergence** of the series, and is the least upper bound of the set of points at which the series converges. The convergence is absolute at every point in the interior of the interval. If a power series converges absolutely for all values of x, we say that its **radius of convergence is infinite** (case 2 above). If it converges only at the **radius of convergence is zero**.

Theorem 2 (The Term-by-Term Differentiation Theorem)

If $\sum c_n(x-a)^n$ converges in the interval $a-R<x<a+R$ some $R>0$, it defines a function f :

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n, \quad a-R<x<a+R,$$

Such a function f has derivatives of all orders inside the interval of convergence. We can obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every interior point of the interval of convergence of the original series.

Problem Find series for $f'(x)$ and $f''(x)$ if

$$f(x)=\frac{1}{1-x}=1+x+x^2+x^3+x^4+\cdots+x^n+\cdots$$

$$=\sum_{n=0}^{\infty} x^n, \ -1<x<1$$

Answer

$$f'(x)=\frac{1}{(1-x)^2}=1+2x+3x^2+4x^4+\cdots+nx^{n-1}+\cdots$$

$$=\sum_{n=1}^{\infty} nx^{n-1}, \ -1<x<1$$

$$f''(x) = \frac{2}{(1-x)^2} = 2 + 6x + 12x^2 + \cdots + n(n-1)x^{n-2} + \cdots$$

$$= \sum_{n=2}^{\infty} n(n-1)x^{n-2}, \ -1 < x < 1$$

Theorem 3 The Term-by-Term Integration Theorem

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R (R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Problem Identify the function

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, \qquad -1 \le x \le 1 \qquad \ldots (7)$$

Answer

We differentiate the original series term by term and get

$$f'(x) = 1 - x^2 + x^4 - x^6 + \cdots, \qquad -1 < x < 1.$$

This is a geometric series with first term 1 and ratio $-x^2$, so

$$f'(x) = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$$

We can now integrate $f'(x) = \frac{1}{1+x^2}$ to get

$$f(x) = \int f'(x)\, dx = \int \frac{dx}{1+x^2} = \tan^{-1} x + C \qquad \ldots (8)$$

The series (7) for $f(x)$ is zero when $x = 0$, so from (8), $C = 0$. Hence

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \tan^{-1} x, \quad -1 < x < 1 \qquad \ldots (9)$$

Problem Discuss the interval of convergence and radius of convergence of $\sum_{n=0}^{\infty} n! x^n$.

Answer

The power series $\sum_{n=0}^{\infty} n! x^n = 1 + x + 2x^2 + 6x^3 + \cdots$ converges only at $x = 0$ (and limit at that point is 1), but diverges for every $x \neq 0$; this follows from the ratio test since $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(n+1)|x|}{n! x^n}\right| = (n+1)|x| \to \infty$ as $n \to \infty$ (for fixed $x, x \neq 0$). Here the radius of convergence is $R = 0$.

Problem Determine the interval of convergence and radius of convergence for the series

$$\sum_{n=1}^{\infty}(-1)^n(n+1)\frac{(x+1)^n}{2^n}.$$

Answer

Let $\quad u_n=(-1)^n(n+1)\dfrac{(x+1)^n}{2^n}$

Then $\quad \lim_{n\to\infty}\left|\dfrac{u_{n+1}}{u_n}\right|=\lim_{n\to\infty}\left|\dfrac{(n+2)(x+1)^{n+1}}{2^{n+1}}\cdot\dfrac{2^n}{(n+1)(x+1)^n}\right|$

$$=\lim_{n\to\infty}\left|\frac{n+2}{n+1}\right|\left|\frac{(x+1)}{2}\right|\lim_{n\to\infty}\left|\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right|\cdot\left|\frac{x+1}{2}\right|=\left|\frac{x+1}{2}\right|$$

So for convergence, $\left|\dfrac{x+1}{2}\right|$ must be less than 1. i.e., $\left|\dfrac{x+1}{2}\right|<1$

That is $-1<\dfrac{x+1}{2}<1$ or $-2<x+1<2$ or $-3<x<1$

Hence the given series converges for $-3<x<1$

At x = –3, we have the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n n}{2^n}=\sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n(2)^n n}{2^n}$$

$$=\sum_{n=1}^{\infty}n \text{ which is divergent}$$

At x = 1, we have the divergent alternating series $\sum_{n=1}^{\infty}(-1)^n$ .Hence the interval of convergence is $-3<x<1$. Also the series converges absolutely for $|x-(-1)|<2$ and therefore radius of convergence is 2.

Theorem (The Series Multiplication Theorem for Power Series)

If $A(x)=\sum_{n=0}^{\infty}a_nx^n$ and $B(x)=\sum_{n=0}^{\infty}b_nx^n$ converge absolutely for $|x|<R$, and

$c_n=a_0b_n+a_1b_{n-1}+a_2b_{n-2}+a_{n-1}b_1+a_nb_0=\sum_{k=0}^{n}a_kb_{n-k}$ then

$\sum_{n=0}^{\infty}c_nx^n$ converges absolutely to $A(x)\,B(x)$ for $|x|<R$. That is,

$$\left(\sum_{n=0}^{\infty}a_nx^n\right)\cdot\left(\sum_{n=0}^{\infty}b_nx^n\right)=\sum_{n=0}^{\infty}c_nx^n$$

Problem Multiply the geometric series

$\sum_{n=0}^{\infty}x_n=1+x+x^2+\cdots+x^n+\cdots=\dfrac{1}{1-x}$, for $|x|<1$.

by itself to get a power series for $\frac{1}{(1-x)^2}$, when $|x|<1$

Answer

Let $$A(x)=\sum_{n=0}^{\infty}a_n x^n = 1+x+x^2+\cdots+x^n+\cdots=\frac{1}{(1-x)}$$

$$B(x)=\sum_{n=0}^{\infty}b_n x^n = 1+x+x^2+\cdots+x^n+\cdots=\frac{1}{(1-x)}$$

and $$c_n = \underbrace{a_0b_n + a_1b_{n-1}+\cdots a_kb_{n-k}+\cdots+a_nb_0}_{n+1\,\text{terms}}$$

$$=\underbrace{1+1+\cdots+1}_{n+1\,\text{ones}}=n+1.$$

Then, by the Series Multiplication Theorem,

$$A(x)\cdot B(x)=\sum_{n=0}^{\infty}c_n x^n=\sum_{n=0}^{\infty}(n+1)x^n$$

$$=1+2x+3x^2+4x^3+\cdots+(n+1)x^n+\cdots$$

is the series for $\frac{1}{(1-x)^2}$. The series converge absolutely for $|x|<1$ Notice that Problem 5 gives the same answer because

$$\frac{d}{dx}\left(\frac{1}{1-x}\right)=\frac{1}{(1-x)^2}$$

Exercises

In Exercises 1-16, (a) find the series, radius and interval of convergence. For what values of x does the series converge (b) absolutely, (c) conditionally?

1. $\sum_{n=0}^{\infty}(x+5)^n$ 2. $\sum_{n=1}^{\infty}\frac{(3x+2)^n}{n}$ 3. $\sum_{n=0}^{\infty}(2x)^n$ 4. $\sum_{n=1}^{\infty}\frac{(-1)^n(x+2)^n}{n}$ 5. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{\sqrt{n}}$

6. $\sum_{n=0}^{\infty}\frac{3^n x^n}{n!}$ 7. $\sum_{n=0}^{\infty}\frac{(2x+3)^{2n+1}}{n!}$ 8. $\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{\sqrt{n^2+3}}$ 9. $\sum_{n=0}^{\infty}\frac{nx^n}{4^n(n^2+1)}$ 10. $\sum_{n=1}^{\infty}\sqrt{n}(2x+5)^n$

11. $\sum_{n=1}^{\infty}(\ln n)\,x^n$ 12. $\sum_{n=1}^{\infty}n!(x-4)^n$ 13. $\sum_{n=0}^{\infty}(-2)^n(n+1)(x-1)^2$ 14. $\sum_{n=2}^{\infty}\frac{x^n}{n\ln n}$

15. $\sum_{n=1}^{\infty}\frac{(3x+1)^{n+1}}{2n+2}$ 16. $\sum_{n=0}^{\infty}\frac{(x-\sqrt{2})^{2n+1}}{2^n}$

In Exercises 17-19, find the series' interval of convergence and, within this interval, the sum of the series as a function of x.

17. $\sum_{n=0}^{\infty}\frac{(x+1)^{2n}}{9^n}$ 18. $\sum_{n=0}^{\infty}(\ln x)^n$ 19. $\sum_{n=0}^{\infty}\left(\frac{x^2-1}{2}\right)^n$

CHAPTER.12 TAYLOR AND MACLAURIN'S SERIES

There are various methods or formulae by which we can expand a given function in ascending integral powers of x. The methods (formulae) are based on the following assumptions:

(i) The expansion of the function in ascending powers of the variable is possible.

(ii) All the higher derivatives of the function exist and finite.

(iii) The infinite series is convergent .

The Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots$$

In most of the cases, the Taylor's series converges to $f(x)$ at every x and we often write the **Taylor's series** at $x = a$ as

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a) + \ldots \qquad \ldots (1)$$

The polynomial

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

$$+\frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a)$$

is called **Taylor's polynomial of degree *n*.**

The *alternate form of Taylor series* is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + \cdots, \qquad \ldots (2)$$

where h is small.

Problem Find the Taylor series and Taylor polynomials generated by the exponential function $f(x) = e^x$ at $a = 0$.

Answer

Let $f(x) = e^x$ then $f(0) = 1$

Differentiating successively and putting $x = 0$, we get

$f'(x) = e^x;$ $\qquad f'(0) = 1$

$f''(x) = e^x;$ $\qquad f''(0) = 1$

$f'''(x) = e^x;$ $\qquad f'''(0) = 1$

$\ldots$

$f^{(n)}(x) = e^x;$ $\qquad f^{(n)}(0) = 1$

The Taylor series generated by f at $x = 0$ is

$$f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots + \frac{f^{(n)}}{n!}(x-0)^n + \ldots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots.$$

i.e., $\qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

The above series is known as **exponential series**. Later in this chapter, we will see by the definition of Maclaurin series, that the above is also the Maclaurin series for e^x.

The Taylor polynomial of order n at $x = 0$ is

$$P_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}.$$

Problem Give an example of a function whose Taylor series converges at every x but converges to $f(x)$ only at $x = 0$ (i.e., at $a = 0$).

Answer Consider the function

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

whose derivatives of all orders exist at $x = 0$ and that $f^{(n)}(0) = 0$ for all n. Hence the Taylor series generated by f at $x = 0$ (i.e., at $a = 0$).

$$f(0) + f'(0)\left(x-0\right) + \frac{f''(0)}{2!}\left(x-0\right)^2 + \ldots + \frac{f^{(n)}(0)}{n!}(x-0)^n + \ldots$$

$$= 0 + 0 \cdot x + 0 \cdot x^2 + \ldots + 0 \cdot x^n + \ldots$$

$$= 0.$$

The above series converges for every x (its sum is 0) but converges to $f(x)$ only at $x = 0$.

Problem Using Taylor's series, show that

$$\sin(x+h) = \sin x + h \cdot \cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \ldots$$

Answer

Let $\qquad f(x) = \sin x,$ $\qquad$ then $\qquad f(x+h) = \sin(x+h)$

Also $f'(x) = \cos x,$

$f''(x) = -\sin x,$

$f'''(x) = -\cos x$ etc.

Using the Taylor's series given by (2), we obtain

$$\sin(x+h) = \sin x + h.\cos x - \frac{h^2}{2!}\sin x - \frac{h^3}{3!}\cos x + \ldots$$

Problem Using Taylor's series, show that

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \frac{h^5}{5x^5} - \cdots.$$

Deduce that

$$\log\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \ldots$$

Answer

Take $f(x) = \log x,$ so that $f(x+h) = \log(x+h);$

Also $f'(x) = \frac{1}{x},$ $f''(x) = -\frac{1}{x^2},$ $f'''(x) = \frac{2}{x^3},$ etc.

Substituting these values in Taylor's series (2), we obtain

$$\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4} + \frac{h^5}{5x^5} - \cdots.$$

Now putting h =1 in the above series we get

$$\log(x+1) = \log x + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \ldots.$$

implies $\log(x+1) - \log x = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \ldots$

implies $\log\left(\frac{x+1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \ldots$

implies $\log\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \frac{1}{4x^4} + \frac{1}{5x^5} - \ldots$

Exercises

In Exercises 1-4, find the Taylor polynomials of orders 0, 1, 2 and 3 generated by f at a.

