Calculus

Chapter 4A Approximation using the derivative

It's an algebraic take on the limit rule

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

** that 'formula' is familiar to you isn't it … you will never forget it will you ☺ **

and through the magic of Algebra it becomes

$$h\,f'(x) = \lim_{h\to 0} f(x+h) - f(x)$$

$$\lim_{h\to 0} f(x+h) = h\,f'(x) + f(x)$$

and

$$f(x+h) \approx f(x) + hf'(x)$$

… notice the limit has been removed, so therefore h does not equal zero, but a small actual value … therefore the solution can only be approximate, as the original function was based on setting h=0, but here, h is NOT zero! **

eg. Approximate the value of $\sqrt{37}$ … Note that this is very close to $\sqrt{36}$, so we can approximate.

Set $f(x) = \sqrt{x}$ $\quad \to f'(x) = \frac{1}{2\sqrt{x}}$

Using the rule $f(x+h) \approx f(x) + h\times f'(x)$

Becomes $f\left(\sqrt{x+h}\right) \approx f\sqrt{x} + h\frac{1}{2\sqrt{x}}$

Now we say, $x = 36$, which leaves $h = 1$

$$f\left(\sqrt{36+1}\right) \approx f\sqrt{36} + 1\times\frac{1}{2\sqrt{36}}$$

$$\sqrt{36+1} \approx 6 + \frac{1}{12}$$

$$\therefore \sqrt{37} \approx 6\frac{1}{12}$$

clearly the bigger h is, the more approximate your answer.

Do Exercise 4A

Note, only do the marked questions … NO need to do any Trig questions here

Integration

We should have covered some of this in Maths B, so just as a refresher …

Integration is the Opposite of Differentiation. We could say that Integration is Anti-Differentiation !

We shall take more of a mathematical approach to Integration in Maths C, than in Maths B.

If we know our Derivatives, then we already know some integrals … it's just the reverse process !

If we have a known derivative, then we know to just work backwards to know the Integral.

Table of known Derivatives

f(x)	f ’(x)
x^n	nx^{n-1}
$ae^{g(x)}$	$g'(x)ae^{g(x)}$
$\ln g(x)$	$\frac{g'(x)}{g(x)}$
$\sin ax$	$a\cos ax$
$\cos ax$	$-a\sin ax$

etc etc

Lets get some terminology established.

$\int$ is the sign for Integral. As a verb we would say we need to Integrate.

$\int (\quad)\, dx$ means calculate the Integral of ( ) with respect to x

and the thing inside the brackets is called the “Integrand”

So, firstly lets set $\int f(x) = F(x)$

Integration

f(x)	F(x)
ax^n	$\dfrac{ax^{n+1}}{n+1} + c$
$f(x) \pm g(x)$	$\int f(x) \pm \int g(x)$
$kf(x)$	$k\int f(x)$
$(ax+b)^n$	$\dfrac{(ax+b)^{n+1}}{a(n+1)} + c$
$\dfrac{1}{ax+b}$ or $(ax+b)^{-1}$	$\dfrac{1}{a}\ln(ax+b) + c$
e^{kx}	$\dfrac{1}{k}e^{kx} + c$
$\ln x$	$x \ln x - x + c$
$\sin kx$	$\dfrac{\cos kx}{k} \quad C$
$\cos kx$	$\dfrac{\sin kx}{k} \quad C$
$sec^2 kx$	$\dfrac{\tan kx}{k} \quad C$
$\dfrac{1}{\sqrt{a^2 \quad x^2}}$	$sin \;\; 1 \, \dfrac{x}{a} \quad C$
$\dfrac{1}{\sqrt{a^2 \quad x^2}}$	$\cos \;\; 1 \, \dfrac{x}{a} \quad C$
$\dfrac{a}{a^2 \quad x^2}$	$tan \;\; 1 \, \dfrac{x}{a} \quad C$

Don’t forget to ADD the “c”

Make sure you understand why there is a “C” ... and Never forget to add it!

Trig Identities

Advanced Periodic Functions skills will be helpful when Integrating some Trig functions as some manipulation of trigonometric Identities will be required.

