Calculus II

Vladimir V. Kisil

22nd May 2003

Chapter 1

General Information

This is an online manual is designed for students. The manual is available at the moment in HTML with frames (for easier navigation), HTML without frames and PDF formats. Each from these formats has its own advantages. Please select one better suit your needs.

There is on-line information on the following courses:

- Calculus I.
- Calculus II.
- Geometry.

1.1 Web page

There is a Web page which contains this course description as well as other information related to this course. Point your Web browser to

`http://maths.leeds.ac.uk/ kisilv/courses/math152.html`

1.2 Course description and Schedule

Dates	**Topics** **1 General Information** **1.1** *Web page* **1.2** *Course description and Schedule* **1.3** *Warnings and Disclaimers* **9 Infinite Series** **9.5** *A brief review of series* **9.6** *Power Series* **9.7** *Power Series Representations of Functions* **9.8** *Maclaurin and Taylor Series* **9.9** *Applications of Taylor Polynomials* **11 Vectors and Surfaces** **11.2** *Vectors in Three Dimensions* **11.3** *Dot Product* **11.4** *Vector Product* **11.5** *Lines and Planes* **11.6** *Surfaces* **12 Vector-Valued Functions** *Vector-Valued Functions* **12.1** *Limits, Derivatives and Integrals* **13 Partial Differentiation** **13.1** *Functions of Several Variables* **13.2** *Limits and Continuity* **13.3** *Partial Derivatives* **13.4** *Increments and Differentials* **13.5** *Chain Rules* **13.6** *Directional Derivatives* **13.7** *Tangent Planes and Normal Lines* **13.8** *Extrema of Functions of Several Variables* **13.9** *Lagrange Multipliers* **14 Multiply Integrals** **14.1** *Double Integrals* **14.2** *Area and Volume* **14.3** *Polar Coordinates* **14.4** *Surface Area* **14.5** *Triple Integrals* **14.7** *Cylindrical Coordinates* **14.8** *Spherical Coordinates* **15 Vector Calculus** **15.1** *Vector Fields* **15.2** *Line Integral* **15.3** *Independence of Path* **15.4** *Green's Theorem* **15.5** *Surface Integral* **15.6** *Divergence Theorem* **15.7** *Stoke's Theorem*

1.3 Warnings and Disclaimers

Before proceeding with this interactive manual we stress the following:

- These Web pages are designed in order to help students as a source of *additional information.* They are **NOT** an obligatory part of the course.
- The main material introduced during *lectures* and is contained in *Textbook*. This interactive manual is **NOT** a substitution for any part of those primary sources of information.
- It is **NOT** required to be familiar with these pages in order to pass the examination.

- The entire contents of these pages is continuously improved and updated. Even for material of lectures took place weeks or months ago changes are made.

Contents

Chapter 9

Infinite Series

9.5 A brief review of series

We refer to the chapter Infinite Series of the course **Calculus I** for the review of the following topics.

(i). Sequences of numbers

(ii). Convergent and Divergent Series

(iii). Positive Term Series

(iv). Ratio and Root Test

(v). Alternating Series and Absolute Convergence

9.6 Power Series

It is well known that polynomials are simplest functions, particularly it is easy to differentiate and integrate polynomials. It is desirable to use them for investigation of other functions. Infinite series reviewed in the previous sections are very important because they allow to represent functions by means of power series, which are similar to polynomials in many respects. An example of such representations is harmonic series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Definition 9.6.1 Let x be a variable. A *power series in* x is a series of the form

$$\sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots,$$

where each b_k is real number.

A power series turns to be infinite (constant term) series if we will substitute a constant c instead of the variable x. Such series could converge or diverge. All power series converge for $x = 0$. The convergence of power series described by the following theorem.

Theorem 9.6.2 *(i). If a power series $\sum b_n x^n$ converges for a nonzero number c, then it is absolutely convergent whenever $|x| < |c|$.*

(ii). If a power series $\sum b_n x^n$ diverges for a nonzero number d, then it diverges whenever $|x| > |d|$.

PROOF. The proof follows from the Basic Comparison Test of the power series for $|x|$ and convergent geometric series with $r = \left|\frac{x}{c}\right|$. $\square$

From this theorem we could conclude that

Theorem 9.6.3 *If $\sum b_n x^n$ is a power series, then exactly one of the following true:*

(i). The series converges only if $x = 0$.

(ii). The series is absolutely convergent for every x.

(iii). There is a number r such that the series is absolutely convergent if x is in open interval $(-r, r)$ and divergent if $x < -r$ or $x > r$.

The number r from the above theorem is called *radius of convergence*. The totality of numbers for which a power series converges is called its *interval of convergence*. The interval of convergence may be any of the following four types: $[-r, r]$, $[-r, r)$, $(-r, r]$, $(-r, r)$.

There is a more general type of power series

Definition 9.6.4 Let b be a real number and x is a variable. A *power series in* $x - d$ is a series of the form

$$\sum_{n=0}^{\infty} b_n (x-d)^n = b_0 + b_1(x-d) + b_2(x-d)^2 + \cdots + b_n(x-d)^n + \cdots,$$

where each b_n is a real number.

This power series is obtained from the series in Definition 9.6.1 by replacement of x by $x-d$. We could obtain a description of convergence of this series by replacement of x by $x-d$ in Theorem 9.6.3.

The following exercises should be solved in the following way:

(i). Determine the radius r of convergence, usually using Ratio test or Root Test.

(ii). If the radius r is finite and nonzero determine if the series is convergent at points $x=-r$, $x=r$. Note that the series could be alternating at one of them and apply Alternating Test.

Exercise 9.6.5 Find the interval of convergence of the power series:

$$\begin{aligned}\sum \frac{1}{n^2+4}x^n; &\quad \sum \frac{1}{\ln(n+1)}x^n;\\ \sum \frac{10^{n+1}}{3^{2n}}x^n; &\quad \sum \frac{(3n)!}{(2n)!}x^n;\\ \sum \frac{10^n}{n!}x^n; &\quad \sum \frac{1}{2n+1}(x+3)^n;\\ \sum \frac{n}{3^{2n-1}}(x-1)^{2n}; &\quad \sum \frac{1}{\sqrt{3n+4}}(3x+4)^n;\end{aligned}$$

9.7 Power Series Representations of Functions

As we have seen in the previous section a power series $\sum b_n x^n$ could define a convergent infinite series $\sum b_n c^n$ for all $c \in (-r, r)$ which has a sum $f(c)$. Thus the power series define a function $f(x) = \sum b_n x^n$ with domain $(-r, r)$. We call it the *power series representation of* $f(x)$. Power series are used in calculators and computers.

Example 9.7.1 Find function represented by $\sum (-1)^k x^k$.

The following theorem shows that integration and differentiations could be done with power series as easy as with polynomials:

Theorem 9.7.2 *Suppose that a power series $\sum b_n x^n$ has a radius of convergence $r > 0$, and let f be defined by*

$$f(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots$$

for every $x \in (-r, r)$. *Then for* $-r < x < r$

$$\begin{aligned} f'(x) &= b_1 + b_2 x + b_3 x^2 + \cdots + n b_n x^{n-1} + \cdots && (9.7.1) \\ &= \sum_{n=1}^{\infty} n b_n x^{n-1}; \\ \int_0^x f(t)\,dt &= b_0 x + b_1 \frac{x^2}{2} + b_2 \frac{x^3}{3} + \cdots + b_n \frac{x^{n+1}}{n+1} + \cdots && (9.7.2) \\ &= \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}. \end{aligned}$$

Example 9.7.3 Find power representation for

(i). $\frac{1}{(1+x)^2}$.

(ii). $\ln(1+x)$ and calculate $\ln(1.1)$ to five decimal places.

(iii). $\arctan x$.

Theorem 9.7.4 *If x is any real number,*

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

PROOF. The proof follows from observation that the power series $f(x) = \sum \frac{x^n}{n!}$ satisfies to the equation $f'(x) = f(x)$ and the only solution to this equation with initial condition $f(0) = 1$ is $f(x) = e^x$. $\square$

Corollary 9.7.5

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

Example 9.7.6 Find a power series representation for $\sinh x$, xe^{-2x}.

Exercise 9.7.7 Find a power series representation for $f(x)$, $f'(x)$, $\int_0^x f(t)\,dt$.

$$f(x) = \frac{1}{1+5x}; \qquad f(x) = \frac{1}{3-2x}.$$

Exercise 9.7.8 Find a power series representation and specify the radius of convergence for:

$$\frac{x}{1-x^4}; \qquad \frac{x^2-3}{x-2}.$$

Exercise 9.7.9 Find a power series representation for

$$f(x) = x^2 e^{(x^2)}; \qquad f(x) = x^4 \arctan(x^4).$$

9.8 Maclaurin and Taylor Series

We find several power series representation of functions in the previous section by a variety of different tools. *Could it be done in a regular fashion?* Two following theorem give the answer.