1. $f(x) = \ln(1+x),\ a = 0$
2. $f(x) = 1/(x+2),\ a = 0$
3. $f(x) = \cos x,\ a =$ ® $/4$
4. $f(x) = \sqrt{x+4},\ a = 0$

In Exercises 5-9, find the Taylor series generated by f at $x = a$

5. $f(x) = 2x^3 + x^2 + 3x - 8, \ \ a = 1$
6. $f(x) = 3x^5 - x^4 + 2x^3 + x^2 - 2, \ \ a = -1$
7. $f(x) = x/(1-x), \ a = 0$
8. $f(x) = 2^x, \ a = 1$ 9. e^x at $x = 1$.

MACLAURIN'S SERIES

If we take $a = 0$ in (1), we get the **Maclaurin's series** expansion

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(0) + \ldots \qquad \ldots (3)$$

Problem Find the expansion of $(1+x)^m$, using Maclaurin's series.

Answer Let $f(x) = (1+x)^m$ then $f(0) = 1$

Differentiating successively and putting $x = 0$, we get

$f'(x) = m(1+x)^{m-1}$ $f'(0) = m$

$f''(x) = m(m-1)(1+x)^{m-2}$ $f''(0) = m(m-1)$

$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$ $f'''(0) = m(m-1)(m-2)$

In general, $f^{(n)}(x) = m(m-1)(m-2)\ldots(m-n+1)(1+x)^{m-n}$

and $f^{(n)}(0) = m(m-1)(m-2)\ldots(m-n+1)$

Substituting these values in the Maclaurin's series, we get

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots .$$

$$\ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n + \ldots$$

Remark: The above series is known as **binomial series.**

Problem Using Maclaurin's series expand $\tan x$ up to the term containing x^5.

Answer

Let $f(x) = y = \tan x$ then $f(0) = y(0) = \tan 0 = 0;$

$f'(x) = y_1 = \sec^2 x$

Note that, since $\sec^2 x = 1 + \tan^2 x$, $f'(x) = y_1 = 1 + y^2$.

y $\quad f'(0) = y_1(0) = 1 + y^2(0) = 1 + 0 = 1;$

Differentiating $f'(x) = y_1 = 1 + y^2$ successively, we obtain

$$f''(x) = y_2 = 2yy_1 \qquad f''(0) = y_2(0) = 2y(0)y_1(0) = 0;$$

$$f'''(x) = y_3 = 2y_1^2 + 2yy_2$$

$$f'''(0) = y_3(0) = 2[y_1(0)]^2 + 2y(0)y_2(0) = 2;$$

$$f''''(x) = y_4 = 4y_1y_2 + 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3;$$

$$f''''(0) = y_4(0) = 0;$$

$$f^{(5)}(x) = y_5 = 6y_2^2 + 6y_1y_3 + 2y_1y_3 + 2yy_4 = 6y_2^2 + 8y_1y_3 + 2yy_4;$$

$$f^{(5)}(0) = 16; \qquad 12 + 4 + 0 =$$

Putting these values in the Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

we obtain

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15} x^5 + \ldots$$

Problem Expand $e^{\sin x}$ up to the term containing x^4 using Maclaurin's series.

Answer

Let $\quad f(x) = y = e^{\sin x} \quad$ then $\quad f(0) = y(0) = e^{\sin 0} = e^0 = 1;$

$$f'(x) = y_1 = e^{\sin x} \cos x = y \cos x$$

$$f'(0) = y_1(0) = y(0)\cos 0 = 1 \cdot 1 = 1;$$

$$f''(x) = y_2 = y_1 \cos x - y \sin x$$

$$f''(0) = y_1(0)\cos 0 - y(0)\sin 0 = 1 \cdot 1 - 1 \cdot 0 = 1$$

$$f'''(x) = y_3(x) = y_2 \cos x - y_1 \sin x - y_1 \sin x - y \cos x$$

$$= y_2 \cos x - 2y_1 \sin x - y \cos x$$

$$f'''(0) = y_3(0) = y_2(0) \cos 0 - 2y_1(0) \sin 0 - y(0) \cos 0 = 0;$$

$$f^{(4)}(x) = y_4(x) = y_3 \cos x - y_2 \sin x - 2y_2 \sin x - 2y_1 \cos x$$

$$- y_1 \cos x + y \sin x.$$

i.e., $f^{(4)}(x) = y_4(x) = y_3 \cos x - 3y_2 \sin x - 3y_1 \cos x + y \sin x;$

$$f^{(4)}(0) = y_4(0) = y_3(0) \cos 0 - 3y_2(0) \sin 0 - 3y_1(0) \cos 0 + y(0) \sin 0 \qquad = 0 - 0 - 3 \cdot 1 + 0 = -3;$$

Putting these values in the Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots$$

we obtain

$$e^{\sin x} = 1 + x \cdot 1 + \frac{x^2}{2!} + \frac{x^3}{3!} \times 0 + \frac{x^4}{4!}(-3) + \ldots$$

i.e., $$e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \ldots$$

Exercises

Find the Maclaurin's series, for the functions in Exercises 1-16.

1. 2^x 2. e^{-x} 3. $\frac{1}{1+x}$ 4. $\sin 3x$ **5.** $7\cos(-x)$ **6. sec x**

7. $\cosh x = \frac{e^x + e^{-x}}{2}$ **8.** $x^4 - 2x^3 - 5x + 4$ 9. $\log(1+e^x)$

10. $\log(1 - x + x^2)$ 11. $\log\cosh x$ 12. $\log\cos x$

13. $e^x \sin x$14. $e^{x\cos\alpha} \cos(x\sin\alpha)$ 15. $e^{ax}\cos bx$ 16. $e^x \cos x$

In Exercises 17-24, using Maclaurin's theorem, prove the expansions.

17. $\tan x = 1 + \frac{1}{3}.x^3 + \frac{2}{15}.x^5 + \ldots$

18. $\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \ldots$

19. $\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \ldots$

20. $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \ldots$

21. $e^x \sec x = 1 + x + 2x^2/\,2! + 4x^3\,/3! + \ldots$

22. $\log(1 + \sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \ldots$

23. $\log\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7 7}{} + \cdots\right], \ -1 < x < 1.$

24. $e^{x\cos\sim}\cos(x\sin\sim) = 1 + \cos\sim + \frac{x^2}{2!}\cos 2\sim + \ldots$

Find the Maclaurin series for the functions in the following Exercises.

25. $e^{x/2}$ 26. $\frac{1}{1-x}$ 27. $\sin\frac{x}{2}$

28. $5\cos® x$ 29. $\sinh x = \frac{e^x - e^{-x}}{2}$ 30. $\cos^2 x$ *31.* $(x+1)^2$

32. $e^x \sin^2 x$ 33. $e^{m\sin^{-1} x}$ 34. $e^x \log(1+x)$

CHAPTER.13 CONVERGENCE OF TAYLOR SERIES: ERROR ESTIMATES

Theorem 1 : Taylor's Theorem If f and its n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on $[a, b]$ or on $[b, a]$, and $f^{(n)}$ is differentiable on (a, b) or on (b, a), then there exists a number c between a and b such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots$$

$$+ \frac{(b-a)^n}{n!} f^{(n)}(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{(n+1)}(c)$$

Corollary to Taylor's Theorem : Taylor's Formula If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots$$

$$+ \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x), \qquad \ldots(1)$$

where

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c) \qquad \ldots(2)$$

for some c between a and x. When we state Taylor's theorem this way, it says that for each x in I,

$$f(x) = P_n(x) + R_n(x).$$

Equation (1) is called **Taylor's formula.** The function $R_n(x)$ is called the **remainder of order *n*** or the **error term** for the approximation of f by $P_n(x)$ over I. If $R_n(x) \to 0$ as $n \to \infty$ for all x in I, we say that the Taylor series generated by f at $x = a$ **converges to** f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} f^{(k)}(a).$$

Theorem 2 The Remainder Estimation Theorem If there are positive constants M and r such that $\left|f^{(n+1)}(t)\right| \le Mr^{n+1}$ for all t between a and x, inclusive, then the remainder term $R_n(x)$ in Taylor's theorem satisfies the inequality

$$\left|R_n(x)\right| \le M \frac{r^{n+1}\left|x-a\right|^{n+1}}{(n+1)!}.$$

If these conditions hold for every n and all the other conditions of Taylor's theorem are satisfied by f, then the series converges to $f(x)$.

Problem Show that the Maclaurin series for $\sin x$ converges to $\sin x$ for all x.

Answer

The function and its derivatives are

$$f(x)=\sin x, \qquad f'(x)=\cos x,$$
$$f''(x)=-\sin x, \qquad f'''(x)=-\cos x,$$
$$\vdots \qquad \vdots$$
$$f^{(2k)}(x)=(-1)^k\sin x, \qquad f^{(2k+1)}(x)=(-1)^k\cos x,$$

so

$$f^{(2k)}(0)=0, \quad \text{and} \quad f^{(2k+1)}(0)=(-1)^k.$$

The series has only odd-powered terms and, for $n=2k+1$, Taylor's Theorem gives

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+(-1)^k\frac{x^{2k+1}}{(2k+1)!}+R_{2k+1}(x).$$

All the derivatives of sin x have absolute values less than or equal to 1, so we can apply the Remainder Estimation Theorem with $M=1$ and $r=1$ to obtain

$$|R_{2k+1}(x)|\le 1\cdot\frac{|x|^{2k+2}}{(2k+2)!}.$$

Since $\frac{|x|^{2k+2}}{(2k+2)!)}\to 0$ as $k\to\infty$, whatever be the value of x, $R_{2k+1}(x)\to 0$, and hence the Maclaurin series for $\sin x$ converges to $\sin x$ for all x.

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \qquad \ldots(3)$$

Truncation Error

The Maclaurin series for e^x converges to e^x for all x. But we still need to decide how many terms to use to approximate e^x to a given degree of accuracy. We get this information from the Remainder Estimation Theorem.

Problem Calculate e with an error of less than 10^{-6}.

Answer

Using Example 1 with $x=1$, we obtain

$$e=1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+R_n(1),$$

where

$$R_n(1)=e^c\frac{1}{(n+1)!} \quad \text{for some } c \text{ between 0 and 1.}$$

e is an irrational number lying between 2 and 3. Hence $e<3$, and also noting that $e^0=1$, we are certain that

$$\frac{1}{(n+1)!} < R_n(1) < \frac{3}{(n+1)!}$$

because $1 < e^c < 3$ for $0 < c < 1$.

We note that $\frac{1}{9!} > 10^{-6}$, while $\frac{3}{10!} < 10^{-6}$. Thus we should take $(n+1)$ to be at least 10, or n to be at least 9. With an error of less than 10^{-6},

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots + \frac{1}{9!} \approx 2.7182.82.$$

Euler's Formula

A complex number is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. If we substitute $x = i¯$ (¯ real) in the Maclaurin series for e^x and use the relations

$i^2 = -1,\qquad i^3 = i^2 i = -i,\quad i^4 = i^2 i^2 = 1,\ i^5 = i^4 i = i,$

and so on, to simplify the result, we obtain

$$e^{i¯} = 1 + \frac{i¯}{1!} + \frac{i^{2}¯^{2}}{2!} + \frac{i^{3}¯^{3}}{3!} + \frac{i^{4}¯^{4}}{4!} + \frac{i^{5}¯^{5}}{5!} + \frac{i^{6}¯^{6}}{6!} + \cdots$$

$$= \left(1 - \frac{¯^2}{2!} + \frac{¯^4}{4!} + \frac{¯^6}{6!} + \cdots\right) + i\left(¯ - \frac{¯^3}{3!} + \frac{¯^5}{5!} - \cdots\right) = \cos¯ + i\sin¯.$$

Definition For any real number θ,

$$e^{i\theta} = \cos\theta + i\sin\theta \ldots(5)$$

Eq.(5), called Euler's formula, enables us to define e^{a+bi} to be $e^a \cdot e^{bi}$ for any complex number $a + bi$

Exercises

Find the Maclaurin series of the functions in Exercises 1-3.

