Calibration of Inductance Standards in the Maxwell-Wien Bridge Circuit*

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This paper discusses the errors from residuals in the Maxwell-Wien bridge and the effect of these on the measurement of inductors in a bridge not having a Wagner ground. Particular attention is given to the use of substitution methods for accurate measurements and especially to the "equal-substitution" (comparison) method, which can yield excellent precision in the calibration of inductance standards.

1. Introduction

Self inductors and mutual inductors can be constructed to have inductance that is computable from their geometry and dimensions. Although the inductance of certain inductors having measurable mechanical dimensions and simple geometrical form can be computed with excellent accuracy, such inductors are not often used as reference standards for inductance measurements. In practice, inductance measurements are more conveniently made with reference to noncomputable reactances in the form of compact and stable reference standards of inductance or capacitance. Accurate values of inductance can be determined by comparing the reactance of the inductor with the reactance of either standard capacitors or standard inductors by use of appropriate bridge circuits. However, the most precise measurement of an unknown quantity (in this case inductance) is made by comparing it with a standard of like kind and magnitude, the small difference being measured by a corresponding small change in one element of the bridge circuit. When comparison methods are used, it is often possible to reduce the detrimental effects of residuals in the measuring circuit to such an extent that the precision of comparison is much better than the accuracy with which the calibration standard is known.

The Maxwell-Wien bridge circuit has long been used for the accurate measurement of inductance. For measurements of best accuracy with any alternating-current bridge circuit, even for comparison measurements, it is necessary to consider the effects of residuals, the ways of reducing these effects, and the handling of corrections to offset the net errors.

2. Maxwell-Wien Bridge

A bridge circuit originally developed by J. C. Maxwell [1] [1] for use with ballistic detectors was adapted by M. Wien [2] for a-c measurements. For many years the circuit has been used widely by standardizing laboratories for measurements of inductance. Figure 1 shows a schematic of a well-shielded Maxwell-Wien bridge with Wagner arms for elimination of errors resulting from current diverted to ground through leakage impedances. The inductance, L, to be measured is placed in series with a resistance, r_L. If the components of the bridge arms shown in figure 1 were pure, the inductance measured when the bridge is balanced would be

$$L = CR_P R_S, \qquad (1)$$

and to obtain the balance it is necessary that

$$\frac{R}{R_S} = \frac{R_P}{r_L}. \qquad (2)$$
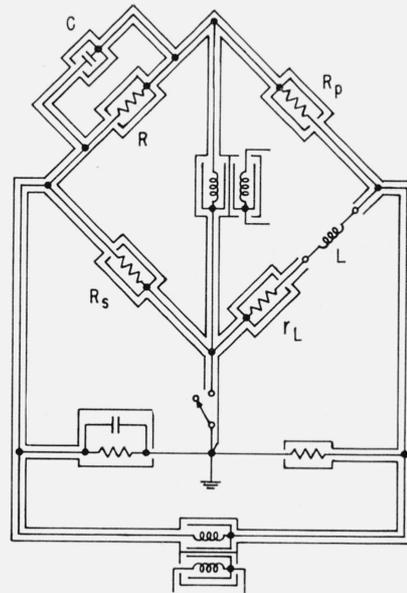
FIGURE 1. *The Maxwell-Wien bridge circuit with complete shielding and a Wagner ground.*

In practical situations it is not possible to obtain completely pure components. The residuals in the components contribute to the systematic errors, and it is necessary to analyze the circuit sufficiently to correct for these errors or to determine the maximum uncertainty in the results if the errors are neglected. Figure 2 shows the components of figure 1 with the addition of appropriate residual components, but without the Wagner arms. For example, the series resistor, r_L of figure 1, in the inductance arm is represented in figure 2 by a pure resistance, r_L, in series with an inductance, l_L. The inductance, l_L, is a lumped constant that may be either positive or negative depending upon the relative magnitudes of inductive and capacitive effects. Although lumped constant residuals are not independent of frequency, the lumped constant concept provides a useful equivalent circuit at low frequencies. The other resistive components are similarly represented by a pure resistance and an effective series inductance. The variable capacitance, C, is presumed to contain an effective series resistance, r. The resistance R_L in figure 2 is the equivalent series resistance of the inductor.