Theorem 9.8.1 *If a function f has a power series representation*

$$f(x) = \sum_{k=0}^{\infty} b_n x^n$$

with radius of convergence $r > 0$, then $f^{(k)}(0)$ exists for every positive integer k and

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Theorem 9.8.2 *If a function f has a power series representation*

$$f(x) = \sum_{k=0}^{\infty} b_n (x-d)^n$$

with radius of convergence $r > 0$, then $f^{(k)}(d)$ exists for every positive integer k and

$$f(x) = f(d) + \frac{f'(d)}{1!}(x-d) + \frac{f''(d)}{2!}(x-d)^2 + \cdots + \frac{f^{(n)}(d)}{n!}(x-d)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(d)}{n!}(x-d)^n$$

Exercise 9.8.3 Find Maclaurin series for:

$$f(x) = \sin 2x; \qquad f(x) = \frac{1}{1-2x}.$$

Remark 9.8.4 It is easy to see that linear approximation formula is just the Taylor polynomial $P_n(x)$ for $n = 1$.

The last formula could be split to two parts: the *nth-degree Taylor polynomial* $P_n(x)$ of f at d:

$$P_n(x) = f(d) + \frac{f'(d)}{1!}(x-d) + \frac{f''(d)}{2!}(x-d)^2 + \cdots + \frac{f^{(n)}(d)}{n!}(x-d)^n$$

and the *Taylor remainder*

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-d)^{n+1},$$

where $z \in (d, x)$. Then we could formulate a sufficient condition for the existence of power series representation of f.

Theorem 9.8.5 *Let f have derivatives of all orders throughout an interval containing d, and let $R_n(x)$ be the Taylor remainder of f at d. If*

$$\lim_{n\to\infty} R_n(x) = 0$$

for every x in the interval, then $f(x)$ is represented by the Taylor series for $f(x)$ at d.

Example 9.8.6 Let f be the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0, \end{cases}$$

then f cannot be represented by a Maclaurin series.

Exercise 9.8.7 Show that for function $f(x) = e^{-x}$

$$\lim_{n\to\infty} R_n(x) = 0$$

and find the Maclaurin series.

The important Maclaurin series are:

Function	Maclaurin series	Convergence
e^x	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	$(-\infty, \infty)$
$\ln(1+x)$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$	$(-1, 1]$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\sinh x$	$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$	$(-\infty, \infty)$
$\cosh x$	$\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$	$(-\infty, \infty)$
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$	$[-1, 1]$

Exercise 9.8.8 Find Maclaurin series for $\sin^2 x$.

Exercise 9.8.9 Find a series representation of $\ln x$ in powers of $x - 1$.

Exercise 9.8.10 Find first three terms of the Taylor series for f at d:

$$f(x) = \arctan x, \quad d = 1; \qquad f(x) = \csc x, \quad d = \pi/3.$$

9.9 Applications of Taylor Polynomials

We could use the Taylor polynomial $P_n(x)$ for an approximation of a function $f(x)$ in a neighborhood of point x_0. The important observation is: to keep amount of calculation on a low level we prefer to consider polynomials $P_n(x)$ with small n. But for such n the obtained accuracy is tolerable only for a small neighborhood of x_0. If x is remote from x_0 to obtain a reasonably good approximation with $P_n(0)$ for a small n we need to take the Taylor expansion in another point x_0' which is closer to x.

Exercise 9.9.1 Find the Maclaurin polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$ for $f(x)$, sketch their graphs. Approximate $f(a)$ to four decimal places by means of $P_3(x)$ and estimate $R_3(x)$ to estimate the error.

$$f(x) = \ln(x+1) \qquad a = 0.9.$$

Exercise 9.9.2 Find the Taylor formula with remainder for the given $f(x)$, d and n.

$$\begin{aligned} f(x) &= e^{-1}; \qquad d = 1, \quad n = 3. \\ f(x) &= \sqrt[3]{x}; \qquad d = -8, \quad n = 3. \end{aligned}$$

Chapter 11

Vectors and Surfaces

11.2 Vectors in Three Dimensions

Similarly to Cartesian coordinates on the Euclidean plane we could introduce *rectangular coordinate system* or *xyz-coordinate system* in three dimensions. The origin is usually denoted by O and three axises are OX, OY, OZ. The positive directions are selected in the way to form the *right-handed coordinate system*. In this system the *coordinate of a point* is an ordered triple of real numbers (a_1, a_2, a_3). Points with all three coordinates being positive form the *first octant*.

Similarly to two dimensional case we have the following formulas

Theorem 11.2.1 *(i). The* distance *between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is*

$$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

(ii). The midpoint of the line segment $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$

(iii). An equation of a sphere of radius r and center $P_0(x_0, y_0, z_0)$ is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

We define *vector* $\mathbf{a} = (a_1, a_2, a_3)$ in the three dimensional case as a transformation which maps point (x, y, z) to $(x + a_1, y + a_2, z + z_3)$. Vectors could be added and multiplied by a scalar according to the rules:

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (a_1 + b_1, a_2 + b_2, a_3 + b_3); \\ c\mathbf{a} &= (ca_1, ca_2, ca_3); \end{aligned}$$

There is a special *null vector* $\mathbf{0} = (0, 0, 0)$ and *inverse vector* $-\mathbf{a} = (-a_1, -a_2, -a_3)$ for any vector a.

We have the following properties:

(i). $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

(ii). $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

(iii). $\mathbf{a} + \mathbf{0} = \mathbf{a}$.

(iv). $\mathbf{a} + -\mathbf{a} = \mathbf{0}$.

(v). $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + a\mathbf{b}$.

(vi). $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$.

(vii). $(cd)\mathbf{a} = c(d\mathbf{a}) = d(c\mathbf{a})$.

(viii). $1\mathbf{a} = \mathbf{a}$.

(ix). $0\mathbf{a} = \mathbf{0} = c\mathbf{0}$.

We define *subtraction of vectors* (or *difference of vectors*) by the rule:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

Definition 11.2.2 Nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ have

(i). the *same direction* if $\mathbf{b} = c\mathbf{a}$ for some scalar $c > 0$.

(ii). the *opposite direction* if $\mathbf{b} = c\mathbf{a}$ for some scalar $c < 0$.

Definition 11.2.3 We define vectors:

$$\mathbf{i} = (1, 0, 0), \qquad \mathbf{j} = (0, 1, 0), \qquad \mathbf{k} = (0, 0, 1).$$

It is obvious that

$$\mathbf{a} = (a_1, a_2, a_3) = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

The *magnitude of vector* is defined to be

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

11.3 Dot Product

Besides addition of vectors and multiplication by the scalar there two different operation which allows to multiply vectors.

Definition 11.3.1 The *dot product* (or *scalar product*, or *inner product*) $\mathbf{a}\cdot\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Theorem 11.3.2 Properties of the dot product *are:*

(i). $\mathbf{a}\cdot\mathbf{a} = \|a\|^2$.

(ii). $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$.

(iii). $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$.

(iv). $(m\mathbf{a})\cdot\mathbf{b} = m\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot(m\mathbf{b})$.

(v). $\mathbf{0}\cdot\mathbf{a} = 0$.

Definition 11.3.3 Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors.

(i). If $\mathbf{b} \neq c\mathbf{a}$ then *angle θ between a and b* is the angle of triangle defined by them.

(ii). If $\mathbf{b} = c\mathbf{a}$ then $\theta = 0$ if $c > 0$ and $\theta = \pi$ if $c < 0$.

Vectors are *orthogonal* or *perpendicular* if $\theta = \pi/2$. By a convention $\mathbf{0}$ is orthogonal and parallel to any vector.

Theorem 11.3.4 *For nonzero* $\mathbf{a}$ *and* $\mathbf{b}$*:*

$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos\theta.$$

Corollary 11.3.5 *For nonzero* $\mathbf{a}$ *and* $\mathbf{b}$*:*

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\,\|\mathbf{b}\|}.$$

Corollary 11.3.6 *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal if and only if* $\mathbf{ab} = 0$.

