Castigliano's Theorem

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To Use This Method…

- You should have a some background with:
- Deflection of a beam/cylinder due to:
  - Axial loading
  - Bending
  - Torsion
- Calculating normal and polar moments of inertia.
- Deriving equations for linear changes in quantities.
- Using singularity functions (for more advanced applications; no examples here explicitly show it, but it is often used in conjunction with Castigliano's Theorem.

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Definition

- Determining the deflection of beams typically requires repeated integration of singularity functions.
- Castigliano's Theorem lets us use strain energies at the locations of forces to determine the deflections.
- The Theorem also allows for the determining of deflections for objects with changing cross sectional areas.

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Definition

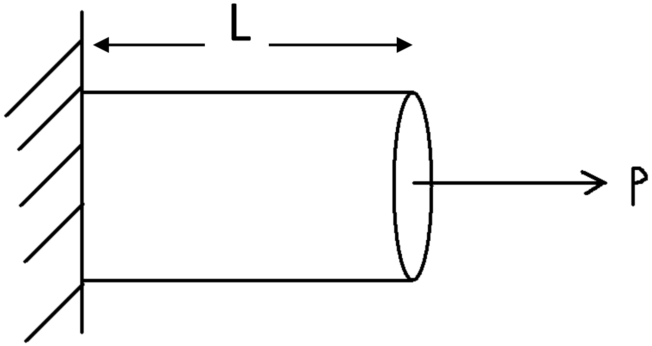
- Castigliano's Theorem is given as:

$$\delta = \frac{\partial U}{\partial P}$$

- Where δ is the deflection, U is the strain energy and P is the force (or torque) at a certain point.

Variations

- Different loading conditions require different strain energies. For axial loading:



$$U = \int_0^L \frac{P^2 dx}{2EA}$$

- Where P is the load, E is the material's Young's Modulus (usually either in GPa or ksi), A is the cross sectional area, and L is the length.

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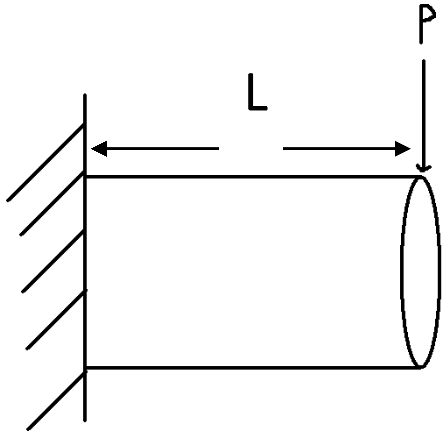
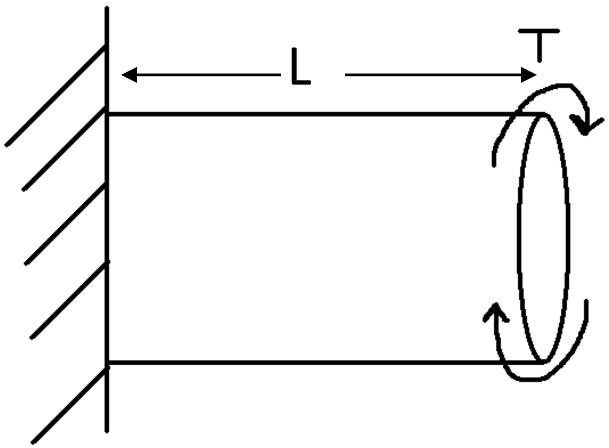
Variations

- For a material in bending:
- 

$$U = \int_0^L \frac{M^2 dx}{2EI}$$

- Where M is the moment applied, and I is the area moment of inertia.

Variations

- For a material in torsion:
- 

$$U = \int_0^L \frac{T^2 dx}{2GJ}$$

- Where T is the torque applied, G is the Modulus of Rigidity, and J is the polar moment of inertia.

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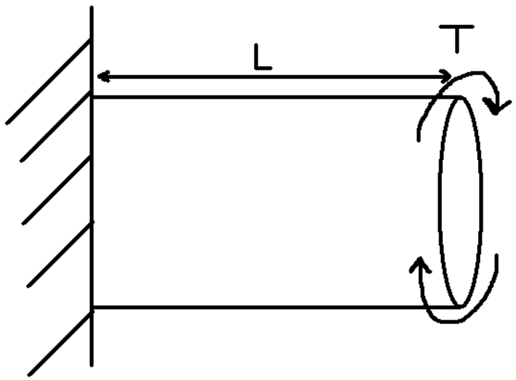
Variations

- Note: except for the Young's Modulus and Modulus of Rigidity (E and G), it is not guaranteed that the other variables are not functions of x.

- Sometimes dimensions of the material change as functions of x, and thus the moments of inertia change; and sometimes the forces applied may vary with x.