3. Equation of Balance

The impedances of the four bridge arms are

$$Z_{RC} = \frac{[(R+j\omega l)(1+\omega^2 r^2 C^2)}{1+\omega^2 r^2 C^2 + \omega C(\omega rC + j)(R+j\omega l)}, \quad (3)$$

$$Z_P = R_P + j\omega l_P, \quad (4)$$

$$Z_L = R_L + r_L + j\omega(L+l_L), \quad (5)$$

and

$$Z_S = R_S + j\omega l_S. \quad (6)$$

The derivation of eq (3) is given in the appendix. When the bridge is balanced

$$Z_{RC} Z_L = Z_P Z_S. \quad (7)$$

After the multiplication indicated by eq (7) is performed, the equation of imaginary components can be written

$$(1+\omega^2 r^2 C^2)[R(L+l_L)+(R_L+r_L)l - R_S l_P - R_P l_S]$$

$$= C[RR_P R_S + \omega(RR_P l_S + RR_S l_P + R_P R_S l)\omega rC$$

$$-\omega^2(Rl_P l_S + R_P ll_S + R_S ll_P) - \omega^3 ll_P l_S \omega rC]. \quad (8)$$

Considerable simplification of eq (8) is possible if the residuals are expressed as time constants and dissipation factors, as follows. Let

$$\omega rC = D_C, \quad (9)$$

$$\frac{l}{R} = \tau_R, \quad (10)$$
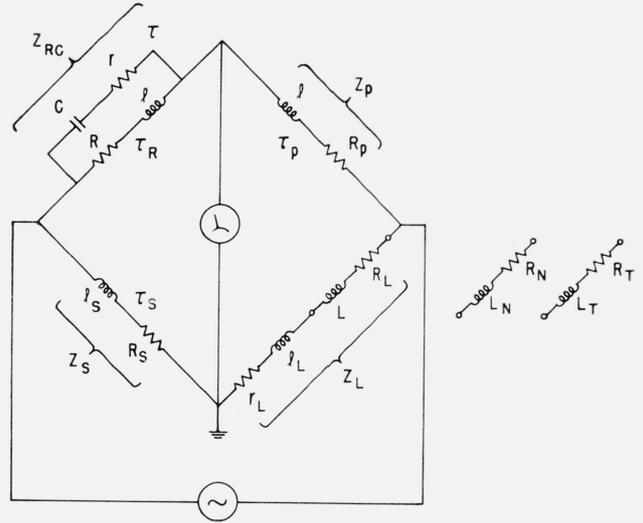
FIGURE 2. *The Maxwell-Wien bridge circuit showing the residuals in the arms as lumped constants.*

$$\frac{l_P}{R_P} = \tau_P, \quad (11)$$

and

$$\frac{l_S}{R_S} = \tau_S. \quad (12)$$

With negligible error it will be found that

$$L = CR_P R_S[1 - \omega^2(\tau_P \tau_S + \tau_R \tau_P + \tau_R \tau_S) - D_C^2$$
$$+ \omega(\tau_P + \tau_S + \tau_R)D_C - \omega^3 \tau_P \tau_S \tau_R D_C]$$
$$- (R_L + r_L)\tau_R + \frac{R_P R_S}{R}(\tau_P + \tau_S) - l_L, \quad (13)$$

and a convenient simplification results if the small terms in brackets are replaced by the symbol K. Then the equation of balance becomes

$$L = CR_P R_S(1+K) - (R_L + r_L)\tau_R + \frac{R_P R_S}{R}(\tau_P + \tau_S) - l_L, \quad (14)$$

where

$$K = -\omega^2(\tau_P \tau_S + \tau_R \tau_P + \tau_R \tau_S) - D_C^2$$
$$+ \omega(\tau_P + \tau_S + \tau_R)D_C - \omega^3 \tau_P \tau_S \tau_R D_C. \quad (15)$$

