

CATEGORICAL QUANTUM MECHANICS

Samson Abramsky and Bob Coecke

1 INTRODUCTION

Our aim is to revisit the mathematical foundations of quantum mechanics from a novel point of view. The standard axiomatic presentation of quantum mechanics in terms of Hilbert spaces, essentially due to von Neumann [1932], has provided the mathematical bedrock of the subject for over 70 years. Why, then, might it be worthwhile to revisit it now?

First and foremost, the advent of *quantum information and computation* (QIC) as a major field of study has breathed new life into basic quantum mechanics, asking new kinds of questions and making new demands on the theory, and at the same time reawakening interest in the foundations of quantum mechanics.

As one key example, consider the changing perceptions of *quantum entanglement* and its consequences. The initial realization that this phenomenon, so disturbing from the perspective of classical physics, was implicit in the quantum-mechanical formalism came with the EPR *Gedanken*-experiment of the 1930's [Einstein *et al.*, 1935], in the guise of a “paradox”. By the 1960's, the paradox had become a *theorem* — Bell's theorem [Bell, 1964], demonstrating that non-locality was an essential feature of quantum mechanics, and opening entanglement to experimental confirmation. By the 1990's, entanglement had become a *feature*, used in quantum teleportation [Bennett *et al.*, 1993], in protocols for quantum key distribution [Ekert, 1991], and, more generally, understood as a computational and informatic *resource* [Bouwmeester *et al.*, 2001].

1.1 *The Need for High-Level Methods*

The current tools available for developing quantum algorithms and protocols, and more broadly the whole field of quantum information and computation, are *deficient* in two main respects.

Firstly, they are too *low-level*. One finds a plethora of ad hoc calculations with ‘bras’ and ‘kets’, normalizing constants, matrices etc. The arguments for the benefits of a high-level, conceptual approach to designing and reasoning about quantum computational systems are just as compelling as for classical computation. In particular, we have in mind the hard-learned lessons from Computer Science of the importance of *compositionality*, *types*, *abstraction*, and the use of tools from algebra and logic in the design and analysis of complex informatic processes.

At a more fundamental level, the standard mathematical framework for quantum mechanics is actually *insufficiently comprehensive* for informatic purposes. In describing a protocol such as teleportation, or any quantum process in which *the outcome of a measurement is used to determine subsequent actions*, the von Neumann formalism leaves feedback of information from the classical or macroscopic level back to the quantum *implicit* and *informal*, and hence not subject to rigorous analysis and proof. As quantum protocols and computations grow more elaborate and complex, this point is likely to prove of increasing importance.

Furthermore, there are many fundamental issues in QIC which remain very much open. The current low-level methods seem unlikely to provide an adequate basis for addressing them. For example:

- What are the precise structural relationships between superposition, entanglement and mixedness as quantum informatic resources? Or, more generally,
- Which features of quantum mechanics account for differences in computational and informatic power as compared to classical computation?
- How do quantum and classical information interact with each other, and with a spatio-temporal causal structure?
- Which quantum control features (e.g. iteration) are possible and what additional computational power can they provide?
- What is the precise logical status and axiomatics of No-Cloning and No-Deleting, and more generally, of the quantum mechanical formalism as a whole?

These questions gain additional force from the fact that a variety of different quantum computational architectures and information-processing scenarios are beginning to emerge. While at first it seemed that the notions of Quantum Turing Machine [Deutsch, 1985] and the quantum circuit model [Deutsch, 1989] could supply canonical analogues of the classical computational models, recently some very different models for quantum computation have emerged, e.g. Raussendorf and Briegel's *one-way quantum computing* model [Raussendorf and Briegel, 2001; Raussendorf *et al.*, 2003] and *measurement based quantum computing* in general [Jozsa, 2005], *adiabatic quantum computing* [Farhi *et al.*, 2000], *topological quantum computing* [Freedman *et al.*, 2004], etc. These new models have features which are both theoretically and experimentally of great interest, and the methods developed to date for the circuit model of quantum computation do not carry over straightforwardly to them. In this situation, we can have no confidence that a comprehensive paradigm has yet been found. It is more than likely that we have overlooked many new ways of letting a quantum system compute.

Thus there is a need to design structures and develop methods and tools which apply to these *non-standard quantum computational models*. We must also address

the question of how the various models compare — can they be interpreted in each other, and which computational and physical properties are preserved by such interpretations?

1.2 *High-Level Methods for Quantum Foundations*

Although our initial motivation came from quantum information and computation, in our view the development of high-level methods is potentially of great significance for the development of the foundations of quantum mechanics, and of fundamental physical theories in general. We shall not enter into an extended discussion of this here, but simply mention some of the main points:

- By identifying the fundamental mathematical structures at work, at a more general and abstract level than that afforded by Hilbert spaces, we can hope to gain new structural insights, and new ideas for how various physical features can be related and combined.
- We get a new perspective on the logical structure of quantum mechanics, radically different to the traditional approaches to quantum logic.
- We get a new perspective on “No-Go” theorems, and new tools for formulating general results applying to whole classes of physical theories.
- Our structural tools yield an *effective calculational formalism* based on a diagrammatic calculus, for which automated software tool-support is currently being developed. This is not only useful for quantum information and computation, it may also yield new ways of probing key foundational issues. Again, this mirrors what has become the common experience in Computer Science. In the age of QIC, *Gedanken*-experiments turn into programs!

We shall take up some of these issues again in the concluding sections.

1.3 *Outline of the Approach*

We shall use *category theory* as the mathematical setting for our approach. This should be no surprise. Category theory is the language of modern structural mathematics, and the fact that it is not more widely used in current foundational studies is a regrettable consequence of the sociology of knowledge and the encumbrances of tradition. Computer Science, once again, leads the way in the applications of category theory; abstract ideas can be very practical!

We shall assume a modest familiarity with basic notions of category theory, including symmetric monoidal categories. Apart from standard references such as [MacLane, 1998], a number of introductions and tutorials specifically on the use of monoidal categories in physics are now available [Abramsky and Tzevelekos, 2008; Baez and Stay, 2008; Coecke and Paquette, 2008]. More advanced textbooks in the area are [Kock, 2003; Street, 2007].

We shall give an axiomatic presentation of quantum mechanics at the abstract level of *strongly compact closed categories with biproducts* — of which the standard von Neumann presentation in terms of Hilbert spaces is but one example. Remarkably enough, all the essential features of modern quantum protocols such as *quantum teleportation* [Bennett *et al.*, 1993], *logic-gate teleportation* [Gottesman and Chuang, 1999], and *entanglement swapping* [Żukowski *et al.*, 1993] — which exploit quantum mechanical effects in an essential way— find natural counterparts at this abstract level. More specifically:

- The basic structure of a symmetric monoidal category allows *compound systems* to be described in a resource-sensitive fashion (cf. the ‘no cloning’ [Dieks, 1982; Wootters and Zurek, 1982] and ‘no deleting’ [Pati and Braunstein, 2000] theorems of quantum mechanics).
- The compact closed structure allows *preparations and measurements of entangled states* to be described, and their key properties to be proved.
- The strong compact closed structure brings in the central notions of adjoint, unitarity and sesquilinear inner product —allowing an involution such as complex conjugation to play a role— and it gives rise to a two-dimensional generalization of Dirac’s *bra-ket* calculus [Dirac, 1947], in which the structure of compound systems is fully articulated, rather than merely implicitly encoded by labelling of basis states.
- Biproducts allow *probabilistic branching* due to measurements, *classical communication* and *superpositions* to be captured. Moreover, from the combination of the—apparently purely qualitative—structures of strong compact closure and biproducts there emerge *scalars* and a *Born rule*.

