CALCULUS
IN
BUSINESS AND ECONOMICS

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Mathematics 112

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Contents

Preface

Like Math 111, Math 112 does not follow the example/problem/example/problem model where you simply work problems exactly like given examples, only with different numbers. There are very few drill problems. You will often be asked questions that need to be answered with words instead of numbers or formulas. You will be challenged to think about ideas rather than plugging numbers into formulas. This makes some people uncomfortable at first, but most people find that they get used to it with practice.

How to Use This Text

The heart of this text consists of twenty-four worksheets. Most worksheets begin with one or more **Key Questions**. The Key Questions are followed by a list of exercises.

You should begin each worksheet by reading and thinking about the Key Questions. You will not always be able to answer the Key Questions right away; that is the purpose of the exercises. They will lead you to an understanding of the issues involved and, we hope, to answers to the Key Questions.

Each exercise is numbered, like $\boxed{8}$ You should work *all* of the exercises carefully. Sometimes, you might not see right away what it is that they are getting at, but go ahead and play along anyway. **Write up your solutions to ALL of the exercises**. Do not just jot down computations. If your answers involve drawing on given graphs, there is an appendix of Graphs and Tables at the back of the book that you may draw on and rip out to turn in with your homework.

Interspersed with the exercises, you will find bits of exposition. They will be formatted like this paragraph. They do not require answers, but you will certainly want to read them.

At the end of most worksheets, you'll find one or more exercises marked with an arrow. These are problems which (a) provide some review of the material covered in that portion of the course, (b) add a little bit of *new material*, and (c) try to tie things together.

Math Study Center (MSC)

Because of the challenging nature of this course, the Mathematics Department offers a Study Center for the students in Math 111 and 112. The Math Study Center provides a supposrtive atmosphere for you to work on your math either individually or in groups. Details regarding the time and location of the Math Study Center will be announced in class and on your instructor's website.

Comments, Suggestions, Typos, . . .

This text is a work in progress. It probably contains mistakes. We will keep an updated list of corrections posted on the web at

`http://www.math.washington.edu/~m112/m112typos.html`

If you find any mistakes that are not already mentioned there, please report them via email to

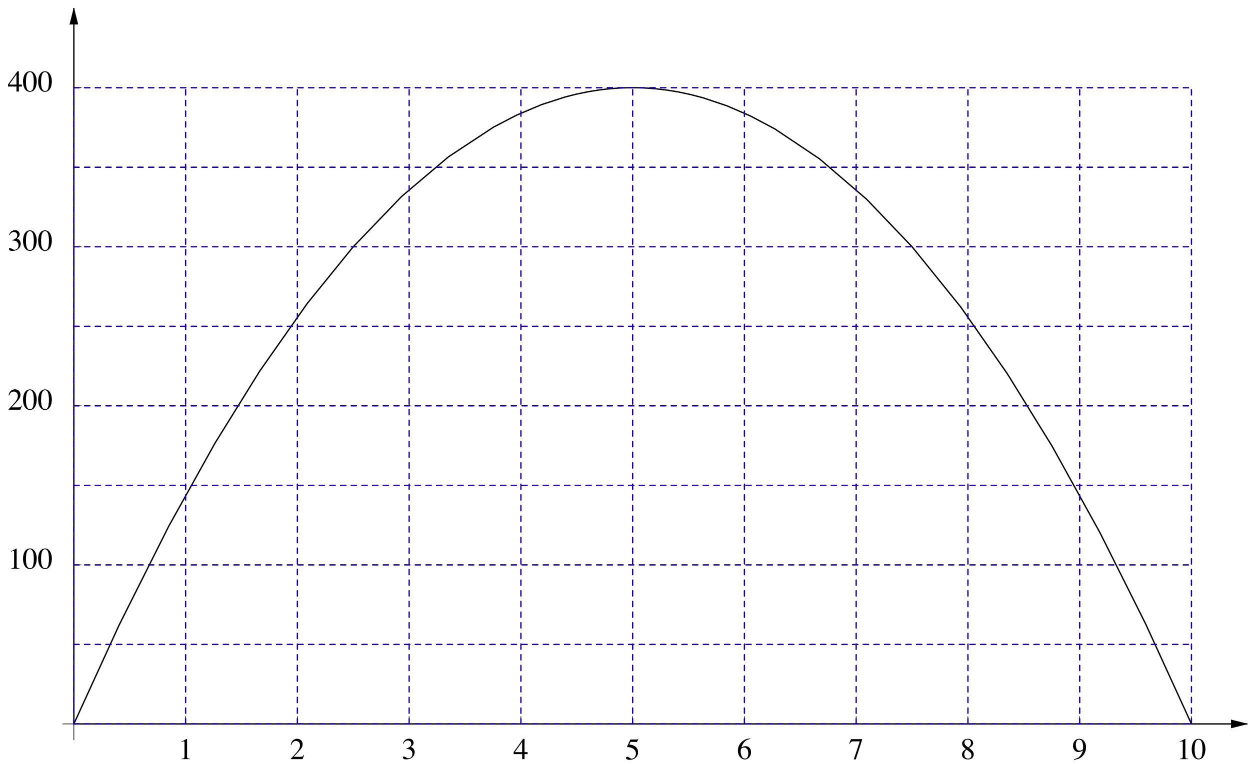
`taggart@math.washington.edu`

Chapter 1

Tangents and Derived Graphs

Worksheet #1 *The Vertical Speed of a Shell*

A shell is fired straight up by a mortar. The graph below shows its altitude as a function of time.



Imagine that there is a tiny speedometer in the shell that records the shell's velocity just as the speedometer in your car records the car's velocity. The function describing the shell's altitude (A) *vs*. time will be denoted by $A = f(t)$, where A is in feet and t is in seconds.

Key Questions

I. What is the reading on the shell's speedometer at time $t = 2$?

II. What is the reading on the shell's speedometer when it reaches the highest point of its path?

III. Does the shell stop at the top?

IV. What is the reading on the shell's speedometer at time $t = 0$, just as it is being fired?

$\boxed{1}$ Use the altitude graph to find the following quantities:

a) The distance covered from time $t = 2$ to $t = 4$.

b) $f(6.5) - f(4.5)$

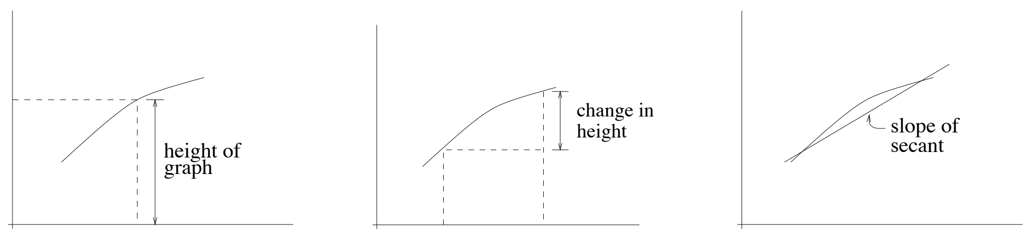
c) The average speed of the shell from time $t = 0$ to time $t = 3$.

d) The average speed of the shell from time $t = 1$ to time $t = 4$.

e) $\dfrac{f(5) - f(2)}{3}$

f) The time required for the shell to reach the altitude 300 ft.

$\boxed{2}$ Recall that the following quantities can be read from the altitude graph.



First read the following quantities from the graph and then tell what this quantity is in terms of a shell being fired.

a) The height of the graph at $t = 3$ seconds; $t = 7$ seconds.

b) The change in the height of the graph from $t = 2$ to $t = 4.5$ seconds.

c) The slope of the secant to the graph from $t = 3$ to $t = 3.5$ seconds.

The formula that corresponds to the altitude graph is

$$A = f(t) = 160t - 16t^2.$$

To use the formula to get the information you have been getting from the graph, you have to recall

The Substitution Rule. To find $f(4)$ from the formula you substitute 4 in for t everywhere in the formula: $f(4) = 160(4) - 16(4)^2 = 384$ feet. To find $f(5) - f(4)$, you substitute and follow the recipe:

$$f(5) - f(4) = [160(5) - 16(5)^2] - [160(4) - 16(4)^2] = 400 - 384 = 16 \textit{ ft}.$$

To find $\dfrac{f(5) - f(3)}{2}$, you substitute and again follow the recipe:

$$\frac{f(5) - f(3)}{2} = \frac{[160(5) - 16(5)^2] - [160(3) - 16(3)^2]}{2} = 32 \textit{ ft/sec}.$$

$\boxed{3}$ Use the formula for $A = f(t)$ to determine the quantities in questions 1(a)–(f) and 2. Check your answers against the answers you got from the graph.

Thus far the questions have required that you get information about average speeds and distances covered from the graph or formula. There is an accuracy problem with the graph, but there should not be any great confusion about how these quantities are to be found. However, we have not yet dealt with the issue of the readings on the imaginary speedometer in the shell. There can be great confusion about this issue—in fact, at the beginning, it is a matter of personal opinion and preference.

One approach to the question of what is a speedometer reading is the one we used in Math 111:

> *Math 111 version. The speedometer reading is the average speed over a very tiny time interval—for instance, one that is 0.1 or 0.01 seconds long.*

$\boxed{4}$ The average speed is measured on the graph by the slope of the secant over the time interval. Draw your best approximation to the secant line to the altitude graph from $t = 2$ to $t = 2.01$. Draw a nice long line with a plastic ruler and compute its slope. Would you say that this number is a good approximation of the speedometer reading at $t = 2$, and therefore an answer to Key Question I?

$\boxed{5}$ By measuring the slope of the secant to the graph over a very tiny time interval, make your best estimate of the speed of the shell when it reaches the peak of its path, at time $t = 5$. Thus you have a graphical answer to Key Question II.

$\boxed{6}$ When I answer Question 5 with my clear plastic ruler I draw a line that is absolutely horizontal. Its slope is 0, so I conclude that the speed of the shell at $t = 5$ seconds is 0. Do you agree? Does this mean that the shell stops at time $t = 5$? As you answer this question, think carefully about the way the word "stops" is used in everyday English.

$\boxed{7}$ Many people would say that at time $t = 0$, just as the shell is fired, it has no speed, or that its speedometer is not reading, or that its speedometer reading is 0. However, if you adopt the Math 111 version of speedometer readings—that they are approximated by secants over tiny time intervals—what would you say the speedometer reading of the shell is at $t = 0$? This is your first answer to Key Question IV.

The graph gives relatively quick answers to the speedometer questions. But there is always the issue of accuracy with graphs. So let's use the formula to get the same answers.

$\boxed{8}$ Again take the Math 111 version of the approximate speedometer reading at $t = 2$ to be the average speed from from $t = 2$ to $t = 2.01$. That is $\dfrac{f(2.01) - f(2.00)}{0.01}$. Use the Substitution Rule in the formula for $f(t)$ to determine this speed. Compare the answer you get with the answer you got from the graph in Question 4.

$\boxed{9}$ Again, take the approximate speedometer reading to be the average speed over a tiny time interval and use the formula to approximate the reading of the speedometer at $t = 5$ seconds by computing the average speed from $t = 5$ to $t = 5.01$ seconds.

$\boxed{10}$ The speedometer reading at $t = 5$ seconds that I got from the graph was 0 ft/sec. Presumably, your answer to Question 9 is different. How do you explain the difference? Which answer would you say is "more correct?" Do you believe that the speedometer would actually read 0 at $t = 5$ seconds?

$\boxed{11}$ Use the technique applied in questions 8 and 9 to make an estimate of the speedometer reading at time $t = 0$ by computing average speed from $t = 0$ to $t = 0.01$ seconds.

$\boxed{12}$ You could make even better approximations of the speedometer reading by computing average speed over an even shorter interval.

(a) Find a better approximation of the speedometer reading at $t = 2$ by computing average speed from $t = 2$ to $t = 2.001$. Compare this to your answer to question 8.

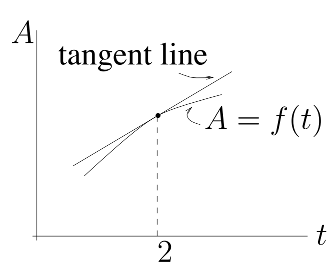
(b) Find a better approximation of the speedometer reading at $t = 5$ by computing average speed from $t = 5$ to $t = 5.001$. Compare this to your answer to question 9.

(c) Find a better approximation of the speedometer reading at $t = 0$ by computing average speed from $t = 0$ to $t = 0.001$. Compare this to your answer to question 11.

Thus far we have used the Math 111 approach that says that the approximate speedometer reading is the average speed over a tiny time interval. This is all right on the graph, because graphs are somewhat inaccurate anyway. But when we use the formula it leads to the confusion that we get different answers—depending on how short a time interval we use. The answers the formula gives us are all pretty close, so we could just round them off and say "that's the answer." But rounding off throws away the accuracy that comes from the formula.

Here is a different approach that many people find appealing (and others do not). Whatever your feelings toward this approach, follow it and see where it leads. The graphical version of this approach is as follows:

<u>The Tangent Approach</u>. The slope of the secant to the curve from $t = 2$ to $t = 2.01$ is a good approximation to the speedometer reading, but we will do better if we take a smaller time interval. In fact, there is one line that just touches the graph at $t = 2$ (and nowhere else) that best represents the speed. This line is called the tangent line.



$\boxed{13}$ Draw your best attempt at a tangent line to the graph of altitude *vs*. time at $t = 2$. Use a plastic ruler and draw a long line. (For many of you the line may not be any different from the secant line you drew earlier.) Measure the slope of the tangent line you drew. Would you say this is a better approximation to the speedometer reading at $t = 2$ than the approximation you found in question 4?

$\boxed{14}$ Draw the tangent to the graph at $t = 5$ and use it to approximate the speedometer reading at that time. Draw a tangent line to the graph at $t = 0$ and use it to make your best estimate of the speedometer reading at time $t = 0$.

The tangent line approach to the speedometer problem is based on the belief that at any one point on the graph, there is a unique line that captures the steepness of the graph at the point in question. The slope of this line is then taken to be the official speedometer reading of the shell at the given time. People have three distinct reactions to this approach:

a) *There is no such line.*
b) *Of course, there is such a line, and anybody can see it.*
c) *There sort of is such a line, but it really is just a secant line for two very, very close points.*

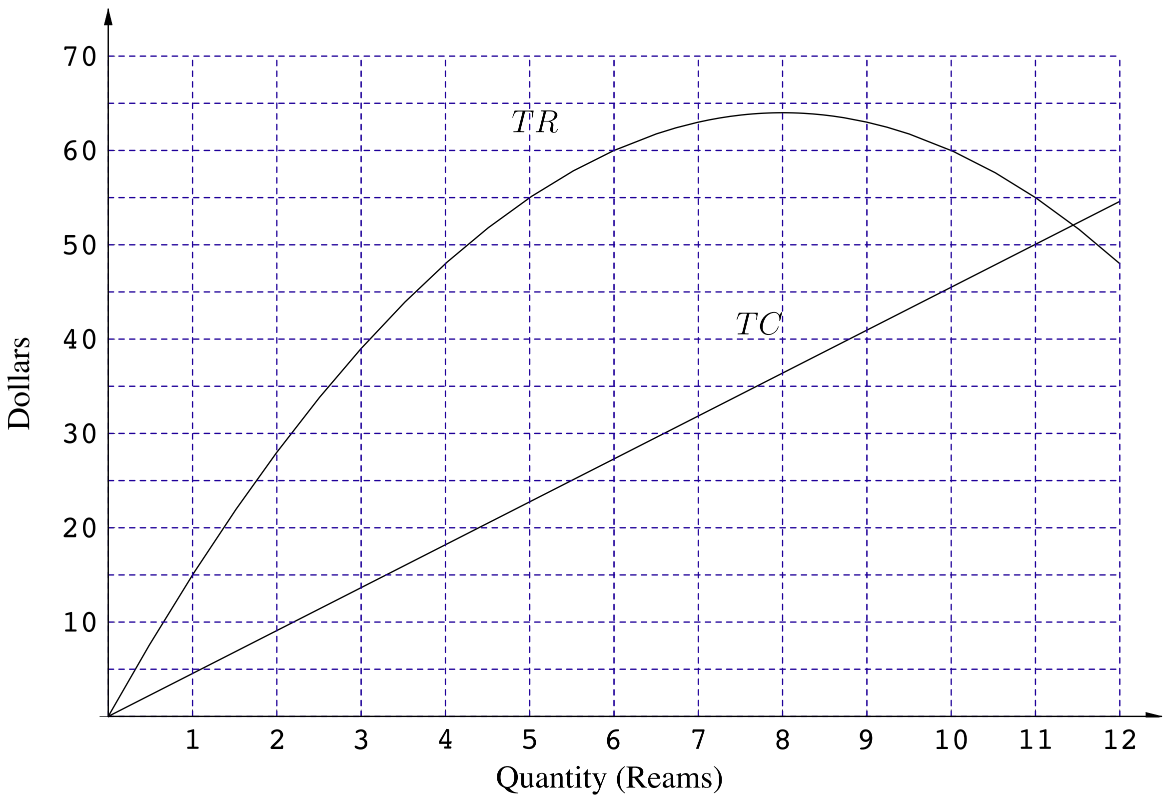
If you do not believe in such a line, then I would think you would have a tough time believing in speedometer readings. If you do believe there is such a line, but it is just another secant line, then you are saying your car's speedometer measures a type of average speed—which I do not believe. I am going to try to persuade you that, with some mind-stretching, you can imagine that there is exactly one tangent line to the curve at each point, but it is not *a secant line. It really does only touch at one point.*

The interesting question is: How can we use the accuracy of the formula to precisely describe such a line and get a really accurate reading of its slope?

Worksheet #2

Two Versions of Marginal Revenue and Cost

Below are the Total Revenue and Total Cost graphs for small orders in your print shop.



The official economics text definition of Marginal Revenue is:

> **Increment.**The Marginal Revenue at q items is the increment in Revenue that comes when we increase q by 1 item.

In Math 111 we found it easier to read Marginal Revenue from a graph as follows:

> **Slope.** The Marginal Revenue at q items is the slope of the secant between the points on the TR graph at q items and one more item.

Key Question

Now suppose that things have gotten more complicated at the print shop, and you take orders by the Hundreds of Pages (1 Hundred Pages = 0.2 Reams). Thus “one more item” is 0.2 Reams. Then what is the Marginal Revenue at $q = 3$ Reams by the increment definition and by the slope definition? What would happen if you took orders by the page, so that “one more item” meant one more page (OR 0.002 Reams)?

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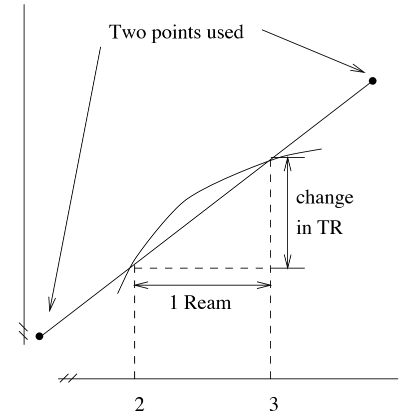
1 Suppose first that you fill orders by the Ream. Approximate the change in the height of the TR graph from $q = 2$ to $q = 3$ Reams (i.e., estimate the Marginal Revenue at $q = 2$ Reams).

2 Draw a nice long secant line connecting the points on the TR graph at $q = 2$ and $q = 3$. Use two points on this line that are quite far apart to measure its slope. Your numerical answer here should be close to your numerical answer to Question 1, except that, technically speaking, the units in the first question are dollars, whereas the units here are dollars/Ream. Explain why they give the same numerical answer.



3 Whatever holds for Marginal Revenue should hold for Marginal Cost. Use the following economist's definition of Marginal Cost to approximate the MC at $q = 2$ (taking an item to be 1 Ream).

> **Increment.** The Marginal Cost at q items is the increment in Total Cost that comes when we increase q by one more item.

4 The slope definition of Marginal Cost is

> **Slope.** The Marginal Cost at q items is the slope of the secant between the points on the TC graph at q items and one more item.

But since the TC graph given above is straight, secant lines to it are always the graph itself. Estimate the slope of the TC graph and check that this is numerically close to the answer you got to Question 3.

Now suppose that, as in the Key Question, we decided to take the basic unit of printing to be one Hundred Pages (=0.2 Reams). Then by the economist's definitions, the MR *and* MC *should get much smaller, because "one more item" is actually a smaller quantity. On the other hand, if you take the slope definitions,* MR *and* MC *are not going to change much, if at all, because moving the points closer along the graph is not going to change very much the secant line you actually draw. The confusion is in the units.*

One way to deal with the units problem is, if we are going to print orders in Hundreds of Pages, then we should re-label the horizontal axis in hundreds of pages. Thus 1 Ream is relabeled 5 Hundred Pages, 2 Reams is relabeled 10 Hundred Pages and so forth.

5 Relabel the horizontal axis so that the units are Hundreds of Pages. (Do not obliterate the old units, because we really are going to go back to them.)

6 Since "one more item" = 1 Hundred Pages, the increment definition of MR at $q = 15$ Hundred Pages requires that you calculate the change in the height of the TR graph from 15 to 16 Hundred Pages. Estimate the change in TR that results from increasing quantity from 15 to 16 Hundred Pages.

7 The slope definition of MR at $q = 15$ Hundred Pages requires that you draw a nice long secant line from $q = 15$ to $q = 16$ Hundred Pages and measure its slope, using two points that are far apart. Do so and use your answer to give the MR at $q = 15$ Hundred Pages. Then use this answer to tell what the increment in Revenue would be as you move from 15 to 16 Hundred Pages.

8 What is the MC at $q = 15$ Hundred Pages?

Thus everything works out—the two types of definition agree—if you relabel axes. But that's a tedious thing to have to do. The way to deal with this problem is to leave the units alone and always be prepared to convert (by arithmetic) to new units. But mainly, we will avoid the problem by focusing on the slope definition and not changing units.

9 We want the MR at 20 Hundred Pages when "one more item" means 1 Hundred Pages. That is, we want the change in TR that results from changing quantity from 20 to 21 Hundred Pages. But we want to use the old units, Reams, on the horizontal axis. Since 1 Ream is 5 Hundred Pages, 20 Hundred Pages is 4 Reams and 21 Hundred Pages is 4.2 Reams. This means we want a secant line through the TR graph from $q = 4$ Reams to $q = 4.2$ Reams. Draw this secant line and measure its slope. Use a similar method to compute the MC at 20 Hundred Pages.

10 Your answer to Question 9 says that an increase of 0.2 Reams leads to a *rate* of increase of a certain number of dollars/Ream. Convert this rate to a certain number of dollars per Hundred Pages and then tell what the MR and MC would be at 20 Hundred Pages.

Presumably, by now you see that when "one item" is a small difference on the horizontal scale, we have to use the slope definition of MR and MC in order to read numbers from the graph. If we are very careful and keep the units straight, this definition is basically the same as the economics text (increment) version of MR and MC. But we would rather not have to be so careful. The nice thing is that we can use the slope definition and not be so careful, and everything works out just fine.

11 Approximate the MR at 17 Hundred Pages by computing the slope of the secant line through TR from $q = 3.4$ to $q = 3.6$ Reams. Leave the units in dollars per Ream.

12 The profit for printing any size order is the difference between the Total Revenue and Total Cost for that quantity. This is measured by the vertical distance between the two graphs. Find the quantity q (in Reams) at which profit is maximized.

13 Now take one item to mean 0.2 Reams (=1 Hundred Pages) and check that, at the quantity q you gave in Question 12, MR is equal to MC. Use the slope definitions of these, of course.

Thus, regardless of units or which definition you use, the basic principle for economics still holds: Profit is maximized when Marginal Revenue is equal to Marginal Cost.

Next suppose we take one item to be one page =0.002 Reams. This means that the MR (or MC) at 3 Reams is the slope of the secant from $q = 3$ to $q = 3.002$ Reams. Even though we are taking one item to be one page, we still measure slopes in the units dollars/Ream.

$\boxed{14}$ What is the MC at $q = 3$ Reams when one item means one Page? Convert the units of this slope into dollars per Page and give the official economics text version of the MC at $q = 3$ Reams (1500 Pages).

$\boxed{15}$ The secant line to the TC graph from $q = 3$ to $q = 3.002$ Reams is not much of a mystery. But the secant line to the TR graph from $q = 3$ to $q = 3.002$ Reams is a bit more difficult to visualize. Do your best to draw this secant line and measure its slope. Give your answer in units of dollars per Ream and then convert it into dollars per Page to give the official economics version of MR at 3 Reams (1500 Pages).

$\boxed{16}$ When pressed to draw the line asked for in Question 15, students describe the line they drew in one of the following ways:

a) ("Touches") It touches the graph just at $q = 3$ and nowhere else.

b) ("Magic secant") It really is connecting two points on the graph that are very, very close.

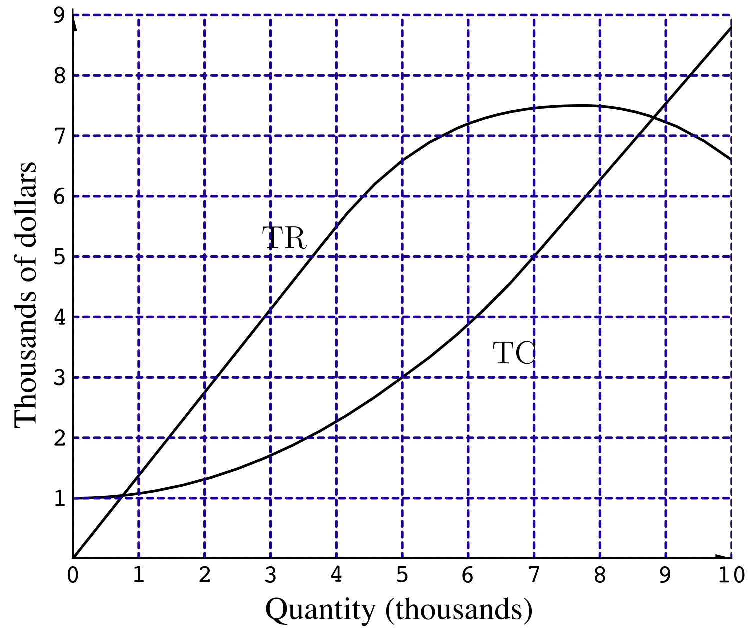
c) ("Balanced") The graph bends off equally on both sides.

d) ("Sliding secants") Imagine drawing secant lines between $q = 3$ and points nearby, and then imagine the nearby points getting closer to $q = 3$. This line is the secant line they are getting close to.

How would you describe the line you drew in Question 15—by one of the above descriptions? If none of the above descriptions is the same as yours, write out your own description. Can you give yours a caption name, like the ones in parentheses?

$\boxed{17}$ Draw the line necessary to measure the MR at $q = 4.6$ Reams, when one item means 1 page = 0.002 Reams. Measure its slope using units of dollars per page.

→ $\boxed{18}$ The graphs below are of Total Revenue and Total Cost of manufacturing Framits.



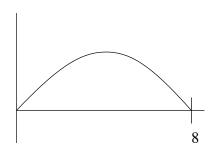
a) Find the Marginal Cost and Marginal Revenue of manufacturing 3 thousand Framits.

b) Find the value q at which the MC and MR graphs cross.

c) The Average Cost of producing q Framits is defined by $\frac{TC(q)}{q}$. Find the quantity q such that the Average Cost of producing q thousand Framits is \$8 per Framit.

d) What is the cost of producing one more Framit, if you are already producing 4 thousand Framits?

e) Which of the following statements are true of the graph shown to the right?



(i) It is shaped like the MR graph.

(ii) It is shaped like the MC graph.

(iii) It is shaped like the graph of $\mathrm{MR}-\mathrm{MC}$.

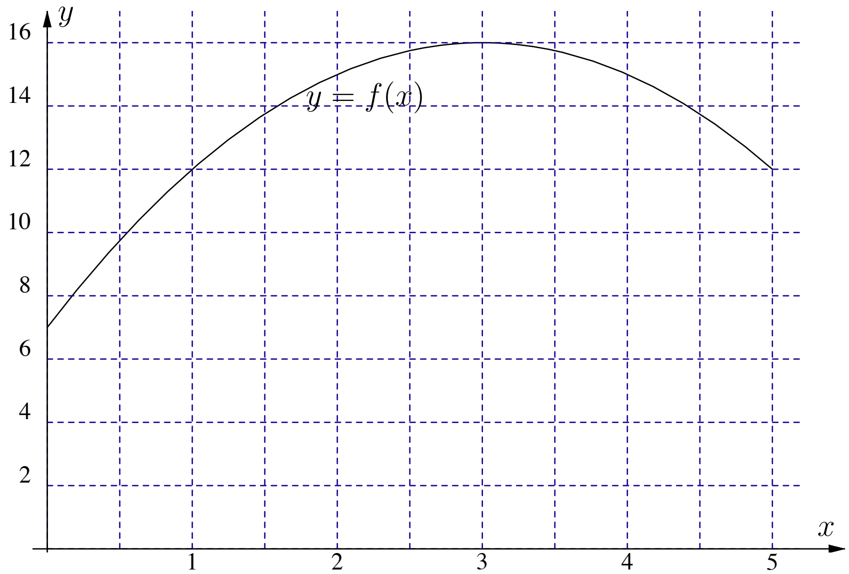
(iv) None of the above.

f) Imagine that the Fixed Costs of manufacturing Framits goes up by \$2,000. What is the MC of manufacturing 6 thousand Framits under this condition?

g) Find a range of values of q over which TR goes up but MR goes down. If none such exists, say so, and tell why.

Worksheet #3 *Tangents and Secants*

The graph below is of the function $f(x) = -x^2 + 6x + 7$



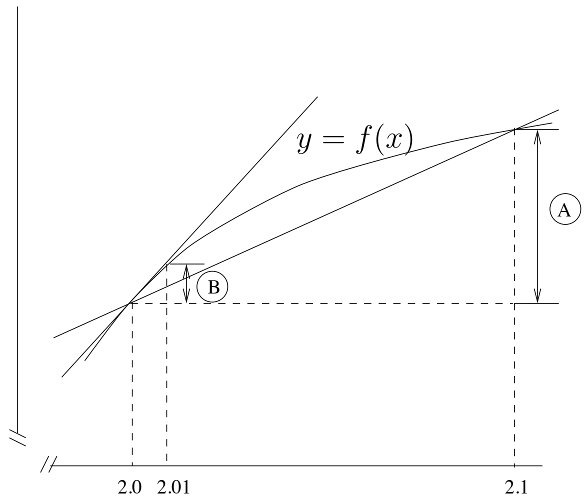
If the graph were viewed as a graph of altitude *vs*. time, then a reasonable question to ask would be to use the graph to determine the speedometer reading at time $t = 2$ (as in Worksheet #1). If the graph were viewed as a graph of Total Revenue *vs*. quantity, then a reasonable question to ask would be to determine the MR at $x = 2$. Both of these values would be given by the slope of the tangent line to the graph at 2. We can approximate this slope by drawing the tangent line, selecting two points on it, and computing the slope. But it would be nice to get an answer that is not subject to approximation error.

Key Question

What is the exact slope of the line that is tangent to this graph at $x = 2$?

$\boxed{1}$ Draw a tangent line to the graph of $f(x)$ at $x = 2$ and compute its slope.

$\boxed{2}$ Below is a large blow-up of the piece of the graph near $x = 2.0$. Mark the following quantities on the y-axis: $f(2.0)$, $f(2.01)$, $f(2.1)$.



$\boxed{3}$ On the blow-up there are dimensions labeled A and B. Express A and B in terms of $f(2.0)$, $f(2.01)$, and $f(2.1)$. Use the formula for $f(x)$ to determine the numerical value of A and B.

$\boxed{4}$ There are two secant lines on the blow-up above. Compute their slopes. For practical purposes I have drawn the two secant lines as having quite different steepnesses. Is this consistent with the numerical answers you got? How would these secant lines look on the original graph of $f(x)$ given at the beginning of the worksheet?

$\boxed{5}$ If you did Exercise 4 correctly, the two slopes you computed are 1.9 and 1.99. Which is closer to the slope of the tangent line at $x = 2$?

You just did the work to fill in the first two rows of the following table, which gives the slope of the secant line through the graph of $f(x)$ at $x = 2$ and $x = 2 + h$ for several values of h.

Table of Slopes at 2

$2+h$	$f(2+h)$	$f(2)$	$f(2+h)-f(2)$	$\frac{f(2+h)-f(2)}{h}$
2.1	*15.19*	*15*	*0.19*	*1.9*
2.01	*15.0199*	*15*	*0.0199*	*1.99*
2.001	*15.001999*	*15*	*0.001999*	*1.999*
2.0001	*15.00019999*	*15*	*0.00019999*	*1.9999*

$\boxed{6}$ Draw a blow-up of the graph of $f(x)$ near $x = 2.0$ and show how to obtain the third line of the table. The number 1.999 from the third row of the table represents the slope of a secant line through the graph of $f(x)$. Describe how this secant line would look if you tried to draw it on the original graph of $f(x)$ at the beginning of the worksheet.

The next question is, of course, "How would you illustrate the fourth line of the **Table of Slopes at 2** *on the full size graph of $y = f(x)$?" And the answer is that once you get past the first line of the table, all the secant lines look pretty much the same on the regular graph. If there are differences among the secant lines, you would have to make bigger and bigger blow-ups to see them. Thus, as far as the graph is concerned, you can take the tangent to be a magical secant—a* *secant between two points so close together you can not see the difference—and you can take the slope for this particular graph at that point to be 2. The formula and the graph give exactly the same information.*

But is "slope = 2" really the correct answer?

$\boxed{7}$ There is a distinct pattern to the answers in the first four lines of the **Table of Slopes at 2**. Complete the following computations by following the patterns established in the first four lines of the **Table of Slopes at 2**. (Do not use a calculator. Just follow the pattern.)

$$f(2.00001) = \qquad\qquad f(2.000001) =$$

$$f(2.00001) - f(2) = \qquad\qquad f(2.000001) - f(2) =$$

$$\frac{f(2.00001) - f(2)}{0.00001} = \qquad\qquad \frac{f(2.000001) - f(2)}{0.000001} =$$

Do these calculations again — this time use the formula for $f(x)$ and a calculator. For some of you, the calculator will round and the pattern established in the table will not continue.

It looks like, if we had a super-large calculator that never rounded off, then we would get a different answer every time we took a new point to the right of, but closer to, 2. And it does not look like any of these answers is 2. They get closer, but they never quite get there.

Calculators are a convenience, but this rounding-off error makes them too inaccurate for our present purposes. But you know enough algebra to do without one.

$\boxed{8}$ The formula for $f(x)$ is $f(x) = -x^2 + 6x + 7$. Then by the substitution Rule

$$f(2 + h) = -(2 + h)^2 + 6(2 + h) + 7.$$

Simplify the expression on the right hand side of the above equation so that the equation reads

$$f(2 + h) = (\qquad)h^2 + (\qquad)h + (\qquad).$$

Then write out a formula of the form

$$f(2 + h) - f(2) = (\qquad)h^2 + (\qquad)h + (\qquad),$$

and then use your formula to show that

$$\frac{f(2 + h) - f(2)}{h} = 2 - h.$$

$\boxed{9}$ This last formula, which came from the Substitution Rule and algebraic simplifications, is very powerful. If you give it a particular value of h, say $h = 0.1$, it gives you the corresponding value of $\frac{f(2+h)-f(2)}{h}$ which is the slope of the secant from 2 to $2 + h$. (If $h = 0.1$, then the slope of the secant line from 2 to $2 + h$ is $= 2 - 0.1 = 1.9$.) Use this formula to check the second line of the **Table of Slopes at 2**. Then use the formula to find the value of $\frac{f(2+h)-f(2)}{h}$ when $h = 0.00001$. Check this answer against your answer to Question 7.

The formula $\frac{f(2+h)-f(2)}{h} = 2 - h$ tells us that no matter how small we make h (no matter how close the two points are), the slope of the secant line from 2 to $2 + h$ is never going to reach 2, although it can be made as close as we like to 2. You cannot set h equal to 0, because if you do, then you do not have two different points, and you are not talking about a secant line. The whole idea collapses. Thus, mathematicians say: "The tangent line to this graph at $x = 2$ is the line that touches the graph there and has slope 2." This does not tell you how to draw the tangent line directly from the graph, and it does not really say what kind of line the tangent line is—it just says that it is a particular line with slope 2. On the other hand, once you draw the line you probably will agree that:

A. The tangent line is very, very close to the secant line connecting the two points very close to 2.

B. The tangent line is an indicator of the steepness of the graph.

A good deal of this course involves the notion of a tangent line. But I cannot quite tell you what sort of a creature it is. On the other hand, if we have the formula for the graph, I can tell you with great accuracy what the slope of the tangent line is.

$\boxed{10}$ Go back to the graph of $f(x)$ at the beginning of this worksheet. Draw your own version of the tangent to the graph at $x = 1$.

$\boxed{11}$ Using the Substitution Rule and some algebra, complete the following equations, and then check the last line.

$$f(1+h) = (\qquad)h^2 + (\qquad)h + (\qquad)$$
$$f(1+h) - f(1) = (\qquad)h^2 + (\qquad)h + (\qquad)$$
$$\frac{f(1+h) - f(1)}{h} = 4 - h.$$

$\boxed{12}$ Use the last formula in Question 11 to find the slope of the secant to the graph from $x = 1$ to $x = 1.000002$.

A mathematician would use the last line of Question 11 to say that the slope of the tangent to the graph at $x = 1$ is 4. This is because 4 is the slope that the secant lines are getting closer to as you let the two points get closer. But this also says that the tangent line is different from all of the secant lines, because the secant lines all have slope different from 4.

If the graph we started with were a distance vs. time graph, then we would say that the instantaneous speed at time $t = 1$ is 4 ft/sec.

$\boxed{13}$ Imitate the procedure used in questions 11 and 12 to find the slope of the tangent to the graph of $f(x)$ at $x = 2.5$. Then check your answer by drawing the tangent line on the graph and measuring its slope.

→ $\boxed{14}$ The graph below is of the function $y = f(x)$.

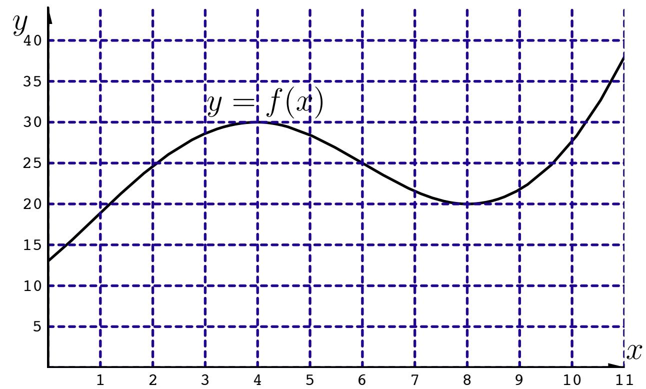
a) Find a value of x at which the tangent line to $f(x)$ has slope 6.

b) Determine the value of $\frac{f(2+h)-f(2)}{h}$ when $h = 3$.

c) Find a value of x for which

$$f(x+2) - f(x) = 7.$$

d) Find two values u and v such that the tangent line to $f(x)$ at $x = u$ is parallel to the tangent line to $f(x)$ at $x = v$.

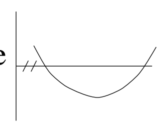


Let $g(x)$ be a function defined by $g(x) =$the slope of the tangent line to f at x.

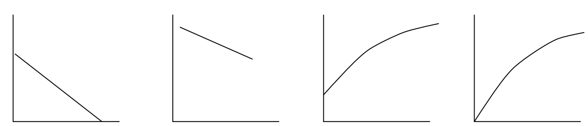
e) Give an interval over which the graph of $g(x)$ is

(i) positive and increasing

(ii) negative and decreasing.

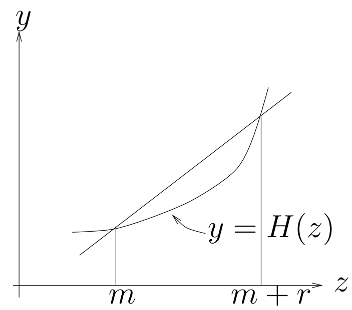
f) Find an interval over which the graph of $g(x)$ is shaped like 

g) Which of the graphs to the right is closest to the shape of the graph of $g(x)$ over the interval 0 to 2?



$\longrightarrow$ $\boxed{15}$ Below is a rough sketch of the graph of the function $y = H(z)$. The slope of the secant line shown is

$$\frac{H(m+r) - H(m)}{r} = 4m + 2r - 3$$



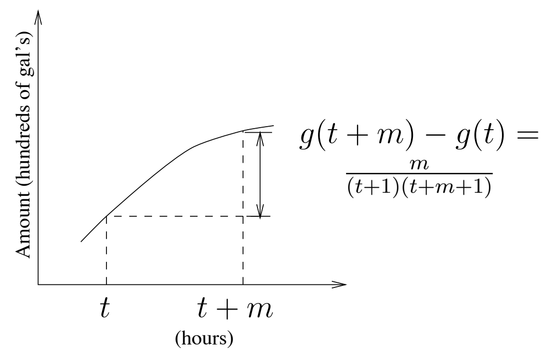
a) Determine the value of $H(5) - H(3)$.

b) Write out a formula in terms of k for $H(2 + k) - H(2)$.

c) Find the value of $\frac{H(4.002) - H(4)}{0.002}$.

$\longrightarrow$ $\boxed{16}$ Below is a rough sketch of part of the graph of the amount (in <u>hundreds</u> of gallons) of water flowing into a reservoir as a function of time. Denote the function by $A = g(t)$. We are not given the formula for $g(t)$, but instead are told that

$$g(t+m) - g(t) = \frac{m}{(t+1)(t+m+1)}.$$



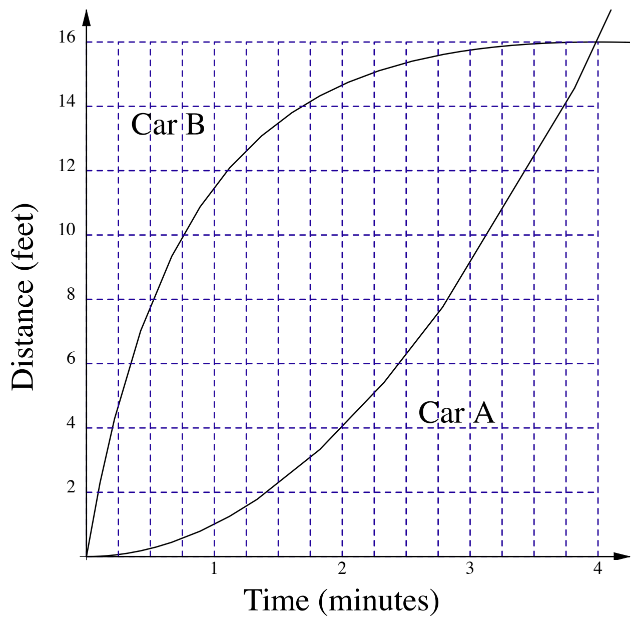
a) Find the average rate of flow of water into the reservoir from $t = 2$ to $t = 5$ hours.

b) Write out a formula in terms of h for $\frac{g(2+h) - g(2)}{h}$.

c) Suppose that there are 50 gallons in the reservoir at time $t = 1$. Use the formula for $g(t + m) - g(t)$ to find the amount of water in the reservoir at time $t = 3$.

Worksheet #4 *Drawing Speed Graphs*

The graphs below give the distance *vs*. time for two electronically controlled cars moving along parallel tracks. At time $t = 0$ the cars are next to one another.



Key Question

I. Find the time t at which the cars are farthest apart.

II. Find the time t at which the two cars are going the same speed.

III. Find a time t at which Car B is going 1 ft/min faster than Car A.

$\boxed{1}$ Key Question I is just a question about graph reading. You are being asked for the value of t at which the two graphs are farthest apart. What is your answer?

$\boxed{2}$ Key Question II asks for the value of t for which the slopes of the tangents to the two graphs shown are equal. You can either find this directly—which is difficult, since the two graphs are so far apart for the values of t you are looking at. Or you can recall a basic principle about such graphs—that they are furthest apart when their tangents are parallel. What is your answer?

3 Key Question III can be answered by a trial-and-error method. But it is much more difficult than Key Question II, because looking for tangent lines that have a specific numerical relationship between their slopes is much more difficult than looking for parallel tangent lines. Furthermore, when you are looking for parallel tangent lines you are also looking for the place where the graphs are furthest apart. You have no such visual clues for Key Question III. We will approach Key Questions II and III more systematically, but for now you should make your best guess for the answer to Key Question III.

What we need to solve Key Questions II and III is a very accurate table of slopes of the tangents to the two graphs for many values between 0 and 4. Then we could just scan the table for the place where the readings for A and B are the same (to answer Key Question II), and then scan the table to find the value of t where the reading for B is 1 ft/min greater than the reading for A (to answer Key Question III). Our next step is to make such a table and then use it to answer the Key Questions. We will make a complete table with t ranging from 0 to 4, because later on we will want to answer the question: For what value of t is Car A going 3 ft/min faster than Car B (which is like Key Question III, but different)?

4 With the techniques we have available to us at this stage, finding the slopes of tangents to the two graphs and filling in the tables is going to be tedious. Fortunately for you, we've done most of the work for you. Fill in the remaining entries of the following table by drawing the appropriate tangent lines and approximating their slopes.

t	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Speed of Car A at time t	0	0.4	0.8	1.2	1.6		2.4	2.8	3.2	3.6	
Speed of Car B at time t	24	16.36	11.76	8.77	6.72		4.17	3.34	2.7	2.19	1.78

t	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Speed of Car A at time t	4.4	4.8	5.2	5.6	6	6.4	6.8	7.2	7.6	8
Speed of Car B at time t	1.44	1.16	0.93	0.73		0.42	0.29	0.18	0.09	0

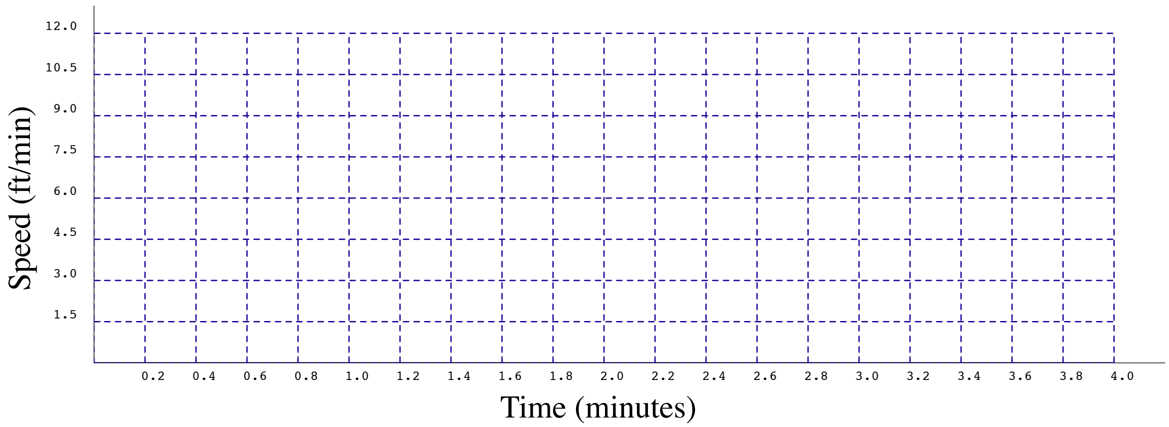
5 Now that you have the tables of speeds *vs*. time for the two cars, you should be able to answer Key Questions II and III with great ease. Do so, and compare your answers to the answers you got in questions 2 and 3. However, as you see, the tables are not perfect either, since they have holes in them and still must be used to estimate.

6 Use the speed *vs*. time tables to find the time t when Car A is going 3 ft/min faster than Car B.

The tables are quite efficient for answering speed vs. time questions for the two cars. But looking at them tells nothing about the patterns *of speed vs. time of the two cars. Now that we have the tables, we might as well take the next step and draw the graphs of speed vs. time for the two cars. But before doing this, you should decide in your own mind what the patterns of speed vs. time for the two cars are.*

7 First use the two distance *vs*. time graphs to give an overall description of speed *vs*. time for each of the two cars. (E.g., "Car X speeds up the whole time," or "Car Y speeds up and then goes at a constant speed.")

8 Using your table draw, on the axes that follow, the speed *vs*. time graph of Car A ONLY.



9 Now use the graph you drew in Question 8 to describe in everyday words the speed *vs*. time pattern of Car A.

10 Now it is time to plot B's speed *vs*. time graph on the axes above. You will not be able to include the first two or three entries of your table, because the speed axis does not go high enough, so start at about $t = 0.4$.

We started with two Distance vs. time graphs and asked speed vs. time questions. This led us to making speed vs. time tables. That led us to making speed vs. time graphs. As you can see, the speed vs. time graphs have different shapes. In fact, even though the two distance graphs are not that different, the speed vs. time graphs pick up and describe differences between distance graphs that the naked eye would not ordinarily see.

But the speed vs. time graphs are also handy for answering practical questions.

11 Use the speed *vs*. time graphs to determine the time when Car A is going 1 ft/min faster than Car B. Check your answer on the two distance *vs*. time graphs at the beginning of this worksheet.

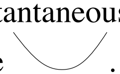
→ $\boxed{12}$ The graphs below are of distance *vs*. time for two cars, Car A and Car B. We denote the distance functions by $A(t)$ and $B(t)$, respectively.

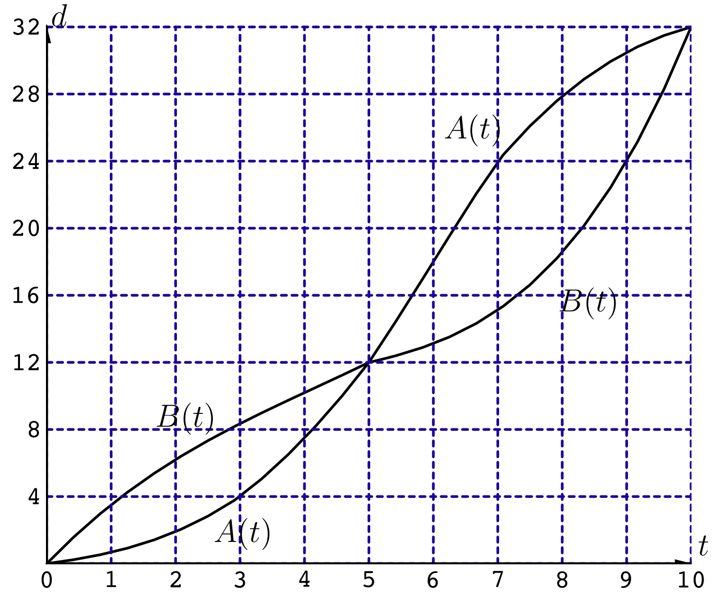
a) Find *two* times when the speeds of the two cars are equal.

b) Find a time t such that the average speed of Car A over the next three minutes is 4.0 ft/min.

c) Find a 4-minute time interval over which Car B travels 7 ft.

d) Find the longest time interval you can over which Car A's speed graph is higher than 5 ft/min.

e) Find a time interval of at least 3 minutes when the graph of B's instantaneous speed is shaped like  .



f) Give a time when Car B is ahead of Car A and the instantaneous speed graph of $A(t)$ is at least 1 ft/min higher than the instantaneous speed graph of Car B. If no such time exists, tell why.

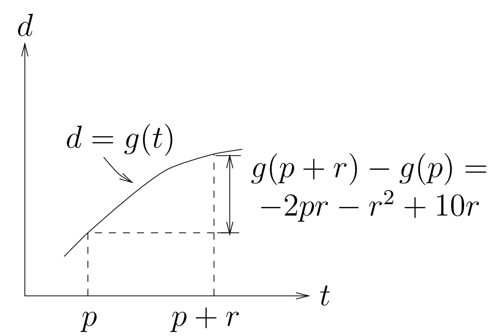
→ $\boxed{13}$ Below is a rough sketch of a portion of the distance *vs*. time graph for a car. As indicated in the picture, the distance covered by the car from time $t = p$ to time $t = p + r$ is

$$-2pr - r^2 + 10r.$$

a) What is the average speed of the car from time $t = 3$ to time $t = 3.8$?

b) Write out a formula in terms of s for the distance covered by the car from time $t = 1$ to time $t = s$.

c) Find the value of $\frac{g(4.01)-g(4)}{0.01}$.

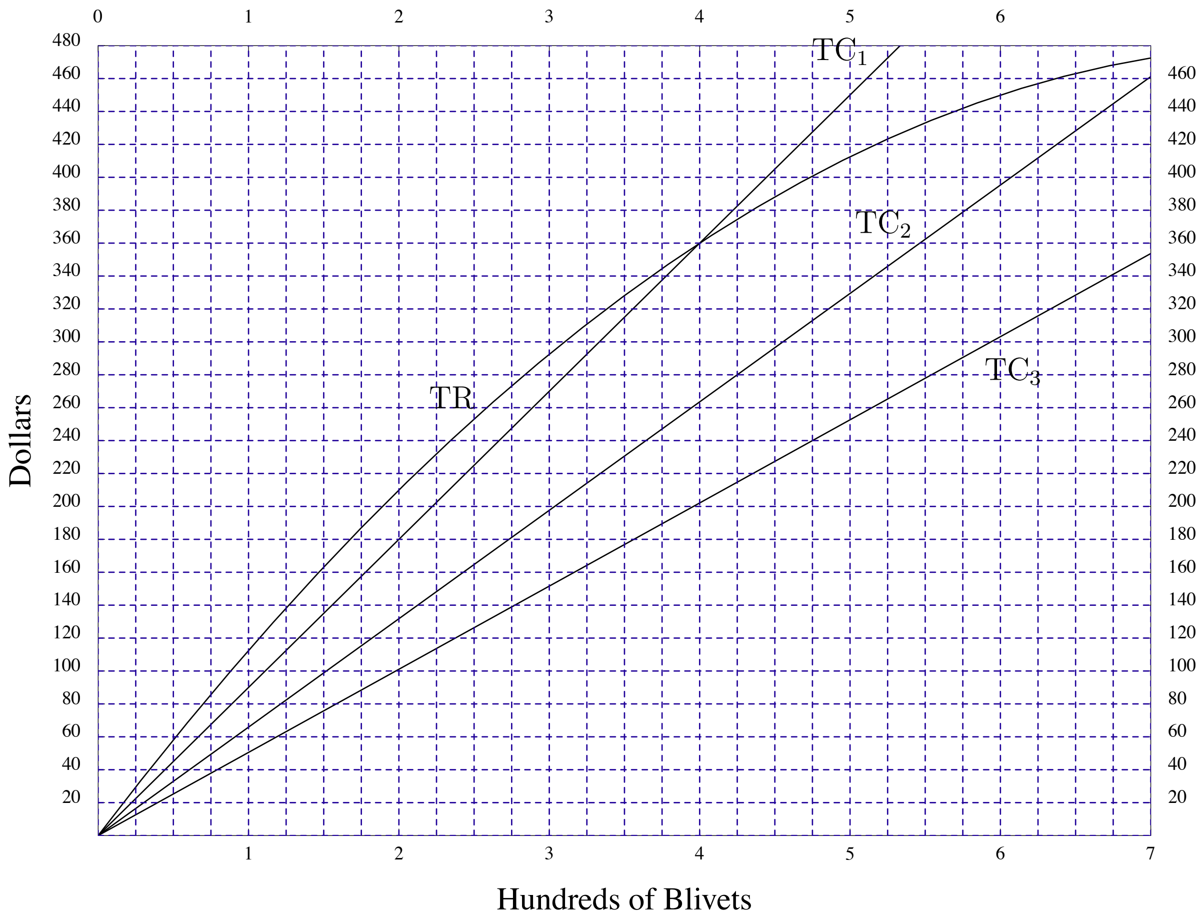


Worksheet #5

Drawing and Using a Marginal Revenue Graph

Key Question

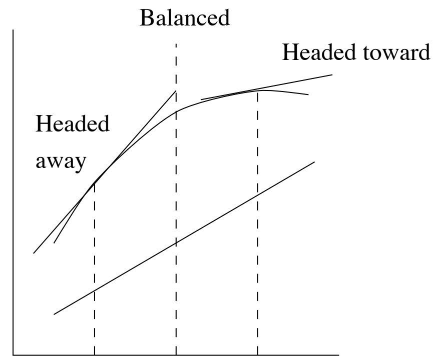
The curvy graph below is of Total Revenue of selling Blivets. The straight-line graphs are three possible Total Cost graphs of manufacturing Blivets (TC_1, TC_2 and TC_3). We wish to determine the quantity q at which the profit is maximized for each of the three TC graphs. One approach is to find the value at which the distance between the TR graph and a particular TC graph is greatest. Another is to find the value at which the TR and a particular TC graph have the same slope. Use this second approach on the given graphs to find the three desired values of q. Then tell how you would find the three values of q, if, instead of the given graphs, you had graphs of Marginal Revenue and Marginal Cost.



