CHAPTER 6

CAPACITANCE, INDUCTANCE, AND MUTUAL INDUCTANCE

CONTENTS

6.1 The Capacitor

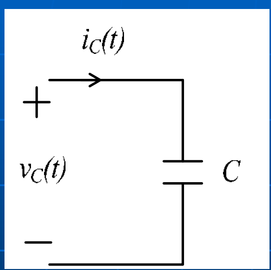
In this chapter, two new and important passive linear components are introduced.

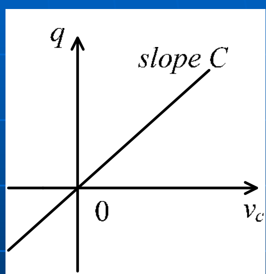
They are ideal models.

Resistors dissipate energy but capacitors and inductors are energy storage components.

6.1 The Capacitor

Circuit symbol and component model.





$q \triangleq C \cdot v_C \;,\quad q(t) = \int^{t} i_C(t)\, dt$

q : charge

C : capacitance , in F(Farad)

6.1 The Capacitor

$$v_C(t) = \frac{1}{C}\int i_C(t)\, dt = \frac{1}{C}\int_{-\infty}^{t_0} i_C(t)\, dt + \frac{1}{C}\int_{t_0}^{t} i_C(t)\, dt$$

$$v_C(t) @ v_C(t_0) + \frac{1}{C}\int_{t_0}^{t} i_C(t)\, dt \quad \text{.......... (A)}$$

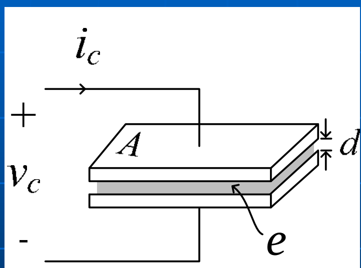
$$i_C(t) = \frac{dq}{dt} = C\frac{dv_C}{dt} \quad \text{..................... (B)}$$

1 farad = 1 Coulomb/Volt

The unit of capacitance is chosen to be farad in honor of the English physicist , Michael Faraday(1791-1867).

6.1 The Capacitor

Example 1 : A parallel-plate capacitor



$$C = e\frac{A}{d}$$

e : the permittivity of the dielectric material between the plates

6.1 The Capacitor

Example 1 : (cont.)

If $v_c > 0$ *and* $i_c > 0$ *, or* $v_c < 0$ *and* $i_c < 0$

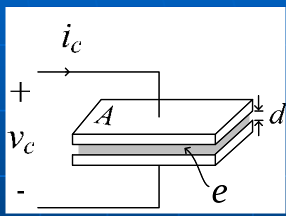
the capacitor is being charged.

If $v_c \cdot i_c < 0$ *, the capacitor is discharging.*

$$p(t) = v_C(t)\ i_C(t) = v_C(t)\ C\frac{dv_c(t)}{dt}$$

Energy in a capacitor

$$w = \int_{-\infty}^{t} p(t)\ dt = \frac{1}{2}C\ v_c^2(t) = \frac{q^2}{2C} \geq 0$$



6.1 The Capacitor

(a) *When* v_C *is constant, then* $i_c = 0.$

ie., equivalent to open circuit

(b) $v_C(t)$ *is a continuous function*

if there is only finite strength energy sources inside the circuit.

$$\mathbf{Q}\ v_C(t+e) = v_C(t) + \frac{1}{C}\int_{t}^{t+e} i_C(t)dt$$

$$\lim_{e\to 0} v_C(t+e) = v_C(t) \quad , \quad v_C(0^+) = v_C(0^-)$$

i.e. $v_C(t)$ *can not change abruptly for finite* $i_C(t)$

6.1 The Capacitor

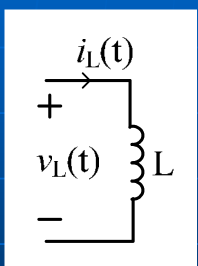
(c) An ideal capacitor does not dissipate energy . It stores energy in the electrical field .

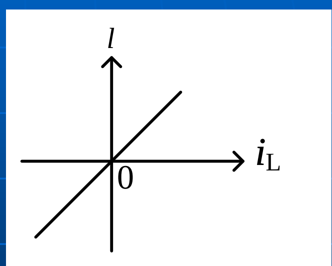
(d) A nonideal capacitor has a leakage resistance

ESR : equivalent series resistance

6.2 The Inductor

Circuit symbol and component model.