1. $e^{-x/2}$ 2. $\sin\left(\frac{®x}{2}\right)$ 3. $\cos(x^{3/2}\sqrt{2})$

Find Maclaurin series for the functions in Exercises 4-9.

4. $x^2 \sin x$ 5. $\sin x - x + \frac{x^3}{3!}$ 6. $x^2\cos(x^2)$

7. $\sin^2 x$ 8. $x\ln(1+2x)$ 9. $\frac{2}{(1-x)^3}$

10. If $\cos x$ is replaced by $1-(x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1-(x^2/2)$ tend to be too large, or too small? Give reasons for your answer.

11. The estimate $\sqrt{1+x} = 1 + (x/2)$ is used when x is small. Estimate the error when $|x| < 0.01$.

Each of the series in Exercises 12-13 is the value of the Maclaurin series of a function f(x) at some point. What function and what point? What is the sum of the series?

12. $1-\frac{®^2}{4^2\cdot 2!}+\frac{®^4}{4^4\cdot 4!}-\cdots+\frac{(-1)^k(®)^{2k}}{4^{2k}\cdot(2k!)}+\cdots$ 13. $®-\frac{®^2}{2}+\frac{®^3}{3}-\cdots+(-1)^{k-1}\frac{®^k}{k}+\cdots$

14. Multiply the Maclaurin series for e^x and $\cos x$ together to find the first five nonzero terms of the Maclaurin series for $e^x \cos x$.

MODULE IV

CHAPTER.14 CONIC SECTIONS AND QUADRATIC EQUATIONS

A **circle** is the set of points in a plane whose distance from a given fixed point in the plane is constant. The fixed point is the **center** of the circle; the constant distance is the **radius.**

The standard form of the circle of radius a centered at the origin is

$$x^2+y^2=a^2.$$

The standard form of the circle of radius a centered at the point (h, k) is

$$(x-h)^2+(y-k)^2=a^2$$

Definitions A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.

Attention! If the focus F lies on the directrix L, the parabola is the line through F perpendicular to L. We consider this to be a **degenerate case** and assume henceforth that F does not lie on L.

Standard forms of Parabolas

A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0,p)$ on the positive $y-$axis and that the directrix is the line $y=-p$. A point $P(x,y)$ lies on the parabola if and only if $PF=PQ$. From the distance formula, we have

$$PF=\sqrt{(x-0)^2+(y-p)^2}=\sqrt{x^2+(y-p)^2}$$

$$PQ=\sqrt{(x-x)^2+(y-(-p))^2}=\sqrt{(y+p)^2}$$

When we equate the above expressions, square and simplify, we obtain

$$y=\frac{x^2}{4p} \qquad \text{or} \qquad x^2=4py \qquad \ldots(1)$$

From Eq. (1), we note that parabola is symmetric about the y-axis. In other words, the axis of symmetry of the parabola given by Eq.(1) is the y-axis. We call the y-axis the **axis** of the parabola $x^2=4py$.

The point where a parabola crosses its axis is the **vertex.** The vertex of the parabola $x^2 = 4py$ lies at the origin. The positive number p is the parabola's **focal length.**

If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then Eqs. (1) become

$$y = -\frac{x^2}{4p} \quad \text{and} \quad x^2 = -4py \qquad \ldots(2)$$

Similarly, we obtain similar equations for parabolas opening to the right or to the left and are given in the following Table.

Table: Standard-form equations for parabolas with vertices at the origin $(p > 0)$

Equation	Focus	Directrix	Axis	Opens
$x^2 = 4py$	$(0,\ p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0,\ -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p, 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p, 0)$	$x = p$	x-axis	To the left

Problem Find the vertex, focus, directrix, and axis of the parabola $x^2 = -6y$.

Answer

Comparing $x^2 = -6y$ with $x^2 = -4py$, we obtain $p = \frac{3}{2}$. Hence $x^2 = -6y$ represents a parabola opens to downward and whose vertex is $(0,\ 0)$ and focus is $(0,\ -p) = (0,\ -\frac{3}{2})$. Equation of the directrix $y = p$ is $y = \frac{3}{2}$ or $y - \frac{3}{2} = 0$ The axis is the y axis.

Definition An **ellipse** is the set of points in a plane whose distance from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The quickest way to construct an ellipse uses the definition. Put a loop of string around two tacks F_1 and F_2, pull the string taut with a pencil point P, and move the pencil around to trace a closed curve. The curve is an ellipse because the sum $PF_1 + PF_2$, being the length of the loop minus the distance between the tacks, remains constant. The ellipse's foci lie at F_1 and F_2.

Definition The line through the foci of an ellipse is the ellipse's **focal axis.** The point on the axis halfway between the foci is the **center.** The points where the focal axis and ellipse cross are the ellipse's **vertices**.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$, and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point $P(x, y)$ on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \qquad \ldots(4)$$

The Major and Minor Axes of an Ellipse

The **major axis** of the ellipse is the line segment of length $2a$ joining the points $(\pm a,\ 0)$. The **minor axis** is the line segment of length $2b$ joining the points $(0,\ \pm b)$. The number a itself is the **semi major axis,** the number c, found from Eq. (5) as

$$c = \sqrt{a^2 - b^2}$$

is **the center-to-focus distance** of the ellipse.

Problem The ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ intercepts the co-ordinate axes at the points $(5, 0)$, $(-5, 0)$, $(0, 4)$, and $(0, -4)$. The distance between $(5, 0)$ and $(-5, 0)$ is larger than the distance between $(0, 4)$ and $(0, -4)$ and therefore the *major axis is horizontal* and:

Semi major axis: $a = \sqrt{25} = 5$,

Semi minor axis: $b = \sqrt{16} = 4$

Center-to-focus distance: $c = \sqrt{25 - 16} = \sqrt{9} = 3$

Foci: $(\pm c, 0) = (\pm 3, 0)$

Vertices: $(\pm a, 0) = (\pm 5, 0)$

Center: $(0, 0)$

The ellipse is shown in Fig.10.

Definition A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ and the constant difference is $2a$, then a point $P(x, y)$ lies on the hyperbola if and only if

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, yields

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1$$

Definitions The line through the foci of a hyperbola is the **focal axis.** The point on the axis halfway between the foci is the hyperbola's **center.** The points where the focal axis and hyperbola cross are the **vertices .**

Asymptotes of Hyperbolas

The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has two asymptotes, the lines

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x.$$

The fastest way to find the equations of the asymptotes is to replace the 1 in the above equation by 0 and solve the new equation for y and get $y = \pm\frac{b}{a}x.$

Problem Given the equation

$$\frac{x^2}{3} - \frac{y^2}{4} = 1$$

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{3+4} = \sqrt{7}$

Foci: $(\pm c,\ 0) = (\pm\sqrt{7},\ 0)$, Vertices: $(\pm a,\ 0) = (\pm\sqrt{3},\ 0)$

Center: $(0,\ 0)$

Asymptotes: $\frac{x^2}{3} - \frac{y^2}{4} = 0$ or $y = \pm\frac{2}{\sqrt{3}}x$

Exercises

The following exercises give equations of parabolas. Find each parabola's focus and directrix. Then sketch the parabola. Include the focus and directrix in your sketch.

1. $y^2 = 12x$ 2. $x^2 = -8y$ 3. $y^2 = 4x^2$ 4. $x = -3y^2$

The following exercises give equations of ellipses. Put each equation in standard form. Then sketch the ellipse. Include the foci in your sketch.

5. $16x^2 + 25y^2 = 400$ 6. $2x^2 + y^2 = 2$ 7. $3x^2 + 2y^2 = 6$ 8. $6x^2 + 9y^2 = 54$

CHAPTER.15 CLASSIFYING CONIC SECTIONS BY ECCENTRICITY

Definition The **eccentricity** of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a>b)$ is

$$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$$

Problem Locate the vertices of an ellipse of eccentricity 0.8 whose foci lie at the points $(0, \pm 4)$.

Solution

Here $e=0.8$ and $c=4$.

Since $e=c/a$, the vertices are the points $(0, \pm a)$ where

$$a=\frac{c}{e}=\frac{4}{0.8}=5$$

or $(0, \pm 5)$.

Problem The orbit of Halley's comet is an ellipse 36.18 astronomical units long by 9.12 astronomical unit wide. Its eccentricity is

$$e=\frac{\sqrt{a^2-b^2}}{a}=\frac{\sqrt{(36.18/2)^2-(9.12/2)^2}}{(1/2)(36.18)}=\frac{\sqrt{(18.09)^2-(4.56)^2}}{18.09}\approx 0.97$$

Definition The eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$e=\frac{c}{a}=\frac{\sqrt{a^2+b^2}}{a}.$$

Problem Find the eccentricity of the hyperbola $9x^2-16y^2=144$.

Solution

We divide both sides of the hyperbola's equation by 144 to put it in standard form, obtaining

$$\frac{9x^2}{144}-\frac{16y^2}{144}=1$$

or $$\frac{x^2}{16}-\frac{y^2}{9}=1.$$

With $a^2=16$ and $b^2=9$, we find that $c=\sqrt{a^2+b^2}=\sqrt{16+9}=5$, so the eccentricity is given by

$$e = \frac{c}{a} = \frac{5}{4}.$$

Definition The **eccentricity** of a parabola is $e = 1$.

Problem Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line $x = 1$ as the corresponding directrix.

Solution

The focus is $(c, 0) = (3, 0)$, so $c = 3$.

Suppose $(a, 0)$ is the vertex to the right of the focus (3, 0). Then the directrix is the line

$$x = \frac{a}{e} = 1, \qquad \text{so} \quad a = e.$$

As $e = c/a$ defines eccentricity, we have

$$e = \frac{c}{a} = \frac{3}{e}, \text{ so } e^2 = 3 \text{ and } e = \sqrt{3}.$$

Knowing *e*, we can now derive the equation we want from the equation $PF = e \cdot PD$

$$\sqrt{(x-3)^2 + (y-0)^2} = \sqrt{3}\,|x-1|, \text{ as } \; e = \sqrt{3}$$

Squaring both sides, we obtain

$$x^2 - 6x + 9 + y^2 = 3(x^2 - 2x + 1).$$

i.e., $\qquad 2x^2 - y^2 = 6, \qquad$ or $\qquad \dfrac{x^2}{3} - \dfrac{y^2}{6} = 1$.

Exercises

In Exercise 1-4, find the eccentricity, foci and directrices of the ellipse.

1. $7x^2 + 16y^2 = 112$ 2. $2x^2 + y^2 = 4$

3. $9x^2 + 10y^2 = 90$ 4. $169x^2 + 25y^2 = 4225$

Exercises 5-6 give the foci or vertices and the eccentricities of ellipses centered at the origin of the *xy*-plane. In each case, find the ellipse's standard-form equation.

5. Foci: $(\pm 8, 0)$; Eccentricity: 0.2

6. Vertices: $(\pm 10, 0)$; Eccentricity: 0.24

Exercise 7-8 give foci and corresponding directrices of ellipses centered at the origin of the *xy*-plane. In each case, use the dimensions in Fig.5 to find the eccentricity of the ellipse. Then find the ellipse's standard-form equation.

7. Focus: $(4, 0)$; Directrix: $x = \dfrac{16}{3}$

8. Focus: $(-\sqrt{2}, 0)$; Directrix: $x = -2\sqrt{2}$

9. Draw the orbit of Pluto (eccentricity 0.25) to scale. Explain your procedure.

10. Find an equation for the ellipse of eccentricity 2/3 that has the line $x = 9$ as a directrix and the point (4,0) as the corresponding focus.

11. An ellipse is revolved about its major axis to generate an ellipsoid. The inner surface of the ellipsoid is silvered to make a mirror. Show that a ray of light emanating from one focus will be reflected to the other focus. (*Hint:* Place the ellipse in standard position in the *xy*-plane and show that the lines from a point *P* on the ellipse to the two foci make congruent angles with the tangent to the ellipse at *P*.) Sound waves also follow such paths, and this property is used in constructing "whispering galleries."

In Exercises 12-15, find the eccentricity foci and directrices of the hyperbola.

12. $9x^2 - 16y^2 = 144$ 13. $y^2 - x^2 = 4$

14. $y^2 - 3x^2 = 3$ 15. $64x^2 - 36y^2 = 2304$

Exercises 16-17 give the eccentricities and the vertices or foci of hyperbolas centered at the origin of the *xy*-plane. In each case, find the hyperbola's standard-form equation.