We are then able to use this abstract setting to give precise formulations of quantum teleportation, logic gate teleportation, and entanglement swapping, and to prove correctness of these protocols — for example, proving correctness of teleportation means showing that the final state of Bob’s qubit equals the initial state of Alice’s qubit.

1.4 Development of the Ideas

A first step in the development of these ideas was taken in [Abramsky and Coecke, 2003], where it was recognized that compact-closed structure could be expressed in terms of bipartite projectors in Hilbert space, thus in principle enabling the structural description of information flows in entangled quantum systems. In [Coecke, 2003] an extensive analysis of a range of quantum protocols was carried out concretely, in terms of Hilbert spaces, with a highly suggestive but informal graphical notation of information-flow paths through networks of projectors. The decisive step in the development of the categorical approach was taken in [Abramsky and Coecke, 2004], with [Abramsky and Coecke, 2005] as a supplement improving the

definition of strongly compact closed category. The present article is essentially an extended and revised version of [Abramsky and Coecke, 2004]. There have been numerous subsequent developments in the programme of categorical quantum mechanics since [Abramsky and Coecke, 2004]. We shall provide an overview of the main developments in Section 7, but the underlying programme as set out in [Abramsky and Coecke, 2004] still stands, and we hope that the present article will serve as a useful record of this approach in its original conception.

1.5 Related Work

To set our approach in context, we compare and contrast it with some related approaches.

Quantum Logic

Firstly, we discuss the relationship with quantum logic as traditionally conceived, *i.e.* the study of lattices abstracted from the lattice of closed linear subspaces of Hilbert space [Birkhoff and von Neumann, 1936].

We shall not emphasize the connections to logic in the present article, but in fact our categorical axiomatics can be seen as the algebraic or semantic counterpart to a *logical type theory* for quantum processes. This type theory has a resource-sensitive character, in the same sense as Linear logic [Girard, 1987] — and this is directly motivated by the no-cloning and no-deleting principles of quantum information. The correspondence of our formalism to a logical system, in which a notion of *proof-net* (a graphical representation of multiple-conclusion proofs) gives a diagrammatics for morphisms in the free strongly compact closed category with biproducts, and simplification of diagrams corresponds to *cut-elimination*, is developed in detail in [Abramsky and Duncan, 2006].

This kind of connection with logic belongs to the proof-theory side of logic, and more specifically to the Curry-Howard correspondence, and the three-way connection between logic, computation and categories which has been a staple of categorical logic, and of logical methods in computer science, for the past three decades [Lambek and Scott, 1986; Abramsky and Tzevelekos, 2008].

The key point is that we are concerned with the direct mathematical representation of *quantum processes*. By contrast, traditional quantum logic is concerned with *quantum propositions*, which express *properties* of quantum systems. There are many other differences. For example, compound systems and the tensor product are central to our approach, while quantum logic has struggled to accommodate these key features of quantum mechanics in a mathematically satisfactory fashion. However, connections between our approach and the traditional setting of orthomodular posets and lattices have been made by John Harding [2007; 2008].

Categories in Physics

There are by now several approaches to using category theory in physics. For comparison, we mention the following:

- [Baez and Dolan, 1995; Crane, 2006]. Higher-dimensional categories, TQFT's, categorification, etc.
- [Isham and Butterfield, 1998; Doëring and Isham, 2007]. The topos-theoretic approach.

Comparison with the topos approach The topos approach aims ambitiously at providing a general framework for the formulation of physical theories. It is still in an early stage of development. Nevertheless, we can make some clear comparisons.

Our approach		Topos approach
monoidal	<i>vs.</i>	cartesian
linear	<i>vs.</i>	intuitionistic
processes	<i>vs.</i>	propositions
geometry of proofs	<i>vs.</i>	geometric logic

Rather as in our comparison with quantum logic, the topos approach is primarily concerned with quantum propositions, whereas we are concerned directly with the representation of quantum processes. Our underlying logical setting is linear, theirs is cartesian, supporting the intuitionistic logic of toposes. It is an interesting topic for future work to relate, and perhaps even usefully combine, these approaches.

Comparison with the n -categories approach The n -categories approach is mainly motivated by the quest for quantum gravity. In our approach, we emphasize the following key features which are essentially absent from the n -categories work:

- operational aspects
- the interplay of quantum and classical
- compositionality
- open *vs.* closed systems.

These are important for applications to quantum informatics, but also of foundational significance.

There are nevertheless some intriguing similarities and possible connections, notably in the rôle played by *Frobenius algebras*, which we will mention briefly in the context of our approach in Section 7.

1.6 Outline of the Article

In Section 2, we shall give a rapid review of quantum mechanics and some quantum protocols such as teleportation. In Sections 3, 4 and 5, we shall present the main ingredients of the formalism: compact and strongly compact categories, and biproducts. In Section 6, we shall show how quantum mechanics can be axiomatized in this setting, and how the formalism can be applied to the complete specification and verification of a number of important quantum protocols. In Section 7 we shall review some of the main developments and advances made within the categorical quantum mechanics programme since [Abramsky and Coecke, 2004], thus giving a picture of the current state of the art.

2 REVIEW OF QUANTUM MECHANICS AND TELEPORTATION

In this paper, we shall only consider *finitary* quantum mechanics, in which all Hilbert spaces are finite-dimensional. This is standard in most current discussions of quantum computation and information [Nielsen and Chuang, 2000], and corresponds physically to considering only observables with finite spectra, such as *spin*. (We refer briefly to the extension of our approach to the infinite-dimensional case in the Conclusion.)

Finitary quantum theory has the following basic ingredients (for more details, consult standard texts such as [Isham, 1995]).

1. The *state space* of the system is represented as a finite-dimensional Hilbert space \mathcal{H} , *i.e.* a finite-dimensional complex vector space with a ‘sesquilinear’ inner-product written $\langle \phi | \psi \rangle$, which is conjugate-linear in the first argument and linear in the second. A *state* of a quantum system corresponds to a one-dimensional subspace \mathcal{A} of \mathcal{H} , and is standardly represented by a vector $\psi \in \mathcal{A}$ of unit norm.
2. For informatic purposes, the basic type is that of *qubits*, namely 2-dimensional Hilbert space, equipped with a *computational basis* $\{|0\rangle, |1\rangle\}$.
3. *Compound systems* are described by tensor products of the component systems. It is here that the key phenomenon of *entanglement* arises, since the general form of a vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is

$$\sum_{i=1}^n \alpha_i \cdot \phi_i \otimes \psi_i$$

Such a vector may encode *correlations* between the first and second components of the system, and cannot simply be resolved into a pair of vectors in the component spaces.