You may be called upon to use any of the identities from the previous chapter again here … and you must have recall of any Trig Identity mentioned in the text Book Chapters …

I also provided a Huge list of extra Trig identities that you do not need to remember, but it may be an idea to have your sheet at hand to assist here.

OK … so lets look at Integration …

Chapter 4B Integration by substitution

They key is … look for the derivative of part of the function, within another part of the integrand … you will see what I mean after a few practice questions.

The text book makes this look confusing … but it really isn't!

I call it Integration by "U" substitution.

Best we look at an example … !

$$\int 2x(x^2+1)^3 \; dx$$

(note the relationship between inside and outside the brackets)

So we set, $u = x^2 + 1$ $\frac{du}{dx} = 2x$ $\rightarrow \; du = 2x\,dx$

Back to the Integral $\int 2x(x^2+1)^3 dx$

Rearrange to $\int (x^2+1)^3 2x\,dx$

By substitution (twice) we now get $\int u^3 du$

Which easily evaluates to $\int u^3 = \frac{u^4}{4}$

And again by substitution, we can finally say

$$\int 2x(x^2+1)^3 dx = \boldsymbol{\frac{1}{4}(x^2+1)^4 + C}$$

** take note of the Integral Table on page 132. I will not guarantee your calculator will enable you to solve your exam questions appropriately. I suggest you Memorise this table. There really are only 4 that you won't already know **

Do Exercise 4B

Chapter 4C multiples of a derivative in the integrand

Here we use the Linear property of differentials AND integrals to solve for integral questions where the derivative within the integrand is not exactly in the right ‘multiple’.

The Linear law of Derivatives and Integrals:

$$\int kg(x) = k\int g(x) \qquad\qquad \frac{d}{dx}kf(x) = k\frac{d}{dx}f(x)$$

See it in action: $\int 6x^2(x^3-2)^5\ dx$

set $u = x^3 - 2$ $\quad \frac{du}{dx} = 3x^2 \quad \to 2\frac{du}{dx} = 6x^2 \quad \to 2du = 6x^2dx$

back to Integral …

$$\begin{aligned}\int 6x^2(x^3-2)^5\,dx &= \int (x^3-2)^5\,6x^2dx \\ &= \int u^5 2du \\ &= 2\int u^5 du \\ &= 2\frac{u^6}{6} \\ &= \frac{1}{3}(x^3-2)^6 + C\end{aligned}$$

ALTERNATE solution:

$$\int 6x^2(x^3-2)^5\ dx$$

You notice that its not a straight substitution, and think that it would be "nice" if the first function was simply $3x^2$ … then manipulate it so that it is … !

$$\int 6x^2(x^3-2)^5\,dx = 2\int 3x^2(x^3-2)^5\ dx$$

set $\quad u = x^3 - 2 \qquad \frac{du}{dx} = 3x^2 \qquad \rightarrow du = 3x^2dx$

$$2\int 3x^2(x^3-2)^5 = 2\int (x^3-2)^5\,3x^2dx$$

$$= 2\int u^5\,du$$

$$= 2\frac{u^6}{6}$$

$$= \frac{1}{3}(x^3-2)^6 + C$$

Does it matter which way you do it … NO … Just get it right, and make sure your Solution shows Perfect communication … Don't skip any steps, or Panel will think you are just using your calculator … ☺

Seems simple hey … the algebra does get interesting in the harder ones!

Just like derivatives, the preferred form of your solution should be "Factor Form", and this gives you some interesting algebra!

Take Question 1 p)

$$\int \frac{3x}{\sqrt{8-x}}\,dx$$

set $u = 8 - x$ so, clearly $x = 8 - u$

and $\frac{du}{dx} = -1$ so, $du = -dx$

by substitution, we get

$$\int \frac{3(8-u)}{u^{\frac{1}{2}}} \times -du$$

$$= \int \frac{3u - 24}{u^{\frac{1}{2}}} du$$

$$= \int 3u^{\frac{1}{2}} - 24u^{\frac{-1}{2}}\, du$$

$$= 2u^{\frac{3}{2}} - 48u^{\frac{1}{2}}$$

$$= 2(8-x)^{\frac{3}{2}} - 48(8-x)^{\frac{1}{2}}$$

$$= 2(8-x)(8-x)^{\frac{1}{2}} - 24(8-x)^{\frac{1}{2}}$$

$$= 2\{\, (8-x) - 24 \,\}\, (8-x)^{\frac{1}{2}}$$

$$= -2(x+16)\sqrt{8-x} + C$$

Algebra operations still hold in Integral equations … See Worked Example 8 for some innovative algebra work to obtain an integral. If given a question, keep looking at ALL the equations you have. Manipulate them and see where you can substitute to obtain the correct ‘form’.