4. Effect of Stray Capacitance

Before analyzing this equation in detail it is profitable to consider the effect of stray capacitance across the terminals of the inductor. Such capacitance can be introduced by the leads connecting the inductor to the bridge (see fig. 3). For this analysis, whatever capacitance exists within the case of the inductor is considered not a part of C_S, but rather part of the inductor. It should be understood that this internal shunt capacitance contributes to the
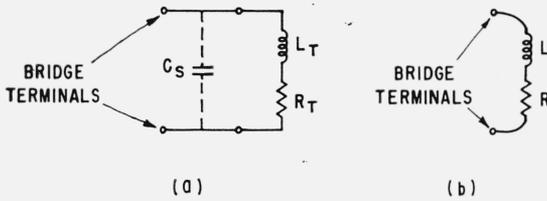
FIGURE 3. *The inductor under test, L_T, is connected to bridge terminals with leads having stray capacitance, C_S, between them.*

The bridge measures the equivalent inductance, L, which differs from L_T because of the stray capacitance.

effective series inductance measured at the terminals of the inductor, thereby causing the effective series inductance to vary with frequency to some extent.

Figure 3a shows the equivalent series inductance under test, L_T, in series with the internal resistance of the inductor, R_T. The stray capacitance, C_S, external to the inductor is in parallel with L_T and R_T. The bridge measurement does not completely separate these elements but indicates the effective inductance, L, as shown in figure 3b. The effective series inductance, L_T, at the terminals of the inductor is related to the measured value, L, at the ends of the leads by the approximation

$$L_T \approx L\left(1 - \omega^2 L_T C_S + \frac{R_T^2 C_S}{L_T} + \omega^2 R_T^2 C_S^2\right). \quad (16)$$

The derivation of eq (16) is given in the appendix. The last three terms in parentheses constitute the small correction terms resulting from stray capacitance. The first and the last of these vary as the square of the frequency. The last term is usually much the smaller of the two in practical situations. The second correction term is not dependent upon frequency and must never be overlooked as a source of error in measurements of the highest accuracy. For example, in the measurement of a 10 h inductor having an internal resistance of 10,000 ohms, at the end of a coaxial cable having a capacitance of 100 pf, the error from neglecting the second correction term is 0.01 h, equivalent to 0.1 percent of the quantity measured. Often the first correction term is significant while the other terms are negligible, and the effective series inductance then can be written

$$L_T \approx L(1 - \omega^2 L_T C_S). \quad (17)$$

However, in this paper both the first and second correction terms will be carried in the analysis to follow.

5. Substitution Methods

The existence of the residuals and the consequent correction terms in eqs (14) and (16) are detrimental to good accuracy for residuals are often very difficult to measure or estimate to better than an order of magnitude. To eliminate the effect on measurements of some of these residuals, substitution

methods are used. For example, if a standard inductor of equal nominal value is substituted for the test inductor and the difference of inductance is measured by changing C and r_L (or R) slightly, then the residuals in the resistance arms of the bridge have negligible effect on the measurement. Also, the effect of the stray capacitance across the inductor under test or across the standard inductor is much reduced. The standard inductor must be very accurately known, however.

If accurate measurements of large inductors are contemplated and a shorting plug of negligible, or relatively small, inductance is substituted for the test inductor, C_S must be reduced to a negligible value by complete shielding, and a Wagner ground should be used. The effect of the residual, l_L, in the resistor, r_L, is eliminated by this method only if l_L is the same for the two conditions of balance.

The latter method, in which a relatively small inductance replaces the inductor under test, can be designated the "zero-substitution" method to distinguish it from the "equal-substitution" method, in which the test inductor is replaced by a standard of equal nominal inductance.