$\boxed{1}$ First assume that the TC graph is TC_1, and use the method of finding the greatest vertical distance between the TR graph and the TC_1 graph to find the value of q that maximizes profit. The TR graph bends so little that it may be hard to find a unique best quantity q. If so, give an interval of values of q that you think lead to maximum profits.

2 Check that the value (or values) of q that you chose in Question 1 has the property that: *at* that value of q the tangent to the TR graph is parallel to the TC_1 graph. In fact, it may be that by just using slopes of tangents you can limit the possible range of q's better than by using the vertical distance between the graphs.

You should be clear on why the profit is maximized when the slope of the TR graph is equal to the slope of the TC_1 graph. One way to think about this is that, for points to the left of this value of q, the TR graph is steeper and therefore "headed away" from the TC_1 graph. For points to the right of this value of q, the TR graph is less steep and therefore is "headed toward" the TC_1 graph. The point q that you chose, then, is a kind of "balancing point," where the TR graph is neither headed away from nor headed toward the TC_1 graph. That is the place where the TR graph is furthest away from the TC_1 graph.

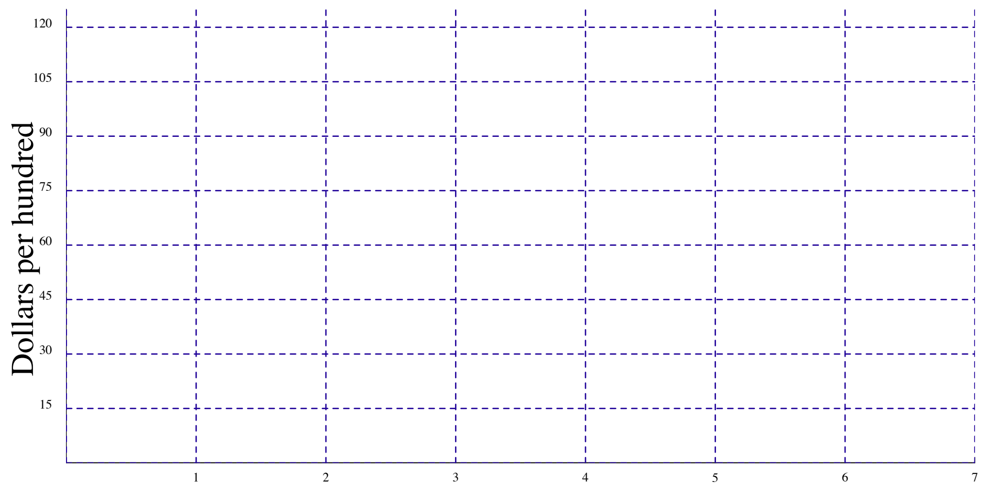


3 Find the value of q (or a small range of values of q) where the slope of the tangent to the TR graph is the same as the slope of the TC_2 graph—and therefore find the value of q that maximizes profit, when the Total Cost graph is TC_2. Repeat this problem when the Total Cost graph is TC_3.

4 If we take the Total Cost graph to be TC_1, then the Marginal Cost is constant and is equal to the slope of TC_1. Compute this slope and call it MC_1. If the units of this slope are in dollars/hundred Blivets, then (in the terminology of Worksheet #2), you are giving the slope definition of MC. Convert the units to tell how much more money it costs to produce one more Blivet if the Total Cost graph is TC_1.

Again, as in Worksheet #2, we will use the slope definition of Marginal Revenue and Cost. It is easier to deal with and differs from the economist's definition only in units. Also, as in Worksheet #2, we will take the slope of the tangent to the TR graph to represent Marginal Revenue—whereas, technically, we should use a secant line which connects two points one Blivet (not one hundred Blivets) apart. Thus, in Question 3, above, you have maximized profit by finding q where Marginal Revenue and Marginal Cost are equal. This would be an easy problem if we had an MR graph instead of a TR graph. That is what we will get next.

5 We want a graph that gives Marginal Revenue at all values of q between 0 and 7. For that we should imitate the preceding worksheet and find the tangents to the TR graph at 20 or 30 points, and then plot their slopes. However, along with most economics texts, I have drawn a "very nice" TR graph. Using the graph make the most accurate readings you can of the MR at $q = 2$, 4 and 6 (in Dollars per Hundred Blivets). Then plot these values on the axes that follow.



6 If you have done Question 5 with much care, the three points you plotted will be pretty close to a straight line. Draw a straight line that comes closest to going through these three points.

7 The graph you drew in Question 6 is a candidate for the MR *vs.* q graph. Determine the values it gives for $q = 1$ and $q = 5$. Then, by drawing the tangents to the TR graph, check to see that this straight line is the correct MR graph (within reasonably tolerable levels of accuracy for graph reading).

8 In Question 2 you were looking for the value of q at which the slope of the TR graph is equal to 90—the slope of the TC_1 graph. The slope of the TR graph is MR. So you are looking for the value of q at which the MR graph you drew takes the value 90. Use the MR graph to find this value of q and check against your answer to Question 2.

9 Use the MR graph you drew to find when the MR is equal to the slope of TC_2. Check your answer against Question 3.

10 Use your MR graph to find the quantity q at which profit is maximized when the Total Cost graph is TC_3. Check against your answer to the last part of Question 3.

11 Suppose you wanted to draw an MC graph if the Total Cost graph is given by TC_2. Then you would choose a quantity q and measure the slope of the tangent line to the TC_2 graph at that point. This is weird, but to a mathematician, the tangent to the TC_2 graph at any value of q is just the straight-line TC_2 graph. Thus you would get the same MC all along the TC_2 graph. On the axes following Question 5, draw the MC graph that goes with the TC_2 graph. Check that the profit is maximized when the MC and MR graphs cross.

12 As a final twist, suppose that you had an added Revenue of \$120—even if you sold no Blivets at all—so that your entire TR graph was raised by \$120. Then what would the resulting MR graph look like? How would this affect the quantities at which profits are maximized for the various TC graphs?

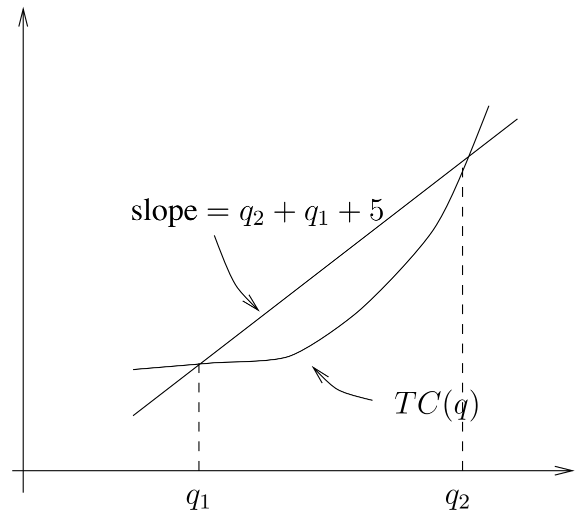
→ $\boxed{13}$ Below is a rough sketch of a segment of the graph of Total Cost *vs*. quantity of manufacturing Blivets. The variable q is given in thousands of Blivets, and the Cost is given in Thousands of dollars. We do not have a formula for Total Cost *vs*. quantity, but only the formula for the slope of the secant shown:

$$\frac{C(q_2) - C(q_1)}{q_2 - q_1} = q_2 + q_1 + 5.$$

a) Find the increase in Total Cost that comes with changing quantity from 3 thousand to 6 thousand Blivets.

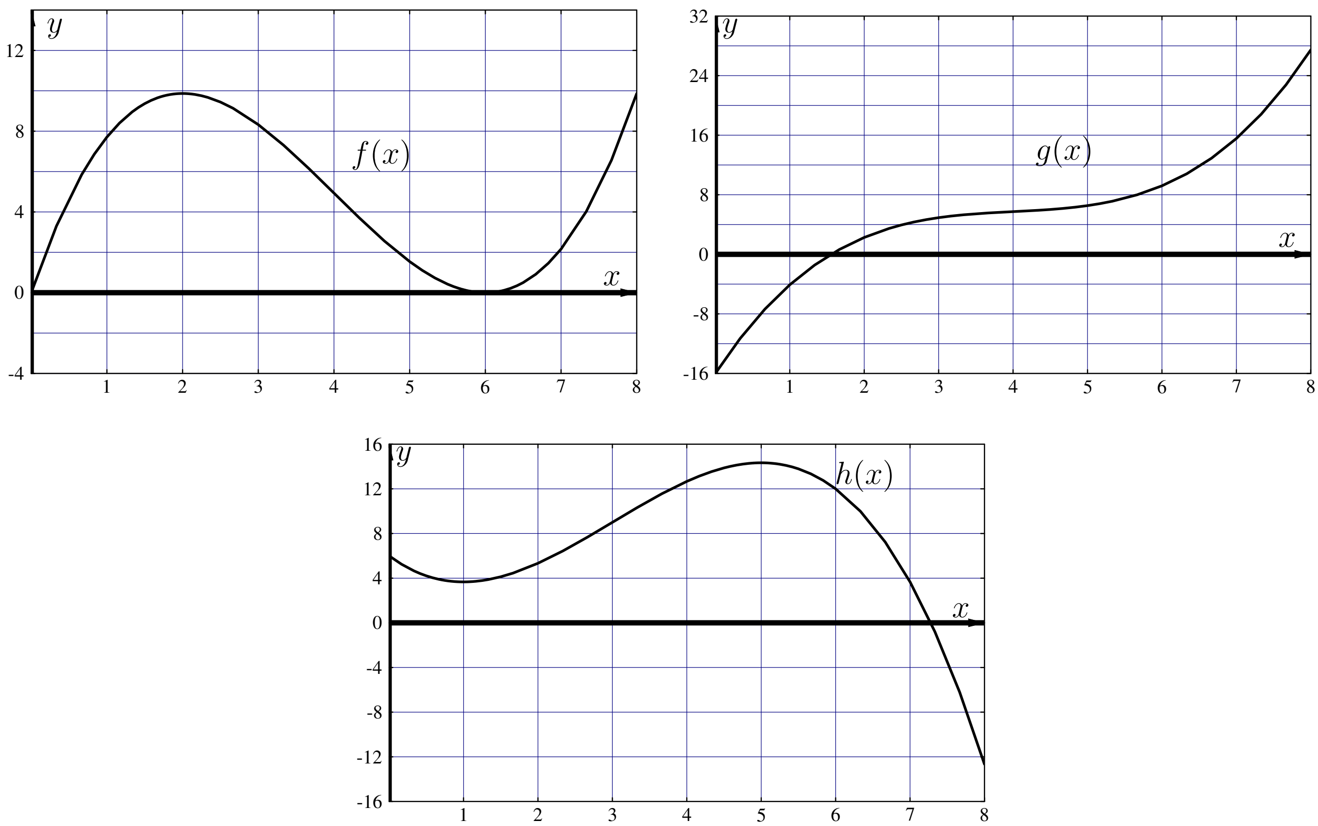
b) Determine the value of $\frac{C(4.001)-C(4)}{0.001}$.

c) Suppose that the Total Cost of producing 4 thousand Blivets is \$36,000. What is the Total Cost of producing 5 thousand Blivets?



Worksheet #6 *Derived Graphs*

We have below the graphs of three functions, $y = f(x)$, $y = g(x)$ and $y = h(x)$. For each of the graphs we could follow a procedure similar to the one used to get the speed graphs from the distance graphs in Worksheet #4 and similar to the one used to get the Marginal Revenue graph from the Total Revenue graph in Worksheet #5. That is, we could: Measure the slope of the tangent at various points, say, $x = 0, 1, 2, 3, \ldots, 8$, and, on a new set of axes, plot these slopes of tangents. By this procedure we would have formed three new graphs, called the *Derived Graphs* of the graphs shown. (Do not draw the derived graphs yet, though.)



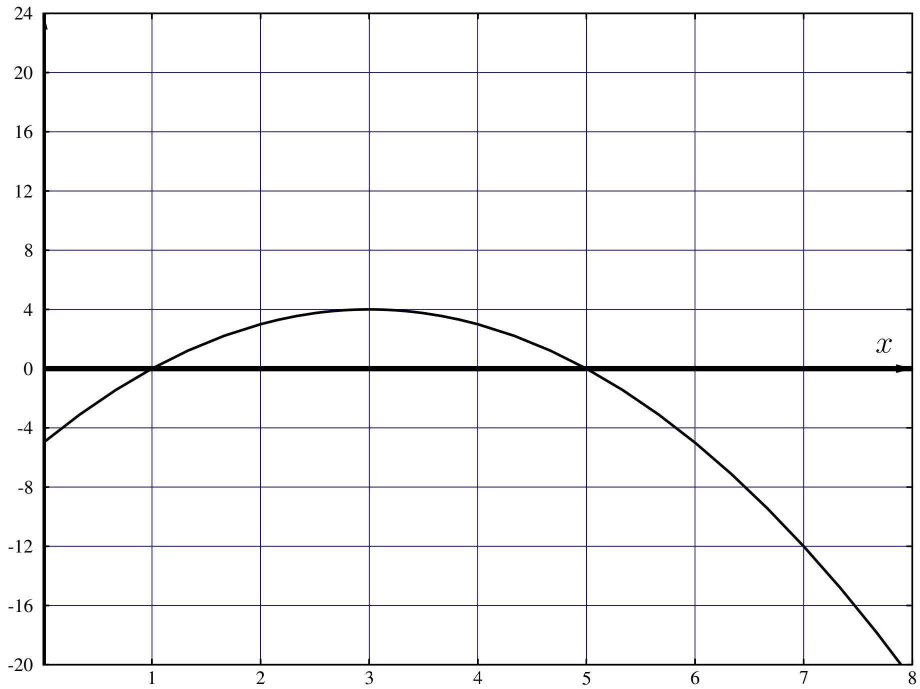
Key Question

I. For each of the graphs, tell when its derived graph crosses the x-axis.

II. Give the longest interval you can over which the derived graph of the graph of $g(x)$ is positive but decreasing.

III. Is there an interval of values of x over which the derived graph of the graph of $h(x)$ is higher than the derived graph of the graph of $f(x)$—if both were placed on the same axes?

$\boxed{1}$ Carefully measure the slopes of the tangent lines to $y = f(x)$ at $x = 1, 2, 3, \ldots, 8$ and plot the values on the axes below.



$\boxed{2}$ The graph you drew in Question 1 should be below the x-axis on the interval $x = 2$ to $x = 6$. How does this fact appear on the graph of $y = f(x)$?

$\boxed{3}$ The graph of $y = f(x)$ gets less steep as you go from $x = 0$ to $x = 2$. How does this fact show up in the derived graph of the graph of $f(x)$?

$\boxed{4}$ Now look at the graph of $g(x)$ and describe in words what the pattern would be if you measured the slopes of the tangent lines at $x = 0, 1, 2, \ldots, 8$. The easiest way to read off this pattern is to let a straight-edge run along the graph, watching how the slope of the straight-edge goes up and down.

$\boxed{5}$ Use the pattern you came up with in Question 4 to draw a rough sketch of the derived graph of the graph of $g(x)$. (Do not do it on the axes above.)

Notation. When you answered Question 1, you measured the slope of the tangent line to the graph of $f(x)$ at various points. We need a name for this.

$f'(1)$ means the slope of the tangent line to the graph of $f(x)$ at $x = 1$.
$f'(2)$ means the slope of the tangent line to the graph of $f(x)$ at $x = 2$.

In general,

$f'(a)$ means the slope of the tangent line to the graph of $f(x)$ at $x = a$.

$\boxed{6}$ By my measurements, $g'(2) = 4.2$. Draw the appropriate tangent line and check my answer. Likewise, check that $g'(6) = 4.2$ and $g'(4) = 0.5$.

$\boxed{7}$ Determine the numerical values of $g'(0), g'(1), g'(3), g'(5), g'(7)$, and $g'(8)$. Then use these numbers and the values given in Question 6 to plot on the axes below Question 1 the values of the derived graph of $g(x)$ at $x = 0, 1, \ldots, 8$. Sketch in the derived graph of the graph of $g(x)$. Check your graph against the rough sketch you made in Question 5.

$\boxed{8}$ Give, in your own words, the relationship between the derived graphs of the graphs of $f(x)$ and $g(x)$. Most people would say that $f(x)$ and $g(x)$ have very different shapes, but that their derived graphs are quite similar. Do you agree?

$\boxed{9}$ Suppose you drew a new graph on the same set of axes as the graph of $f(x)$ in such a way that the new graph were parallel to $f(x)$, but exactly 5 units higher. How would the derived graph of this new graph be related to the derived graph of $f(x)$? Caution: Think carefully about this.

We are drawing new graphs in this worksheet and calling them things like "the derived graph of the graph of $f(x)$," or the "derived graph of the graph of $g(x)$." We need an abbreviation. But we already sort of have one.

$f'(1)$ means the slope of the tangent to the graph of $f(x)$ at $x = 1$. But it also is the height of the derived graph at $x = 1$. Therefore, we will give the derived graph the name $f'(x)$. Then, $f'(2)$ means two things:

$$f'(2) = \begin{cases} \nearrow \text{The slope of the tangent to the graph of } f(x) \text{ at } x = 2. \\ \searrow \text{The height of the graph of } f'(x) \text{ at } x = 2. \end{cases}$$

But since these two different things are really the same number, no confusion will arise by giving them the same name.

$\boxed{10}$ Look at the axes below Question 1. I have already drawn a graph there. This is the graph of $h'(x)$. That is, what you have there is the derived graph of the graph of $h(x)$. Use this graph to find the values of $h'(0)$, $h'(3)$, and $h'(5)$. But then $h'(0)$ is the slope of the tangent to the graph of $h(x)$ at $x = 0$. By drawing a tangent line on the graph of $h(x)$, check the reading you just gave for $h'(0)$. Similarly, check the readings of $h'(3)$, and $h'(5)$, by measuring the appropriate slopes of tangents to the graph of $h(x)$.

$\boxed{11}$ You should now have the accurately drawn derived graphs $f'(x)$, $g'(x)$ and $h'(x)$. Use them to answer Key Question I.

$\boxed{12}$ Use your answer to Question 11 and the derived graphs to tell whether either (or both) of the following principles is true.

- When a graph has a horizontal tangent, its derived graph crosses the horizontal axis.
- When a derived graph has a horizontal tangent, the graph you started with crosses the horizontal axis.

13 Using the derived graph $g'(x)$, you should be able to easily answer Key Question II. But now tell if there is an interval over which $h'(x)$ is negative but increasing.

14 In Question 13 you found an interval over which $h'(x)$ is negative but increasing. Now look back at the graph of $h(x)$. Is the slope of $h(x)$ increasing or decreasing over this interval?

Mathematician's Quirk. Over the interval $x = 0$ to $x = 1$ the graph of $h(x)$ is getting less steep, but its slope is increasing. This is because, over that interval, the slopes are negative numbers getting closer to 0. For a mathematician, the number -1 is higher than the number -4. Thus the slopes are increasing over that interval.

15 By looking at the derived graphs $h'(x)$ and $f'(x)$ you can now answer Key Question III. Do so. Would you say that $h(x)$ is steeper than $f(x)$ over the entire interval you gave?

The derived graph of a graph is a record of slopes of tangents. If the graph we had started with had been of distance vs. time, then the derived graph would be a record of speeds, so that it would be a speed graph. Likewise, the derived graph of a Total Revenue graph is a Marginal Revenue graph.

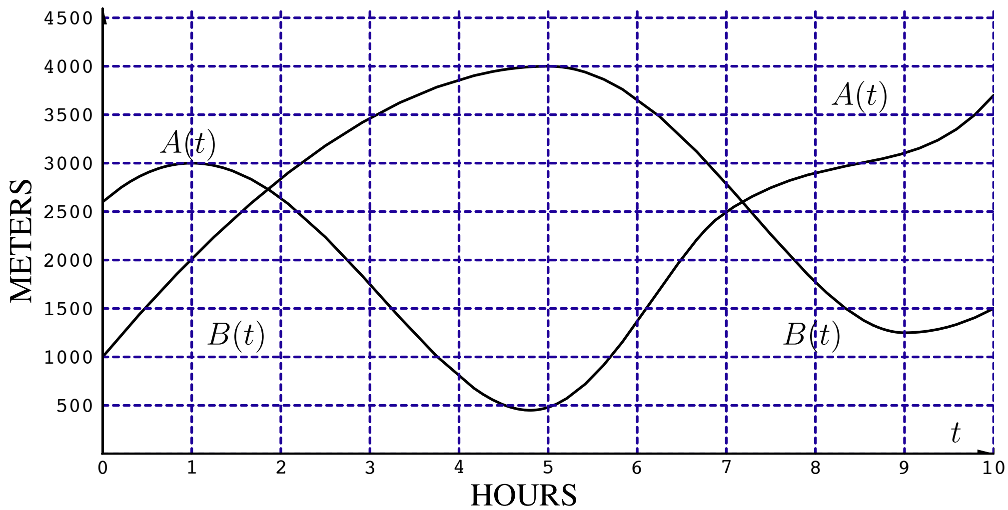
16 Suppose that $f(x)$ were a graph of distance (in feet) *vs*. time (in minutes) of an electronically controlled car (so that the car goes forward for 2 minutes and then backs up for 4 minutes). Now suppose that you had a different car that was always 5 feet further away from the starting point than the given car. What would the distance *vs*. time graph of this new car look like? How would the speed graphs of the two cars be related? Check your answer against Question 9.

The moral of questions 9 and 16 is that moving a graph up and down does not affects its derived graph. But what if you move the derived graph up and down?

17 Imagine another function, $A(x)$ whose graph is to be drawn on the same set of axes as the graph of $g(x)$, but suppose that the derived graph of $A(x)$ is three units lower than $g'(x)$. How will this affect the shape of $A(x)$ in relation to the shape of $g(x)$?

If you are in doubt about your answer to Question 17, go back and look at the derived graphs $f'(x)$ and $g'(x)$. Each is a vertical shift of the other, but the two graphs we started with are very different.

$\longrightarrow$ $\boxed{18}$ The two graphs below are of altitude *vs*. time of two weather balloons. Call them Balloon A and Balloon B.



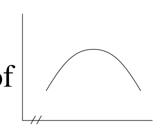
a) Find the time interval during which the derived graph of $A(t)$ is greater than 800 meters/hour.

b) Find a time t such that $B(t+3) - B(t) = 1400$ meters.

c) Find the time interval during which both of the following conditions hold

- Balloon A is below Balloon B.
- The derived graphs of $A(t)$ and $B(t)$ are above the t-axis.

d) Find a time interval at least 2 hours long over which the derived graph of $A(t)$ is shaped like the graph on the right.



e) Give a time interval at least one hour long when the slope of the tangent to the graph of $A(t)$ is negative but increasing.

→ $\boxed{19}$ Below are the *derived* graphs of two other graphs. The "original graphs" from which these graphs came are not shown. The original graphs are given by the functions

$$\text{Original graphs: } y = f(x), \qquad y = g(x).$$

The derived graphs that are shown are given by the formulas

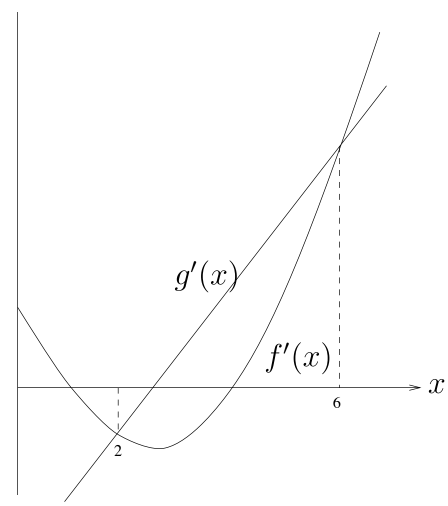
$$f'(x) = 2x^2 - 10x + 8 \qquad g'(x) = 6x - 16.$$

a) Find the values of x where the graph of $f(x)$ has a horizontal tangent.

b) Find the value(s) of x where $f'(x)$ has a horizontal tangent.

c) Is there an interval of length 2, between $x = 0$ and $x = 8$, over which $f(x)$ is increasing and $g(x)$ is decreasing? If so, name it. If not, tell why.

d) Find the value of x between $x = 2$ and $x = 6$ where the value of $g(x)$ is the lowest. Show your work.



e) Find the value of x between 0 and 1 where $f(x)$ is highest. Show your work.

f) Suppose that the original graphs, of $f(x)$ and $g(x)$, cross at $x = 2$. Tell whether the following statement is true or false. Explain your answer.

The graphs of $f(x)$ and $g(x)$ cross at $x = 6$.

→ $\boxed{20}$ Below are the *speed vs.* time graphs of two cars. At time $t = 0$ the two cars are next to one another. They then travel along parallel tracks. The formulas for the speed graphs are:

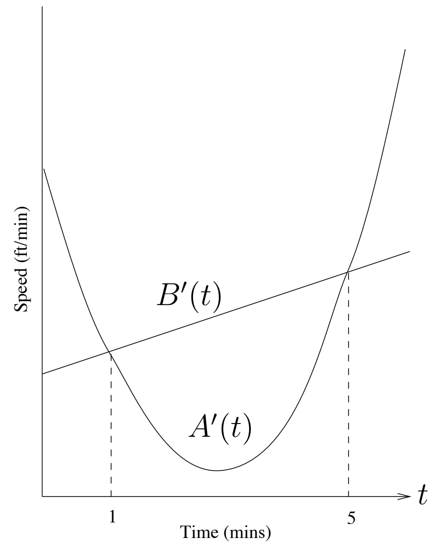
$$\begin{aligned} A'(t) &= 3t^2 - 16t + 23 \\ B'(t) &= 2t + 8. \end{aligned}$$

a) At what time in the first two minutes are the two cars furthest apart?

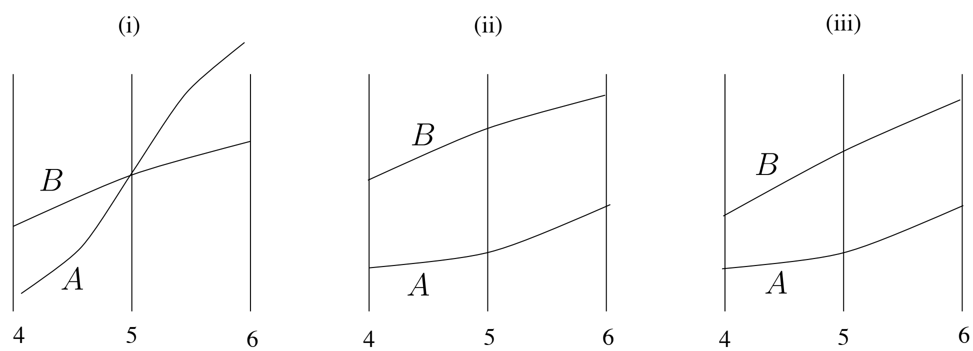
b) At what time in the interval $t = 2$ to $t = 4$ is the speed of Car A lowest?

c) What is the speed of Car B the second time the two cars are traveling at the same speed?

d) At what time in the time interval $t = 1$ to $t = 2$ minutes is the distance between the two cars greatest?



e) Car B is ahead at time $t = 4$ minutes. Which of the three pairs of graphs below is closest to the distance *vs*. time graphs for the two cars in the interval $t = 4$ to $t = 6$?



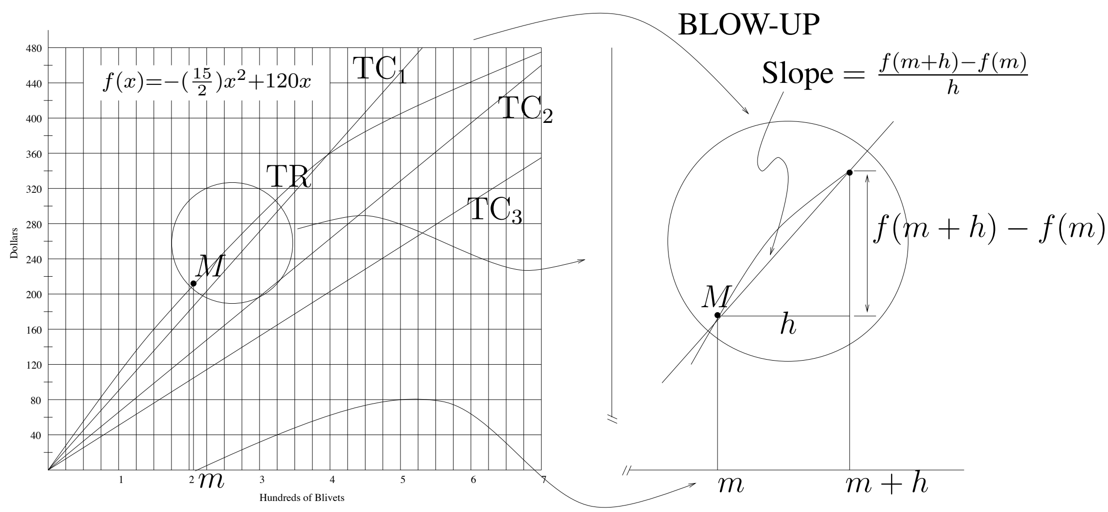
Chapter 2

Tangents and Formulas; The Derivative

Worksheet #7

The Formula for the Slope of the Tangent

In Worksheet #5 you found places where the tangent line to a graph had certain pre-assigned values. The graph used in Worksheet #5 is shown below. But actually there is a formula that belongs to the graph in Worksheet #5 and that is also shown below. We would like to use this formula to get more accurate readings of the slopes of the secant lines and tangent to the graph, and to use algebra to solve the Key Question of Worksheet #5. To the right below we have a blow-up of one tiny segment of the graph shown to the left.



To find the slope of the secant line shown in the blow-up, we just use the Substitution Rule in the formula for $f(x)$ and follow the recipe $\frac{f(m+h)-f(m)}{h}$.

Key Questions

I. Find the slope of the secant line from $x = 3$ to $x = 3.01$.

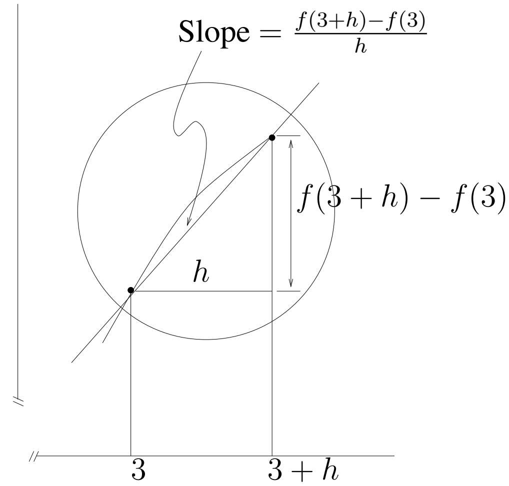
II. Find the slope of the tangent line at $x = 4$.

III. Find the slope of the tangent line at $x = m$.

$\boxed{1}$ Use the formula for $f(x)$ to evaluate $f(2)$ and $f(5)$. Check your answers on the graph.

$\boxed{2}$ Use the formula for $f(x)$ to evaluate $f(3)$. Then use this result and your answer to Question 1 to find the value of $\frac{f(5)-f(3)}{2}$. Then use the graph at the beginning of Worksheet #5 to check your answer. Repeat this with the number $\frac{f(3.5)-f(3)}{0.5}$.

The first two questions are a re-run of questions you answered in Worksheet #5. By using the Substitution Rule and a lot of calculations you get roughly the same answers to questions about slopes of secants that you got with the graph. The formula answers are more accurate.



3 Use the formula for $f(x)$ (and a calculator) to find the slope of the secant line from $x = 3$ to $x = 3.01$.

Thus you have answered Key Question I. In Questions 2 and 3 you found values of $\frac{f(3.5)-f(3)}{0.5}$ *and* $\frac{f(3.01)-f(3)}{0.01}$*. That is, you found slopes of secants between* $x = 3$ *and points nearby to the right of it. If I ask you for* $\frac{f(3.25)-f(3)}{0.25}$*, then you have to go back and do a lot of calculations all over again. Let's use algebra to save some steps. A point* h *units to the right of* $x = 3$ *is called* $x = 3 + h$.

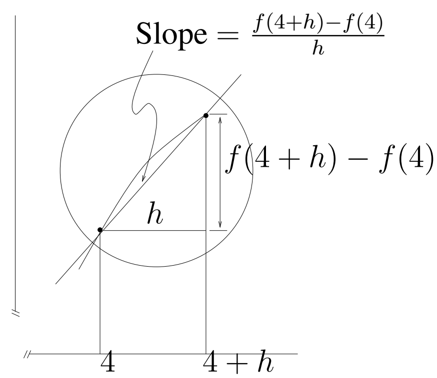
4 The height of the graph at $x = 3 + h$ cannot be found on the graph, because we have not specified h yet, but we can get a formula value for it in terms of h from the formula for $f(x)$. By the Substitution Rule evaluate $f(3 + h)$ and then write it out in the form $f(3+h) = (\quad\quad) + (\quad\quad)h + (\quad\quad)h^2$.

5 Use your answer to Question 4 in the recipe $\frac{f(3+h)-f(3)}{h}$ to write out a formula in terms of h for $\frac{f(3+h)-f(3)}{h} = (\quad\quad) + (\quad\quad)h$.

6 By setting $h = 0.5$ your answer to Question 5 should give you the value of $\frac{f(3.5)-f(3)}{0.5}$. Make the substitution and then check your answer to Question 2.

7 To find $\frac{f(3.01)-f(3)}{0.01}$ you must set $h = 0.01$ in your answer to Question 5. Do so. Check against your answer to Question 3.

As a way of approaching Key Question II, we will repeat at $x = 4$ *what we just did at* $x = 3$*. The point nearby and to the right of 4 will be denoted by* $4 + h$

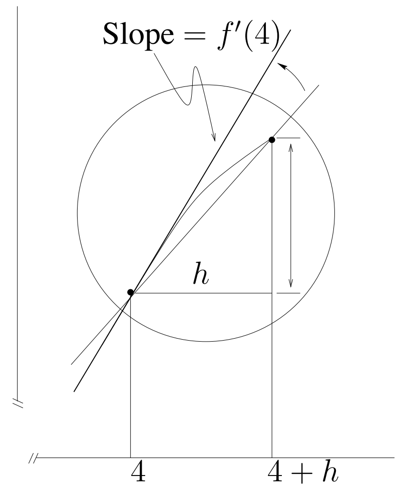


8 Use the Substitution Rule and some algebra to evaluate $f(4 + h)$ and write it in the form $f(4+h) = (\quad\quad) + (\quad\quad)h + (\quad\quad)h^2$.

9 Use your answer to Question 8 to write out a formula for $\frac{f(4+h)-f(4)}{h}$ in terms of h.

10 Use your answer to Question 9 to find the slopes of the secants $\frac{f(4.1)-f(4)}{0.1}$, $\frac{f(4.001)-f(4)}{0.001}$ and $\frac{f(4.00001)-f(4)}{0.00001}$.

The power of doing the algebra here is that we get a formula for slopes of all *secants to the right of $x = 4$—all at once, not just one at a time. We have not yet found the slope of the tangent, but we are close.*



11 As you let h take smaller and smaller values, the slopes of the secants $\frac{f(4+h)-f(4)}{h}$ get closer and closer to one number. This number seems like a good bet for the slope of the tangent at $x = 4$. Do you agree? Then, according to this, what is the value of $f'(4)$, the slope of the tangent at $x = 4$?

Thus we have answered Key Question II.

Suppose that, instead of giving you the specific points $x = 3$ or $x = 4$, I said, let's deal with a point $x = m$, and decide later on what its specific value will be. Then we would have the same picture and the same algebra, except that we would have to drag along the symbol "m" instead of computing with the numbers 3 or 4. But this is just a complication, not a whole new approach.

12 Imitate your work in Questions 4 and 8 to complete the formula

$$f(m+h) = (\qquad)m^2 + (\qquad)mh + (\qquad)h^2 + (\qquad)m + (\qquad)h.$$

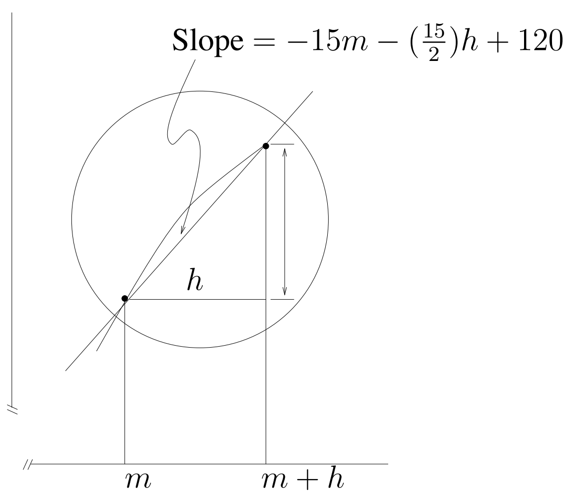
13 Imitate your work in Questions 5 and 9 to complete the formula

$$\frac{f(m+h)-f(m)}{h} = \frac{(\qquad)m^2 + (\qquad)mh + (\qquad)h^2 + (\qquad)m + (\qquad)h}{h}.$$

Then simplify your answer by canceling out some h's.

Thus you have shown that the slope of the secant line in the picture to the right is

$$-15m - (\frac{15}{2})h + 120.$$



14 In Questions 12 and 13 you worked with a general, unspecified point $x = m$, instead of the numerical values $x = 3$ or $x = 4$. Your answer to Question 13 should tell you about specific values when $x = 3$ or $x = 4$. Replace m by 3 in your answer to Question 13 and, after simplifying, check against your answer to Question 5. Then repeat this process with $x = 4$ to check your answer to Question 9.

$\boxed{15}$ In Question 11 you used the formula $\frac{f(4+h)-f(4)}{h} = 60 - (\frac{15}{2})h$ to deduce that $f'(4) = 60$. Now we have replaced $x = 4$ by $x = m$, and we have the formula

$$\frac{f(m+h) - f(m)}{h} = -15m - (\frac{15}{2})h + 120.$$

Imitate what you did at $x = 4$ to tell what $f'(m)$, the slope of the tangent at $x = m$, would be.

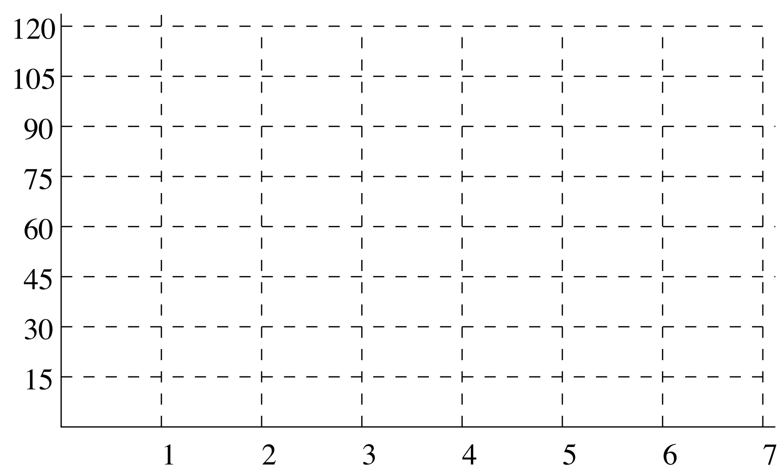
$\boxed{16}$ Your answer to Question 15 should be $f'(m) = -15m + 120$. If we replace m by 4, we should get $f'(4)$. Do this and check your answer to Question 11.

$\boxed{17}$ The formula $f'(m) = -15m + 120$ gives the slope of the tangent to the graph of $f(x)$ at the general, unspecified point m. If we let $m = 3$ or $m = 6$, we get particular slopes of tangents, i.e., $f'(3)$ and $f'(6)$. Use the formula to find these values and then check the answers by drawing tangent lines on the graph at the beginning of Worksheet #5.

$\boxed{18}$ The table below plots values of $f'(m)$ *vs.* m for various values of m.

m	0	1	2	3	4	5	6	7
$f'(m)$				75	60		30	

Complete the table using the formula $f'(m) = -15m + 120$. Then on the axes below sketch the graph that goes with the table.



Even though you used the formula to get the values of the table, the table is a table of slopes of tangents to the graph of $f(x)$. You would get the same table if you had drawn a lot of tangents and measured their slopes. Thus the graph you just drew is the derived graph of the graph we started with. The formula $f'(m) = -15m + 120$ is the formula for the derived graph.

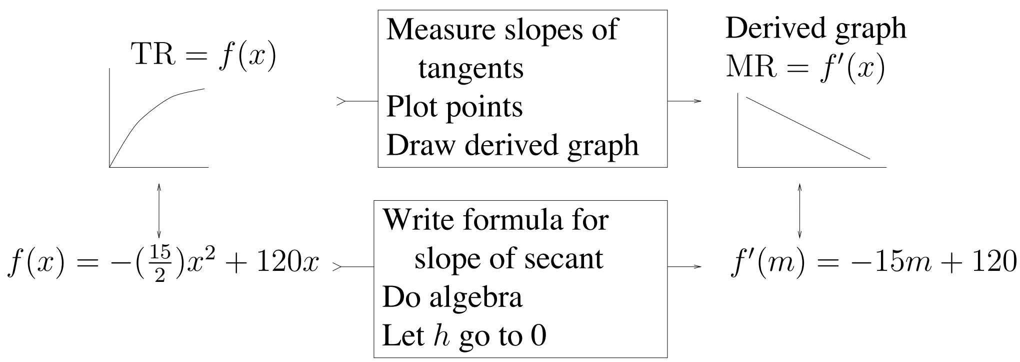
$\boxed{19}$ In Question 5 of Worksheet #5 you drew the MR graph that belongs to the TR graph of that worksheet. That is, you drew the derived graph of the graph given there. Check that the derived graph that you drew in Worksheet #5, by plotting slopes of tangents, is the same as the derived graph that you drew in Question 18 by using the formula.

Recall that the graph of $f(x)$ given at the beginning of this worksheet is the same as the graph of TR given in Worksheet #5. So, $f(x)$ and $TR(q)$ have the same formula (with different variables):

$$f(x) = \left(\frac{15}{2}\right) x^2 + 120x \text{ and } TR(q) = \left(\frac{15}{2}\right) q^2 + 120q.$$

Recall further that the slope of a tangent line to the TR graph is a value of MR. So, the graph of MR is the derived graph of TR.

Below is a schematic diagram of your work in Worksheet #5 and this worksheet. In Worksheet #5 you worked with graphs alone. In this worksheet you started with the formula and, using ideas from algebra, worked your way over to the formula for the derived graph.



Since $f(x)$ and $TR(q)$ have the same formulas (with different variables), all the algebra you did to find $f'(m)$ would apply to TR. That means that the formula for $TR'(m)$ would be $TR'(m) = -15m + 120$. But TR' is the same as MR. And we usually use the variable q, not m, in the formula for MR. That means that the formula for $MR(q)$ is

$$MR(q) = -15q + 120.$$

$\boxed{20}$ In Worksheet #5 we wanted values of q where the Marginal Revenue is equal to the Marginal Cost for each of the three Total Cost graphs—because those are the places where Profit is maximized for the various TC graphs. The three Marginal Costs are just the slopes of the three TC graphs. I read them as \$90/Hundred, \$66/Hundred and \$50/Hundred. Thus all the work of that worksheet was devoted to the following three questions:

a) For what value of q is $\text{MR} = 90$?

b) For what value of q is $\text{MR} = 66$?

c) For what value of q is $\text{MR} = 50$?

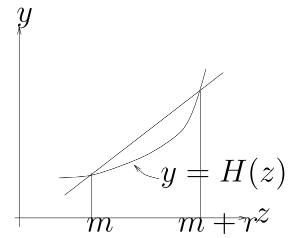
Use the formula for MR(q) to answer these three questions. Then check your answers against Questions 8, 9, and 10 of Worksheet #5.

→ $\boxed{21}$ a) To the right is a rough sketch of the graph of the function $y = H(z)$.

The slope of the secant line shown is

$$\frac{H(m+r) - H(m)}{r} = 4m + 2r - 3$$

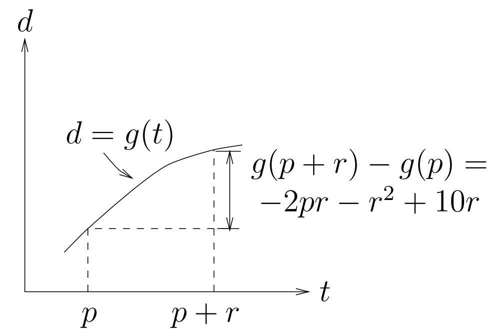
Write out a formula in terms of p for $H'(p)$.



b) To the right is a rough sketch of a portion of the distance *vs*. time graph for a car. As indicated in the picture, the distance covered by the car from time $t = p$ to time $t = p + r$ is

$$-2pr - r^2 + 10r.$$

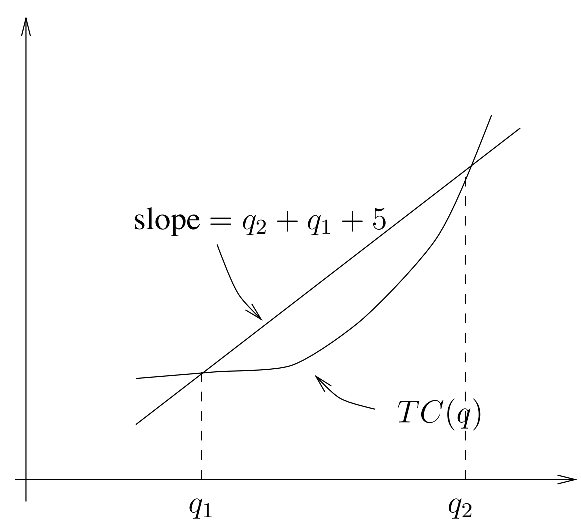
Write out a formula in terms of t for the instantaneous speed of the car at time t.



c) To the right is a rough sketch of a segment of the graph of Total Cost *vs*. quantity of manufacturing Blivets. The variable q is given in thousands of Blivets, and the Cost is given in Thousands of dollars. We do not have a formula for Total Cost *vs*. quantity, but only the formula for the slope of the secant shown:

$$\frac{TC(q_2) - TC(q_1)}{q_2 - q_1} = q_2 + q_1 + 5.$$

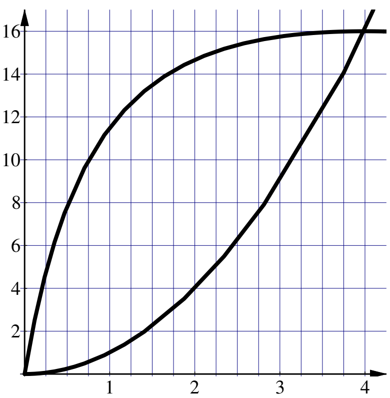
Write out a formula for $TC'(q)$, the Marginal Cost, in terms of q.

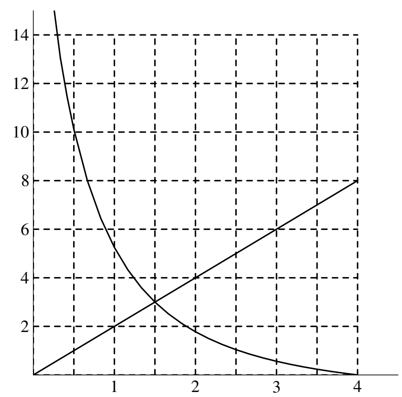


Worksheet #8

The Formulas for Two Speed Graphs

The two graphs to the left below are the distance *vs*. time graphs of Worksheet #4. To the right below are the speed *vs*. time graphs that you laboriously plotted in that worksheet.





Needless to say, the graphs I drew in Worksheet #4 are the graphs of functions with formulas. The formulas are:

$$\textit{Car A} : A(x) = x^2 \qquad\qquad \textit{Car B} : B(x) = -x - \frac{25}{x+1} + 25.$$

Since the speed graphs to the right are derived graphs, you ought to be able to follow the pattern of Worksheet #7 to compute the formula for the speed graphs (and thereby avoid that awful job of measuring and plotting thirty slopes of tangents). Below is a description of a general procedure for starting with a function $K(x)$, given by a formula, and finding the slope of the tangent, $K'(m)$, at a particular point $x = m$.

Procedure for Finding the Formula for $K'(m)$ from the Formula for $K(x)$

I. *Use the Substitution Rule to find $K(m+h)$. Then simplify.*

II. *Use algebra to write simple formula for $K(m+h) - K(m)$.*

III. *Write formula for slope of secant $\frac{K(m+h)-K(m)}{h}$.*

IV. *Simplify the answer to III. in such a way that you can tell what happens to it when h is made infinitely small. (This boils down to getting that h in the denominator to cancel out.)*

V. *Use IV to write out a formula for $K'(m)$.*

Key Question

In this worksheet your job will be to follow the procedure with the particular functions $A(x)$ and $B(x)$ connected with the graphs in Worksheet #4.

$\boxed{1}$ Follow Steps I to V, with $K(x)$ replaced with $A(x) = x^2$, to arrive at the conclusion that $A'(m) = 2m$.

> *We use the letter m to give the impression of working with a fixed value of x while we do all the algebra in the procedure for finding $A'(m)$. But there is no* real *difference between using the letter m and using the letter x. Thus the formula $A'(m) = 2m$ can be rewritten as $A'(x) = 2x$. Check that this is indeed the formula for the speed graph of Car A given at the beginning of this worksheet.*
>
> *Thus by algebra we are getting a formula that describes exactly the speed graph—rather than measuring lots of slopes of tangents and still getting a rough graph. The procedure for finding $B'(m)$ from $B(x)$ follows the general pattern. But, since this formula is a little more complicated, you may need some guidance with the algebra.*

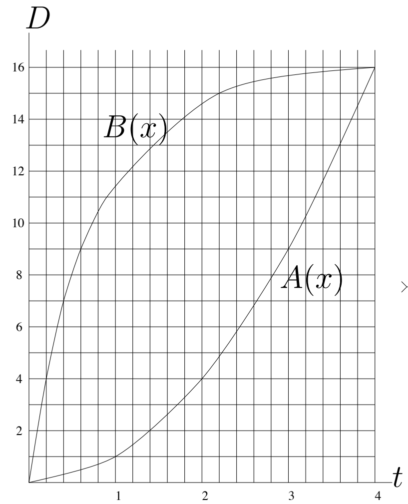
$\boxed{2}$ Perform Step I on $B(x)$. In performing Step II, use parentheses a lot to make sure that you do not lose track of the negative signs. Your answer to Step II should be $-h - \frac{25}{m+h+1} + \frac{25}{m+1}$. To make Step III easier, get a common denominator and combine the fractions. (Don't do Step III just yet.)

$\boxed{3}$ Step III just amounts to dividing by h. Remember that dividing by h is the same as multiplying by $\frac{1}{h}$. Multiply your simplified result from the previous problem by $\frac{1}{h}$. Your result is the slope of the secant line through the graph of $B(x)$ at $x = m$ and $x = m + h$.

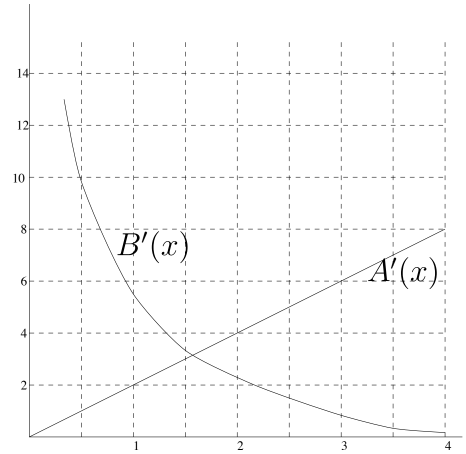
$\boxed{4}$ You should be all set to do Steps IV and V: tell what happens to the slopes as h gets very, very small? That is, what is the formula for $B'(m)$?

$\boxed{5}$ Switch the name of the variable from m to x in the formula for $B'(m)$, obtaining a formula for $B'(x)$, and find $B'(2)$, $B'(3)$, $B'(6)$, and $B'(7)$ by evaluating the new formula. Then check these values against the speed graphs given at the beginning of the worksheet.

> *Thus, by pure algebra, we get the formula for the derived graph of the graph of $B(x)$. A schematic representation of what we have done in Worksheet #4 and the present worksheet is shown in the next figure.*



Measure slopes of tangents
Plot Points
Draw derived graphs



$$A(x) = x^2$$
$$B(x) = -x - \frac{25}{x+1} + 25$$

Write formula for slope of secant
Do algebra
Let h go to 0

$$A'(x) = 2x$$
$$B'(x) = -1 + \frac{25}{(x+1)^2}$$

$\boxed{6}$ But algebra is good for much more than drawing graphs easily. Use the formula for $A'(x)$ to find the time at which the speed of Car A is 5 ft/min. Find the time at which the speed of Car B is 3 ft/min.

$\boxed{7}$ Key Question II of Worksheet #4 asks for the time at which the speeds of the two cars are identical. Set up the equation that one would solve in order to answer this question. Unfortunately, there are no elementary techniques for solving this equation. You might try to see how far you get. On the other hand, you can use the equation to check the answer you got to Key Question II in Worksheet #4. The answer I got was $x = 1.5$, and this checks perfectly.

$\boxed{8}$ Key Question III in Worksheet #4 asked for the value of x at which $B'(x) = A'(x) + 1$. Replace $B'(x)$ and $A'(x)$ by the formulas we have for them and, by using algebra, show that Key Question III of Worksheet #4 reduces to solving the equation $\frac{25}{(x+1)^2} = 2(x+1)$. Change this equation into $\frac{25}{2} = (x+1)^3$, and solve it. Check your answer against the answer given to Key Question III in Worksheet #4.