16. Eccentricity: 2; Vertices: $(\pm 2, 0)$

17. Eccentricity: 1.25; Foci: $(0, \pm 5)$

Exercises 18-19 give foci and corresponding directrices of hyperbolas centered at the origin of the *xy*-plane. In each case, find the hyperbola's eccentricity. Then find the hyperbola's standard-form equation.

18. Focus: $(\sqrt{10}, 0)$; Directrix: $x = \sqrt{2}$

19. Focus: $(-6, 0)$; Directrix: $x = -2$

CHAPTER.16 QUADRATIC EQUATIONS AND ROTATIONS

In this chapter, we examine that the Cartesian graph of any equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad \dots (1)$$

in which A, B, and C are not all zero, is nearly always a conic section. The exceptions are the cases in which there is no graph at all or the graph consists of two parallel lines. It is conventional to call all graphs of the above equation, curved or not, **quadratic curves.**

The Cross Product Term

We note that the term Bxy in Eq.(1) did not appear in the equations for the conic sections discussed in Chapter B7. This happened because the axes of the conic sections ran parallel to (in fact, coincided with) the coordinate axes. In the next example we see what happens when the parallelism is absent.

Problem Determine equation for the hyperbola with $a = 3$ and foci at $F_1(-3,-3)$ and $F_2(3,3)$.

Solution

Let $P(x, y)$ be a point on the given hyperbola. Then, the equation $|PF_1 - PF_2| = 2a$ becomes $|PF_1 - PF_2| = 2(3) = 6$ and

$$\sqrt{(x+3)^2 + (y+3)^2} - \sqrt{(x-3)^2 + (y-3)^2} = \pm 6$$

We move one radical to the right hand side, square, solve for the radical that still appears and square again, and then the above equation reduces to

$$2xy = 9,$$

in which the cross-product term is present. The asymptotes of the hyperbola in Eq. are the x- and y- axes, and the focal axis makes an angle of $\pi/4$ radians with the positive x-axis.

Rotating the Coordinate Axes to Eliminate the Cross Product Term xy.

Consider the usual Cartesian coordinates system with mutually perpendicular x- and y- axes. Let P be a point with the Cartesian coordinates (x, y) based on this x- and y-axes. We now form new axes obtained by rotating the x and y –axes at an angle α in the counter clockwise rotation.

Then

$$x = OM = OP\cos(\beta + \theta) = OP\cos\beta\cos\theta - OP\sin\beta\sin\theta$$

$$y = MP = OP\sin(\beta + \theta) = OP\cos\beta\sin\theta + OP\sin\beta\cos\theta$$

Since

$$OP\ \cos\beta = OM' = x'$$

and

$$OP\ \sin\beta = M'P = y',$$

the equations in reduce to the following:

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x'\sin\theta + y'\cos\theta$$

Eq (4) is the **equations for rotating Coordinate axes**

Problem The x- and y- axes are rotated through an angle of $\pi/4$ radians about the origin. Find an equation for the hyperbola $2xy = 9$ in the new coordinates.

Answer

Since $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$, using Eqs. (4a) and (4b), we obtain

$$x = \frac{x' - y'}{\sqrt{2}}, \quad y = \frac{x' + y'}{\sqrt{2}}$$

substituting these into the equation $2xy = 9$, we obtain

$$2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) = 9$$

$$x'^2 - y'^2 = 9$$

$$\frac{x'^2}{9} - \frac{y'^2}{9} = 1.$$

Problem Find a quadratic equation that is absent of *xy*-term and represent the curve $2x^2 + \sqrt{3}xy + y^2 - 10 = 0$.

Answer

The coordinate axes are to be rotated through an angle θ to produce an equation for the curve

$$2x^2 + \sqrt{3}xy + y^2 - 10 = 0$$

that has no cross product term. We find θ and the new equation.

Comparing with Eq. (1), the equation $2x^2 + \sqrt{3}xy + y^2 - 10 = 0$ has $A = 2, B = \sqrt{3},$ and $C = 1$. We substitute these values into Eq. (7) to find θ :

$$\cot 2\theta = \frac{A - C}{B} = \frac{2 - 1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

From the right triangle in Fig.4, we see that one appropriate choice of angle is $2\theta = \pi/3$, so we take $\theta = \pi/6$. Substituting $\theta = \pi/6, A = 2,$

$B=\sqrt{3}$, $C=1,\ D=E=0,$ and $F=-10$ into Eqs. (6a) to (6f) gives

$$A'=\frac{5}{2},\quad B'=0,\ C'=\frac{1}{2},\ D'=E'=0,\ F=-10.$$

Equation (5) then gives

$$\frac{5}{2}x'^2+\frac{1}{2}y'^2-10=0,$$

or $$\frac{x'^2}{2}+\frac{y'^2}{20}=1.$$

The curve is an ellipse with foci $(0,\pm 2\sqrt{5})$ on the new $y'-$axis .

The Discriminant Test

Method of determining the nature of the quadratic curve - The Discriminant Test

The quadratic curve $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is

a) a parabola if $B^2-4AC=0$,

b) an ellipse if $B^2-4AC<0$

c) a hyperbola if $B^2-4AC>0$.

provided degenerate cases may not arise.

Problem

a) $4x^2-8xy+4y^2+5x-3=0$ represents a parabola because

$$B^2-4AC=(-8)^2-4\cdot 4\cdot 4=64-64=0.$$

b) $2x^2+xy+y^2-1=0$ represents an ellipse because

$$B^2-4AC=(1)^2-4\cdot 2\cdot 1=-7<0$$

c) $3xy-y^2-5y+1=0$ represents a hyperbola because

$$B^2-4AC=(3)^2-4(0)(-1)=9>0.$$

Exercises

Use the discriminant B^2-4AC to decide whether the equations in Exercise 1-8 represent parabolas, ellipse, or hyperbola.

1. $3x^2-18xy+27y^2-5x+7y=-4$
2. $2x^2-\sqrt{15}\,xy+2y^2+x+y=0$
3. $2x^2-y^2+4xy-2x+3y=6$
4. $x^2+y^2+3x-2y=10$
5. $3x^2+6xy+3y^2-4x+5y=12$

6. $2x^2 - 4.9xy + 3y^2 - 4x = 7$

7. $25x^2 + 21xy + 4y^2 - 350x = 0$

8. $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0$

In Exercise 9-13, rotate the coordinate axes to change the given equation into an equation that has no cross product (xy) term. Then identify the graph of the equation. (The new equation will vary with the size and direction of the rotation you use.)

9. $x^2 + xy + y^2 = 1$ 10. $x^2 - \sqrt{3}xy + 2y^2 = 1$

11. $3x^2 - 2\sqrt{3}xy + y^2 = 1$ 12. $xy - y - x + 1 = 0$

13. $3x^2 + 4\sqrt{3}xy - y^2 = 7$

14. Find the sine and cosine of an angle through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$4x^2 - 4xy + y^2 - 8\sqrt{5}x - 16\sqrt{5}y = 0 .$$

Do not carry out the rotation.

In Exercises 15-17, use a calculator to find an angle ~ through which the coordinate axes can be rotated to change the given equation into a quadratic equation that has no cross product term. Then find sin~ and cos~ to 2 decimal places and use Eqs.(6a) to (6f) to find the coefficients of the new equation to the nearest decimal place. In each case, say whether the conic section is an ellipse, a hyperbola, or a parabola.

15. $2x^2 + xy - 3y^2 + 3x - 7 = 0$ 16. $2x^2 - 12xy + 18y^2 - 49 = 0$

17. $3x^2 + 5xy + 2y^2 - 8y - 1 = 0$

18. What effect does a 180° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.

- The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$
- The hyperbola $(x^2/a^2) - (y^2/b^2) = 1$
- The circle $x^2 + y^2 = a^2$
- The line $y = mx$
- The line $y = mx + b$

CHAPTER.17 PARAMETRIZATIONS OF PLANE CURVES

Definitions If x and y are given as continuous functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **curve** in the coordinate plane. The equations are **parametric equations** for the curve. The variable t is a **parameter** for the curve and its domain I is the parameter interval. If I is a closed interval, $a \le t \le b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminal point** of the curve. When we give parametric equations and a parameter interval for a curve in the plane, we say that we have **parameterized** the curve. The equations and interval constitute a **parameterization** of the curve.

In many applications t denotes time, but in some applications denote an angle or the distance a particle has traveled along its path from its starting point.

Problem Give the parametrization of the circle $x^2 + y^2 = 1$.

Solution

The equations and parameter interval

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2®$$

describe the position $P(x, y)$ of a particle that moves counter clockwise around the circle $x^2 + y^2 = 1$ as t increases.

The point lies on this circle for every value of t, because

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

But how much of the circle does the point $P(x, y)$ actually traverse?

To find out, we track the motion as t runs from 0 to $2®$. The parameter t is the radian measure of the angle that radius OP makes with the positive x – axis. The particle starts at $(1, 0)$, moves up and to the left as t approaches $® / 2$, and continues around the circle to stop again at $(1, 0)$ when $t = 2®$. The particle traces the circle exactly once.

Problem The equations and parameter interval

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le ®$$

describe the position $P(x, y)$ of a particle that moves counter clockwise around the circle $x^2 + y^2 = 1$ as t increases and traces only half of the circle.

Problem Verify that the equations and parameter interval

$$x = \cos t, \qquad y = -\sin t, \qquad 0 \le t \le \pi$$

describe the position $P(x, y)$ of a particle that moves clockwise around the circle $x^2 + y^2 = 1$ as t increases from 0 to π and covers only half of the circle.

Solution

We know that the point P lies on this circle for all t because its coordinates satisfy the circle's equation. How much of the circle does the particle traverse? To find out, we track the motion as t runs from 0 to π. As in Example 1, the particle starts at (1,0). But now as t increases, y becomes negative, decreasing to -1 when $t = \pi/2$ and then increasing back to 0 as t approaches π. The motion stops at $t = \pi$ with only the lower half of the circle covered.

Problem Describe the motion of a particle whose position $P(x, y)$ at time t is given by

$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$. *Solution*

We find a Cartesian equation for the particle's coordinates by eliminating t between the equations

$$\cos t = \frac{x}{a}, \qquad \sin t = \frac{y}{b}.$$

We accomplish this with the identity $\cos^2 t + \sin^2 t = 1$, which gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The particle's coordinates (x, y) satisfy the equation

$(x^2/a^2) + (y^2/b^2) = 1$, so the particle moves along this ellipse. When $t = 0$, the particle's coordinates are

$$x = a\cos(0) = a, \qquad y = b\sin(0) = 0,$$

so the motion starts at $(a, 0)$. As t increases, the particle rises and moves towards the left, moving counter clockwise. It traverses the ellipse once, returning to its starting position $(a, 0)$ at time $t = 2\pi$ (Fig.6).

Exercises

Exercises 1-12 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

1. $x = \cos 2t$, $y = \sin 2t$, $0 \le t \le \pi$
2. $x = \cos(\pi - t))$, $y = \sin(\pi - t)$, $0 \le t \le \pi$
3. $x = 4\sin t$, $y = 2\cos t$, $0 \le t \le \pi$
4. $x = 4\sin t$, $y = 5\cos t$, $0 \le t \le 2\pi$

5. $x = -\sqrt{t}$, $y = t$, $t \geq 0$

6. $x = \sec^2 t - 1$, $y = \tan t$, $-® / 2 < t < ® / 2$

7. $x = \csc t$, $y = \cot t$, $0 < t < ®$

8. $x = 1 - t$, $y = 1 + t$, $-\infty < t < \infty$

9. $x = 3 - 3t$, $y = 2t$, $0 \leq t \leq 1$

10. $x = t$, $y = \sqrt{4 - t^2}$, $0 \leq t \leq 2$

11. $x = \sqrt{t + 1}$, $y = \sqrt{t}$, $t \geq 0$

12. $x = 2\sinh t$, $y = 2\cosh t$, $-\infty < t < \infty$

13. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $((x^2 / a^2) + (y^2 / b^2) = 1$

    a) once clockwise, b) once counterclockwise,

    b) twice clockwise, d) twice counterclockwise.

14. Find parametric equations for the circle $x^2 + y^2 = a^2$ using as parameter the arc length s measured counterclockwise from the point $(a, 0)$ to the point (x, y).