$\lambda \triangleq L \cdot i_L \ , \ \lambda(t) = \int^{t} v_L(t)\,dt$

λ : *flux linkage , in web - turns*

L : inductance , in H (Henry)

6.2 The Inductor

$$i_L = \frac{1}{L}\int_{-\infty}^{t} v_L(t)\,dt = \frac{1}{L}\int_{-\infty}^{t_0} v_L(t)\,dt + \frac{1}{L}\int_{t_0}^{t} v_L(t)\,dt$$

$$i_L(t) = i_L(t_0) + \frac{1}{L}\int_{t_0}^{t} v_L(t)\,dt \quad ----------(A)$$

$$v_L(t) = \frac{d\,l}{dt} = L\frac{di_L(t)}{dt} \quad -------------(B)$$

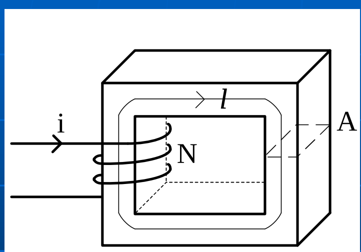
$$p(t) = v_L i_L = L i_L(t)\frac{di_L(t)}{dt}$$

Energy in an inductor

$$w(t) = \int^{t} p(t)\;dt = \frac{1}{2}L i_L^{\,2}(t) \quad \geq 0$$

6.2 The Inductor

Example 2 : An Inductor



H : magnetic field intensity

B : flux density

$f : \int B \; d\,A$ flux

$\oint H\,d\,l = N\,i$

$H l = N i \;,\; H = \frac{N i}{l}$

$B = m H = \frac{m N i}{l}$, m permeability of the core

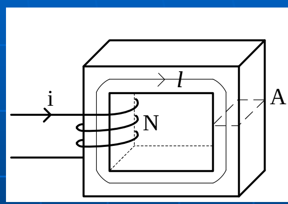
6.2 The Inductor

Example 2 : (cont.)

$$\phi = BA = \frac{\mu ANi}{l}$$

$$\lambda = N\phi = \frac{\mu AN^2 i}{l}, \text{ flux linkage}$$

$$\therefore L = \frac{\lambda}{i} = \frac{\mu AN^2}{l}$$



6.2 The Inductor

The unit of inductance is the henry (H) , named in honor of the American inventor Joseph Henry (1797-1878) .

1H = 1 volt-second / ampere

(a) when i_L is constant , then v_L=0 .
i.e. , equivalent to short circuit

(b) i_L(t) is a continuous function if there is only finite strength source inside the circuit .

$$i_L(t+\epsilon) = i_L(t) + \frac{1}{L}\int_t^{t+\epsilon} v_L(t)\,dt$$

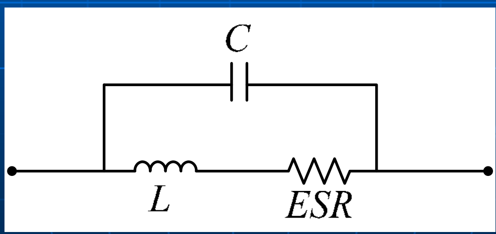
$$\lim_{\epsilon \to 0} i_L(t+\epsilon) = i_L(t) \quad , \quad or \quad i_L(0^+) = i_L(0^-)$$

i.e. $i_L(t)$ can not change abruptly for finite $v_L(t)$.

6.2 The Inductor

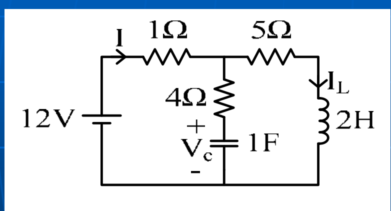
(c) An ideal inductor does not dissipate energy .
It stores energy in the magnetic field .

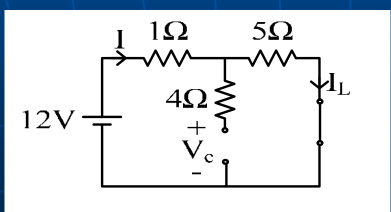
(d) A nonideal inductor contains winding resistance and parasitic capacitance .