$\longrightarrow$ $\boxed{9}$ a) Suppose that $g(z) = 3z^2 + 2z$.

i. Write out a formula in terms of z and h for $g(z+h) - g(z)$. Write it in the form: $(\quad)h^2 + (\quad)h + (\quad)$.

ii. Find a formula for $g'(z)$.

iii. Plug $z = 1$ into the formulas for $g(z)$ and $g'(z)$ to find the values of $g(1)$ and $g'(1)$. Then use the approximation

$$g'(1) \approx \frac{g(1.0003) - g(1)}{0.0003}$$

to estimate the value of $g(1.0003)$ *without* plugging 1.0003 into the formula for $g(z)$.

b) Suppose the distance *vs*. time function for a car is given by

$$d = f(t) = -2t^2 + 12t.$$

i. Write out a formula for the distance covered by this car in the time $t = 3$ to time $t = 3 + h$. Simplify.

ii. Find the formula for the car's instantaneous speed: $f'(t)$.

iii. Use the approximation

$$f'(2) \approx \frac{f(2.003) - f(2)}{0.003}$$

to estimate $f(2.003) - f(2)$ *without* plugging 2.003 and 2 into the formula for $f(z)$.

c) The Total Revenue in thousands of dollars for selling q thousand Blivets is given by the formula

$$TR(q) = -q^2 + 20q.$$

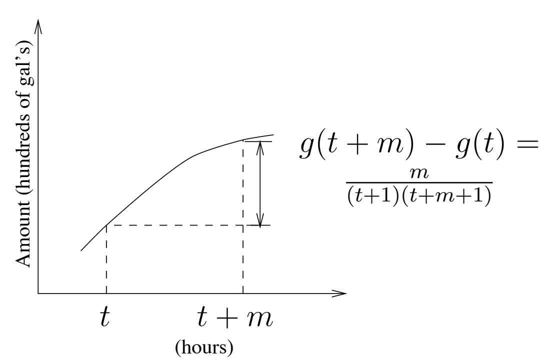
i. Write out a formula in terms of h for $TR(3 + h) - TR(3)$. Write your formula in the form $(\quad)h^2 + (\quad)h + (\quad)$.

ii. Approximate the additional total revenue that results from selling the 3,001-st Blivet.

$\longrightarrow$ $\boxed{10}$ Below is a rough sketch of part of the graph of the amount $g(t)$ of water (in <u>hundreds</u> of gallons) flowing into a reservoir as a function of time. We are not given the formula for $g(t)$, but instead are told that

$$g(t + m) - g(t) = \frac{m}{(t + 1)(t + m + 1)}.$$

a) Write out a formula in terms of t for the function that gives the instantaneous rate of flow of water at time t.



Now suppose that $A = H(t) = 2t^2 - 3t + 4$ gives amount of water *vs*. time for another reservoir.

b) Write out a formula in terms of r for the slope of the secant to the graph of $H(t)$ from $t = 1$ to $t = 1 + r$.

c) Find a formula for $H'(t)$.

d) Evaluate $H'(2)$ and $H(2)$. Then use them to approximate $H(2.0003)$.

Worksheet #9 *Taking Derivatives*

The point of this worksheet is for you to learn a mechanical process for simply writing down the formula for the derived graph, when you are given the formula for the graph you started with. There are no big-dome intellectual, mathematical problems here. What we have is a mindless technique that you should become very quick and accurate at applying. Thus far, we have worked out the following examples.

	<u>Worksheet #7</u>	<u>Worksheet #4, 8</u>	<u>Worksheet #4, 8</u>
Formula for "original" graph	$f(x) = -(\frac{15}{2})x^2 + 120x$	$A(x) = x^2$	$B(x) = -x - \frac{25}{(x+1)} + 25$
$\Big\downarrow$ D D D	$\Big\downarrow$ D D D	$\Big\downarrow$ D D D	$\Big\downarrow$ D D D
Formula for derived graph	$f'(x) = -15x + 120$	$A'(x) = 2x$	$B'(x) = -1 + \frac{25}{(x+1)^2}$

In each of these examples we began with the formula for the "original" graph, wrote out the formula for the slope of the secant, did some algebra, let h go to 0, and then got a new formula. Here is this procedure worked out for the function $C(x) = x^3$, keyed to the steps described in the Key Question to Worksheet #8.

I. $C(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

II. $C(x+h) - C(x) = 3x^2h + 3xh^2 + h^3$

III. $\frac{C(x+h)-C(x)}{h} = \frac{3x^2h+3xh^2+h^3}{h}$

IV. $\frac{(C(x+h)-C(x)}{h} = 3x^2 + 3xh + h^2$

V. $C'(x) = 3x^2$.

******* *The function that we get, the formula for the derived graph, is called the <u>derivative</u> of the function we started with. Thus, $3x^2$ is the derivative of x^3. Similarly, the derivative of $f(x) = -(\frac{15}{2})x^2 + 120x$ is $f'(x) = -15x + 120$.*

If you worked through this procedure for several dozen examples, you would notice patterns emerging in the relationship between the formula for the function and the formula for the derivative. They can be described as a series of rules, *which you will have to memorize*.

<u>THE POWER RULE</u> *If*

$$\begin{array}{ccc} f(x) & = & x^m \\ \Big\downarrow D\,D\,D & & \Big\downarrow D\,D\,D \\ f'(x) & = & m \cdot x^{m-1} \end{array}$$

, then

For example, if $A(x) = x^2$, then $A'(x) = 2x$, and if $C(x) = x^3$, then $C'(x) = 3x^2$. Likewise, if $g(x) = x^{13}$, then $g'(x) = 13x^{12}$. The rule says that to take the derivative of $f(x) = x^m$, put the exponent down in front of the x, and give x a new exponent that is one less than the old one.

$$f(x) = x^{m}$$

$$f'(x) = m \cdot x^{m-1}$$

In writing the Power Rule down, we think of it as being about powers of x like x^{13} and x^2. But it is about other powers of x as well:

$$\begin{array}{lcl} g(x) = \frac{1}{x^2} & = & x^{-2} \\ \Big\downarrow D\,D\,D & & \Big\downarrow D\,D\,D \\ g'(x) = & & -2 \cdot x^{-3} = -\frac{2}{x^3} \end{array} \qquad \begin{array}{lcl} h(x) = \sqrt{x} & = & x^{1/2} \\ \Big\downarrow D\,D\,D & & \Big\downarrow D\,D\,D \\ h'(x) = & & \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}} \end{array}$$

$\boxed{1}$ Use the Power Rule to find the derivatives of the following functions.

a) $f(x) = x^{11}$

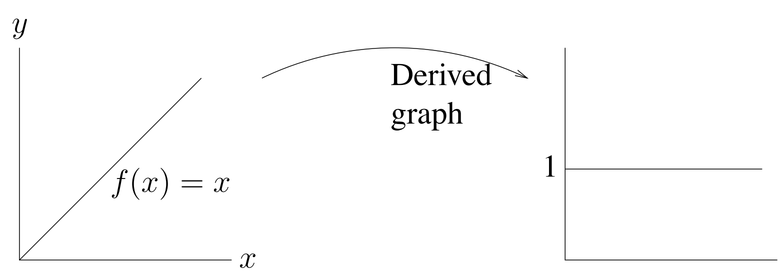
b) $g(x) = \frac{1}{x}$

c) $h(x) = \sqrt[3]{x}$

d) $k(x) = x$

e) $m(x) = 1$ (HINT: $1 = x^0$)

In part (d) of the last exercise, you found that the derivative of x is 1. That is not such a surprising fact, though, when you consider the graphs.



THE COEFFICIENT RULE *If* $f(x)=c \cdot g(x)$, *then*

$$\begin{array}{ccc} f(x) & = & c \cdot g(x) \\ \Big\downarrow D\,D\,D & \Big\downarrow & \Big\downarrow D\,D\,D \\ f'(x) & = & c \cdot g'(x) \end{array}$$

This rule says that if you are trying to write down a derivative, and there is some real number sitting in front of the formula, then just carry the real number along in your answer. For example:

$$\begin{array}{ccccc} d(x) = & 3 & \cdot & x^2 \\ \Big\downarrow D & \Big\downarrow & & \Big\downarrow D \\ d'(x) = & 3 & \cdot & 2x & = 6x \end{array} \qquad \begin{array}{ccccc} E(x) = & 15 & \cdot & x^5 \\ \Big\downarrow D & \Big\downarrow & & \Big\downarrow D \\ E'(x) = & 15 & \cdot & 5\cdot x^4 & = 75x^4 \end{array} \qquad \begin{array}{ccccc} L(x) = \frac{4}{x^2} = & 4 & \cdot & x^{-2} \\ \Big\downarrow D & \Big\downarrow & & \Big\downarrow D \\ L'(x) = & 4 & \cdot & (-2)x^{-3} & = -\frac{8}{x^3} \end{array}$$

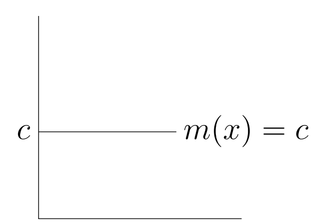
$\boxed{2}$ Use the Power Rule and the Coefficient Rule to compute the derivatives of the following functions:

a) $f(x) = 9x^8$
b) $g(x) = \frac{4}{x^3}$
c) $h(x) = 5\sqrt{x}$
d) $k(x) = 10x$
e) $m(x) = 12$ (HINT: $12 = 12x^0$)

In part (e) of this last exercise, there is nothing special about the number 12. You would have arrived at the same answer if $m(x)$ was *any* constant, say $m(x) = c$.

$$\begin{array}{ccccc} m(x) = & c\cdot 1 = & c & \cdot & x^0 \\ \Big\downarrow D & & \Big\downarrow & & \Big\downarrow D \\ m'(x) = & & c & & (0\cdot x^{-1}) = 0. \end{array}$$

That is, the derivative of c is 0, for any constant c. But this fact is also not too surprising, if you consider the graph. The graph of $m(x) = c$ is a horizontal straight line at $y = c$. The slope of the tangent to this graph is always 0. So the derived graph of $m(x)$ is the line $y = 0$, the x-axis.



<u>*THE SUM RULE*</u> *If*

$$\begin{array}{ccccc} f(x) & = & g(x) & + & h(x) \\ \Big\downarrow D & & \Big\downarrow D & & \Big\downarrow D \\ f'(x) & = & g'(x) & + & h'(x) \end{array}$$

, then

This rule says that if you begin with two functions, $g(x)$ and $h(x)$, and you want to write down the derivative of their sum, $f(x) = g(x) + h(x)$, then write down the derivatives of the two functions, and add them. For example:

$$
\begin{array}{ccccc}
d(x) & = & 3x^2 & + & \frac{5}{x} \\
\Bigg\downarrow D & & \Bigg\downarrow D & & \Bigg\downarrow D \\
d'(x) & = & 3 \cdot 2x & + & 5 \cdot (-\frac{1}{x^2}) \\
 & = & 6x - \frac{5}{x^2} & &
\end{array}
\qquad\qquad
\begin{array}{cccc}
f(x) = & -(\frac{15}{2})x^2 & + & 120x \\
\Bigg\downarrow D & \Bigg\downarrow D & & \Bigg\downarrow D \\
f'(x) = & -(\frac{15}{2})(2x) & + & 120x^0 \\
 & = -15x + 120 & &
\end{array}
$$

(This second formula should look familiar. You got this same formula for $f'(x)$ in Worksheet #7, but there you had to do it the hard way.)

$\boxed{3}$ Use the Power, Coefficient and Sum Rules to find derivatives of the following functions.

a) $f(x) = 3x^2 - 2x + 1$

b) $g(x) = 3\sqrt{x} + \frac{2}{x}$

c) $h(x) = 3x^4 - 5x^3 + \frac{2}{x^2}$

d) $k(x) = \frac{4}{\sqrt{x}} - \frac{3}{\sqrt{x^3}} + 12$

These three rules are just the start of the rules of differentiation—the rules for finding, or "taking," derivatives. The later rules get pretty bizarre. Notice that the function $B(x) = -x - \frac{25}{(x+1)} + 25$ does not yet fit our rules, because $\frac{25}{(x+1)}$ cannot be written as a power of x. Thus you cannot yet find its derivative by this mechanical technique.

This process of taking derivatives has been around for a long time and another notational system has grown up to describe it. You will need to know this system. Suppose that you write the function $f(x) = 3x^5 + 4x^2 + 5$ as $y = 3x^5 + 4x^2 + 5$ (that is, you use the variable y instead of functional notation), then instead of writing $f'(x) = 15x^4 + 8x$, you write $\frac{dy}{dx} = 15x^4 + 8x$. That is, the symbol $\frac{dy}{dx}$ replaces $f'(x)$. This symbol, $\frac{dy}{dx}$ is a single ("corporate") symbol. It does *not* mean some number dy divided by some number dx, or anything like that. Thus we would write

$$
\begin{array}{ccccc}
y & = & x^{3/2} & + & 4x \\
\Bigg\downarrow D & & \Bigg\downarrow D & & \Bigg\downarrow D \\
\frac{dy}{dx} & = & (\frac{3}{2})x^{1/2} & + & 4 \cdot 1 = \frac{3}{2}x^{1/2} + 4
\end{array}
$$

$\boxed{4}$ Find the derivatives of the functions described below. In some cases you may have to first do algebra to put the function in a form to be able to use the differentiation rules.

a) $f(t) = \frac{1+t^2}{t^3}$
b) $y = (1 + x^3)(1 - x)$
c) $z = \sqrt{u^3} - \sqrt{5}$
d) $g(x) = (1 + 2x^2)^2$
e) $w = (4t)^2 - 4 + \frac{1}{t}$
f) $f(z) = 16$
g) $y = \frac{16}{t^2}$

$\boxed{5}$ The three graphs in Worksheet #6 are given by the three formulas:

$$\begin{aligned} f(x) &= \frac{x^3}{3} - 4x^2 + 12x \\ g(x) &= \frac{x^3}{3} - 4x^2 + 16.5x - 16 \\ h(x) &= -\frac{x^3}{3} + 3x^2 - 5x + 5. \end{aligned}$$

Use the rules for differentiation to find $f'(x)$, $g'(x)$, and $h'(x)$. Then check that these are the formulas for the three derived graphs of these functions that you drew below Question 1 in Worksheet #6.

$\longrightarrow$ $\boxed{6}$ a) Find $\frac{dy}{dx}$, if $y = \sqrt{x} - \frac{x^3}{4}$.
b) Find $A'(r)$ if $A(r) = \frac{3}{r^2} + 5r^7$.
c) Find $F'(t)$ if $F(t) = (t - 6)(t + \frac{1}{t})$.
d) Find $\frac{dz}{dw}$ if $z = w^2(1 + w + \frac{1}{w})$.
e) Find $g'(t)$ if $g(t) = \frac{2}{\sqrt{t^3}} - \frac{3}{\sqrt[3]{t^2}}$.
f) Find $H'(w)$ if $H(w) = \sqrt{w} \cdot (1 - \frac{3}{w^2})$.
g) Find $\frac{ds}{dt}$ if $s = \frac{4t^3 - 3t^4 + t^5}{t^2}$.

$\longrightarrow$ $\boxed{7}$ Below are rough sketches of the graphs of two functions.

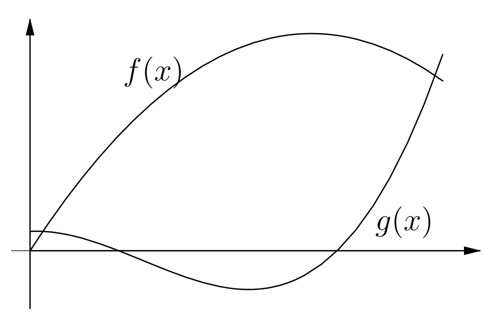
$$\begin{aligned} f(x) &= -2x^2 + 20x \\ g(x) &= \frac{x^3}{3} - 3x^2 + x + 10. \end{aligned}$$

a) Write out formulas for $f'(x)$ and $g'(x)$.

b) Find the value of x at which the slope of the tangent to the graph of $f(x)$ is 10.

c) Find the value of x at which the *derived* graph of $g(x)$ reaches its lowest value.

d) Find the positive value of x at which the function $P(x) = f(x) - g(x)$ is at its maximum value.



e) What is the maximum value reached by the function $P(x)$?

f) Give a value of x at which $f(x)$ is greater than $g(x)$, but $g'(x)$ is greater than $f'(x)$.

g) Write down an equation you would solve in order to find the value of x at which the slope of the graph of $f(x)$ is 4 units greater than the slope of the graph of $g(x)$. Do not solve the equation. Write it in the form: $(\quad)x^3 + (\quad)x^2 + (\quad)x + (\quad) = 0$. (Some of the coefficients may be 0.)

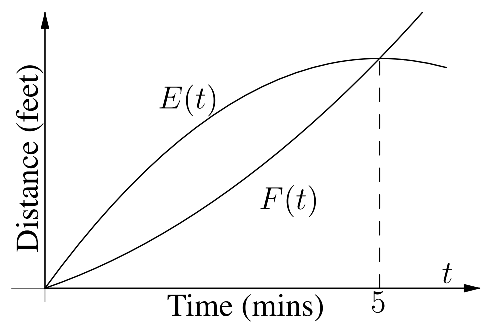
h) Give an interval of values of x over which $f'(x)$ is greater than 5 and less than 15.

$\longrightarrow$ $\boxed{8}$ Below are Distance *vs*. time graphs for two electronically controlled cars. The formulas for the graphs are:

$$E(t) = 20t - 2t^2 \qquad\qquad F(t) = t^2 + 5t.$$

a) Write an equation of the form $At^2 + Bt + C = 0$ that you would solve to determine the two times when the E-car is ahead of the F-car by 20 feet.

b) Find the longest interval you can over which the instantaneous speed of the E-car is greater than the instantaneous speed of the F-car.



c) True or False: (Give reasons.) The E-car is never ahead of the F-car by 45 feet.

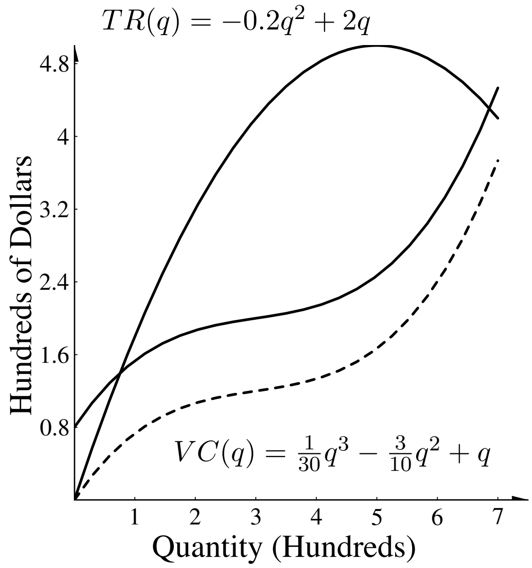
d) The average trip speed at time t of the E-car is defined by $E(t)/t$ = (Distance covered/time elapsed). Find a particular time t at which the average trip speed of the E-car at time t is equal to the instantaneous speed of the E-car at (the same) time t. If none such exists, tell why.

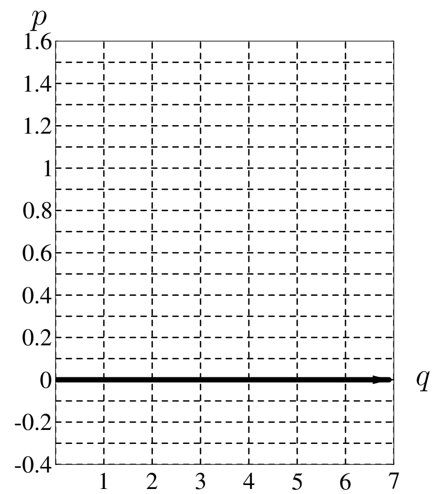
e) Draw a rough sketch of the speed graphs of the two cars. Indicate at least two particular points on each graph.

f) Find the time t at which the derived graph of the graph of the E-car is 3 ft/min higher than the derived graph of the graph of the F-car.

Worksheet #10
Using the Marginal Revenue and Marginal Cost Formulas

To the left below are the Total Revenue, Total Cost, and Variable Cost graphs for manufacturing Framits.





The formulas for TR and VC are given, so that, by taking derivatives, you can find the formulas for Marginal Revenue and Marginal Cost.

Key Questions

Find each of the following quantities.

A. The quantity q at which TR is greatest.

B. The quantity q at which MC is least.

C. The quantity q at which Profit is greatest.

$\boxed{1}$ First imagine that you had large, clear graphs of TR and TC, and you were going to measure slopes of many tangents and then sketch the derived graphs of these graphs (to get the MR and MC graphs). What would be the shapes of these two graphs? (Describe them in words.)

$\boxed{2}$ Now, in terms of the MR and MC graphs you described in Question 1, what would you be looking for to find each of the quantities asked for in the Key Questions?

$\boxed{3}$ Use derivative rules to write out the formula for $TR'(q)$. This is the formula for marginal revenue.

$\boxed{4}$ Look at the formula for MR. You should be able to tell, by just looking at it, what the MR graph looks like. Use the formula for MR to draw a very accurate MR graph on the blank axes given at the beginning of the Worksheet.

$\boxed{5}$ In Question 2 you described what to look for on the MR graph to find the value q at which TR is greatest. Use your accurate MR *graph* to determine this value.

$\boxed{6}$ Use the *formula* for MR to find the quantity q at which TR is greatest.

$\boxed{7}$ The formula for TR is a quadratic. Therefore, to find the highest point on the TR graph you should be able to use the Vertex Formula on the formula for TR: $TR(q) = -0.2q^2 + 2q$. Do so, and check your answer to Question 6.

$\boxed{8}$ According to the graphs, the TC graph is parallel to the VC graph, but 0.8 units higher. Thus the two formulas are:

$$\begin{aligned} VC(q) &= \tfrac{q^3}{30} - \tfrac{3}{10}q^2 + q \\ TC(q) &= \tfrac{q^3}{30} - \tfrac{3}{10}q^2 + q + 0.8 \end{aligned}$$

Using the rules of differentiation find the derivatives of $VC(q)$ and $TC(q)$. Tell, in terms of the graphs, why both formulas give the same MC formula.

$\boxed{9}$ Tell what the shape of the MC graph is.

$\boxed{10}$ Plot the values $VC'(0), VC'(1), VC'(2), \ldots, VC'(7)$ on the same axes as the MR graph above and then use these points to sketch the MC graph.

$\boxed{11}$ Look at your answer to Question 2 to find out what to look at on the MC graph to find the quantity q at which MC is lowest. Use your *sketch* of the MC graph to determine this quantity.

$\boxed{12}$ The formula for MC is a quadratic. Use the *Vertex Formula* on it to determine the quantity q at which MC is lowest.

$\boxed{13}$ The value q you found in Question 12 can be found using calculus instead of the Vertex Formula. You are looking for the value of q at which the graph of MC is at its low point. At this point the slope of the tangent to the MC graph is 0. Thus, to find the desired value, you should:

a) Find the derivative of the formula for MC. (That's right, it is a derivative of a derivative.)

b) Set the derivative equal to 0.

c) Solve the resulting equation.

Follow this procedure to find the value q at which MC is least. Check against your answer to Question 12.

$\boxed{14}$ The profit of manufacturing Framits is maximized when MR is equal to MC. Use your MR and MC *graphs* to estimate the value q at which MR is equal to MC.

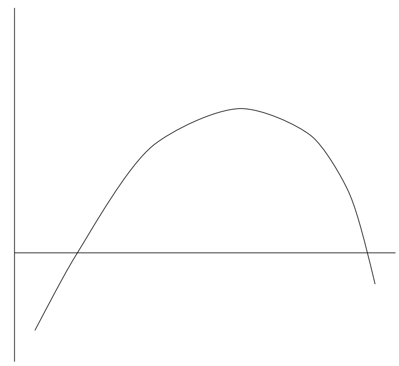
$\boxed{15}$ Next use the *formulas* for MR and MC to find the precise value at which profit is maximized. To solve this algebraically, you will have to set the formula for MR equal to the formula for MC, then simplify, and then use the Quadratic Formula to solve for q.

$\boxed{16}$ Here is another way to get the same answer you just got in Question 15. The graph of profit *vs*. quantity looks like the sketch below. Using the recipe

$$\text{Profit } = TR - TC$$

and the formulas for TR and TC, write out a formula for profit in terms of quantity q.

The formula for profit in terms of q is third degree. So we can not use something like the Vertex Formula on it to find where the high-point is. This is a problem that can only be solved using calculus. If we want the high-point of the graph shown, we want the value of q at which the graph has a horizontal tangent. The slope of the tangent to this graph is given by the formula for the derivative $\frac{dP}{dq}$.



$\boxed{17}$ Find the formula $\frac{dP}{dq}$ for the derivative of P with respect to q.

$\boxed{18}$ We want the value of q at which the slope of the tangent to the profit *vs*. q graph is 0. So we want q at which $\frac{dP}{dq} = 0$. When you set up this equation, notice its similarity to one of the equations you came up with in Question 15. By using the Quadratic Formula, solve your equation that came from $\frac{dP}{dq} = 0$. Compare your answer to your answer to Question 15.

$\longrightarrow$ $\boxed{19}$ The Total Revenue (in thousands of dollars) for selling q thousand Blivets is given by the formula $R(q) = -q^2 + 20q$.

a) Write out the formula for $R(3+h) - R(3)$. Put your formula in the form

$$(\quad)h^2 + (\quad)h + (\quad).$$

b) Find the additional Revenue that results from selling the 3,001-st Blivet.

$\longrightarrow$ $\boxed{20}$ The Total Revenue and Total Cost functions for manufacturing Items are:

$$TR(q) = -5q^2 + 80q \qquad\qquad TC(q) = q^3 - 12q^2 + 60q + 60,$$

where q is given in thousands of Items and Revenue and Cost are given in thousands of dollars.

a) Determine the quantity q at which MR is \$70.

b) Marginal Profit is defined as $\mathrm{MR} - \mathrm{MC}$. Write out an equation of the form

$$Aq^3 + Bq^2 + Cq + D = 0$$

that you would solve in order to find the quantity q at which Marginal Profit is \$10. (Do not solve the equation.)

c) Find the MR for *two* quantities q at which $\mathrm{MC} = \$39$.

d) For what quantity q is MC lowest?

e) What is the largest value of TR?

f) Give the longest possible interval of values for which MR is greater than \$40 and less than \$60.

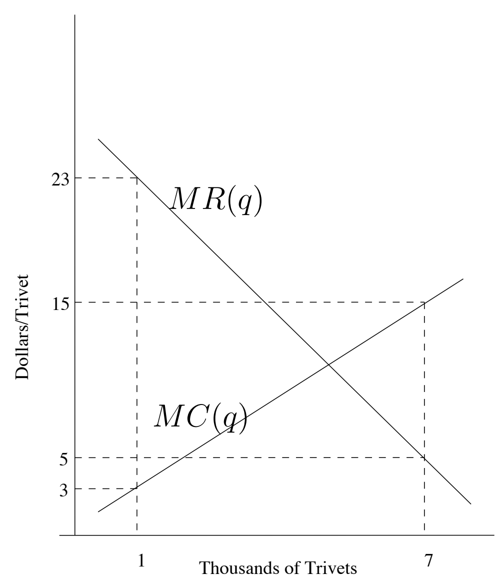
→ $\boxed{21}$ Below are the graphs of Marginal Cost and Marginal Revenue of manufacturing Trivets. Notice the units on the axes, but also recall that (dollars/trivet) is equal to (thousands of dollars/thousand trivets).

a) Write out the formulas for Marginal Revenue and Marginal Cost.

b) Write out likely candidates for the formulas for Total Revenue and Total Cost. (Assume that TR and TC of manufacturing 0 Trivets is \$0.)

c) Determine the value q at which the profit of manufacturing Trivets is greatest.

d) Choose the quantity q between 1 and 3 thousand Trivets that gives the greatest profit. Show your work and explain your answer.

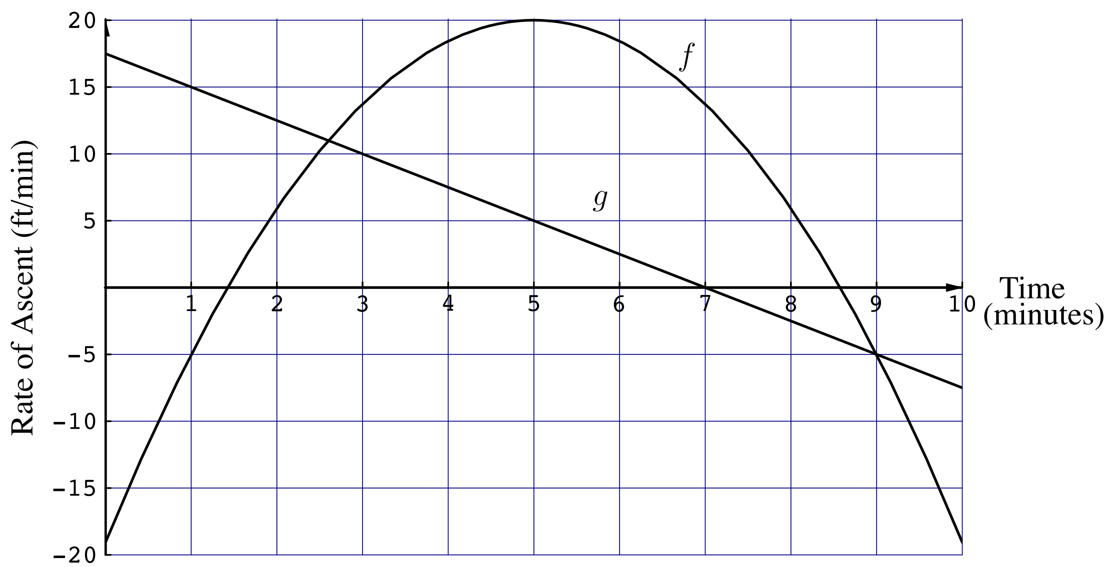


e) Suppose that you could increase Marginal Revenue by \$1.00 for every quantity of Trivets sold, so that your entire $MR(q)$ graph was raised by \$1.00. Would this influence the quantity at which profits are maximized? Explain.

f) What is the Cost of manufacturing the 3,001st Trivet?

Worksheet #11 ***Two Balloons***

The graphs below are of the instantaneous *rates* of ascent of two passenger balloons. (E.g., at time $t = 1$ min the g-balloon is ascending at a rate of 15 ft/min; the f-balloon is ascending at an approximate rate of -5 ft/min, which means that it is descending.) When we start watching the balloons at time $t = 0$, they are next to one another at 30 feet above the ground. We should be able to use these graphs to give us information about the elevations of the two balloons.



Key Question

Give a play-by-play description of the *relative* positions of the two balloons (e.g., "they are coming together," "the g-balloon is above the f-balloon at time …")

$\boxed{1}$ Right when we start watching the balloons, they are next to one another, 30 feet above the ground. The reading on the g-graph is 17.5 ft/min and the reading on the f-graph is -20 ft/min. What would you see the two balloons doing over the next few seconds?

$\boxed{2}$ The readings on the g-graph go down over the first 1.5 minutes. Does this mean that the g-balloon is going down? If not, what does it mean?

$\boxed{3}$ The readings on the f-graph go from -20 ft/min to 0 ft/min over the first 1.5 minutes. Interpret this fact in terms of what the f-balloon is actually doing.

$\boxed{4}$ Would you say that the f-balloon and the g-balloon are growing farther apart or closer together over the first 1.5 minutes? Justify your answer in terms of readings of the two graphs.

$\boxed{5}$ Now suppose I tell you that at 1.5 minutes the g-balloon is 54 feet higher than the f-balloon. Over the next 1.1 minutes both balloons are rising and the g-graph is always above the f-graph over that time interval. Would you say that the g balloon is getting farther away from the f-balloon, or that the two balloons are coming together?

The two rate graphs cross at 2.6 minutes. That means only that the two balloons have the same rate *of ascent at that time. It does* not *mean that the two balloons are next to one another. In fact, since the g-balloon has always been rising faster than the f-balloon, the two balloons are farther apart at 2.6 minutes than they have been at any time previously.*

$\boxed{6}$ If you started looking at the two balloons at 2.6 minutes, you would see the g-balloon quite far above the f-balloon. On the other hand, the g-graph is *below* the f-graph for the next few minutes. This means that the g-balloon is ascending more slowly than the f-balloon. Are the two balloons still getting farther apart, or are they getting closer together then?

By now you should see that, since these are rate *graphs, the usual ways of looking at graphs do not tell us very much about the actual elevations of the balloons, or of their relative positions.*

$\boxed{7}$ In each cell of the following table, put a "U" if the balloon is ascending (going up) and a "D" if the balloon is descending (going down) during the indicated time interval.

time interval	0–1.5	1.5–2.6	2.6–5	5–7	7–8.6	8.6–9	9–10
f							
g							

$\boxed{8}$ Explain why the following statement is true: "When the rate-of-ascent graph is crossing the horizontal axis, the balloon is changing direction (from up to down or from down to up)."

$\boxed{9}$ Explain why the following statement is true: "When the rate-of-ascent graph is crossing the horizontal axis, the balloon is either reaching a low point or a high point."

The two graphs given at the beginning of the worksheet are rate-of-ascent graphs. The questions would have been really easy if you had been given graphs of the altitudes vs. time of the two balloons over the 10-minute interval.

$\boxed{10}$ Suppose you had been given elevation *vs*. time graphs for the two balloons, and suppose you measured the slope of the tangent to the elevation graph of the g-balloon at time $t = 1$. Use the rate-of-ascent graph for balloon g to tell what the slope of that tangent would be.

$\boxed{11}$ The actual readings of the rate-of-ascent graph of the g-balloon are positive until $t = 7$, and then they become negative. How would this fact appear on the elevation graph of the balloon?

The moral is that since the two graphs that we started with are rate-of-ascent graphs, they are the derived graphs of the elevation vs. time graphs of the two balloons. That is, the information about the balloons is the reverse of what you are used to. For most of the previous worksheets, *we started with a particular graph (or graphs) and found the derived graph (or graphs). In this worksheet, we started with the derived graphs, and we asked questions about the graphs they must have come from.*

$\boxed{12}$ The rate-of-ascent graph of the f-balloon cuts through the t-axis twice. Tell what the elevation *vs*. time graph of f looks like near these two points. HINT: Go back and look at Questions 8 and 9.

$\boxed{13}$ The f-graph shown at the beginning of the worksheet is the derived graph of the elevation *vs*. time graph of the f-balloon. Thus, when the f-graph is positive, the elevation graph has positive slope, and when the f-graph is negative, the elevation graph has negative slope. Use this fact to give an overall description of the shape of the elevation graph of the f-balloon. After you have done this, compare your answer to your entries in the f-row of the chart from exercise #7.

$\boxed{14}$ The rate-of-ascent graph of the f-balloon reaches its peak at $t = 5$ minutes. What does this say about the elevation *vs*. time graph of the f-balloon at $t = 5$ minutes?

You have a systematic procedure for going from the elevation vs. time graphs of the two balloons to the rate-of-ascent graphs. You would simply draw the derived graphs. But you do not as yet have a systematic procedure for going the other way. What you need to be able to do now is make general statements about the elevation graphs from the rate-of-ascent graphs—the type of thing you did in Questions 5, 6, 8, and 9.

On the other hand, the following approach should make sense to you, based on what you have learned so far.

The rate-of-ascent graphs are the derived graphs of the elevation graphs. The rate-of-ascent graphs are given by the following formulas:

$$f(t) = -1.6t^2 + 16t - 20 \qquad\qquad g(t) = -2.5t + 17.5.$$

Therefore, these formulas must be the derivatives of the formulas connected with the elevation graphs. If we denote the elevation vs. time function for the f-balloon by $F(t)$, and the elevation vs. time function for the g-balloon by $G(t)$, then it must be that

$$F'(t) = -1.6t^2 + 16t - 20 \qquad\qquad G'(t) = -2.5t + 17.5.$$

But for instance, what must the formula for $G(t)$ look like, if $G'(t) = -2.5t + 17.5$? It must be that $G(t) = \frac{-2.5}{2}t^2 + 17.5t$. But that's not quite correct, because we know that $G(0) = 30$ feet. Thus, it must be that $G(t) = \frac{-2.5}{2}t^2 + 17.5t + 30$.

$\boxed{15}$ Check that, for this formula for $G(t)$, it is true that $G'(t) = -2.5t + 17.5$.

$\boxed{16}$ Making an educated guess, I write $F(t) = \frac{-1.6}{3}t^3 + \frac{16}{2}t^2 - 20t + 30$. Take the derivative of this formula to check that this is indeed a good guess.

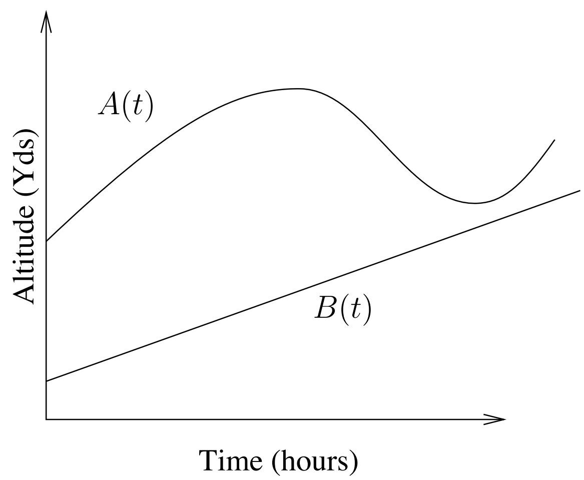
$\boxed{17}$ Draw very rough sketches of $G(t)$ and $F(t)$, and check the statements you made about relative distances between the two balloons. (e.g. Check your answers to Questions 5, 6, 8, and 9.)

$\longrightarrow$ $\boxed{18}$ The graphs below are of altitude (A, in yards) *vs*. time (t, in hours) for two weather balloons. The formulas for the graphs are

$$A(t) = t^3 - \tfrac{27}{2}t^2 + 54t + 25,$$
$$B(t) = 7t + 10.$$

a) Write out an equation of the form $Kt^3 + Nt^2 + Pt + Q = 0$ that you would solve in order to find the times t at which the instantaneous rates of ascent of the two balloons are equal.

b) Find the longest interval you can over which the A-balloon is descending.



c) Find the lowest altitude the A-balloon reaches in the time interval $t = 2$ to $t = 8$.

d) Are the balloons getting closer together or farther apart at time $t = 3$? Give reasons, using calculus.

e) Find the time t at which the (instantaneous) rate of ascent of the A-balloon is lowest.

f) Use the fact that $A'(6.5) = 5.25$ yds/hr and the derivative of $B(t)$ to tell which, if any, of the following statements is/are true.

   i) The two balloons are both rising and getting closer together at $t = 6.5$.

   ii) The two balloons are both rising but getting farther apart at $t = 6.5$.

   iii) The A-balloon is rising, but the A-balloon is lower than the B-balloon at $t = 6.5$ (so the picture is wrong!).

→ $\boxed{19}$ The graphs below are rate-of-ascent graphs of two balloons. The altitude *vs*. time functions will be denoted by $H(t)$ and $J(t)$. The two balloons are next to one another, 50 feet up, at time $t = 0$.

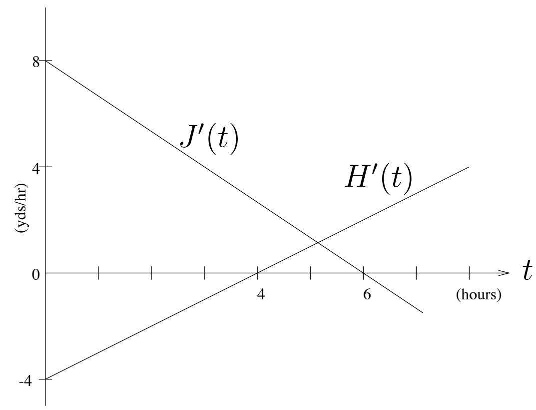
a) Each of the following statements is either true (T), false (F), or you cannot decide on the basis of the given information (DK). Circle the correct choice for each.

(i) $H(1) > H(2)$ T F DK

(ii) $J(2) > 0$ T F DK

(iii) At time $t = 5$ the J-balloon is above the H-balloon T F DK

(iv) The distance between the balloons is greater at 6 than at 5.5 hours T F DK



b) Find an interval of length 2 during which both balloons are going up. If none such exists, tell why.

c) Find a time when the slope of the graph of $H(t)$ is 2. If none exists, tell why.

d) Find the time in the first 6 hours when the two altitude graphs ($H(t)$ and $J(t)$) are farthest apart.

e) Find the time in the time interval $t = 2$ to $t = 4$ hours when the rate-of-ascent of the J-balloon is lowest.

Chapter 3

Differentiation Technique

Worksheet #12 *A Second Set of Rules*

In Worksheet #9 you learned the rules for finding derivatives of the most elementary functions. In this worksheet (and the following one) you will learn a set of rules for finding derivatives of just about any function you will ever encounter.

Part I. The Chain Rule

A. The examples $y = (x^2+3x+1)^4$, $y = \sqrt{x^2+3x+1}$, $y = \frac{1}{x^2+3x+1}$ all fall into the category $y = [f(x)]^n$, where $f(x) = x^2+3x+1$ and $n = 4$ in the first case, $n = 1/2$ in the second case, and $n = -1$ in the third case. The differentiation rule for this category is

GENERALIZED POWER RULE
If $y = [f(x)]^n$, then $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$

In the above examples $f(x) = x^2+3x+1$ and $f'(x) = 2x+3$. Thus

If $y = (x^2+3x+1)^4$, then $\frac{dy}{dx} = 4(x^2+3x+1)^3(2x+3)$.
If $y = (x^2+3x+1)^{1/2}$, then $\frac{dy}{dx} = \frac{1}{2}(x^2+3x+1)^{-1/2}(2x+3)$.
If $y = (x^2+3x+1)^{-1}$, then $\frac{dy}{dx} = (-1)(x^2+3x+1)^{-2}(2x+3)$.

Recall that $\frac{dy}{dx}$ is only one notation for derivatives. The first example above could be rewritten as

If $g(x) = (x^2+3x+1)^4$, then $g'(x) = 4(x^2+3x+1)^3(2x+3)$

or simply as

$$\frac{d}{dx}[(x^2+3x+1)^4] = 4(x^2+3x+1)^3(2x+3).$$

1 a) Find $\frac{dz}{dx}$, if $z = \sqrt{3x+\frac{1}{x}}$.

b) Find $\frac{d}{dt}[\frac{1}{3t^2-\sqrt{t}}]$.

c) Find $f'(v)$, if $f(v) = (\sqrt{v}+2v^2)^{1/3}$.

d) How does the Generalized Power Rule apply to the function $y = x^n$?

B. Recall the number "e" that came up in continuous compounding. It comes up in many other situations as well. The examples $y = e^x$, $y = e^{(0.01)x}$, and $y = e^{3x^2+1}$ all fall into the category $y = e^{g(x)}$, where $g(x) = x$ in the first example, $g(x) = (0.01)x$ in the second, and $g(x) = 3x^2+1$ in the third. The differentiation rule here is

EXPONENTIAL FUNCTION RULE

If $y = e^{g(x)}$, then $\frac{dy}{dx} = e^{g(x)}g'(x)$.

Thus in the above examples we have

If $y = e^x$, then $\frac{dy}{dx} = e^x \cdot \frac{d}{dx}(x) = e^x \cdot (1) = e^x$.
If $y = e^{(0.01)x}$, then $\frac{dy}{dx} = e^{(0.01)x} \cdot \frac{d}{dx}(0.01x) = e^{(0.01)x}(0.01)$.
If $y = e^{3x^2+1}$, then $\frac{dy}{dx} = e^{3x^2+1} \cdot \frac{d}{dx}(3x^2+1) = e^{3x^2+1}(6x)$.

$\boxed{2}$ a) Find $\frac{dz}{dx}$ if $z = e^{3x-\frac{1}{x}}$.
b) Find $\frac{d}{dt}[e^{3t^2-\sqrt{t}}]$.
c) Find $f'(v)$ if $f(v) = e^{\sqrt{v}+2v^2}$

C. Recall the logarithm function that came up at the end of Math 111. The examples

$$y = \ln(x) \qquad y = \ln((0.01)x) \qquad y = \ln(3x^2+1)$$

all fit into the category $y = \ln(g(x))$, where $g(x) = x$ in the first example, $g(x) = (0.01)x$ in the second, and $g(x) = 3x^2 + 1$ in the third. The differentiation rule here is

LOGARITHMIC FUNCTION RULE

If $y = \ln(g(x))$, then $\frac{dy}{dx} = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$.

Thus, in the above examples we have

If $y = \ln(x)$, then $\frac{dy}{dx} = \frac{1}{x} \cdot \frac{d}{dx}(x) = \frac{1}{x} \cdot (1) = \frac{1}{x}$.
If $y = \ln((0.01)x)$, then $\frac{dy}{dx} = \frac{1}{(0.01)x} \cdot \frac{d}{dx}(0.01x) = \frac{1}{(0.01)x} \cdot (0.01) = \frac{1}{x}$.
If $y = \ln(3x^2+1)$, then $\frac{dy}{dx} = \frac{1}{3x^2+1} \cdot \frac{d}{dx}(3x^2+1) = \frac{1}{3x^2+1} \cdot (6x) = \frac{6x}{3x^2+1}$.

$\boxed{3}$ a) Find $\frac{dz}{dx}$ if $z = \ln(3x - \frac{1}{x})$.
b) Find $\frac{d}{dt}[\ln(3t^2 - \sqrt{t})]$.
c) Find $f'(v)$ if $f(v) = \ln(\sqrt{v} + 2v^2)$.

We used an arrow notation to write out derivatives in Worksheet #9. E.g.,

$$\begin{array}{ccccc} y & = & 3x^2 & + & 4x \\ \Big\downarrow D\,D\,D & & \Big\downarrow D\,D\,D & & \Big\downarrow D\,D\,D \\ \frac{dy}{dx} & = & 6x & + & 4 \end{array}.$$

We can use the same notation here to describe the three rules we have given thus far, but we will embellish the notation somewhat. An arrow $\Big\downarrow C\,C\,C$ means "copy the same thing below" (as opposed to an arrow with D's which means "write down the derivative below." Then we have

Generalized Power Rule

$$y = [f(x)]^{n}$$

$$D\,D\,D \qquad C\,C\,C \qquad D\,D\,D$$

$$\frac{dy}{dx} = n[f(x)]^{n-1} f'(x)$$

Exponential Function Rule

$$y = e^{g(x)}$$

$$D\,D\,D \qquad C\,C\,C \qquad D\,D\,D$$

$$\frac{dy}{dx} = e^{g(x)} g'(x)$$

Logarithmic Function Rule

$$y = \ln(g(x))$$

$$D\,D\,D \qquad D\,D\,D \qquad C\,C\,C$$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)}$$

Notes

1. You might well ask where these rules came from. The answer is that they came from the same place as the rules that you had in Worksheet #9. That is, you start with a particular function $y = f(x)$ (e.g., $f(x) = e^{g(x)}$), you write out the formula for the slope of the secant, $\frac{f(x+h)-f(x)}{h}$, you do some strange fiddling, and then you let h go to 0. When you play this game with functions like $y = x^{12}$, you can effectively *set* $h = 0$ at the right moment. That does not work with these new functions.

2. The three rules given thus far are special cases of an even more general rule, the *chain rule*, which says

$$y = A\,(f(x))$$

$$D\,D\,D \qquad D\,D\,D \quad C\,C\,C \qquad D\,D\,D$$

$$\frac{dy}{dx} = A'(f(x)) \cdot f'(x)$$

We have no need of the general chain rule here, other than the three particular cases given thus far. But when we are applying any of these three particular rules we will say that we are applying the chain rule.

$\boxed{4}$ Use the above rules to find the derivatives of the following functions.

a) $y = e^{\sqrt{x}+2x}$

b) $f(t) = (t^2 - \frac{1}{t})^{1/4}$

c) $z = \frac{1}{(2t+1)^4}$

d) $g(w) = \ln[\frac{1}{w} - \frac{1}{w^2}]$

e) $y = (x + e^x)^2$

f) $z = e^{\sqrt{1+x}}$

g) $h(r) = \ln(r + e^r)$

h) $w = (1 + \ln(z) + e^z)^4$

Part II. The Product and Quotient Rules

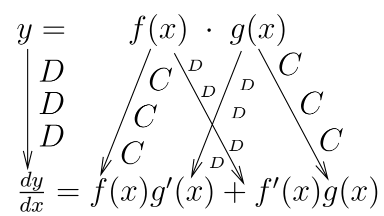
Now we have rules of a different type. In Exercise #4 of Worksheet #9 you found the following derivatives:

(a) If $f(t) = \frac{1+t^2}{t^3}$, then $f(t) = t^{-3} + t^{-1}$, and $f'(t) = -\frac{3}{t^4} - \frac{1}{t^2}$.

(b) If $y = (1 + x^3)(1 - x)$, then $y = 1 + x^3 - x - x^4$, and $\frac{dy}{dx} = 3x^2 - 1 - 4x^3$.

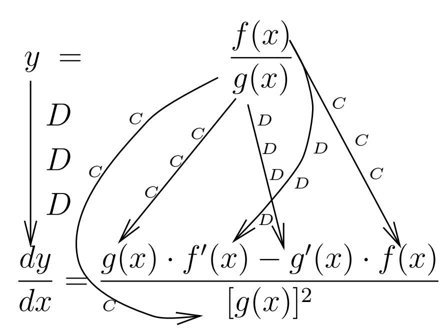
Exercise (b) falls into the category $y = f(x)g(x)$ (where $f(x) = 1+x^3$, and $g(x) = 1-x$). That is, it is a *product* of two functions. Exercise (a) falls into the category $y = \frac{f(t)}{g(t)}$, (where $f(t) = 1 + t^2$, and $g(t) = t^3$). That is, it is the *quotient* of two functions. We have a rule for each of these two types of situations. I will state the rules with the arrow notation of Worksheet #9 and also in the more conventional derivative notations.

PRODUCT RULE



If $y = f(x) \cdot g(x)$, then $\frac{dy}{dx} = f(x)g'(x) + f'(x)g(x)$.

QUOTIENT RULE



If $y = \dfrac{f(x)}{g(x)}$, then $\dfrac{dy}{dx} = \dfrac{g(x) \cdot f'(x) - g'(x) \cdot f(x)}{[g(x)]^2}$

Observe that with both the product and quotient rules you copy each function and combine it with the derivative of the other. The order in which you do this does not matter in the Product Rule, but it makes a big difference in the Quotient Rule. However, I have found that the best way to avoid making mistakes when applying these rules is to adopt one particular order in which you do things and never vary it. Applying these rules has to become automatic, almost unconscious, for you. Now we can apply the rules to the exercises we began this section with.

(b) $y = (1+x^3)(1-x)$

$$\frac{dy}{dx} = (1+x^3)(-1) + (3x^2)(1-x)$$

(a) $f(t) = \dfrac{1+t^2}{t^3}$

$$f'(t) = \frac{t^3(2t) - (3t^2)(1+t^2)}{(t^3)^2} = \frac{2t^4 - 3t^2 - 3t^4}{t^6} = \frac{-3}{t^4} - \frac{1}{t^2}$$

Here are four more worked examples. The first two could have been done in Worksheet #9 and the last two not.

$$y = \sqrt{t}\ (t+3t^2)$$

$$\frac{dy}{dt} = \sqrt{t}\ (1+6t) + (\tfrac{1}{2})t^{-1/2}(t+3t^2)$$

$$f(t) = \frac{\sqrt{t}}{t+3t^2}$$

$$f'(t) = \frac{(t+3t^2)(\frac{1}{2}t^{-1/2}) - (1+6t)(\sqrt{t})}{(t+3t^2)^2}$$

$$h(t) = \sqrt{t+1}\cdot(2t^2-4t^3)$$

$$h'(t) = \sqrt{t+1}(4t-12t^2) + (\tfrac{1}{2})(t+1)^{-1/2}(2t^2-4t^3)$$

$$z = \frac{e^x}{\sqrt{x}}$$

$$\frac{dz}{dx} = \frac{\sqrt{x}\cdot e^x - (\frac{1}{2})x^{-1/2}\cdot e^x}{(\sqrt{x})^2}$$

5 Find the derivatives of the following functions.

a) $y = (1 + x^2 - x^3)(\sqrt{x} - 2)$

b) $f(r) = \dfrac{1 - 3r^2 + r}{3r + r^3}$

c) $g(z) = (1 + e^z + z^2)(1 + z^3)$

d) $w = \dfrac{e^t + 2t}{t^{1/3}}$

e) $A(w) = \dfrac{e^w + \sqrt{w}}{\ln(w)}$

f) $y = (\ln(x) + \sqrt{x})(\sqrt{x + 2})$

g) $f(t) = t^2(t^4 + t^2 + 1) + 3t$

h) $y = \dfrac{x^2 + 2x}{x^2 + 4x + 2} + 3x^2$

i) $g(z) = \ln(z^2 + 1)$

j) $h(w) = e^{3w^2+4w+2} + w^3$

k) $z = (u^2 + 3u + 4)^4$

l) $f(x) = (x^4 - 3x^2 + 2x)(2 + x^3 - 5x^5)$

m) $y = (-3t^2 + 4t + 5)e^t + \ln t$

n) $g(x) = \dfrac{1}{(x^4 - x^3 + 2x)^4}$

Worksheet #13 *Combining Differentiation Rules*

You now have all the rules necessary to find all the derivatives of all the functions you are ever likely to encounter. The only difficulty is that some functions are put together in such a way that it is not clear how to use the rules to find their derivatives.

Part I. Multiple Use of the Product and Quotient Rules.