CHAPTER.18 CALCULUS PARAMETRIZED CURVES

Slopes of Parameterized Curves

Consider a parameterized curve $x = f(t),\ y = g(t)$. The curve is **differentiable at $t = t_0$** if f and g are differentiable at $t = t_0$. The curve is **differentiable** if it is differentiable at every parameter value. The curve is **smooth** if f' and g' are continuous and not simultaneously zero.

Formula for Finding $\frac{dy}{dx}$ from $\frac{dy}{dt}$ and $\frac{dx}{dt}$ $\left(\frac{dx}{dt} \neq 0\right)$

At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dy}{dx}$ are related by the Chain Rule equation

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}.$$

If $\frac{dx}{dt} \neq 0$, we may divide both sides of the above equation by $\frac{dx}{dt}$ to solve for $\frac{dy}{dx}$ and obtain

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Problem Find the tangent to the right-hand hyperbola branch

$$x = \sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

at the point $(\sqrt{2}, 1)$, where $t = \frac{\pi}{4}$

Solution

The slope of the curve at t is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t}.$$

Hence the slope at the point $t = \frac{\pi}{4}$ is obtained by setting $t = \frac{\pi}{4}$ in the above equation and is given by

$$\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}} = \frac{\sec(\pi/4)}{\tan(\pi/4)} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

The point-slope equation of the tangent is

$$y - y_0 = m(x - x_0)$$

$$y-1=\sqrt{2}(x-\sqrt{2})$$
$$y=\sqrt{2}x-2+1$$
$$y=\sqrt{2}x-1.$$

The Parametric Formula for $\frac{d^2y}{dx^2}$

If the parametric equations for a curve define y as a twice-differentiable function of x, then $\frac{d^2y}{dx^2}$ can be calculated as follows:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(y')=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}$$

Problem Find $\frac{d^2y}{dx^2}$ if $x=2t-t^3,\quad y=t-t^2.$

Answer

To use (3), we have to evaluate $\frac{dy'}{dt}$ and $\frac{dx}{dt}$. Also, to evaluate $y'=\frac{dy}{dx}$ we have to evaluate $\frac{dy}{dt}$ also.

Now $\quad \frac{dy}{dt}=\frac{d}{dt}(t-t^2)=1-2t$

and $\quad \frac{dx}{dt}=\frac{d}{dt}(2t-t^3)=2-3t^2.$

Hence $\quad y'=\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{1-2t}{2-3t^2}.$

Also, $\quad \frac{d^2y}{dx^2}=\frac{d}{dx}(y')=\frac{\frac{d}{dt}\left(\frac{1-2t}{2-3t^2}\right)}{\frac{dx}{dt}}.$

Using quotient rule,

$$\frac{d}{dt}\left(\frac{1-2t}{2-3t^2}\right)=\frac{\left(2-3t^2\right)(-2)-\left(1-2t\right)(-6t^2)}{\left(2-3t^2\right)^2}=\frac{-4+12t^2-12t^3}{\left(2-3t^2\right)^2}.$$

Hence, $\quad \frac{d^2y}{dx^2}=\frac{\frac{-4+12t^2-12t^3}{\left(2-3t^2\right)^2}}{2-3t^2}=\frac{-4+12t^2-12t^3}{\left(2-3t^2\right)^3}.$

Lengths of Parametrized Curves. Centroids

The length of a smooth curve $x = f(t),\ y = g(t),\ a \le t \le b,$ is given by the following integral:

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt .$$

Problem Find the length of the arc of the curve

$$x = a\sin 2t(1+\cos 2t), \qquad y = a\cos 2t(1-\cos 2t)$$

measured from the origin to any point.

Solution

The origin corresponds to the point $(x, y) = (0,0)$, which implies $x = 0$ and hence $0 = a\sin 2t(1+\cos 2t)$, which gives a value for t as t = 0. In other words, t = 0 is the parametric value corresponding to the origin. Also, the parametric value of an arbitrary point is t. Since it is required to find length of the arc from the origin to any point on the curve, the limits of integration are t = 0 and t = t.

Now differentiating $x = a\sin 2t + \frac{1}{2}a\sin 4t$ with respect to t, we get

$$\frac{dx}{dt} = 2a\cos 2t + 2a\cos 4t = 4a\cos 3t\cos t$$

Also, differentiating $y = a\left(\cos 2t - \frac{1+\cos 4t}{2}\right)$, with respect to t, we obtain

$$\frac{dy}{dt} = 2a(\sin 4t - \sin 2t) = 4a\sin t\cos 3t$$

Hence

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^t \sqrt{(4a\cos 3t.\cos t)^2 + (4a\sin t.\cos 3t)^2}\ dt$$

$$= \int_0^t 4a\cos 3t\ dt = \frac{4}{3}a\sin 3t.$$

Problem Find the centroid of the first-quadrant arc of the astroid

$$x = \cos^3 t, y = \sin^3 t, 0 \le t \le 2®.$$

Solution

We take the curve's density to be ¢ $=1$ and calculate the curve's mass and moments about the coordinate axes.

The distribution of mass is symmetric about the line $y = x$, so $\bar{x} = \bar{y}$. A typical segment of the curve has mass

$dm = \delta \cdot ds = 1 \cdot ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt = 3\cos t \sin t\, dt$, using the previous example.

The curve's mass is

$M = \int_0^{\pi/2} dm = \int_0^{\pi/2} 3\cos t \sin t\, dt = \dfrac{3}{2}$, again using the previous example

The curve's moment about the $x-$axis is

$$M_x = \int \tilde{y}\, dm = \int_0^{\pi/2} \sin^3 t \cdot 3\cos t \sin t\, dt$$

$$= 3\int_0^{\pi/2} \sin^4 t \cos t\, dt = 3 \cdot \frac{\sin^5 t}{5}\Bigg]_0^{\pi/2} = \frac{3}{5}, \text{ using reduction formula}$$

Hence,

$$\bar{y} = \frac{M_x}{M} = \frac{3/5}{3/2} = \frac{2}{5}.$$

Therefore, $\bar{x}$ also equals to $\dfrac{2}{5}$ and the centroid is the point (2/5, 2/5).

The Area of a Surface of Revolution

If a smooth curve $x = f(t),\ y = g(t),\ a \le t \le b$, is traversed exactly once as t increases from a to b, then:

(i). the area of the surface generated by revolving the curve about the $x-$axis $(y \ge 0)$ is given by

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(ii). the area of the surface generated by revolving the curve about the $y-$axis $(x \ge 0)$ is given by

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Problem The standard parameterization of the circle of radius 1 centered at the point (0,1) in the $xy-$plane is

$x = \cos t,\ \ y = 1 + \sin t,\ \ 0 \le t \le 2\pi$

Use this parametrization to find the area of the surface swept out by revolving the circle about the $x-$axis.

Solution

We evaluate the formula

$$S = \int_a^b 2® \, y \sqrt{\left(\frac{dx}{dy}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \qquad = \int_0^{2®} 2® \, (1+\sin t)\sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$= 2® \int_0^{2®} (1+\sin t)dt$, since $\sin^2 t + \cos^2 t = 1$

$= 2® \left[t - \cos t\right]_0^{2®} = 4®^2$.

Exercises

In Exercise 1-6, find an equation for the line tangent to the curve at the point defined by the given value of *t*. Also, find the value of d^2y/dx^2 at this point.

1. $x = \sin 2® \, t, \; y = \cos 2® \, t, \; t = -1/6$
2. $x = \cos t, \; y = \sqrt{3}\cos t, \; t = 2® /3$
3. $x = \sec^2 t - 1, \; y = \tan t, \; t = -® /4$
4. $x = -\sqrt{t+1}, \; y = \sqrt{3t}, \; t = 3$
5. $x = 1/t, \; y = -2 + 1\text{n}\, t, \; t = 1$
6. $x = \cos t, \; y = 1 + \sin t, \; t = ® /2$

Assuming that the equations in Exercises 7-8 define *x* and *y* implicitly as differentiable functions $x = f(t), \; y = g(t),$ find the slope of the curve $x = f(t), \; y = g(t),$ as the given value of *t*.

7. $x = \sqrt{5 - \sqrt{t}}, \; y(t-1) = 1\text{n}\, y, \; t = 1$
8. $x\sin t + 2x = t, \; t\sin t - 2t = y, \; t = ®$

Find the lengths of the curves in Exercises 9-11

9. $x = t^3, \; y = 3t^2/2, \; 0 \le t \le \sqrt{3}$
10. $x = (2t+3)^{3/2}/3, \; y = t + t^2/2, \; 0 \le t \le 3$
11. $x = 1\text{n}\,(\sec t + \tan t) - \sin t, \; y = \cos t, \; 0 \le t \le ® /3$

Find the areas of the surfaces generated by revolving the curves in Exercises 12-13 above indicated axes.

12. $x = (2/3)t^{3/2}, \; y = 2\sqrt{t}, \; 0 \le t \le \sqrt{3}: \; y-\text{axis}$
13. $x = 1\text{n}\,(\sec t + \tan t) - \sin t, \; y = \cos t, \; 0 \le t \le ® /3; \; x-\text{axis}$
14. The line segment joining the origin to the point (h, r) is revolved about the $x-$axis to generate a cone of height *h* and base radius *r*. Find the cone's surface area with the parametric equations $x = ht, \; y = rt, \; 0 \le t \le 1$. Check your result with the geometry formula: $\text{Area} = ® r$ (Slant height).
15. a) Find the coordinates of the centroid of the curve

$$x = e^t \cos t, \; y = e^t \sin t, \; 0 \le t \le ®$$

b) Sketch the curve. Find the centroids coordinate to the nearest tenth and add the centroid to your sketch.

16. Most centroid calculations for curves are done with a calculator or computer that has an integral evaluation program. As a case in point, find to the nearest hundredth, the coordinates of the centroid of the curve

$$x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}\,.$$

CHAPTER.19 POLAR COORDINATES

To define polar co-ordinates we first fix an origin O called **pole** (or **origin)** and a horizontal line originating at O, called **initial ray** (or **polar axis**). Corresponding to each point P in the plane one can assign polar co-ordinates (r, θ) in which the first number 'r' gives the directed distance from O to P and the second number θ gives the directed angle from the initial line to the segment OP (Fig. 1).

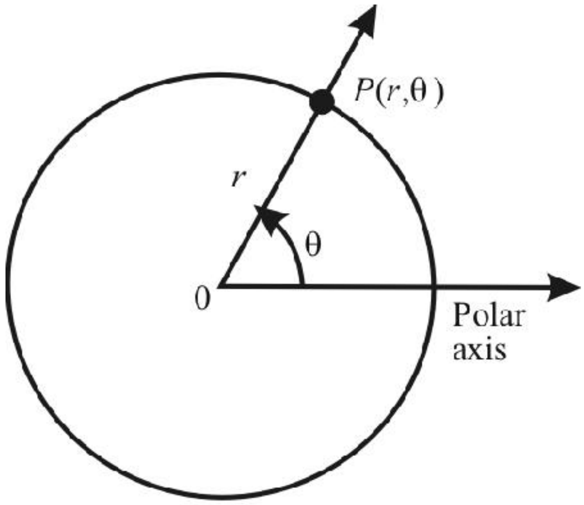

Fig.1 Polar Coordinates

***Problem* 1** Find all polar coordinates corresponding to the point P with a polar coordinate $(2, \pi/6)$.

Answer

For $r = 2$, the complete list of angles is

$$\frac{®}{6},\ \frac{®}{6} \pm 2®,\ \frac{®}{6} \pm 4®,\ \frac{®}{6} \pm 6®,\ \ldots$$

For $r = -2$, the complete list of angles is

$$\frac{-5®}{6},\ \frac{-5®}{6} \pm 2®,\ \frac{-5®}{6} \pm 4®,\ \frac{-5®}{6} \pm 6®,\ \ldots$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{®}{6} + 2n®\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

and $$\left(-2, \frac{-5®}{6} + 2n®\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

Polar equations - Elementary Coordinate Equations

A polar equation is an equation involving polar co-ordinates**.**

Circle

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Fig. 5).

The equation $r = a$ represents the circle of radius $|a|$ centered at O.