Corollary 11.3.7 (Cauchy-Schwartz-Bunyakovskii Inequality)

$$|\mathbf{a}\cdot\mathbf{b}| \leq \|\mathbf{a}\|\,\|\mathbf{b}\|$$

Theorem 11.3.8 (Triangle Inequality)

$$\|\mathbf{a}+\mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\| \,.$$

We define *component of* $\mathbf{a}$ *along* $\mathbf{b}$

$$\mathbf{comp_b a} = \mathbf{a}\cdot\frac{1}{\|\mathbf{b}\|}\mathbf{b}$$

Definition 11.3.9 The *work done by a constant force* $\mathbf{a}$ as its point of application moves along the vector $\mathbf{b}$ is $\mathbf{a}\cdot\mathbf{b}$.

11.4 Vector Product

Definition 11.4.1 A *determinant of order 2* is defined by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

A *determinant of order 3* is defined by

$$\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} c_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} c_1.$$

Definition 11.4.2 The *vector product* (or *cross product*) $\mathbf{a}\times\mathbf{b}$ is

$$\begin{aligned} \mathbf{a}\times\mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}. \end{aligned}$$

Theorem 11.4.3 *The vector* $\mathbf{a}\times\mathbf{b}$ *is orthogonal to both* $\mathbf{a}$ *and* $\mathbf{b}$.

Theorem 11.4.4 *If* θ *is the angle between nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$*, then*

$$\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin\theta.$$

Corollary 11.4.5 *Two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a}\times\mathbf{b} = \vec{0}$.

Exercise 11.4.6 Compile the multiplication table for vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

Be careful, because:

$$\begin{aligned}\mathbf{i}\times\mathbf{j} &\neq \mathbf{j}\times\mathbf{i}\\ (\mathbf{i}\times\mathbf{j})\times\mathbf{j} &\neq \mathbf{i}\times(\mathbf{j}\times\mathbf{j}).\end{aligned}$$

Theorem 11.4.7 Properties of the vector product *are*

(i). $\mathbf{a}\times\mathbf{b} = -(\mathbf{b}\times\mathbf{a})$.

(ii). $(m\mathbf{a})\times\mathbf{b} = m(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(m\mathbf{b})$.

(iii). $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.

(iv). $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$.

(v). $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$.

(vi). $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Dot and vector products related to geometric properties.

Exercise 11.4.8 Prove that the distance from a point R to a line l is given by

$$d = \frac{\left\|\vec{PQ}\times\vec{PR}\right\|}{\left\|\vec{PQ}\right\|}.$$

Exercise 11.4.9 Prove that the volume of the oblique box spanned by three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ is $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$.

11.5 Lines and Planes

Theorem 11.5.1 *Parametric equation for the line through $P_1(x_1, y_1, z_1)$ parallel to $\mathbf{a} = (a_1, a_2, a_3)$ are*

$$x = x_1 + a_1 t, \quad y = y_1 + a_2 t, \quad z = z_1 + a_3 t; \qquad t \in \mathbb{R}^{\cdot}$$

Note that we obtain the same line if we use any vector $\mathbf{b} = c\mathbf{a}$, $c \neq 0$.

Corollary 11.5.2 *Parametric equation for the line through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are*

$$x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t, \quad z = z_1 + (z_2 - z_1)t; \qquad t \in \mathbb{R}^{\cdot}$$

Exercise 11.5.3 Find equations of the lines:

(i). $P(1,2,3)$; $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

(ii). $P_1(2,-2,4)$, $P_2(2,-2,-3)$.

Exercise 11.5.4 Determine whether the lines intersect: $x = 2 - 5t$, $y = 6 + 2t$, $z = -3 - 2t$; $x = 4 - 3v$, $y = 7 + 5v$, $z = 1 + 4v$.

Definition 11.5.5 Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ and let l_1 and l_2 be lines that are parallel to the position vectors of $\mathbf{a}$ and $\mathbf{b}$.

(i). The *angles between lines* l_1 and l_2 are θ and $\pi - theta$.

(ii). The lines l_1 and l_2 are *parallel* iff $\mathbf{b} = c\mathbf{a}$ for $c \in \mathbb{R}$.

(iii). The lines l_1 and l_2 are *orthogonal* iff $\mathbf{a} \cdot \mathbf{b} = 0$ for $c \in \mathbb{R}$.

The *plane* through P_1 with normal vector $\vec{P_1P_2}$ is the set of all points P such that $\vec{P_1P}$ is orthogonal to $\vec{P_1P_2}$.

Theorem 11.5.6 *An equation of the plane through $P_1(x_1, y_1, z_1)$ with normal vector $\mathbf{a} = (a_1, a_2, a_3)$ is*

$$a_1(x - x_1) + a_2(y - y_1) + a_3(z - z_1) = 0.$$

Theorem 11.5.7 *The graph of every linear equation $ax + by + cz + d = 0$ is a plane with normal vector (a, b, c).*

Exercise 11.5.8 Find an equation of the plane through $P(4, 2, -6)$ and normal vector $\vec{OP}$.

Exercise 11.5.9 Sketch the graph of the equation

(i). $y = -2$;

(ii). $3x - 2z - 24 = 0$;

Definition 11.5.10 Two planes with normal vectors $\mathbf{a}$ and $\mathbf{b}$ are

(i). *parallel* if $\mathbf{a}$ and $\mathbf{b}$ are parallel;

(ii). *orthogonal* if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal;

Exercise 11.5.11 Find an equation of the plane through $P(3, -2, 4)$ parallel to $-2x + 3y - z + 5 = 0$.

Theorem 11.5.12 (Symmetric Form for a Line)

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{a_2} = \frac{z - z_1}{a_3}.$$

Exercise 11.5.13 Show that *distance* from a point $P_0(x_0, y_0, z_0)$ to the plane $ax + by + cz + d = 0$ is

$$h = \left|\mathbf{comp_n}\overrightarrow{P_0P_1}\right|,$$

where $\mathbf{n} = (a, b, c)$ and P_1—any point on the plane.

Exercise 11.5.14 Show that planes $3x+12y-6z = -2$ and $5x+20y-10z = 7$ are parallel and find distance between them.

Exercise 11.5.15 Find an equation of the plane that contains the point $P(4, -3, 0)$ and line $x = t + 5$, $y = 2t - 1$, $z = -t + 7$.

Exercise 11.5.16 Show that *distance* between two lines defined by points P_1, Q_1 and P_2, Q_2 is given by the formula

$$d = \left|\mathbf{comp_n}\overrightarrow{P_1P_2}\right|, \qquad \mathbf{n} = \frac{\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}}{\left\|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}\right\|}.$$

Exercise 11.5.17 Find the distance between point $P(3, 1, -1)$ and line $x = 1 + 4t$, $y = 3 - t$, $z = 3t$.

11.6 Surfaces

It is important to represent different surfaces (not only planes) from 3d space into our two dimensional drawing. Some useful technique is given by *trace on a surface* S in a plane, namely by intersection of S an the plane.

There are several classic important types of surfaces. To follows given examples you need to remember equations of conics in Cartesian coordinates.

Example 11.6.1 $z = x^2 + y^2$ define *circular paraboloid* or *paraboloid of revolution.*

Definition 11.6.2 Let C be a curve in a plane, and let l be a line that is not in a parallel plane. The set of points on all lines that are parallel to l and intersect C is a *cylinder*. The curve C called is called *directrix of the cylinder*.

Example 11.6.3 The *right circular cylinder* is given by the equation $x^2 + y^2 = r^2$.

Similarly to quadratic equations equations defining conics the equation

$$Ax^2 + By^2 + cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

defines *quadric surface*. We consider simplest cases with $D = E = F = G = H = I = 0$.

Definition 11.6.4 *Ellipsoid*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Definition 11.6.5 The *hyperboloid of one sheet*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Definition 11.6.6 The *hyperboloid of two sheets*:

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Definition 11.6.7 The *cone*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$$

Definition 11.6.8 The *paraboloid*:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = cz.$$

Definition 11.6.9 The *hyperbolic paraboloid*:

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = cz.$$

Chapter 12

Vector-Valued Functions

Definition 12.0.10 Let D be a set of real numbers. A *vector-valued function* $\mathbf{r}$ with domain D is a correspondence that assigns to each number t in D exactly one vector $\mathbf{r}(t)$ in $\mathbb{R}^3$.

Theorem 12.0.11 *If D is a set of real numbers, then $\mathbf{r}$ is a vector-valued function with domain D if and only if there are scalar function f, g, and h such that*

$$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}.$$

Exercise 12.0.12 Sketch the two vectors

$$\mathbf{r}(t) = t\,\mathbf{i} + 3\sin t\mathbf{j} + 3\cos t\,\mathbf{k}, \quad \mathbf{r}(0),\ \mathbf{r}(\pi/2).$$

Set of *endpoints* of all vectors $\vec{OP} = \mathbf{r}(t)$ define a *space curve* C. A *parameter equation* of the curve C is

$$x = f(t), \quad y = g(t), \quad z = z(t).$$

The *orientation* of C is the direction determined by increasing values of t.