Just to remind you, here's the Product Rule again:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\cdot\frac{d}{dx}[g(x)] + g(x)\cdot\frac{d}{dx}[f(x)].$$

And the Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}[f(x)] - f(x)\cdot\frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Suppose you have the function $y = (x^2+2x)(e^x)(\sqrt{x})$. You have the Product Rule that covers the case of *two* expressions being multiplied together, but here we have *three* expressions multiplied together. The secret is to take two of the expressions among x^2+2x, e^x, and $\sqrt{x}$, lump them together, and momentarily think of them as one thing. I will lump x^2+2x and e^x, but any other pair will do.

$$\begin{aligned} y &= [(x^2+2x)(e^x)](\sqrt{x}) \\ \frac{dy}{dx} &= [(x^2+2x)(e^x)]\cdot\frac{d}{dx}(\sqrt{x}) + (\sqrt{x})\cdot\frac{d}{dx}[(x^2+2x)(e^x)] \end{aligned}$$

Now, the derivative of $\sqrt{x}$ is $\frac{1}{2}x^{-1/2}$. But, in order to take the derivative of $(x^2+2x)(e^x)$, we must apply the product rule for a second time:

$$\begin{aligned} \frac{dy}{dx} &= [(x^2+2x)(e^x)](\frac{1}{2}x^{-1/2}) + (\sqrt{x})\left[(x^2+2x)\cdot\frac{d}{dx}(e^x) + (e^x)\cdot\frac{d}{dx}(x^2+2x)\right] \\ &= (x^2+2x)(e^x)(\frac{1}{2}x^{-1/2}) + (\sqrt{x})[(x^2+2x)(e^x) + (e^x)(2x+2)] \end{aligned}$$

The common feature of the differentiation problems of this worksheet is this: *You are in the middle of applying one rule and you have to apply a different (or the same) rule*. Here is another worked example:

$$\begin{aligned} f(t) &= \frac{(t^2+1)(\ln t)}{t^3} \\ f'(t) &= \frac{(t^3)\cdot\frac{d}{dt}[(t^2+1)(\ln t)] - [(t^2+1)(\ln t)]\cdot\frac{d}{dt}(t^3)}{(t^3)^2} \end{aligned}$$

Like last time, the derivative of t^3 is easy, but taking the derivative of $(t^2+1)(\ln t)$ requires using the product rule in the middle of the quotient rule:

$$\begin{aligned} f'(t) &= \frac{(t^3)[(t^2+1)\cdot\frac{d}{dt}(\ln t)+(\ln t)\cdot\frac{d}{dt}(t^2+1)]-(t^2+1)(\ln t)(3t^2)}{t^6} \\ &= \frac{(t^3)[(t^2+1)(\frac{1}{t})+(\ln t)(2t)]-(t^2+1)(\ln t)(3t^2)}{t^6} \end{aligned}$$

$\boxed{1}$ Below are two differentiation problems. In each case I have grouped the expressions in two different ways. For each problem use the two groupings to find the derivatives in two ways, and then use algebra to show that both answers are the same.

a) $g(t) = t^3(\sqrt{t+1})(e^t)$
First way: $g(t) = [t^3(\sqrt{t+1})](e^t)$
Second way: $g(t) = (t^3)[(\sqrt{t+1})(e^t)]$

b) $y = \dfrac{e^t}{t^3\sqrt{t+1}}$
First way: $y = \dfrac{e^t}{[t^3(\sqrt{t+1})]}$
Second way: $y = \dfrac{1}{t^3}\cdot\left[\dfrac{e^t}{\sqrt{t+1}}\right]$

<u>Answers</u>

i) First way: $g'(t) = [t^3(\sqrt{t+1})]e^t + [3t^2\sqrt{t+1}+t^3\cdot\frac{1}{2}(t+1)^{-1/2}]e^t$

ii) First way: $\frac{dy}{dt} = \dfrac{t^3(\sqrt{t+1})e^t - [3t^2\sqrt{t+1}+t^3\frac{1}{2}(t+1)^{-1/2}]e^t}{[t^3(\sqrt{t+1})]^2}$

Part II. Multiple Use of the Chain Rule.

We want to find the derivatives of such functions as $y = e^{[(2x^2+1)^{1/2}]}$ and $f(t) = [\ln(1+t^2)]^4$. Both of these problems require the Chain Rule, but there are so many symbols and letters flying around that it is difficult to figure out where to start. Here is some advice to get started with such a problem.

Try to describe the *essence* of the function. For example, the function $f(t) = [\ln(1+t^2)]^4$ could be described as "something raised to the fourth power." Therefore, you'd start the derivative process with the Generalized Power Rule:

$$\frac{d}{dt}[g(t)]^4 = 4[g(t)]^3\cdot\frac{d}{dt}[g(t)].$$

In this case,

$$\begin{aligned} f(t) &= [\ln(1+t^2)]^4 \\ f'(t) &= 4[\ln(1+t^2)]^3 \cdot \frac{d}{dt}[\ln(1+t^2)]. \end{aligned}$$

Now you have to take the derivative of $\ln(1+t^2)$. Again, try to describe the *essence* of $\ln(1+t^2)$: it's the natural log of something. The rule for taking derivatives of natural logs is:

$$\frac{d}{dt}[\ln(g(t))] = \frac{1}{g(t)} \cdot \frac{d}{dt}[g(t)].$$

So far we have:

$$\begin{aligned} f'(t) &= 4[\ln(1+t^2)]^3 \cdot \frac{d}{dt}[\ln(1+t^2)] \\ &= 4[\ln(1+t^2)]^3 \cdot \frac{1}{1+t^2} \cdot \frac{d}{dt}(1+t^2). \end{aligned}$$

All we have left to do is to take the derivative of $1+t^2$, which isn't too hard. So,

$$f'(t) = 4[\ln(1+t^2)]^3 \cdot \frac{1}{1+t^2} \cdot (2t).$$

Let's go through the same process to take the derivative of $y = e^{\sqrt{2x^2+1}}$. You would have to describe this function as "e to the something." The derivative rule for exponential functions is:

$$\frac{d}{dx}[e^{g(x)}] = e^{g(x)} \cdot \frac{d}{dx}[g(x)].$$

So, if $y = e^{\sqrt{2x^2+1}}$, then

$$\begin{aligned} \frac{dy}{dx} &= e^{\sqrt{2x^2+1}} \cdot \frac{d}{dx}[\sqrt{2x^2+1}] \\ &= e^{\sqrt{2x^2+1}} \cdot \frac{d}{dx}[(2x^2+1)^{1/2}]. \end{aligned}$$

Now, $(2x^2+1)^{1/2}$ is "something to the $1/2$ power" and, thus, you should apply the Generalized Power Rule next:

$$\begin{aligned} \frac{dy}{dx} &= e^{\sqrt{2x^2+1}} \cdot \frac{1}{2}(2x^2+1)^{-1/2} \cdot \frac{d}{dx}(2x^2+1) \\ &= e^{\sqrt{2x^2+1}} \cdot \frac{1}{2}(2x^2+1)^{-1/2} \cdot (4x). \end{aligned}$$

(Are you getting an idea of why this is called the Chain Rule?)

$\boxed{2}$ Find the derivatives.

a) $g(t) = \ln[(1+t^2)^4]$
b) $h(t) = [1+(\ln t)^2]^4$
c) $y = e^{[(3x+x^2)^2]}$
d) $f(t) = \sqrt{\ln(t^2+1)}$
e) $z = \ln[\sqrt{e^x}]$

Part III. Combining All the Rules.

Finally consider the following two functions:

$$y = (e^{\sqrt{x}} \cdot x^2)^3 \text{ and } f(t) = \sqrt{\frac{t^4+1}{3t-1}}.$$

We want to find their derivatives, but again, we need to apply several rules, and it is not clear which order to apply the rules in. For instance, the first function requires the Product Rule (because it involves $e^{\sqrt{x}} \cdot x^2$), and it requires the Generalized Power Rule (because it contains a third power). Again, to know which order to apply the rules in, we think of the *essence* of the function. The first function should be viewed as "something to the third power," and we start by using the Generalized Power Rule. As we are doing that, we will have to use the Product Rule. While we are in the midst of using the Product Rule we will have to find the derivative of $e^{\sqrt{x}}$, and this will require the Exponential Functions Rule. That is

$$\begin{aligned}
\frac{dy}{dx} &= 3(e^{\sqrt{x}} \cdot x^2)^2 \frac{d}{dx}(e^{\sqrt{x}} \cdot x^2) \\
&= 3(e^{\sqrt{x}} \cdot x^2)^2 \left[e^{\sqrt{x}} \cdot \frac{d}{dx}(x^2) + (x^2) \cdot \frac{d}{dx}(e^{\sqrt{x}}) \right] \\
&= 3(e^{\sqrt{x}} \cdot x^2)^2 \left[e^{\sqrt{x}}(2x) + (x^2)(e^{\sqrt{x}}) \cdot \frac{d}{dx}(\sqrt{x}) \right] \\
&= 3(e^{\sqrt{x}} \cdot x^2)^2 \left[e^{\sqrt{x}}(2x) + (x^2)(e^{\sqrt{x}}) \cdot (\frac{1}{2}x^{-1/2}) \right]
\end{aligned}$$

Similarly, $f(t) = \sqrt{\frac{t^4+1}{3t-1}} = \left(\frac{t^4+1}{3t-1}\right)^{1/2}$ must be thought of as "something to the $1/2$ power." So, we'll start with the Generalized Power rule. But in the midst of doing that, we'll discover that we must use the Quotient Rule as well:

$$\begin{aligned}
f'(t) &= \frac{1}{2}\left(\frac{t^4+1}{3t-1}\right)^{-1/2} \cdot \frac{d}{dt}\left(\frac{t^4+1}{3t-1}\right) \\
&= \frac{1}{2}\left(\frac{t^4+1}{3t-1}\right)^{-1/2} \cdot \left[\frac{(3t-1)\frac{d}{dt}(t^4+1) - (t^4+1)\frac{d}{dt}(3t-1)}{(3t-1)^2} \right] \\
&= \frac{1}{2}\left(\frac{t^4+1}{3t-1}\right)^{-1/2} \cdot \left[\frac{(3t-1)(4t^3) - (t^4+1)(3)}{(3t-1)^2} \right]
\end{aligned}$$

3 Find the derivatives of the following functions.

a) $y = e^x(\ln(x+1))$

b) $f(t) = \dfrac{(1+x^2)^3}{x^4}$

c) $z = e^{2t^2+3t}$

d) $g(v) = (v^2+1)(\ln v)e^v$

e) $w = \dfrac{u^2+1}{u\ln(u)}$

f) $h(x) = \ln(e^x + 2x)$

g) $z = \sqrt{x^2 \cdot e^x + 1}$

h) $f(w) = (w\ln(w)+1)^4$

i) $y = \dfrac{(\ln x)\sqrt{x+1}}{x}$

j) $g(z) = (\sqrt{z} + 2z^2) \cdot e^2$

k) $g(v) = [\ln(v+1)]e^{v+1} + v^2$

l) $z = e^{\sqrt{2x^2+1}}$

m) $y = \dfrac{(t^2+1)(t^3+1)}{t}$

n) $h(x) = e^{\sqrt{x}} \cdot (x^2+1)$

o) $z = \sqrt{e^{t^2}+1}$

p) $f(t) = \dfrac{\ln(t^2+1)}{t^{1/2}}$

q) $g(x) = e^{\sqrt{x}} \cdot (x + \ln(x))$

r) $y = \dfrac{\ln(t^2+1)}{3t+1}$

s) $z = \dfrac{x \cdot e^x}{\sqrt{x+1}}$

t) $y = x\sqrt{x+1} \cdot (\ln x)$

u) $g(z) = \dfrac{z^2}{\sqrt{z^2+1}}$

v) $h(v) = \dfrac{v^4(v+1)^{4/3}}{v^{2/3}}$

w) $z = (\ln x)(x^2 + 3x^4 - e^x)$

x) $R(q) = \dfrac{q^2+2q+1}{e^q(q^3+3q)}$

y) $Q = (t^2+2t)(t^3-3t) + e^t \cdot t$

z) $f(x) = \dfrac{x^2}{\sqrt{x+1}} - x^3\ln(x)$

Chapter 4

Optimization

Worksheet #14 *Local and Global Optima*

Key Question

The total revenue and total cost (both in hundreds of dollars) for selling q hundred Shrubnods are given by:

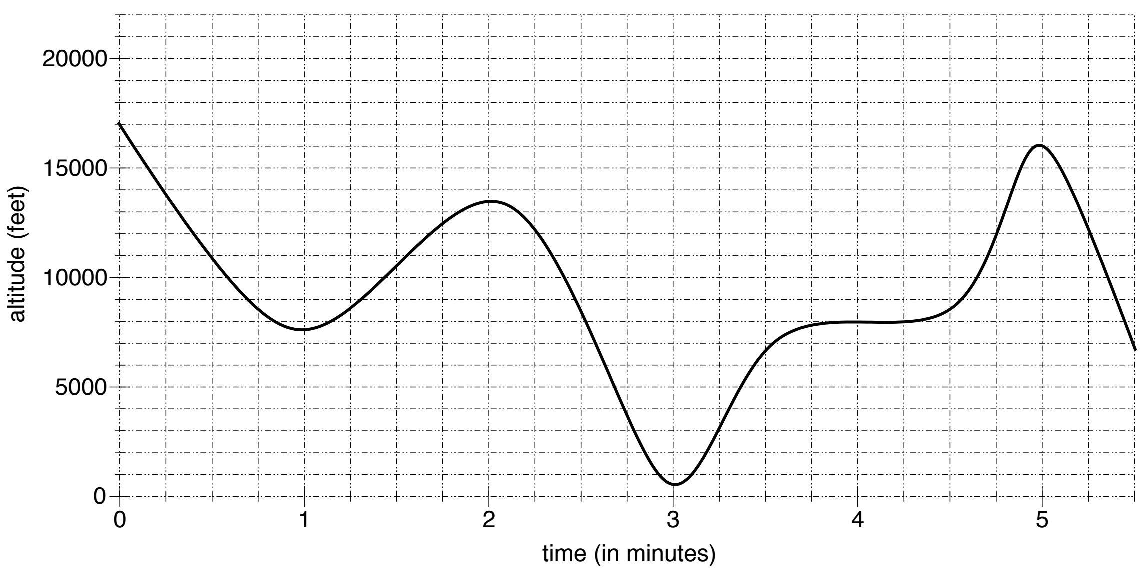
$$TR(q) = -0.08q^2 + 2.35q \qquad \text{and} \qquad TC(q) = 0.01q^3 - 0.3q^2 + 3q + 4.$$

A. If you produce between 8 and 15 hundred Shrubnods, what production level will yield the largest profit?

B. If you produce between 5 and 10 hundred Shrubnods, what production level will yield the largest profit?

C. If you can produce any number of Shrubnods, what production level will yield the largest profit? What is the largest possible profit for producing Shrubnods?

We'll come back to the scenario described in the Key Question. Below is the graph of the altitude, $A(t)$, for a balloon that is rising and falling.



$\boxed{1}$ Name all times at which the altitude graph has a horizontal tangent line. What can you say about the graph of the balloon's instantaneous rate of ascent (its vertical speed) at those times?

If a function $f(x)$ changes from increasing to decreasing at $x = a$, then we say that $f(x)$ has a <u>local maximum</u> at $x = a$. Similarly, if $f(x)$ changes from decreasing to increasing at $x = a$, then we say that $f(x)$ has a <u>local minimum</u> at $x = a$.

$\boxed{2}$ Name all times at which $A(t)$ has a local maximum. Name all the times at which $A(t)$ has a local minimum.

If $f'(a) = 0$, then we say that $f(x)$ has a critical number at $x = a$.

$\boxed{3}$ Is it possible for a function $f(x)$ to have a critical number at $x = a$, but neither a local maximum nor a local minimum there? Explain.

In exercise #2, you should have said that the altitude graph has local maxima at $t = 2$ and $t = 5$. (Note: maxima=the plural of maximum.) The graph of $A(t)$ is reaching "high points" at these two times. But neither of these two times yields the highest *point shown on the graph. The time at which the balloon is at its highest is $t = 0$. We say that the global maximum value of the altitude function on the interval from $t = 0$ to $t = 5.5$ occurs at $t = 0$.*

$\boxed{4}$ The balloon reaches its global maximum altitude at $t = 0$. What is that altitude?

$\boxed{5}$ We define the global minimum value of altitude similarly. The altitude graph has local minima at $t = 1$ and $t = 3$. Those are both "low points" on the graph, but which value of t gives *the* lowest point shown on the graph? What is the global minimum value of the altitude function on the interval from $t = 0$ to $t = 5.5$?

$\boxed{6}$ a) On its own set of axes, sketch the graph of $A'(t)$, the balloon's instantaneous rate of ascent at time t.

b) $A(t)$ has local maxima at $t = 2$ and $t = 5$. How can you see that on the graph of $A'(t)$?

c) How can you tell when $A(t)$ has local minima by looking at the graph of $A'(t)$?

So far, here's what we know:

- *$A(t)$ has critical numbers at $t = 1$, $t = 2$, $t = 3$, $t = 4$, and $t = 5$. (That's where $A'(t) = 0$.)*
- *$A(t)$ has local maxima at $t = 2$ and $t = 5$. (That's where $A(t)$ changes from increasing to decreasing OR where $A'(t)$ changes from positive to negative.)*
- *$A(t)$ has local minima at $t = 1$ and $t = 3$. (That's where $A(t)$ changes from decreasing to increasing OR where $A'(t)$ changes from negative to positive.)*
- *On the interval from $t = 0$ to $t = 5.5$, the global maximum value of $A(t)$ occurs at $t = 0$ and the global minimum value of $A(t)$ occurs at $t = 3$.*

$\boxed{7}$ We're now going to restrict our attention to smaller time intervals. Fill in the following chart with the values of t_{max}, the value of t that gives the global maximum altitude on the given interval, and t_{min}, the value of t that gives the global minimum altitude on the given interval.

Interval	t_{max}	t_{min}
from $t = 0$ to $t = 5.5$	0	3
from $t = 0.75$ to $t = 2.25$		
from $t = 1.75$ to $t = 2.75$		
from $t = 4.5$ to 4.75		

The previous exercise demonstrates the following fact: On the interval from $x = a$ to $x = b$, the global maximum value of a function $f(x)$ occurs either at a critical number or at one of the endpoints of the interval. That is, the global maximum value of $f(x)$ is either a local maximum value of $f(x)$ OR it is $f(a)$ or $f(b)$. Similarly, the global minimum value of $f(x)$ occurs either at a critical number or at one of the endpoints of the interval. The global minimum value of $f(x)$ is either a local minimum value of $f(x)$ OR it is $f(a)$ or $f(b)$.

This gives a convenient process to find the local and global maximum and minimum values of a function $f(x)$ on the interval from $x = a$ to $x = b$.

Step 1: *Compute $f'(x)$, set $f'(x) = 0$, and solve for x. This gives the critical numbers of f.*

Step 2: *Evaluate f at each of the critical numbers that lie between a and b and evaluate f at the endpoints a and b.*

Step 3: *Use the information gathered in Steps 1 and 2 to sketch a rough graph of $f(x)$ on the interval from a to b. You should be able to see on your graph all local and global optima.*

You can use this process to answer the Key Question.

$\boxed{8}$ a) Find the formula for $P(q)$, profit from selling q hundred Shrubnods.

b) Compute $P'(q)$, set $P'(q) = 0$, and solve for q. This gives the critical numbers of P.

c) Key Question A involves profit on the interval from $q = 8$ to $q = 15$. Any critical numbers that lie outside of this interval are of no interest to us at this point. Evaluate the profit function at the critical numbers that lie between 8 and 15. Also, compute $P(8)$ and $P(15)$.

d) Sketch a rough graph of $P(q)$ on the interval from $q = 8$ to $q = 15$. What production level (i.e. quantity) gives the global maximum value of P on this interval?

$\boxed{9}$ Repeat steps c) and d) from the previous exercise with the interval from $q = 5$ to $q = 10$, thus answering Key Question B.

Key Question C is a little different since we are not given an interval to consider. Certainly we need q to be positive, but there are no restrictions on q beyond that. So we're considering the interval from $q = 0$ to ∞. This makes Step 2 of the process impossible, since we can't evaluate a function at ∞. But we have enough information about the function to find its global maximum value.

$\boxed{10}$ Find the value of $P(q)$ at every positive critical number of P. (You've already done this for one of the critical numbers.) These are the only quantities at which P has a horizontal tangent; that is, the graph of P cannot "change direction" (from increasing to decreasing or from decreasing to increasing) at any other quantities. Also, compute $P(0)$. This should be enough information to sketch a rough graph of $P(q)$ and pick off the production level that maximizes profit and the profit at that value of q.

$\boxed{11}$ What is the global minimum value of profit on the interval from $q = 0$ to $q = 5$ hundred Shrubnobs? (NOTE: Your answer should be negative. We can call negative profit *loss*. If the smallest possible profit is negative, then its absolute value is the largest possible loss.)

12 If you can produce any number of Shrubnods, what production level will cause you to lose the most money? What is the largest possible loss for producing Shrubnods?

→ **13** The total cost in dollars for selling q Blivets is given by the function

$$TC(q) = 0.0005q^2 + 50q + 10125.$$

a) Write out formulas for average cost and marginal cost at q.

b) What is the smallest value of average cost on the interval from $q = 1$ to $q = 5000$?

c) Which of the following best describes the graph of average cost on the interval from $q = 1$ to $q = 3000$? Justify your choice.

   i) always increasing
   ii) increasing then decreasing
   iii) always decreasing
   iv) decreasing then increasing

d) At what quantity is marginal cost equal to average cost?

e) What is the smallest value of marginal cost on the interval from $q = 1$ to $q = 5000$?

→ **14** The total revenue in dollars for selling q Framits is given by the function

$$TR(q) = -0.05q^2 + 600q.$$

The variable cost in dollars for selling q Framits is given by the function

$$VC(q) = 0.000002q^3 - 0.03q^2 + 400q.$$

a) If you sell 1000 Framits, your profit is \$103,000. What is the value of your fixed cost?

b) Find the largest interval on which the total revenue function is positive.

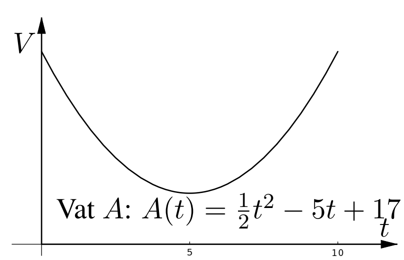
c) How many Framits must you sell in order to maximize profit?

d) What is the smallest value of average variable cost?

→ **15** The volume of water in two vats can increase or decrease.

The graph and formula for the volume of water (in gallons) in vat A are shown at the right for times between $t = 0$ and $t = 10$ minutes. The formula for the volume of water in vat B at time t (minutes) is

$$B(t) = \frac{1}{3}t^3 - \frac{11}{2}t^2 + 24t + 5$$



a) Find all values of t at which the graph of $B(t)$ reaches a local maximum or local minimum.

b) What is the lowest volume of water in vat B over the period $t = 0$ to $t = 10$ minutes?

c) Determine the longest interval you can (in the interval from 0 to 10) over which the volume of water in vat A is increasing and the volume of water in vat B is decreasing. ("None" is a possible answer.)

d) Let $D(t) = B(t) - A(t)$. Find the largest value of $D(t)$ on the interval from $t = 0$ to $t = 10$.

Worksheet #15 ***Demand Curves***

Key Question

You sell Gizmos on a sliding price scale. The price per Gizmo is given by the function

$$p = h(q) = q^2 - 50q + 625 \text{ dollars.}$$

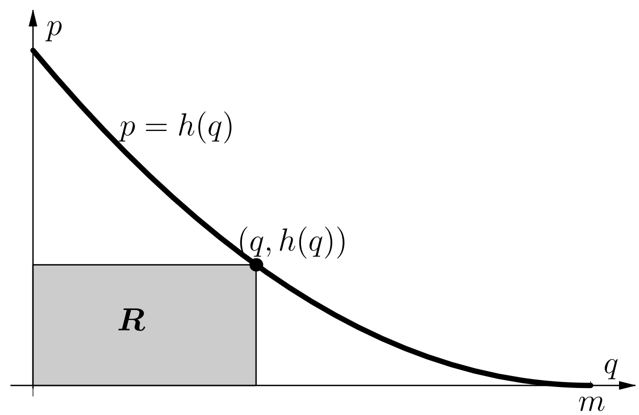
What quantity should you sell in order to maximize your total revenue?

The price per item function is called the demand function for Gizmos. The graph of the demand function is called the demand curve.

$\boxed{1}$ From the point of view of a business owner, explain why it only makes sense to use a demand function to set the price per item if the demand curve is positive and decreasing.

$\boxed{2}$ Find the largest interval on which the demand function $p = h(q) = q^2 - 50q + 625$ is positive and decreasing. (HINT: It is a quadratic function. Find its vertex.) Sketch an accurate graph of the demand curve on that interval.

One way to think of the total revenue for selling q Gizmos is as the area of the rectangle with one corner at the origin and the diagonally opposite corner at the point $(q, h(q))$ as shown at right.



$\boxed{3}$ On your graph of $p = h(q)$ from exercise 2, draw the rectangle with one corner at the origin and the diagonally opposite corner at the point $(10, h(10))$. Find the area of this rectangle. Describe the width, the height, and the area of this rectangle in the context of selling Gizmos.

$\boxed{4}$ Describe the "rectangle" with one corner at the origin and the diagonally opposite corner at the point $(0, h(0))$. What is the area of this "rectangle"? Describe the width, the height, and the area in the context of selling Gizmos.

$\boxed{5}$ Describe the "rectangle" with one corner at the origin and the diagonally opposite corner at the point $(25, h(25))$. What is the area of this "rectangle"? Describe the width, the height, and the area in the context of selling Gizmos.

If you think of total revenue at q as the area of the rectangle with one corner at the origin and the diagonally opposite corner at the point $(q, h(q))$ on the demand curve, then your work on the previous three exercises should convince you that total revenue is \$0 at $q = 0$ and at $q = 25$ and, for quantities in between, total revenue is positive.

6 Use the graph of the demand function from exercise 2 to make a guess as to which, of all the rectangles of this type, has the largest possible area.

7 The other way we've been thinking of total revenue is as price per Gizmo times quantity. That is, $TR(q) = pq$. Use the formula $p = h(q) = q^2 - 50q + 625$ to find a formula for total revenue.

8 Since the price scale only makes sense for values of q between 0 and 25 (the interval you found in exercise 2), your formula for total revenue only makes sense on that interval as well. Use the three-step process from Worksheet #14 to find the maximum value of total revenue on the interval from $q = 0$ to $q = 25$.

Step 1: Compute $TR'(q)$, set it equal to 0, and solve for q to find your critical numbers.

Step 2: Evaluate TR at each of the critical numbers that lie between 0 and 25. Also compute $TR(0)$ and $TR(25)$.

Step 3: Sketch a rough graph of $TR(q)$ on the interval from 0 to 25. What is the global maximum value of TR on that interval? What production level yields the largest TR? (Compare to the guess you made in exercise 6.)

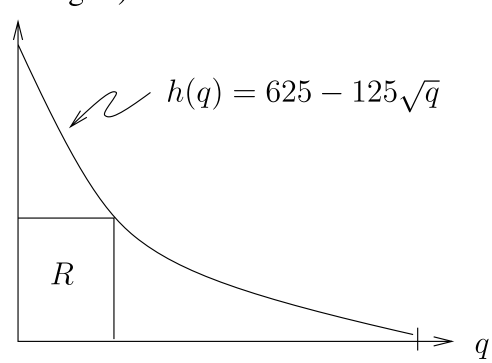
9 For each of the following demand functions:

- Find the interval on which the demand function makes sense (i.e. the demand curve is positive and decreasing).
- Find the production level that yields the largest possible value of total revenue on that interval.
- State the largest possible value of total revenue on that interval.

a) $p = h(q) = q^2 - 50q + 700$

b) $p = h(q) = mq + b$ (a line with $m < 0$ and $b > 0$)

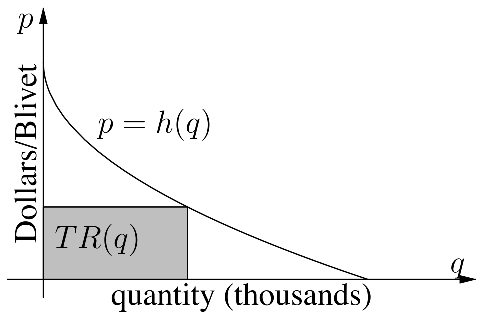
c) $p = h(q) = 625 - 125\sqrt{q}$ (Graph shown at right:)



→ $\boxed{10}$ The Demand Curve for Blivets (shown below) has the formula

$$p = h(q) = 12 - 4\sqrt{q}\,,$$

where q is measured in thousands of Blivets and p is measured in dollars. We get Total Revenue (measured in thousands of dollars) from this curve by the recipe $TR(q) = q \cdot h(q)$. The Total Cost (measured in thousands of dollars) of manufacturing Blivets is given by the formula $TC(q) = q+1$. Due to the limited supply of raw material, we can not manufacture more than 3500 Blivets ($q = 3.5$).



a) Write the formulas for $TR(q)$ and $TR'(q)$. (*Simplify your answers as much as possible*.)

b) Find the value of q in the interval between $q = 0$ and $q = 3.5$ thousand Blivets at which $TR(q)$ reaches its largest value.

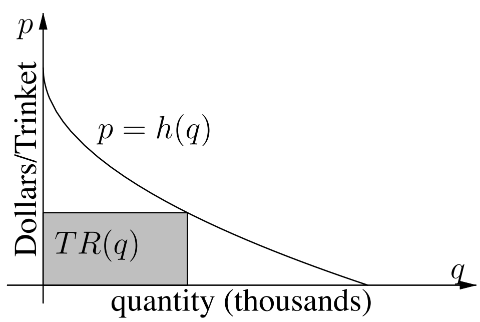
c) Write the formula for the Total Profit $P(q)$ of selling q thousand Blivets. (*Simplify your answer as much as possible*.)

d) Find the value of q in the interval between $q = 0$ and $q = 3.5$ thousand at which the profit is greatest.

→ $\boxed{11}$ Suppose that in order to achieve monthly sales of q thousand Trinkets you have to sell them at $p = h(q) = q^2 - 30q + 225$ dollars a piece.

a) Write down the formula for $TR(q)$, the total monthly revenue (in thousands of dollars) of selling q thousand Trinkets.

b) Suppose that your monthly sales are between 1000 and 10,000 Trinkets. (so q is between 1 and 10) What is the price per Trinket that you must charge in order to maximize your total monthly revenue?



c) Suppose that you want your monthly sales to be between 1000 and 10,000 Trinkets. What is the maximum price per Trinket that you can charge?

Worksheet #16 *The Second Derivative Test*

Key Question

If q can be any positive number, what is the largest possible value of the following function?

$$R(q) = \frac{1}{q} - \frac{1}{q^2}$$

$\boxed{1}$ Compute $R'(q)$, the derivative of the function in the Key Question. Find all positive critical numbers of $R(q)$.

$\boxed{2}$ If the Key Question had specified an interval to consider, we could follow the three-step process of Worksheet #14 to find the maximum value of $R(q)$. Explain why, in this case, this "recipe" doesn't work.

We know that the graph of $R(q)$ has a horizontal tangent line at $q = 2$. There is a tool that may tell us whether this horizontal tangent line occurs at a local maximum or a local minimum of $R(q)$. This tool involves taking the derivative of $R'(q)$.

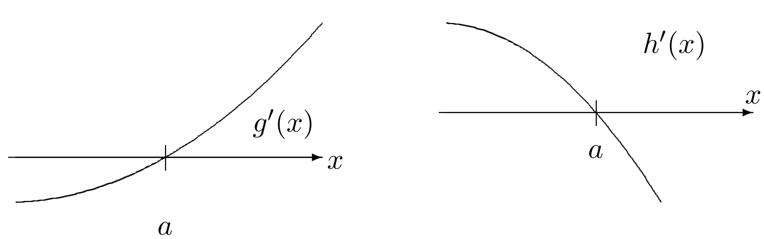
Given a function $f(x)$, we can take its derivative $f'(x)$. But $f'(x)$ is just another function with its own derivative. The derivative of $f'(x)$ is called the <u>second derivative</u> of $f(x)$. We denote it $f''(x)$. (We could continue to take derivatives forever. The derivative of $f''(x)$ is $f'''(x)$, the <u>third derivative</u> of $f(x)$. For this reason, we often call $f'(x)$ the <u>first derivative</u> of $f(x)$.)

$\boxed{3}$ For each of the following, compute the derivative and the second derivative.

a) $g(x) = 2x^3 + 3x^2 - 9x + 176$
b) $h(t) = \frac{1}{5}t^5 - \frac{3}{4}t^4 + \frac{1}{2}t^2$
c) $k(u) = u + \frac{1}{u}$
d) $l(y) = y^2 + \ln(y^2 + 1)$
e) $m(z) = e^z \cdot z^4$

The first derivative $f'(x)$ tells us where $f(x)$ is increasing and where it is decreasing. Since $f''(x)$ is the derivative of $f'(x)$, it tells us where $f'(x)$ is increasing and decreasing. But what does it tell us about the original function, $f(x)$?

Consider the first derivatives of two functions $g(x)$ and $h(x)$. (Note: These are not the graphs of $g(x)$ and $h(x)$. These are their derived *graphs.)*

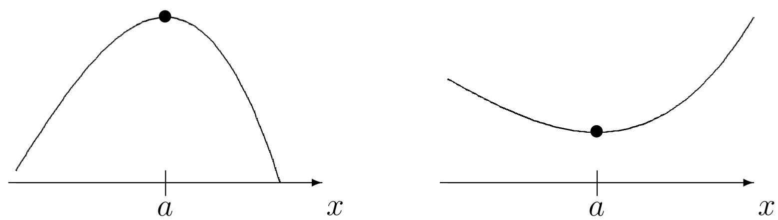


$\boxed{4}$ Both derivatives are crossing the x-axis at $x = a$. That is, $g'(a) = 0$ and $h'(a) = 0$. What does that imply about $g(x)$ and $h(x)$?

$\boxed{5}$ The graph of $g'(x)$ is increasing on the interval shown. What does that imply about $g''(a)$, the slope of the tangent line to $g'(x)$ at $x = a$?

$\boxed{6}$ Similarly, what can you say about $h''(a)$?

$\boxed{7}$ Here are the graphs of $g(x)$ and $h(x)$, but which is which?

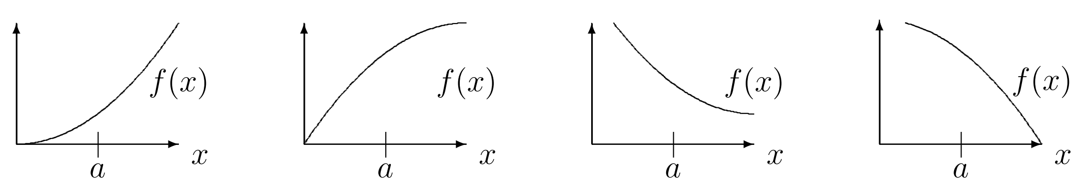


Describe how to tell which graph is $g(x)$ and which graph is $h(x)$ and label the graphs. Draw the tangent line to each at $x = a$.

Notice that the tangent line to $g(x)$ at $x = a$ sits below *the graph of $g(x)$. We say that a function with this property is <u>concave up</u> at $x = a$. The tangent line to $h(x)$ at $x = a$ sits* above *the graph of $h(x)$. We say that a function with this property is <u>concave down</u> at $x = a$.*

$\boxed{8}$ In each of the following situations, determine whether $f'(x)$ is positive or negative on the interval shown and whether $f'(x)$ is increasing or decreasing on that interval. Also, determine whether $f''(a)$ is positive or negative.

$f(x)$ increasing concave up	$f(x)$ increasing concave down	$f(x)$ decreasing concave up	$f(x)$ decreasing concave down



By now, I am hoping you're convinced that a function $f(x)$ is concave up when $f'(x)$ is increasing. Further, you know that $f'(x)$ is increasing when its derivative $f''(x)$ is positive. Therefore, it's not a stretch to say that $f(x)$ is concave up precisely when $f''(x)$ is positive. Similarly, $f(x)$ is concave down when $f''(x)$ is negative. Further, you should believe that, if $f(x)$ has a horizontal tangent line at $x = a$ and $f(x)$ is concave up at $x = a$, then $f(x)$ has a local minimum at $x = a$. Similarly, if $f(x)$ has a horizontal tangent line at $x = a$ and $f(x)$ is concave down at $x = a$, then $f(x)$ has a local maximum at $x = a$.

$\boxed{9}$ Back to the Key Question. You've shown that $R(q)$ has a horizontal tangent line at $q = 2$.

a) Compute $R''(q)$, $R''(2)$, and determine whether R has a local maximum or a local minimum at $q = 2$.

b) Since $q = 2$ is the only positive critical number of $R(q)$, explain why this local maximum or minimum must also be a global maximum or minimum.

c) Answer the Key Question.

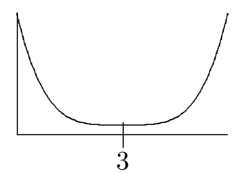
We've developed the first two parts of:

<u>The Second Derivative Test</u>

- *If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at $x = a$.*
- *If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at $x = a$.*

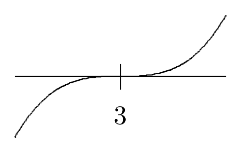
There is one other possibility: what happens if $f'(a) = 0$ and $f''(a) = 0$? The next three exercises show that, if $f''(a) = 0$, then anything goes!

$\boxed{10}$ Consider the function $f(x) = (x - 3)^4 + 7$, whose graph is given at right. The graph shows a local (and global) minimum at $x = 3$.



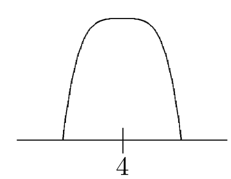
a) Compute $f'(x)$ and find all the critical numbers of f.

b) The graph of $f'(x)$ is given at right. By looking at the graph of $f'(x)$ what would you guess is the slope of the tangent line to $f'(x)$ at $x = 3$?



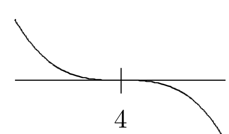
c) Compute $f''(x)$ and evaluate it at $x = 3$. (Did you guess correctly in part (b)?)

$\boxed{11}$ Consider the function $g(x) = 0.1 - (x - 4)^4$, whose graph is given at right. The graph shows a local (and global) maximum at $x = 4$.



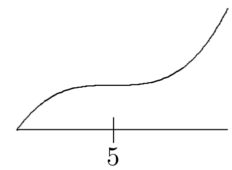
a) Compute $g'(x)$ and find all the critical numbers of g.

b) The graph of $g'(x)$ is given at right. By looking at the graph of $g'(x)$ what would you guess is the slope of the tangent line to $g'(x)$ at $x = 4$?



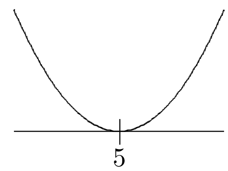
c) Compute $g''(x)$ and evaluate it at $x = 4$. (Did you guess correctly in part (b)?)

$\boxed{12}$ Consider the function $h(x) = (x-5)^3 + 9$, whose graph is given at right. The graph shows a horizontal tangent line at $x = 5$, but no local optimum there. (We say that $h(x)$ has a “seat” at $x = 5$. Can you see why?)



a) Compute $h'(x)$ and find all the critical numbers of h.

b) The graph of $h'(x)$ is given at right. By looking at the graph of $h'(x)$ what would you guess is the slope of the tangent line to $h'(x)$ at $x = 5$?



c) Compute $h''(x)$ and evaluate it at $x = 5$. (Did you guess correctly in part (b)?)

We can now complete the statement of:

<u>The Second Derivative Test</u>

- *If $f'(a) = 0$ and $f''(a) > 0$, then $f(x)$ has a local minimum at $x = a$.*
- *If $f'(a) = 0$ and $f''(a) < 0$, then $f(x)$ has a local maximum at $x = a$.*
- *If $f'(a) = 0$ and $f''(a) = 0$, then the test fails — f may have a local maximum, a local minimum, or a seat at $x = a$.*

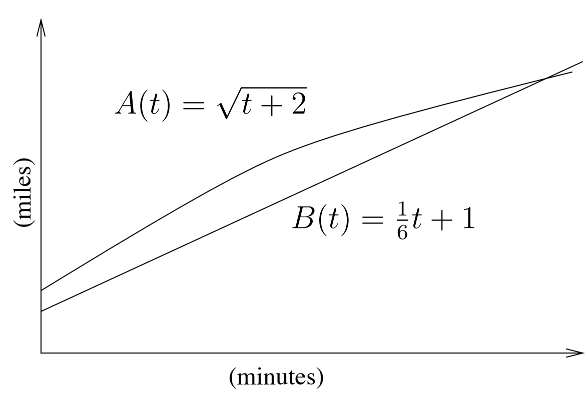
$\longrightarrow$ $\boxed{13}$ We wish to graph the function $f(x) = -\frac{4}{3}x^3 + 40x^2 - 375x + 1500$ over the interval from $x = 0$ to $x = 15$.

a) Find all values of x in the interval from $x = 0$ to $x = 15$ at which the graph of $f(x)$ has a horizontal tangent line.

b) Apply the Second Derivative Test to each value that you found in part (a). What does the test tell you?

c) Evaluate the function $f(x)$ at the two points you gave in (a) and the two end-points, $x = 0$ and $x = 15$. Then draw a rough sketch of the function over the interval $x = 0$ to $x = 15$.

$\longrightarrow$ $\boxed{14}$ To the right are the distance *vs*. time graphs for two cyclists. Let $D(t)$ be the distance by which Cyclist A is ahead of Cyclist B.



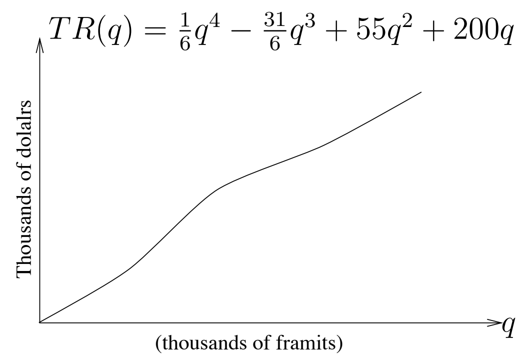
a) Determine the time at which Cyclist B passes Cyclist A.

b) Give the largest and smallest values of $D(t)$ in the interval $t = 0$ to $t = 12$.

c) Find the times t at which B reaches his highest and lowest overall average trip speed in the time interval $t = 1$ to $t = 12$. Explain your answer.

d) If we define the function $F(t) = B(t) - A(t)$, then $F(t)$ has positive values after a while. Moreover $F(t)$ is increasing after $t = 40$ minutes. Is the shape of the graph of $F(t)$ concave up or down for values of t greater than $t = 40$?

$\longrightarrow$ $\boxed{15}$ The graph to the right is of Total Revenue of selling Framits. (q is in thousands of Framits, and $TR(q)$ is in thousands of dollars.)



a) Write the formula for Marginal Revenue of manufacturing Framits. (The units will be dollars.)

b) Find *all* values of q at which the MR graph has a horizontal tangent.

c) Apply the Second Derivative Test to each value that you gave in part (b). What does the test tell you?

d) Determine the global maximum value and global minimum value of MR over the interval $q = 0$ to $q = 12$ (thousand).

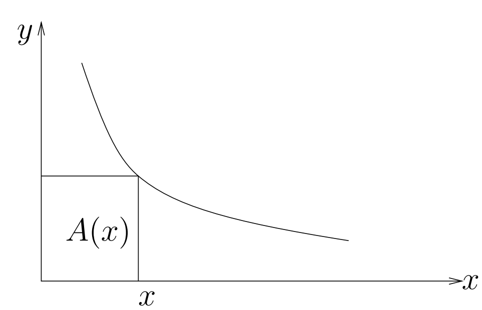
e) Use the second derivative to tell whether the *TR* graph is concave up or concave down at $q = 12$.

$\longrightarrow$ $\boxed{16}$ The formula for the graph shown to the right is $y = 2 + \dfrac{4}{3x - 1}$. We defined a new function

$$A(x) = \text{ The area of the rectangle determined by } x.$$

a) Write out formulas for $A(x)$ and $A'(x)$.

b) Apply the Second Derivative Test to each value of x in the interval $x = \frac{1}{3}$ to $x = 3$ at which $A'(x) = 0$. What does the test tell you?

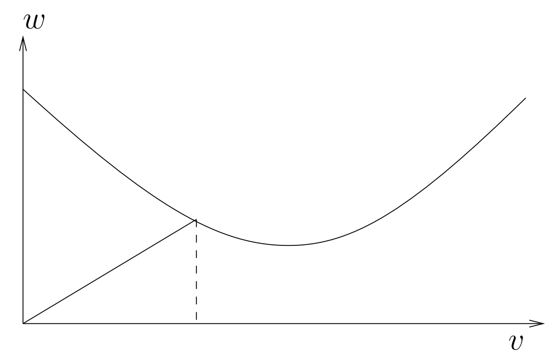


c) Draw a rough sketch of the graph of $A(x)$ for x greater than 20. (Hint: Use $A'(x)$ to tell whether $A(x)$ is increasing or decreasing, and use $A''(x)$ to tell whether the graph is concave up or down.)

$\longrightarrow$ $\boxed{17}$ The formula for the parabola to the right is

$$w = g(x) = \frac{1}{4}x^2 - 4x + 36.$$

We define a new function by $S(x) =$ the slope of the diagonal to the point $(x, g(x))$. Thus $S(x) = \frac{g(x)}{x}$.



a) Write out formulas for $S(x)$ and $S'(x)$.

b) Find all the positive values of x at which the graph of $S(x)$ has a horizontal tangent.

c) Apply the Second Derivative Test to each of the values x at which $S(x)$ has a horizontal tangent. What does the test tell you?

d) (i) Find the value of x in the interval $x = 1$ to $x = 6$ at which $S(x)$ reaches a global maximum.

(ii) Find the value of x in the interval $x = 1$ to $x = 6$ at which $S(x)$ reaches a global minimum.

Show your work and explain your answer in both parts.

e) Give a positive number y for which the following is true: There is no positive value of x for which $S(x) = y$.

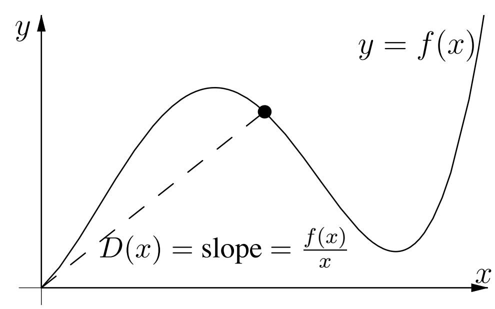
f) Give an interval of length 1 over which $S(x)$ is increasing and $g(x)$ is decreasing.

$\longrightarrow$ $\boxed{18}$ The graph to the right is of the function

$$y = f(x) = \frac{1}{6}x^4 - \frac{5}{2}x^3 + 8x^2 + 12x\,.$$

We use this graph to define a new function

$$D(x) = \text{ Slope of diagonal to } f(x) \;=\; \frac{f(x)}{x}\,.$$



a) Write out the formulas for the functions $D(x)$ and $D'(x)$.

b) Find all values of x at which the graph of $D(x)$ has a horizontal tangent. Use the Second Derivative Test to determine whether $D(x)$ reaches a local maximum or a local minimum at each of these points.

c) Determine the lowest value of $D(x)$ on the interval $x = 1$ to $x = 12$. Show your work and tell why your answer is correct.

d) We define a new function

$$T(x) = \text{ Slope of the tangent to graph of } f(x)\,.$$

Determine all positive values of x at which $T(x)$ has a local minimum or local maximum.

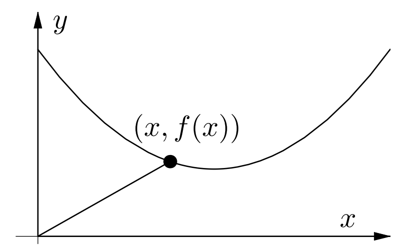
$\longrightarrow$ $\boxed{19}$ The graph to the right is of the function $f(x) = x^2 - 8x + 25$.

a) Let $S(x)$ to be the slope of the diagonal from $(0,0)$ to $(x, f(x))$ (as shown). Write out the formula for $S(x)$.

b) Use the first derivative to find all positive critical numbers of $S(x)$.

c) Use the second derivative to determine whether the critical number you have found gives a local minimum or a local maximum.

d) Explain how you know that this local optimum is also a global optimum.



Chapter 5

Functions of Several Variables

Worksheet #17 *Multivariable Functions*

In the previous sections of the course, you worked with functions that consisted of only one variable. For example, $TR(q) = 25q - 0.5q^2$ is a formula for Total Revenue. Its only input is quantity q. In this section, you will explore functions that have multiple inputs. Since each input is represented by a different variable, these functions are called multivariable functions.

Introduction to Multivariable Functions

Example: During 2001, Puget Sound Energy utilized two methods for calculating the electrical bill for residential users: flat rate and time-of-use.

With the first method, a customer pays a flat rate for each kilowatt-hour of electricity used, no matter what time of day the usage occurs. Puget Sound Energy charged \$0.05 per kilowatt-hour for flat rate usage.

$\boxed{1}$ Customer 1 chooses the flat rate method.

a) Determine the charge for 2000 kilowatt-hours to Customer 1.

b) Develop a formula $C_1(x)$ that gives the charge to Customer 1 for x kilowatt-hours.

On the other hand, the time-of-use method charges different amounts per kilowatt-hour depending on when the electricity is used. During peak times, when the demand is highest, the charge per kilowatt-hour is higher than that at off-peak times. The time-of-use system has three rates:

peak hours \$0.062 per kilowatt-hour for usage between 6 a.m. and 10 a.m. and between 5 p.m. and 9 p.m.

daytime hours \$0.0536 per kilowatt-hour for usage between 10 a.m. and 5 p.m.

off-peak hours \$0.047 per kilowatt-hour for usage between 9 p.m. and 6 a.m.

$\boxed{2}$ Determine the charge for Customer 2, who chose the time-of-use method and who had the following usage during a one-month period.

Time of Usage	Kilowatt-hours	Rate (\$)	Charge
Peak Hours	1000	0.062	
Daytime Hours	700	0.0536	
Off-Peak Hours	300	0.047	
		Total Charge:	

In order to make a formula for the charge to Customer 2, one variable will not be enough. We'll need three variables, one for each time category.

$\boxed{3}$ Let x be the electricity usage during peak hours, y be the usage during daytime hours, and z be the usage during off-peak hours, all for a one-month period. Develop a formula for $C_2(x, y, z)$, the charge to Customer 2.

$\boxed{4}$ Suppose Customer 2 purchases smart appliances for a total of \$3000. These smart appliances (dishwasher, washer/dryer, water heater, etc.) can be programmed to operate during off-peak hours. Customer 2 also adjusts patterns of electrical usage so that the meter reading per month is now given by the following.

Time of Usage	Kilowatt-hours	Rate (\$)	Charge
Peak Hours	500	0.062	
Daytime Hours	300	0.0536	
Off-Peak Hours	1200	0.047	
		Total Charge:	

How many months will it take for Customer 2 to recoup the investment in smart appliances? That is, how long will it take before Customer 2 has saved a total of \$3000 on electric bills?

Example: Another example of a function that has multiple inputs is the formula used to compute the balance of a certificate of deposit compounded continuously, $A = Pe^{rt}$. This function has three inputs: the principal P, the interest rate r, and the time of maturation t. To emphasize the fact that this is a function with three variables, we could instead write: $A(P, r, t) = Pe^{rt}$.

$\boxed{5}$ Compute each of the following and describe your answer in terms of the certificate of deposit. (For example, $A(500, 0.02, 1) = \$510.10$ is the value of a CD that was purchased for \$500 and earned 2% annual interest, compounded continuously, for one year.)

a) $A(5000, 0.06, 8)$

b) $A(P, 0.05, 0.5)$

c) $A(700, r, 4)$

d) $A(85000, 0.072, t)$

Rates of Change of Multivariable Functions

Rates of change (not to be confused with interest rates) can be found in multivariable functions in a manner similar to that for functions with only one variable. The technique is to look at the change in only one variable at a time. That means you have to "fix" the other variables and think about what happens to the function as the remaining variable changes.

As an example, consider the function in part (d) of the previous exercise: $A(85000, 0.072, t)$. With the other two variables fixed at $P = 85000$ and $r = 0.072$, the value of a CD is a function only of the variable t. If we call this single-variable function $v(t)$, then $v(t) = A(85000, 0.072, t) = 85000e^{0.072t}$.

$\boxed{6}$ Recall the three types of rate of change of $v(t)$.

overall: $\dfrac{v(t) - v(0)}{t}$

incremental: $\dfrac{v(b) - v(a)}{b - a}$

instantaneous: $v'(t)$

a) Compute the overall rate of change in the value of the CD at $t = 7$ years.
b) Compute the incremental rate of change in the value of the CD during the six-month period, beginning at $t = 5$ years. (Give your answer in units of dollars per year.)
c) Compute the instantaneous rate of change in the value of the CD at $t = 10$ years.

To remind you how we defined overall and incremental rates of change of a single-variable function, we introduced a new letter v. However, that wasn't really necessary. Since $v(t) = A(85000, 0.072, t)$, the overall rate of change in the value of the CD at time t is:

$$\frac{A(85000, 0.072, t) - A(85000, 0.072, 0)}{t}.$$

Similarly, the incremental rate of change in the value of the CD from $t = a$ to $t = b$ is:

$$\frac{A(85000, 0.072, b) - A(85000, 0.072, a)}{b - a}.$$

The instantaneous rate of change in the value of the CD at time t is the derivative of the function $A(85000, 0.72, t)$. If A was always a function of only the variable t, we'd denote its derivative $A'(t)$. To indicate that we are differentiating a multi-variable function, we use slightly different notation for this derivative. The instantaneous rate of change in the value of this CD is denoted $A_t(85000, 0.072, t)$.

$\boxed{7}$ If you fix the principal at \$10,000 and the annual interest rate at 6.3%, compute each of the following:

a) The overall rate of change in the value of the CD at $t = 15$.
b) The incremental rate of change in the value of the CD over the h-year interval beginning at $t = 3$.
c) The instantaneous rate of change in the value of the CD at $t = 4.2$ years.

$\boxed{8}$ Now let's fix the values of r and t and let the value of the principal vary. Fix the interest rate at 8.4% and the time of maturation at 5 years. Then the value of the CD is a function only of P: $A(P, 0.084, 5) = Pe^{0.084(5)}$.

a) Compute $\dfrac{A(1500, 0.084, 5) - A(500, 0.084, 5)}{1500 - 500}$. (We call this number the average change in the value of the CD with respect to P. For each dollar you increase the principal by, the value of the CD increases by this much on average.)

b) Compute the derivative of the function $Pe^{0.084(5)}$. (We'd denote this derivative $A_P(P, 0.084, 5)$.)

Derivatives of Multivariable Functions

In the exercises above, we were able to treat multivariable functions just like regular functions when evaluating them and determining rates of change. We just had to take the precaution of dealing with only one variable at a time. A similar approach is used when taking the derivative of a multivariable function. The derivative rules for single-variable functions work here as well, but the rules are applied to one of the variables at a time, while the others are considered fixed. However, unlike in the previous exercises, we need not specify the values at which we fix these other variables.

For example, let's consider the function $A(P, r, t) = Pe^{rt}$. We'll start by thinking of the principal and interest rate as fixed and time as the only variable.

- First, we need to indicate which of our multiple variables we're allowing to change. Here, we are thinking of P and r as fixed and we are allowing the time to change. We take the derivative of A *with respect to* t. The derivative of A with respect to t is denoted $A_t(P, r, t)$.

- Next, think of the other variables as being fixed. Although we will not be assigning the variables P and r numerical values, we treat them like constants. Some people like to indicate that a variable is fixed by marking the variable with an asterisk. We could write the formula as $A(P^*, r^*, t) = P^*e^{r^*t}$. The asterisks are a reminder that the principal and the interest rate are fixed and should be treated like constants.