Problem $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

Line

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞ the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

Relation to Cartesian Coordinates

We suppose that the polar axis coincides with the positive x-axis of the Cartesian system. Then the polar coordinates (r, θ) of a point P and the Cartesian coordinates (x, y) of the same point are related by the following equations:

$$x = r\cos\theta$$

$$y = r\sin\theta,$$

where $\quad x^2 + y^2 = r^2 \quad$ and $\quad \dfrac{y}{x} = \tan\theta$.

Problem Find the Cartesian equivalent to the polar equation

$$r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}.$$

Answer The given equation can be written as

$r\left(\cos\theta\cos\dfrac{\pi}{4} + \sin\theta\sin\dfrac{\pi}{4}\right) = \sqrt{2}$, using the trigonometry identity

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

i.e., $\quad r\left(\cos\theta \cdot \dfrac{1}{\sqrt{2}} + \sin\theta \cdot \dfrac{1}{\sqrt{2}}\right) = \sqrt{2}$

i.e., $\quad \dfrac{1}{\sqrt{2}}x + \dfrac{1}{\sqrt{2}}y = \sqrt{2}$, $\quad$ since $x = r\cos\theta$ and $y = r\sin\theta$

i.e., $\quad x + y = 2$.

Problem Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

a) $r\cos\theta = -4$ b) $r^2 = 4r\cos\theta$ c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Answer

We use the substitutions $r\cos\theta = x,\ r\sin\theta = y,\ r^2 = x^2 + y^2$

a) Using the above, the Cartesian equation corresponding to the polar equation $r\cos\theta = -4$ is $x = -4$. Hence the graph is the vertical line $x = -4$ passing through the point $(-4, 0)$ on the x - axis.

b) The Cartesian equation corresponding to the polar equation $r^2 = 4r\cos\theta$ is obtained as follows:

$$x^2 + y^2 = 4x$$

$$x^2 - 4x + y^2 = 0$$

$$x^2 - 4x + 4 + y^2 = 4 \qquad \text{Completing the square}$$

$$(x-2)^2 + y^2 = 4$$

The graph is the circle having radius 2 and centered at (2, 0).

c) The Cartesian equation corresponding to the polar equation $r = \dfrac{4}{2\cos\theta - \sin\theta}$ is obtained as follows:

$$r(2\cos\theta - \sin\theta) = 4$$

$$2r\cos\theta - r\sin\theta = 4$$

$$2x - y = 4$$

$$y = 2x - 4$$

The graph is the straight line having slope $m = 2$ and $y-$intercept $b = -4$.

Problem By changing to Cartesian coordinates, show that $r = 8\sin\theta$ is a circle and $r = \dfrac{2}{1-\cos\theta}$ is a parabola.

Answer

If we multiply $r = 8\sin\theta$ by r, we get

$$r^2 = 8r\sin\theta$$

which, in Cartesian coordinates, is

$$x^2 + y^2 = 8y$$

and may be written successively as

$$x^2 + y^2 - 8y = 0$$

$$x^2 + y^2 - 8y + 16 = 16$$

$$x^2 + (y-4)^2 = 16,$$

the equation of the circle of radius 4 centered at (0, 4). Also,

$$r = \frac{2}{1-\cos\theta}$$

implies

$$r - r\cos\theta = 2$$

$$r - x = 2$$

$$r = x + 2$$

$$r^2 = x^2 + 4x + 4$$

$$x^2 + y^2 = x^2 + 4x + 4$$

$$y^2 = 4(x+1),$$

the equation of a parabola with vertex at (–1, 0) and focus at the origin.

Problem Find the polar equivalent of the curve whose Cartesian equation is $x^2 - y^2 = 1$.

Answer

We have $x = r\cos\theta$ and $y = r\sin\theta$.

Replacing x and y by these values in $x^2 - y^2 = 1$, we get

$$r^2\cos^2\theta - r^2\sin^2\theta = 1$$

implies $$r^2\left(\cos^2\theta - \sin^2\theta\right) = 1$$

implies $$r^2\cos 2\theta = 1,$$

which is the equivalent polar equation.

Exercises

1. Which polar coordinate pairs label the same point?

   a) $(-2, \pi/3)$ b) $(2, -\pi/3)$ c) (r, θ)

   d) $(r, \theta+\pi)$ e) $(-r, \theta)$ f) $(2, -2\pi/3)$

   g) $(-r, \theta+\pi)$ h) $(-2, 2\pi/3)$

2. Plot the following points. Then find all the polar coordinates of each points.

   a) $(3, \pi/4)$ b) $(-3, \pi/4)$ c) $(3, -\pi/4)$ d) $(-3, -\pi/4)$

3. Find the Cartesian coordinates of the following points (given in polar coordinates.).

   a) $\left(\sqrt{2}, \pi/4\right)$ b) $(1, 0)$ c) $(0, \pi/2)$ d) $\left(-\sqrt{2}, \pi/4\right)$

e) $(-3, 5\pi/6)$ f) $(5, \tan^{-1}(4/3))$ g) $(-1, 7\pi)$ h) $\left(2\sqrt{3}, 2\pi/3\right)$

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 4-11.

4. $0 \le r \le 2$
5. $1 \le r \le 2$
6. $\theta = 2\pi/3,\ r \le -2$
7. $\theta = 11\pi/4, \quad r \ge -1$
8. $\theta = \pi/2, \quad r \le 0$
9. $0 \le \theta \le \pi, \quad r = -1$
10. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$
11. $0 \le \theta \le \pi/2, \quad 1 \le |r| \le 2$

Replace the polar equations in Exercises 12-24 by equivalent Cartesian equations. Then describe or identify the graph.

12. $r\sin\theta = -1$
13. $r\cos\theta = 0$
14. $r = -3\sec\theta$
15. $r\sin\theta = r\cos\theta$
16. $r^2 = 4r\sin\theta$
17. $r^2\sin 2\theta = 2$
18. $r = 4\tan\theta\sec\theta$
19. $r\sin\theta = \ln r + \ln\cos\theta$
20. $\cos^2\theta = \sin^2\theta$
21. $r^2 = -6r\sin\theta$
22. $r = 3\cos\theta$
23. $r = 2\cos\theta - \sin\theta$
24. $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Replace the Cartesian equations in Exercises 25-31 by equivalent polar equations.

25. $y = 1$
26. $x - y = 3$
27. $x^2 - y^2 = 1$
28. $xy = 2$
29. $x^2 + xy + y^2 = 1$
30. $(x-5)^2 + y^2 = 25$
31. $(x+2)^2 + (y-5)^2 = 16$

CHAPTER.20 GRAPHING IN POLAR COORDINATES

Problem Graph the cardiod $r = 1 - \cos\theta$.

Answer

The equation $r = 1 - \cos\theta$ remains unchanged when θ is changed to $-\theta$. Hence the curve is symmetric about the *x*-axis.

As θ increases from 0 to π , $\cos\theta$ decreases from 1 to –1 and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ increases from π to 2π , $\cos\theta$ increases from –1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π .

The curve leaves the origin with slope $\tan(0) = 0$ and return to the origin with slope $\tan(2\pi) = 0$. Hence tangent at the origin is the *x*- axis.

We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the $x-$ axis to complete the graph. The curve is called a cardioid because of its heart shape.

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

Problem 7 Trace the cardioid $r = a(1 + \cos\theta)$

Answer

(i) The curve is symmetric about the initial line, since the change of θ by $-\theta$ does not alter the given equation.

(ii) Plot certain points as follows. From the Table we observe that when θ increases from 0 to π, the values of r goes on decreasing from $2a$ to 0 .

θ	$r = a(1 + \cos\theta)$

0	$2a$
$\frac{\pi}{3}$	$\frac{3a}{2}$
$\frac{\pi}{2}$	a
$\frac{2\pi}{3}$	$\frac{a}{2}$
π	0

Problem Trace $r^2 = a^2 \cos 2\theta$. (Lemniscate of Bernoulli)

(i) The curve is symmetric about the initial line.

(ii) The curve is symmetric about the line $\theta = \pi/2$.

(iii) Plot certain points as follows. We connect them from 0 to $\frac{\pi}{4}$ and complete the remaining portions using the symmetry about the line $\theta = \frac{\pi}{2}$ and about the initial line .

θ	$r = \pm a\sqrt{\cos 2\theta}$
0	$\pm a$
$\frac{\pi}{6}$	$\pm\frac{a}{\sqrt{2}}$
$\frac{\pi}{4}$	0

Exercises

Identify the symmetries of the curves in Exercises 1-6, then sketch the curves

1. $r = 1 + \cos\theta$ 2. $r = 1 - \sin\theta$ 3. $r = 2 + \sin\theta$
4. $r = \sin(\theta/2)$ 5. $r^2 = \cos\theta$ 6. $r^2 = -\sin\theta$

Graph the lemniscates in Exercises 7-8. What symmetrics do these curves have?

7. $r^2 = 4\cos 2\theta$ 8. $r^2 = -\sin 2\theta$

Find the slopes of the curves in Exercises 9-10 at the given points. Sketch the curves along with their tangents at these points.

9. (Cardioid) $r = -1 + \cos\theta;\ \theta = \pm\pi/2$
10. (Four leaved rose) $r = \sin 2\theta;\ \theta = \pm\pi/4, \pm 3\pi/4$

Limacons is Old French for “snail Equations for limacons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes. Graph the limacons in Exercises 11-14.

11. (Limacons with an inner loop)

 $a)\, r = \frac{1}{2} + \cos\theta$ $\qquad$ $b)\, r = \frac{1}{2} + \sin\theta$

12. (Dimpled limacons)

 $a)\, r = \frac{3}{2} + \cos\theta$ $\qquad$ $b)\ \ r = \frac{3}{2} - \sin\theta$

13. Oval limacons

 $a)\ r = 2 + \cos\theta$ $\qquad$ $b)\ r = -2 + \sin\theta$

14. Cardioids

 $a)\ r = 1 - \cos\theta$ $\qquad$ $b)\ r = -1 + \sin\theta$

15. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

16. Sketch the region defined by the inequality

 $0 \le r \le 2 - 2\cos\theta$

17. Show that the point $(2, 3\pi/4)$ lies on the curve $r = 2\sin 2\theta$

 Find the points of intersection of the pairs of curves in Exercises 18-21.

18. $r = 1 + \cos\theta,\quad r = 1 - \cos\theta$ $\qquad$ 19. $r = 2\sin\theta,\ r = 2\sin\theta$

20. $r = \sqrt{2};\ r^2 = 4\sin\theta$ $\qquad$ 21. $r = 1,\ r^2 = 2\sin 2\theta$

CHAPTER.21 POLAR EQUATIONS FOR CONIC SECTIONS

Introduction

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets move can all be described with a single relatively simple coordinate equation. We develop that equation here.

EQUATION FOR A LINE IN POLAR COORDINATES

To determine the equation of a line we consider the following two cases.

Case (i) *When the line passes through the pole*: Then the equation of the line in polar form is

$$\theta = \theta_0 ,$$

where θ_0 is a constant.

Case (ii) *When the line does not pass through the pole* :

Let $P_0(r_0, \theta_0)$ be the point on the line such that it is the foot of the perpendicular from origin. Then if $P(r, \theta)$ is any other point on the line, then from the right angled triangle OP_0P, we have

$$\cos(\theta - \theta_0) = \frac{r_0}{r}$$

or $\quad r\cos(\theta - \theta_0) = r_0. \qquad \ldots (1)$

The standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line *L*, and $r_0 \geq 0$, then the equation for *L* is given by Eq. (1) above.

Problem Write the polar equation for the line in Fig.3. Use the identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$ to find its Cartesian equation.

Answer

Here $\theta_0 = \frac{\pi}{3}$ and $r_0 = 2$, so using Eq. (1), we obtain

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 2$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 2$$

$$x + \sqrt{3}y = 4.$$

Problem Find the angle between the lines whose equations are $d = r\cos(\theta - \alpha)$ and $d_1 = r\cos(\theta - \alpha_1)$. Deduce the condition for the lines to be perpendicular.

Answer

$$d = r\cos(\theta - \alpha) = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$

i.e.,
$$d = x\cos\alpha + y\sin\alpha$$

$\therefore$ slope of this line is $m = -\dfrac{\cos\alpha}{\sin\alpha} = -\cot\alpha$

Similarly, slope of the second line is $m_1 = -\cot\alpha_1$.