Exercise 12.0.13 Sketch the curve and indicate orientation:

$$\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + 3\,\mathbf{k}; \quad 0 \le t \le 4.$$

The following theorem is completely analogous to arc length of a plane curve:

Theorem 12.0.14 *If a curve C has a smooth parameterization*

$$x = f(t), \quad y = g(t), \quad z = z(t), \quad a \le t \le b$$

and if C does not intersect itself, except possibly for $t = a$ and $t = b$, then the length L of C is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt.$$

Exercise 12.0.15 Find the arc length:

$$x = e^t \cos t, \quad y = e^t, \quad z = e^t \sin t; \quad 0 \le t \le 2\pi;$$
$$x = 2t, \quad y = 4\sin 3t, \quad z = 4\cos 3t; \quad 0 \le t \le 2\pi;$$

12.1 Limits, Derivatives and Integrals of Vector-valued Functions

All definitions and results in this section are in close relation with the theory of scalar-valued function **Calculus I**. We advise to refresh Chapters on Limits and Derivative from **Calculus I** course.

Definition 12.1.1 Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. The *limit* $\mathbf{r}(t)$ as t approaches a is

$$\lim_{t\to a} \mathbf{r}(t) = \left[\lim_{t\to a} f(t)\right]\mathbf{i} + \left[\lim_{t\to a} g(t)\right]\mathbf{j} + \left[\lim_{t\to a} h(t)\right]\mathbf{k}.$$

provides f, g, and h have limits as t approaches a.

The next definition coincides with definition of continuity for scalar-valued function:

Definition 12.1.2 A vector valued function $\mathbf{r}$ is *continuous* at a if

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a).$$

Particularly $\mathbf{r}(t)$ is continuous iff $f(t)$, $g(t)$, and $h(t)$ are continuous. Similarly we define derivative

Definition 12.1.3 Let $\mathbf{r}$ be a vector-valued function. The *derivative* is the vector-valued function $\mathbf{r}'$ defined by

$$\mathbf{r}'(t) = \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)]$$

for every t such that the limit exists.

Exercise 12.1.4 Find the domain, first and second derivatives of the functions:

$$\begin{aligned}\mathbf{r}(t) &= \sqrt[3]{t}\,\mathbf{i} + \frac{1}{t}\,\mathbf{j} + e^{-t}\,\mathbf{k};\\ \mathbf{r}(t) &= \ln(1-t)\,\mathbf{i} + \sin t\,\mathbf{j} + t^2\,\mathbf{k}.\end{aligned}$$

Theorem 12.1.5 *Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ and f, g, and h are differentiable, then*

$$\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}.$$

The *geometric meaning* is as expected—this is tangent vector to the curve defined by $\mathbf{r}$.

Exercise 12.1.6 Find parameter equation for the tangent line to C at P:

$$x = e^t, \quad y = te^t, \quad z = t^2 + 4; \qquad P(1, 0, 4).$$

The properties of the derivative are as follows:

Theorem 12.1.7 *If $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector-valued functions and c is a scalar, then*

(i). $[\mathbf{u}(t) + \mathbf{v}(t)]' = \mathbf{u}'(t) + \mathbf{v}'(t)$;

(ii). $[c\mathbf{u}(t)]' = c\mathbf{u}'(t)$;

(iii). $[\mathbf{u}(t) \cdot \mathbf{v}(t)]' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$;

(iv). $[\mathbf{u}(t) \times \mathbf{v}(t)]' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$;

As a consequence of these properties we could easily prove the following

Theorem 12.1.8 *If $\mathbf{r}$ is differentiable and $\|\mathbf{r}\|$ is constant, then $\mathbf{r}'$ is orthogonal to $\mathbf{r}'(t)$ for every t in the domain of $\mathbf{r}'$.*

Finally we define integrals of vector-valued functions using integrals of scalar-valued functions:

Definition 12.1.9 Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ and f, g, and h are integrable, then

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

If $\mathbf{R}'(t) = \mathbf{r}(t)$, then $\mathbf{R}(t)$ is an *antiderivative* of $\mathbf{r}(t)$.

Theorem 12.1.10 *If* $\mathbf{R}(t)$ *is an antiderivative of* $\mathbf{r}(t)$ *on* $[a, b]$, *then*

$$\int_a^b \mathbf{r}(t)\, dt = \mathbf{R}(t)]_a^b = \mathbf{R}(b) - \mathbf{R}(a).$$

Exercise 12.1.11 Find $\mathbf{r}(t)$ subject to the given conditions:

$$\mathbf{r}'(t) = 2\mathbf{i} - 4t^3\,\mathbf{j} + 6\sqrt{t}\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

Chapter 13

Partial Differentiation

13.1 Functions of Several Variables

It is common that real-world quantities depend from many different parameters. Mathematically we describe them as functions of several variables. We start from definition of functions of two variables.

Definition 13.1.1 Let D be a set of ordered pairs of real numbers. A *function of two variables* f is a correspondence that assigns to each pair (x, y) in D exactly one real number, denoted by $f(x, y)$. The set D is the *domain* of f. The *range* of f consists of all real numbers $f(x, y)$, where $(x, y) \in D$.

Exercise 13.1.2 Describe domain of f and find its values:

$$\begin{aligned} f(r, s) &= \sqrt{1 - r} - e^{r/s}; \quad f(1, 1), f(0, 4), f(-3, 3) \\ f(x, y, z) &= 2 + \tan x + y \tan z; \quad f(\pi/4, 4, \pi/6), f(0, 0, 0). \end{aligned}$$

Exercise 13.1.3 Sketch graph of f:

$$f(x, y) = \sqrt{2 - 2x - x^2 - y^2}, \qquad f(x, y) = 3 - x - 3y.$$

Exercise 13.1.4 Sketch the *level curves* for f:

$$f(x, y) = xy, \qquad k = -4, 1, 4.$$

Exercise 13.1.5 (i). Find the equation of *level surface* of f that contains the point P.

$$f(x, y, z) = z^2 y + x; \quad P(1, 4, -2).$$

(ii). Describe the level surface of f for given k:

$$f(x, y, z) = z + x^2 + 4y^2, \qquad k = -6, 6, 12.$$

13.2 Limits and Continuity

The fundamental notion of limit could be introduced for a function of two variables as follows

Definition 13.2.1 Let a function f of two variables be defined throughout the interior of a circle with center (a, b), except possibly at (a, b) itself. The statement

$$\lim_{(x,y)\to(a,b)} f(x, y) = L \quad \text{or} \quad f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta, \text{ then } |f(x, y) - L| < \epsilon.$$

Exercise 13.2.2 Find limits

$$\lim_{(x,y)\to(2,1)} \frac{4+x}{2-y}, \qquad \lim_{(x,y)\to(-1,3)} \frac{y^2+x}{(x-1)(y+2)}.$$

Theorem 13.2.3 (Two-Path Rule) *If two different paths to a point $P(a, b)$ produce two different limiting values for f, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.*

Exercise 13.2.4 Show that the limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 2xy + 5y^2}{3x^2 + 4y^2}, \qquad \lim_{(x,y)\to(0,0)} \frac{3xy}{5x^4 + 2y^4}.$$

Definition 13.2.5 A function f of two variables is *continuous* at an interior point (a, b) of its domain if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

Exercise 13.2.6 Describe the set of all points at which f is continuous

$$f(x, y) = \frac{xy}{x^2 - y^2}, \qquad f(x, y) = \sqrt{xy} \tan z.$$

Definition 13.2.7 Let a function f of two variables be defined throughout the interior of a circle with center (a, b, c), except possibly at (a, b, c) itself. The statement

$$\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = L \quad \text{or} \quad f(x, y, z) \to L \text{ as } (x, y, z) \to (a, b, c)$$

means that for every $\epsilon > 0$ there is a $\delta > 0$ such that if

$$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta, \text{ then } |f(x, y, z) - L| < \epsilon.$$

Theorem 13.2.8 (Composition of Continuous Functions) *If a function f of two variables is continuous at (a, b) and a function g of one variables is continuous at $f(a, b)$, then the function $h(x, y) = g(f(x, y))$ is continuous at (a, b).*

Exercise 13.2.9 Use Theorem on Composition of Continuous Functions to determine where h is continuous.

$$f(x, y) = 3x + 2y - 4, \qquad g(t) = \ln(t + 5).$$

13.3 Partial Derivatives

For functions of several variables the concept of derivative could modified as follows:

Definition 13.3.1 Let f be a function of two variables. The *first partial derivatives* of f with respect to x and y are functions f'_x and f'_y such that

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &= f'_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}, \\ \frac{\partial}{\partial x} f(x, y) &= f'_x(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}. \end{aligned}$$

Exercise 13.3.2 Find first partial derivatives of f

$$\begin{aligned} f(x, y) = (x^3 - y^2)^5; &\qquad f(x, y) = e^x \ln xy; \\ f(r, s, v, p) = r^3 \tan s + \sqrt{s} e^{(v^2)} - v \cos 2p; &\qquad f(x, y, z) = xyz\, e^{xyz}. \end{aligned}$$

This notion has a geometrical meaning which is very close to geometrical meaning of usual derivative derivative.