- Finally, we take the *partial derivative* of A with respect to the variable t using rules from previous worksheets. Remember we treat t as the variable and P^* and r^* like constants. Applying the derivative rule for exponential functions, we get

  $$A_t(P^*, r^*, t) = P^*e^{r^*t} \cdot (r^*).$$

  The last part comes from applying the chain rule to an exponential function. (Recall, for example, that the derivative of $5e^{3x}$ is $5e^{3x} \cdot 3$, since 3 is the derivative of $3x$. Since we are treating r^* just like a number, the derivative of r^*t is simply r^*.)

$\boxed{9}$ Compute $A_P(P, r^*, t^*)$ and $A_r(P^*, r, t^*)$.

Recall that we can denote the derivative of $y = f(x)$ in two ways: $\frac{dy}{dx}$ and $f'(x)$. The notation A_t for the partial derivative of A with respect to t is the multi-variable analog of single-variable notation f'. Prime notation doesn't work with a multi-variable function because there is no indication of which variable is actually being treated as a variable. There is also an analog to $\frac{d}{dx}$ notation. The partial derivative of A with respect to t can be denoted $\frac{\partial A}{\partial t}$. (The derivatives you computed in the previous exercise would be, respectively, $\frac{\partial A}{\partial P}$ and $\frac{\partial A}{\partial r}$.)

Once you get the hang of treating a variable like a constant, you can skip all that asterisk business. As another example, let's look at the multivariable function we computed for the time-of-use electricity charges:

$$C_2(x, y, z) = 0.062x + 0.0536y + 0.047z.$$

To compute $\dfrac{\partial C_2}{\partial x}$, think of everything without an x in it as a number. Even though the terms $0.0536y$ and $0.047z$ have a variable in them, they don't have an x in them. So, we treat those two terms like constants. Thus,

$$\frac{\partial C_2}{\partial x} = 0.062.$$

$\boxed{10}$ Compute $\dfrac{\partial C_2}{\partial y}$ and $\dfrac{\partial C_2}{\partial z}$.

For practice, consider the function:

$$E(b, m) = b^2 + 3.4m^2 + 2.833bm - 5.6b - 15.4m + 102.3.$$

Then, the partial derivative of E with respect to m is computed by treating all of the b's as numbers:

$$\frac{\partial E}{\partial m} = 6.8m + 2.833b - 15.4.$$

$\boxed{11}$ Compute $\dfrac{\partial E}{\partial b}$.

All of the derivative rules still apply (the Sum Rule, Product Rule, Quotient Rule, and the Chain Rule). The following exercise will give you lots of practice taking partial derivatives.

$\boxed{12}$ Find the indicated partial derivatives.

a) $\dfrac{\partial z}{\partial x}$ if $z = x^3y^2 + 3y^3 - 6e^xy$

b) $f_t(t, m)$ if $f(t, m) = te^m + t^2(m^2 + 2m)$

c) $\dfrac{\partial w}{\partial y}$ if $w = \dfrac{x^3y}{y+1}$

d) $\dfrac{\partial t}{\partial s}$ if $t = (s^2 + rs)(r + s^3)$

e) $g_v(u, v)$ if $g(u, v) = e^{u^2v}$

f) $\dfrac{\partial p}{\partial y}$ if $p = x^3y^2 + 3x^2y + 4x^2 - 5y^2$

g) $h_u(u, v)$ if $h(u, v) = \dfrac{u+v}{u-v}$

h) $\dfrac{\partial z}{\partial m}$ if $z = (m^2p + p^2m)(m + p)^3$

One final note: recall that the single-variable function $f(x)$ can have a local minimum or a local maximum only at an x-value that makes $f'(x) = 0$. Similarly, a multi-variable function $f(x, y)$ may have a local minimum or maximum at those x- and y-values that make its partial derivatives equal to zero. That is, if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then the pair (a, b) is a candidate for a local maximum or local minimum of $f(x, y)$. (There actually is a Second Derivative Test for functions of two variables that would allow us to check if this point gives a max, a min, or neither, but we will not study it in this course.)

$\boxed{13}$ Consider the function $f(x,y) = -2x^2 + 12x - 6y^2 + 4y + 4xy + 24$.

a) Write out the two partial derivatives $f_x(x,y)$ and $f_y(x,y)$.

b) Find all values of (x,y) that are candidates for a local maximum or a local minimum of $f(x,y)$ by setting both of your partial derivatives equal to 0 and solving the system of equations that results.

c) If you fix y to be 0, then $f(x,0)$ becomes a function only of x.

(i) Let $g(x) = f(x,0)$. Write out a formula for $g(x)$ and $g'(x)$.

(ii) Explain why $\dfrac{g(2.0001) - g(2)}{0.0001} \approx g'(2)$. (Talk about secant lines and tangent lines.) Then use the derivative $g'(x)$ to approximate the value of $\dfrac{g(2.0001) - g(2)}{0.0001}$.

(iii) Express $g'(2)$ as a partial derivative of f.

(iv) Use partial derivatives to tell which of the following numbers is bigger.

$$\frac{f(2.0001,0) - f(2,0)}{0.0001} \qquad \frac{f(2,0.0001) - f(2,0)}{0.0001}.$$

$\longrightarrow$ $\boxed{14}$ Suppose $z = f(x,y) = -6x^2 + 2x - 4y^2 - 3y + 8xy + 30$.

a) Write out formulas for $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$.

b) Find all values of (x,y) which are candidates for a local maximum or local minimum.

c) If you fix x to be 2, then $z = f(2,y)$ becomes a function of only one variable, the variable y. Find the largest and smallest values this function on the interval from $y = -4$ to $y = 0$.

d) Suppose $(x,y) = (-6,-7)$. Circle the correct choice: A small increase in x (with y held fixed) leads to a (larger smaller) increase in z than a small increase in y (with x held fixed).

e) Consider the following four functions of x: $f(x,4)$, $f(x,6)$, $f(x,8)$, $f(x,10)$. Use the partial derivative to tell which of these functions has the steepest graph at $x = 1$.

$\longrightarrow$ $\boxed{15}$ Suppose $p = h(r,s) = 12 + rs + \frac{27}{r} + \frac{8}{s}$.

a) Write out formulas for $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial s}$.

b) Find all values (r,s) which are candidates for local maximum or local minimum.

c) Find a value of (r,s) at which both of the following occur:

If you fix r and increase s slightly, then p increases.
If you fix s and increase r slightly, then p decreases.

$\longrightarrow$ $\boxed{16}$ Suppose $z = f(x,y) = -x^3 + 8y^3 + 4xy + 6$.

a) Write out formulas for $f_x(x,y)$ and $f_y(x,y)$.

b) Find the largest and smallest values of $f(x,-1)$ over the interval from $x=-2$ to $x=2$.

c) Use partial derivatives to tell which of the following numbers is bigger:

$$\frac{f(1.00003,1)-f(1,1)}{0.00003} \qquad \frac{f(1,1.00003)-f(1,1)}{0.00003}.$$

d) Consider the following four functions of y: $f(-2,y)$, $f(-1,y)$, $f(0,y)$, $f(1,y)$. Use partial derivatives to tell which of these functions has the steepest graph at $y=-1$.

$\longrightarrow$ $\boxed{17}$ Suppose that $z = g(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - xy + 7$.

a) Write out the two partial derivatives of z.

b) Find all points (x,y) which are candidates for local minima or local maxima of z.

c) Which graph is steeper $h(t) = g(-2,t)$ at $t=3$ or $k(t) = g(t,3)$ at $t=-2$?

$\longrightarrow$ $\boxed{18}$ Suppose $p = P(r,s) = r^2s + rs^2 - 5rs + 2r$.

a) Write out formulas for $\frac{\partial p}{\partial r}$ and $\frac{\partial p}{\partial s}$.

b) Which graph is steeper: $P(3,s)$ at $s=2$ or $P(r,2)$ at $r=3$?

c) Assume that $r=r^*$ is some fixed positive value of r. This makes p a function of s. From your knowledge of quadratic functions, give a formula for the value of s that gives the smallest value of p (for this r^*). Your formula should involve r^*, of course.

$\longrightarrow$ $\boxed{19}$ Suppose $z = f(x,y) = 6x^2 - 2x + 2y^2 - 3y + 7xy + 3$.

a) Write out formulas for $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$.

b) Find all values of (x,y) which are candidates for a local maximum

c) Find the largest and smallest values of z for $y=-1$ and x ranging from 0 to 3.

d) Suppose $(x,y)=(3,4)$. Circle the correct answer: A small increase in y (with x held fixed) leads to a (larger smaller) increase in z than a small increase in x (with y held fixed).

e) Suppose a and b are fixed positive numbers and that a is greater than b. Which of the following numbers is larger:

- the slope of $f(a,y)$ at $y=3$ OR
- the slope of $f(b,y)$ at $y=3$?

Worksheet #18 ***Linear Programming***

Key Question

Your company makes two fruit juices, Apple-Cranberry and Cranberry-Apple. Apple-Cranberry is 60% apple juice and 40% cranberry juice, while Cranberry-Apple is 30% apple juice and 70% cranberry juice. You make \$0.40 profit on each gallon of Apple-Cranberry juice that you sell and \$0.50 profit on each gallon of Cranberry-Apple juice that you sell. Your daily supply is limited to 12,000 gallons of pure apple juice and 11,000 gallons of pure cranberry juice.

I. How much Apple-Cranberry juice and Cranberry-Apple juice should you produce daily so as to maximize your profit?

II. What is your maximum daily profit?

III. How much apple juice and how much cranberry juice should you buy so as to maximize your profit?

IV. Suppose that your production capacity is reduced so that, you cannot produce more than 10,000 gallons of Cranberry-Apple juice daily, and you cannot produce more than 15,000 gallons of Apple-Cranberry juice daily. How should you adjust your production so as to still maximize your profit?

Before trying to answer these questions, let's set up some notation. Let x denote your daily production of Apple-Cranberry juice (measured in thousands of gallons), and let y denote your daily production of Cranberry-Apple juice (again measured in thousands of gallons). Notice that neither x nor y can be negative, so your daily production of both juices corresponds to a point in the first quadrant of the (x, y)-plane.

$\boxed{1}$ How much apple juice do you need to make 10,000 gallons of Apple-Cranberry juice ($x = 10$) and 20,000 gallons of Cranberry-Apple juice ($y = 20$)?

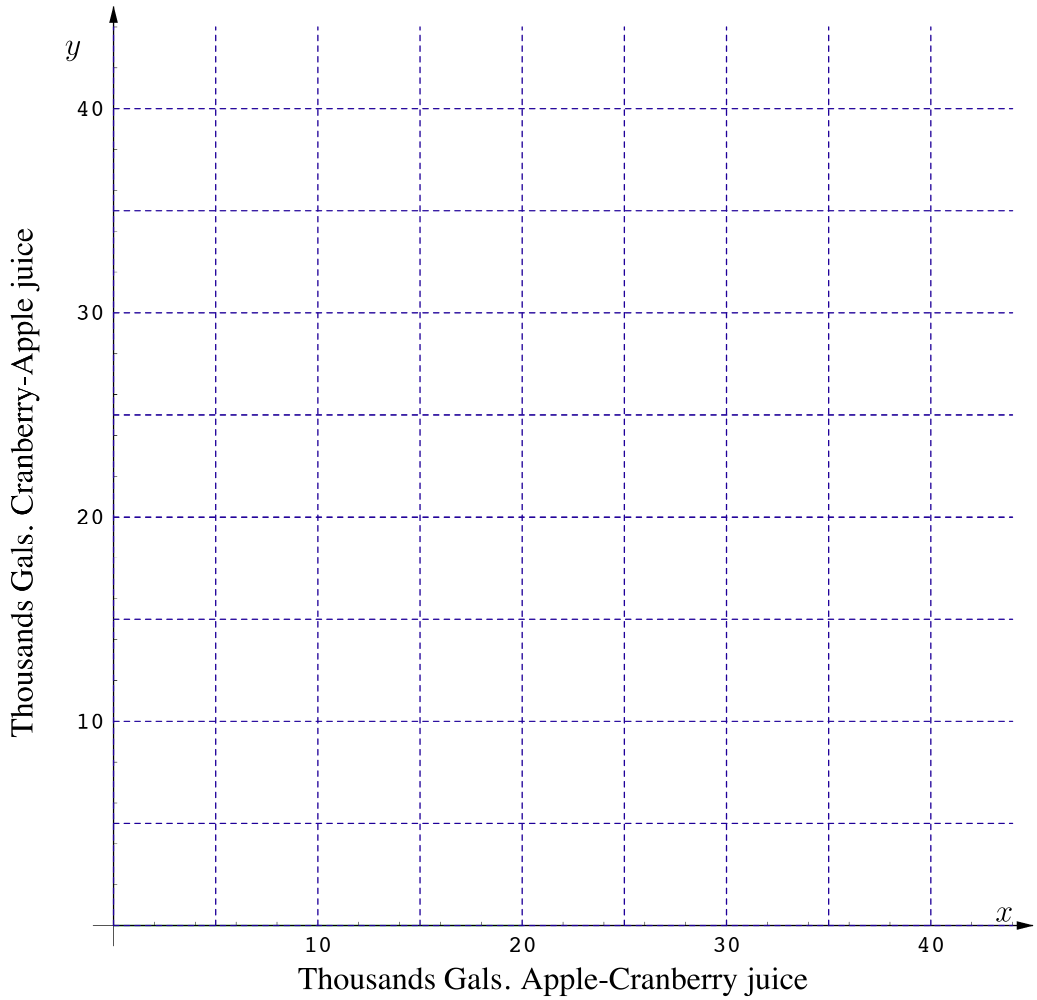
$\boxed{2}$ Let $A(x, y)$ denote the amount of apple juice (in thousands of gallons) that you need to make x thousand gallons of Apple-Cranberry juice and y thousand gallons of Cranberry-Apple juice. Find a formula for $A(x, y)$.

$\boxed{3}$ Since your daily supply of apple juice is limited to 12,000 gallons, your daily production has to satisfy the inequality

$$A(x, y) \leq 12\,.$$

Draw the line $A(x, y) = 12$ on the axes on the next page. Notice that the point $(x, y) = (10, 20)$ is on the line. Explain in ordinary English what it means for the point $(10, 20)$ to be on the line.

$\boxed{4}$ Notice that the line $A(x, y) = 12$ divides the first quadrant into two parts: points (x, y) where $A(x, y) > 12$ and points where $A(x, y) \leq 12$. Shade in the portion of the quadrant where $A(x, y) > 12$.



$\boxed{5}$ Is it possible to achieve a daily production that corresponds to a point in the shaded region? Explain why or why not.

$\boxed{6}$ Now let $C(x, y)$ denote the amount of cranberry juice (in thousands of gallons) that you need to make x thousands gallons of Apple-Cranberry and y thousand gallons of Cranberry-Apple. Find the formula for $C(x, y)$.

$\boxed{7}$ Since your daily supply of cranberry juice is limited to 11,000 gallons, your daily production has to satisfy the inequality

$$C(x, y) \leq 11 .$$

On the same set of axes, draw the line $C(x, y) = 11$. That line divides the octant into two parts: the part where $C(x, y) > 11$, and the part where $C(x, y) \leq 11$. Shade in all points in the first quadrant where $C(x, y)$ is bigger than 11.

$\boxed{8}$ Is it possible to achieve a daily production that corresponds to a point in the region where $C(x, y) > 11$? Explain why or why not.

> *The points in first quadrant that remain unshaded are the important ones—they correspond to daily productions that you could actually achieve under your restrictions.*
>
> *The overlap of the regions $A(x, y) \leq 12$ and $C(x, y) \leq 11$ (i.e. the unshaded region) is called the **region of feasibility**. Your daily production must correspond to a point in the region of feasibility.*

$\boxed{9}$ Now let's get look at your daily profit, which we are going to measure in thousands of dollars. What is your daily profit if your daily production is 10,000 gallons of Apple-Cranberry and 10,000 gallons of Cranberry-Apple?

$\boxed{10}$ Your profit depends on your daily production of Apple-Cranberry juice and Cranberry-Apple juice. That is, your profit is a function of the two variables x and y. Let's denote this function by $P(x, y)$. Find a formula for $P(x, y)$. Since profit is the thing that we are interested in maximizing, we call this our **objective function**.

> *We can now rephrase Key Question I:* "Find the point (x, y) in the feasible region at which the objective function is greatest."
>
> ***Important Fact:*** *The region of feasibility is a polygon and the point (x, y) at which profit is greatest is one of its corners (also known as* vertices*).*
>
> *We'll see more later about* why *this fact is true. Let's just accept it for now and use it to solve Key Questions I, II, and III.*

$\boxed{11}$ The feasible region you've drawn should have **four** vertices. The origin is one vertex. One vertex sits on the x-axis and another on the y-axis. The fourth vertex is the point of intersection of the two lines you drew when you were drawing the feasible region. Find the exact coordinates (x, y) of each of the four vertices.

> *Our **Important Fact** says that the maximum profit occurs at one of the vertices of the feasible region. Therefore, if we compute the profit at each of the vertices, the largest value we see will be the maximum possible profit.*

$\boxed{12}$ Evaluate $P(x, y)$ at each of your four vertices and answer Key Questions I, II, and III.

> *We now have a method for maximizing an objective function $P(x, y)$ subject to one or more constraints.*
>
> **Step 1:** *Use the constraints to graph the feasible region.*
>
> **Step 2:** *Find the exact coordinates (x, y) of each vertex of the feasible region.*
>
> **Step 3:** *Plug the coordinates of each vertex into the objective function $P(x, y)$. The largest value you obtain is the maximum possible value of $P(x, y)$.*

$\boxed{13}$ We're now ready to justify the **Important Fact**.

a) To get a feel for how profit depends on x and y, find all points in the first quadrant where your daily profit is \$2,500. That is, find all solutions of the equation

$$P(x, y) = 2.5$$

This corresponds to a line in the (x, y)-plane. Draw that line on the grid that contains your feasible region. Let's gradually increase our profit. Draw the constant profit lines $P(x, y) = 5$, $P(x, y) = 7.5$, and $P(x, y) = 10$. For future reference, label each line by the profit to which it corresponds (i.e. $P = 2.5$, $P = 5.0$, etc.).

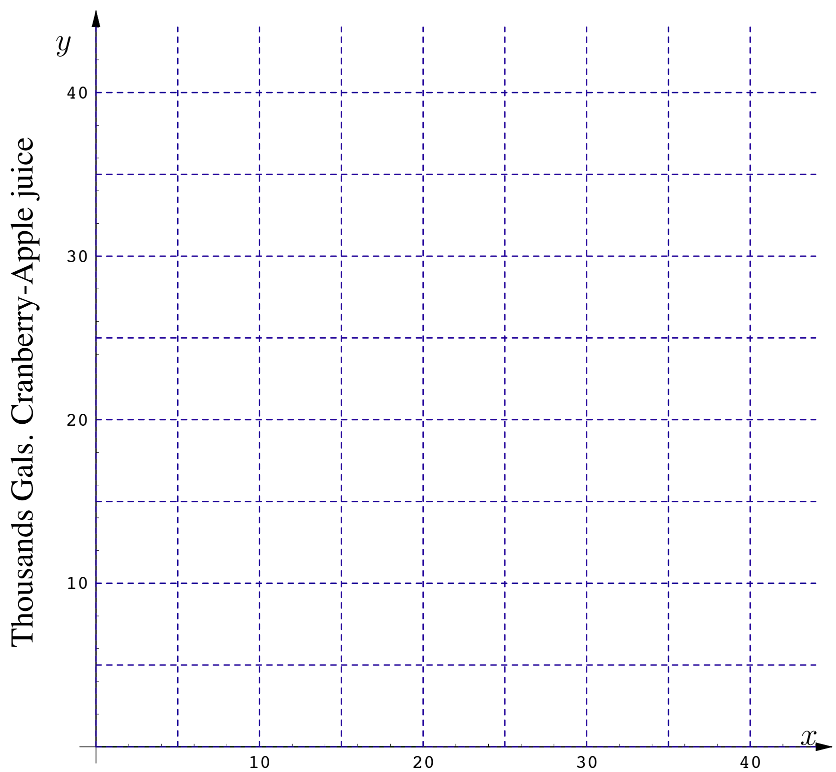
b) What do you notice about the lines that you drew?

c) Without computing anything, guess what the line $P(x, y) = 12.5$ might be.

d) Use the graph and parts (a)-(c) to explain why the maximum possible profit occurs at a vertex of the feasible region. (Hint: Look at how the constant profit lines change as profit increases and remember that only points in the feasible region matter.)

$\boxed{14}$ Now consider the situation described in Key Question IV. The additional conditions restrict the region of feasibility still further. Express each of the constraints described in IV as an inequality.

$\boxed{15}$ In the grid below, draw the region of feasibility corresponding to the conditions given in IV. (Don't forget to include the constraints $A(x, y) \leq 12$ and $C(x, y) \leq 11$.)



$\boxed{16}$ Find the exact coordinates of all of the vertices and answer Key Question IV.

→ $\boxed{17}$ The *Grass is Greener* lawn care company produces two different lawn fertilizers, Regular and Deluxe. The profit on each bag of Regular is \$0.75, while the profit on each bag of Deluxe is \$1.20. Each bag of Regular contains 3 pounds of active ingredients and 7 pounds of inert ingredients. In contrast, each bag of Deluxe contains 4 pounds of active ingredients and 6 pounds of inert ingredients. Due to limited warehouse facilities, the company can stock up to 8,400 pounds of active ingredients and 14,100 pounds of inert ingredients. Let x denote the number of bags of Regular, and let y denote the number of bags of Deluxe.

a) What is the formula for $P(x, y)$, the profit earned in producing x bags of Regular and y bags of Deluxe?

b) What is the formula for $A(x, y)$, the number of pounds of active ingredients needed to produce x bags of Regular and y bags of Deluxe? What is the formula for $I(x, y)$, the number of pounds of inert ingredients needed to produce x bags of Regular and y bags of Deluxe?

c) Sketch the region of feasibility. Find the (x, y) coordinates of all vertices of the region of feasibility?

d) How many bags of each product should the company produce to maximize profit?

→ $\boxed{18}$ Your company makes two household cleaners: "Miracle Bathtub Cleaner" and "Speedex Floor Cleaner". Your daily production of both cleaners combined is limited to 2,000 gallons. Your daily sales of Miracle Bathtub Cleaner never exceed 1,200 gallons, and your daily sales of Speedex Floor Cleaner never exceed 1,400 gallons. Finally, you make \$1.00 profit on each gallon of Miracle Bathtub Cleaner that you sell and \$2.00 on each gallon of Speedex Floor Cleaner that you sell.

Let x be your daily production (in thousands of gallons) of Miracle Bathtub Cleaner, and let y be your daily production (in thousands of gallons) of Speedex Floor Cleaner. The goal of the problem is to find the daily production schedule that will yield the greatest profit. You will do this by formulating the problem as a linear programming problem.

a) Find the formula for your daily profit, $P(x, y)$ (expressed in thousands of dollars).

b) Sketch the feasible region.

c) Find the exact coordinates of all vertices of the feasible region.

d) How much of each cleaner should you produce daily in order to maximize your profit? Explain your answer.

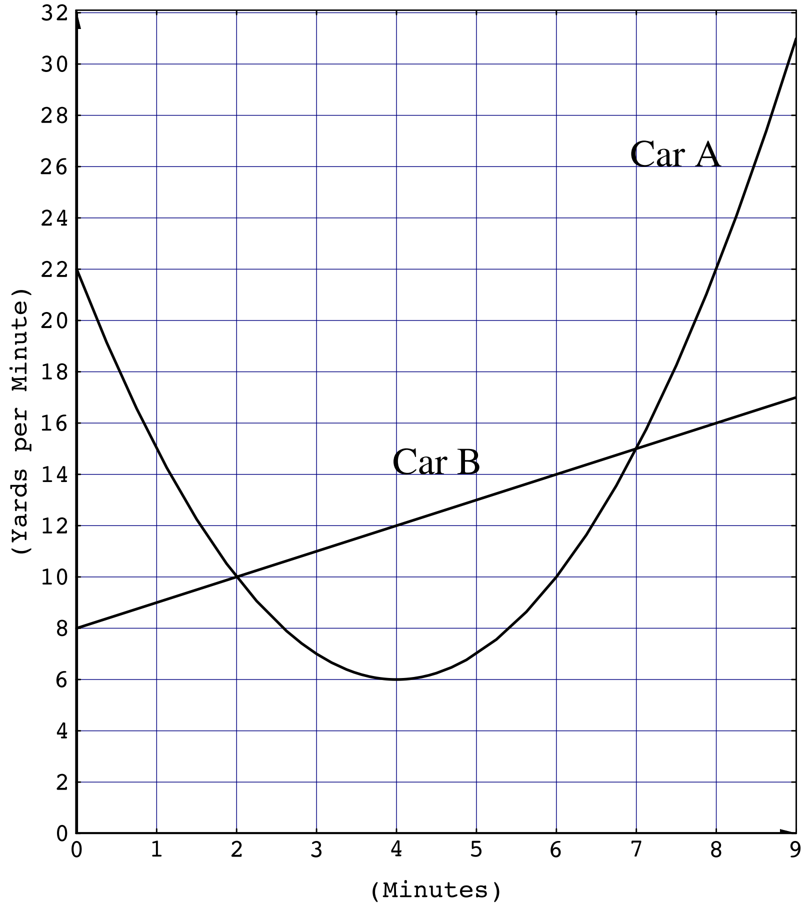
e) What is your maximum profit?

Chapter 6

Going Backwards: Integration

Worksheet #19 *Distances from Speed Graphs*

The graphs below are of speed *vs*. time for two cars traveling along a straight track. The cars are next to one another at time $t = 0$.



Key Questions

I. What is the distance between the two cars (and who is ahead) when the speed graphs cross?

II. At what time does Car B first catch up to and pass Car A?

III. Does Car A overtake Car B in the period 7 to 9 minutes?

---

1 The readings of the two graphs for the first 2 minutes are:

t	0	.5	1	1.5	2
Car A	22	18	15	12	10
Car B	8	8.5	9	9.5	10

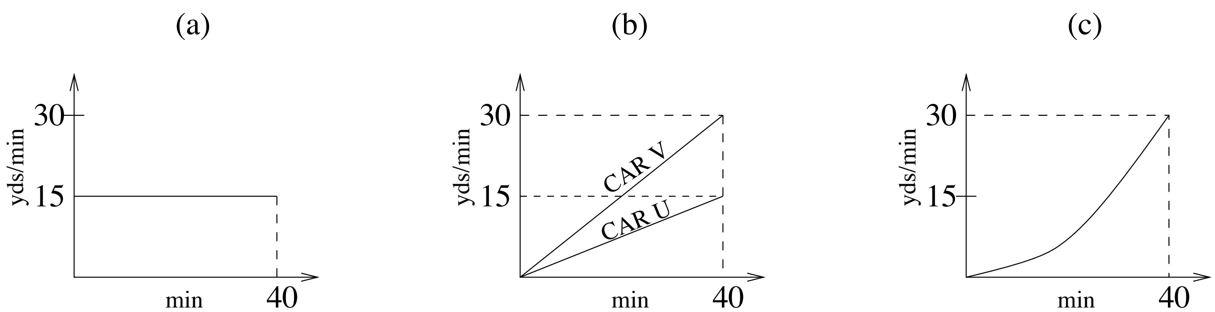
Does this mean that the two cars are getting farther apart or closer together? Explain.

2 Starting right after $t = 2$ minutes, Car B is going faster than Car A. Describe the relative positions of the two cars over this time. Car A goes faster than Car B for the first two minutes; Car B goes faster than Car A for the next two minutes. Does this mean that Car B catches up to Car A at time $t = 4$? Explain your answer in everyday terms.

To answer the two numerical Key Questions we will have to find a reasonable way to read distances from the speed graphs.

3 We know that average speed = (distance traveled)/(time elapsed). That means that distance traveled = (average speed)×(time elapsed). Use this formula to determine the distance covered by a car if it travels, on average, 20 yards/min for 30 minutes.

4 In graph (a) below, a car travels at a constant speed of 15 yards per minute. How far does this car travel in 40 minutes?



5 To apply the formula given in Question #3 to either of the graphs (b) above, you need an average speed to multiply the 40 minutes by. But both of the cars, Car U and Car V, are speeding up the whole time. Car U's speed goes from 0 to 15 yards per minute over the 40 minutes. Half-way between 0 and 15 yards per minute is the speed 7.5 yards per minute. Car U spends exactly half of the 40 minutes traveling at a rate less than 7.5 yards per minute and the other half traveling at a rate faster than 7.5 yards per minute. In this case, since Car U's speed is linear, the average speed of Car U over this forty-minute period actually is 7.5 yards per minute. How far does Car U travel during this forty-minute period?

6 In graph (b) above, what is Car V's average speed during the 40 minutes shown? How far does Car V travel in that forty minutes?

$\boxed{7}$ Now consider the car whose speed graph is given in graph (c) above. Does this car cover more or less distance than Car V in graph (b)? Explain.

The moral is that we can easily compute an average speed for straight line speed graphs, but it becomes more difficult for curvy graphs. It will turn out that we can get around this difficulty.

$\boxed{8}$ The speed graph of Car A (in the beginning of this worksheet) is quite close to being a straight line for the first two minutes, while the graph of Car B *is* a straight line. Estimate the average speeds of the two cars over the first two minutes and use the average speeds to compute the distances covered by the two cars in the first two minutes. Then you can answer Key Question I for the first time the speed graphs cross: What is the distance between the cars at $t = 2$ minutes?

Answering Key Question I for the second time the speed graphs cross is going to be much more complicated than what we just did, because the graph of Car A is not anywhere close to being a straight line for an interval lasting more than two minutes. The key to finding the distance covered by Car A in the first seven minutes is to find how far Car A has gone in each *of the 1-minute intervals 0 to 1, 1 to 2, 2 to 3, etc. and then add these distances up. We can find these smaller distances because for each of the 1-minute intervals the graph of Car A is close to being a straight line.*

$\boxed{9}$ By estimating the average speed over each interval, fill in the chart below. Each entry should be a distance covered by the given car in the given 1-minute interval.

Interval	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9
Distance covered in interval by Car A	18.5	12.5	8.5						
Distance covered in interval by Car B	8.5	9.5	10.5						

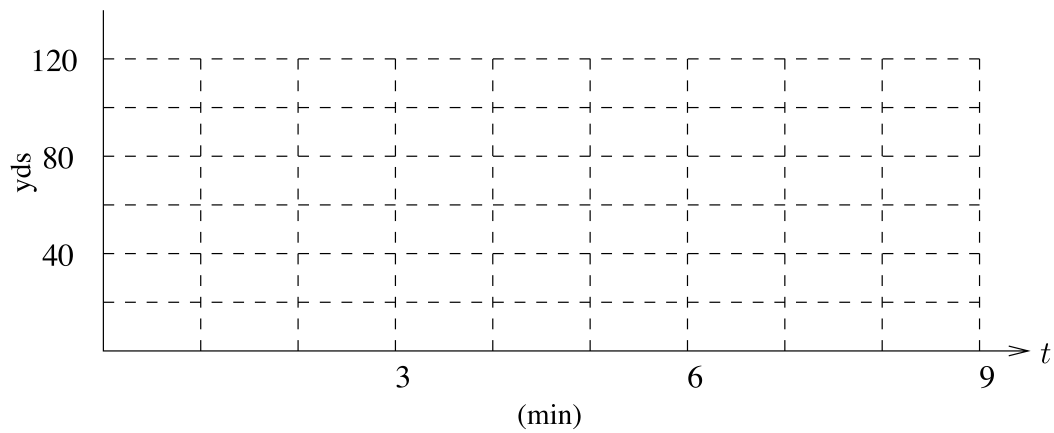
$\boxed{10}$ First use the chart in Question #9 to find out how far the two cars have gone in the first 2 minutes and check your answer to Question #8. Then use the chart to find out how far the two cars have gone in the first 7 minutes. Answer Key Question I for the second time the speed graphs cross.

The chart in Question #9 gives lots of distance information, but for instance, it is not particularly helpful for answering Key Questions II or III (try them). The difficulty is that these questions are about distances covered overall from the beginning of the "race" to a given time, and the chart gives distances covered in one minute intervals. We need another kind of chart.

$\boxed{11}$ For each of the times in the chart below, fill in the distance covered by each car from time $t = 0$ to that time by adding up the appropriate entries in the chart in Question #9.

Time t	0	1	2	3	4	5	6	7	8	9
Distance covered by time t by Car A	0	18.5	31	39.5						
Distance covered by time t by Car B	0	8.5	18	28.5						

$\boxed{12}$ The chart in Question #11 is just the usual table of distances covered *vs*. time for 2 cars. Using this cart, sketch smooth distance *vs*. time graphs for the two cars on the axes below.



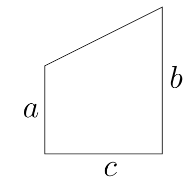
$\boxed{13}$ Use the distance *vs*. time graphs or chart to answer Key Questions II and III.

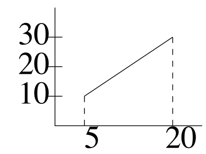
You just drew the distance vs. time graphs for the two cars. We started with the speed graphs. Thus the graphs we started with are the derived graphs, and we have found the "original graphs."

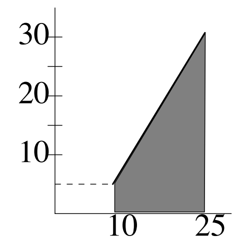
$\boxed{14}$ Use your distance *vs*. time graphs to tell when Car A is farthest ahead of Car B. According to a standard principle, the two cars should have the same speed at this time. Use the speed graphs to check to see if this is true. Repeat this question with the time at which Car B is farthest ahead of Car A.

By careful graph reading and breaking the graph of Car A into one-minute intervals we have managed to answer the Key Questions and also draw the distance vs. time graphs for the two cars. But it would really be nice to have a visual, pictorial way of reading the distances covered by the cars without all the computations—the way we can read speed from a distance graph by looking at the slope of the tangent. It turns out that this can be done.

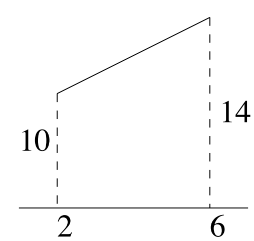
There is a formula from geometry that says that if you have a figure shaped like the one to the right (it is called a trapezoid), then the area of this figure is equal to $\frac{1}{2}(a+b)c$*. On the other hand, if you have a speed graph like the one shown to the right, then to calculate the distance covered by the car in this interval, you calculate the speed half-way between the 10 and the 30 (to get the average speed), and then multiply this speed by the time elapsed to get the distance covered in the interval. That is distance* $= \frac{1}{2}(10+30)15$*, because* $\frac{1}{2}(10+30)$ *gives the speed half-way between 10 and 30. Thus, what we have is the odd coincidence that in order to calculate the distance covered by the car, you do the same computations as you would do to calculate the area of the trapezoid that has the speed graph as its top. Here is another example. The speed graph just to the right here forms the top of a trapezoid—the shaded one. To get the distance covered by the car in that time interval you multiply* $\frac{1}{2}(5+30)$ *(which is the average speed) by 15, the time elapsed, to get 262.5. To get the area of that trapezoid you compute* $\frac{1}{2}(5+30)15$*. So the two quantities are the same.*







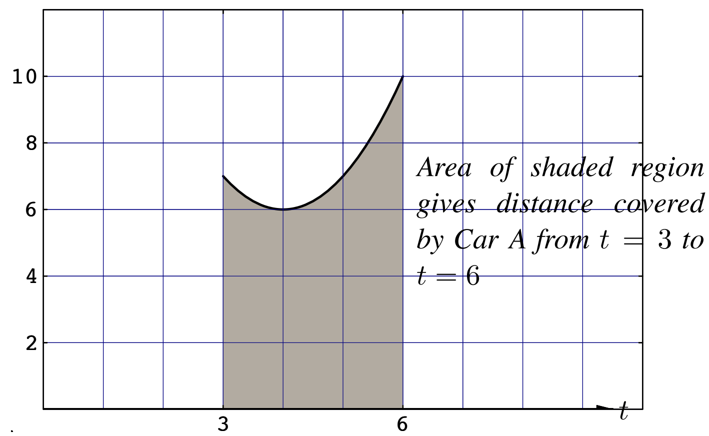
15 The speed graph for Car B for the time $t = 2$ to $t = 6$ is the top of a trapezoid with dimensions as shown to the right. Calculate the area of the trapezoid to find the distance covered by Car B over this time interval. Then check your answer using the table in Question #9.



16 The speed graph of Car A for the time interval $t = 7$ to $t = 9$ is (close enough to) the top of a trapezoid. Use the formula for the area of a trapezoid to determine the distance covered by Car A in this time interval. Check your answer using the table in Question #9.

17 Suppose you had just the first four minutes of the two speed graphs we started with, and you could alter the speed graph of Car B for the time interval $t = 3$ to $t = 4$ minutes. How would you change the graph so that Car B catches up to Car A at time $t = 4$?

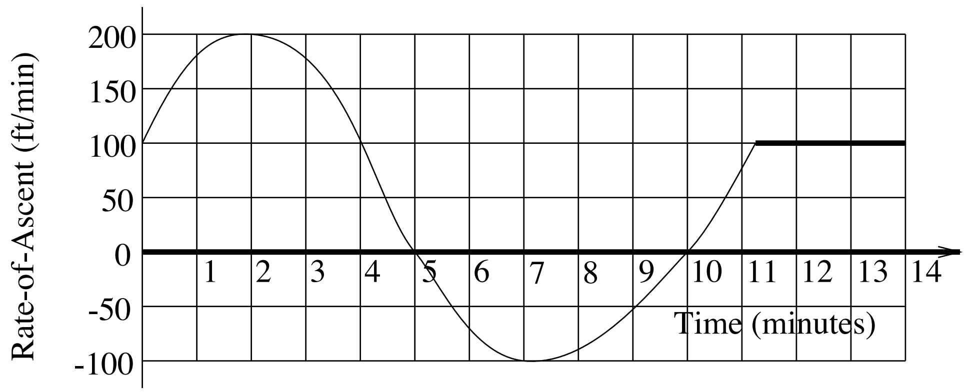
If we wanted the distance covered by Car A from time $t = 3$ *to time* $t = 6$*, we could, as usual, break the graph up into a bunch of one-minute-long straight line graphs. Then we could calculate the area under these graphs by using the formula, and then we would add. What we would be doing, then, is computing the area under the speed graph of Car A from time* $t = 3$ *to* $t = 6$ *minutes. Thus*



$\boxed{18}$ You can measure this area by counting the squares and partial squares in the region. I count 9 whole squares and then some partial partial squares. The area of each of these squares is 2 yds/min times 1 minute. So each square is worth 2 yards. First tell how many squares there are in the region shown above, and then use this number to tell how far Car A goes in the given time interval. Then use the table in Question #9 to check your answer.

$\boxed{19}$ The area under the speed graph of Car A from time $t = 0$ to time $t = 2$ indicates the distance covered by Car A in the first two minutes. The area under B's speed graph from $t = 0$ to $t = 2$ indicates the distance Car B goes in the first two minutes. Thus the area *between* the two graphs should indicate how far Car A is ahead of Car B at time $t = 2$ minutes. By counting up squares and partial squares between the two graphs from $t = 0$ to $t = 2$ estimate how far Car A is ahead of Car B at time $t = 2$. Then check your answer by using the table in Question #11. Give a similar interpretation to the area between the two speed graphs from $t = 2$ to $t = 6$. Compute this area. Explain how you could use these two areas between the graphs to tell which car is ahead at time $t = 6$.

→ $\boxed{20}$ The Altitude *vs*. time graph of a balloon is given by the function $A = F(t)$. We are not told the Altitude of the balloon at time $t = 0$. The graph of Rate-of-Ascent of the balloon is shown below. We call this function $r(t)$.



a) Name all the times when the graph of $F(t)$ has a horizontal tangent.

b) Give the time in the first <u>10</u> minutes when the balloon is highest.

c) Give the 2-minute interval during which the balloon gains the greatest altitude.

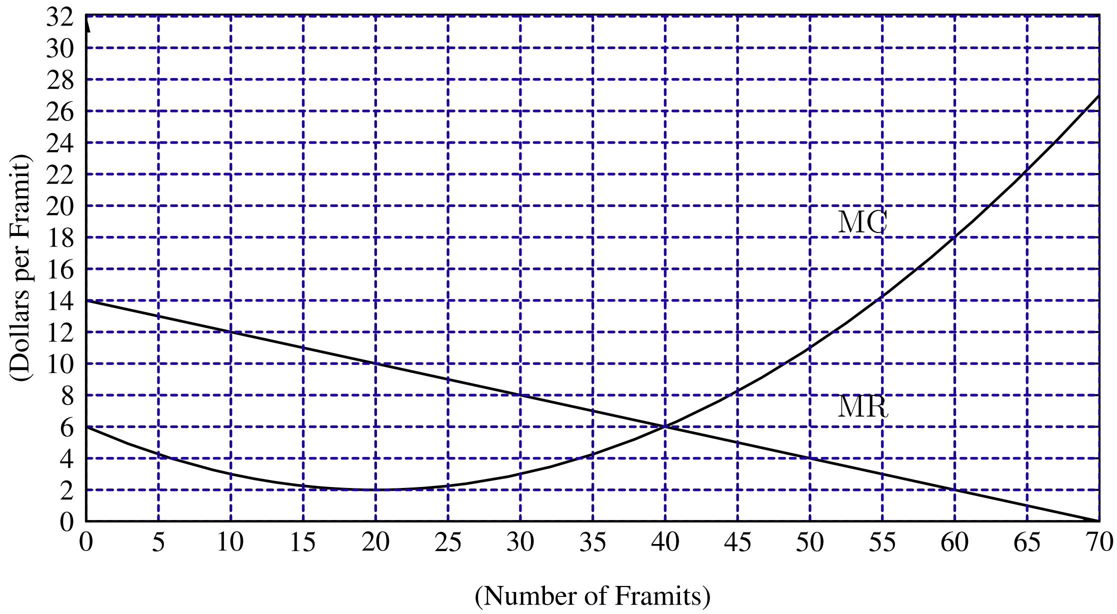
d) Give a 1-minute interval over which the balloon is ascending at a decreasing rate. ("None" is a possible answer.)

e) Find a 3-minute interval during which the balloon rises 550 feet.

f) What is the average rate of ascent of the balloon over the first 3 minutes?

Worksheet #20 *Total Revenue and Cost from Marginal Revenue and Cost*

The graphs below are of the Marginal Revenue and Marginal Cost of producing Framits.



Recall that the MR and MC graphs are the derived graphs of the TR and TC graphs, respectively, so that out there in the world someplace are TR and TC graphs which have the above graphs as derived graphs. Assume that Fixed Costs are 0, so that the TC graph goes through the origin.

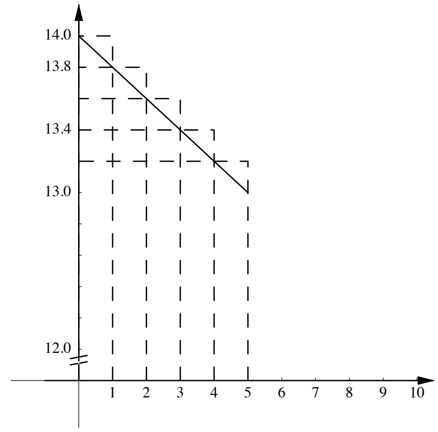
Key Questions

I. For what quantity q is the Profit of producing Framits maximized?

II. What is the largest value of q at which a profit can be made by manufacturing Framits?

III. What is the largest Profit that can be made by manufacturing Framits?

You should be able to answer Key Question I already from your knowledge of Marginal Revenue and Marginal Cost. But the other two Key Questions require that you read quantitative information from the MR and MC graphs in a way that you have not done before.

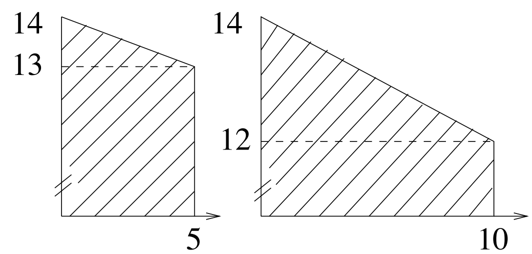


1 The official definition of MR is: The Marginal Revenue at a quantity q is the *additional* Total Revenue obtained by selling one more Framit. A very accurate blow-up of the beginning of the MR graph is shown to the right. Since the MR at $q = 0$ is 14, this says that the first item brings in Total Revenue of \$14. How much is the TR if you sell 2 Framits? What is the TR of selling 3, 4, 5 Framits? [Careful: to get the TR for 5 Framits you add up the MR for 0, 1, 2, 3, and <u>4</u> Framits.]

2 The MR at 5 Framits is \$13, and the MR at 10 Framits is \$12. Use these facts to draw an exact blow-up of the MR graph from $q = 5$ to $q = 10$ on the axes above.

3 By my calculations, the TR of selling 5 Framits is \$68.00. Use your accurate blow-up to calculate the TR of selling 6, then 7, then 8, then 9, and then 10 Framits.

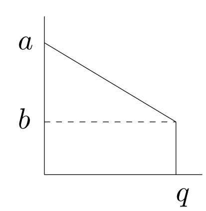
Clearly, we cannot continue making blow-ups of graphs and adding long columns of figures. We need a gimmick to short-cut these calculations—area is going to be the gimmick.



4 Calculate the area of the trapezoid that has the first 5 minutes of the MR graph as its top. Compute the area of the trapezoid that has the first 10 minutes of the MR graph as its top. Check that these areas are numerically very close to the Total Revenues of selling 5 and 10 Framits, respectively.

It certainly seems like another odd coincidence—that we can get TR either by adding a lot of numbers or by computing an area. But it is not really a coincidence; there is a theoretical justification for this fact. There is even a justification for why there is a slight discrepancy between the answers we get from the two methods. Unfortunately, these justifications are quite theoretical, and they do not seem to persuade students of anything. As you will see, computing TR by area works well—this is the best justification I know of.

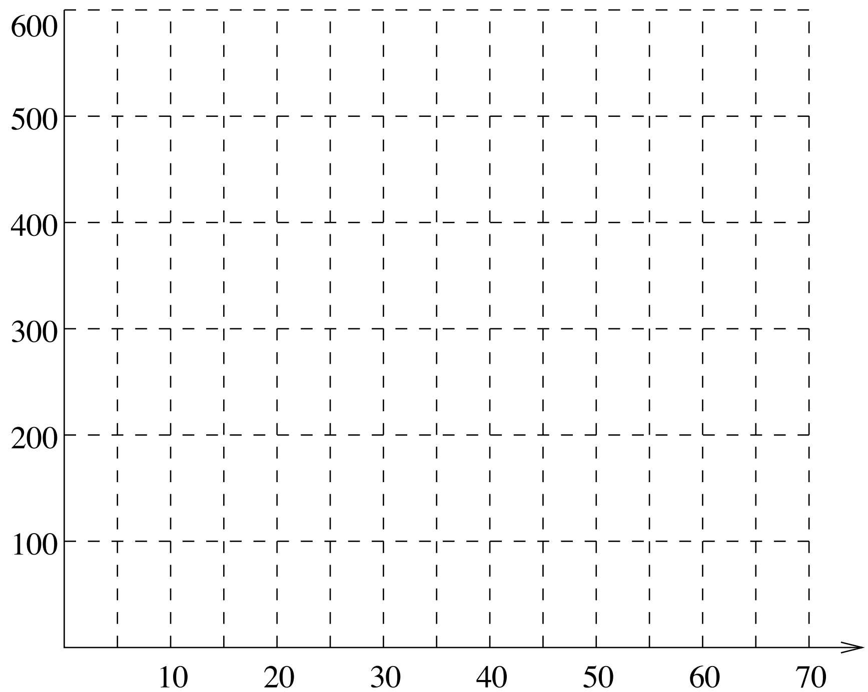
Thus, for now, we will take the following principle as accepted. Given a straight-line MR graph, as pictured at right, the Total Revenue of selling q items is the area of a trapezoid, and is given by $\frac{1}{2}(a + b)q$. A similar fact holds for the straight-line MC graph.



5 The chart below is of TR *vs.* q for selling Framits. Complete the chart by computing areas.

q	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70
TR	0	67.5	130			287.5			400		450				490

6 Use the chart above to draw a smooth graph of TR *vs*. q on the axes below.



7 The best check on this idea of computing TR by measuring areas is to see if it gives the correct TR graph—or more accurately, if it gives the TR graph that looks like it goes with the MR graph. Use the graph that you just drew to measure the Marginal Revenue of selling $q = 2, 4, 6$ Framits. (That is, measure the slope of the tangent to your graph at these places.) Check your readings against the readings on the MR graph given at the beginning of this worksheet.

It would be nice if we could find values of TC as easily as we have found values of TR, but, unfortunately, the MC graph is not a straight line. To get values of TC we have to use the same trick we used in the previous worksheet when faced with curvy graphs. We observe that even though the MC graph is not a straight line, short segments of it are close enough to straight lines for us to work with them as if they were straight lines. To keep things straight as we do so, we will need functional notation. Let $TC(q)$ be the Total Cost of manufacturing q Framits.

8 To my eyes the MC at $q = 0$ is 6, the MC at $q = 5$ is 4.2, and the graph is straight between $q = 0$ and $q = 5$. Use these facts to compute the TC at $q = 5$. (That is, find $TC(5)$.) Remember that Fixed Costs are assumed to be \$0.

9 To my eyes the MC graph from $q = 0$ to $q = 10$ has a little too much sag in it to be called a straight line, but the MC graph from $q = 5$ to $q = 10$ is close enough to being a straight line. Compute the area under that part of the MC graph (from $q = 5$ to $q = 10$).

I get 18 as the area under the MC graph from 5 to 10. But this does not mean that the Total Cost of manufacturing 10 Framits is \$18. It means that as you change q from 5 to 10, then your costs increase by \$18. That is, $TC(10) - TC(5) = \$18$. To get $TC(10)$ you have to add the \$18 to the TC at $q = 5$. By this method I get that $TC(10) = \$43.50$.

Thus, calculating TC from the curvy MC graph is a two-step process—just as getting the distances from the curvy speed graph was. First you need the areas of the slices that have straight lines as their tops. Then you must add.

$\boxed{10}$ The chart below is of *changes* in Total Cost as you go from one value of q to another. For example, it says that $TC(35) - TC(30) = \$18$. Complete the chart by measuring areas over the 5-Framit intervals.

Interval	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–60	60–65	65–70
Change in TC	25.5	18	13				18			48	63			122.5

$\boxed{11}$ By adding the entries in the above chart, complete the chart below of Total Cost *vs.* q for Framits.

q	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70
TC	0	25.5	43.5	56.5											

$\boxed{12}$ Use the chart in Question 11 to draw a smooth graph of TC *vs.* q on the axes below Question 6—on the same axes that have your TR graph.

$\boxed{13}$ By measuring some slopes on your TC graph, check that the MC graph we started with is the derived graph of your TC graph.

$\boxed{14}$ Use the TR and TC graphs you drew to answer Key Questions I, II, and III.

In the terminology of the first part of this course, the TC graph is the original graph, and the MC graph is the derived graph. Thus, as in the previous worksheet, we have constructed the original graph from the derived graph by using area.

When we use this process of constructing a new graph by measuring areas, we will say that we are forming an accumulated graph. *Forming the accumulated graph is, then, the opposite of forming the derived graph.*

We have used the fact that area under an MR (or MC) graph measures TR (or TC) when the graph is a straight line or is very close to it. To get answers from the curvy MC graph we broke it into straight-line segments. But the fact is that area under an MR (or MC) graph always *measures TR (or TC), as you can see by working the following question.*

15 To get the TC at $q = 30$, you found the areas of the six trapezoids: from 0 to 5, from 5 to 10, from 10 to 15, etc., and then added these areas up. Were you not just finding the area under the MC graph from $q = 0$ to $q = 30$, then? To check this, find the area under the MC graph from $q = 0$ to $q = 30$ by adding squares and partial squares. The area of each square is $2 \times 5 = \$10$.

16 Let $TR(q)$ be the Total Revenue of selling q Framits. By counting squares, find $TR(30)$.

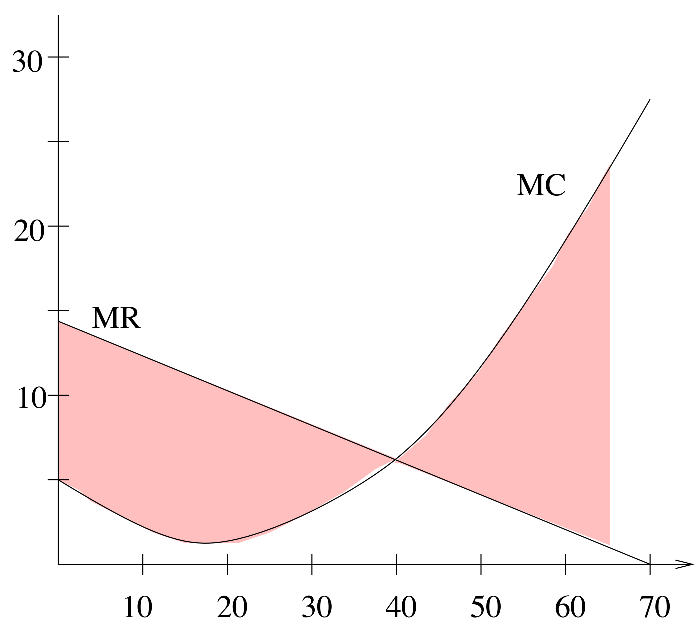
Now let $P(q)$ be the Profit that results from manufacturing and selling q Framits, so that $P(q) = TR(q) - TC(q)$.

17 Using your answers to questions 15 and 16, find $P(30)$. Check this answer against your TR and TC graphs.

Since $P(q) = TR(q) - TC(q)$, and $TR(q)$ is the area under the MR graph between 0 and q, while $TC(q)$ is the area under the MC graph from 0 to q, it must be that $P(q)$ is the area between *the two graphs from 0 to q.*

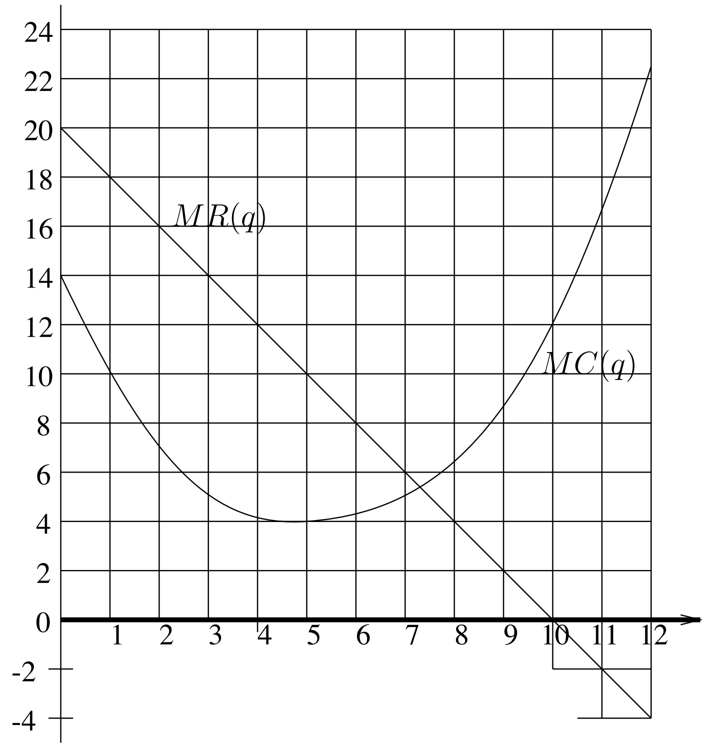
18 By counting squares, determine the area between the MC and MR graphs from $q = 0$ to $q = 15$. This should be $P(15)$. Check your answer against your TR and TC graphs.