A general formula for inductance measurements by substitution methods with the Maxwell-Wien bridge may be derived from the two balance equations, the first with the inductor, L_T, in the circuit, and the second with a standard, L_N, in its place. If the small correction terms represented by K in eq (14) are constant for both conditions of balance and if C_S is changed only slightly (to C_{SN}) when the standard inductor or shorting plug is connected, then with certain justifiable approximations

$$L_T = L_N + (C - C_N)R_P R_S[1 - \omega^2(L_T + L_N)C_S + K]$$
$$- \omega^2 L_T L_N (C_S - C_{SN})$$
$$+ (R_T^2 - R_N^2)C_S - (l_L - l_{LN}). \quad (18)$$

The derivation of eq (18) is given in the appendix. This equation is useful in showing the effect of uncertainties in the magnitude of circuit components on the determination of inductance.

If good quality components are used and the bridge is well designed, the time constants that contribute to K will be small. However, K cannot be determined with certainty and is somewhat variable (dependent upon actual circuit constants). The uncertainty in K sets an ultimate practical limit to the accuracy of measurement of inductance by the Maxwell-Wien bridge.

6. Zero-Substitution Method

When the zero-substitution method is employed the small inductance, L_N, need not be accurately known, but the capacitor, C, and the resistance product, $R_P R_S$, must be known with better accuracy than that desired for the inductance, L_T. It is possible to construct very small inductances that are computable from geometry and dimensions. A shorting bar between two terminals has a finite inductance

that can often be calculated or estimated well enough to serve as the "zero-reference", L_N, for bridge measurements of inductors of much larger magnitude. If the measured inductance is defined as the increase in inductance when a shorting-link or switch associated with the inductor is manipulated in a specified manner, then the actual inductance of the short circuit, L_N, is immaterial. The term $(l_L - l_{LN})$ can be minimized if the resistance, r_L, is inductance compensated; i.e., if designed so that the effective series inductance remains constant as the resistance is varied. The uncertainties in this term may limit the accuracy of measurement of small inductors. If C_S is not extremely small, it is evident that the uncertainty in C_S can have a significant effect on the accuracy of measurement by this method. In eq (18) the term in brackets containing C_S can become objectionably large when large values of inductance are being measured. For example, if the inductance being measured is approximately 10 h, if $C_S = 100$ pf ± 10 pf, and if $\omega = 10,000$ radians per second, then $\omega^2 L C_S = 0.10 \pm 0.01$. Thus the correction term is 10 percent of the measured inductance, and the measurement uncertainty is 1 percent from this cause alone, disregarding the probable lack of validity of the assumptions (when the correction terms are so large) that were used to derive eq (18). The effect of capacitance C_S thus places a definite limitation on the accuracy which can be obtained by the zero-substitution method with a grounded bridge. In order to obtain better accuracy when $\omega^2 L$ is large it is necessary to use the "equal-substitution method" (described below) or to resort to separate shielding on the inductor leads with an ungrounded bridge using Wagner arms. This latter possibility involves considerations beyond the scope of this paper.

7. Equal-Substitution Method

If an accurately known standard inductance, L_N, is substituted for the test inductor, L_T, the capacitance, C, and the resistance product, $R_P R_S$, need not be so accurately calibrated. Between the two conditions of balance the only circuit components changed are C and r_L, both by small amounts. The residuals D_C and l_L are essentially unchanged in this case, and all the other residuals are identical for both conditions of balance. For the equal-substitution method the general eq (18) can be rewritten with negligible additional error as

$$L_T = L_N + (C - C_N) R_P R_S (1 - 2\omega^2 L_T C_S + K)$$
$$- \omega^2 L_T^2 (C_S - C_{SN}) + (R_T^2 - R_N^2) C_S - (l_L - l_{LN}). \quad (19)$$

In eqs (18) and (19) the stray capacitances, C_S and C_{SN}, are not considered to be exactly equal; however, it is expected that they will be maintained nearly equal. The importance of this, even in equal-substitution measurements, is evident from eq (19). In the comparison of 10 h inductors at an angular frequency of 10,000 radians per second, if the difference $C_S - C_{SN}$ is only 1 pf, then $\omega^2 L_T^2 (C_S - C_{SN})$ is 0.01 h. This error of 0.1 percent shows the importance of maintaining C_S nearly constant during the substitution.