6.2 The Inductor

Example 3 : Under dc and steady state conditions, find (a) I , V_C & I_L , (b) W_C and W_L





$$I = I_L = \frac{12}{1+5} = 2A$$

$$V_C = 5I_L = 10V$$

$$W_C = \frac{1}{2}\times 1\times 10^2 = 50J$$

$$W_L = \frac{1}{2}\times 2\times 2^2 = 4J$$

6.2 The Inductor

(a) The capacity of C and L to store energy makes them useful as temporary voltage or current sources , i.e. , they can be used for generating a large amount of voltage or current for a short period of time.

(b) The continuity property of $V_C(t)$ and $i_L(t)$ makes inductors useful for spark or arc suppression and for converting pulsating voltage into relatively smooth dc voltage.

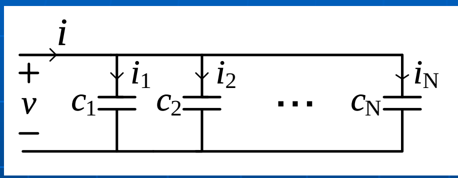
6.2 The Inductor

(c) The frequency sensitive property of L and C makes them useful for frequency discrimination.

(eg. LP , HP , BP filters)

6.3 Series-Parallel Combinations of Capacitance and Inductance

N capacitors in parallel

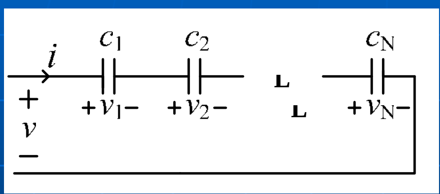


$$v_1(0) = v_2(0) = \cdots = v_N(0)$$
$$v_1 = v_2 = \cdots = v_N = v$$

$$\because i = i_1 + i_2 + \cdots + i_N$$
$$= c_1 \frac{dv}{dt} + c_2 \frac{dv}{dt} + \cdots + c_N \frac{dv}{dt}$$
$$= \left(\sum_{k=1}^{N} c_k \right) \frac{dv}{dt} \quad = C_{eq} \frac{dv}{dt}$$
$$\therefore C_{eq} = c_1 + c_2 + \cdots + c_N$$
$$v(0) = v_k(0)$$

6.3 Series-Parallel Combinations of Capacitance and Inductance

N capacitors in series

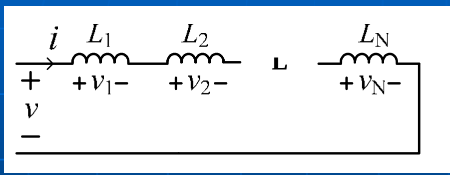


$$i_1 = i_2 = \cdots = i_N$$

$$\because v_k(t) = \frac{1}{c_k} \int_{t_0}^{t} i(t)\, dt + v_k(t_0)$$
$$\therefore v = \left(\sum_{k=1}^{N} \frac{1}{c_k} \right) \int_{t_0}^{t} i(t)\, dt + \sum_{k=1}^{N} v_k(t_o)$$
$$= \frac{1}{C_{eq}} \int_{t_0}^{t} i(t)\, dt + v(t_0)$$
$$\therefore \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_N}$$
$$v(t_0) = v_1(t_0) + v_2(t_0) + \cdots + v_N(t_0)$$

6.3 Series-Parallel Combinations of Capacitance and Inductance

N inductors in series



$$i_1(0) = i_2(0) = \cdots = i_N(0)$$

$$i_1(t) = i_2(t) = \cdots = i_N(t) = i(t)$$

$$\because v = v_1 + v_2 + \cdots + v_N$$

$$= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \cdots + L_N \frac{di}{dt}$$

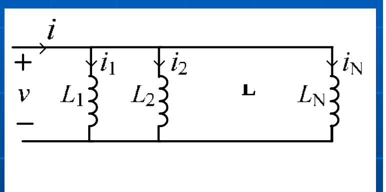
$$= \left(\sum_{k=1}^{N} L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

$$\therefore L_{eq} = L_1 + L_2 + \cdots + L_N$$

$$i(0) = i_1(0) = i_2(0) \cdots = i_N(0)$$

6.3 Series-Parallel Combinations of Capacitance and Inductance

N inductors in parallel



$$\because v_1 = v_2 = \cdots = v_N = v$$

$$\therefore i = i_1 + i_2 + \cdots + i_N$$

$$i = i_1(t_0) + \frac{1}{L_1}\int_{t_0}^{t} v\,dt + i_2(t_0) + \frac{1}{L_2}\int_{t_0}^{t} v\,dt + \cdots + i_N(t_0) + \frac{1}{L_N}\int_{t_0}^{t} v\,dt$$