As q takes the values 5, 10, 15, 20, etc., the area between the graphs increases, so that Profit increases. But then starting beyond $q = 40$, the MC graph is above the MR graph, so that additional items cost more than they bring in, and Profits decline. We take care of this by assigning areas negative value when MC is above *MR. Thus Profits reach their peak when the MC graph crosses the MR graph from below, as it does at $q = 40$. Furthermore, Profits will cease altogether when the area between the graphs to the right of $q = 40$ equals the area between the graphs to the left of $q = 40$, as in the picture below.*



$\boxed{19}$ In Question 14 you answered Key Question II and found the value of q where Profit drops to 0. Check that for this value of q the area between MR and MC from $q = 0$ to $q = 40$ balances out the area between MR and MC from $q = 40$ to your answer from Key Question II, as in the picture above.

$\longrightarrow$ $\boxed{20}$ The graphs to the right are of $MR(q)$ and $MC(q)$ (both in **dollars**) for producing q **thousand** Trivets.



a) Determine the increase in the Total Revenue that results from increasing q from 2 to 5 thousand Trivets.

b) Give the lowest value of MC in the interval 0 to 9 thousand Trivets.

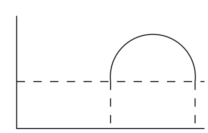
c) By how much does Profit change when q is changed from 3 to 6 thousand Trivets? Is this an increase or a decrease?

d) What is the largest possible Total Revenue that results from selling Trivets?

e) Is there a net loss from producing 12 thousand Trivets. (Assume that Fixed Costs $= \$0$).

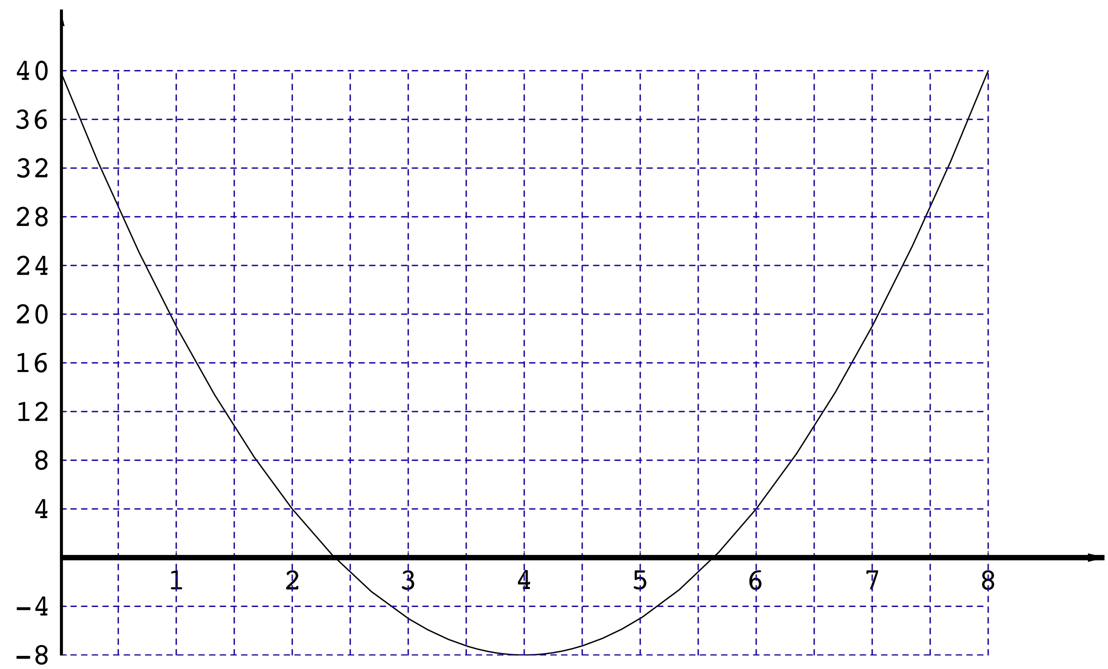
f) Determine the value of $TC(6) - TC(4)$.

g) Give a 3 thousand Trivet interval over which the graph of Profit *vs*. quantity is roughly shaped like the sketch to the right. (“None” is a possible answer.)



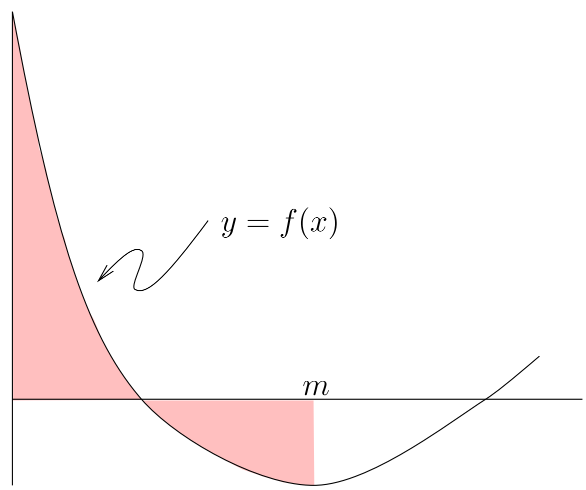
Worksheet #21 ***Area Under Graphs***

The graph below is of the function $y = f(x)$.



For any value of m between 0 and 8 we use this graph to define another function, which we shall call $A(m)$, as follows:

$$A(m) = \left(\begin{array}{l}\text{The area below the graph}\\ \text{of } f(x) \text{ and above the } x\text{-axis}\\ \text{from } x = 0 \text{ to } x = m\end{array}\right) - \left(\begin{array}{l}\text{The area above the graph}\\ \text{of } f(x) \text{ and below the } x\text{-axis}\\ \text{from } x = 0 \text{ to } x = m\end{array}\right)$$



Key Questions

For each of the statements below tell whether it is true (T), false (F), or there is not enough information to tell (DK).

I. $A(2) > A(1)$

II. $A(4) < A(5)$

III. $A(5.7) = 0$

IV. $A(7) = A(1)$

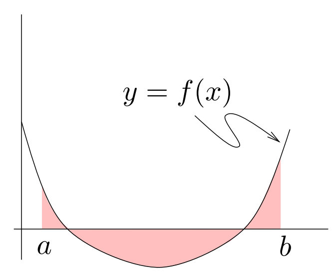
$\boxed{1}$ From $x = 0$ to $x = 0.5$, the region between $f(x)$ and the x-axis is all *above* the x-axis. So, $A(0.5)$ is simply the area of the region between $f(x)$ and the x-axis. On the graph of $f(x)$ given at the beginning of the worksheet, shade in the region whose area is $A(0.5)$ and compute $A(0.5)$.

$\boxed{2}$ On the graph of $f(x)$ given at the beginning of the worksheet, shade in the regions whose areas are $A(1)$ and $A(2)$. (Use different colors or shading patterns (for example, stripes vs. dots) to mark the two different areas.) Answer Key Question I.

$\boxed{3}$ We would like to decide on the value of $A(0)$. To do so, look at the definition of $A(m)$ and replace m by 0. Do you agree that $A(0)$ is the area of a region that has width 0? What would you say is the area of such a rectangle?

To go further with these questions we will introduce a notation for describing areas related to graphs.

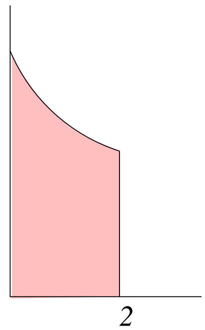
$$\int_a^b f(x)\,dx = \left(\begin{array}{l}\textit{The area below the graph}\\ \textit{of } f(x) \textit{ and above the } x\textit{-axis}\\ \textit{from } x=a \textit{ to } x=b\end{array}\right) - \left(\begin{array}{l}\textit{The area above the graph}\\ \textit{of } f(x) \textit{ and below the } x\textit{-axis}\\ \textit{from } x=a \textit{ to } x=b\end{array}\right)$$



For example, each of the symbols below stands for the indicated area.

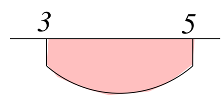
$$\int_0^2 f(x)\,dx$$

is the area of this region



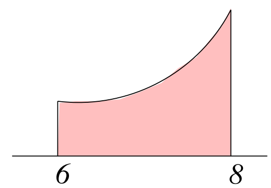
$$\int_3^5 f(x)\,dx$$

is the negative of the area of this region



$$\int_6^8 f(x)\,dx$$

is the area of this region



This symbol is a complicated one and requires some explaining. There are four parts:

(i) *The "$\int$" tells you that you are finding an area. This symbol is called an* integral sign.

(ii) *The "$f(x)$" tells you that we are measuring areas between the graph of the particular function $f(x)$ and the x-axis. This helps keep things straight when there are several graphs present, as in previous worksheets.*

(iii) *The letters or numbers at the bottom and the top of the integral sign tell you where along the x-axis you will start and stop measuring the area.*

(iv) *The "dx" tells you that the integral expression is over. Think of the $\int$ and the dx like bookends or parentheses. You shouldn't have one without the other.*

The entire symbol $\int_a^b f(x)\,dx$ *is called the* definite integral *of $f(x)$ from a to b.*

$\boxed{4}$ Using the graph of $f(x)$ given at the beginning of the worksheet, count squares and partial squares to compute each of the following definite integrals. (Notice that in this case, the area of each square is $4 \times 0.5 = 2$.)

a) $\displaystyle\int_0^2 f(x)\,dx$

b) $\displaystyle\int_3^5 f(x)\,dx$

c) $\displaystyle\int_6^8 f(x)\,dx$

Back to the Key Questions: the function we are working with in this worksheet can be written as

$$A(m) = \int_0^m f(x)\,dx.$$

For example, $A(1) = \int_0^1 f(x)\,dx$ *and* $A(2) = \int_0^2 f(x)\,dx$. $A(1)$ *and* $A(2)$ *are both given by areas under a curvy graph. We use our old method of breaking the region up into chunks that look like trapezoids. We'll use chunks that are each 0.5 units wide. Notice that the area of each chunk can be expressed as its own definite integral:*

$$\begin{aligned} A(1) &= \int_0^1 f(x)\,dx = \int_0^{0.5} f(x)\,dx + \int_{0.5}^1 f(x)\,dx \\ A(2) &= \int_0^2 f(x)\,dx = \int_0^{0.5} f(x)\,dx + \int_{0.5}^1 f(x)\,dx + \int_1^{1.5} f(x)\,dx + \int_{1.5}^{2.0} f(x)\,dx \end{aligned}$$

$\boxed{5}$ Assume that for each of these 0.5-unit intervals the graph of $f(x)$ is a straight line, so that you can use the formula for the area of a trapezoid to determine the value of the four areas required to find $A(2)$ and $A(1)$. Then use these values (of the four areas) to find the value of $A(1)$ and $A(2)$.

Suppose that you wanted to find the value of $A(2.5)$. *What you would want to do is take your value of* $A(2)$ *and add to it the area between the graph and the* x*-axis between 2.0 and 2.5. But over that interval, part of the graph is above the* x*-axis and part is below. Remember that area above the* x*-axis is added and area below the* x*-axis is subtracted. I would say that between 2 and 2.5 there are 0.4 squares above the* x*-axis and 0.1 below, so that the net contribution to the area between 2 and 2.5 is 0.3 squares* $= 0.3 \times 2$ *units* $= 0.6$ *units. Thus, by my eyes* $A(2.5) = 40.9$.

$\boxed{6}$ Compute the value of $\int_{2.5}^{3.0} f(x)\ dx$ by using the formula for a trapezoid. (This integral should be a negative number since $f(x)$ is below the x-axis from $x = 2.5$ to $x = 3.0$.) Then use your value to compute $A(3.0)$.

$\boxed{7}$ Before we get caught up in minute computations, go back to the Key Questions. Key Question II asks if $A(5)$ is greater than $A(4)$. If you knew $A(4)$ and were going to compute $A(5)$, what would you do? Would you be producing a larger or smaller number? Answer Key Question II.

$\boxed{8}$ Key Question III asks if the area below the x-axis and above $f(x)$ between $x = 2.3$ and $x = 5.7$ balances out the area above the x-axis and below $f(x)$ from $x = 0$ to $x = 2.3$. What do you think?

$\boxed{9}$ Key Question IV asks you to compare the areas $\int_0^1 f(x)dx$ and $\int_0^7 f(x)dx$. Are they equal? Justify your answer.

$\boxed{10}$ The obvious next thing to do is to plot the graph of the function $A(m)$. But before doing this, describe what you think it will look like. It goes through the origin since $A(0) = 0$. For what values is $A(m)$ increasing/decreasing? Does it ever go below the horizontal axis?

To draw the graph of $A(m)$ we will use the method of drawing an accumulated graph that we used in the two previous worksheets. That is, we will use the formula for area of a trapezoid to get areas of slices and then add the areas of slices to get areas under the curvy graph. Below is a table of "areas" of the slices between the curve and the x-axis for 0.5 unit intervals. We put "area" in quotation marks because some of the numbers are negative (when the graph of $f(x)$ sits below the x-axis). When we say the "area" under the curve on the interval from a to b, we really mean the number $\int_a^b f(x)\,dx$, which may be negative.

Interval	*0–0.5*	*0.5–1*	*1–1.5*	*1.5–2*	*2–2.5*	*2.5–3*	*3–3.5*	*3.5–4*
Area under the curve for that interval	*17.2*	*12*	*7.4*	*3.7*	*0.6*	*-1.6*	*-3.1*	*-3.8*

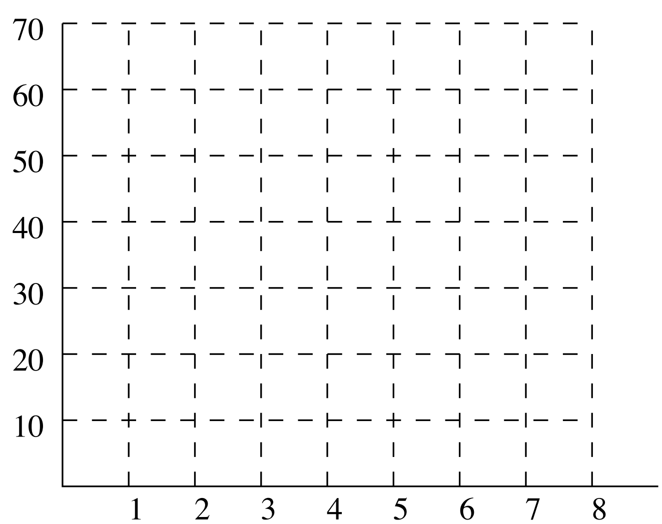
Interval	*4–4.5*	*4.5–5*	*5–5.5*	*5.5–6*	*6–6.5*	*6.5–7*	*7–7.5*	*7.5–8*
Area	*-3.8*	*-3.1*	*-1.6*	*0.6*	*3.7*	*7.4*	*12*	*17.2*

$\boxed{11}$ Use the above chart to complete the following chart of values of $A(m)$.

m	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
$A(m)$	0	17.2	29.2	36.6	40.3	40.9	39.3	

m	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
$A(m)$									

$\boxed{12}$ On the axes below plot the graph of $A(m)$ *vs.* m. Check that once again we have constructed the original graph from a derived graph.



$\boxed{13}$ In Question 4, you computed the value of $\displaystyle\int_6^8 f(x)\ dx$ by counting squares. Here are two other ways to compute this integral.

a) You could compute $\displaystyle\int_6^8 f(x)\ dx$ by adding four of the areas in the table preceding Question 11. Add the appropriate areas from that table to compute $\displaystyle\int_6^8 f(x)\ dx$ and compare to the value you found in Question 4.

b) By looking at the graph of $f(x)$, you should be able to see that

$$\int_6^8 f(x)dx = \int_0^8 f(x)dx - \int_0^6 f(x)dx.$$

But $\displaystyle\int_0^8 f(x)\ dx$ is what we're calling $A(8)$ and $\displaystyle\int_0^6 f(x)\ dx$ is $A(6)$. So,

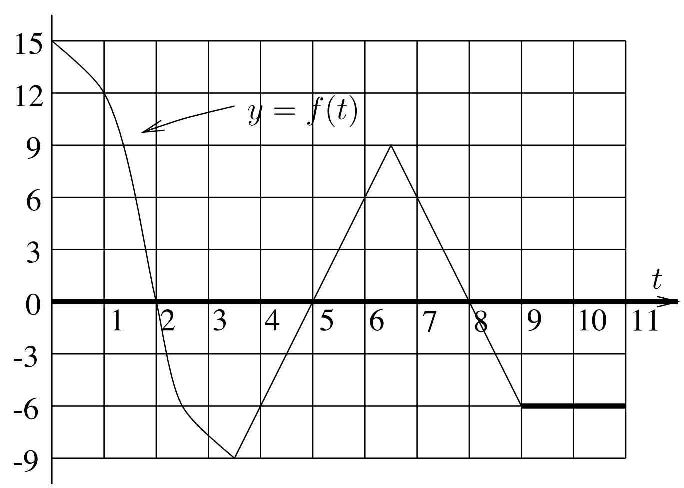
$$\int_6^8 f(x)dx = A(8) - A(6).$$

Use the table you compiled in Question 11 to compute $A(8) - A(6)$ and compare your answer to the value you found in Question 4.

$\boxed{14}$ Use a method similar to the one used in part (b) of the previous question to determine the value of $\displaystyle\int_2^6 f(x)dx$.

$\longrightarrow$ $\boxed{15}$ The graph to the right is of the function $y = f(t)$. Using it, we define another function:

$$A(m) = \int_0^m f(t)\ dt.$$



a) Find the value of m between 0 and 11 (including 0 and 11) at which $A(m)$ reaches a global maximum; at which $A(m)$ reaches a global minimum.

b) Find all values of m at which $A'(m) = 6$.

c) Give all values of m greater than 3.5 at which $A(m) = A(3.5)$.

d) Which of the following are true?

(i) The graph shown above is the derived graph of the graph of $A(m)$.

(ii) The graph of $A(m)$ is the derived graph of the graph shown above.

(iii) If we had formulas for $f(t)$ and $A(m)$ (and we changed the name of the variable m), then $f'(t) = A(t)$.

(iv) The graph of $A(m)$ is always above the m-axis.

e) Give the interval of length 1 over which $A(m)$ increases most. Give an interval of length 3 over which $A(m)$ increases most.

f) Determine the value of $\dfrac{A(7) - A(5)}{2}$

g) Determine the value of $A'(6.5)$.

h) Circle the correct answer: On the interval from from $m = 3.5$ to $m = 6.5$, $A(m)$:

Increases all the time	Increases then decreases	Decreases all the time	Decreases then increases	Remains constant	None of the preceeding

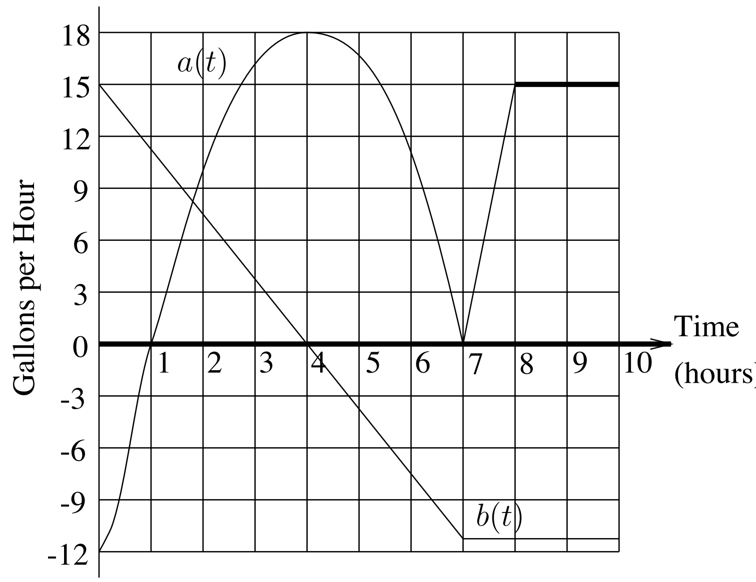
i) Imagine another function $g(t)$ drawn on the same axes as the graph of $f(t)$ above, but drawn parallel to $f(t)$ 3 units higher. Suppose we define $B(m) = \int_0^m g(t)\, dt$. Which of the following are true?

(i) $B(2) = A(2)$ (ii) $B'(2) = A'(2)$

(iii) $B(2) = A(2) + 6$ (iv) $B(2) = A(2) + 3$

(v) $B'(2) = A'(2) + 6$ (vi) $B'(2) = A'(2) + 3$

$\longrightarrow$ $\boxed{16}$ The graphs to the right are of rate-of-flow of water in and out of two vats. We will denote the various functions for this problem as follows:



$$\begin{aligned}
&\text{Amount of water in Vat A at time } t &= A(t)\\
&\text{Amount of water in Vat B at time } t &= B(t)\\
&\text{Rate-of-flow at Vat A at time } t &= a(t)\\
&\text{Rate-of-flow at Vat B at time } t &= b(t)
\end{aligned}$$

At $t = 0$, each vat contains 30 gallons.

a) Determine the value of $\int_1^3 b(t)dt$. Determine the value of $A(3) - A(1)$.

b) We define the function $D(t)$ by $D(t) = A(t) - B(t)$. That is, $D(t)$ is the difference in the amounts of water in the two vats. Give the largest value of $D(t)$ in 10 hours.

c) Give the highest value of $A'(t)$ in the interval 0 to 10 hours.

d) Give the time t when $A(t) = A(0)$ (t not equal to 0). Give the time t (t not equal to 2) when $B(t) = B(2)$.

e) Give a 2-hour interval over which the graph of $A(t)$ increases and then decreases. ("None" is a possible answer.)

f) Which of the following are true?

(i) $A'(7.2) > A'(3)$ (ii) $A'(9) = 0$

(iii) $B(9) < B(8)$ (iv) $A(6) < A(5)$

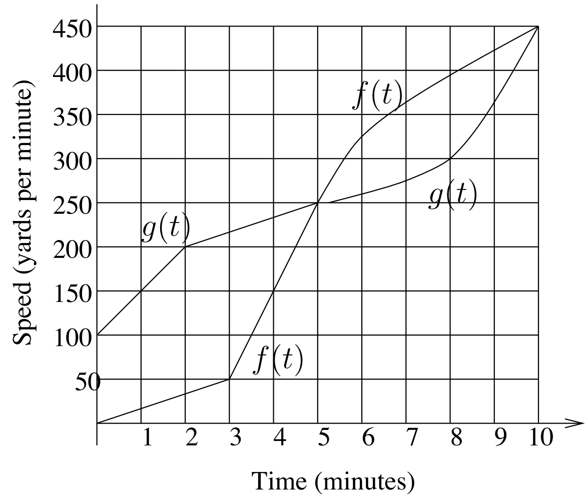
g) Give the global minimum of $A(t)$ in the interval 0 to 10.

h) Give the global minimum of $B(t)$ in the interval 0 to 10.

i) Give all *local* minima of $A(t)$ in the interval 0 to 10.

→ 17 The graphs to the right are of speed *vs*. time of two cars. The two cars are next to one another at time $t = 0$. We denote the functions for this problem as follows:

Distance covered by Car F by time t	$=$	$F(t)$
Distance covered by Car G by time t	$=$	$G(t)$
Speed of Car F at time t	$=$	$f(t)$
Speed of Car G at time t	$=$	$g(t)$



a) Tell which of the following statements are true.

(i) The graph of $F(t)$ is the derived graph of the graph of $f(t)$.

(ii) $G(m) = \int_0^m g(t)dt$

(iii) The graph of $G(t)$ is the accumulated graph of the graph of $g(t)$.

(iv) $F'(t) = f(t)$

b) Give a *precise* value for the distance between the two cars at time $t = 5$. (Hint: Use areas of trapezoids.)

c) Give a 3-minute time interval over which the F-Car is slowing down. ("None" is a possible answer.)

Let $D(t)$ be the distance by which Car G is ahead of Car F at time t, i.e., $D(t) = G(t)-F(t)$.

d) Determine the value of $D(9) - D(8)$.

e) Give a time after $t = 3$ when $D(t) = D(3)$.

f) Give a 3-minute interval between $t = 0$ and $t = 10$ over which $D(t)$ decreases and then increases. ("None" is a possible answer.)

g) Give the average speed of Car G over the first 2 minutes.

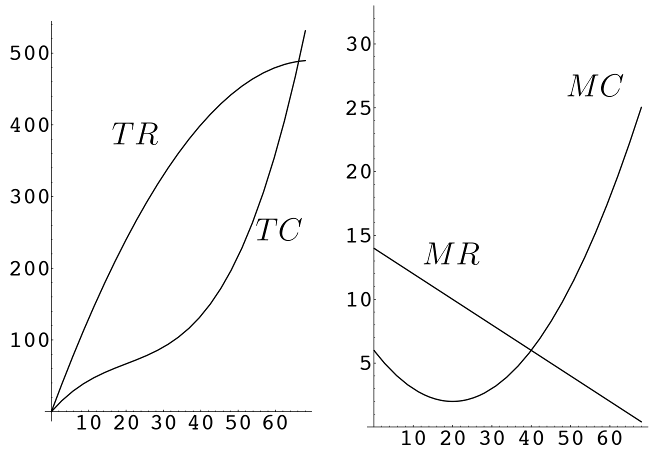
h) Give a time interval over which the distance between the cars increases by 200 yards.

i) Give *all* times (if any) for which the graph of $D(t)$ has a horizontal tangent.

Worksheet #22 *Areas by Formula*

Key Question

In Worksheet #20 we plotted TR and TC graphs from MR and MC graphs by computing areas. Assuming $FC = \$0$, we got the graphs on the left of TR and TC from the graphs on the right of MR and MC.



Key Questions

I. How do we get *formulas* for TR and TC from *formulas* for MR and MC?

II. How do we deal with a more general value of FC?

$\boxed{1}$ We found $TR(10)$ by computing the area under the MR graph from $q = 0$ to $q = 10$. Use the notation from the previous worksheet to express $TR(10)$ as a definite integral. Do the same for $TR(50)$ and $TR(0)$.

$\boxed{2}$ The graph of MR in Worksheet #20 is a straight line. Go back to the beginning of that worksheet and find the formula for $MR(q)$.

$\boxed{3}$ In Exercise #2, you should have found that

$$MR(q) = -0.2q + 14.$$

You know that $MR(q) = TR'(q)$. Since MR is *linear*, the formula for TR must be quadratic:

$$TR(q) = aq^2 + bq + c.$$

Use the fact that $TR'(q) = MR(q)$ to find the values of a and b. Use the fact that the graph of TR always goes through the origin to find the value of c.

$\boxed{4}$ Use the formula you just found for $TR(q)$ to compute $TR(10), TR(50)$, and $TR(0)$. Compare these values to the chart in #5 in Worksheet #20.

$\boxed{5}$ In Worksheet #20, we assumed $FC = \$0$ and found $TC(q)$ by computing the area under MC from 0 to q. If $FC = \$0$, then there is no difference between TC and VC. The graph of TC shown at the beginning of this worksheet is also the graph of VC.

a) If FC is increased to \$100, how would the graphs of TC, VC, and MC change?

b) If FC is increased to \$200, how would the graphs of TC, VC, and MC change?

So, different graphs of TC can have the same graph of MC. Starting with the graph of MC, how do we determine which of these is the correct TC graph?

Let $A(m) = \int_0^m MC(q)\, dq$, the area under the marginal cost graph from $q = 0$ to $q = m$. Then we know:

- *MC is the derivative of A; and*
- *$A(0) = 0$ (That is, the graph of A goes through the origin.)*

But MC is also the derivative of TC. So, TC and A have the same derived graph. That means that the graph of TC must look exactly like the graph of A, with (possibly) a vertical shift. Since the graph of A looks just like the graph of TC AND it goes throught the origin, the graph of A must be the graph of VC.

That is, in general, it is $VC(q)$, not *$TC(q)$, that is given by the area under MC from 0 to q. But TC and VC have the same derived graph, MC. So, $VC'(q) = MC(q)$.*

$\boxed{6}$ The formula for $MC(q)$ is quadratic: $MC(q) = 0.01q^2 - 0.4q + 6$. Since $VC'(q) = MC(q)$, the formula for variable cost must be cubic:

$$VC(q) = aq^3 + bq^2 + cq + d.$$

Find the values of a, b, c, and d that give the correct formula for $VC(q)$.

$\boxed{7}$ In Worksheet #20, we assumed $FC = \$0$, so $TC(q) = VC(q)$. Use the formula you just found to compute $TC(0)$, $TC(10)$, and $TC(15)$. Compare to the chart in Exercise #11 in Worksheet #20.

$\boxed{8}$ Use the formula you found for $VC(q)$ to find a formula for $TC(q)$ if $FC = \$100$.

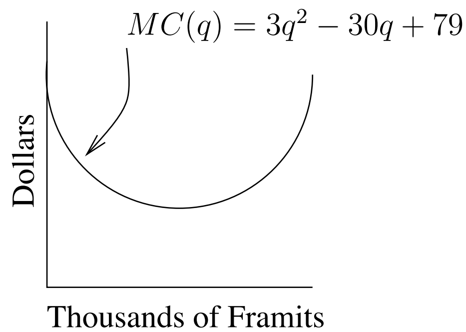
$\boxed{9}$ Write out the formula for profit $P(q)$ if $FC = \$100$.

We now know:

- *$TR(q)$ = the area under MR from 0 to q*
- *$VC(q)$ = the area under MC from 0 to q*
- *$TC(q)$ = (the area under MC from 0 to q) + FC*

$\boxed{10}$ In Worksheet 20, we said that profit $P(q)$ was the area between MR and MC from 0 to q. But that was assuming that FC was \$0. Describe how to compute $P(q)$ from the graphs of MR and MC if you must consider a non-zero value of fixed cost.

→ 11 To the right is the graph of the Marginal Cost of producing Framits. The formula for MC is $MC(q) = 3q^2 - 30q + 79$. The formula for the Total Revenue of selling Framits is $TR(q) = -(3/2)q^2 + 55q$, q is given in thousands of Framits and Total Revenue is given in thousands of dollars.



a) Write out the formula for $\mathrm{VC}(q)$.

b) Assume that Fixed Costs are 100 thousand dollars. Write out the formula for TC.

c) Write out the formula for the Marginal Revenue of selling Framits.

d) Find the value of q at which the Profit for producing Framits is greatest.

e) What is the largest possible Profit that can be made from producing Framits?

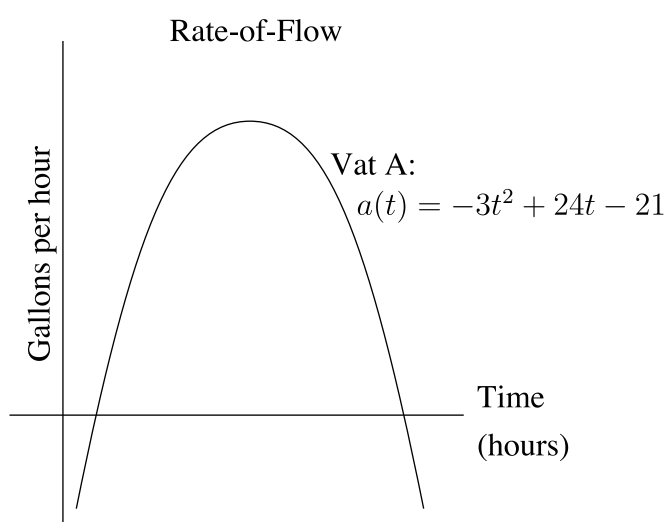
f) For what value of q is the TR from selling Framits largest?

→ 12 Water is flowing in and out of two vats. Below is the *rate* of flow graph for Vat A. Denote the Amount *vs*. time function for Vat A by $A(t)$ and its rate of flow by $a(t)$.

You are given that the *Amount vs*. time function for Vat B is given by the formula

$$B(t) = -2t^2 + 16t + 40.$$

The two vats have the same amount of water in them at time $t = 0$.



a) Determine all values of t at which the function $A(t)$ reaches a local maximum or local minimum on the interval 0 to 10 hours.

b) Determine all values of t at which the function $B(t) - A(t)$ reaches a local maximum or minimum on the interval 0 to 10 hours.

c) Write out the formula for the Amount *vs*. time function for Vat A. Use the fact that the vats have equal amounts of water in them at time $t = 0$.

d) At what time is the water level in Vat A rising most rapidly?

e) What is the largest amount of water in Vat A in the period $t = 0$ to $t = 10$ hours?

f) What is the highest rate at which water is coming into Vat B in the period $t = 0$ to $t = 10$ hours?

g) How much water comes into Vat A from time $t = 1$ to time $t = 8$ hours?

$\longrightarrow$ $\boxed{13}$ The speed *vs*. time graphs of two rocket cars are given by the following formulas:

$$\text{Car A} : s = a(t) = t^2 - 8t + 18 \qquad\qquad \text{Car B} : s = b(t) = -2t + 13$$

where t is in minutes and speeds are in miles per minute. Let $A(t)$ and $B(t)$ represent the distances covered by time t for the cars. Since we measure distances covered since time $t = 0$, we have $A(0) = 0$, and $B(0) = 0$.

a) Write out the formulas for $A(t)$ and $B(t)$.

b) Find the time when Car B is farthest ahead of Car A in the first 10 minutes.

c) Determine the greatest distance by which B gets ahead of A in the time period 0 to 10.

d) Determine the lowest speed at which Car A ever travels.

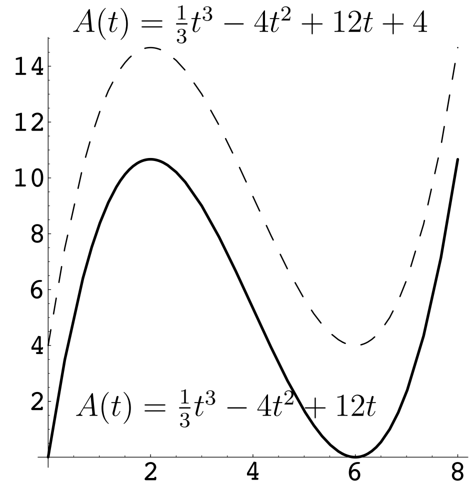
e) Write out the equation you would solve in order to find the times when one car passes the other. Your equation should be in the form $Pt^3 + Rt^2 + St + U = 0$.

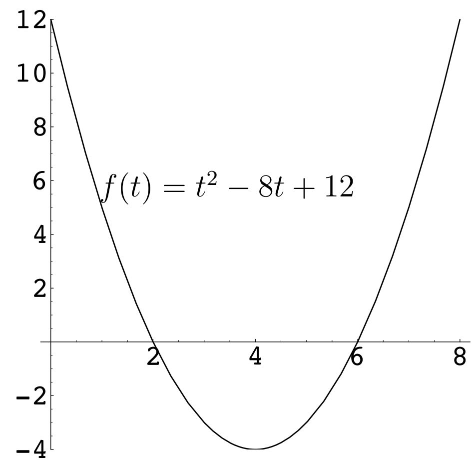
f) Determine the *two* times at which the speed of Car A is 8 mpm.

g) How far does Car B travel in the time $t = 2$ to $t = 5$?

Worksheet #23 *Anti-Derivatives*

The graph to the right below is of *rate-of-ascent vs.* time for a balloon. Its formula is $f(t) = t^2 - 8t + 12$.





If we denote the altitude *vs.* time function of this balloon by $A(t)$, then, since $A'(t) = f(t)$, we "guess backwards" that the formula for $A(t)$ must be $A(t) = \frac{1}{3}t^3 - 4t^2 + 12t$, which has the solid graph to the left above. But $\frac{1}{3}t^3 - 4t^2 + 12t$ is not the only formula that has $t^2 - 8t + 12t$ as its derivative. The derivative of $\frac{1}{3}t^3 - 4t^2 + 12t + 4$ is also $t^2 - 8t + 12$, so it could also be that $A(t) = \frac{1}{3}t^3 - 4t^2 + 12t + 4$, which is the dashed graph to the left above.

Key Questions

Is one of these formulas "more correct" than the other? If they are "equally correct", are there other "equally correct" formulas for $A(t)$? If so, how will we ever decide when we have found all the formulas?

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1. First check that $\frac{d}{dt}(\frac{1}{3}t^3 - 4t^2 + 12t) = t^2 - 8t + 12$ and that $\frac{d}{dt}(\frac{1}{3}t^3 - 4t^2 + 12t + 4) = t^2 - 8t + 12$, so that, for all we can tell, either one of these formulas could be the correct formula for $A(t)$.

2. Name at least one other formula that would be a candidate for the formula for $A(t)$, given the fact that $A'(t) = t^2 - 8t + 12$. How would the graph of your formula be related to the graphs of the two possible altitude *vs.* time graphs shown above?

3. Just playing with formula does not seem to tell us which altitude formula to choose, perhaps the graphs will tell us. The rate-of-ascent graph should be the derived graph of the altitude graph. Does this tell you which is the "more correct" altitude graph among the graphs shown above? Does one of those altitude graphs look like it should be preferred over others, given that we started with the rate-of-ascent graph shown?

$\boxed{4}$ There is a very subtle verbal distinction that casts some light on this confusing situation. Suppose that I told you that the graph to the left were not of altitude *vs*. time, but was of altitude *gained since time 0*. If you plotted altitude gained since time 0, then where would you start; what value would you plot at time 0? Do you agree that if the graph to the left were to be of altitude gained since time 0, then there would be only one correct graph to the left above?

Suppose that there were a clear grid on the graph of rate-of-ascent, and instead of working with formulas, we had plotted the accumulated graph of the rate-of-ascent graph by measuring areas. Then again we would have started from 0, and there would be only one correct graph to the left above. The reason for this is that, technically, we would have been plotting altitude gained *since time 0. There is no way we could actually plot the altitude unless we knew what altitude the balloon was at at time 0.*

We can guess backwards to find the formula for $A(t)$, given that $A'(t) = t^2 - 8t + 12$, but there are infinitely many correct guesses which differ from one another in the constant term, i.e. any of the following candidates is correct:

$$A(t) = \begin{cases} \frac{1}{3}t^3 - 4t^2 + 12t \\ \frac{1}{3}t^3 - 4t^2 + 12t + 5 \\ \frac{1}{3}t^3 - 4t^2 + 12t + 15 \end{cases}$$

Thus to cover all *possible answers we write:*

> *If $A'(t) = t^2 - 8t + 12$, then $A(t) = \frac{1}{3}t^3 - 4t^2 + 12t + k$, where k is an as yet undetermined constant.*

Similarly, If $f'(x) = 3x^2 + 4$, then $f(x) = x^3 + 4x + k$.

$\boxed{5}$ Suppose $G'(t) = 6t^2 + 2t - 5$. Write out the formula for $G(t)$, including the undetermined constant.

Here is a notation and terminology for this guessing backwards process. In Question 5 you are told that $G'(t) = 6t^2 + 2t - 5$, and you conclude that $G(t) = 2t^3 + t^2 - 5t + k$. This is because

> $6t^2 + 2t - 5$ *is the derivative of* $2t^3 + t^2 - 5t + k$.

We say that

> $2t^3 + t^2 - 5t + k$ *is the* anti-derivative *of* $6t^2 + 2t - 5$.

Similarly, the expression

> $3x^2 + 4$ *is the derivative of* $x^3 + 4x + k$

can be restated as

> $x^3 + 4x + k$ *is the* anti-derivative *of* $3x^2 + 4$.

We write $2t^3 + t^2 - 5t + k = \displaystyle\int (6t^2 + 2t - 5)\, dt$ *and* $x^3 + 4x + k = \int (3x^2 + 4)\, dx$.

The symbol $\int (\quad)\, dx$ *is read "the anti-derivative of." This symbol* $\int (\quad)\, dx$ *is part of the definite integral symbol, but you have to learn to keep the two symbols separate. They will be connected in the next worksheet, but they are different things. Observe that*

> $\int_a^b f(t)\, dt$ *stands for an area, and when you have a specific function* $f(t)$ *and specific values for* a *and* b*, it gives a specific number.*

On the other hand

> $\int f(t)\, dt$ *stands for a whole family of functions whose derivative is the given function* $f(t)$*. It always has the undetermined constant in it.*

The symbol $\int (\quad)\, dx$ *is sometimes called "an indefinite integral."*

$\boxed{6}$ By taking derivatives check that the following anti-derivatives are correct.

$$\int (4x^{1/3} + 4x^{-1/2})\, dx = 3x^{4/3} + 8x^{1/2} + k,$$

$$\int \frac{2}{\sqrt{w}}\, dw = 4\sqrt{w} + k,$$

$$\int \left[3t^4 + \frac{1}{8}t^{-3/2}\right]\, dt = \frac{3}{5}t^5 - \frac{1}{4}t^{-1/2} + k$$

It is time to develop techniques for writing out anti-derivatives, just as we had techniques for finding derivatives. Unfortunately, things do not work out so neatly for anti-derivatives as they did for derivatives. For instance, it is fairly difficult to find the anti-derivative $\int x(\ln(x))\, dx$*; it is extremely difficult to find the anti-derivative* $\int \frac{1}{1+x^2}\, dx$*, and it is impossible to find the anti-derivative* $\int e^{(x^2)}\, dx$*. Thus, in this book when dealing with anti-derivatives, we will consider only functions that can be written by adding expressions of the form* ax^m*, where* a *and* m *are any numbers at all. For example, we will find such anti-derivatives as*

$$\int \left[(\frac{3}{x^2}) + \sqrt{x}\right]\, dx \qquad\qquad \int \left[(\frac{4}{t^{2/3}} - 5t^3 + 4\right]\, dt.$$

To find the anti-derivative of a sum we proceed just as we did with derivatives—we consider each of the terms separately and then carry the plus signs down. Thus, to do the problems above, we will need to find the following anti-derivatives, and then we will need to add the answers:

$$\int \frac{3}{x^2}\, dx \qquad \int \sqrt{x}\, dx \qquad \int \frac{4}{t^{2/3}}\, dt \qquad \int (-5t^3)\, dt \qquad \int 4\, dt.$$

In general, we need to find $\int ax^m\, dx$*. By guessing backwards, the answer to this comes out as*

$$\int ax^m dx = \frac{a}{m+1}x^{m+1} + k$$

(except when $m = -1$, where this does not make sense).

To convince yourself that this anti-derivative formula is correct, take the derivative of

$$\frac{a}{m+1}x^{m+1} + k$$

and see if you get ax^m.

7 Using the above formula, find the following anti-derivatives. Then check your answers by taking derivatives.

a) $\displaystyle\int \frac{3}{x^2}\,dx = \int 3x^{-2}\,dx$ b) $\displaystyle\int \sqrt{x}\,dx = \int x^{1/2}\,dx$ c) $\displaystyle\int \frac{4}{t^{2/3}}\,dt = \int (4t^{-2/3})\,dt$

d) $\displaystyle\int (-5t^3)\,dt$ e) $\displaystyle\int 4\,dt = \int 4t^0\,dt$

Thus, you should now see that $\displaystyle\int \left(\frac{3}{x^2} + \sqrt{x}\right)\,dx = -3x^{-1} + \frac{2}{3}x^{3/2} + k$.

$$\int \left(\frac{4}{t^{2/3}} - 5t^3 + 4\right)\,dt = 12t^{1/3} - \frac{5}{4}t^4 + 4t + k$$

8 Find the following anti-derivatives. As always, check anti-derivatives by differentiating your answer.

a) $\displaystyle\int \left[3x^4 - \frac{2}{3x^2} + e\right]\,dx$ b) $\displaystyle\int \left(\frac{4}{\sqrt{t}} + \frac{5}{\sqrt{t^3}}\right)\,dt$

c) $\displaystyle\int \left[\frac{1}{6u^4} - \frac{3}{2u^3} - \frac{1}{2}\right]\,du$ d) $\displaystyle\int \left[\sqrt{e}w^2 + e\sqrt{w}\right]\,dw$

There is one thread we have left dangling. In the general formula for $\int ax^m\,dx$ we did not deal with the case $m = -1$. That is, we have not written out the anti-derivative $\int \frac{1}{x}\,dx$. The question is: What function as $\frac{1}{x}$ as its derivative? and the answer is that $[\ln(x)] + k$ has $\frac{1}{x}$ as its derivative. Thus

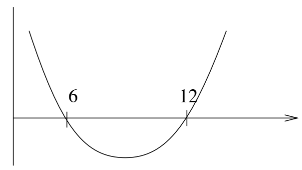
$$\int \frac{a}{x}\,dx = a\ln(x) + k.$$

$\longrightarrow$ **9** Evaluate the following anti-derivatives.

a) $\displaystyle\int \left(\sqrt{x} - \frac{3}{x^2}\right)\,dx$ b) $\displaystyle\int (u^{-2} + 4u^{-3} + 3)\,du$ c) $\displaystyle\int \left(2 - \frac{3}{\sqrt{z}} + \frac{4}{z}\right)\,dz$

d) $\displaystyle\int \left(w^{2/3} - \frac{3}{4w^{3/2}} - 6\right)\,dw$ e) $\displaystyle\int \left(\frac{1}{4}t^{1/4} - 2 + \frac{1}{t^{1/4}}\right)\,dt$ f) $\displaystyle\int \left(5x - \frac{3}{x^5} + 2\sqrt{x}\right)\,dx$

$\longrightarrow$ $\boxed{10}$ To the right is the graph of the function $f(x) = x^2 - 18x + 72$. We use this graph to define another function:



$$F(m) = \int_0^m f(x)\, dx = \int_0^m (x^2 - 18x + 72)\, dx.$$

a) Determine all values of m in the interval 0 to 15 at which $F(m)$ has a local minimum or local maximum. Tell which points give minima and which give maxima.

b) Determine the global maximum and global minimum of the function $F(m)$ on the interval 0 to 15.

c) Now suppose that we have another function $G(m)$ given by the formula $G(m) = -2m^2 + 48m$, and suppose we that plotted $G(m)$ and $F(m)$ on the same axes. Determine the value of m in the interval 0 to 15 at which the difference $G(m) - F(m)$ is greatest.

d) Give the longest interval starting at 0 that you can over which both functions $F(m)$ and $G(m)$ are increasing.

e) Give the lowest value of $F'(m)$ over the interval 0 to 15.

f) Give the highest value of $G(m)$ over the interval 0 to 15.

$\longrightarrow$ $\boxed{11}$ Two weather balloons are next to one another at time $t = 0$. You are given the following information about them:

Altitude *vs*. time for Balloon R	:	$R(t) = -\frac{2}{3}t^3 + 9t^2 - 36t + 150$
Rate-of-ascent for Balloon R	:	$R'(t) =$
Altitude *vs*. time for Balloon S	:	$S(t) =$
Rate-of-ascent for Balloon S	:	$S'(t) = 2t - 22$

(t is given in hours, and Altitude is given in hundreds of feet.)

a) Write out all the candidates (to within an undetermined constant) for the formula for Altitude *vs*. time for Balloon S.

b) Use the fact that $S(0) = R(0) = 150$ to determine the formula for $S(t)$ without any undetermined constant.

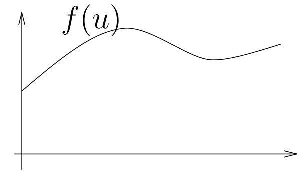
c) Give all the times at which the altitude of the Balloon R reaches a local maximum or minimum in the first 9 hours. Tell which is which.

d) What is the greatest altitude that Balloon R reaches in the first 9 hours? What is the lowest altitude that Balloon R reaches in the first 9 hours?

e) What is the lowest altitude Balloon S reaches in the first 9 hours?

f) Give the two times at which the two balloons are ascending (or descending) at the same instantaneous rate.

g) What is the change in the altitude of the S-Balloon in the time interval $t = 2$ to $t = 5$ hours?



$\longrightarrow$ $\boxed{12}$ To the right is a sketch of the graph of the function $f(u) = \frac{1}{6}u^3 - \frac{9}{2}u^2 + 36u + 14$. We define another function $g(u)$ by: $g'(u) = 4u - 16$ and $g(0) = 12$.

a) Find all values for which the graphs of $f(u)$ and $g(u)$ have parallel tangents.

b) First draw very rough sketches of the graphs of $f'(u)$ and $g'(u)$, and then give the value of u for which $f(u)$ and $g(u)$ are furthest apart over the interval 0 to 12.

c) Write out the formula for $g(u)$. There should be no undetermined constant in the formula.

d) Determine the value of u for which the slope of the tangent to the graph of $g(u)$ is 12.

e) Find all values of u at which the function $f(u)$ reaches a local maximum or minimum on the interval 0 to 15. Tell which is which.

f) Determine the global minimum of $f(u)$ on the interval 0 to 15.

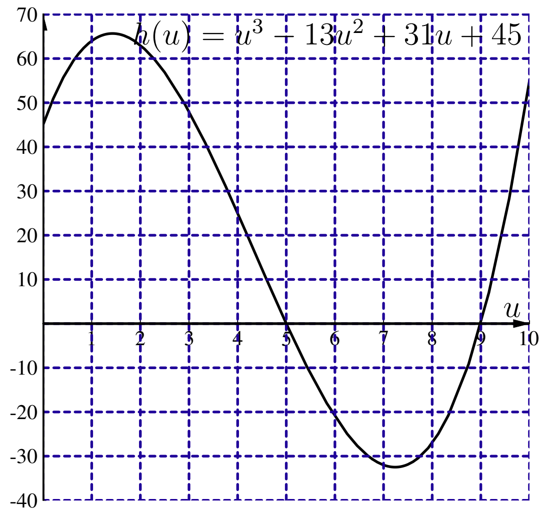
g) Find all values of u at which the function $g(u)$ reaches a local maximum or minimum on the interval 0 to 15.

Worksheet #24

The Fundamental Theorem of Integral Calculus

The graph below is of the function $h(u) = u^3 - 13u^2 + 31u + 45$. We learned that, by using derivative rules, we could find the slope of a tangent line to a function without even looking at the graph of that function. We wish to find the area of the region between the graph of $h(u)$ and the u-axis from $u = 1$ to $u = 5$. That is, we wish to evaluate

$$\int_1^5 (u^3 - 13u^2 + 31u + 45)\, du.$$



Key Question

Can we find the area *without* using the graph?

$\boxed{1}$ Either by counting squares or using trapezoids, find the area under the graph of $h(u)$ from $u = 1$ to $u = 5$. That is, compute $\displaystyle\int_1^5 h(u)\, du$.

Once again, let $A(m) = \displaystyle\int_0^m h(u)\, du$.

$\boxed{2}$ Which of the following statements are true?

(a) The function A is the derivative of the function h.

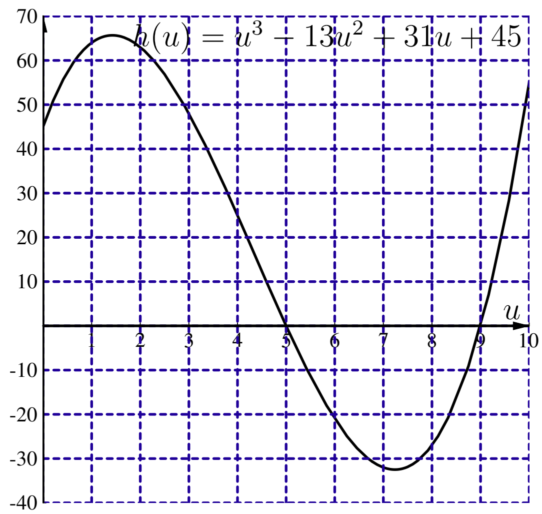
(b) The function h is the derivative of the function A.

(c) The function A is an anti-derivative of the function h.

(d) The function h is an anti-derivative of the function A.

3 Describe the graph of $A(m)$. For which values of m is it increasing; for which is it decreasing? For which values of m does $A(m)$ have local maxima or local minima? What is the "y"-intercept of $A(m)$?

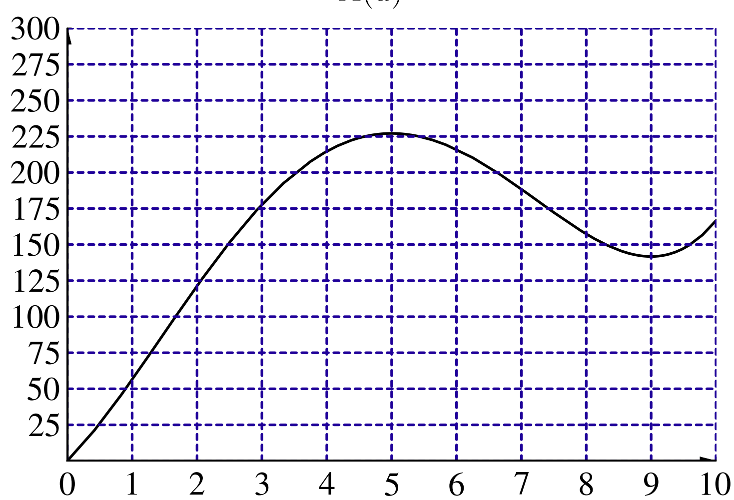
4 (a) On the graph below, shade the region whose area is $A(5)$. Use a different color or different shading (dots vs. stripes, for example) to shade the region whose area is $A(1)$.



(b) If you subtract $A(1)$ from $A(5)$, you get the area of a region. Express this area as a definite integral: $A(5) - A(1) = ...$

5 By counting a lot of squares or using many trapezoids, I could find the graph of $A(u)$. This graph is given below. Use the graph of $A(u)$ to approximate $A(5) - A(1)$. Compare your answer to Question 1.

$A(u)$



$\boxed{6}$ Recall from Question 2 that h is the derived graph of A. That is, $A'(u) = h(u)$. This means that A is an anti-derivative of h. That is, $A(u) = \int h(u)\ du$. We know the formula for $h(u)$: $h(u) = u^3 - 13u^2 + 31u + 45$. Find a formula for $A(u)$ that includes an undetermined constant.

$\boxed{7}$ The graph of A goes through the origin. That is, $A(0) = 0$. Use this fact to determine the undetermined constant in the formula you just found for $A(u)$.

$\boxed{8}$ You should now have a formula for $A(u)$ that has no undetermined constant. Use this formula to compute $A(5) - A(1)$. Compare your answer to your answers to Questions 1 and 5.

$\boxed{9}$ Look back to the graph of $h(u)$ given at the beginning of this worksheet.

(a) Explain why $\displaystyle\int_5^9 h(u)\ du$ is negative.

(b) Either by counting squares or using trapezoids, approximate $\displaystyle\int_5^9 h(u)\ du$ from the graph of $h(u)$.

(c) Explain why $\displaystyle\int_5^9 h(u)\ du = A(9) - A(5)$. (Draw some pictures. Shade some regions.)