The term $(l_L - l_{LN})$ can be reduced to a minimum by employing an adjustable resistor, r_L, having inductance compensation. Uncertainties in this term may limit accuracy if the resistances of the test and standard inductors differ greatly, and the inductance compensation of the resistor, r_L, is inadequate. The adjustable resistor is in series with the inductor and is connected to the grounded corner of the bridge as shown in figure 2. If, instead, this resistor were connected to the ungrounded end of the inductance arm of the bridge, the capacitance to ground from the several decades of this resistor would shunt the inductor, and as the resistance is varied to accommodate inductors having different internal resistance the change in stray capacitance would cause an error that would be difficult to correct. If one end of the resistor is grounded, and if care is taken to keep the resistance as low as possible, the capacitance to ground within the resistor merely shunts the resistance, contributing to the effective residual inductance, l_L, which is generally insignificant.

The term $(R_T^2 - R_N^2) C_S$ deserves special attention because it is directly dependent upon the effective series resistances of the inductors. As an example of the effect of this term, consider that, in the measurement of a 10 h inductor, $R_T = 10,000$ ohms, $R_N = 8,000$ ohms, and $C_S = 100$ pf. Then the term $(R_T^2 - R_N^2) C_S = 0.0036$ h, nearly 0.04 percent of the quantity measured. This correction, being independent of frequency (if skin effect may be neglected) exists even at low frequencies, at which most of the other correction terms are negligible.

If the time constants, τ_P, τ_S, and τ_R, are 1 μsec or less and D_C is 0.001 or less (these are conservative but reasonable estimates for commercially available components), and the angular frequency is 10,000 radians per second, the magnitude of K is less than 0.0003. For a 1 percent difference in inductors, if this term were neglected the resulting error would be only 3 ppm of the measured inductance.

Thus, the equal-substitution method can be employed without the necessity of using a Wagner ground providing care is taken to keep the stray capacitance, C_S, reasonably small and nearly constant and to apply corrections to the measured values to offset errors resulting from the difference in the internal resistances of the inductors.

8. Effect of Frequency Difference

It has been assumed that the frequency of the a-c supply is stable. If the frequency, ω_N, when the standard inductor is in the circuit is not equal to the frequency, ω, when the test inductor is connected, there can be an error resulting from the stray capacitance, C_S. A difference of frequency between the test balance and the standard balance will necessitate the addition of the term

$$-C_S L_T L_N (\omega^2 - \omega_N^2)$$

to the right side of eq (18), and since $\omega \approx \omega_N$ in any practical case, this can be factored to produce

$$-2\omega C_S L_T L_N (\omega - \omega_N).$$

Since $L_T \approx L_N$ for the equal-substitution method, the additional term to be added to eq (19) is

$$-2\omega C_S L_T^2 (\omega - \omega_N).$$

If $\omega = 10,000$ radians per second, $\omega_N = 10,010$ radians per second (a 0.1% difference), and $C_S = 100$ pf, for the measurement of a 10 h inductor by the equal-substitution method the magnitude of the correction term is 0.002 h, an error of 0.02 percent. It is important to realize that this analysis of the effect of frequency variation does not account for the change of effective series inductance with frequency resulting from eddy currents, skin effect, or distributed capacitance within the inductors.

9. Calibration of Inductors

In recent years standard inductors having good stability have become available in a wide range of nominal values. These have made accurate measurements by the equal-substitution method feasible and convenient. Several complete sets of standards are maintained in the laboratories of NBS both at Washington and at Boulder, and are used regularly for calibrating similar standards submitted for certification. The values assigned to NBS working standards ultimately depend upon a computable inductor or capacitor.

The equal-substitution method can be utilized with a variety of a-c bridge circuits. The essential requirements are that the bridge have good short-time stability, adequate resolution, and means for externally equalizing the storage factor (Q) of the inductors being compared. An advantage of the equal-substitution method is the lessened need for accuracy of adjustment of the bridge components because the bridge is used merely to measure small differences. A number of commercially available bridges embody the Maxwell-Wien bridge circuit to which the analysis given in this paper is primarily devoted. It is almost invariably necessary, however, to improve their resolution by adding a calibrated variable capacitor in parallel with those built into the bridge.