$$= \left(\sum_{k=1}^{N} \frac{1}{L_k} \right) \int_{t_0}^{t} v\,dt + \sum_{k=1}^{N} i_k(t_0)$$

$$= \frac{1}{L_{eq}} \int_{t_0}^{t} v\,dt + i(t_0)$$

$$\therefore \frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$

$$i(t_0) = i_1(t_0) + i_2(t_0) + \cdots + i_N(t_0)$$

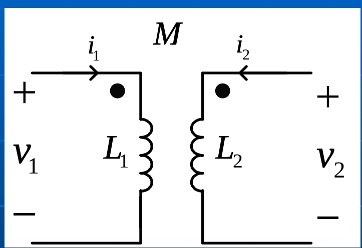
6.3 Series-Parallel Combinations of Capacitance and Inductance

In summary

	Resistor	Capacitor	Inductor
V-I	$V = RI$	$v = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i\,dt$	$v = L\frac{di}{dt}$
I-V	$I = \frac{1}{R}V$	$i = C\frac{dv}{dt}$	$i = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v\,dt$
P or W	$P = \frac{V^2}{R} = I^2 R$	$W = \frac{1}{2}Cv^2$	$W = \frac{1}{2}Li^2$
series	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
parallel	$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$
dc case	*same*	*open circuit*	*short circuit*

6.4 Mutual Inductance

Circuit symbol and model of coupling inductors



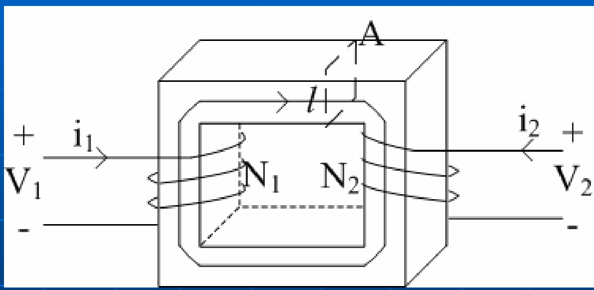
$$v_1 = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = M\frac{di_1}{dt} + L_2\frac{di_2}{dt}$$

L_1 , L_2 : self inductances
M : mutual inductance
unit : in H

6.4 Mutual Inductance

Example 4 : Mutual inductance



Apply I_1 , with $i_2=0$

$$\oint \vec{H} \cdot d\vec{l} = N_1 \cdot I_1$$

6.4 Mutual Inductance

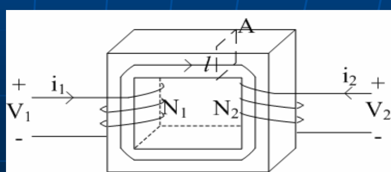
Assume uniform magnetic field intensity H

$$H = \frac{N_1 I_1}{l} \ , \ \therefore \ B = \mu H = \frac{\mu N_1 I_1}{l}$$

$$\phi = \int B \cdot dA = \frac{\mu A N_1 I_1}{l}$$

$$\lambda_1 = N_1 \phi = \frac{\mu A N_1^2 I_1}{l} \ ; \ \lambda_2 = N_2 \phi = \frac{\mu A N_1 N_2 I_1}{l}$$

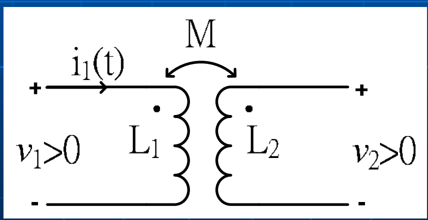
$$L_1 \triangleq \frac{\lambda_1}{I_1} = \frac{\mu A N_1^2}{l} \ , \ M_{21} \triangleq \frac{\lambda_2}{I_1} = \frac{\mu A N_1 N_2}{l}$$

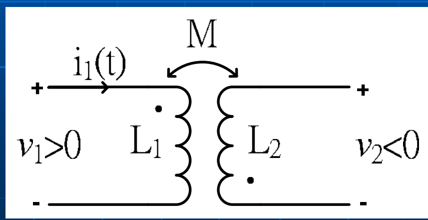


6.4 Mutual Inductance

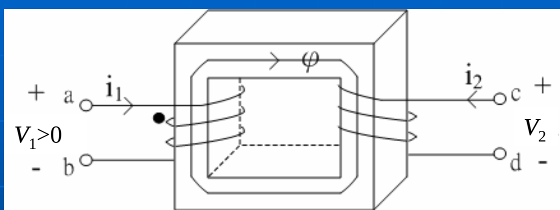
Dot convention for mutually coupled inductors:

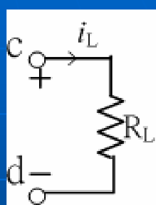
When the reference direction for a current enters the dotted terminal of a coil , the polarity of the induced voltage in the other coil is positive at its corresponding dotted terminal.