The *adjoint* to a linear map $f : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is the linear map $f^\dagger : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ such that, for all $\phi \in \mathcal{H}_2$ and $\psi \in \mathcal{H}_1$,

$$\langle \phi | f(\psi) \rangle_{\mathcal{H}_2} = \langle f^\dagger(\phi) | \psi \rangle_{\mathcal{H}_1}.$$

Unitary transformations are linear isomorphisms $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$U^{-1} = U^\dagger : \mathcal{H}_2 \rightarrow \mathcal{H}_1.$$

Note that all such transformations *preserve the inner product* since, for all $\phi, \psi \in \mathcal{H}_1$,

$$\langle U(\phi) | U(\psi) \rangle_{\mathcal{H}_2} = \langle (U^\dagger U)(\phi) | \psi \rangle_{\mathcal{H}_1} = \langle \phi | \psi \rangle_{\mathcal{H}_1}.$$

Self-adjoint operators are linear transformations $M : \mathcal{H} \rightarrow \mathcal{H}$ such that $M = M^\dagger$.

4. The *basic data transformations* are represented by unitary transformations. Note that all such data transformations are necessarily *reversible*.
5. The *measurements* which can be performed on the system are represented by self-adjoint operators.

The act of measurement itself consists of two parts:

- 5a. The observer is informed about the measurement outcome, which is a value x_i in the spectrum $\sigma(M)$ of the corresponding self-adjoint operator M . For convenience we assume $\sigma(M)$ to be *non-degenerate* (linearly independent eigenvectors have distinct eigenvalues).
- 5b. The state of the system undergoes a change, represented by the action of the projector P_i arising from the *spectral decomposition*

$$M = x_1 \cdot P_1 + \dots + x_n \cdot P_n$$

In this spectral decomposition the projectors $P_i : \mathcal{H} \rightarrow \mathcal{H}$ are idempotent, self-adjoint, and mutually orthogonal

$$P_i \circ P_i = P_i \quad P_i = P_i^\dagger \quad P_i \circ P_j = 0, \quad i \neq j.$$

This spectral decomposition always exists and is unique by the *spectral theorem* for self-adjoint operators. By our assumption that $\sigma(M)$ was non-degenerate each projector P_i has a one-dimensional subspace of \mathcal{H} as its fixpoint set (which equals its image).

The probability of $x_i \in \sigma(M)$ being the actual outcome is given by the *Born rule* which does not depend on the value of x_i but on P_i and the system state ψ , explicitly

$$\text{Prob}(P_i, \psi) = \langle \psi | P_i(\psi) \rangle.$$

The status of the Born rule within our abstract setting will emerge in Section 8. The derivable notions of *mixed states* and *non-projective measurements* will not play a significant rôle in this paper.

The values x_1, \dots, x_n are in effect merely labels distinguishing the projectors P_1, \dots, P_n in the above sum. Hence we can abstract over them and think of a

measurement as a list of n mutually orthogonal projectors (P_1, \dots, P_n) where n is the dimension of the Hilbert space.

Although real-life experiments in many cases destroy the system (e.g. any measurement of a photon’s location destroys it) measurements always have the same shape in the quantum formalism. When distinguishing between ‘measurements which preserve the system’ and ‘measurements which destroy the system’ it would make sense to decompose a measurement explicitly in two components:

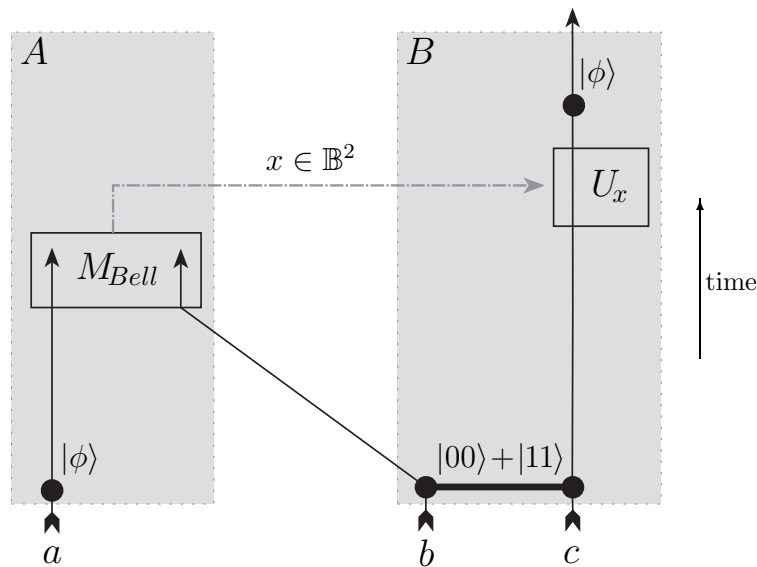
- *Observation* consists of receiving the information on the outcome of the measurement, to be thought of as specification of the index i of the outcome-projector P_i in the above list. Measurements which destroy the system can be seen as ‘observation only’.
- *Preparation* consists of producing the state $P_i(\psi)$.

In our abstract setting these arise naturally as the two ‘building blocks’ which are used to construct projectors and measurements.

We now discuss some important quantum protocols which we chose because of the key rôle entanglement plays in them — they involve both initially entangled states, and measurements against a basis of entangled states.

2.1 Quantum teleportation

The quantum teleportation protocol [Bennett *et al.*, 1993] (see also [Coecke, 2003]§2.3&§3.3) involves three qubits a , b and c and two spatial regions A (for “Alice”) and B (for “Bob”).



Qubit a is in a state $|\phi\rangle$ and located in A . Qubits b and c form an ‘EPR-pair’, that is, their joint state is $|00\rangle + |11\rangle$. We assume that these qubits are initially in B e.g. Bob created them. After *spatial relocation* so that a and b are located in A , while c is positioned in B , or in other words, “Bob sends qubit b to Alice”, we can start the actual teleportation of qubit a . Alice performs a *Bell-base measurement* on a and b at A , that is, a measurement such that each P_i projects on one of the one-dimensional subspaces spanned by a vector in the *Bell basis*:

$$b_1 := \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad b_2 := \frac{|01\rangle + |10\rangle}{\sqrt{2}} \quad b_3 := \frac{|00\rangle - |11\rangle}{\sqrt{2}} \quad b_4 := \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

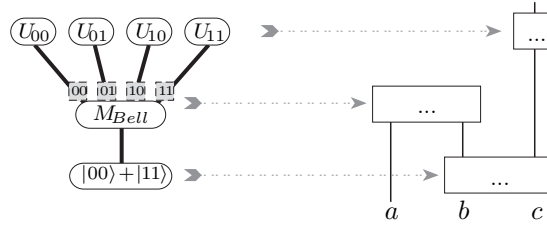
This measurement can be of the type ‘observation only’. Alice observes the outcome of the measurement and “sends these two classical bits ($x \in \mathbb{B}^2$) to Bob”. Depending on which classical bits he receives Bob then performs one of the unitary transformations

$$\beta_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \beta_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \beta_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \beta_4 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