(d) Use the formula you found for $A(u)$ in Questions 6 and 7 to compute $A(9) - A(5)$. Compare your answer to your answer from part (b).

Let's sum up our method for computing $\displaystyle\int_a^b h(u)\ du$*:*

Define the function $A(m) = \displaystyle\int_0^m h(u)\ du$*. We know this function* A *is an anti-derivative of the function* h*. So, we know we can find a formula for* $A(u)$*:*

$$A(u) = \int h(u)\ du.$$

This formula has an undetermined constant. We use the fact that $A(0) = 0$ *to find the undetermined constant. Then* $\displaystyle\int_a^b h(u)\ du$ *is simply* $A(b) - A(a)$*.*

This method actually contains an unnecessary step: determining the undetermined constant.

$\boxed{10}$ Go back to the formula for $A(u)$ that you found in Question 6. This formula should have a "$+k$" tacked on to the end. Use this formula for $A(u)$ to compute $A(5) - A(1)$ and $A(9) - A(5)$. Explain what happens to the "k".

The lesson here is that, it's not just this specific function $A(u) = \frac{1}{4}u^4 - \frac{13}{3}u^3 + \frac{31}{2}u^2 + 45u$ *that works. We've shown that*

$$\int_1^5 h(u)\ du = A(5) - A(1).$$

But it's also true that

$$\int_1^5 h(u)\,du = B(5) - B(1),$$

if $B(u) = \frac{1}{4}u^4 - \frac{13}{3}u^3 + \frac{31}{2}u^2 + 45u + 20$. *(The 20s cancel each other out.) It's also true that*

$$\int_1^5 h(u)\,du = C(5) - C(1),$$

if $C(u) = \frac{1}{4}u^4 - \frac{13}{3}u^3 + \frac{31}{2}u^2 + 45u + 109$. *(The 109s cancel each other out.)*
As long as $H(u)$ *is any anti-derivative of* $h(u)$,

$$\int_1^5 h(u)\,du = H(5) - H(1).$$

Since all anti-derivatives of $h(u)$ *look like* $\frac{1}{4}u^4 - \frac{13}{3}u^3 + \frac{31}{2}u^2 + 45u + k$, *we might as well use the simplest* k *there is: 0.*

We now have the following recipe to compute

$$\int_a^b f(x)\,dx.$$

Step I. *Find a function* $F(x)$ *that is an anti-derivative of* $f(x)$. *(And remember, you don't need to include a "*$+k$*".)*

Step II. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

(You may see this notation: $\displaystyle\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.
$[F(x)]_a^b$ *just means* $F(b) - F(a)$.*)*

Using this notation, the problem we've been working on becomes:

$$\begin{aligned}
\int_1^5 u^3 - 13u^2 + 31u + 45\,du &= \left[\frac{1}{4}u^4 - \frac{13}{3}u^3 + \frac{31}{2}u^2 + 45u\right]_1^5 \\
&= \left(\frac{1}{4}\cdot 5^4 - \frac{13}{3}\cdot 5^3 + \frac{31}{2}\cdot 5^2 + 45\cdot 5\right) \\
&\qquad - \left(\frac{1}{4}\cdot 1^4 - \frac{13}{3}\cdot 1^3 + \frac{31}{2}\cdot 1^2 + 45\cdot 1\right) \\
&= 227.083333333 - 56.4166666667 \\
&= 170.6666667
\end{aligned}$$

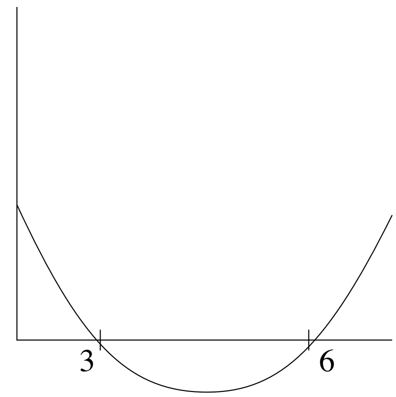
11 The graph to the right is of the function

$$f(x) = \frac{1}{3}x^2 - 3x + 6.$$

Use the recipe given above to compute

$$\int_1^6 f(x)\,dx.$$

What does this number represent?

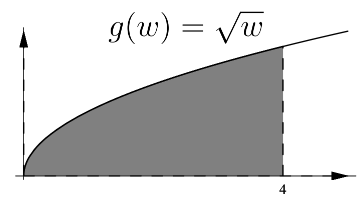


12 Use the recipe to compute

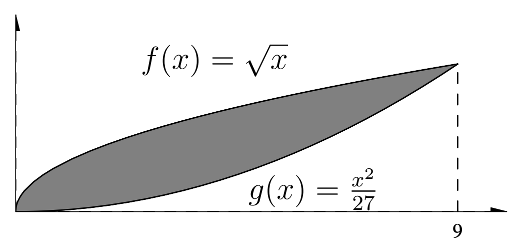
$$\int_2^6 3x^2 - 24x + 40\,dx.$$

What does this number represent?

13 The graph at right is of the function $g(w) = \sqrt{w}$. It's shaded from $w = 0$ to $w = 4$. Use the recipe to compute the area of the shaded region.



14 We wish to find the area of the shaded region shown to the right. First do this by finding the area below $f(x)$ between 0 and 9, and finding the area below $g(x)$ between 0 and 9, and then subtracting. I.e., use the fact that

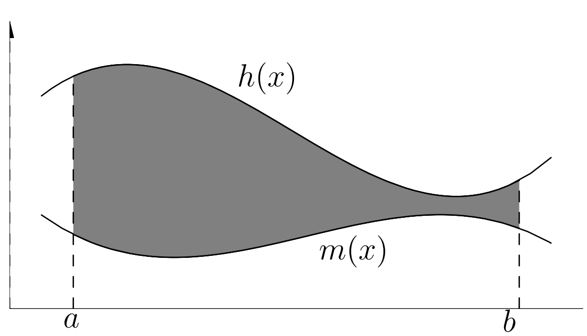


$$\text{shaded area } = \int_0^9 f(x)\,dx - \int_0^9 g(x)\,dx.$$

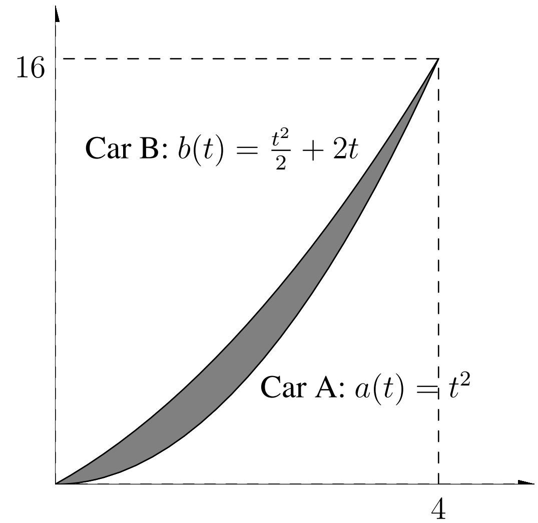
Then try another approach by evaluating the definite integral

$$\int_0^9 [f(x) - g(x)]\,dx = \int_0^9 \left[\sqrt{x} - \frac{1}{27}x^2\right]\,dx.$$

In general, if you have two graphs $h(x)$ and $m(x)$, then to find the area between them and also between $x = a$ and $x = b$, you evaluate the definite integral $\int_a^b [h(x) - m(x)]\,dx$.



15 The two graphs at the right are of speed *vs.* time for two cars. The areas $\int_0^4 a(t)\,dt$ and $\int_0^4 b(t)\,dt$ represent the distances covered by the two cars in the first 4 minutes. The difference in the areas is equal to the difference between the distances that Car A and Car B have gone in the first 4 minutes. Find this distance by evaluating the definite integral $\int_0^4 (b(t) - a(t))\ dt$.



→ 16 Evaluate the following definite integrals.

(a) $\displaystyle\int_1^{10} 3x^2 - 4x\,dx$

(b) $\displaystyle\int_e^{e^2} \frac{1}{x}\,dx$

(c) $\displaystyle\int_1^{64} \frac{3}{2\sqrt{x}} + \frac{1}{\sqrt[3]{x}}\,dx$

→ 17 (a) Suppose $f(x) = x^2 - 18x + 72$. Compute:

i. $\displaystyle\int_2^5 f(x)\,dx$

ii. $\displaystyle\int_{18}^{20} f'(x)\,dx$

(b) Suppose $R(t) = -\frac{2}{3}t^3 + 9t^2 - 36t + 150$ and $S'(t) = 2t - 22$. Compute

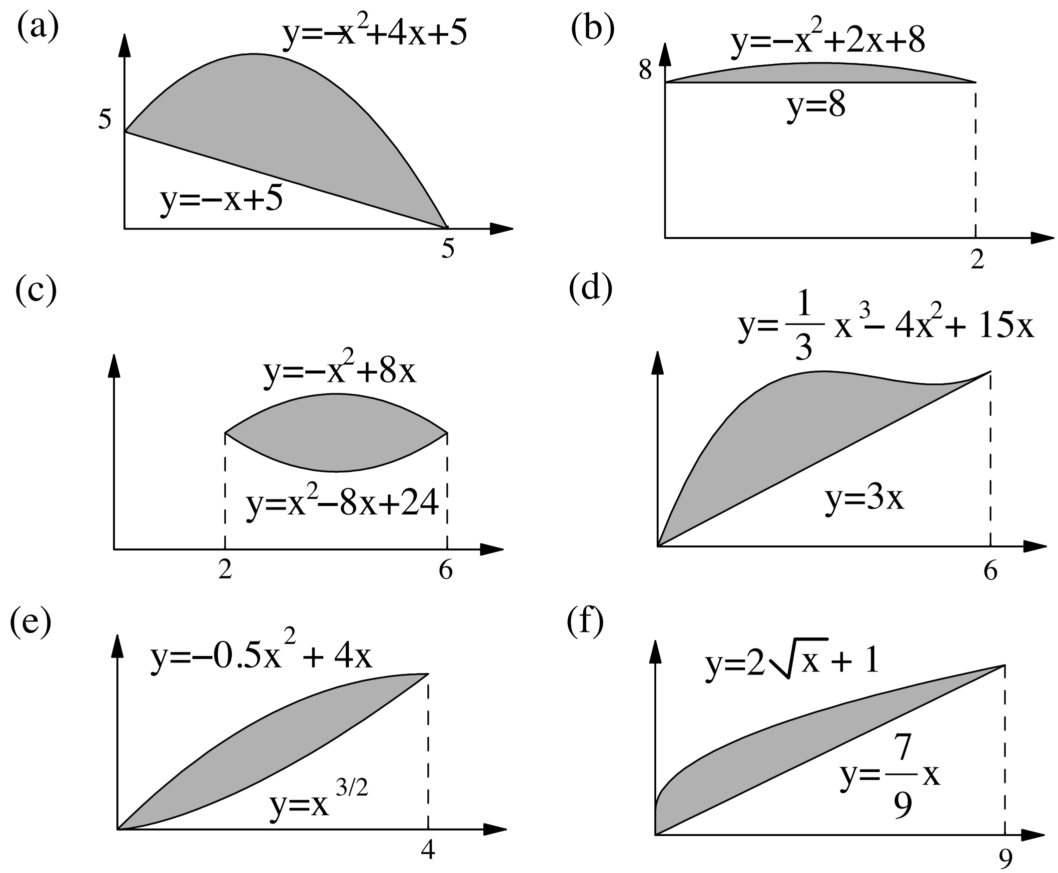
$$\int_1^2 [R'(t) - S'(t)]\ dt.$$

(c) Suppose $g'(u) = 4u - 16$ and $h(u) = \frac{1}{6}u^3 - \frac{9}{2}u^2 + 36u + 14$. Compute:

i. $\displaystyle\int_2^4 g'(u)\,du$

ii. $\displaystyle\int_0^3 [h'(u) - g'(u)]\ du$

→ $\boxed{18}$ Find the areas indicated in the following diagrams.



Answers to Selected Exercises

WORKSHEET #1

$\boxed{1}$ (a) about 125 feet; (b) about -30; (c) about 112 feet per second; (d) about 80 feet per second; (e) about 50; (f) about 2.5 seconds and 7.5 seconds.

$\boxed{2}$ (a) The height of the graph at $t = 3$ seconds is about 335. This means that the height of the shell after 3 seconds is about 335 feet. (b) The change in the height of the graph from $t = 2$ to $t = 4.5$ seconds is about 140. This means that the shell traveled approximately 140 feet in the 2.5 seconds from $t = 2$ to $t = 4.5$. (c) The slope of the secant from $t = 3$ to $t = 3.5$ is about 60. This is the shell's average speed during the 0.5 seconds from $t = 3$ to $t = 3.5$.

$\boxed{3}$ (a) 128 feet; (b) -32; (c) 112 feet per second; (d) 80 feet per second; (e) 48; (f) 2.5 seconds and 7.5 seconds.

$\boxed{4}$ The slope of the secant line should be about 95. This is a reasonable approximation of the actual speed of the shell at $t = 2$: 95 feet per second.

$\boxed{7}$ aprroximately 175 feet per second

$\boxed{8}$ $\dfrac{f(2.01) - f(2)}{0.01} = 95.84$

$\boxed{9}$ -0.16

$\boxed{11}$ 159.84 feet per second

WORKSHEET #2

$\boxed{1}$ about \$11

$\boxed{3}$ about \$4.55

$\boxed{7}$ about \$2

$\boxed{8}$ about \$0.91

$\boxed{9}$ MR is about \$7.50 per Ream; MC is still \$4.55 per Ream

$\boxed{10}$ MR is about \$1.56 per Hundred Pages; MC is about \$0.91 per Hundred Pages

$\boxed{11}$ MR at 17 Hundred Pages is about \$9.22 per Ream (or \$1.84 per Hundred Pages)

$\boxed{12}$ $q = 6$ Reams

$\boxed{14}$ MC is stil \$4.55 per Ream, which is \$0.0091 per Page

$\boxed{15}$ MR at $q = 3$ Reams is about \$9.80 per Ream (or \$0.0196 per Page)

$\boxed{17}$ MR at $q = 4.6$ Reams is about \$6.50 per Ream (or \$0.013 per Page)

$\boxed{18}$ (all answers approximate) a) $\mathrm{MC} = \$0.48$, $\mathrm{MR} = \$1.38$; b) 5.0 thousand Framits; c) 0.1 thousand Framits; d) \$0.65; e) (iv); f) \$1.00; g) from $q = 3$ to $q = 7.5$ thousand framits.

WORKSHEET #3

$\boxed{3}$ $A = 0.19$; $B = 0.0199$

$\boxed{8}$ $f(2+h) = (-1)h^2 + (2)h + (15)$; $f(2+h) - f(2) = (-1)h^2 + (2)h + (0)$.

$\boxed{11}$ $f(1+h) = (-1)h^2 + (4)h + (12)$; $f(1+h) - f(1) = (-1)h^2 + (4)h + (0)$.

$\boxed{12}$ 3.999998

$\boxed{13}$ 1

$\boxed{14}$ a) 1.2 or 9.5; b) 1.25; c) 1 .2 or 8.0; d) many answers, e.g. 1.2, 9.5; e) (i) from $x = 8$ to $x = 11$, (ii) from $x = 4$ to $x = 6$; f) from $x = 3.5$ to $x = 8.5$; g) second from left

$\boxed{15}$ a) 26; b) $5k + 2k^2$; c) 13.004

$\boxed{16}$ a) $\frac{1}{18}$ hundred gallons/hr.; b) $\frac{g(2+h)-g(2)}{h} = \frac{1}{9+3h}$; c) 75 gals

WORKSHEET #4

$\boxed{1}$ $t = 1.5$ minutes

$\boxed{6}$ $t = 2.2$ minutes

$\boxed{11}$ at approximately $t = 1.7$ minutes

$\boxed{12}$ a) 2.6, 7.6; b) 2.5,6.15; c) from $t = 1.9$ to $t = 5.9$; d) from $t = 5$ to $t = 7$; e) from $t = 3$ to $t = 7$; f) anything from 3.0 to 5.0

$\boxed{13}$ a) 3.2; b) $8(s-1) - (s-1)^2$ OR $-s^2 + 10s - 9$; c) 1.99

WORKSHEET #5

$\boxed{1}$ $q = 2$ Hundred Blivets

$\boxed{3}$ The slope of the tangent to TR is parallel to TC_2 at about $q = 3.6$ Hundred Blivets. The slope of the tangent to TR is parallel to TC_3 at about $q = 4.6$ Hundred Blivets.

$\boxed{4}$ $MC_1 = 90$ dollars per Hundred Blivets. This is \$0.90 per Blivet.

$\boxed{5}$ $MR(2) = 90, MR(4) = 60, MR(6) = 30$

$\boxed{12}$ The MR graph wouldn't change because, even though the TR graph would be higher, the slopes of the tangent lines would all stay the same. The quantities that maximize profit also would not change.

$\boxed{13}$ a) $42,000; b) 13.001; c) $50,000

WORKSHEET #6

$\boxed{1}$

x	1	2	3	4	5	6	7	8
$f'(x)$			-2.8	-3.8	-2.8	0	4.69	11.3

(I left a few for you to do.)

$\boxed{2}$ The graph of $f(x)$ is decreasing from $x = 2$ to $x = 6$.

$\boxed{3}$ The derived graph of $f(x)$ is decreasing from $x = 0$ to $x = 2$.

$\boxed{4}$ The derived graph of $g(x)$ would always be positive since the tangent lines to the graph of $g(x)$ all have positive slope. The derived graph of $g(x)$ would decrease from $x = 0$ to $x = 4$ and then increase from $x = 4$ to $x = 8$.

$\boxed{7}$

x	1	2	3	4	5	6	7	8
$g'(x)$	8.8	4.2		0.5		4.2	8.8	15.3

(I left a few for you to do.)

$\boxed{9}$ The derived graph of this new graph would be exactly the same as the derived graph of $f(x)$.

$\boxed{10}$ $h'(0) = -5, h'(3) = 4, h'(5) = 0$

$\boxed{11}$ The derived graph of $f(x)$ crosses the x-axis at $x = 2$ and $x = 6$. The derived graph of $g(x)$ never crosses the x-axis. The derived graph of $h(x)$ crosses the x-axis at $x = 1$ and $x = 5$.

$\boxed{12}$ The first statement is always true. The second statement is not.

$\boxed{13}$ The derived graph of $g(x)$ is positive and decreasing from $x = 0$ to $x = 4$. The derived graph of $h(x)$ is negative and increasing from $x = 0$ to $x = 1$.

$\boxed{15}$ The derived graph of $h(x)$ is higher than the derived graph of $f(x)$ from approximately $x = 1.6$ to approximately $x = 5.4$.

$\boxed{18}$ a) from $t = 5.4$ to $t = 6.9$; b) $t = 1.5$; c) from $t = 4.8$ to $t = 5$; d) from $t = 5$ to $t = 7$; e) from $t = 3.1$ to $t = 4.8$

$\boxed{19}$ a) $1, 4$; b) 5/2; c) no; d) 8/3; e) $x = 1$; f) false;

$\boxed{20}$ a) $t = 1$; b) $t = 8/3$; c) 18 ft/min; d) $t = 1$; e) (ii)

WORKSHEET #7

$\boxed{2}$ $\dfrac{f(5)-f(3)}{2} = 60$; $\dfrac{f(3.5)-f(3)}{0.5} = 71.25$

$\boxed{3}$ 74.925

$\boxed{4}$ $f(3+h) = (292.5) + (75)h + (-7.5)h^2$

$\boxed{5}$ $\dfrac{f(3+h)-f(3)}{h} = (75) + (-7.5)h$

$\boxed{8}$ $f(4+h) = (360) + (60)h + (-7.5)h^2$

$\boxed{9}$ $\dfrac{f(4+h)-f(4)}{h} = 60 + (-7.5)h$

$\boxed{10}$ $\dfrac{f(4.1)-f(4)}{0.1} = 59.25$, $\dfrac{f(4.001)-f(4)}{0.001} = 59.9925$, $\dfrac{f(4.00001)-f(4)}{0.00001} = 59.999925$

$\boxed{11}$ $f'(4) = 60$

$\boxed{12}$ $f(m+h) = (-7.5)m^2 + (-15)mh + (-7.5)h^2 + (120)m + (120)h$

$\boxed{13}$ $\dfrac{f(m+h)-f(m)}{h} = \dfrac{(0)m^2 + (-15)mh + (-7.5)h^2 + (0)m + (120)h}{h} = -15m - 7.5h + 120$

$\boxed{15}$ $f'(m) = -15m + 120$

$\boxed{17}$ $f'(3) = 75$, $f'(6) = 30$

$\boxed{20}$ (a) $q = 2$; (b) $q = 3.6$; (c) $q = 4.67$

$\boxed{21}$ (a) $4p - 3$; (b) $10 - 2t$; (c) $C'(q) = 2q + 5$

WORKSHEET #8

$\boxed{2}$ Step I: $B(m+h) = -(m+h) - \dfrac{25}{m+h+1} + 25$; Step II simplified: $B(m+h) - B(m) = -h + \dfrac{25h}{(m+h+1)(m+1)}$;

$\boxed{3}$ Step III: $\dfrac{B(m+h)-B(m)}{h} = \dfrac{1}{h}\left[-h + \dfrac{25h}{(m+h+1)(m+1)}\right]$

$\boxed{4}$ Step IV: $\dfrac{B(m+h)-B(m)}{h} = -1 + \dfrac{25}{(m+h+1)(m+1)}$; Step V: $B'(m) = -1 + \dfrac{25}{(m+1)^2}$

$\boxed{6}$ 2.5 min; 1.5 min

$\boxed{8}$ 1.32 min

$\boxed{9}$ (a) i. $3h^2 + (6z + 2)h$; ii. $g'(z) = 6z + 2$; iii. 5.0024
(b) i. $-2h^2$; ii. $f'(t) = -4t + 12$; iii. 0.012
(c) i. $TR(3 + h) - TR(3) = (-1)h^2 + (14)h + (0)$; ii. \$14

$\boxed{10}$ (a) $g'(t) = \frac{1}{(t+1)^2}$; (b) $\frac{H(1+r)-H(1)}{r} = 1 + 2r$; (c) $H'(t) = 4t - 3$; (d) $H'(2) = 5$, $H(2) = 6$, $H(2.003) \approx 6.0015$

WORKSHEET #9

$\boxed{1}$ a) $f'(x) = 11x^{10}$; b) $g'(x) = -\frac{1}{x^2}$; c) $h'(x) = \frac{1}{3x^{2/3}}$; d) $k'(x) = 1$; e) $m'(x) = 0$;

$\boxed{2}$ a) $f'(x) = 72x^7$; b) $g'(x) = -\frac{12}{x^4}$; c) $h'(x) = \frac{5}{2\sqrt{x}}$; d) $k'(x) = 10$; e) $m'(x) = 0$;

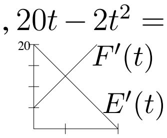
$\boxed{3}$ a) $f'(x) = 6x-2$; b) $g'(x) = \frac{3}{2\sqrt{x}} - \frac{2}{x^2}$; c) $h'(x) = 12x^3 - 15x^2 - \frac{4}{x^3}$; d) $k'(x) = -\frac{2}{x^{3/2}} + \frac{9}{2x^{5/2}}$;

$\boxed{4}$ a) $f'(t) = -3t^{-4} - t^{-2}$; b) $\frac{dy}{dx} = 3x^2 - 1 - 4x^3$; c) $\frac{dz}{du} = \frac{3}{2}u^{1/2}$; d) $g'(x) = 8x + 16x^3$; e) $\frac{dw}{dt} = 32t - t^{-2}$; f) $f'(z) = 0$; g) $\frac{dy}{dt} = -\frac{32}{t^3}$;

$\boxed{5}$ $f'(x) = x^2 - 8x + 12$; $g'(x) = x^2 - 8x + 16.5$; $h'(x) = -x^2 + 6x - 5$

$\boxed{6}$ a) $\frac{1}{2\sqrt{x}} - \frac{3}{4}x^2$; b) $-\frac{6}{r^3} + 35r^6$; c) $2t - 6 + \frac{6}{t^2}$; d) $2w + 3w^2 + 1$; e) $-3t^{-5/2} + 2t^{-5/3}$; f) $\frac{1}{2\sqrt{w}} + \frac{9}{2}w^{-5/2}$; g) $4 - 6t + 3t^2$.

$\boxed{7}$ a) $f'(x) = -4x + 20$, $g'(x) = x^2 - 6x + 1$; b) $x = 2.5$; c) $x = 3$; d) $x = 5.472$; e) $P(5.472) = 69.295$; f) Any value after 5.472 for which $P(x)$ is still positive (such as $x = 6$); g) $(0)x^3 + (-1)x^2 + (2)x + (15) = 0$; h) 1.25 to 3.75.

$\boxed{8}$ a) $-3t^2 + 15t - 20 = 0$; b) 0 to 2.5; c) True because at such a time, $20t - 2t^2 = t^2 + 5t + 45$, which has no roots; d) no such time exists; e) see picture to right 

; f) $t = 2$.

WORKSHEET #10

$\boxed{2}$ A. the quantity q at which MR crosses the q-axis
B. the quantity q at which the graph of MC (a parabola) has its vertex
C. the quantity q at which MR and MC intersect

$\boxed{3}$ $MR = R'(q) = -0.4q + 2$

$\boxed{5}$ $q = 5$

$\boxed{11}$ $q = 3$

$\boxed{14}$ $q = 4.32$

$\boxed{16}$ Profit$=-\frac{1}{30}q^3 + 0.1q^2 + q - 0.8$

$\boxed{17}$ $\frac{dP}{dq} = -0.1q^2 + 0.2q + 1$

$\boxed{19}$ a) $R(3+h) - R(3) = (-1)h^2 + (14)h + (0)$; b) \$14;

$\boxed{20}$ a) $q = 1$; b) $-3q^2 + 14q + 10 = 0$; c) MR at $q = 1$ is \$70, at $q = 7$ is \$10; d) $q = 4$; e) \$320,000; f) 2 to 4;

$\boxed{21}$ a) $R'(q) = -3q + 26$, $C'(q) = 2q + 1$; b) $R(q) = -\frac{3}{2}q^2 + 26q$; $C(q) = q^2 + q$; c) When $R'(q) = C'(q)$, $q = 5$ thousand; d) 3 thousand, because $R'(q)$ is greater than $C'(q)$ everywhere between 1 and 3, so make as many as possible; e) Yes, because $R'(q)$ would meet $C'(q)$ further to the right; f) $C'(3) = 7$ dollars.

WORKSHEET #11

$\boxed{1}$ The f-balloon would be descending and the g-balloon would be ascending.

$\boxed{2}$ No. The g-balloon is ascending during that time interval. But its speed is decreasing. That is, it's getting slower.

$\boxed{3}$ The f-balloon is descending, but it too is getting slower. In fact, at $t = 1.5$, the f-balloon is stopped for an instant, ready to change direction.

$\boxed{4}$ They're getting further apart.

$\boxed{5}$ They are getting further apart.

$\boxed{6}$ They're getting closer together.

$\boxed{10}$ 15

$\boxed{11}$ The elevation graph would have a horizontal tangent line at $t = 7$ and would be changing from increasing to decreasing.

$\boxed{12}$ The elevation graph would have horizontal tangent lines at those two times. At the first time (about $t = 1.5$), the elevation graph would change from decreasing to increasing. At the second (about $t = 8.5$), the elevation graph would change from increasing to decreasing.

$\boxed{14}$ The elevation graph is steepest at $t = 5$.

$\boxed{18}$ a) $3t^2 - 27t + 47 = 0$; b) $t = 3$ to $t = 6$; c) 79 yds; d) closer together. A is above, but B is rising faster, since $B'(3)$ is greater than $A'(3) = 0$; e) $t = 4.5$; f) (i) is true.

$\boxed{19}$ a) i) T; ii) T; iii) T; iv) F; b) 4 to 6; c) $t = 6$; d) $t = 36/7$; e) $t = 4$.

WORKSHEET #12

$\boxed{1}$ a) $\frac{dz}{dx} = \frac{1}{2}(3x + \frac{1}{x})^{-1/2}(3 - \frac{1}{x^2})$.

b) $-(3t^2 - \sqrt{t})^{-2}(6t - \frac{1}{2}t^{-1/2})$.

c) $f'(v) = \frac{1}{3}(\sqrt{v} + 2v^2)^{-2/3}(\frac{1}{2}v^{-1/2} + 4v)$.

$\boxed{2}$ a) $\frac{dz}{dx} = e^{3x - \frac{1}{x}}(3 + \frac{1}{x^2})$.

b) $e^{3t^2 - \sqrt{t}}(6t - \frac{1}{2}t^{-1/2})$.

c) $f'(v) = e^{\sqrt{v} + 2v^2} \cdot (\frac{1}{2}v^{-1/2} + 4v)$.

$\boxed{3}$ a) $\frac{dz}{dx} = \frac{3 + \frac{1}{x^2}}{3x - \frac{1}{x}}$.

b) $\frac{6t - \frac{1}{2}t^{-1/2}}{3t^2 - \sqrt{t}}$.

c) $f'(v) = \frac{\frac{1}{2}v^{-1/2} + 4v}{\sqrt{v} + 2v^2}$.

$\boxed{4}$ a) $\frac{dy}{dx} = (e^{\sqrt{x} + 2x})(\frac{1}{2}x^{-1/2} + 2)$

c) $\frac{dz}{dt} = -4(2t + 1)^{-5}(2)$

e) $\frac{dy}{dx} = 2(x + e^x)(1 + e^x)$

g) $h'(r) = \frac{1 + e^r}{r + e^r}$

$\boxed{5}$ a) $\frac{dy}{dx} = (1 + x^2 - x^3)(\frac{1}{2}x^{-1/2}) + (2x - 3x^2)(\sqrt{x} - 2)$

c) $g'(z) = (1 + e^z + z^2)(3z^2) + (1 + z^3)(e^z + 2z)$

e) $A'(w) = \dfrac{\ln(w) \cdot (e^w + (\frac{1}{2})w^{-1/2}) - (e^w + \sqrt{w})\frac{1}{w}}{[\ln(w)]^2}$

g) $f'(t) = t^2(4t^3 + 2t) + (2t)(t^4 + t^2 + 1) + 3$

i) $g'(z) = \dfrac{2z}{z^2 + 1}$

k) $\frac{dz}{du} = 4(u^2 + 3u + 4)^3(2u + 3)$

m) $\frac{dy}{dt} = (-3t^2 + 4t + 5)e^t + (-6t + 4)e^t + (1/t)$

WORKSHEET #13

$\boxed{1}$ a) First way: $g'(t) = [t^3(\sqrt{t + 1})]e^t + [3t^2\sqrt{t + 1} + t^3 \cdot \frac{1}{2}(t + 1)^{-1/2}]e^t$

b) First way: $\frac{dy}{dt} = \dfrac{t^3(\sqrt{t + 1})e^t - [3t^2\sqrt{t + 1} + t^3 \cdot \frac{1}{2}(t + 1)^{-1/2}]e^t}{[t^3(\sqrt{t + 1})]^2}$

$\boxed{2}$ a) $g'(t) = (\frac{1}{(1+t^2)^4})(4(1+t^2)^3)(2t)$

b) $h'(t) = 4(1+(\ln t)^2)^3)(2 \cdot \ln t)(1/t)$

c) $\frac{dy}{dx} = e^{[(3x+x^2)^2]}2(3x+x^2)(3+2x)$

d) $f'(t) = \frac{1}{2}(\ln(t^2+1))^{-1/2}(\frac{2t}{t^2+1})$

e) $\frac{dz}{dx} = \frac{(1/2)(e^x)^{-1/2}e^x}{\sqrt{e^x}}$

$\boxed{3}$ a) $\frac{dy}{dx} = e^x \cdot \left(\frac{1}{x+1}\right) + e^x[\ln(x+1)]$

c) $\frac{dz}{dt} = e^{2t^2+3t}(4t+3)$

e) $\frac{dw}{du} = \dfrac{u(\ln(u))(2u) - [u(1/u) + \ln(u)](u^2+1)}{[u\ln(u)]^2}$

g) $\frac{dz}{dx} = (\frac{1}{2})(x^2e^x+1)^{-1/2}(x^2e^x + 2x \cdot e^x)$

i) $\frac{dy}{dx} = \dfrac{x[(\ln x)(\frac{1}{2})(x+1)^{-1/2} + (1/x)(\sqrt{x+1})] - (\ln x)\sqrt{x+1}}{x^2}$

k) $g'(v) = [\ln(v+1)]e^{v+1} + e^{v+1}\left[\frac{1}{v+1}\right] + 2v$

m) $\frac{dy}{dt} = \dfrac{t[t^2+1)(3t^2) + (2t)(t^3+1)] - (t^2+1)(t^3+1)}{t^2}$

o) $\dfrac{dz}{dt} = (1/2)(e^{t^2}+1)^{-1/2} \cdot e^{t^2} \cdot (2t)$

q) $g'(x) = e^{\sqrt{x}} \cdot (1 + (1/x)) + e^{\sqrt{x}}(\frac{1}{2})x^{-1/2}(x + \ln(x))$

s) $\frac{dz}{dx} = \dfrac{\sqrt{x+1}[x \cdot e^x + e^x] - (x \cdot e^x)(\frac{1}{2})(x+1)^{-1/2}}{(\sqrt{x+1})^2}$

u) $g'(z) = \dfrac{(\sqrt{z^2+1})(2z) - (1/2)(z^2+1)^{-1/2} \cdot 2z \cdot z^2}{z^2+1}$

w) $\frac{dz}{dx} = (\ln x)(2x + 12x^3 - e^x) + (1/x)(x^2 + 3x^4 - e^x)$

y) $\dfrac{dQ}{dt} = (t^2+2t)(3t^2-3) + (2t+2)(t^3-3t) + e^t + (e^t)(t)$

WORKSHEET #14

$\boxed{1}$ $t = 1, 2, 3, 4, 5$; The rate of ascent graph is crossing (or touching) the t-axis at these times.

$\boxed{2}$ local max at $t = 2$ and 5; local min at $t = 1$ and 3;

$\boxed{3}$ Yes! The altitude graph has a critical number at $t = 4$, but neither a local maximum nor a local minimum there.

$\boxed{4}$ 17,000 feet

$\boxed{5}$ $t = 3$ gives the lowest point; 600 feet

$\boxed{6}$ (b) $A'(t)$ is changing from positive to negative at $t = 2$ and $t = 5$.

(c) $A(t)$ has local minima when $A'(t)$ is changing from negative to positive.

$\boxed{8}$ (a) $P(q) = -0.01q^3 + 0.22q^2 - 0.65q - 4$

(b) critical numbers are at $q = 1.67$ and 13;

(d) Profit is maximized at $q = 13$.

$\boxed{9}$ Profit is maximized at $q = 10$.

$\boxed{10}$ Profit is maximized at $q = 13$.

$\boxed{11}$ The minimum value of profit is -4.52 hundred dollars.

$\boxed{12}$ There is no largest possible loss. Eventually, profit will keep decreasing, getting more and more negative.

$\boxed{13}$ (a) $AC(q) = 0.0005q + 50 + \frac{10125}{q}$; $MC(q) = 0.001q + 50$;

(b) 54.5 dollars per Blivet

(c) (iii)

(d) $q = 4500$ Blivets

(e) 50.001 dollars

$\boxed{14}$ (a) \$75,000

(b) from $q = 0$ to $q = 12000$

(c) $q = 3333.33$ Framits

(d) smallest value of AVC is $AVC(7500) = \$287.50$

$\boxed{15}$ (a) $t = 3, 8$

(b) 5 gallons

(c) from $t = 5$ to $t = 8$

(d) $D(3.35) = 30.35$ gallons

WORKSHEET #15

$\boxed{2}$ from $q = 0$ to $q = 25$

$\boxed{3}$ area=2250; The width (10) is the quantity; the height ($h(10) = 225$) is the price per item when you sell 10 Gizmos; the area is the total revenue for selling 10 Gizmos.

$\boxed{4}$ “area”=0

$\boxed{5}$ “area”=0

$\boxed{7}$ $TR(q) = q^3 - 50q^2 + 625q$

$\boxed{8}$ critical numbers are $q = 8.33$ and $q = 25$; Max TR is $TR(8.33) = \$2314.81$.

$\boxed{9}$ (a) Maximum TR on the interval from $q = 0$ to $q = 25$ is $TR(10) = \$3000$.

(b) Maximum TR on the interval from $q = 0$ to $q = -\frac{b}{m}$ is $TR(-\frac{b}{2m}) = -\frac{b^2}{4m}$.

(c) Maximum TR on the interval from $q = 0$ to $q = 25$ is $TR(11.11) = \$2314.81$.

$\boxed{10}$ (a) $TR(q) = 12q - 4q^{3/2}$; $TR'(q) = 12 - 6^{1/2}$

(b) $q = 3.5$

(c) $P(q) = -4q^{3/2} + 11q - 1$

(d) $q = 3.36$

$\boxed{11}$ (a) $TR(q) = q^3 - 30q^2 + 225q$

(b) \$100 per Trinket

(c) \$196 per Trinket

WORKSHEET #16

$\boxed{1}$ $R'(q) = -\frac{1}{q^2} + \frac{2}{q^3}$. $q = 2$ is the only critical number.

$\boxed{3}$ (a) $g'(x) = 6x^2 + 6x - 9$; $g''(x) = 12x + 6$

(c) $k'(u) = 1 - u^{-2}$; $k''(u) = 2u^{-3}$

(e) $m'(z) = 4e^z \cdot z^3 + z^4 \cdot e^z$; $m''(z) = 12z^2 \cdot e^z + 8z^3 \cdot e^z + z^4 \cdot e^z$ (simplified)

$\boxed{4}$ g and h both have horizontal tangents at $x = a$

$\boxed{5}$ $g''(a)$ is positive

$\boxed{6}$ $h''(a)$ is negative

$\boxed{8}$ Leftmost graph: $f'(x)$ is positive and increasing; $f''(a)$ is positive

$\boxed{9}$ (a) $R''(q) = \frac{2}{q^3} - \frac{6}{q^4}$; $R''(2) = -\frac{1}{8}$; $R(q)$ has a local maximum at $q = 2$.

(c) The largest possible value of $R(q)$ is $R(2)$: $\frac{1}{4}$.

$\boxed{10}$ (a) $f'(x) = 4(x - 3)^3$; $x = 3$ is the only critical number of f.

(c) $f''(x) = 12(x - 3)^2$; $f''(3) = 0$

$\boxed{11}$ (a) $g'(x) = -4(x - 4)^3$; $x = 4$ is the only critical number of g.

(c) $g''(x) = -12(x - 4)^2$; $g''(4) = 0$

$\boxed{12}$ (a) $h'(x) = 3(x - 5)^2$; $x = 5$ is the only critical number of h.

(c) $h''(x) = 6(x-5)$; $h''(5) = 0$

$\boxed{13}$ (a) $x = 7.5$ and $x = 12.5$

(b) $f(x)$ has a local minimum at $x = 7.5$ and a local maximum at $x = 12.5$.

$\boxed{14}$ (a) $t = 25.4$

(b) largest $D(7) = \frac{5}{6} = 0.83$, smallest $D(0) = 0.41$

(c) Average trip speed is constant. Highest and lowest values are both $\frac{1}{6}$ mpm.

(d) Concave up.

$\boxed{15}$ (a) $MR(q) = \frac{2}{3}q^3 - \frac{31}{2}q^2 + 110q + 200$

(b) $q = 5.5$ and $q = 10$

(c) MR has a local maximum at $q = 5.5$ and a local minimum at $q = 10$.

(d) The global minimum is \$200 ($MR(0)$) and the global maximum is \$447.04 ($MR(5.5)$).

(e) concave up

$\boxed{16}$ (a) $A(x) = 2x + \dfrac{4x}{3x-1} \qquad A'(x) = 2 + \dfrac{(3x-1)4 - (4x)(3)}{(3x-1)^2} = 2 - \dfrac{4}{(3x-1)^2}$

(b) $A(x)$ has a local minimum at $x = 0.80$.

$\boxed{17}$ (a) $S(x) = \frac{1}{4}x - 4 + \frac{36}{x}$, $S'(x) = \frac{1}{4} - 36x^{-2}$

(b) $x = 12$

(c) $S(x)$ has a local minimum at $x = 12$.

(d) (i) $x = 1$

(ii) $x = 6$

(e) Any value of y between 0 and 2 is correct.

(f) No such interval exists.

$\boxed{18}$ (a) $D(x) = \frac{1}{6}x^3 - \frac{5}{2}x^2 + 8x + 12$, $D'(x) = \frac{1}{2}x^2 - 5x + 8$.

(b) $D(x)$ has a horizontal tangent at $x = 2$ and at $x = 8$. $D(x)$ has a local maximum at $x = 2$ and a local minimum at $x = 8$.

(c) The lowest value of $D(x)$ on the interval from $x = 1$ to $x = 12$ is $\frac{4}{3}$ ($D(8)$).

(d) $T(x) = \frac{2}{3}x^3 - \frac{15}{2}x^2 + 16x + 12$. $T(x)$ has a local maximum at $x = 1.29$ and a local minimum at $x = 6.21$.

$\boxed{19}$ (a) $S(x) = x - 8 + \frac{25}{x}$

(b) $x = 5$ is the only positive critical number of $S(x)$.

(c) $S(x)$ has a local minimum at $x = 5$.

WORKSHEET #17

$\boxed{1}$ (a) \$100; (b) $C_1(x) = 0.05x$

$\boxed{2}$ \$113.62

$\boxed{3}$ $C_2(x, y, z) = 0.062x + 0.0536y + 0.047z$

$\boxed{4}$ 296 months (almost 25 years)

$\boxed{5}$
a) $A(5000, 0.06, 8) = \$8080.37$
b) $A(P, 0.05, 0.5) = 1.025315121P$
c) $A(700, r, 4) = 700e^{4r}$
d) $A(85000, 0.072, t) = 85000e^{0.072t}$

$\boxed{6}$
a) During the first 7 years, the account grows on average by \$7957.57 per year.
b) During the six months from $t = 5$ to $t = 5.5$ years, the account grows at a rate of \$8931.78 per year.
c) At $t = 10$ years, the account is growing at a rate of \$12,573.13 per year.

$\boxed{7}$
a) During the first 15 years, the account grows on average by \$1048.54 per year.
b) During the h years beginning at time t, the account grows at a rate of

$$\frac{10000e^{0.063(3+h)} - 10000e^{0.063(3)}}{h} \text{ dollars per year.}$$

c) At $t = 4.2$ years, the account is growing at a rate of \$820.83 per year.

$\boxed{8}$
a) For each dollar added to the principal, the amount in the account at the end of 5 years grows by \$1.52.
b) $A_P(P, 0.084, 5) = e^{0.084(5)} = 1.521961556$ (Note: The function $A(P, 0.084, 5) = Pe^{0.084(5)} = 1.521961556P$ is a straight line.)

$\boxed{9}$ $A_P(P, r^*, t^*) = e^{r^*t^*}$, $A_r(P^*, r, t^*) = P^*e^{rt^*} \cdot t^*$

$\boxed{10}$ $\dfrac{\partial C_2}{\partial y} = 0.0536$; $\dfrac{\partial C_2}{\partial z} = 0.047$

$\boxed{11}$ $\dfrac{\partial E}{\partial b} = 2b + 2.833m - 5.6$

$\boxed{12}$
a) $\frac{\partial z}{\partial x} = 3x^2y^2 - 6e^xy$
b) $f_t(t, m) = e^m + 2t(m^2 + 2m)$
c) $\dfrac{\partial w}{\partial y} = \dfrac{(y+1)x^3 - x^3y}{(y+1)^2}$

d) $\frac{\partial t}{\partial s} = (s^2 + rs)(3s^2) + (2s + r)(r + s^3)$

e) $g_v(u, v) = e^{u^2 v}(u^2)$

f) $\frac{\partial p}{\partial y} = x^3 2y + 3x^2 - 10y$

g) $h_u(u, v) = \dfrac{(u - v) - (u + v)}{(u - v)^2}$

h) $\frac{\partial z}{\partial m} = (m^2 p + p^2 m)3(m + p)^2 + (2mp + p^2)(m + p)^3$

13

a) $f_x(x, y) = -4x + 4y + 12$, $f_y(x, y) = 4x - 12y + 4$.

b) $(5, 2)$;

c) i. $g(x) = -2x^2 + 12x + 24$; $g'(x) = -4x + 12$

ii. $g'(2) = 4$. So, $\dfrac{g(2.0001) - g(2)}{0.0001} \approx 4$

iii. $g'(2) = f_x(2, 0)$

iv. $\frac{f(2,0.0001) - f(2,0)}{0.0001}$ is bigger. (Explain!)

14

a) $\dfrac{\partial z}{\partial x} = -12x + 2 + 8y$ $\qquad \dfrac{\partial z}{\partial y} = -8y - 3 + 8x$

b) $(-\frac{1}{4}, -\frac{5}{8})$

c) Largest: 10; Smallest: -106

d) larger

e) $f(x, 10)$

15

a) $\dfrac{\partial p}{\partial r} = s - \dfrac{27}{r^2} \quad \dfrac{\partial p}{\partial s} = r - \dfrac{8}{s^2}$

b) $(\frac{9}{2}, \frac{4}{3})$

c) There are many answers. One acceptable answer is $(4, \frac{3}{2})$. Find your own.

16

a) $f_x(x, y) = -3x^2 + 4y$, $f_y(x, y) = 24y^2 + 4x$

b) Largest: 14, Smallest: -18

c) second is bigger

d) $f(1, y)$

17

a) $\dfrac{\partial z}{\partial x} = x^2 - y$, $\dfrac{\partial z}{\partial y} = y^2 - x$,

b) $(0, 0)$ and $(1, 1)$.

c) $h(t)$ at $t = 3$.

18

a) $\dfrac{\partial p}{\partial r} = 2rs + s^2 - 5s + 2 \quad \dfrac{\partial p}{\partial s} = r^2 + 2rs - 5r$

b) $P(r,2)$ at $r=3$

c) $s=\dfrac{5-r^*}{2}$

$\boxed{19}$ a) $\dfrac{\partial z}{\partial x}=12x-2+7y$; $\dfrac{\partial z}{\partial y}=4y-3+7x$

b) $(13,-22)$

c) global max: 35; global min: 4.625

d) smaller

e) the first one is larger

WORKSHEET #18

$\boxed{1}$ 12 thousand gallons

$\boxed{2}$ $A(x,y)=0.6x+0.3y$

$\boxed{5}$ No, it is not possible because you would run out of apple juice.

$\boxed{6}$ $C(x,y)=0.4x+0.7y$

$\boxed{9}$ 9 thousand dollars

$\boxed{10}$ $P(x,y)=0.4x+0.5y$

$\boxed{11}$ $(0,0)$, $(20,0)$, $(0,15.71)$, and $(17,6)$

$\boxed{12}$ KQI: 17 thousand gallons of Apple-Cranberry; 6 thousand gallons of Cranberry-Apple;

KQII: 9.8 thousand dollars

KQIII: 12 thousand gallons of apple; 11 thousand gallons of cranberry

$\boxed{13}$ (b) All the lines you drew should be parallel. As profit gets larger, the lines should get farther away from the origin.

$\boxed{14}$ $x\leq 15$; $y\leq 10$

$\boxed{16}$ The vertices of your feasible region are $(0,0)$, $(0,10)$, $(15,0)$, $(10,10)$, and $(15,\frac{50}{7})$. The profit is largest at the vertex $(15,\frac{50}{7})$. So, you should produce 15 thousand gallons of Apple-Cranberry and $\frac{50}{7}=7.143$ thousand gallons of Cranberry-Apple in order to maximize profit.

$\boxed{17}$ a) $P(x,y)=0.75x+1.20y$

b) $A(x,y)=3x+4y$, $I(x,y)=7x+6y$

c) Vertices: (0,0), (600,1650), (0,2100),(2014.29,0)

d) Compute profit at each vertex: $P(0,0) = 0$, $P(0,2100) = 2520$, $P(2014.29,0) = 1510.71$, $P(600,1650) = 2430$. Max profit achieved when company produces 0 bags of Regular and 2100 bags of Deluxe

$\boxed{18}$ a) $P(x,y) = x + 2y$.

b) The feasible region is a polygon with 5 vertices.

c) Vertices: $(0,0)$, $(1.2,0)$, $(0,1.4)$, $(0.6,1.4)$, $(1.2,0.8)$.

d) The maximum profit occurs at a vertex of the feasible region. To find the production level that gives maximum profit, we check all vertices. We find that maximum profit is achieved at $(x,y) = (0.6,1.4)$. So you should produce 600 gallons of Miracle Bathroom Cleaner, and 1,400 gallons of Speedex Floor Cleaner

e) Maximum profit is $P(0.6,1.4) = 3.4$ thousand dollars.

WORKSHEET #19

$\boxed{1}$ farther apart

$\boxed{3}$ 600 yards

$\boxed{4}$ 600 yards

$\boxed{5}$ 300 yards

$\boxed{6}$ ave speed = 15 yds/min; dist trav = 600 yards

$\boxed{7}$ The car in graph (c) covers less distance than car V.

$\boxed{8}$ Car A: typical speed=16 yds/min; distance traveled=32 yds;

Car B: typical speed=9 yds/min; distance traveled=18 yds;

distance between = 14 yds.

$\boxed{10}$ At $t = 7$, car B is ahead by approximately 7 miles.

$\boxed{13}$ KQII: Car B catches up to car A around $t = 5$.

KQIII: Car A overtakes car B at some time between $t = 8$ and $t = 9$.

$\boxed{15}$ 48 yards

$\boxed{16}$ approximately 46 yds

$\boxed{18}$ approximately 21 yds

$\boxed{20}$ a) $t = 5$ and $t = 10$

b) $t = 5$

c) from $t = 1$ to $t = 3$

d) from $t = 3$ to $t = 4$

e) from $t = 0.6$ to $t = 3.6$

f) 170 ft/min

WORKSHEET #20

$\boxed{1}$ $TR(2) = \$27.80$; $TR(5) = 68$.

$\boxed{3}$ $TR(10) = \$131$

$\boxed{4}$ first 5 minutes: area=67.5; first 10 minutes: area=130

$\boxed{8}$ $TC(5) = 25.5$

$\boxed{9}$ area=18

$\boxed{14}$ KQI: $q = 40$;
KQII: $q = 69$ (approximately);
KQIII: approximately \$266.60

$\boxed{20}$
a) \$39,000
b) \$4
c) profit increases by \$20,000
d) \$100,000
e) Yes
f) \$8,300
g) from $q = 5.6$ to $q = 8.6$ thousand Trivets

WORKSHEET #21

$\boxed{4}$ a) about 40; b) about -14; c) about 40

$\boxed{5}$ $A(1) = 29.2$, $A(2) = 40.3$

$\boxed{6}$ $\displaystyle\int_{2.5}^{3.0} f(x)\,dx \approx -1.5$; $A(3.0) \approx 39.4$

$\boxed{7}$ If you knew $A(4)$, you would subtract an area to get $A(5)$. You'd be producing a smaller number.

$\boxed{10}$ $A(m)$ increases from 0 to 2.4, decreases from 2.4 to 5.6, and increases again from 5.6 to 8. It does not go below the horizontal axis.

$\boxed{13}$ $\displaystyle\int_6^8 f(x)\,dx = 40.3$

$\boxed{14}$ $\displaystyle\int_2^6 f(x)\,dx = -15.8$

$\boxed{15}$

a) global max at $m = 2$, global min at $m = 0$

b) $m = 1.6$, 6, and 7

c) $m = 6.5$ and $m = 9.625$

d) (i) and (iv) are true

e) Length 1: from $m = 0$ to $m = 1$; Length 3: from $m = 0$ to $m = 3$ or from $m = 5$ to $m = 8$

f) 5.25

g) 9

h) decreases then increases

i) (iii) and (vi) are true

$\boxed{16}$

a) $\displaystyle\int_1^3 b(t)\,dt = 15$, $A(3) - A(1) = 19$

b) 128

c) 18

d) $t = 2.1$, $t = 6$

e) None

f) (iii) T

g) $A(1) = 24$

h) $B(10) = 9.125$

i) Same as (g)

$\boxed{17}$

a) (ii), (iii), and (iv) are true

b) 600 yds

c) None

d) -85

e) $t = 7.5$

f) None

g) 150 yds/min

h) from $t = 0$ to $t = 1.3$

i) $t = 5$ and $t = 10$

WORKSHEET #22

$\boxed{1}$ $TR(10) = \int_0^{10} MR(q)\, dq$

$\boxed{3}$ $TR(q) = -0.1q^2 + 14q$

$\boxed{4}$ $TR(10) = 130$

$\boxed{5}$ a) TC would shift up by \$100; VC and MC would not change.

$\boxed{6}$ $VC(q) = 0.00333333q^3 - 0.2q^2 + 6q$

$\boxed{7}$ $TC(10) = \$43.33$

$\boxed{8}$ $TC(q) = 0.00333333q^3 - 0.2q^2 + 6q + 100$

$\boxed{9}$ $P(q) = -0.00333333q^3 + 0.1q^2 + 8q - 100$

$\boxed{11}$

a) $\text{VC}(q) = q^3 - 15q^2 + 79q$
b) $C(q) = q^3 - 15q^2 + 79q + 100$
c) $R'(q) = -3q + 55$
d) $q = 8$
e) \$60,000
f) $q = 18.333$ thousand Framits

$\boxed{12}$

a) local max at $t = 7$; local min at $t = 1$
b) local min at $t = 7.74$; local max at $t = 1.59$
c) $A(t) = -t^3 + 12t^2 - 21t + 40$
d) $t = 4$
e) 138 gallons
f) 16 gallons per hour
g) 98 gallons

$\boxed{13}$

a) $A(t) = \frac{1}{3}t^3 - 4t^2 + 18t$, $B(t) = -t^2 + 13t$
b) $t = 5$
c) 8.33 miles
d) 2 miles per minute
e) $\frac{1}{3}t^3 - 3t^2 + 5t + 0 = 0$
f) $t = 1.55$ and $t = 6.45$
g) 18 miles

WORKSHEET #23

$\boxed{2}$ $A(t)$ could be any function of the form $\dfrac{t^3}{3} - 4t^2 + 12t + k$, where k is any constant. It would look just like the other two graphs, but shifted vertically.

$\boxed{4}$ NOTE: The graph of *altitude gained* would go through the origin.