Examples

- Imagine a cylinder attached to a fixed wall, with constant diameter d=4 cm and length L=2 m, and a torque of 8 N·m is applied. Assume G=120 GPa.



- To find the displacement of the cylinder, we use Castigliano's Theorem with the strain energy for torsion.

Examples

- Continuing example:

$$\delta = \frac{\partial}{\partial T}\left[\int_0^L \frac{T^2 dx}{2GJ}\right]$$

- With $J = \frac{\pi}{32} d^4$

- The area and moment of inertia are not changing, so we can easily find the displacement.

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Examples

- Continuing example:

- Because differentiating and integrating are linear operations, the partial derivative can be placed inside the integral:

$$\delta = \frac{\partial}{\partial T}\left[\int_0^L \frac{T^2 dx}{2GJ}\right] = \int_0^L \frac{\partial}{\partial T}\left[\frac{T^2}{2GJ}\right] dx = \int_0^L \frac{T dx}{GJ}$$

$$\delta = \frac{TL}{GJ} = \frac{8 \cdot 2}{120 \times 10^9 \cdot \frac{\pi}{32} .04^4} = .5305 \text{ mm}$$

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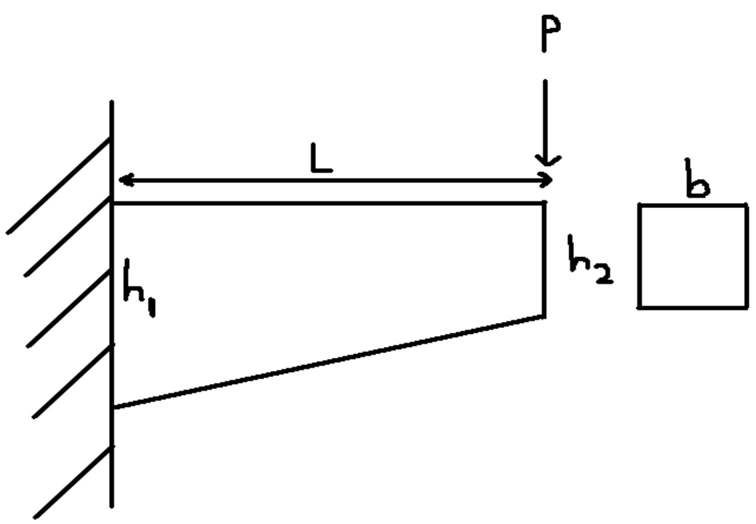
Examples

- Imagine having a beam with a changing cross section shown below, with an initial height of 3 m and a final height of 1 m, with a constant base length of 2 m. The beam has a length of 6 m, with a Young's Modulus of 120 GPa, and a force is applied with magnitude P=10 kN.



Examples

- We will use Castigliano's Theorem applied for bending to solve for the deflection where M is applied.

$$\delta = \frac{\partial}{\partial P}\left[\int_0^L \frac{M^2 dx}{2EI}\right]$$

- To find M, we need to consider the circumstances. At the wall (x=0) the moment felt is the maximum moment or PL, but at the end of the beam, the moment is zero because moments at the locations do not contribute to the overall moments.

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Examples

- Continuing example:
- And so: $M(x) = PL - Px$

- The height is also a function of x, and the initial and final heights can be used to formulate an equation:

$$h(x) = \frac{h_f - h_i}{L} x + h_i = -\frac{1}{3} x + 3$$

Examples

- Continuing example:
- And so the moment of inertia, as a function of x, is:

$$I(x) = \frac{1}{12} bh^3 = \frac{1}{12}(2)(-\frac{1}{3}x + 3)^3$$

- Substituting the functions we have derived into the equation for the displacement:

$$\delta = \int_0^L \frac{\partial}{\partial P}\left[\frac{(PL - Px)^2}{2E\left[\frac{1}{6}(-\frac{1}{3}x + 3)^3\right]}\right]dx$$

Examples

- Continuing example:

$$\delta = \int_0^L \frac{(PL - Px)(L - x)dx}{E\left[\frac{1}{6}(-\frac{1}{3}x + 3)^3\right]}$$

$$\delta = \int_0^6 \frac{(60000 - 10000x)(6 - x)dx}{E\left[\frac{1}{6}(-\frac{1}{3}x + 3)^3\right]}$$

$\delta =$ 1.311 µm

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Summary

- After viewing this tutorial, you should be confident in:
- Identifying a situation (whether in axial loading, bending, or torsion) where Castigliano's Theorem may be applied to solve for the deflection in a beam or cylinder.
- Generate equations for the changes in height, base, or even force across the length of a beam/cylinder.

References

- Mechanics of Materials – Beer/Johnson 5th Edition
  - Section 11.13 “Deflections by Castigliano’s Theorem”

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