The resistor, r_L, is partially inductance compensated and is connected in series with the cable connecting the inductors to the bridge. It has been determined that the existing variations of inductance are negligible relative to the inductance being measured by this method. This resistor may be regarded as serving the purpose of externally equalizing the Q of the standard and test inductors.

It is seldom necessary to apply a correction for the term $(R_T^2 - R_N^2)C_S$ in eq (19). On rare occasions when this term is significant, a crude measurement of R_T and R_N is adequate.

The resistance product (range) in the bridge is set so that the difference between the test and the standard inductors can be accommodated by adjustment of the externally connected variable capacitor and resistor, the other bridge controls being left unchanged.

The analysis of errors in the Maxwell-Wien bridge circuit described in this paper was carried out as part of the investigation of the feasibility of adopting the equal-substitution method for the rapid and convenient measurement of inductance.

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10. Appendix: Derivation of Equations

Equation (3) is derived from the equation of the impedance of Z_C and Z_R in parallel.

$$\frac{1}{Z_{RC}} = \frac{1}{R+j\omega l} + \frac{\omega C}{\omega r C - j}$$

$$= \frac{1}{R+j\omega l} + \frac{\omega C(\omega rC + j)}{\omega^2 r^2 C^2 + 1}$$

$$= \frac{1+\omega^2 r^2 C^2 + \omega C(\omega rC+j)(R+j\omega l)}{(R+j\omega l)(1+\omega^2 r^2 C^2)}.$$

Therefore

$$Z_{RC} = \frac{(R+j\omega l)(1+\omega^2 r^2 C^2)}{1+\omega^2 r^2 C^2 + \omega C(\omega rC+j)(R+j\omega l)}.$$

Equation (8) is obtained by substituting eqs (3), (4), (5), and (6) into eq (7), giving

$$(1+\omega^2 r^2 C^2)(R+j\omega l)[(R_L+r_L)+j\omega(L+l_L)]$$

$$= [1+\omega^2 r^2 C^2 + \omega C(\omega rC+j)(R+j\omega l)]$$

$$(R_P+j\omega l_P)(R_S+j\omega l_S),$$

from which the equation of imaginary components is

$$(1+\omega^2 r^2 C^2)[R(L+l_L)+(R_L+r_L)l - R_S l_P - R_P l_S]$$

$$= C[RR_P R_S + \omega(RR_P l_S + RR_S l_P + R_P R_S l)\omega rC$$

$$-\omega^2(Rl_P l_S + R_P ll_S + R_S ll_P) - \omega^3 ll_P l_S \omega rC].$$

Equation (16) describes the effect of stray capacitance. Figure 3a represents an inductor, L_T, having an internal resistance, R_T, connected to the bridge terminals by a cable having shunt capacitance, C_S.

The impedance of this circuit, Z, is derived as follows

$$\frac{1}{Z}=\frac{1}{R_T+j\omega L_T}+j\omega C_S,$$

$$Z=\frac{(R_T+j\omega L_T)(1-\omega^2 L_T C_S-j\omega R_T C_S)}{(1-\omega^2 L_T C_S+j\omega R_T C_S)(1-\omega^2 L_T C_S-j\omega R_T C_S)},$$

$$Z=\frac{R_T+j\omega[L_T(1-\omega^2 L_T C_S)-R_T^2 C_S]}{(1-\omega^2 L_T C_S)^2+\omega^2 R_T^2 C_S^2}.$$

In the equivalent circuit shown in figure 3b

$$Z=R+j\omega L,$$

and from the two preceding equations it is evident that

$$L=\frac{L_T(1-\omega^2 L_T C_S)-R_T^2 C_S}{(1-\omega^2 L_T C_S)^2+\omega^2 R_T^2 C_S^2}$$

and, with negligible error,

$$L=\frac{L_T\left(1-\omega^2 L_T C_S-\dfrac{R_T^2 C_S}{L_T}\right)}{(1-2\omega^2 L_T C_S+\omega^2 R_T^2 C_S^2)}.$$