Theorem 13.3.3 *Let S be the graph of $z = f(x, y)$, and let $P(a, b, f(a, b))$ be a point on S at which f'_x and f'_y exists. Let C_1 and C_2 be the traces of S on the planes $x = a$ and $y = b$, respectively, and let l_1 and l_2 be the tangent lines to C_1 and C_2 at P.*

(i). The slope of l_1 in the plane $x = a$ is $f'_y(a, b)$.

(ii). The slope of l_1 in the plane $y = b$ is $f'_x(a, b)$.

We could define *second partial derivatives* by repetition. There are four of them:

$$
\begin{aligned}
f''_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right); \\
f''_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right); \\
f''_{xy} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right); \\
f''_{yx} &= \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).
\end{aligned}
$$

Exercise 13.3.4 If $v = y\ln(x^2+z^2)$, find v'''_{zzy}.

Theorem 13.3.5 *Let f be a function of two variables x and y. If f, f'_x, f'_y, f''_{xy}, and f''_{yx} are continuous on an open region R, then $f''_{xy} = f''_{yx}$ through R.*

Exercise 13.3.6 Verify that $f''_{xy} = f''_{yx}$.

$$f(x,y) = \frac{x^2}{x+y}; \qquad f(x,y) = \sqrt{x^2+y^2+z^2}.$$

Review

Exercise 13.3.7 Find the interval of convergence of the power series:

$$\sum (-1)^n \frac{3^n}{n!}(x-4)^n; \qquad \sum (-1)^n \frac{e^{n+1}}{n^n}(x-1)^n.$$

Exercise 13.3.8 Obtain a power series representation for the function

$$f(x) = x^2\ln(1+x^2); \qquad f(x) = \arctan\sqrt{x}.$$

Exercise 13.3.9 Find all values of c such that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal $\mathbf{a} = 4\mathbf{i} + 2\mathbf{j} + c\mathbf{k}$, and $\mathbf{b} = \mathbf{i} + 22\mathbf{j} - 3c\mathbf{k}$.

Exercise 13.3.10 Find the volume of the box having adjacent sides AB, AC, AD: $A(2,1,-1)$, $B(3,0,2)$, $C(4,-2,1)$, $D(5,-3,0)$.

Exercise 13.3.11 Find an equation of the plane through $P(-4,1,6)$ and having the same trace in xz-plane as the plane $x+4y-5z=8$.

Exercise 13.3.12 Find arc length of the curve: $x = 2t$, $y = 4\sin 3t$, $z = 4\cos 3t$; $0 \leq t \leq 2\pi$.

Exercise 13.3.13 Find a parametric Al equation of the tangent line to curve $x = t\sin t$, $y = t\cos t$, $z = t$; at $P(\pi/2, 0, \pi/2)$.

Exercise 13.3.14 Show that limit does not exist.

$$\lim_{(x,y,z)\to(2,0,0)} \frac{(x-2)yz^2}{(x-2)^4 + y^4}.$$

13.4 Increments and Differentials

Definition 13.4.1 Let $w = f(x,y)$, and let Δx and Δ be increments of x and y, respectively. The *increment of function* w is

$$\Delta w = f(x + \Delta x, y + \Delta y) - f(x,y).$$

Theorem 13.4.2 *Let $w = f(x,y)$, where the function f is defined on a rectangular region $R = \{(x,y) :\ a < x < b,\ c < y < d\}$. Suppose f'_x and f'_y exist throughout R and are continuous at (x_0, y_0). Then*

$$\Delta w = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y.$$

A function w is *differentiable* if its increment could be represented as above.

Definition 13.4.3 The *differential of function* w is

$$dw = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta x.$$

13.5 Chain Rules

Among different rules of derivation most powerful is the

Theorem 13.5.1 (Chain rules) *If $w = f(u,v)$, with $u = g(x,y)$, $v = h(x,y)$, and if f, g, and h are differentiable, then*

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}; \\ \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y}. \end{aligned}$$

PROOF. It follows from the Theorem on Increment. □

This formulas could be better understood and remembered if we will draw a tree representing dependence of variables.

Exercise 13.5.2 Find $\partial w/\partial x$, $\partial wpartialy$ if $w = uv + v^2$, $u = x \sin y$, $v = y \sin x$.

Similar formulas are true for different number of variables

Exercise 13.5.3 Find $\partial z/\partial x$, $\partial z/\partial y$ if $z = pq + qw$, $p = 2x - y$, $q = x - 2y$, $w = -2x + 2y$.

Chain rules could be used to derive already known formulas in a new way.

Exercise 13.5.4 Derive formula $(uv)' = u'v + uv'$ using chain rules.

Exercise 13.5.5 Derive from chain rules the following formula for implicit derivatives of y defined by $F(x, y) = 0$:

$$y' = -\frac{F'_x(x, y)}{F'_y(x, y)}.$$

13.6 Directional Derivatives

We could give a definition generalizing partial derivatives.

Definition 13.6.1 Let $w = f(x, y)$ and $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ be a unit vector. The *directional derivative* of f at $P(x, y)$ in the direction $\mathbf{u}$, denoted $D_{\mathbf{u}}f(x, y)$, is

$$D_{\mathbf{u}} = \lim_{s \to 0} \frac{f(x + su_1, y + su_2) - f(x, y)}{s}.$$

Partial derivatives are particular cases of directional derivatives: $\partial/\partial x = D_{\mathbf{i}}$ and $\partial/\partial y = D_{\mathbf{j}}$. It is interesting that we could calculate any directional derivative if we know only partial ones.

Theorem 13.6.2 *If f is a differentiable function of two variables, then*

$$D_{\mathbf{u}}f(x, y) = f'_x(x, y)u_1 + f'_y(x, y)u_2.$$

PROOF. It is follows from the Chain Rules. □

Exercise 13.6.3 Find directional derivative

$$f(x, y) = x^3 - 3x^2y - y^3, \quad P(1, -2), \quad \mathbf{u} = \frac{1}{2}(-\mathbf{i} + \sqrt{3}\mathbf{j}).$$

Definition 13.6.4 Let f be a function of two variables. The *gradient* of f is the vector valued function

$$\nabla f(x,y) = f'_x(x,y)\mathbf{i} + f'_y(x,y)\mathbf{j}.$$

Directional derivative in *gradient form* is

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u}.$$

Exercise 13.6.5 Find gradient

$$f(x,y) = e^{3x} \tan y, \quad P(0, \pi/4).$$

From gradient form of directional derivative easily follows the following theorem:

Theorem 13.6.6 *Let f be a function of two variables that is differentiable at the point $P(x,y)$.*

(i). The maximum value of $D_{\mathbf{u}}$ is $\|\nabla f(x,y)\|$.

(ii). The maximum rate of increase of $f(x,y)$ occurs in direction of $\nabla f(x,y)$.

(iii). The minimum value of $D_{\mathbf{u}}$ is $-\|\nabla f(x,y)\|$.

(iv). The minimum rate of increase of $f(x,y)$ occurs in direction of $-\nabla f(x,y)$.

Similarly directional derivatives and gradients could be defined for functions of three variables.

Exercise 13.6.7 Find directional derivative at P in the direction to Q. Find directions of maximal and minimal increase of f.

$$f(x,y,z) = \frac{x}{y} - \frac{y}{z}, \quad P(0,-1,2), \quad Q(3,1,-4).$$

13.7 Tangent Planes and Normal Lines

Theorem 13.7.1 *Suppose that $F(x,y,z)$ has continuous first partial derivatives and that S is the graph of $F(x,y,z) = 0$. If P_0 is a point on S and if F'_x, F'_y, F'_z are not all 0 at P_0, then the vector $\nabla F]_{P_0}$ is normal to the tangent plane to S at P_0. And equation of the tangent plane is*

$$F'_x(x_0,y_0,z_0)(x-x_0) + F'_y(x_0,y_0,z_0)(y-y_0) + F'_z(x_0,y_0,z_0)(z-z_0) = 0.$$

Theorem 13.7.2 *An equation for the tangent plane to the graph of* $z = f(x, y)$ *at the point* (x_0, y_0, z_0) *is*

$$z - z_0 = f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0)$$

Exercise 13.7.3 Find equation for the tangent plane and normal line to the graph.

$$9x^2 - 4y^2 - 25z^2 = 40; \quad P(4, 1, -2).$$

13.8 Extrema of Functions of Several Variables

The definition of *local maximum*, *local minimum*, which are *local extrema*, are the same as for function of one variable.