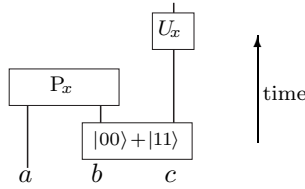
on c — $\beta_1, \beta_2, \beta_3$ are all self-inverse while $\beta_4^{-1} = -\beta_4$. The final state of c proves to be $|\phi\rangle$ as well. (Because of the measurement, a no longer has this state — the information in the source has been ‘destroyed’ in transferring it to the target). Note that the state of a constitutes continuous data — an arbitrary pair of complex numbers (α, β) satisfying $|\alpha|^2 + |\beta|^2 = 1$ — while the actual physical data transmission only involved two classical bits. We will be able to derive this fact in our abstract setting. Teleportation is simply the most basic of a family of quantum protocols, and already illustrates the basic ideas, in particular the use of *preparations of entangled states* as carriers for information flow, performing *measurements* to propagate information, using *classical information* to control branching behaviour to ensure the required behaviour despite quantum indeterminacy, and performing local data transformations using *unitary operations*. (Local here means that we apply these operations only at A or at B , which are assumed to be spatially separated, and not simultaneously at both).

Since in quantum teleportation a continuous variable has been transmitted while the actual *classical communication* involved only two bits, besides this *classical information flow* there has to exist some kind of *quantum information flow*. The nature of this quantum flow has been analyzed by one of the authors in [Coecke, 2003; Coecke, 2004], building on the joint work in [Abramsky and Coecke, 2003]. We recover those results in our abstract setting (see Subsection 3.5), which also reveals additional ‘fine structure’. To identify it we have to separate it from the classical information flow. Therefore we decompose the protocol into:

1. a *tree* with the operations as nodes, and with *branching* caused by the indeterminism of measurements;
2. a *network* of the operations in terms of the order they are applied and the subsystem to which they apply.



The nodes in the tree are connected to the boxes in the network by their temporal coincidence. Classical communication is encoded in the tree as the dependency of operations on the branch they are in. For each path from the root of the tree to a leaf, by ‘filling in the operations on the included nodes in the corresponding boxes of the network’, we obtain an *entanglement network*, that is, a network



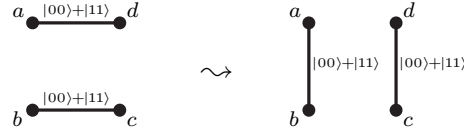
for each of the four values x takes. A component P_x of an observation will be referred to as an *observational branch*. It will be these networks, from which we have removed the classical information flow, that we will study in Subsection 3.5. (There is a clear analogy with the idea of unfolding a Petri net into its set of ‘processes’ [Petri, 1977]). The classical information flow will be reintroduced in Section 5.

2.2 Logic gate teleportation

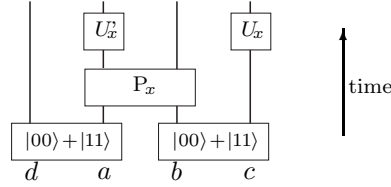
Logic gate teleportation [Gottesman and Chuang, 1999] (see also [Coecke, 2003]§3.3) generalizes the above protocol in that b and c are initially not necessarily an EPR-pair but may be in some other (not arbitrary) entangled state $|\Psi\rangle$. Due to this modification the final state of c is not $|\phi\rangle$ but $|f_\Psi(\phi)\rangle$ where f_Ψ is a linear map which depends on Ψ . As shown in [Gottesman and Chuang, 1999], when this construction is applied to the situation where a , b and c are each a pair of qubits rather than a single qubit, it provides a universal quantum computational primitive which is moreover fault-tolerant [Shor, 1996] and enables the construction of a quantum computer based on single qubit unitary operations, Bell-base measurements and only one kind of prepared state (so-called GHZ states). The connection between Ψ , f_Ψ and the unitary corrections $U_{\Psi,x}$ will emerge straightforwardly in our abstract setting.

2.3 Entanglement swapping

Entanglement swapping [Żukowski *et al.*, 1993] (see also [Coecke, 2003]§6.2) is another modification of the teleportation protocol where a is not in a state $|\phi\rangle$ but is a qubit in an EPR-pair together with an ancillary qubit d . The result is that after the protocol c forms an EPR-pair with d . If the measurement on a and b is non-destructive, we can also perform a unitary operation on a , resulting in a and b also constituting an EPR-pair. Hence we have ‘swapped’ entanglement:



In this case the entanglement networks have the shape:



Why this protocol works will again emerge straightforwardly from our abstract setting, as will generalizations of this protocol which have a much more sophisticated compositional content (see Subsection 3.5).

3 COMPACT CLOSED CATEGORIES AND THE LOGIC OF ENTANGLEMENT

3.1 Monoidal Categories

Recall that a *symmetric monoidal category* consists of a category \mathbf{C} , a bifunctorial *tensor*

$$- \otimes - : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C},$$

a *unit* object I , and natural isomorphisms

$$\lambda_A : A \simeq I \otimes A \qquad \rho_A : A \simeq A \otimes I$$

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \simeq (A \otimes B) \otimes C$$

$$\sigma_{A,B} : A \otimes B \simeq B \otimes A$$

which satisfy certain coherence conditions [MacLane, 1998].

Examples The following two examples are of particular importance and will recur through this section.

1. The category $\mathbf{FdVec}_{\mathbb{K}}$, of finite-dimensional vector spaces over a field \mathbb{K} and linear maps. The tensor product is the usual construction on vector spaces. The unit of the tensor is \mathbb{K} , considered as a one-dimensional vector space over itself.
2. The category \mathbf{Rel} of sets and relations, with cartesian product as the ‘tensor’, and a one-element set as the unit. Note that cartesian product is *not* the categorical product in \mathbf{Rel} .

The Logic of Tensor Product Tensor can express *independent* or *concurrent* actions (mathematically: bifunctionality):

$$\begin{array}{ccc}
 A_1 \otimes A_2 & \xrightarrow{f_1 \otimes 1} & B_1 \otimes A_2 \\
 \downarrow 1 \otimes f_2 & & \downarrow 1 \otimes f_2 \\
 A_1 \otimes B_2 & \xrightarrow{f_1 \otimes 1} & B_1 \otimes B_2
 \end{array}$$

But tensor is *not* a categorical product, in the sense that *we cannot reconstruct an ‘element’ of the tensor from its components*.

This turns out to comprise the *absence of diagonals and projections*:

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 \text{Cf. } & A \vdash A \wedge A & A_1 \otimes A_2 \xrightarrow{\pi_i} A_i \\
 & & A_1 \wedge A_2 \vdash A_i
 \end{array}$$

Hence monoidal categories provide a setting for *resource-sensitive* logics such as Linear Logic [Girard, 1987]. No-Cloning and No-Deleting are built in! *Any* symmetric monoidal category can be viewed as *a setting for describing processes in a resource sensitive way, closed under sequential and parallel composition*

3.2 The ‘miracle’ of scalars

A key step in the development of the categorical axiomatics for Quantum Mechanics was the recognition that the notion of *scalar* is meaningful in great generality — in fact, in any monoidal (not necessarily symmetric) category.

