Calculus for the Life Sciences

Lecture Notes – Linear Models

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Outline

Snowy Tree Cricket

Chirping Crickets and Temperature

- Folk method for finding temperature (Fahrenheit)
  Count the number of chirps in a minute and divide by 4, then add 40
- In 1898, A. E. Dolbear [3] noted that
  "crickets in a field [chirp] synchronously, keeping time as if led by the wand of a conductor"
- He wrote down a formula in a scientific publication (first?)

$$T = 50 + \frac{N - 40}{4}$$

- Does this formula of Dolbear match the folk method described above?

[3] A. E. Dolbear, The cricket as a thermometer, American Naturalist (1897) 31, 970-971

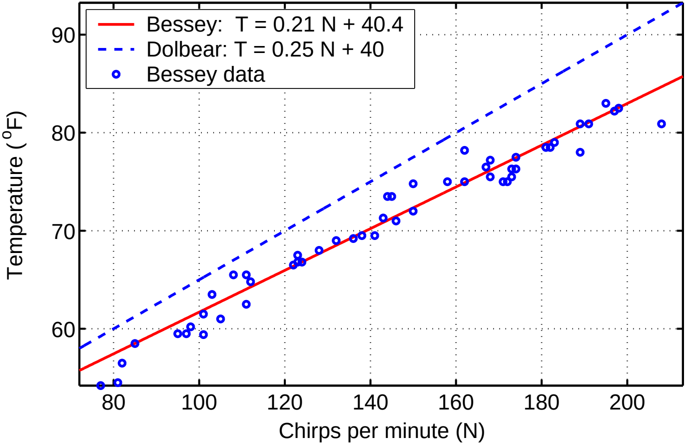
Data Fitting Linear Model

- Mathematical models for chirping of snowy tree crickets, *Oecanthulus fultoni*, are **Linear Models**
- Data from C. A. Bessey and E. A. Bessey [2] (8 crickets) from Lincoln, Nebraska during August and September, 1897 (shown on next slide)
- The least squares best fit line to the data is

$$T = 60 + \frac{N - 92}{4.7}$$

[2] C. A. Bessey and E. A. Bessey, Further notes on thermometer crickets, American Naturalist (1898) 32, 263-264

Bessey Data and Linear Models



Model

Cricket Equation as a Linear Model

The line creates a mathematical model

- The **temperature,** T as a **function** of the rate snowy tree crickets chirp, **Chirp Rate,** N

There are several Biological and Mathematical questions about this **Linear Cricket Model**

There is a complex relationship between the biology of the problem and the mathematical model

Biological Questions – Cricket Model 1

How well does the line fitting the Bessey & Bessey data agree with the Dolbear model given above?

- Graph shows Linear model fits the data well
- Data predominantly below **Folk/Dolbear** model
- Possible discrepancies
  - Different cricket species
  - Regional variation
  - Folk only an approximation
- Graph shows only a few °F difference between models

Biological Questions – Cricket Model 2

When can this model be applied from a practical perspective?

- Biological thermometer has limited use
- Snowy tree crickets only chirp for a couple months of the year and mostly at night
- Temperature needs to be above 50°F

Mathematical Questions – Cricket Model 1

Over what range of temperatures is this model valid?

- Biologically, observations are mostly between 50°F and 85°F
- Thus, limited **range** of temperatures, so limited **range** on the **Linear Model**
- **Range** of **Linear functions** affects its **Domain**
- From the graph, **Domain** is approximately 50–200 **Chirps/min**

Mathematical Questions – Cricket Model 2

How accurate is the model and how might the accuracy be improved?

- Data closely surrounds **Bessey Model** – No more than about 3°F away fom line
- **Dolbear Model** is fairly close though not as accurate – Sufficient for rapid temperature estimate
- Observe that the temperature data trends lower at higher chirp rates – compared against linear model
- Better fit with **Quadratic function** – Is this really significant?

Equation of Line – Slope-Intercept Form

The **Slope-Intercept** form of the **Line**

$$y = m\,x + b$$

- The variable x is the **independent variable**
- The variable y is the **dependent variable**
- The **slope** is m
- The y**-intercept** is b

Equation of Line – Cricket-Thermometer

The folk/Dolbear model for the cricket thermometer

$$T = \frac{N}{4} + 40$$

- The independent variable is N, chirps/min
- The dependent variable is T, the temperature
- Thus, the temperature can be estimated from counting the number of chirps/min
- Equivalently, the temperature (measurement) depends on how rapidly the cricket is chirping

Equation of Line – Point-Slope Form

The **Point-Slope** form of the **Line** is often the most useful form

$$y - y_0 = m(x - x_0)$$

or

$$y = m(x - x_0) + y_0$$

- The **slope** is m
- The given **point** is (x_0, y_0)
- Again the independent variable is x, and the dependent variable is y

Equation of Line – Two Points

Given two points (x_0, y_0) and (x_1, y_1), the **slope** is given by

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

Use the previous **point-slope** form of the line satisfies

$$y = m(x - x_0) + y_0$$

where the slope is calculated above and either point can be used.