Definition 13.8.1 Let f be a function of two variables. A pair (a, b) is a *critical point* of f if either

(i). $f'_x(a, b) = 0$ and $f'_y(a, b) = 0$, or

(ii). $f'_x(a, b)$ or $f'_y(a, b)$ does not exist.

Definition 13.8.2 Let f be a function of two variables that has continuous second partial derivatives. The *discriminant* D of f is given by

$$D(x, y) = f''_{xx}f''_{yy} - [f''_{xy}]^2 = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{vmatrix}.$$

The following result is similar to Second Derivative Test.

Test 13.8.3 (Test for Local Extrema) Let f be a function of two variables that has continuous second partial derivatives throughout an open disk R containing a critical point (a, b). If $D(a, b) > 0$, then $f(a, b)$ is

(i). a local maximum of f if $f''_{xx}(a, b) < 0$.

(ii). a local minimum of f if $f''_{xx}(a, b) > 0$.

If a critical point with existent partial derivatives is not a local extrema then it is called *saddle point*. We could determine them by determinant:

Theorem 13.8.4 *Let f have continuous second partial derivatives throughout an open disk R containing an critical point (a, b) with existent derivatives. If $D(a, b)$ is negative, then (a, b) is a saddle point.*

Exercise 13.8.5 Find extrema and saddle points.

$$\begin{aligned} f(x,y) &= x^2 - 2x + y^2 - 6y + 12 \\ f(x,y) &= -2x^2 - 2xy - \frac{3}{2}y^2 - 14x - 5y \\ f(x,y) &= -\frac{1}{3}x^3 + xy + \frac{1}{2}y^2 - 12y. \end{aligned}$$

Exercise 13.8.6 Find the max and min of f in R.

$$f(x,y) = x^2 - 3xy - y^2 + 2y - 6x; \quad R = \{(x,y) |\, |x| \leq 3, |y| \leq 2\}.$$

Exercise 13.8.7 Find three positive real numbers whose sum is 1000 and whose product is a maximum.

13.9 Lagrange Multipliers

Theorem 13.9.1 *Suppose that f and g are functions of two variables having continuous first partial derivatives and that $\nabla g \neq \mathbf{0}$ throughout a region. If f has an extremum $f(x_0, y_0)$ subject to the constraint $g(x,y) = 0$, then there is a real number λ such that*

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

By other words they are among solution of the system

$$\begin{cases} f'_x(x,y) &= \lambda g'_x(x,y) \\ f'_y(x,y) &= \lambda g'_y(x,y) \\ g(x,y) &= 0 \end{cases}.$$

Exercise 13.9.2 Find the extrema of f subject to the stated constrains

$$f(x,y) = 2x^2 + xy - y^2 + y; \quad 2x + 3y = 1.$$

Chapter 14

Multiply Integrals

We consider the next fundamental operation of calculus for functions of several variables.

14.1 Double Integrals

The definite integral of a function of one variable was defined using using *Riemann sum.* We could apply the same idea for definition of definite integral for a function of several variables.

Definition 14.1.1 Let f be a function of two variables that is defined on a region R. The *double integral* of f over R, is

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} \sum_k f(x_k,y_k)\Delta A,$$

provided the limit exists for the *norm of the partition* tensing to 0.

The following is similar to geometrical meaning of definite integral

Definition 14.1.2 (Geometrical Meaning of Double Integral) Let f be a continuous function of two variables such that $f(x,y)$ is nonnegative for every (x,y) in a region R. The *volume* V of the solid that lies under the graph of $z = f(x,y)$ and over R is

$$V = \iint_R f(x,y)\,dA.$$

Double integral has the following properties (see one variable case).

Theorem 14.1.3 *(i).*

$$\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$$

(ii).

$$\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$$

(iii). If $R = R_1 \cup R_2$ and $R_1 \cap R_2 = \emptyset$

$$\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

(iv). If $(x,y) \geq 0$ throughout R, then $\iint_R f(x,y)\,dA \geq 0$.

Practically double integrals evaluated by means of *iterated integrals* as follows:

Theorem 14.1.4 *Let R be a region of R_x type. If f is continuous on R, then*

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Exercise 14.1.5 Evaluate

$$\int_0^3 \int_{-2}^{-1} (4xy^3 + y)\,dx\,dy \qquad \int_{-1}^1 \int_{x^3}^{x+1} (3x+2y)\,dy\,dx.$$

Exercise 14.1.6 Evaluate $\iint_R e^{x/y}\,dA$ if R bounded by $y = 2x$, $y = -x$, $y = 4$.

Exercise 14.1.7 Sketch the region $x = 2\sqrt{y}$, $\sqrt{3}x = \sqrt{y}$, $y = 2x + 5$ and express the double integral as iterated one.

Exercise 14.1.8 Sketch the region of integration for the iterated integral

$$\int_{-1}^2 \int_{x^2-4}^{x-2} f(x,y)\,dy\,dx.$$

Exercise 14.1.9 Reverse the order of integration and evaluate

$$\int_1^e \int_0^{\ln x} y\,dy\,dx.$$

14.2 Area and Volume

From geometric meaning of double integrals we see that they are usable for finding volumes (and areas).

Exercise 14.2.1 Describe surface and region related to

$$\int_0^1 \int_{3-x}^{1-x^2} (x^2+y^2)\, dy\, dx.$$

Exercise 14.2.2 Find volume under the graph $z = x^2 + 4y^2$ over triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

Exercise 14.2.3 Sketch the solid in the first octant and find its volume $z = y^3$, $y = x^3$, $x = 0$, $z = 0$, $y = 1$.

14.3 Polar Coordinates, Double Integrals in Polar Coordinates

Besides the Cartesian coordinates we could describe a point of the plain by the distance to the preselected point O (*origin* or *pole*) and angle to the ray at origin (*polar axis*). This description is called *polar coordinates*. Here are some interesting curves and their equation in polar coordinates.

(i). circle (O,R): $r = R$.

(ii). circle (a,a): $r = 2a\sin\theta$.

(iii). *cardioid*: $r = a(1+\cos\theta)$.

(iv). *limaçons*: $r = a + b\cos\theta$.

(v). *n-leafed rose*: $r = a\sin n\theta$.

(vi). *spiral of Archimedes*: $r = a\theta$.

Exercise* **14.3.1** Find equation of a straight line in polar coordinates.

Connection between the Cartesian coordinates and polar coordinates is as follows:

Theorem 14.3.2 *The rectangular coordinates (x,y) and polar coordinates (r,θ) of a point P are related as follows:*

(i). $x = r\cos\theta$, $y = r\sin\theta$;

(ii). $r^2 = x^2 + y^2$, $\tan\theta = y/x$ *if* $x \neq 0$.

Theorem 14.3.3 (Test for Symmetry) *(i). The graph of $r = f(\theta)$ is symmetric with respect to the polar axis if $f(-\theta) = f(\theta)$.*

(ii). The graph of $r = f(\theta)$ is symmetric with respect to the vertical line if $f(\pi - \theta) = f(\theta)$ or $f(-\theta) = -f(\theta)$.

(iii). The graph of $r = f(\theta)$ is symmetric with respect to the pole if $f(\pi+\theta) = f(\theta)$.

Theorem 14.3.4 *The slope m of the tangent line to the graph of $r = f(\theta)$ at the point $P(r, \theta)$ is*

$$m = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

The element of area in polar coordinates equal to $\Delta A = \frac{1}{2}(r_2^2 - r_1^2)\Delta\theta = \bar{r}\Delta r\Delta\theta$, where $\bar{r} = \frac{1}{2}(r_2 - r_1)$. Thus *double integral in polar coordinates* could be presented by iterated integral as follows:

$$\begin{aligned}\iint_R f(r,\theta)\,dA &= \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)r\,dr\,d\theta. \\ &= \int_\alpha^\beta \int_{h_1(r)}^{h_2(r)} f(r,\theta)r\,d\theta\,dr.\end{aligned}$$

Exercise 14.3.5 Use double integral to find the area inside $r = 2 - 2\cos\theta$ and outside $r = 3$.

Exercise 14.3.6 Use polar coordinates to evaluate the integral

$$\iint_R x^2(x^2 + y^2)^3\,dA$$

R is bounded by semicircle $y = \sqrt{1 - x^2}$ and the x-axis.