Let $(\mathbf{C}, \otimes, I, \lambda, \alpha, \sigma)$ be a monoidal category. We define a *scalar* in \mathbf{C} to be a morphism $s : I \rightarrow I$, *i.e.* an endomorphism of the tensor unit.

EXAMPLE 1. In $\mathbf{FdVec}_{\mathbb{K}}$, linear maps $\mathbb{K} \rightarrow \mathbb{K}$ are uniquely determined by the image of 1, and hence correspond biuniquely to elements of \mathbb{K} ; composition corresponds to multiplication of scalars. In \mathbf{Rel} , there are just two scalars, corresponding to the Boolean values 0, 1.

The (multiplicative) monoid of scalars is then just the endomorphism monoid $\mathbf{C}(\mathbf{I}, \mathbf{I})$. The first key point is the elementary but beautiful observation by Kelly and Laplaza [Kelly and Laplaza, 1980] that this monoid is always commutative.

LEMMA 2. $\mathbf{C}(\mathbf{I}, \mathbf{I})$ is a commutative monoid

Proof:

$$\begin{array}{ccccccc}
 \mathbf{I} & \xrightarrow{\rho_{\mathbf{I}}} & \mathbf{I} \otimes \mathbf{I} & \equiv & \mathbf{I} \otimes \mathbf{I} & \xrightarrow{\lambda_{\mathbf{I}}^{-1}} & \mathbf{I} \\
 \uparrow s & & \uparrow s \otimes 1 & & \downarrow 1 \otimes t & & \downarrow t \\
 \mathbf{I} & \xrightarrow{\rho_{\mathbf{I}}} & \mathbf{I} \otimes \mathbf{I} & \xrightarrow{s \otimes t} & \mathbf{I} \otimes \mathbf{I} & \xrightarrow{\lambda_{\mathbf{I}}^{-1}} & \mathbf{I} \\
 \downarrow t & & \downarrow 1 \otimes t & & \uparrow s \otimes 1 & & \uparrow s \\
 \mathbf{I} & \xrightarrow{\lambda_{\mathbf{I}}} & \mathbf{I} \otimes \mathbf{I} & \equiv & \mathbf{I} \otimes \mathbf{I} & \xrightarrow{\rho_{\mathbf{I}}^{-1}} & \mathbf{I}
 \end{array}$$

using the coherence equation $\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}$. ■

The second point is that a good notion of *scalar multiplication* exists at this level of generality. That is, each scalar $s : \mathbf{I} \rightarrow \mathbf{I}$ induces a natural transformation

$$s_A : A \xrightarrow{\simeq} \mathbf{I} \otimes A \xrightarrow{s \otimes 1_A} \mathbf{I} \otimes A \xrightarrow{\simeq} A.$$

with the naturality square

$$\begin{array}{ccc}
 A & \xrightarrow{s_A} & A \\
 \downarrow f & & \downarrow f \\
 B & \xrightarrow{s_B} & B
 \end{array}$$

We write $s \bullet f$ for $f \circ s_A = s_B \circ f$. Note that

$$\begin{aligned}
 1 \bullet f &= f \\
 s \bullet (t \bullet f) &= (s \circ t) \bullet f \\
 (s \bullet g) \circ (t \bullet f) &= (s \circ t) \bullet (g \circ f) \\
 (s \bullet f) \otimes (t \bullet g) &= (s \circ t) \bullet (f \otimes g)
 \end{aligned}$$

which exactly generalizes the multiplicative part of the usual properties of scalar multiplication. Thus scalars act globally on the whole category.

3.3 Compact Closure

A category \mathbf{C} is **-autonomous* [Barr, 1979] if it is symmetric monoidal, and comes equipped with a full and faithful functor

$$(\)^* : \mathbf{C}^{op} \rightarrow \mathbf{C}$$

such that a bijection

$$\mathbf{C}(A \otimes B, C^*) \simeq \mathbf{C}(A, (B \otimes C)^*)$$

exists which is natural in all variables. Hence a *-autonomous category is closed, with

$$A \multimap B := (A \otimes B^*)^* .$$

These *-autonomous categories provide a categorical semantics for the multiplicative fragment of linear logic [Seely, 1989].

A *compact closed category* [Kelly, 1972] is a *-autonomous category with a self-dual tensor, i.e. with natural isomorphisms

$$u_{A,B} : (A \otimes B)^* \simeq A^* \otimes B^* \quad u_I : I^* \simeq I .$$

It follows that

$$A \multimap B \simeq A^* \otimes B .$$

A very different definition arises when one considers a symmetric monoidal category as a one-object bicategory. In this context, compact closure simply means that every object A , qua 1-cell of the bicategory, has a specified adjoint [Kelly and Laplaza, 1980].

DEFINITION 3 Kelly-Laplaza. *A compact closed category is a symmetric monoidal category in which to each object A a dual object A^* , a unit*

$$\eta_A : I \rightarrow A^* \otimes A$$

and a counit

$$\epsilon_A : A \otimes A^* \rightarrow I$$

are assigned, in such a way that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\rho_A} & A \otimes I & \xrightarrow{1_A \otimes \eta_A} & A \otimes (A^* \otimes A) \\ \downarrow 1_A & & & & \downarrow \alpha_{A,A^*,A} \\ A & \xleftarrow{\lambda_A^{-1}} & I \otimes A & \xleftarrow{\epsilon_A \otimes 1_A} & (A \otimes A^*) \otimes A \end{array}$$

and the dual one for A^* both commute.

Examples The symmetric monoidal categories (\mathbf{Rel}, \times) of sets, relations and cartesian product and $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$ of finite-dimensional vector spaces over a field \mathbb{K} , linear maps and tensor product are both compact closed. In (\mathbf{Rel}, \times) , we simply set $X^* = X$. Taking a one-point set $\{*\}$ as the unit for \times , and writing R^\cup for the converse of a relation R :

$$\eta_X = \epsilon_X^\cup = \{(*, (x, x)) \mid x \in X\}.$$

For $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$, we take V^* to be the dual space of linear functionals on V . The unit and counit in $(\mathbf{FdVec}_{\mathbb{K}}, \otimes)$ are

$$\eta_V : \mathbb{K} \rightarrow V^* \otimes V :: 1 \mapsto \sum_{i=1}^{i=n} \bar{e}_i \otimes e_i \quad \text{and} \quad \epsilon_V : V \otimes V^* \rightarrow \mathbb{K} :: e_i \otimes \bar{e}_j \mapsto \bar{e}_j(e_i)$$

where n is the dimension of V , $\{e_i\}_{i=1}^{i=n}$ is a basis of V and \bar{e}_i is the linear functional in V^* determined by $\bar{e}_j(e_i) = \delta_{ij}$.