If θ is the angle between the two lines, then

$$\tan\theta = \frac{m - m_1}{1 + m m_1} = -\frac{(\cot\alpha - \cot\alpha_1)}{1 + \cot\alpha\cot\alpha_1} = -\frac{(\tan\alpha_1 - \tan\alpha)}{1 + \tan\alpha_1\tan\alpha}$$

$$\therefore\ \tan\theta = \tan(-(\alpha_1 \mp \alpha))$$

or $$\theta = -(\alpha_1 \mp \alpha).$$

If the lines are perpendicular, then $\theta = \pi/2$, so $\alpha_1 + \alpha = -\pi/2$ or $\alpha_1 - \alpha = -\pi/2$.

Exercises

Sketch the lines in the following exercises and find the Cartesian equations for them.

1. $r\cos\left(\theta + \dfrac{3\pi}{4}\right) = 1$ 2. $r\cos\left(\theta + \dfrac{\pi}{3}\right) = 2$

3. $r\cos\left(\theta - \dfrac{\pi}{4}\right) = \sqrt{2}$ 4. $r\cos\left(\theta - \dfrac{2\pi}{3}\right) = 3$

Find the polar equation in the form $r\cos(\theta - \theta_0) = r_0$ for each of the lines in Exercises 5-

5. $\sqrt{2}x + \sqrt{2}y = 6$ 6. $y = -5$

7. $\sqrt{3}x - y = 1$ 8. $x = -4$

EQUATION FOR A CIRCLE IN POLAR CO-ORDINATES

(i)If the circle of radius a is centered at the pole , its equation is

$r = a$. … (2)

(*ii*) If the circle of radius a is centered at (r_0, θ_0), then using Law of Cosines to the triangle OP_0P , we get

$a^2 = r_0^{\,2} + r^2 - 2r_0 r\cos(\theta - \theta_0)$. … (3)

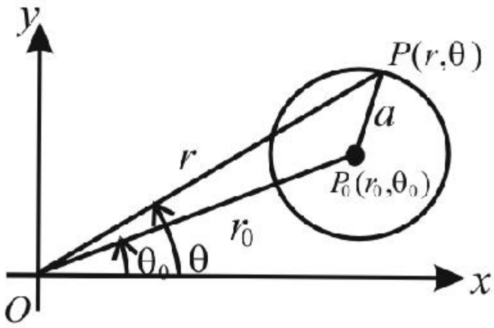

Fig.5

Equations of circles passing through the origin

(*iii*) If the circle of radius a is centered at (r_0, θ_0) and passing through the origin, then $r_0 = a$. Then, by putting $r_0 = a$ in Eq. (3), gives

$$a^2 = a^2 + r^2 - 2ar\cos(\theta - \theta_0),$$

which simplifies to

$$r = 2a\cos(\theta - \theta_0). \qquad \ldots(4)$$

(*iv*) Special Case of (*iii*) : Consider the circle that passes through the origin and whose centre lies on the initial line. Centre lies on the initial line (i.e., on the positive x-axis) implies $\theta_0 = 0$ and in that case Eq. (4) gives (Fig. 7)

$$r = 2a\cos\theta\ . \qquad \ldots(5)$$

(*v*) Special Case of (*iii*) : If the center lies on the positive y-axis, $\theta_0 = \frac{\pi}{2}$, and since $\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$, Eq.(4) becomes (Fig.8)

$$r = 2a\sin\theta. \qquad \ldots(6)$$

(*iv*) If the centre lies on the negative x-axis, then the equation of the circle is obtained by replacing r by $-r$ in Eq.(5) and is (Fig. 9)

$$-r = 2a\cos\theta\ . \qquad \ldots(7)$$

(*v*) If the center lies on the negative y-axis, then the equation of the circle is obtained by replacing r by $-r$ in Eq. (6) and is (Fig. 10)

$$-r = 2a\sin\theta.$$

Problem Find the equations of the circles passing through origin and having radius and center (in polar coordinates) as below:

(*i*) Radius : 3 Center : (3, 0)

(*ii*) Radius : 2 Center : $(2, \pi/2)$

(*iii*) Radius : 1/2 Center : $(-1/2, 0)$

(*iv*) Radius : 1 Center : $(-1, \pi/2)$

Answer

$(i)\ r = 6\cos\theta \quad (ii)\ r = 4\sin\theta \quad (iii)\ r = -\cos\theta \quad (iv)\ r = -2\sin\theta$

Exercises

Find the polar equations for the circles in Exercises 7-10. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

1. $(x-6)^2 + y^2 = 36$
2. $x^2 + (y-5)^2 = 25$
3. $(x+2)^2 + y^2 = 4$
4. $x^2 + (y+7)^2 = 49$
5. $x^2 - 16x + y^2 = 0$
6. $x^2 + y^2 - \frac{4}{3}y = 0$
7. $x^2 + 2x + y^2 = 0$
8. $x^2 + y^2 + y = 0$

ELIPSES, PARABOLAS, AND HYPERBOLAS

To find polar equation for ellipses, parabolas, and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$. This makes

$$PF = r$$

and

$$PD = k - FB = k - r\cos\theta \ .$$

The conic's focus – directrix equation $PF = e \cdot PD$ then becomes

$$r = e(k - r\cos\theta)$$

which can be solved for r to obtain

$$r = \frac{ke}{1 + e\cos\theta} \ .$$

The equation

$$r = \frac{ke}{1 + e\cos\theta} \qquad \ldots(8)$$

represents an **ellipse** if $0 < e < 1$, a **parabola** if $e = 1$ and a **hyperbola** if $e > 1$.

Problem Using Eq.(8), we give equations of some conics:

$e = \frac{1}{3}$: ellipse $r = \frac{k}{3 + \cos\theta}$

$e = 1$: parabola $r = \frac{k}{1 + \cos\theta}$

$e = 3$: hyperbola $r = \frac{3k}{1 + 3\cos\theta}$

Different equations for conic sections with the change of directrix

There are variations of Eq. (8) from time to time, depending on the location of the directrix.

1. If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus) (e replace Eq.(8) by

$$r = \frac{ke}{1 - e\cos\theta} \qquad \ldots(9)$$

the denominator now has a $(-)$ instead of a (+).

2. If the directrix is the line $y = k$ (the origin is still a focus) , we replace Eq.(8) by

$$r = \frac{ke}{1 + e\sin\theta}, \qquad \ldots(10)$$

the equation with sine in them instead of cosine.

3. If the directrix is the line $y = -k$ (the origin is still a focus) , we replace Eq. (8) by

$$r = \frac{ke}{1 - e\sin\theta} \qquad \ldots(11)$$

the equation with sine in them instead of cosine and a $(-)$ instead of a (+).

Problem Find the directrix of the parabola

$$r = \frac{25}{10 + 10\cos\theta}$$

Answer We divide the numerator and denominator by 10 to put the equation in standard form:

$$r = \frac{5/2}{1 + \cos\theta}.$$

This is the equation

$$r = \frac{ke}{1 + e\cos\theta}$$

with $k = 5/2$ and $e = 1$. Hence the equation of the directrix is $x = 5/2$.

Ellipse with Eccentricity *e* and Semimajor Axis *a*

From the ellipse diagram in we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea$$

From this, we find that $ke = a(1 - e^2)$. Replacing ke in Eq. (18) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

The polar equation of an **ellipse** with eccentricity e and semimajor axis a is given by

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \qquad \ldots(13)$$

Notice that when $e = 0$, equation (13) becomes $r = a$, which represents a **circle.**

Eq. (13) is the starting point for calculating planetary orbits.

Problem Find a polar equation for an ellipse with semimajor axis 39.44 AU (astronomical units) and eccentricity 0.25. This is the approximate size of Pluto's orbit around the sun.

Answer

We use Eq. (13) with $a = 39.44$ and $e = 0.25$ to find

$$r = \frac{39.44(1-(0.25)^2)}{1+0.25\cos\theta} = \frac{147.9}{4+\cos\theta}.$$

At its point of closest approach (perihelion), Pluto is

$$r = \frac{147.9}{4+1} = 29.58\,\text{AU}$$

from the sun. At its most distant point (aphelion), Pluto is

$$r = \frac{147.9}{4-1} = 49.3\,\text{AU}$$

from the sun .

Problem Find the distance from one focus of the ellipse in Problem 6 to the associated directrix.

Answer k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea. \qquad \ldots(14)$$

Here $a = 39.44$ and $e = 0.25$, so that

$$k = 39.44\left(\frac{1}{0.25} - 0.25\right) = 147.9\,\text{AU}$$

Exercises

Exercises 1-4 give the eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus. Find a polar equation for each conic section.

1. $e = 1,\ y = 2$
2. $e = 2,\ x = 4$
3. $e = 1/4,\ x = -2$
4. $e = 1/3,\ y = 6$

Sketch the parabolas and ellipses in Exercises 5-8. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

5. $r = \dfrac{6}{2+\cos\theta}$
6. $r = \dfrac{4}{2-2\cos\theta}$
7. $r = \dfrac{12}{3+3\sin\theta}$
8. $r = \dfrac{4}{2-\sin\theta}$

Exercises 1-4 give the eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus. Find a polar equation for each conic section.

1. $e=1,\ x=2$ 2. $e=5,\ y=-6$ 3. $e=1/2,\ x=1$ 4. $e=1/5,\ y=-10$

Sketch the parabolas and ellipses in Exercises 5-8. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

5. $r=\dfrac{1}{1+\cos\theta}$ 6. $r=\dfrac{25}{10-5\cos\theta}$ 7. $r=\dfrac{400}{16+8\sin\theta}$ 8. $r=\dfrac{8}{2-2\sin\theta}$

Sketch the regions defined by the inequalities in Exercise 9-10

9. $0\le r\le 2\cos\theta$ 10. $-3\cos\theta\le r\le 0$

CHAPTER.22 AREA OF POLAR CURVES IN THE PLANE

The area of the sector enclosed by the curve $r = f(\theta)$ and the two radii vectors $\theta = \alpha$ and $\theta = \beta$ is given by

$$A=\frac{1}{2}\int_{\alpha} r^2 d\theta\ .$$

This is the integral of the **area differential**

$$dA=\frac{1}{2}r^2 d\theta.$$

Problem Find the area of the curve $r=a+b\cos\theta,\ a>b.$

Answer

The required area

$$=\frac{1}{2}\int_0^{2\pi} r^2 d\theta=\frac{1}{2}\int_0^{2\pi}\left(a+b\cos\theta\right)^2 d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(a^2+2ab\cos\theta+b^2\cos^2\theta\right)d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(a^2+2ab\cos\theta+b^2\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left[\left(a^2+\frac{b^2}{2}\right)\theta\right]_0^{2\pi}+2ab\left[\sin\theta\right]_0^{2\pi}+\left[\frac{b^2\sin 2\theta}{4}\right]_0^{2\pi}$$

$$=\left(a^2+\frac{b^2}{2}\right)\pi.$$

Problem Find the area enclosed by the curve $r^2 = a^2\cos 2\theta$.

Answer

The given curve is called the *Lemniscate of Bernoulli*.

Since replacing r by $-r$ and θ by $-\theta$ do not alter the equation, the given curve is symmetrical about the initial line and the line given by $\theta=\frac{\pi}{2}$. Hence, to draw the entire curve we need to draw the portion of the curve between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and then apply the symmetries.

The given equation can be written as $\frac{r^2}{a^2} = \cos 2\theta$. Since $\frac{r^2}{a^2}$ is always non-negative, $\cos 2\theta \geq 0$ implies $0 \leq 2\theta \leq \frac{\pi}{2}$ implies $0 \leq \theta \leq \frac{\pi}{4}$.

Hence, in the region $\theta = 0$ and $\theta = \frac{\pi}{2}$, the curve has real portion between $\theta = 0$ and $\theta = \frac{\pi}{4}$ and no real portion between $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$. When θ varies from $\theta = 0$ to $\theta = \frac{\pi}{4}$, r varies from a to 0. Using symmetry about the initial line, we can say that there is a loop between $(r, \theta) = \left(0, -\frac{\pi}{4}\right)$ and $(r, \theta) = \left(0, \frac{\pi}{4}\right)$. Again, using symmetry about $\theta = \frac{\pi}{2}$, there is also one loop to the left of $\theta = \frac{\pi}{2}$. Hence, there are two loops for the given curve.

Hence, the required area is the area enclosed by the two loops.