Example – Slope and Point

Find the equation of a line with a slope of 2, passing through the point $(3, -2)$.
What is the y-intercept?

Skip Example

The point-slope form of the equation gives:

$$\begin{aligned} y - (-2) &= 2(x - 3) \\ y + 2 &= 2\,x - 6 \\ y &= 2\,x - 8 \end{aligned}$$

Example – Two Points

Find the equation of a line passing through the points $(-2, 1)$ and $(3, -2)$

Skip Example

The slope satisfies

$$m = \frac{1-(-2)}{-2-3} = -\frac{3}{5}$$

From the point-slope form of the line equation, using the first point

$$\begin{aligned} y - 1 &= -\frac{3}{5}(x+2) \\ y &= -\frac{3}{5}x - \frac{1}{5} \end{aligned}$$

Parallel and Perpendicular Lines

Consider two lines given by the equations:

$$y = m_1x + b_1 \qquad \text{and} \qquad y = m_2x + b_2$$

The two lines are **parallel** if the slopes are equal, so

$$m_1 = m_2$$

and the y-intercepts are different.

If $b_1 = b_2$, then the lines are the same.

The two lines are **perpendicular** if the slopes are negative reciprocals of each other, that is

$$m_1m_2 = -1$$

Example – Perpendicular Lines 1

Find the equation of the line **perpendicular** to the line

$$5x + 3y = 6$$

passing through the point $(-2, 1)$

Skip Example

Solution: The line can be written

$$\begin{aligned} 3y &= -5\,x + 6 \\ y &= -\frac{5}{3}x + 2 \end{aligned}$$

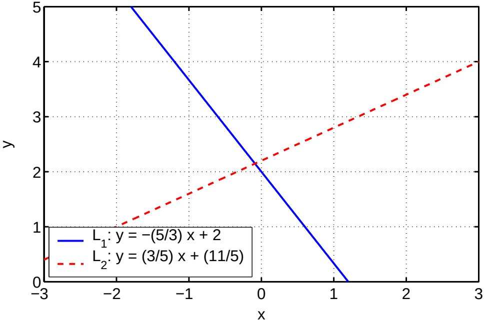
The slope of the perpendicular line (m_2) is the negative reciprocal

$$m_2 = \frac{3}{5}$$

Example – Perpendicular Lines 2

The point slope equation of the perpendicular line is

$$\begin{aligned} y - 1 &= \frac{3}{5}(x+2) \\ y &= \frac{3}{5}x + \frac{11}{5} \end{aligned}$$



Example – Intersection of Lines 1

Find the intersection of the line parallel to the line $y = 2x$ passing through $(1, -3)$ and the line given by the formula

$$3x + 2y = 5$$

Skip Example

Solution: The line parallel to $y = 2x$ has slope $m = 2$, so satisfies

$$\begin{aligned} y + 3 &= 2(x - 1) \\ y &= 2x - 5 \end{aligned}$$

Example – Intersection of Lines 2

Continued: Substitute y into the formula for the second line

$$\begin{aligned} 3x + 2(2x - 5) &= 5 \\ 7x &= 15 \quad \text{or} \quad x = \frac{15}{7} \end{aligned}$$

Substituting the x value into the first line equation gives

$$y = 2\left(\frac{15}{7}\right) - 5 = -\frac{5}{7}$$

The point of intersection is

$$(x, y) = \left(\frac{15}{7}, -\frac{5}{7}\right)$$

Metric System Conversion

All of the conversions for measurements, weights, temperatures, etc. are linear (or affine) relationships

Javascript Conversions

Example – Temperature 1

Convert Temperature Fahrenheit to Celsius

- The freezing point of water is 32°F and 0°C, so take

$$(f_0, c_0) = (32, 0)$$

- The boiling point of water is 212°F and 100°C (at sea level), so take

$$(f_1, c_1) = (212, 100)$$

Example – Temperature 2

Convert Temperature Fahrenheit to Celsius

Solution: The slope satisfies

$$m = \frac{c_1 - c_0}{f_1 - f_0} = \frac{100 - 0}{212 - 32} = \frac{5}{9}$$

The point-slope form of the line gives

$$\begin{aligned} c - 0 &= \frac{5}{9}(f - 32) \\ c &= \frac{5}{9}(f - 32) \end{aligned}$$

The temperature f in Fahrenheit is the independent variable
The equation of the line gives the dependent variable c in Celsius

Example – Weight Conversion 1

Find the weight of a 175 pound man in kilograms.