DEFINITION 4. The name $\lceil f \rceil$ and the coname $\lfloor f \rfloor$ of a morphism $f : A \rightarrow B$ in a compact closed category are

$$\begin{array}{ccc} A^* \otimes A & \xrightarrow{1_{A^*} \otimes f} & A^* \otimes B \\ \eta_A \uparrow & \nearrow \lceil f \rceil & \\ \mathbb{I} & & \end{array} \qquad \begin{array}{ccc} & & \mathbb{I} \\ & \nearrow \lfloor f \rfloor & \uparrow \epsilon_B \\ A \otimes B^* & \xrightarrow{f \otimes 1_{B^*}} & B \otimes B^* \end{array}$$

For $R \in \mathbf{Rel}(X, Y)$ we have

$$\lceil R \rceil = \{(*, (x, y)) \mid xRy, x \in X, y \in Y\} \quad \text{and} \quad \lfloor R \rfloor = \{((x, y), *) \mid xRy, x \in X, y \in Y\}$$

and for $f \in \mathbf{FdVec}_{\mathbb{K}}(V, W)$ with (m_{ij}) the matrix of f in bases $\{e_i^V\}_{i=1}^{i=n}$ and $\{e_j^W\}_{j=1}^{j=m}$ of V and W respectively

$$\lceil f \rceil : \mathbb{K} \rightarrow V^* \otimes W :: 1 \mapsto \sum_{i,j=1}^{i,j=n,m} m_{ij} \cdot \bar{e}_i^V \otimes e_j^W$$

and

$$\lfloor f \rfloor : V \otimes W^* \rightarrow \mathbb{K} :: e_i^V \otimes \bar{e}_j^W \mapsto m_{ij}.$$

Given $f : A \rightarrow B$ in any compact closed category \mathbf{C} we can define $f^* : B^* \rightarrow A^*$

as

$$\begin{array}{ccccc}
 B^* & \xrightarrow{\lambda_{B^*}} & \mathbb{I} \otimes B^* & \xrightarrow{\eta_A \otimes 1_{B^*}} & A^* \otimes A \otimes B^* \\
 \downarrow f^* & & & & \downarrow 1_{A^*} \otimes f \otimes 1_{B^*} \\
 A^* & \xleftarrow{\rho_{A^*}^{-1}} & A^* \otimes \mathbb{I} & \xleftarrow{1_{A^*} \otimes \epsilon_B} & A^* \otimes B \otimes B^*
 \end{array}$$

This operation $(\)^*$ is functorial and makes Definition 3 coincide with the one given at the beginning of this section. It then follows by

$$\mathbf{C}(A \otimes B^*, \mathbb{I}) \simeq \mathbf{C}(A, B) \simeq \mathbf{C}(\mathbb{I}, A^* \otimes B)$$

that every morphism of type $\mathbb{I} \rightarrow A^* \otimes B$ is the name of some morphism of type $A \rightarrow B$ and every morphism of type $A \otimes B^* \rightarrow \mathbb{I}$ is the couname of some morphism of type $A \rightarrow B$. In the case of the unit and the counit we have

$$\eta_A = \lceil 1_A \rceil \quad \text{and} \quad \epsilon_A = \lfloor 1_A \rfloor.$$

For $R \in \mathbf{Rel}(X, Y)$ the dual is the converse, $R^* = R^\cup \in \mathbf{Rel}(Y, X)$, and for $f \in \mathbf{FdVec}_{\mathbb{K}}(V, W)$, the dual is

$$f^* : W^* \rightarrow V^* :: \phi \mapsto \phi \circ f.$$

The following holds by general properties of adjoints and symmetry of the tensor [Kelly and Laplaza, 1980]§6.

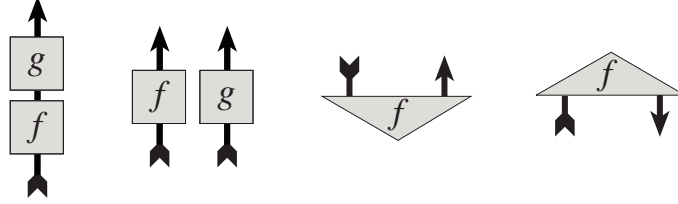
PROPOSITION 5. *In a compact closed category \mathbf{C} there is a natural isomorphism $d_A : A^{**} \simeq A$ and the diagrams*

$$\begin{array}{ccc}
 A^* \otimes A & \xrightarrow{\sigma_{A^*, A}} & A \otimes A^* \\
 \downarrow 1_{A^*} \otimes d_A^{-1} & & \downarrow \epsilon_A \\
 A^* \otimes A^{**} & \xrightarrow{\epsilon_{A^*}} & \mathbb{I}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbb{I} & \xrightarrow{\eta_{A^*}} & A^{**} \otimes A^* \\
 \downarrow \eta_A & & \downarrow d_A \otimes 1_{A^*} \\
 A^* \otimes A & \xrightarrow{\sigma_{A^*, A}} & A \otimes A^*
 \end{array}$$

commute for all objects A of \mathbf{C} .

Graphical representation. Complex algebraic expressions for morphisms in symmetric monoidal categories can rapidly become hard to read. Graphical representations exploit two-dimensionality, with the vertical dimension corresponding to composition and the horizontal to the monoidal tensor, and provide more intuitive presentations of morphisms. We depict objects by wires, morphisms by boxes

with input and output wires, composition by connecting outputs to inputs, and the monoidal tensor by locating boxes side-by-side. We distinguish between an object and its dual in terms of directions of the wires. In particular, $g \circ f$, $f \otimes g$, $\lrcorner f \lrcorner$ and $\lrcorner f \lrcorner$ will respectively be depicted by



Implicit in the use of this graphical notation is that we assume we are working in a *strict* monoidal category, in which the unit and associativity isomorphisms are identities. We can always do this because of the coherence theorem for monoidal categories [MacLane, 1998]. Similarly, strictness is assumed for the duality in compact closed categories:

$$A^{**} = A, \quad (A \otimes B)^* = A^* \otimes B^*, \quad I^* = I.$$

Pointers to references on diagrammatic representations and corresponding calculi are in Section 7.8.

3.4 Key lemmas

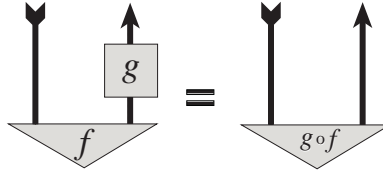
The following Lemmas constitute the core of our interpretation of entanglement in compact closed categories. It was however observed by Radha Jagadeesan [2004] that they can be shown in arbitrary $*$ -autonomous categories using some of the results in [Cockett and Seely, 1997].