6.4 Mutual Inductance

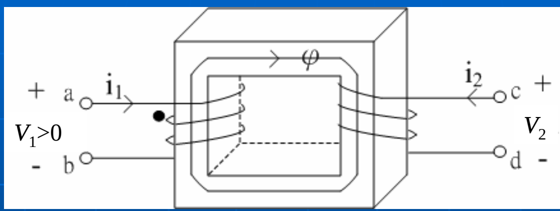


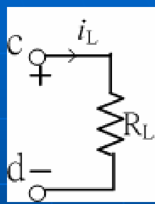


$i_1 \nearrow$, $v_1>0$, $\varphi \nearrow$, ($i_2=0$)

Another dot of coil 2 should be placed in terminal c.

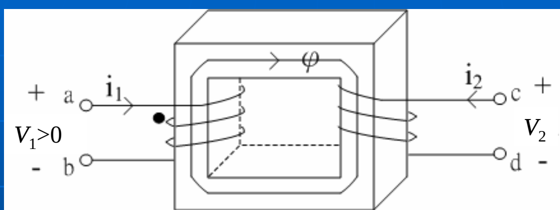
6.4 Mutual Inductance

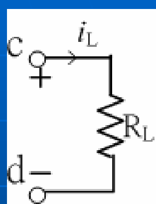




Conceptually , one can connect a resistor at cd terminals.
Then i_2 will be negative.
The generated flux of i_2 will oppose the increasing of φ due to increasing i_1 (Lentz law).
Hence , another dot should be placed at c terminal.

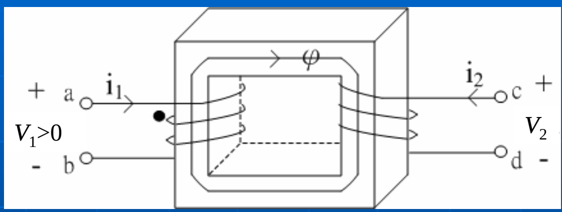
6.4 Mutual Inductance

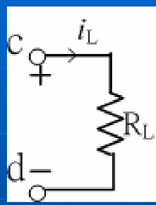




In case , the other dot is placed at d terminal , then i_2 will be positive.
Hence , the generated flux of i_2 will be added to the increasing φ due to i_1.

6.4 Mutual Inductance





Then the induced voltage at coil two will increase and so will i_2.
This will violate the conservation of energy.

6.4 Mutual Inductance

The procedure for determining dot markings

Step1 Assign current direction references for the coils.

Step2 Arbitrarily select one terminal of one coil and mark it with a dot.

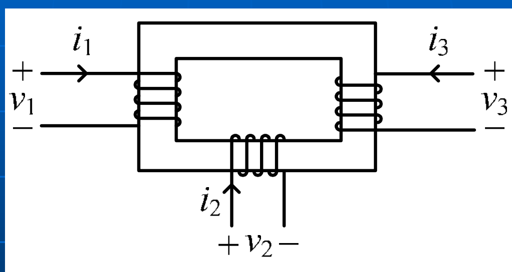
Step3 Use the right-hand rule to determine the direction of the magnetic flux due to the current of the other coil.

6.4 Mutual Inductance

Step4 If this flux direction has the same direction as that of the first dot terminal current , then the second dot is placed at the terminal where the second current enters. Otherwise, the second dot should be placed at the terminal where the second current leaves.

6.4 Mutual Inductance

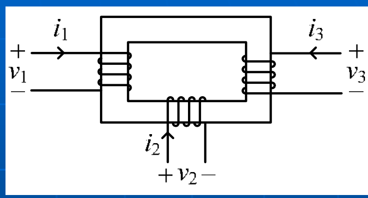
Example 5 : Determining dot markings



Step 1 : Assign i_1 , i_2 , i_3 directions.