Exercise 14.3.7 Evaluate

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} (x^2 + y^2)^{3/2}\,dy\,dx.$$

Exercise 14.3.8 Find volume bounded by paraboloid $z = 4x^2 + 4y^2$, the cylinder $x^2 + y^2 = 3y$, and plane $z = 0$.

14.4 Surface Area

Theorem 14.4.1 *The surface area of the graph $z = f(x, y)$ over the region R is given by*

$$A = \iint_R \sqrt{[f'_x(x,y)]^2 + [f'_y(x,y)]^2 + 1}\, dA.$$

Exercise 14.4.2 Setup a double integral for the surface area of the graph $x^2 - y^2 + z^2 = 1$ over the square with vertices $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(0, -1)$.

Exercise 14.4.3 Find the area of the surface $z = y^2$ over the triangle with vertices $(0, 0)$, $(0, 2)$, $(2, 2)$.

Exercise 14.4.4 Find the area of the first-octant part of hyperbolic paraboloid $z = x^2 - y^2$ that is inside the cylinder $x^2 + y^2 = 1$.

14.5 Triple Integrals

There is no any principal differences to introduce *triple integral*, it could be done using ideas on definite integrals and double integrals.

Definition 14.5.1 Triple integral of f over $3d$-region Q is defined by Riemann sums:

$$\iiint_Q f(x, y, z)\, dV = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k, z_k) \Delta V_k.$$

To evaluate triple integrals we reduce them by iteration to double integrals:

Theorem 14.5.2

$$\begin{aligned} \iiint_Q f(x, y, z)\, dV &= \iint_R \left[\int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\, dz \right] dA \\ &= \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)\, dz\, dy\, dz. \end{aligned}$$

Exercise 14.5.3 Evaluate the iterated integral

$$\int_0^1 \int_{-1}^2 \int_1^3 (6x^2 z + 5xy^2)\, dz\, dx\, dy; \qquad \int_{-1}^2 \int_1^{z^2} \int_{x+z}^{x-z} z\, dy\, dx\, dz.$$

Exercise 14.5.4 Describe region represented by integrals

$$\int_0^1 \int_{z^3}^{\sqrt{z}} \int_0^{4-x} dy\, dx\, dz, \qquad \int_0^1 \int_x^{3x} \int_0^{xy} dz\, dy\, dx.$$

Physical meaning of triple integrals is given by

Theorem 14.5.5 *Mass of a solid with a mass density $\delta(x, y, z)$ is given by*

$$m = \iiint_Q \delta(x, y, z)\, dV$$

Theorem 14.5.6 *Mass of a lamina with an area mass density $\delta(x, y)$ is given by*

$$m = \iiint_R \delta(x, y)\, dA$$

Exercise 14.5.7 Using triple integrals find volume bounded by

(i). $x^2 + z^2 = 4$, $y^2 + z^2 = 4$.

(ii). $z = x^2 + y^2$, $y + z = 2$.

14.7 Cylindrical Coordinates

The *cylindrical coordinates* of a point P is the triple of numbers (r, θ, z), where (r, θ) are the polar coordinates of the projection of P on xy-plane and z is defined as in rectangular coordinates.

Theorem 14.7.1 *The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point are related as follows:*

$$\begin{aligned} x &= r\cos\theta, \quad y = r\sin\theta, \quad z = z, \\ r^2 &= x^2 + y^2, \quad \tan\theta = \frac{x}{y}. \end{aligned}$$

Exercise 14.7.2 Describe the graph in cylindrical ccordinates:

(i). $r = -3\sec\theta$.

(ii). $z = 2r$.

Exercise 14.7.3 Change the equation to cylindrical coordinates:

(i). $x^2 + y^2 = 4z$.

(ii). $x^2 + z^2 = 9$.

Theorem 14.7.4 *Evaluation of triple integral in cylindrical coordinates:*

$$\iiint_Q f(r,\theta,z)\,dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r,\theta,z)\,dz\,dr\,d\theta.$$

Exercise 14.7.5 A solid is bounded by the cone $z = \sqrt{x^2+y^2}$, the cylinder $x^2+y^2=4$, and the xy-plane. Find its volume.

14.8 Spherical Coordinates

The *spherical coordinates* of a point is the triple (ρ, ϕ, θ).

Theorem 14.8.1 *The rectangular coordinates (x, y, z) and the spherical coordinates (ρ, ϕ, θ) of a point related as follows:*

$$\begin{aligned} x &= \rho\sin\phi\cos\theta, x = \rho\sin\phi\sin\theta, \quad z = \rho\cos\theta \\ \rho^2 &= x^2+y^2+z^2. \end{aligned}$$

Exercise 14.8.2 Change coordinates

(i). spherical $(1, 3\pi/4, 2\pi/3)$ to rectangular and cylindrical.

(ii). rectangular $(1, \sqrt{3}, 0)$ to spherical and cylindrical.

Exercise 14.8.3 Describe graphs

(i). $\rho = 5$.

(ii). $\phi = 2\pi/3$.

(iii). $\theta = \pi/4$.

Exercise 14.8.4 Change the equation to spherical coordinates.

$$x^2+y^2=4z; \qquad x^2+(y-2)^2=4; \qquad x^2+z^2=9.$$

Theorem 14.8.5 (Evaluation theorem)

$$\iiint_Q f(\rho,\phi,\theta)\,dV = \int_m^n \int_c^d \int_a^b f(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Exercise 14.8.6 Find volume of the solid that lies outside the cone $z^2 = x^2+y^2$ and inside the sphere $x^2+y^2+z^2=1$.

Exercise 14.8.7 Evaluate integral in spherical coordinates:

$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy.$$

Chapter 15

Vector Calculus

15.1 Vector Fields

We could make one more step after vector valued functions and function of several variables.

Definition 15.1.1 A *vector field in three dimensions* is a function $\mathbf{F}$ whose domain D is a subset of $\mathbb{R}^3$ and whose range is is a subset of $\mathbb{V}^3$. If (x, y, z) is in D, then

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

where M, N, and P are scalar functions.

Exercise 15.1.2 Plot the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.

Example of vector field is as follows:

Definition 15.1.3 Let $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. A vector field $\mathbf{F}$ is an *inverse square field* if

$$\mathbf{F}(x, y, z) = \frac{c}{\|\mathbf{r}\|^3}\mathbf{r}.$$

Examples of inverse square field are given by *Newton's law of gravitation* and *Coulom's law of charge interaction.*

Definition 15.1.4 A vector filed $\mathbf{F}$ is *conservative* if

$$\mathbf{F}(x, y, z) = \nabla f(x, y, z)$$

for some scalar function f. Then f is *potential function* and its value $f(x, y, z)$ is *potential* in (x, y, z).

Exercise 15.1.5 Find a vector field with potential $f(x, y, z) = \sin(x^2 + y^2 + z^2)$.

Theorem 15.1.6 *Every inverse square vector filed is conservative.*

PROOF. The potential is given by $f(r) = \frac{c}{r}$. □

Definition 15.1.7 Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The *curl* of $\mathbf{F}$ is given by

$$\begin{aligned} \text{curl}\mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} \\ &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \end{aligned}$$

Definition 15.1.8 Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$. The *divergence* of $\mathbf{F}$ is given by

$$\text{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

Exercise 15.1.9 Find curl $\mathbf{F}$ and div $\mathbf{F}$ for

$$\mathbf{F}(x, y, z) = (3x + y)\mathbf{i} + xy^2z\mathbf{j} + xz^2\mathbf{k}.$$

Exercise 15.1.10 Prove that for a constant vector $\mathbf{a}$

(i). $\text{curl}\,(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$;

(ii). $\div(\mathbf{a} \times \mathbf{r}) = 0$.

Exercise 15.1.11 Verify the identities:

$$\begin{aligned} \text{curl}\,(\mathbf{F} + \mathbf{G}) &= \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}; \\ \text{div}\,(\mathbf{F} + \mathbf{G}) &= \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}; \\ \text{curl}\,(f\mathbf{F}) &= f(\text{curl}\,\mathbf{F}) + (\nabla f) \times \mathbf{F}; \end{aligned}$$

15.2 Line Integral

We could introduce a new type of integrals for functions of several variables.