LEMMA 6 absorption. For $A \xrightarrow{f} B \xrightarrow{g} C$ we have that

$$(1_{A^*} \otimes g) \circ \lrcorner f \lrcorner = \lrcorner g \circ f \lrcorner.$$

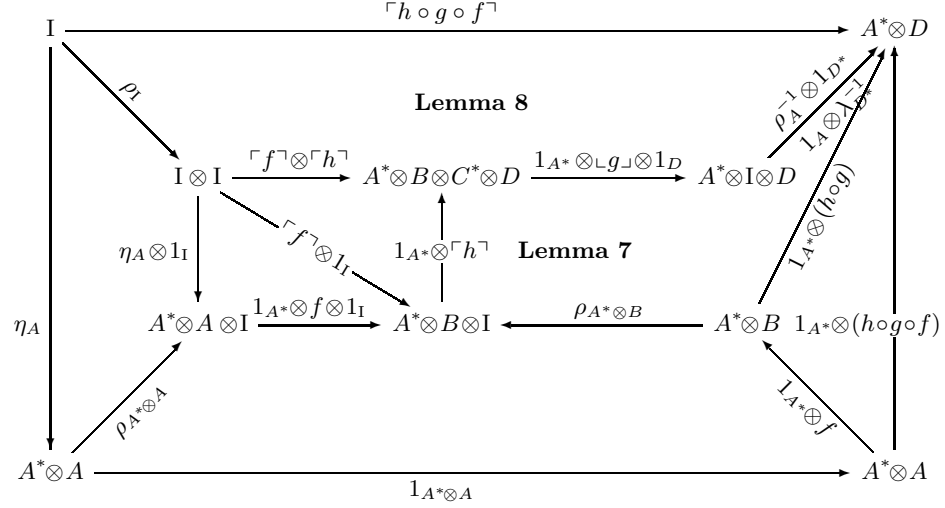
Proof: Straightforward by Definition 4. ■

In a picture,



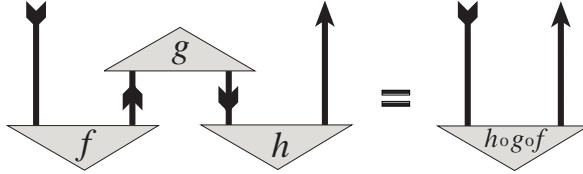
LEMMA 7 Compositionality. For $A \xrightarrow{f} B \xrightarrow{g} C$ we have that

$$\lambda_C^{-1} \circ (\lrcorner f \lrcorner \otimes 1_C) \circ (1_A \otimes \lrcorner g \lrcorner) \circ \rho_A = g \circ f.$$



The top trapezoid is the statement of the Lemma. The diagram uses Lemma 7 and naturality of ρ and λ . ■

In a picture,



Discussion. On the right hand side of Lemma 7 we have $g \circ f$, that is, we first apply f and then g , while on the left hand side we first apply the coname of g , and then the coname of f . In Lemma 8 there is a similar, seemingly ‘acausal’ inversion of the order of application, as g gets inserted between h and f .

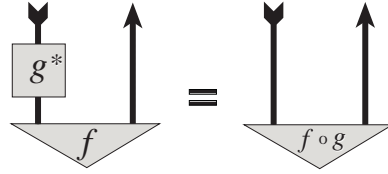
For completeness we add the following ‘backward’ absorption lemma, which again involves a reversal of the composition order.

LEMMA 9 backward absorption. For $C \xrightarrow{g} A \xrightarrow{f} B$ we have that

$$(g^* \otimes 1_{A^*}) \circ \ulcorner f \urcorner = \ulcorner f \circ g \urcorner.$$

Proof: This follows by unfolding the definition of g^* , then using naturality of λ_{A^*} , $\lambda_I = \rho_I$, and finally Lemma 8. ■

In a picture,



The obvious analogues of Lemma 6 and 9 for conames also hold.

3.5 Quantum information flow in entanglement networks

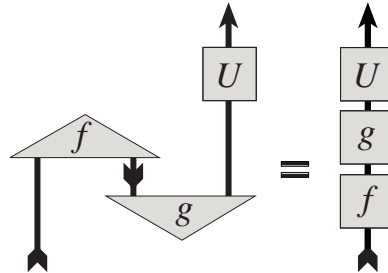
We claim that Lemmas 6, 7 and 8 capture the quantum information flow in the (logic-gate) teleportation and entanglement swapping protocols. We shall provide a full interpretation of finitary quantum mechanics in Section 6 but for now the following rule suffices:

- We interpret *preparation* of an entangled state as a *name* and an *observational branch* as a *coname*.

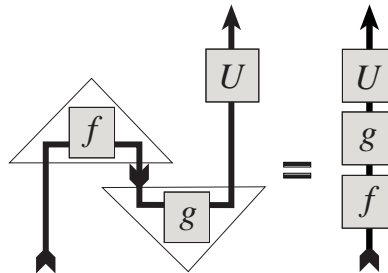
For an entanglement network of teleportation-type shape, applying Lemma 7 yields

$$U \circ (\lambda_C^{-1} \circ (\lrcorner f \lrcorner \otimes 1)) \circ ((1 \otimes \ulcorner g \urcorner) \circ \rho_A) = U \circ g \circ f.$$

In a picture,



We make the information flow more explicit in the following version of the same picture:



Note that the quantum information seems to flow ‘following the line’ while being acted on by the functions whose name or coname labels the boxes (and this fact remains valid for much more complex networks [Coecke, 2003]).

Teleporting the input requires $U \circ g \circ f = 1_A$ — we assume all functions have type $A \rightarrow A$. Logic-gate teleportation of $h : A \rightarrow B$ requires $U \circ g \circ f = h$.

We calculate this explicitly in **Rel**. For initial state $x \in X$ after preparing $\ulcorner S \urcorner \subseteq \{*\} \times (Y \times Z)$ we obtain $\{x\} \times \{(y, z) \mid * \ulcorner S \urcorner(y, z)\}$ as the state of the system. For observational branch $\lrcorner R \lrcorner \subseteq (X \times Y) \times \{*\}$ we have that $z \in Z$ is the output iff $\lrcorner R \lrcorner \times 1_Z$ receives an input $(x, y, z) \in X \times Y \times Z$ such that $(x, y) \lrcorner R \lrcorner *$. Since

$$* \ulcorner S \urcorner(y, z) \Leftrightarrow ySz \quad \text{and} \quad (x, y) \lrcorner R \lrcorner * \Leftrightarrow xRy$$

we indeed obtain $x(R; S)z$. This illustrates that the compositionality is due to a mechanism of imposing constraints between the components of the tuples.

In **FdVec $_{\mathbb{C}}$** the vector space of all linear maps of type $V \rightarrow W$ is $V \multimap W$ and hence by $V^* \otimes W \simeq V \multimap W$ we have a bijective correspondence between linear maps $f : V \rightarrow W$ and vectors $\Psi \in V^* \otimes W$ (see also [Coecke, 2003; Coecke, 2004]):

$$\Psi_f = \frac{1}{\sqrt{2}} \cdot \ulcorner f \urcorner(1) \quad \text{and} \quad \lrcorner f \lrcorner = \langle \sqrt{2} \cdot \Psi_f \mid - \rangle.$$

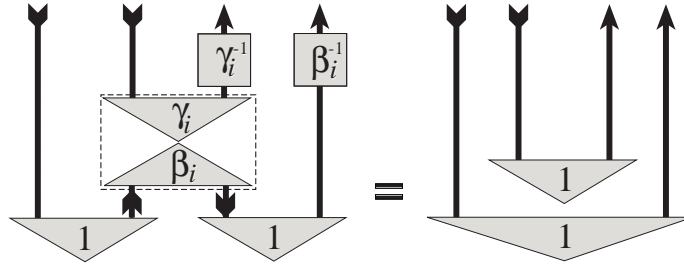
In particular we have for the Bell base:

$$b_i = \frac{1}{\sqrt{2}} \cdot \ulcorner \beta_i \urcorner(1) \quad \text{and} \quad \lrcorner \beta_i \lrcorner = \langle \sqrt{2} \cdot b_i \mid - \rangle.$$