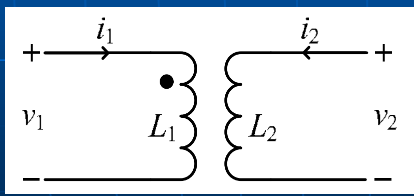
6.4 Mutual Inductance

Example 5 : (cont.)



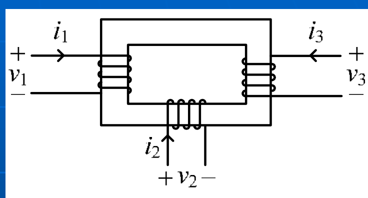
Step 2 :

For coils 1 and 2, choose first dot as follows

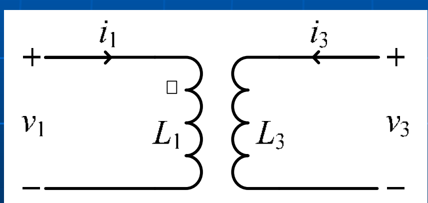


6.4 Mutual Inductance

Example 5 : (cont.)

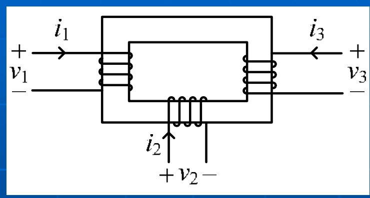


For coils 1 and 3

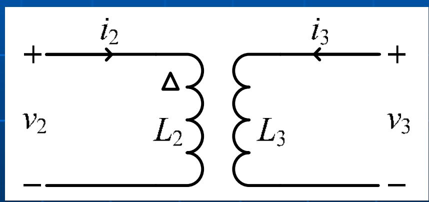


6.4 Mutual Inductance

Example 5 : (cont.)



For coils 2 and 3



6.4 Mutual Inductance

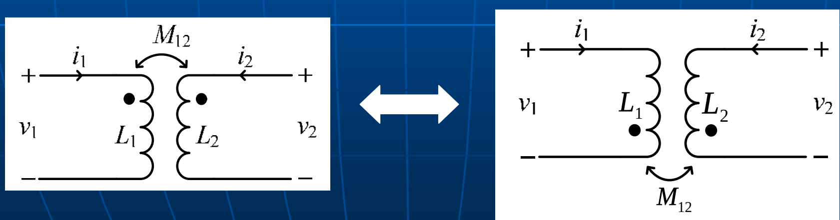
Example 5 : (cont.)

Step 3 :

Check the relative flux directions and determine the dot position at the other coil

For coil 1 and 2

f_1 and f_2 are in the same direction

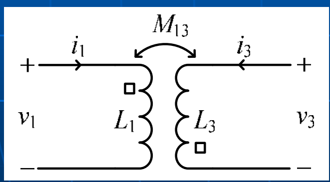


6.4 Mutual Inductance

Example 5 : (cont.)

For coil 1 and 3

f_1 and f_3 are in opposite direction



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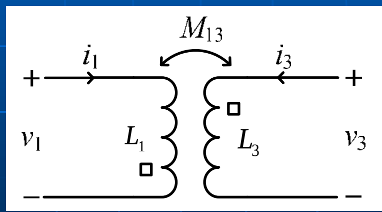
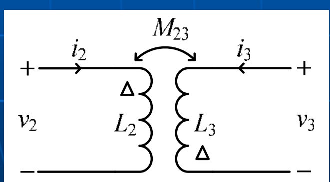


6.4 Mutual Inductance

Example 5 : (cont.)

For coil 2 and 3

f_2 and f_3 are in opposite direction



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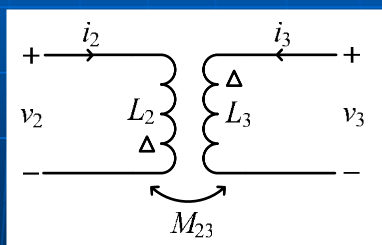


6.4 Mutual Inductance

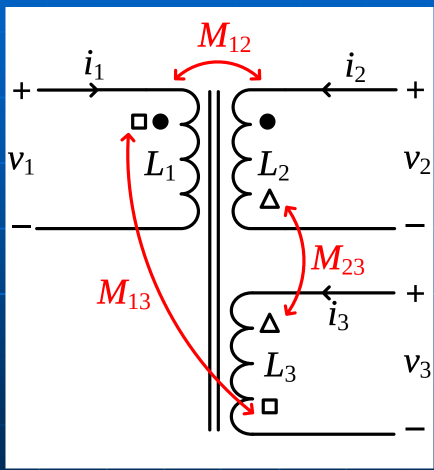
Example 5 : (cont.)