$\boxed{5}$ $G(t) = 2t^3 - t^2 - 5t + k$

$\boxed{7}$ (a) $-\dfrac{3}{x} + k$; (b) $\dfrac{2}{3}x^{3/2} + k$; (c) $12t^{1/3} + k$; (d) $-\frac{5}{4}t^4 + k$; (e) $4t + k$;

$\boxed{8}$ (a)$\dfrac{3}{5}x^5 + \dfrac{2}{3x} + ex + k$; (b) $8\sqrt{t} - \frac{10}{\sqrt{t}} + k$; (c) $-\dfrac{1}{18u^3} + \dfrac{3}{4u^2} - \dfrac{1}{2}u + k$; (d) $\dfrac{\sqrt{e}}{3}w^3 + \dfrac{2e}{3}w^{3/2} + k$

$\boxed{9}$
a) $\frac{2}{3}x^{3/2} + \frac{3}{x} + C$
b) $-u^{-1} - 2u^{-2} + 3u + C$
c) $2z - 6\sqrt{z} + 4\ln(z) + C$
d) $\frac{3}{5}w^{5/3} + \frac{3}{2}w^{-1/2} - 6w + C$
e) $\frac{1}{5}t^{5/4} - 2t + \frac{4}{3}t^{3/4} + C$
f) $\frac{5}{2}x^2 + \frac{3}{4}x^{-4} + \frac{4}{3}x^{3/2} + C$

$\boxed{10}$
a) $m = 6$ gives a local max; $m = 12$ gives a local min
b) max 180, min 0
c) $m = 12$
d) from $m = 0$ to $m = 6$
e) -9
f) 288

$\boxed{11}$
a) $S(t) = t^2 - 22t + k$
b) $S(t) = t^2 - 22t + 150$
c) $t = 3$ gives a local min, $t = 6$ gives a local max
d) 150 hundred feet, 69 hundred feet
e) 33 hundred feet
f) $t = 1$ and $t = 7$
g) -45 hundred feet

$\boxed{12}$
a) $u = 4.94$ and $u = 21.06$
b) $u = 4.94$
c) $g(u) = 2u^2 - 16u + 12$

d) $u = 7$

e) local min at $u = 12$, local max at $u = 6$

f) 14

g) local min at $u = 4$

WORKSHEET #24

$\boxed{1}$ $A(5) - A(1) \approx 170$

$\boxed{2}$ (b) and (c) are true

$\boxed{4}$ (b) $A(5) - A(1) = \displaystyle\int_1^5 h(u)\,du$

$\boxed{6}$ $A(u) = \frac{1}{4}u^4 - \frac{13}{3}u^3 + \frac{31}{2}u^2 + 45u + k$

$\boxed{7}$ $k = 0$

$\boxed{8}$ $A(5) - A(1) = 170.67$

$\boxed{9}$ (d) $\displaystyle\int_5^9 h(u)\,du = -85.333$

$\boxed{11}$ $\displaystyle\int_1^6 f(x)\,dx = 1.39$

$\boxed{12}$ $\displaystyle\int_2^6 3x^2 - 24x + 40\,dx = -16$

$\boxed{13}$ 5.33

$\boxed{14}$ 9

$\boxed{15}$ 5.33

$\boxed{17}$ a) i) 66; ii) 40; b) -5.333; c) i) -8; ii) 102.

$\boxed{18}$

a) $-\frac{125}{3} + \frac{125}{2}$

b) $-\frac{8}{3} + 4$

c) $\left(-\frac{2}{3}\right)(216) + 8(36) - 24(6) + \left(\frac{2}{3}\right)(8) - 8(4) + 24(2)$

d) 36

e) $-\frac{64}{6} + 32 - \frac{64}{5}$

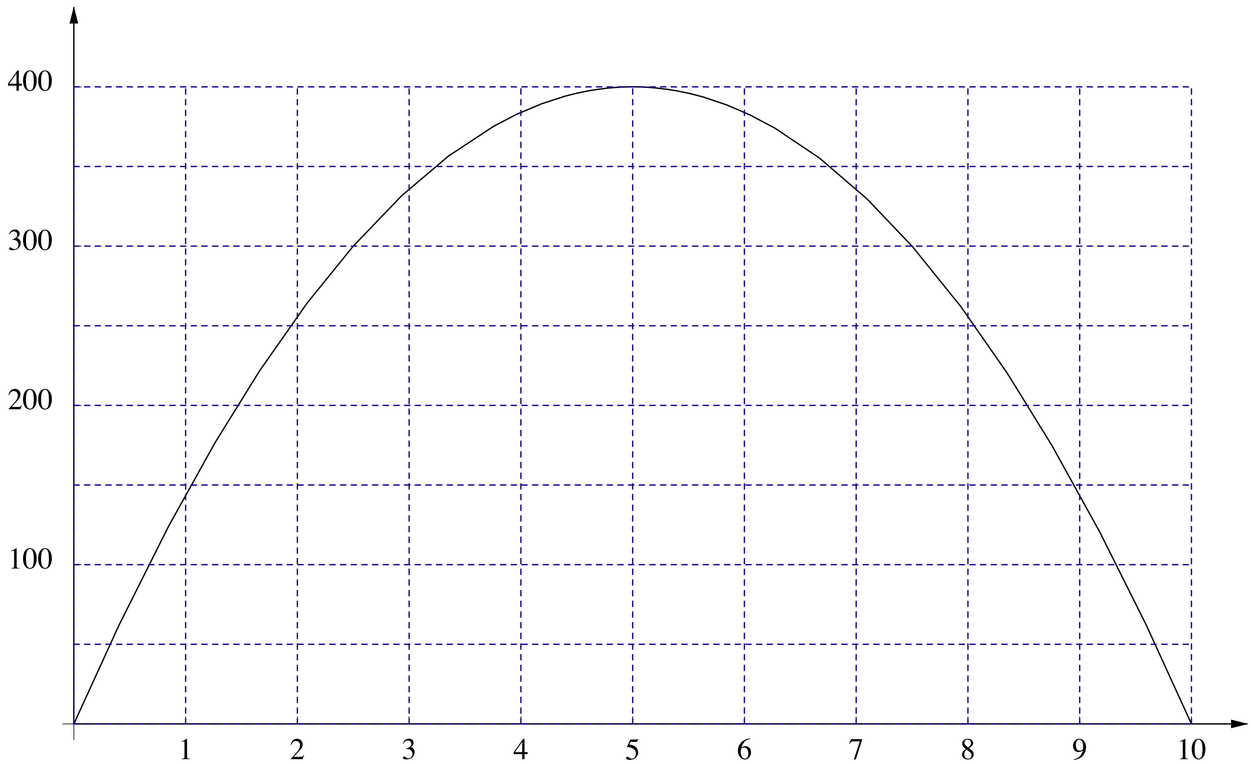
f) $36 + 9 - \frac{63}{2}$

Graphs and Tables

This appendix contains copies of the graphs and tables in the book that you can tear out and include in your homework.

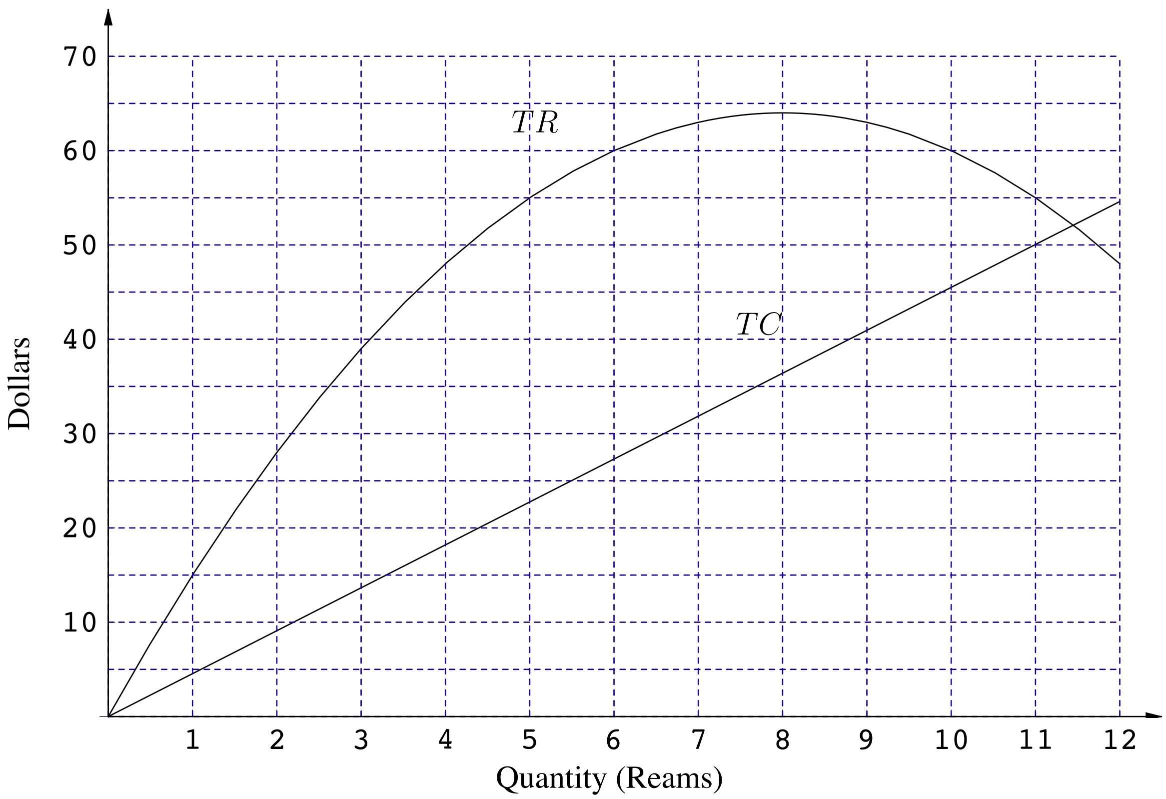
WORKSHEET #1

$\boxed{4}$

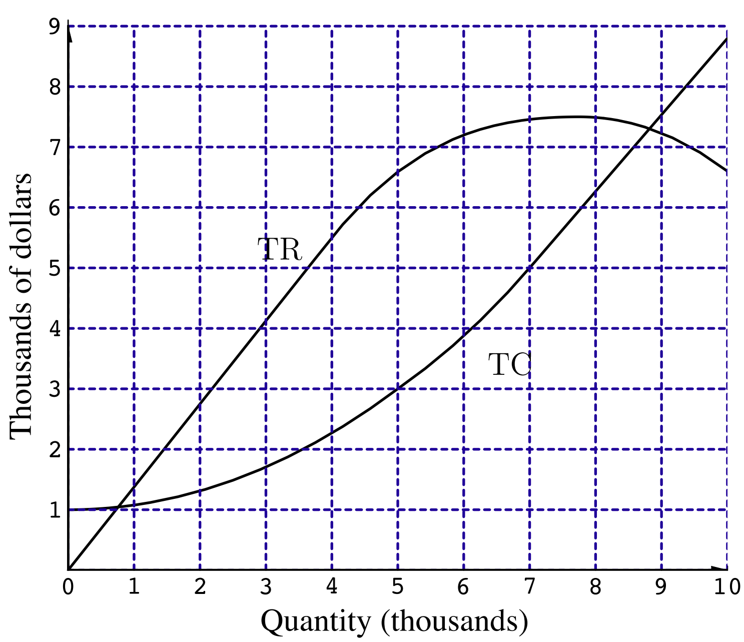


WORKSHEET #2

2

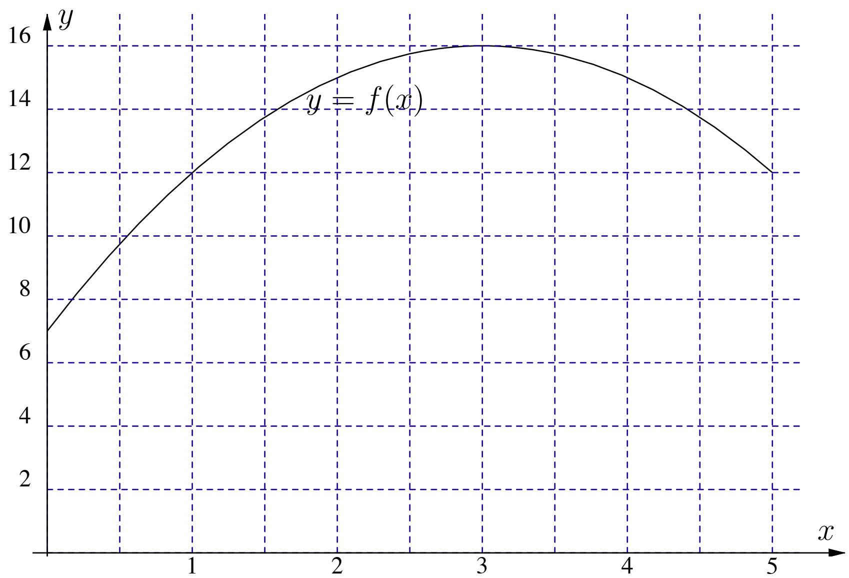


18

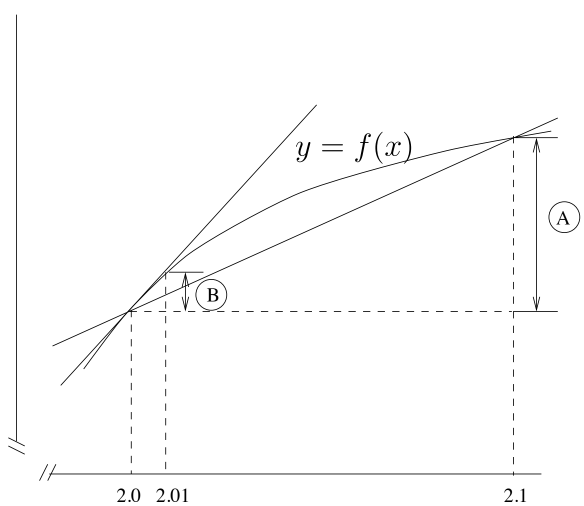


WORKSHEET #3

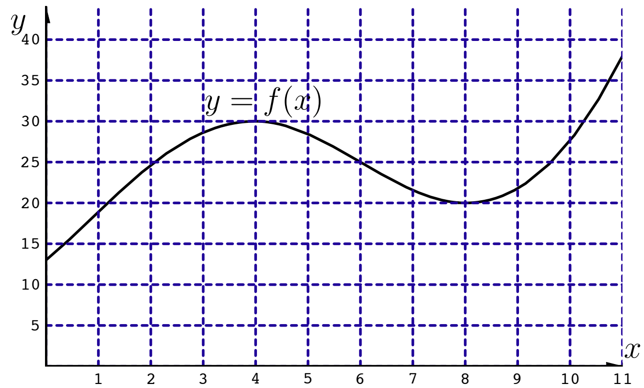
1



2

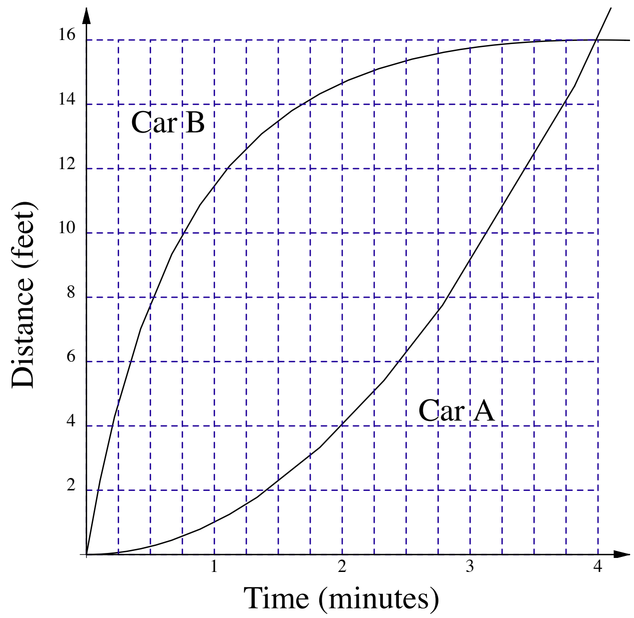


14



WORKSHEET #4

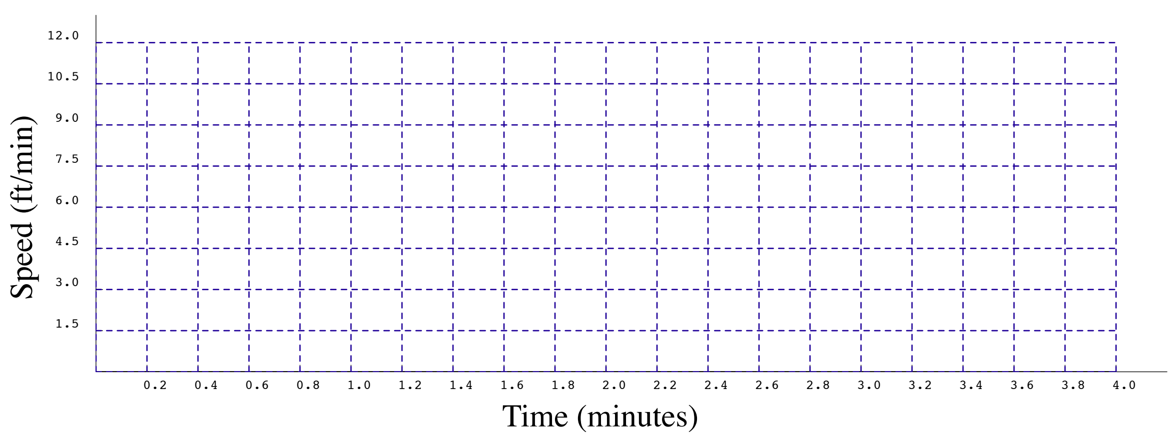
4



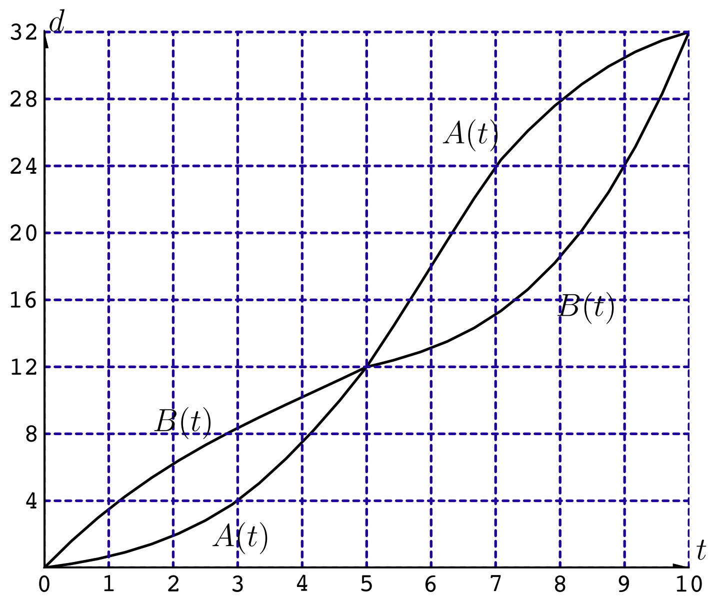
t	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Speed of Car A at time t	0	0.4	0.8	1.2	1.6		2.4	2.8	3.2	3.6	
Speed of Car B at time t	24	16.36	11.76	8.77	6.72		4.17	3.34	2.7	2.19	1.78

t	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Speed of Car A at time t	4.4	4.8	5.2	5.6	6	6.4	6.8	7.2	7.6	8
Speed of Car B at time t	1.44	1.16	0.93	0.73		0.42	0.29	0.18	0.09	0

8

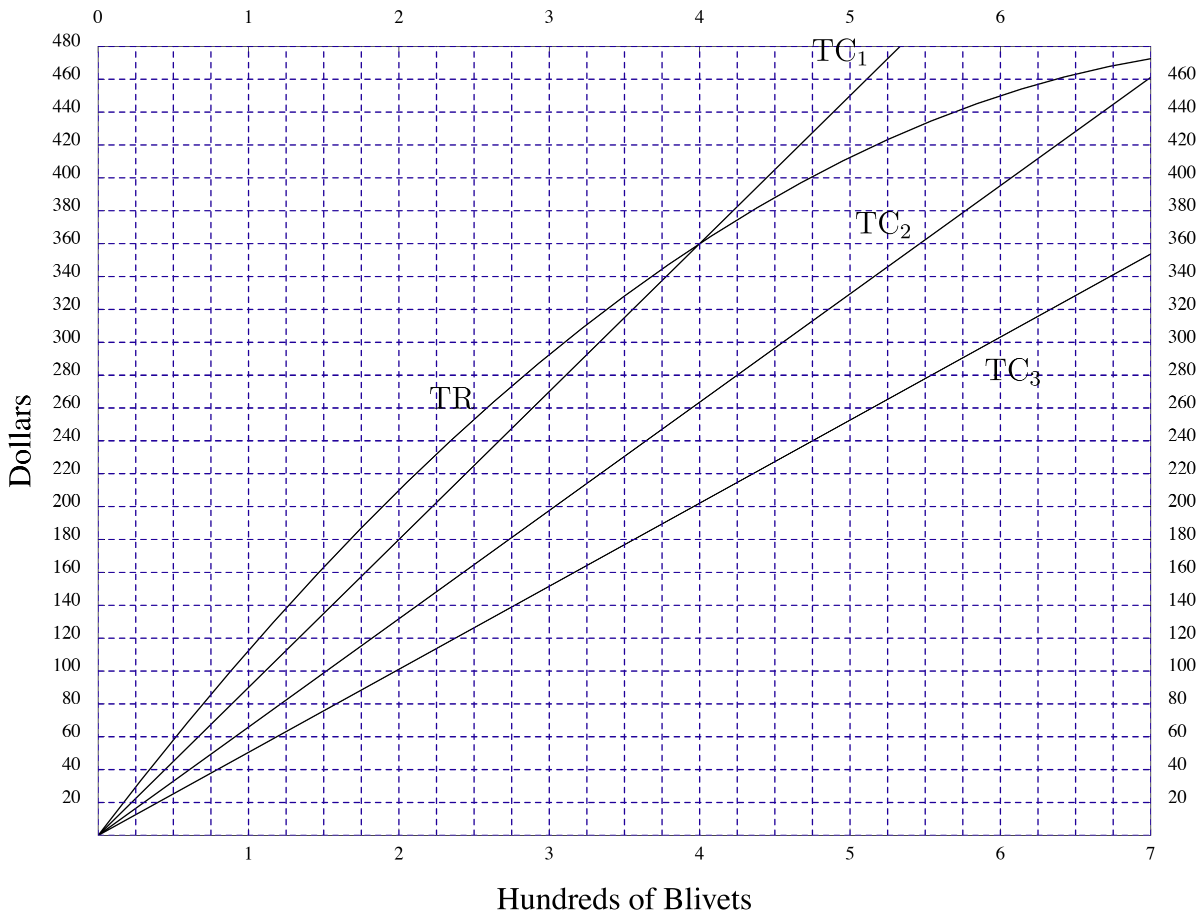


$\boxed{12}$



WORKSHEET #5

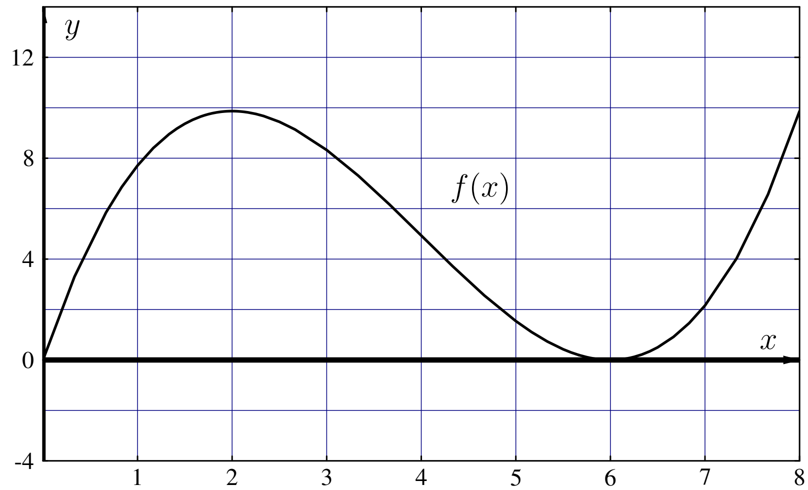
2



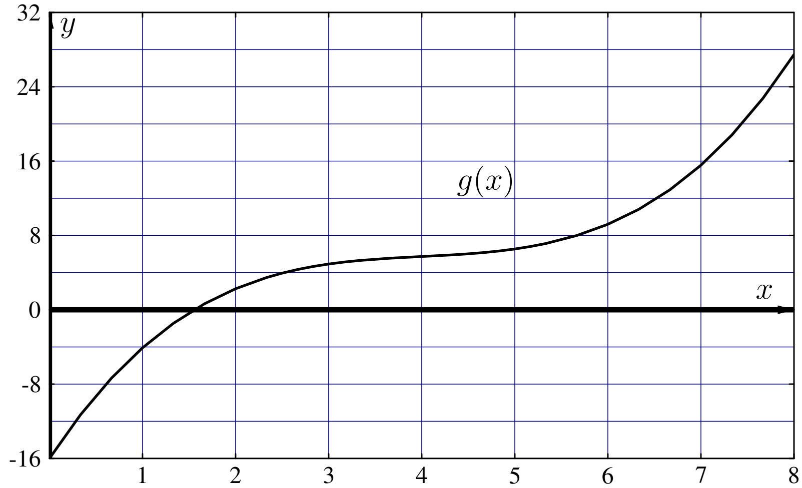
5

WORKSHEET #6

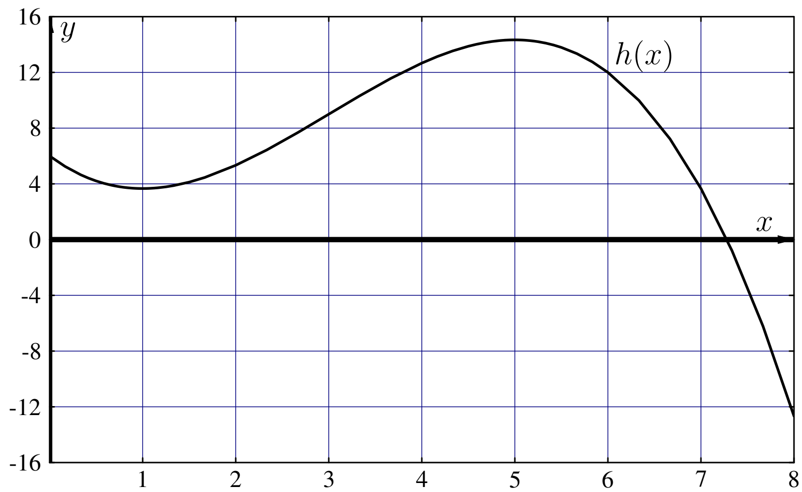
$\boxed{1}$



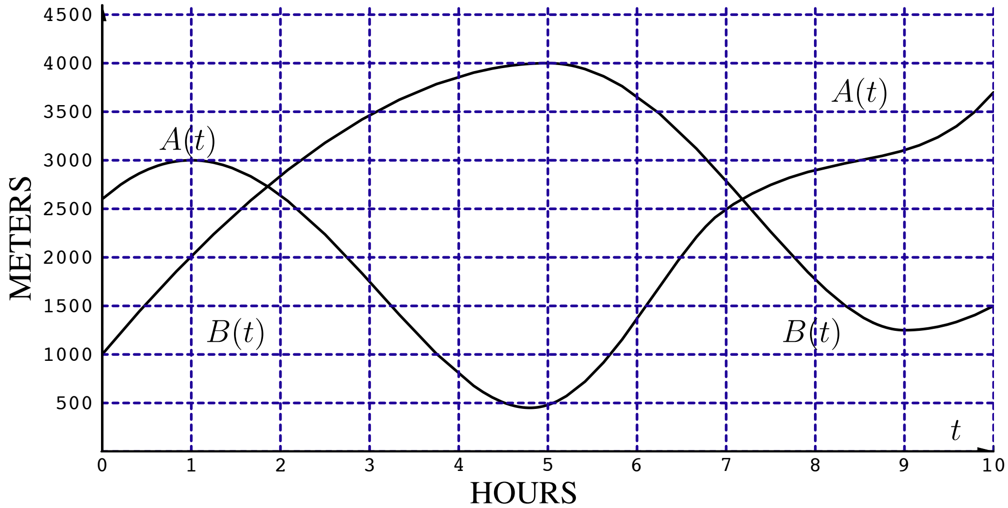
$\boxed{6}$



$\boxed{10}$



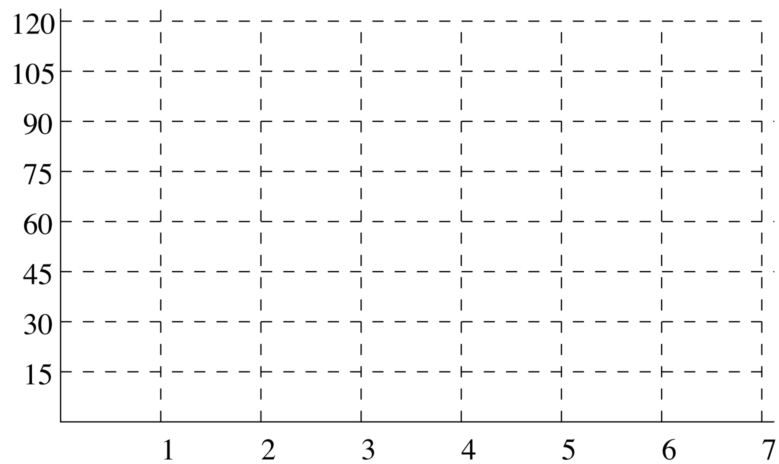
18


4500
4000
3500
3000
2500
2000
1500
1000
500
METERS
$A(t)$
$A(t)$
$B(t)$
$B(t)$
t
0
1
2
3
4
5
6
7
8
9
10
HOURS

WORKSHEET #7

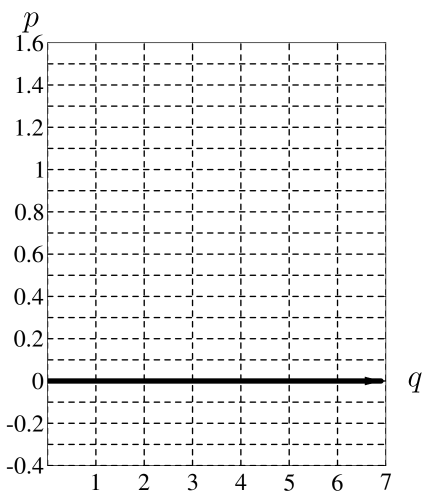
$\boxed{18}$

m	0	1	2	3	4	5	6	7
$f'(m)$				75	60		30	



WORKSHEET #10

$\boxed{4}$



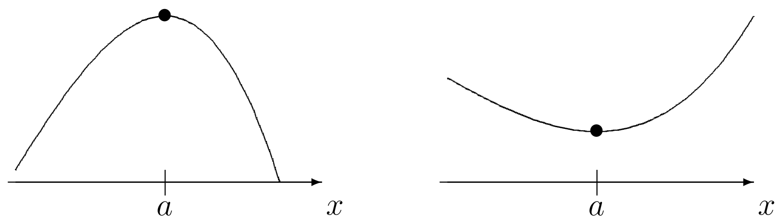
WORKSHEET #11

$\boxed{7}$

time interval	0–1.5	1.5–2.6	2.6–5	5–7	7–8.6	8.6–9	9–10
f							
g							

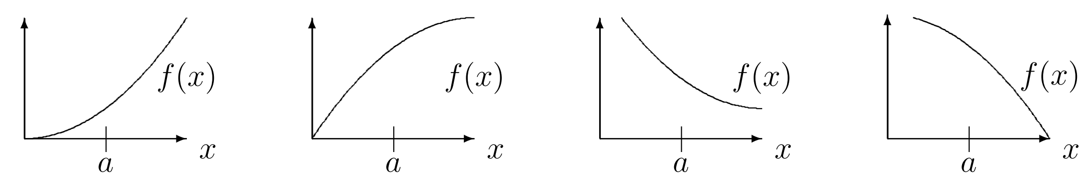
WORKSHEET #16

$\boxed{7}$



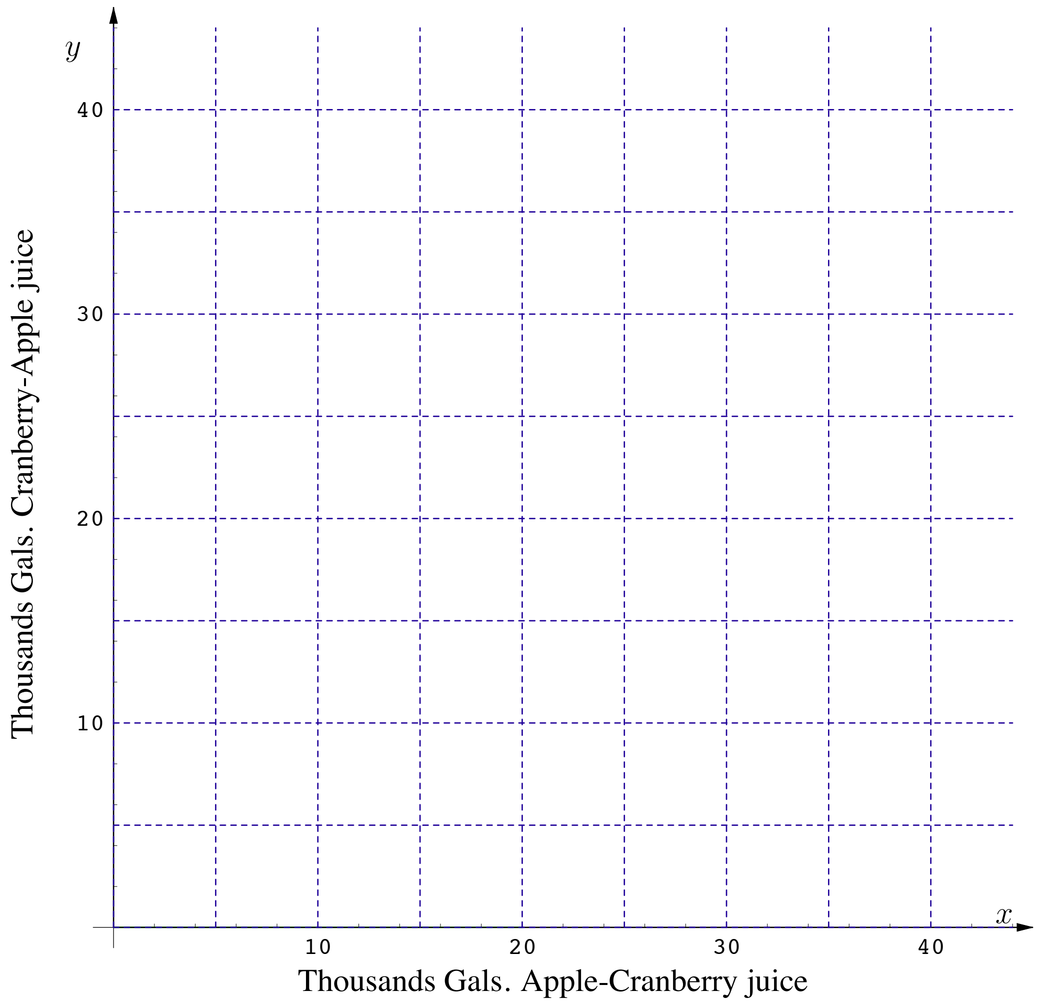
$\boxed{8}$

$f(x)$ increasing concave up | $f(x)$ increasing concave down | $f(x)$ decreasing concave up | $f(x)$ decreasing concave down

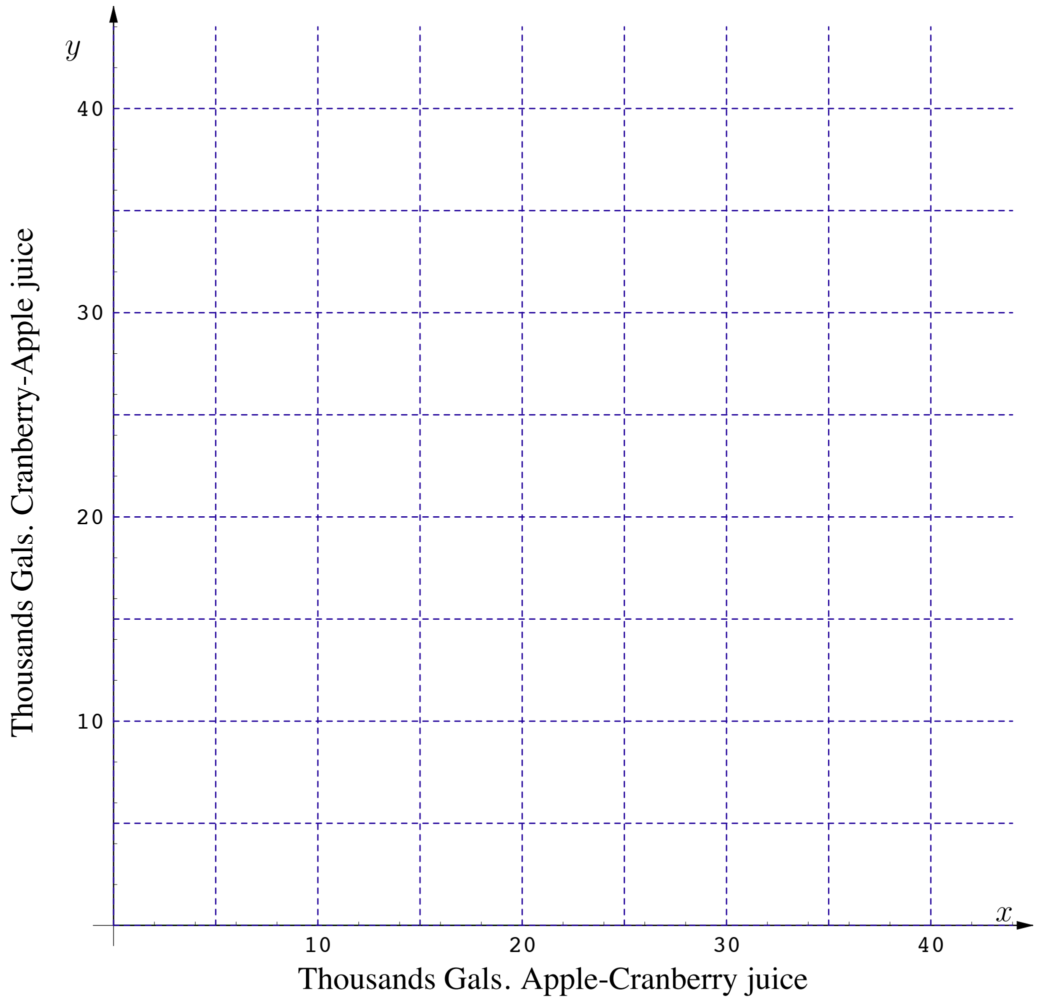


WORKSHEET #18

$\boxed{3}$



15


y
40
30
20
10
Thousands Gals. Cranberry-Apple juice
x
10
20
30
40
Thousands Gals. Apple-Cranberry juice

WORKSHEET #19

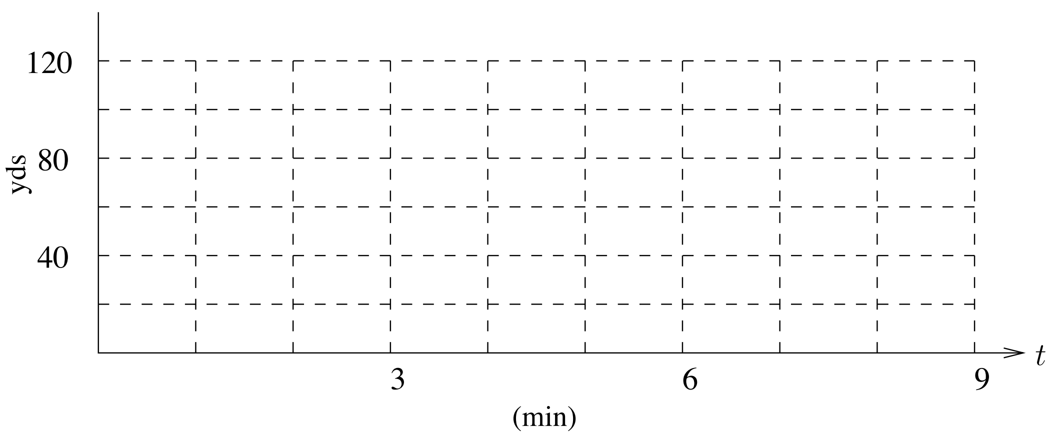
9

Interval	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9
Distance covered in interval by Car A	18.5	12.5	8.5						
Distance covered in interval by Car B	8.5	9.5	10.5						

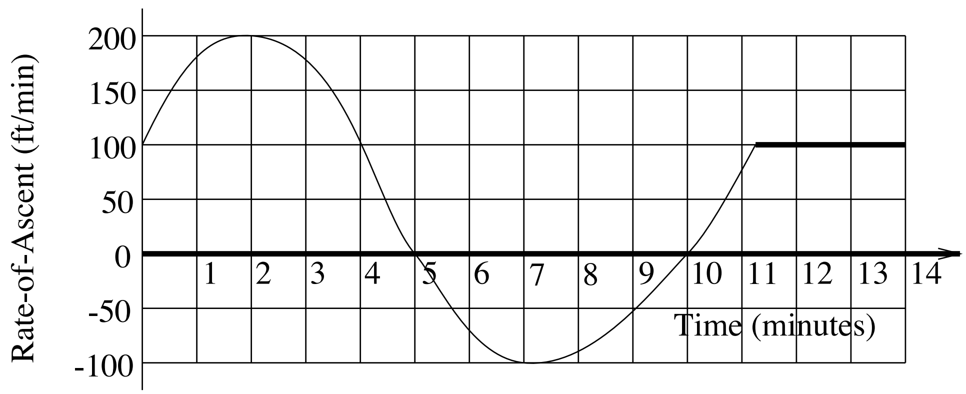
11

Time t	0	1	2	3	4	5	6	7	8	9
Distance covered by time t by Car A	0	18.5	31	39.5						
Distance covered by time t by Car B	0	8.5	18	28.5						

12



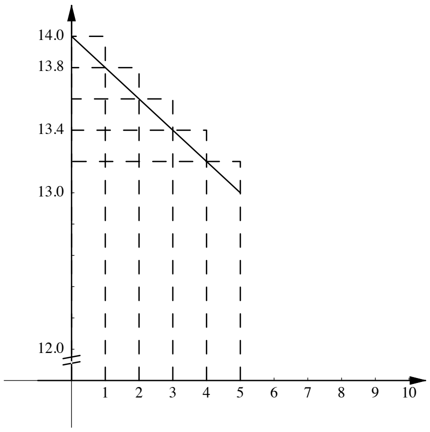
20



Rate-of-Ascent (ft/min)
200
150
100
50
0
-50
-100
1
2
3
4
5
6
7
8
9
10
11
12
13
14
Time (minutes)

WORKSHEET #20

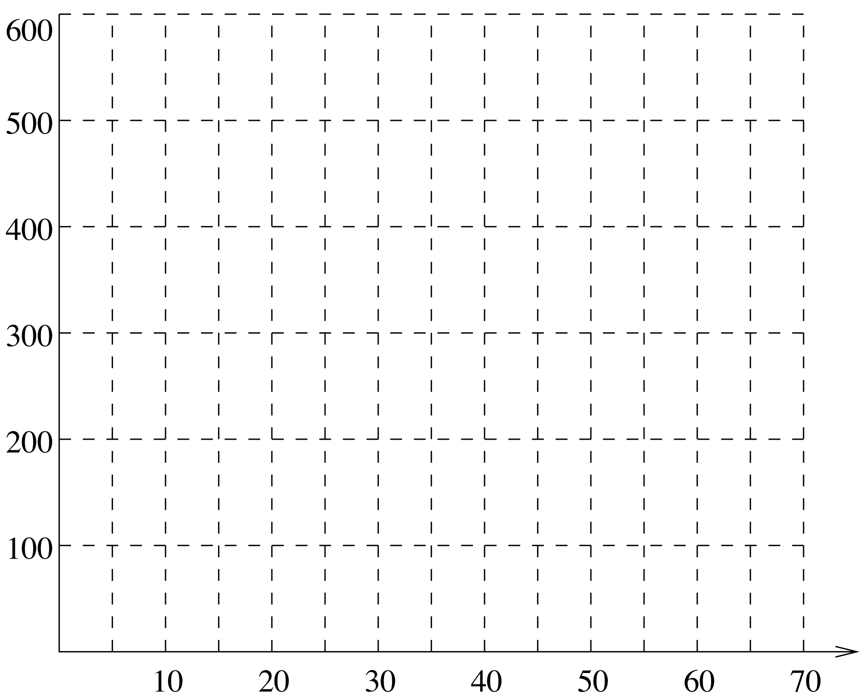
2



5

q	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70
TR	0	67.5	130			287.5			400		450				490

6



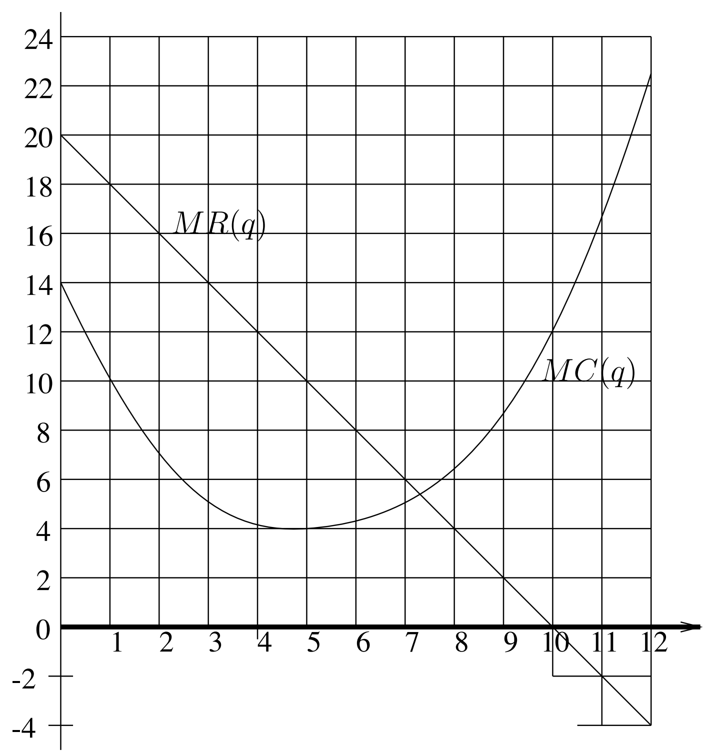
10

Interval	0–5	5–10	10–15	15–20	20–25	25–30	30–35	35–40	40–45	45–50	50–55	55–60	60–65	65–70
Change in TC	25.5	18	13				18			48	63			122.5

11

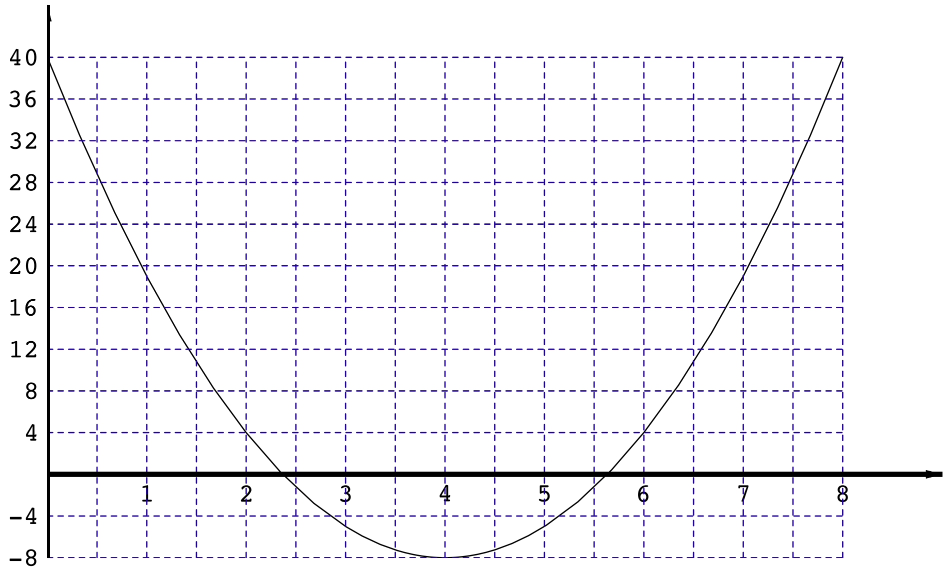
q	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70
TC	0	25.5	43.5	56.5											

20

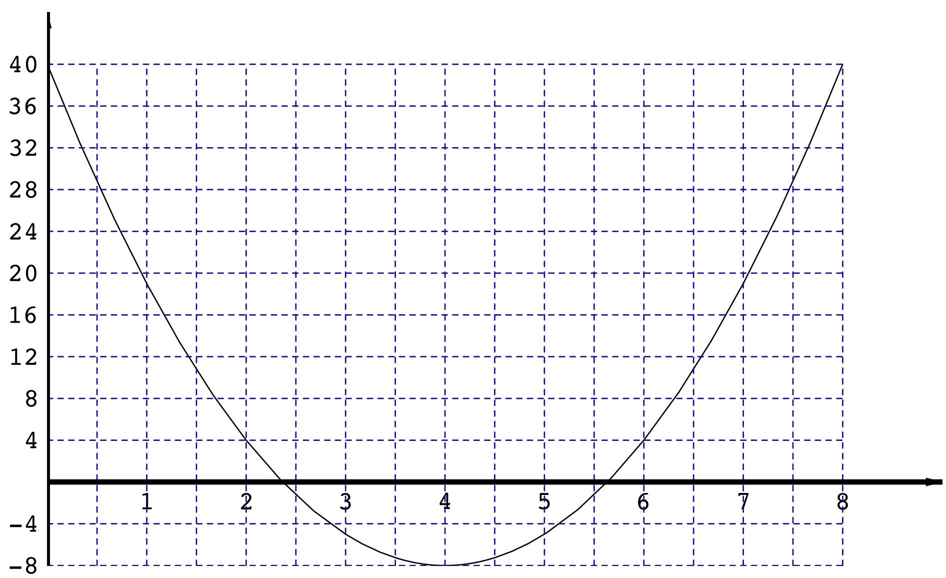


WORKSHEET #21

$\boxed{1}$



$\boxed{2}$

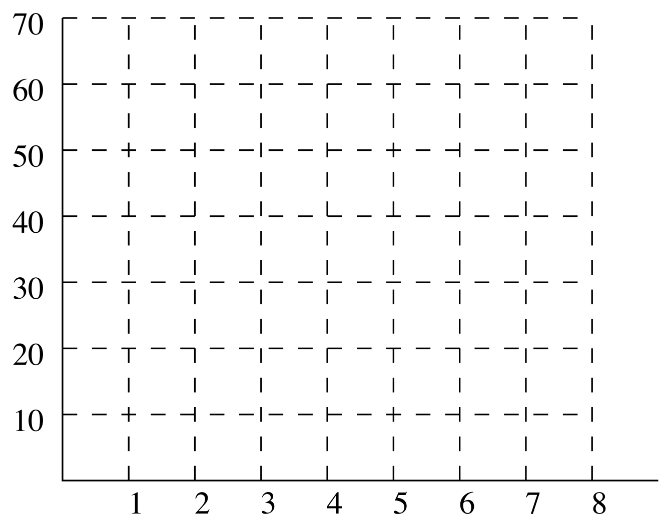


11

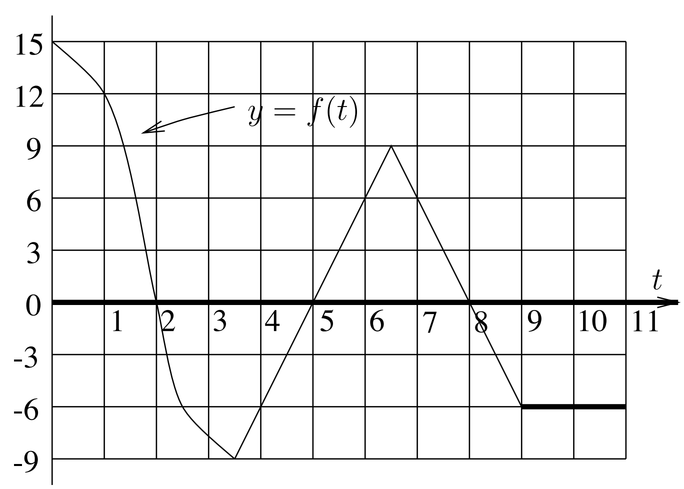
m	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
$A(m)$	0	17.2	29.2	36.6	40.3	40.9	39.3	

m	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0
$A(m)$									

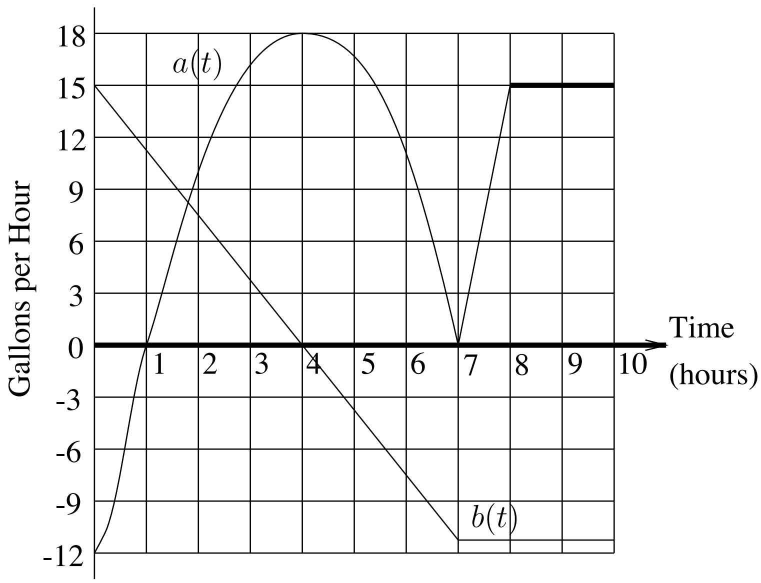
12



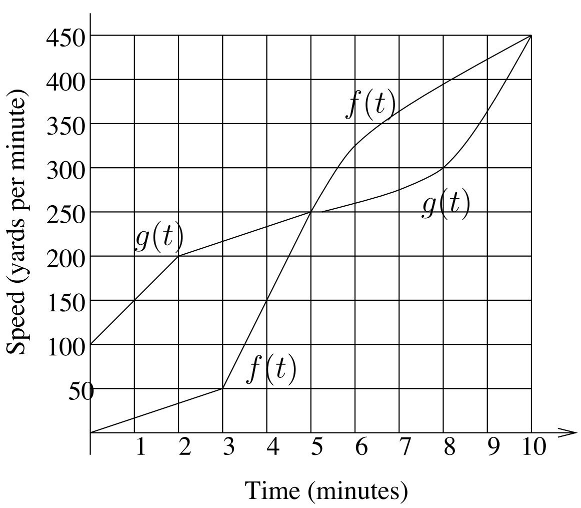
15



16


Gallons per Hour
18
15
12
9
6
3
0
-3
-6
-9
-12
$a(t)$
$b(t)$
1 2 3 4 5 6 7 8 9 10
Time
(hours)

17


Speed (yards per minute)
450
400
350
300
250
200
150
100
50
$f(t)$
$g(t)$
$g(t)$
$f(t)$
1 2 3 4 5 6 7 8 9 10
Time (minutes)

WORKSHEET #24

$\boxed{4}$