Now the area enclosed by one loop of the curve (where θ varies from $\theta = \frac{-\pi}{4}$ to $\theta = \frac{\pi}{4}$, is given by

$$= \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta \; d\theta = \frac{1}{2}a^2\left[\frac{\sin 2\theta}{2}\right]_{-\pi/4}^{\pi/4} = \frac{a^2}{2}.$$

So the required area is two times the area of one loop

$$= 2 \times \frac{a^2}{2} = a^2.$$

Problem Find the area of a loop of the curve $r = a\sin 3\theta$.

Answer

Area of one loop

$$= \frac{1}{2}\int_0^{\pi/3} r^2 d\theta = \frac{1}{2}\int_0^{\pi/3} a^2 \sin^2 3\theta \; d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} \frac{1+\cos 6\theta}{2} d\theta = \frac{a^2}{4}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{\pi a^2}{12}.$$

Exercises ***Set A***

1. Find the area of the circle $r = a\sin 2\theta$.
2. Find the area of the curve $r = 3 + 2\cos\theta$.
3. Find the area enclosed within the curve $r = 4(1+\cos\theta)$.
4. Show that the area of one loop of the three leaved rose $r = a\cos 3\theta$ is $\frac{\pi a^2}{12}$.
5. Find the area of the cardioid $r = a\left(1 - \cos\theta\right)$.

   Find the areas of the regions in Exercises 1-3

6. Inside the oval limacon $r = 4 + 2\cos\theta$
7. Inside one leaf of the four- leaved rose $r = \cos 2\theta$
8. Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$

   Find the areas of the regions in Exercises 9-11

9. Inside the cardioid $r = a(1+\cos\theta)$

10. Inside the lemniscate $r^2 = 2a^2\cos 2\theta$, $a > 0$

11. Inside the six- leaved rose $r^2 = 2\sin 3\theta$

Area between two polar curves

Now we give a formula to find the area of a region like the one in Fig.8 which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$.

Area of the region

$0 \le r_1(\theta) \le r \le r_2(\theta),\ \alpha \le \theta \le \beta$ **is given by**

$$A = \frac{1}{2}\int_\alpha^\beta r_2^2\,d\theta - \frac{1}{2}\int_\alpha^\beta r_1^2\,d\theta = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)d\theta. \qquad \ldots(1)$$

Problem Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1-\cos\theta$.

Answer

We sketch the region to determine its boundaries and find the limits of integration. The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1-\cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, using Eq. (1), is

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2}(r_2^2 - r_1^2)\,d\theta$$

$$= 2\times\frac{1}{2}\int_0^{\pi/2}(r_2^2 - r_1^2)\,d\theta \text{, by symmetry}$$

$$= \int_0^{\pi/2}(1-(1-2\cos\theta+\cos^2\theta))\,d\theta$$

$$= \int_0^{\pi/2}(2\cos\theta - \cos^2\theta)\,d\theta$$

$$= \int_0^{\pi/2}\left(2\cos\theta - \frac{1+2\cos\theta}{2}\right)d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.$$

Exercises

Find the areas of the regions in Exercises 1-5.

1. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$
2. Shared by the circle $r = 2$ and cardioid $r = 2(1-\cos\theta)$
3. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$
4. Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$
5. Inside the circle $r = 6$ above the line $r = 3\csc\theta$
6. a) Find the area of the shaded region in Fig.9.

   b) It looks as if the graph of $r = \tan\theta$, $-\pi/2 < \theta < \pi/2$, could be asymptotic to the lines $x = 1$ and $x = -1$. Is it? Give reasons for your answer.

7. Show that the area of the region included between the cardioids $r = a(1+\cos¯)$ and $r = a(1-\cos¯)$ is $\frac{(3® - 8)a^2}{2}$.

Find the areas of the regions in Exercises 4- 8

1. Shared by the circles $r = 1$ and $r = 2\sin¯$
2. Shared by the cardioids $r = 2(1+\cos¯)$ and $r = 2(1-\cos¯)$
3. Inside the circle $r = 3a\cos¯$ and outside the cardioid $r = a(1+\cos¯)$, $a > 0$
4. a) Inside the outer loop of the limacon $r = 2\cos¯ + 1$ (Ref. Fig. 6)

   b) Inside the outer loop and outside the inner loop of the limacon $r = 2\cos¯ + 1$.
5. Inside the lemniscate $r^2 = 6\cos 2¯$ to the right of the line $r = (3/2)\sec¯$.

CHAPTER.23 LENGTH OF POLAR CURVES

Length of plane curves in polar co-ordinates

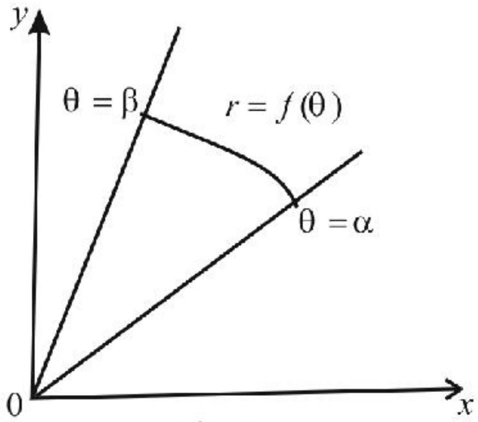

Fig.1

If $r = f(¯)$ has continuous first derivative for $~ \le ¯ \le$ and if the point $P(r,¯)$ traces the curve $r = f(¯)$ exactly once as ¯ runs from ~ to , then the length of the curve is

$$L = \int_{~} \sqrt{r^2 + \left(\frac{dr}{d¯}\right)^2}\, d¯$$

Problem Find the length of the perimeter of the cardioid $r = a(1-\cos¯)$.

Answer

The curve is symmetrical about the initial line, since the change of ¯ by $-¯$ does not alter the given equation.

Hence the perimeter of the given curve is *twice* the length of the arc of the curve lying above the initial line.

Above the initial line, the arc varies from ¯ = 0 to ¯ = ® and hence limits of integration are ¯ = 0 and ¯ = ®.

Hence the required perimeter is given by

$$s = 2\times\int_{\alpha} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = 2\times\int_0^{\pi}\sqrt{\left(a(1-\cos\theta)\right)^2+\left(a\sin\theta\right)^2}\,d\theta \text{ ,since } \frac{dr}{d\theta} = a\sin\theta .$$

$$= 2\int_0^{\pi}\sqrt{2a^2(1-\cos\theta)}\,d\theta = 2\int_0^{\pi} 2a\sin\frac{\theta}{2}\,d\theta = 8a\left[-\cos\frac{\theta}{2}\right]_0^{\pi} = 8a.$$

Problem In an equiangular spiral $r = ae^{\theta\cot\alpha}$, prove that

$$\theta\cot\alpha = \log\left\{\frac{s}{a}\cos\alpha+1\right\},$$

where s is the length of the arc measured from $\theta = 0$ to any arbitrary point.

Answer

Differentiating the given equation with respect to θ, we obtain

$$\frac{dr}{d\theta} = ae^{\theta\cot\alpha}\cot\alpha$$

Hence

$$s = \int_0^{\theta}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{\theta} ae^{\theta\cot\alpha}\sqrt{1+\cot^2\alpha}\,d\theta$$

$$= a\operatorname{cosec}\alpha\int_0^{\theta} e^{\theta\cot\alpha}\,d\theta = a\operatorname{cosec}\alpha\left[\frac{e^{\theta\cot\alpha}}{\cot\alpha}\right]_0^{\theta}$$

$$= a\operatorname{cosec}\alpha\left(\frac{e^{\theta\cot\alpha}-1}{\cot\alpha}\right) = \frac{a}{\cos\alpha}\left(e^{\theta\cot\alpha}-1\right)$$

$$\therefore \quad e^{\theta\cot\alpha} = \frac{s}{a}\cos\alpha+1 \qquad \text{or} \qquad \theta\cot\alpha = \log\left(\frac{s}{a}\cos\alpha+1\right)$$

Exercises

- Find the perimeter of the cardioid $r = a(1+\cos\theta)$
- Find the perimeter of the circle $r = a\cos\theta$. [The given circle is a circle having radius $\frac{a}{2}$, passing through the origin, and centered on the positive x-axis
- Show that the perimeter of the cardioid $r = 4(1-\cos\theta)$ is 32 .

  Find the lengths of the following curves
- The spiral $r = \theta^2,\ 0 \le \theta \le \sqrt{5}$
- The cardioid $r = 1+\cos\theta$
- The parabolic segment $r = 6/(1+\cos\theta),\quad 0 \le \theta \le \pi/2$
- The curve $r = \cos^3(\theta/3),\ 0 \le \theta \le \pi/4$

- The curve $r=\sqrt{1+\cos 2\theta}$, $0\le\theta\le\pi\sqrt{2}$
- The spiral $r=e^{\theta}/\sqrt{2},\ 0\le\theta\le\pi$
- The curve $r=a\sin^2(\theta/2),\ 0\le\theta\le\pi,\ a>0$
- The parabolic segment $r=2/(1-\cos\theta), \pi/2\le\theta\le\pi$
- The curve $r=\sqrt{1+\sin 2\theta},\ 0\le\theta\le\pi\sqrt{2}$

Calculate the circumference of the following circles $(a>0)$

- $r=a$
- $r=a\cos\theta$
- $r=a\sin\theta$

CHAPTER.24 AREA OF SURFACE OF REVOLUTION

Area of surface of revolution in polar coordinates

If $r=f(\theta)$ has a continuous first derivative for $\theta_1\le\theta\le\theta_2$ and if the point $P(r,\theta)$ traces the curve $r=f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the *x*- and *y*- axes are given by the following formlas:

1. Revolution about the *x*- axis $(y\ge 0)$**:** $S=2\pi\int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ …(1)

2. Revolution about the *y*- axis $(x\ge 0)$**:** $S=2\pi\int_{\alpha}^{\beta} r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ …(2)

Problem Find the area of the surface generated by revolving the right- hand loop of the lemniscate $r^2=\cos 2\theta$ about the $y-$axis.

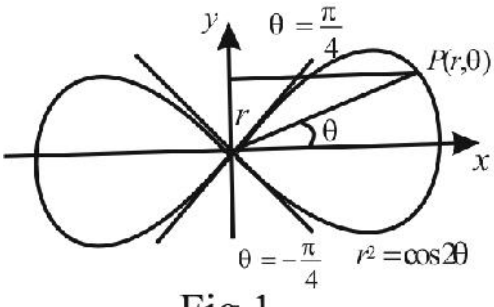

Fig.1

Answer We sketch the loop to determine the limits of integration. The point $P(r,\theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are values we take for α and β. Hence using Eq.(2),

$$S = 2\pi \int_{\alpha} r\cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = 2\pi \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos\theta \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}\, d\theta,$$

as $r\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}$

Evaluation of $2\cos\theta\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}$:

Differentiating $r^2 = \cos 2\theta$, with respect to θ, we obtain

$$2r\frac{dr}{d\theta} = -2\sin 2\theta \quad \Rightarrow \quad r\frac{dr}{d\theta} = -\sin 2\theta \quad \Rightarrow \left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta$$

Also, $r^4 = \left(r^2\right)^2 = \cos^2 2\theta$.

$\therefore \quad \sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1$ and hence $\quad S = 2\pi \int_{-\pi/4}^{\pi/4} \cos\theta \cdot (1) d\theta$

$$= 2\pi \left[\sin\theta\right]_{-\pi/4}^{\pi/4}$$

$$= 2\pi \left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}\ .$$

Exercises

Find the areas of the surfaces generated by revolving the curves in Exercises 1-4 about the indicated axes.

1. $r = \sqrt{\cos 2\theta}$, $0 \le \theta \le \pi/4$, $y-$axis
2. $r^2 = \cos 2\theta$, $x-$axis
3. $r = \sqrt{2}e^{\theta/2}$, $0 \le \theta \le \pi/2$, $x-$axis
4. $r = 2a\cos\theta$, $a > 0$, $y-$axis
5. Show that the area of the surface of the solid formed by the revolution of the cardioids $r = a(1-\cos\theta)$ about its initial line is $\frac{32}{5}\pi a^2$.
6. Show that the area of the surface of the solid formed by the revolution of the lemnisacate $r^2 = a^2\cos 2\theta$ about its initial line is $\left(1-\frac{1}{\sqrt{2}}\right)\pi a^2$.

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