Do Exercise 4C

Chapter 4D

Is about using Trig Identities to manipulate the integrand into something that is "Integratable"?

This is all about practice in using the identities … so… best get to it!

I see limited value in enforcing the memorising of more Trig Identities. If it is not in my list in my worksheets, then I do not expect you to know it.

However, doing this exercise will give you crucial practice at solving integrals … but I confirm there is no need for you to memorise these trig identities, and simply suggest you write down the new identities in this chapter somewhere and refer to them when you work through this chapter.

Do Exercise 4D

Chapter 4E Partial Fractions

Head back to 4C … what if there is a similar term that is the differential of the second term, but it is not an even linear scalar of it … the answer … Partial Fractions!

The book says partial fractions is tedious, but I think it's beautiful!

As it is a longer process, only do it where necessary, but it will be necessary some times!

Lets consider: $\int \frac{6x}{x^2+2x-3} dx$

This is Not 'u' substitution as the 6x is not a linear derivative of the denominator!

What we want to do is to split the fraction into a sum of two fractions.

$$\frac{6x}{x^2+2x-3} = \frac{6x}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

clearly

$$\frac{A}{x+3} + \frac{B}{x-1} = \frac{A(x-1)}{(x+3)(x-1)} + \frac{B(x+3)}{(x-1)(x+3)} = \frac{(A+B)x + 3B - A}{x^2+2x-3}$$

combining the above two lines, we get to:

$$\frac{6x}{x^2+2x-3} = \frac{(A+B)x + 3B - A}{x^2+2x-3}$$

So by 'equating coefficients', CLEARLY, we have

$A + B = 6$ and $3B - A = 0$

solving these simultaneously we get A= 4.5 and B= 1.5

Now it is easy (well easier) to Integrate:

$$\int \frac{4.5}{x+3} + \frac{1.5}{x-1} dx$$

$$= 4.5 \int \frac{1}{x+3}\, dx + 1.5 \int \frac{1}{x-1} dx$$

$$= 4.5 \ln(x+3) + 1.5\ \ln(x-1) + C$$

see, much easier … ☺

There is a slight problem encountered when there is a "square" in the denominator, such as:

$$\int \frac{6x}{x^2+2x+1}\,dx$$

see if you can solve this one as per the previous technique … you end up with simultaneous equations that don't get a solution and you get to a dead end.

When there is a perfect square in the denominator, you split the denominators up differently and establish your Partial Fractions in the form:

$$\frac{A}{(ax+b)^2}+\frac{B}{(ax+b)}$$

So in the above example we would set:

$$\frac{6x}{x^2+2x+1}=\frac{A}{(x+1)^2}+\frac{B}{(x+1)}$$

$$\frac{6x}{x^2+2x+1}=\frac{A}{(x+1)^2}+\frac{B(x+1)}{(x+1)(x+1)}$$

$$\frac{6x}{(x+1)^2}=\frac{A+Bx+B}{(x+1)^2}$$

$$\frac{6x}{(x+1)^2}=\frac{Bx+A+B}{(x+1)^2}$$

to get

$$B=6$$

$$A+B=0$$

so $$A=-6$$

$$\int \frac{6x}{x^2+2x+1}\,dx=\int \frac{-6}{x^2+2x+1}+\frac{6}{x+1}\,dx$$

$$=\int \frac{-6}{(x+1)^2}+\frac{6}{x+1}\,dx$$

$$=-6\int (x+1)^{-2}\,dx+\ 6\int \frac{1}{x+1}\,dx$$

$$=-6\times\frac{(x+1)^{-2+1}1}{-2+1}+6\ln(x+1)$$

$$=\frac{6}{x+1}+6\ln(x+1)+C$$

The creativity of mathematicians is simply amazing. Can you imagine how creative the first person to solve this next one had to be … its so simple, but hard to know what to do unless you have seen it before:

$$\int \frac{2x^3 + x^2 - 5}{x^2 - 1}\, dx$$

Consider the Integrand

$$\frac{2x^3 + x^2 - 5}{x^2 - 1}$$

$$= \frac{2x^3 - 2x + x^2 + 2x - 5}{x^2 - 1}$$

$$= \frac{2x(x^2 - 1) + x^2 + 2x - 5}{x^2 - 1}$$

$$= \frac{2x(x^2 - 1)}{x^2 - 1} + \frac{x^2 + 2x - 5}{x^2 - 1}$$

$$= 2x + \frac{x^2 - 1 + 2x - 5 + 1}{x^2 - 1}$$

$$= 2x + \frac{x^2 - 1}{x^2 - 1} + \frac{2x - 4}{x^2 - 1}$$

$$= 2x + 1 + \frac{2x - 4}{x^2 - 1}$$

and from here you consider just the fraction and set:

$$\frac{2x - 4}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1}$$

and from here you should be right … ?

Chapter 4F Integration by Parts

It's simply using algebra from our Product Derivative rule … see the Textbook for how the algebra works to arrive at the following rule:

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

Use this where there is a PRODUCT in the Integrand. Sometimes one of the terms of the integrand may be set to "1"… as per Worked Example 18.

Further, choose carefully the value for $u\ and\ \frac{dv}{dx}$ … the $\frac{dv}{dx}$ should be set to the part of the Integrand that when Integrated part does-NOT get more Complex … !

You will soon know if you chose wisely. If you choose the wrong way, you go backwards and things get more complicated … If this happens, then just start again.

Typically, for the $\frac{dv}{dx}$ part, you will set it equal to something like … $\sin x$, e^x

Sometimes, no matter which way you choose, you get no-where, but persist, there WILL be a way … check out worked example 19 … that's some nice Maths!

And ALWAYS … where possible, FACTORISE you answer … !

Integration measures the area under a curved function. That is the value calculated of F(x) for the value "x", gives the area between the curve and the x axis between the origin point "0" and point "x" on the x axis.

*** Care … if the value of F(x) is Negative, this represents the area beneath the x-axis. ***

To find the area under a curve, between two points you denote you are looking for a definite area, and we use the term **Definite Integral**

So we have the area between two points a and b:

$$\int_a^b f(x) = \left[F(x)\right]_a^b = F(b) - F(a)$$

*** Care if the function crosses the x axis between these two points.***

And if we are looking for the area between two functions:

$$\int_a^b f(x) - \int_a^b g(x) = \int_a^b \left[f(x) - g(x)\right]$$

again, take care of intersecting points … sketching graphs is most useful, or simply mathematically calculate intersecting points and find the separate areas, and then add them together.

Properties of definite integrals:

$$\int_a^b f(x) = \int_a^c f(x) + \int_c^b f(x)$$

$$\int_a^b kf(x) = k\int_a^b f(x)$$

$$\int_a^b f(x) = -\int_b^a f(x)$$

$$\int_a^b \left[f(x) + g(x)\right] = \int_a^b f(x) + \int_a^b g(x)$$

$$\int f(ax+b) = \frac{1}{a}F(ax+b) + c$$

Definite Integrals … Changing Terminals:

Where an integral is not straight forward and you need to substitute something for ‘x’ to solve, you also need to adjust the terminals, as the variable with respect to which the integral is being calculated has changed. As follows:

$$\int_1^2 \frac{x}{(x+1)^2}\,dx$$

let $u = x + 1$ … so … $x = u - 1$

$\frac{du}{dx} = 1$ … so … $du = dx$

So, BEFORE substituting into the integral, we need to adjust the terminals.