Skip Example

Solution: Tables show **1 kilogram** is **2.2046 pounds**

To convert pounds to kilograms, the slope for the conversion is

$$m = \frac{1 \text{ kg}}{2.2046 \text{ lb}} = 0.45360 \text{ kg/lb}$$

Example – Weight Conversion 2

Solution (cont): Let x be the weight in pounds and y be the weight in kilograms, then

$$y = 0.45360\,x$$

Thus, a 175 lb man is

$$y = 0.45360(175) = 79.38 \text{ kg}$$

Inverse Linear Function

Linear functions always have an Inverse
(Provided $m \neq 0$)

Consider the line

$$y = mx + b$$

Solving for x

$$\begin{aligned} mx &= y - b \\ x &= \frac{y-b}{m} \end{aligned}$$

The **Inverse Line** satisfies

$$x = \left(\frac{1}{m}\right) y - \frac{b}{m}$$

Example - Inverse Line

The equation for converting °F to °C is

$$c = \frac{5}{9}(f - 32)$$

So,

$$f - 32 = \frac{9}{5}c$$

The equation for converting °C to °F is

$$f = \frac{9}{5}c + 32$$

Juvenile Height – Data

The table below gives the **average juvenile height** as a function of age [4]

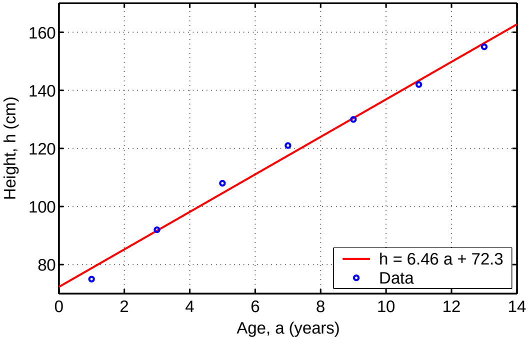
Age (yr)	1	3	5	7	9	11	13
Height (cm)	75	92	108	121	130	142	155

The data almost lie on a line, which suggests a **Linear Model**

[4] David N. Holvey, editor, The Merck Manual of Diagnosis and Therapy (1987) 15th ed., Merck Sharp & Dohme Research Laboratories, Rahway, NJ.

Juvenile Height – Graph

Below is a graph of the data and the besting fitting **Linear Model**



Juvenile Height – Linear Model

The linear least squares best fit to the data is

$$h = ma + b = 6.46a + 72.3$$

The next section will explain finding the linear least squares best fit or linear regression to the data

Model is valid for ages 1 to 13, the range of the data

Juvenile Height – Linear Model

For modeling, it is valuable to place units on each part of the equation

$$h = ma + b = 6.46a + 72.3$$

- The height, h, from our data has units of cm
- Thus, ma and the intercept b must have units cm
- Since the age, a, has units of years, the slope, m, has units of cm/year
- The slope is the rate of growth

Juvenile Height – Linear Model

What questions can you answer with this mathematical model?

What height does the model predict for a newborn baby?

Solution: At $a = 0$, we obtain the h-intercept

The model predicts that the **average newborn** will be **72.3 cm**

However, this is outside the range of the data, which makes its value more suspect

Juvenile Height – Linear Model

What is the average height of an eight year old?

Solution: Let $a = 8$, then

$$h(8) = 6.46(8) + 72.3 = 124.0$$

The model predicts that the **average eight year old** will be **124.0 cm**

What would give a better estimate? (Hint: Local Analysis)

Solution: Average the data at ages 7 and 9

$$h_{ave}(8) = \frac{121 + 130}{2} = 125.5 \text{ cm}$$

Juvenile Height – Linear Model

If a six year old child is 110 cm, then estimate how tall she'll be at age 7

Solution: The model indicates that the growth rate is about 6.5 cm/year

Thus, with average growth she should add about 6.5 cm and be **116.5 cm**

The model predicts the **average 7 year old** is **117.5 cm**
The data shows the **average 7 year old** is **121 cm**

Clearly, the first estimate is the best for this particular girl

Juvenile Height – Model Limitations

What are some of the limitations of the model?