In summary

$$v_1 = L_1 \frac{di_1}{dt} + M_{12}\frac{di_2}{dt} - M_{13}\frac{di_3}{dt}$$

$$v_2 = M_{12}\frac{di_1}{dt} + L_2\frac{di_2}{dt} - M_{23}\frac{di_3}{dt}$$

$$v_3 = -M_{13}\frac{di_1}{dt} - M_{23}\frac{di_2}{dt} + L_3\frac{di_3}{dt}$$



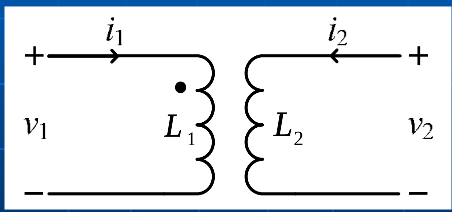
6.4 Mutual Inductance

If the physical arrangement of the coils are not known, the relative polarities of the magnetically coupled coils can be determined experimentally. We need

a dc voltage source V_S
a resistor R : to limit the current
a switch S
a dc voltmeter

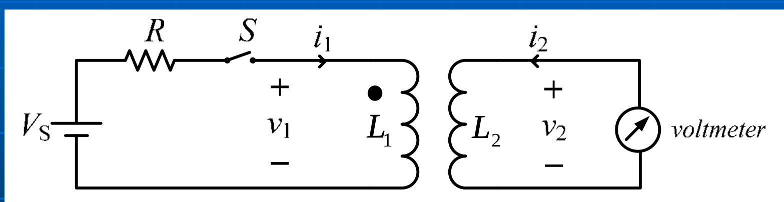
6.4 Mutual Inductance

Step 1 : Assign current directions and arbitrarily assign one dot at coil one.



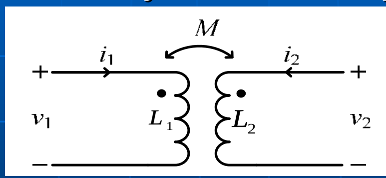
6.4 Mutual Inductance

Step 2 : Connect the setup as follows.

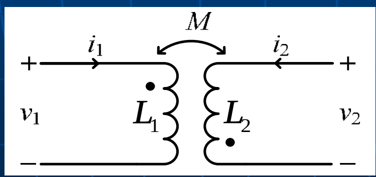


6.4 Mutual Inductance

Step 3 : Determine the voltmeter deflection when the switch is closed.
If the momentary deflection is upscale, then



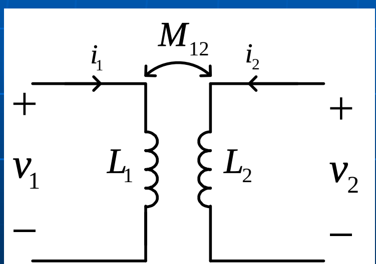
If the momentary deflection is downscale, then



6.4 Mutual Inductance

The reciprocal property of the mutual inductance can be proved by considering the energy relationship .

Step 1 : $i_2=0$, i_1 increased from zero to I .



$$v_1 = L_1 \frac{di_1}{dt}$$

$$v_2 = M_{21} \frac{di_1}{dt}$$

$$\text{input power } p_1 = v_1 i_1 + v_2 i_2$$

$$= L_1 (\frac{di_1}{dt}) i_1$$

$$\text{input energy } w_1 = \int_0^t p_1 dt$$

$$= \int_0^{I_1} L_1 i_1 di_1 = \frac{L_1}{2} I_1^2$$

6.4 Mutual Inductance

Step2 : keep $i_1=I$, i_2 is increased from zero to I_2

$$v_1 = L_1\frac{dI}{dt} + M_{12}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt}$$

$$v_2 = M_{21}\frac{dI}{dt} + L_2\frac{di_2}{dt} = L_2\frac{di_2}{dt}$$

Input power

$$p_2 = v_1 I_1 + v_2 i_2 = M_{12} I_1 \frac{di_2}{dt} + (L_2\frac{di_2}{dt})i_2$$

Input energy

$$w_2 = \int p_2 dt = \int_0^{I_2} M_{12} I_1 di_2 + \int_0^{I_2} L_2 i_2 di_2 = M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2$$