Where $x = 1$, we can see that $u = 1 + 1 = 2$

And where $x = 2$, we can see that $u = 2 + 1 = 3$

So our integral becomes:

$$\int_2^3 \frac{u-1}{u^2}\,du$$

which simplifies to

$$\int_2^3 \frac{1}{u} - u^{-2} \quad du$$

$$= \left[\ln u + \frac{1}{u}\right]_2^3$$

$$= \left(\ln 3 + \frac{1}{3}\right) - \left(\ln 2 + \frac{1}{2}\right)$$

$$= 0.238798$$

You could re-substitute the x values back in place of u and then use the initial terminal values in x … but converting terminals and leaving in terms of u is a bit shorter (maybe/maybe not) … but in any case, conversion of terminals is something we want to show Panel, so do it this way to show off all your mathematical ability!

Chapter 4H Solids of Revolution

*** If and only if you feel up to it, skip to page 17 and investigate Integration more thoroughly before you do Solids of revolution. The stuff at the end is EXTRA and does NOT form part of the curriculum and is NOT necessary for the exam. But it may help you connect some missing pieces in this section. The Curriculum is a bit sparse and there are leaps between steps here that have not been explained. (it is SIX pages of extra work so make the decision that is right for you before you commit to doing this extra learning)***

OK … so welcome back … lets get to Solids of Revolution:

It's hard to draw a good diagram, so please have a look at the figures on page 175 as you read through these notes.

If we consider just a "disc" where $x = a$, we could find its area using $A = \pi r^2$, where the radius is $f(x)$.

And we know that $V = A_{Base} \times h$

Lets now consider a bazillion "discs' at the limit where each disc's $h \to 0$ from point a to point b. Clearly if we added all these volumes up, we have the volume of the solid of revolution … ☺

OK … at the bottom of page 175 the text book "Conveniently" skips from

$$V = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \, \delta x$$

to

$$V = \int_a^b \pi y^2 \, dx$$

but they really didn't say how this happened … ! … that's why I went through all that Riemann Series "guff" … Not that it matters but here is my very Simplistic way of thinking about it …

Go back to that first diagram and we shall think about making a heap of discs all beside each other and we just add up all their volumes.

$$Total\ Volume = V_{Disc} + V_{Disc} + V_{Disc} \ldots$$

because each disc is a "Cylinder", we could say

$$V_{Solid\ of\ Revolution} = \pi r^2 h + \pi r^2 h + \pi r^2 h + \pi r^2 h \ldots$$

let's do some factoring … importantly the radius of each disc is Different so I can't combine these as they are Not "Like terms", but my π and h are constants … so we get

$$V_{Solid\ of\ Revolution} = \pi(r^2 + r^2 + r^2 + r^2 \dots)h$$

lets consider what r actually is. The Radius is the distance from the x-axis to the function … so isn't that just y?

$$V_{Solid\ of\ Revolution} = \pi(y^2 + y^2 + y^2 + y^2 \dots)h$$

the h of each disc is actually the change in x-axis … say Δx … or as we want an infinite of discs where $h \to 0$ we could even say dx … ☺

$$V_{Solid\ of\ Revolution} = \pi(y^2 + y^2 + y^2 + y^2 \dots)\ dx$$

So inside that bracket is the sum of a bazillion values of y^2 where y is a function of x and changes continuously and we are adding them all up. This is very similar to the fundamental theorem of calculus and is where the Riemann Series comes in. Inside that Bracket is the same basis of how Integration works, so that link allows us to say:

$$V_{Solid\ of\ Revolution} = \pi \int y^2\ dx$$

and one last modification is that clearly $y = f(x)$, so we get:

$$\boldsymbol{V_{Solid\ of\ Revolution} = \pi \int f(x)^2\ dx}$$

we can also rotate about the $y - axis$ … and hence that formula is:

$$\boldsymbol{V_{Solid\ of\ Revolution} = \pi \int f(y)^2\ dy}$$

and when we are talking about volumes where there is a region between two functions we have:

$$\boldsymbol{V_{Solid\ of\ Revolution} = \pi \int [\, f(x)^2 - g(x)^2\,]\ dx}$$

TASK: Using Integral Calculus, derive the Volume of a Cylinder and a Cone …

$$V_{Cylinder} = \pi r^2 h$$

$$V_{Cone} = \frac{1}{3}\pi r^2 h$$

*** Extra Stuff : Integration explained … !

Lets take a step back a little … well a LOT! The next pages represent EXTRA work that is NOT technically needed. You do NOT need to understand this … you can still get a VHA without this stuff as it is NOT in the curriculum … but there is a chance that it may connect a few missing pieces and make integration and Calculus a bit more "real".