Definition 15.2.1 The *line integrals along a curve* C with respect to s, x, y, respectively are

$$\begin{aligned}
\int_C f(x,y)\,ds &= \lim_{\|P\|\to 0}\sum_k f(u_k,u_k)\Delta s_k \\
\int_C f(x,y)\,dx &= \lim_{\|P\|\to 0}\sum_k f(u_k,u_k)\Delta x_k \\
\int_C f(x,y)\,dy &= \lim_{\|P\|\to 0}\sum_k f(u_k,u_k)\Delta y_k
\end{aligned}$$

Let a curve C be given parametrically by $x = g(t)$ and $y = h(t)$. Because

$$dx = g'(t)\,dt, \qquad dy = h'(t)\,dt,$$
$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(g'(t))^2 + (h'(t))^2}\,dt.$$

we obtain

Theorem 15.2.2 (Evaluation formula for line integrals) *If a smooth curve* C *is given by*$x = g(t)$ *and* $y = h(t)$*;* $a \le t \le b$ *and* $f(x,y)$ *is continuous in a region containing* C*, then*

$$\begin{aligned}
\int_C f(x,y)\,ds &= \int_C f(g(t),h(t))\sqrt{(g'(t))^2 + (h'(t))^2}\,dt \\
\int_C f(x,y)\,dx &= \int_C f(g(t),h(t))(g'(t)\,dt \\
\int_C f(x,y)\,dy &= \int_C f(g(t),h(t))h'(t))\,dt
\end{aligned}$$

Exercise 15.2.3 Evaluate $\int_C xy^2\,ds$ if C is given by $x = \cos t$, $y = \sin t$; $0 \le t \le \pi/2$.

Exercise 15.2.4 Evaluate $\int_C y\,dy + z\,dy + x\,dz$ if C is the graph of $x = \sin t$, $y = 2\sin t$, $z = sin^2 t$; $0 \le t \le \pi/2$.

Exercise 15.2.5 Evaluate $\int_C xy\,dx + x^2y^3\,dy$ if C is the graph of $x = y^3$ from $(0,0)$ to $(1,1)$.

Exercise 15.2.6 Evaluate $\int_C (x^2+y^2)\,dx+2x\,dy$ along three different paths from $(1,2)$ to $(-2,8)$.

Exercise 15.2.7 Evaluate $\int_C (xy+z)\,ds$ if C is the lime segment from $(0,0,0)$ to $(1,2,3)$.

Theorem 15.2.8 *The* mass of a wire *is given by*

$$m=\int_C \delta(x,y)\,ds,$$

where $\delta(x,y)$ is the linear mass density.

Theorem 15.2.9 *The* work W done by a force F long a path C *is defined as follows*

$$W=\int_C M(x,y,z)\,dx+N(x,y,z)\,dy+P(x,y,z)\,dz.$$

If $\mathbf{T}$ is a unit tangent vector to C at (x,y,z) and $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then

$$W=\int_C \mathbf{F}\cdot\mathbf{T}\,ds=\int_C \mathbf{F}\cdot d\mathbf{r}.$$

15.3 Independence of Path

There is a condition for an integral be independent from the path.

Theorem 15.3.1 *If $\mathbf{F}(x,y)=M(x,y)\mathbf{i}+N(x,y)\mathbf{i}$ is continuous on an open connected region D, then the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is conservative—that is, $\mathbf{F}(x,y)=\nabla f(x,y)$ for some scalar function f.*

Exercise 15.3.2 Show that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path by finding a potential function f for F:

$\mathbf{F}(x,y)=(6xy^2+3y)\mathbf{i}+(6x^2y+2x)\mathbf{j}$; $\quad\mathbf{F}(x,y)=(2xe^{2y}+4y^3)\mathbf{i}+(2x^2e^{2y}+12xy^2)\mathbf{j}$.

In fact we are even able to give a formula for the evaluation:

Theorem 15.3.3 *Let $\mathbf{F}(x,y)=M(x,y)\mathbf{i}+N(x,y)\mathbf{i}$ be continuous on an open connected region D, and C be a piecewise-smooth curve in D with endpoints $A(x_1,y_1)$ and $B(x_2,y_2)$. If $\mathbf{F}(x,y)=\nabla f(x,y)$ for some scalar function f, then*

$$\int_C M(x,y)\,dx+N(x,y)\,dy=\int_{(x_1,y_1)}^{(x_2,y_2)} \mathbf{F}\cdot d\mathbf{r}=[f(x,y)]_{(x_1,y_1)}^{(x_2,y_2)}.$$

Particularly $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every simple closed curve* C.

Exercise 15.3.4 Show that integral is independent of path, and find its value

$$\int_{(0,0)}^{(1,\pi/2)} e^x \sin y\, dx + e^x \cos y\, dy.$$

Theorem 15.3.5 *If F is a conservative force field in two dimensions, then the work done by F along any path C from $A(x_1, y_1)$ to $B(x_2, y_2)$ is equal to the difference in potentials between A and B.*

Theorem 15.3.6 *If $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on a simply connected region D, then the line integral*

$$\int_C M(x, y)\, dx + N(x, y)\, dy$$

is independent of path in D if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Exercise 15.3.7 Use above theorem to show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is not independent of path:

(i). $\mathbf{F}(x, y) = y^3 \cos x\mathbf{i} - 3y^2 \sin x\mathbf{j}$.

(ii). $\int_C e^y \cos x\, dx + xe^y \cos z\, dy + xe^y \sin z\, dz$.

15.4 Green's Theorem

Theorem 15.4.1 (Green's Theorem) *Let G be a piecewise-smooth simple closed curve, and let R be the region consisting of G and its interior. If M and N are continuous functions that have continuous first partial derivatives throughout an open region D containing R, then*

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) da$$

Exercise 15.4.2 Use Green's theorem to evaluate the line integrals

(i). $\oint \sqrt{y}\, dx + \sqrt{x}\, dy$ if C is the tringle with vertices $(1, 1)$, $(3, 1)$, $(2, 2)$.

(ii). $\oint_C y^2\,dx + x^2\,dy$ if C is the boundary of the region bounded by the semicircle $y = \sqrt{4-x^2}$ and x-axis.

As an application we could derive a formula as follows:

Theorem 15.4.3 *If a region R in the xy-plane is bounded by a piece-wise-smooth simple closed curve C, then the area A of R is*

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

The region R could contains holes, provided we integrate over the *entire* boundary and always keep the region R to the left of C.

Exercise 15.4.4 Use the above theorem to find to fine the area bounded by the graphs $y = x^3$, $y^2 = x$.

Theorem 15.4.5 (Vector Form of Green's Theorem)

$$\oint_C \mathbf{F}\cdot\mathbf{T}\,ds = \iint_R (\nabla\times\mathbf{F})\cdot\mathbf{k}\,dA.$$

15.5 Surface Integral

We could define surface integrals in a way similar to definite integral, double, triple, lines integrals by means of Riemann sums:

$$\iint_S g(x,y,z)\,dS = \lim_{\|P\|\to 0}\sum_k g(x_k,y_k,z_k)\Delta T_k.$$

To calculate surface integrals we use

Theorem 15.5.1 *Evaluation formulas for surface integrals are:*

$$\begin{aligned}
\iint_S g(x,y,z)\,dS &= \iint_{R_{xy}} g(x,y,f(x,y))\sqrt{[f'_x(x,y)]^2+[f'_y(x,y)]^2+1}\,dA\\
\iint_S g(x,y,z)\,dS &= \iint_{R_{xz}} g(x,h(x,z)z)\sqrt{[h'_x(x,z)]^2+[h'_z(x,z)]^2+1}\,dA\\
\iint_S g(x,y,z)\,dS &= \iint_{R_{xy}} g(k(y,z),y,z)\sqrt{[k'_y(y,z)]^2+[k'_z(y,z)]^2+1}\,dA
\end{aligned}$$

Exercise 15.5.2 Evaluate surface integral of $g(x,y,z) = x^2+y^2+z^2$ over the part of plane $z = y+4$ that is inside the cylinder $x^2+y^2 = 4$.

Exercise 15.5.3 Express the surface integral $\iint_S (xz + 2y)\, dS$ over the portion of the graph of $y = x^3$ between the plane $y = 0$, $y = 8$, $z = 2$, and $z = 0$ as a double integral over a region in yz-plane.

Definition 15.5.4 The *flux* of vector field F through (or over) a surface S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dx.$$

Exercise 15.5.5 Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dx$ for $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ and S the first octant portion of the sphere $X^2 + y^2 + z^2 = a^2$.

Exercise 15.5.6 Find the flux of $\mathbf{F}(x, y, z) = (x^2+z)\mathbf{i}+y^2z\mathbf{j}+(x^2+y^2+z)\mathbf{k}$ over S is the first-octant portion of paraboloid $z = x^2 + y^2$ that is cut off by the plane $z = 4$.

15.6 Divergence Theorem

15.7 Stoke's Theorem

Bibliography

[1] Earl Swokowski, Michael Olinick, and Dennis Pence. *Calculus.* PWS Publishing, Boston, 6-th edition, 1994.

Index