It should be noted that the discarded term, $\omega^4 L_T^2 C_S^2$, in the denominator is always much smaller than the term, $2\omega^2 L_T C_S$, which is usually much smaller than 1 in practical inductance measurements at low audiofrequencies. Since the terms in parentheses, other than 1, are small, this equation can be solved for L_T and simplified by neglecting higher order terms.

$$L_T\approx L\left(1-\omega^2 L_T C_S+\frac{R_T^2 C_S}{L_T}+\omega^2 R_T^2 C_S^2\right).$$

Equation (18) is obtained from the following two equations which are derived from eq (14) modified according to eq (16)

$$L_T=CR_P R_S\left(1-\omega^2 L_T C_S+\frac{R_T^2 C_S}{L_T}\right)(1+K)$$
$$-(R_L+r_L)\tau_R+\frac{R_P R_S}{R}(\tau_P+\tau_S)-l_L$$

and

$$L_N=C_N R_P R_S\left(1-\omega^2 L_N C_{SN}+\frac{R_N^2 C_{SN}}{L_N}\right)(1+K)$$
$$-(R_{LN}+r_{LN})\tau_R+\frac{R_P R_S}{R}(\tau_P+\tau_S)-l_{LN}.$$

In eq (15) it will be noted that at the higher frequencies the first term in K is the largest term, and that at lower frequencies all terms in K are small and often negligible. The first term in K is independent of D_C, which may vary slightly between

the two conditions of balance; hence, the assumption that K is constant is justifiable. In the above equations since

$$CR_P R_S\approx L_T$$

and

$$C_N R_P R_S\approx L_N$$

and

$$(R_L+r_L)\approx(R_{LN}+r_{LN})$$

$CR_P R_S R_T^2 C_S/L_T\approx R_T^2 C_S$ and

$$C_N R_P R_S R_N^2 C_{SN}/L_N\approx R_N^2 C_{SN}.$$

Therefore,

$$L_T-L_N=[(C-C_N)R_P R_S-R_P R_S\omega^2(CL_T C_S$$
$$-C_N L_N C_{SN})+R_T^2 C_S-R_N^2 C_{SN}](1+K)-(l_L-l_{LN}).$$

The factor $(CL_T C_S-C_N L_N C_{SN})$ can be expanded as follows:

$$(CL_T C_S-C_N L_N C_{SN})=(C-C_N)(L_T C_S+L_N C_{SN})$$
$$+C_N L_T C_S-CL_N C_{SN}.$$

If C_S and C_{SN} can be reduced to zero, this factor becomes zero, and the equation is greatly simplified. However, if C_S is not zero, but if C_S approximately equals C_{SN},

$$CL_T C_S-C_N L_N C_{SN}=(C-C_N)(L_T+L_N)C_S$$
$$+C_N L_T C_S-CL_N C_{SN}$$

and

$$L_T-L_N=\{(C-C_N)R_P R_S[1-\omega^2(L_T+L_N)C_S]$$
$$-R_P R_S\omega^2(C_N L_T C_S-CL_N C_{SN})$$
$$+(R_T^2-R_N^2)C_S\}(1+K)-(l_L-l_{LN}),$$

and with negligible error the substitution $C_N R_P R_S=L_N$ and $CR_P R_S=L_T$ can be made in the small correction terms. The second term in braces can be reduced to $-\omega^2 L_T L_N(C_S-C_{SN})$. If the terms under consideration are not relatively small, the approximations are not valid.

Neglecting second order terms, the above equation can be simplified, giving eq (18):

$$L_T=L_N+(C-C_N)R_P R_S[1-\omega^2(L_T+L_N)C_S+K]$$
$$-\omega^2 L_T L_N(C_S-C_{SN})+(R_T^2-R_N^2)C_S-(l_L-l_{LN}).$$

11. References

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