We have done some pretty nifty mathematics in this chapter, and yet we haven't worked from First Principles … so it is possible we aren't really sure 'how' all this Calculus works (and that's OK).

To learn this thoroughly would take a good portion of a Full Semester of Uni Maths, so I am just picking out some basic concepts rather than trying to fully explain everything!

Lets head back to differentiation:

Recall a derivative has the notation $\frac{dy}{dx}$ … the 'd' doesn't have anything to do with it being the first letter of 'd'erivative, and is not really a mathematical abbreviation for the words (with respect to) … but is short for "Delta". Delta is a Greek Letter and in Capital is written as Δ .Delta is used to note the change in something.

You may see $\frac{dy}{dx}$ sometimes written as $\frac{\Delta y}{\Delta x}$, where Δ=Delta (the change in).

Note that $slope = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

… specifically speaking when we move from $\Delta\, x$ to dx , the difference is that in the dx , that change in value of x approaches zero, recall we are getting the gradient of the function at that instantaneous point, hence the move to differentiation from first principles … Nice hey … it all sort of makes sense … ?

lets move on to Area under a function.

We know that integration gives the Area under a curve (the first theorem of calculus establishes this), but lets look at the area under a curve being the sum of the area of lots of rectangles under the curve:

(maybe get Mr Finney to draw a diagram on the board?)

Consider the line $y = 3$ on a Cartesian Plane.

Now consider the rectangle formed between this function and $x = 1\ and\ 7$.

Clearly the area equals 18!

Lets start here … lets cut this rectangle into 3 smaller rectangles by taking intervals on the x-axis. And then we can add all these areas up and they should total to an area equal to 18 … and clearly this works!

How wide are these changes in x? Oops, can I say Δx ☺

Clearly , the new width of these smaller rectangles are $\frac{7-1}{3} = 2$

So our three rectangles:

$$Area = lw + lw + lw$$

and although our lengths are 3, clearly we will soon be working with a function that is not just parallel to the x-axis, so lets just leave it at l for the moment

$$Area = l2 + l2 + l2$$

$$Area = 2(l + l + l)$$

and now all $l's = 3$

$$Area = 2(3 + 3 + 3) = 18$$

Correct!

What if I wanted you to have more rectangles … lets say n rectangles … we'd have a width of $\frac{7-1}{n}$ … and even though we'd have more rectangles of smaller width, our length of the rectangle would remain as 3.

$$Area = A_1 + A_2 + A_3 + \cdots + A_n$$

$$Area = l_1 w + l_2 w + l_3 w + \cdots + A_n w$$

$$Area = (l_1 + l_2 + l_3 + \cdots + l_n)w$$

$$Area = \sum_{i=1}^{n} l \qquad \times w$$

and $w = \frac{6}{n}$ and $l_1 = l_2 = l_3 = l_n = 3$

$$Area = \frac{6}{n} \times \sum_{i=1}^{n} 3$$

we need to digress a little … consider …

$$\sum_{i=1}^{n} 3$$

we are adding 3 to itself, n times … so the sum of 3, n times is equal to $3n$

now back to

$$Area = \frac{6}{n} \times \sum_{i=1}^{n} 3$$

now becomes

$$Area = \frac{6}{n} \times 3n$$

$$Area = \frac{6}{n} \times \frac{3n}{1} = 18$$

correct … ☺

Now lets look at the area under a general function … and lets consider a number of rectangles filling the area under the curve. Just like when we drew secants to find the slope of the curve, these rectangles aren't a perfect fit. We can do what's called a left hand series, or a right hand series … *** Mr Finney will need to do a diagram on the white board for this ***

Clearly the areas of each individual rectangle is equal to $A = lw$, and here $l = y$ and $w = \Delta x$