- The domain of this function is restricted to some interval around $1 < a < 13$
  - The model predicts the average 20 year old to be 201.5 cm
- Local Analysis
  - Average 8 year old height better predicted from 7 and 9 year olds (125.5 cm)
  - Average newborn better estimated by data for 1 and 3 year old (66.5 cm)

Juvenile Height – Model Improvements

How might the model be improved?

- Growth rates for girls and boys differ – split the data according to sex
- Data show faster growth rates between 0 and 5 and again between 9 and 13
  - Growth spurts occur
  - Design a nonlinear model – Other functions
- Later we study growth models with differential equations

Sea Urchin Growth Model 1

Linear Models are reasonable for estimating growth over short time periods

Consider a population of white sea urchins (*Lytechinus pictus*)

- Mean diameter of 28 mm on June 1 ($t = 0$)
- Mean diameter of 33 mm on July 1 ($t = 30$)

Estimate the mean diameter for the population of *Lytechinus pictus* on June 20 ($t = 19$), July 10 ($t = 39$), August 1 ($t = 61$), and August 15 ($t = 75$)

Which estimates do you trust more and why?

Skip Example

Sea Urchin Growth Model – Solution 2

Solution: The growth model desired has the form:

$$d = mt + d_0$$

where d is the mean diameter (mm) of the urchin and t is the number of days after June 1

The data give two points

$$(t_0, d_0) = (0, 28) \quad \text{and} \quad (t_1, d_1) = (30, 33)$$

The slope is

$$m = \frac{33 - 28}{30 - 0} = \frac{1}{6} \text{ mm/day}$$

Sea Urchin Growth Model – Solution 3

Solution (cont): The growth model satisfies:

$$
\begin{aligned}
d - 28 &= \frac{1}{6}(t - 0) \\
d &= \frac{1}{6}t + 28
\end{aligned}
$$

- The d-intercept is the initial diameter measurement – 28 mm
- The slope is the growth rate – $\frac{1}{6}$ mm/day

Sea Urchin Growth Model – Solution 4

Solution (cont): Model Predictions:

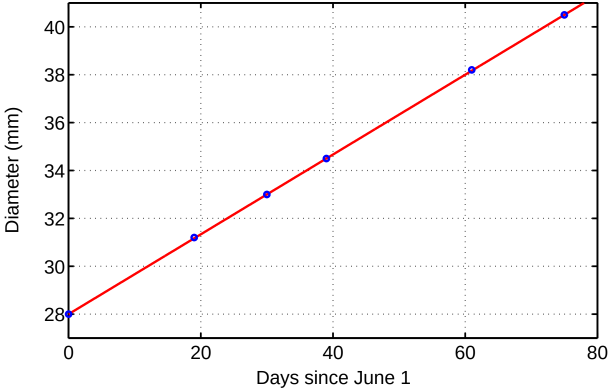
$$d = \frac{1}{6}t + 28$$

June 20 – 31.2 mm (19, 31.2)
July 10 – 34.5 mm (39, 34.5)
August 1 – 38.2 mm (61, 38.2)
August 15 – 40.5 mm (75, 40.5)

- The best estimate is for June 20 – it falls within data measurements
- The others are increasing more suspect
- Growth estimates are more accurate over shorter time intervals

Sea Urchin Growth Model – Graph

Below is a graph of the data and the **Linear Growth Model**



Scuba Diving Model 1

The pressure of air delivered by the regulator to a Scuba diver varies linearly with the depth of the water

The regulator delivers air to a Scuba diver as follows:

- Air Pressure at 29.4 psi when at 33 ft
- Air Pressure at 44.1 psi when at 66 ft

Find the pressure of air delivered at the surface (0 ft.), at 50 ft., and at 130 ft. (the maximum depth for recreational diving).

Scuba Diving Model – Solution 2

Solution: The linear model

$$p = md + p_0$$

where p is the pressure (psi) and d is the depth in feet

The data give two points

$$(d_0, p_0) = (33, 29.4) \quad \text{and} \quad (d_1, p_1) = (66, 44.1)$$

The slope is

$$m = \frac{44.1 - 29.4}{66 - 33} = \frac{14.7}{33} = 0.445 \text{ psi/ft}$$

Scuba Diving Model – Solution 3

Solution (cont): The linear model satisfies

$$\begin{aligned} p - 29.4 &= 0.445(d - 33) \\ p &= 0.445\,d + 14.7 \end{aligned}$$

- At the surface, $d = 0$ and the air pressure is 14.7 psi
- At a depth of $d = 50$ ft, the air pressure is 36.95 psi
- At a depth of $d = 130$ ft, the air pressure is 72.55 psi
- Note these assume we are at sea level and diving in sea water

Scuba Diving Model – Graph

Below is a graph of the data and the **Linear Pressure Model**