6.4 Mutual Inductance

Total energy when $i_1=I_1$, $i_2=I_2$

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + I_1 I_2 M_{12}$$

Similarly , if we reverse the procedure , by first increasing i_2 from zero to I_2 and then increasing i_1 from zero to I_1, the total energy is

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + I_1 I_2 M_{21}$$

Hence , $M_{12}=M_{21}=M$

6.4 Mutual Inductance

Definition of coefficient of coupling k

$$k \triangleq \frac{M}{\sqrt{L_1 L_2}}$$

$$0 \le k \le 1$$

$$0 < k < \frac{1}{2} \text{ , } \textit{loosely coupling}$$

$$\frac{1}{2} \le k < 1 \text{ , } \textit{closely coupling}$$

$$k = 1 \text{ , } \qquad \textit{unity coupling}$$

6.4 Mutual Inductance

According to dot convention chosen , the total energy stored in the coupled inductors should be

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2 \ge 0$$

In particular , consider the limiting case

$$\frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 - M i_1 i_2 = 0$$

The above equation can be put into the following form

$$\left(\sqrt{\frac{L_1}{2}}i_1 - \sqrt{\frac{L_2}{2}}i_2\right)^2 + i_1 i_2\left(\sqrt{L_1 L_2} - M\right) = 0$$

6.4 Mutual Inductance

Thus , $w(t) \geqq 0$ only if

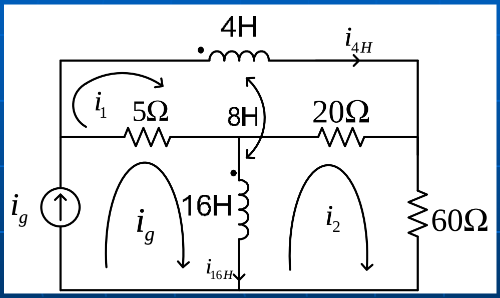
$$\sqrt{L_1 L_2} \geq M$$

when i_1 and i_2 are either both positive or both negative

Hence , $k \leqq 1$

6.4 Mutual Inductance

Example 6 : Finding mesh-current equations for a circuit with magnetically coupled coils.



Three meshes and one current source , only need two unknowns , say i_1 and i_2

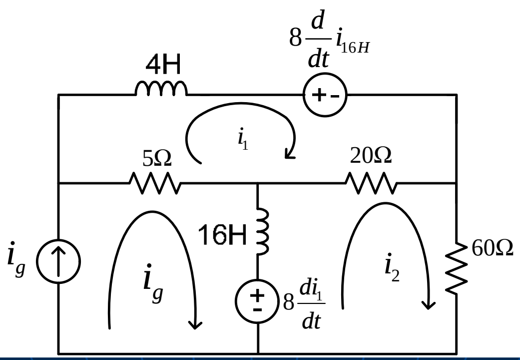
Note : current $i_{4H} = i_1(t)$

current $i_{16H} = i_g - i_2$

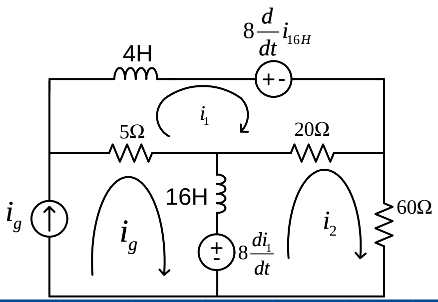
6.4 Mutual Inductance

Due to existence of mutual inductance M=8H , there are two voltage terms for each coil .

One can use dependence source to eliminate the coupling relation as follows.



6.4 Mutual Inductance



Hence , for i_1 mesh

$$4\frac{di_1}{dt}+8\frac{d}{dt}(i_g-i_2)+20(i_1-i_2)+5(i_1-i_g)=0$$

For i_2 mesh

$$20(i_2-i_1)+60i_2+16\frac{d}{dt}(i_2-i_g)-8\frac{d}{dt}i_1=0$$

Summary

- Objective 1 : Know and be able to use the component model of an inductor
- Objective 2 : Know and be able to use the component model of a capacitor.
- Objective 3 : Be able to find the equivalent inductor (capacitor) together with it equivalent initial condition for inductors (capacitors) connected in series and in parallel.

Summary

- Objective 4 : Understand the component model of coupling inductors and the dot convention as well as be able to write the mesh equations for a circuit containing coupling inductors.

Chapter Problems : 6.4
6.19
6.21
6.26
6.40
6.41

Due within one week.