Maybe we could write it as:

$$Area = A_1 + A_2 + A_3 + \cdots + A_n$$

or

$$Area = y_1\Delta x + y_2\Delta x + y_3\Delta x + \cdots + y_n\Delta x$$

and factorise

$$Area = (y_1 + y_2 + y_3 + \cdots + y_n)\Delta x$$

Clearly, the more rectangles we have, the more accurate our area becomes! How about we have an Infinite number of teen tiny rectangles, and hence $\Delta x \to 0 = dx$ … and we have:

$$Area = (y_1 + y_2 + y_3 + \cdots + y_n)\, dx$$

Take a look inside the brackets … wow, that looks like a Series!

$$Area = \sum (y_1 + y_2 + y_3 + \cdots + y_n)\ dx$$

lets neaten that terminology up a little:

$$Area = \sum_{i=1}^{n} f(x)\ \ dx$$

and here we are saying there are an Infinite number of rectangles, so there is an infinite number of values of $f(x)$

$$Area = \lim_{n\to\infty} \quad \sum_{i=1}^{n} f(x)\ dx$$

and by way of the fundamental theorem of calculus we have:

$$Area = \int f(x)\ dx$$

now … lets try and show you the 'link' from first principles

$$\lim_{n\to\infty} \quad \sum f(x)\ dx \qquad to \qquad \int f(x)\ dx$$

I haven't ever been shown, nor found, a General proof that links these above two terms … only some specific examples show it … so lets do that!

Consider … the area under the function

$$f(x) = x + 1$$

between $x = 1\ and\ 7$

$$A = lw + lw + lw + \cdots + lw$$

and $l = y = f(x)$ and are all different

$$A = f(x_1)w + f(x_2)w + f(x_3)w + \cdots + f(x_n)w$$

$$A = \{\ f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\ \}\ w$$

the width of our rectangles become $\frac{7-1}{n} = \frac{6}{n}$

$$A = \frac{6}{n}\{\ f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\ \}$$

$$A = \frac{6}{n}\sum_{i=1}^{n} f(x)$$

$$A = \frac{6}{n}\sum_{i=1}^{n}(x_i + 1)$$

note here that x constantly changes, and we want an infinite number of them, so the value of the function is different in each "iteration".

*** Can you see that $x_i = 'start\ point' + \Delta x \times i$ ... so here $x_i = 1 + \frac{6i}{n}$. ***

We will also take this opportunity to indicate that we want the "limit" of our calculations where $n \to \infty$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\sum_{i=1}^{n}(1 + \frac{6i}{n} \ + 1)$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\sum_{i=1}^{n}(\frac{6i}{n} \ + 2)$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\left(\sum_{i=1}^{n}(\frac{6i}{n}) + \sum_{i=1}^{n}(2)\right)$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\left(\frac{6}{n}\sum_{i=1}^{n}(\ i\ ) + \sum_{i=1}^{n}(2)\right)$$

you may recall from sequences and series that :

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

And if we sum the number 2, n times, we get $2n$:

$$\sum_{i=1}^{n} 2 = 2n$$

*** the use of these "series" short cuts is essential in this process, and these "Series" are called Reimann Series. After you go to Uni, please come back and explain it more thoroughly to me so I can be clearer with next years students! ☺

and we get:

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\left(\frac{6}{n}\,\frac{n(n+1)}{2} \ + \ 2n\right)$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\left(\frac{6(n+1)}{2} \ + \ 2n\right)$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\,(\ 3(n+1) \ + \ 2n\ )$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\,(3n + 3 \ + \ 2n\ )$$

$$A = \lim_{n\to\infty} \quad \frac{6}{n}\,(5n + 3\ )$$

$$A = \lim_{n\to\infty} \quad 30 + \frac{18}{n}$$

now we can set $n = \infty$

$$A = 30 + \frac{18}{\infty}$$

$$A = 30 + 0$$

$$Area = 30$$

OK, so lets check that.

You should be able to draw the function and just verify this area through inspection.

or

You should be able to calculate the area under the function using Yr 9 maths trapezium area formula.

or

verify by calculating the definite integral

$$\int_{1}^{7} x + 1 \quad dx$$

and maybe you can do all three verifications … ?

now … head back to Solids of Revolution.