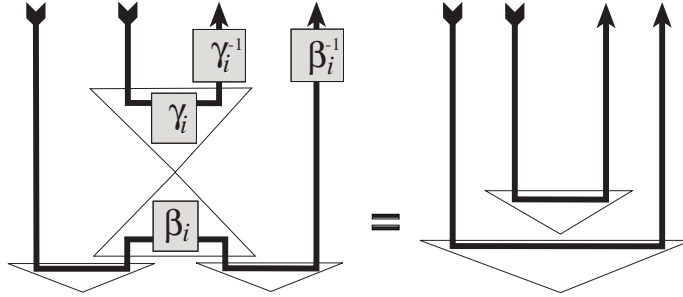
Setting $g := \beta_1 = 1_V$, $f := \beta_i$ and $U := \beta_i^{-1}$ indeed yields $\beta_i^{-1} \circ 1_A \circ \beta_i = 1_A$, which expresses the correctness of the teleportation protocol along each branch.

Setting $g := h$ and $f := \beta_i$ for logic-gate teleportation requires U_i to satisfy $U_i \circ h \circ \beta_i = h$ that is $h \circ \beta_i = U_i^\dagger \circ h$ (since U has to be unitary). Hence we have derived the laws of logic-gate teleportation — one should compare this calculation to the size of the calculation in Hilbert space.

Deriving the swapping protocol using Lemma 6 and Lemma 8 proceeds analogously to the derivation of the teleportation protocol.

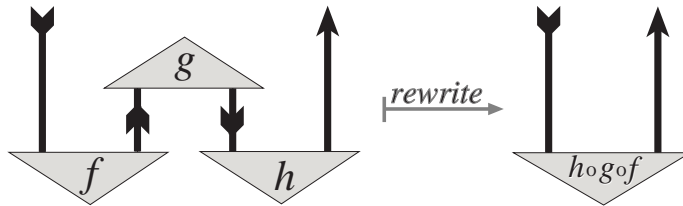


– the two triangles within the dashed line stand for $\ulcorner \gamma_i \urcorner \circ \lrcorner \beta_i \lrcorner$. We obtain two distinct flows due to the fact that a non-destructive measurement is involved.



How γ_i has to relate to β_i such that they make up a true projector will be discussed in Section 6.

For a general entanglement network of the swapping-type (without unitary correction and observational branching) by Lemma 8 we obtain the following ‘reduction’:



This picture, and the underlying algebraic property expressed by Lemma 3.5, is in fact directly related to *Cut-Elimination* in the logic corresponding to compact-closed categories. If one turns the above picture upside-down, and interprets names as Axiom-links and conames as Cut-links, then one has a normalization rule for proof-nets. This perspective is developed in [Abramsky and Duncan, 2006].

4 STRONGLY COMPACT CLOSED CATEGORIES AND 2-DIMENSIONAL DIRAC NOTATION

The key example In Section 3 we analysed the compact closed structure of $\mathbf{FdVec}_{\mathbb{K}}$, where we took the dual of a vector space V to be the vector space of its linear functionals V^* . In the case that V is equipped with an inner product we can refine this analysis. We discuss this for the key example of *finite-dimensional Hilbert spaces*, *i.e.* finite-dimensional complex vector spaces with a *sesquilinear* inner product: the inner product is linear in the second argument, and

$$\langle \phi | \psi \rangle = \overline{\langle \psi | \phi \rangle}$$

which implies that it is *conjugate-linear* rather than linear in its first argument.

We organize these spaces into a category \mathbf{FdHilb} , where the morphisms are linear maps. Note that we do *not* require morphisms to preserve the inner product.

This category provides the basic setting for finite-dimensional quantum mechanics and for quantum information and computation.¹

In the setting of Hilbert spaces, we can replace the dual space by a more elementary construction. In a Hilbert space, each linear functional $\bar{\psi} : \mathcal{H} \rightarrow \mathbb{C}$ is witnessed by some $\psi \in \mathcal{H}$ such that $\bar{\psi} = \langle \psi | \cdot \rangle$. This however does *not* induce an isomorphism between \mathcal{H} and \mathcal{H}^* , due to the conjugate-linearity of the inner product in its first argument. This leads us to introduce the *conjugate space* $\bar{\mathcal{H}}$ of a Hilbert space \mathcal{H} : this has the same additive group of vectors as \mathcal{H} , while the scalar multiplication and inner product are “twisted” by complex conjugation:

$$\alpha \bullet_{\bar{\mathcal{H}}} \phi := \bar{\alpha} \bullet_{\mathcal{H}} \phi \quad \langle \phi | \psi \rangle_{\bar{\mathcal{H}}} := \langle \psi | \phi \rangle_{\mathcal{H}}$$

We can define $\mathcal{H}^* = \bar{\mathcal{H}}$, since \mathcal{H} and $\bar{\mathcal{H}}$ have the same orthonormal bases, and we can define the counit by

$$\epsilon_{\mathcal{H}} : \mathcal{H} \otimes \bar{\mathcal{H}} \rightarrow \mathbb{C} :: \phi \otimes \psi \mapsto \langle \psi | \phi \rangle_{\mathcal{H}}$$

which is indeed (bi)linear rather than sesquilinear! Note that

$$\bar{\bar{\mathcal{H}}} = \mathcal{H}, \quad \overline{A \otimes B} = \bar{A} \otimes \bar{B}.$$

4.1 Why compact closure does not suffice

Note that the categories **FdHilb** and **FdVec_ℂ** are equivalent! This immediately suggests that some additional categorical structure must be identified to reflect the rôle of the inner product.

A further reason for seeking additional categorical structure is to reflect the centrally important notion of *adjoint* in Hilbert spaces:

$$\begin{array}{c} A \xrightarrow{f} B \\ \hline A \xleftarrow{f^\dagger} B \end{array} \quad \langle f\phi | \psi \rangle_B = \langle \phi | f^\dagger\psi \rangle_A$$

This is *not* the same as the dual — the types are different! In “degenerate” CCC’s in which $A^* = A$, e.g. **Rel** or real inner-product spaces, we have $f^* = f^\dagger$. In Hilbert spaces, the isomorphism $A \simeq A^*$ is not linear, but *conjugate linear*:

$$\langle \lambda \bullet \phi | - \rangle = \bar{\lambda} \bullet \langle \phi | - \rangle$$

and hence does not live in the category **Hilb** at all!

¹Much of quantum information is concerned with *completely positive maps* acting on *density matrices*. An account of this extended setting in terms of a general categorical construction within our framework is discussed in Section 7.

4.2 Solution: Strong Compact Closure

A key observation is this: the assignment $\mathcal{H} \mapsto \mathcal{H}^*$ on objects has a *covariant* functorial extension $f \mapsto f_*$, which is essentially identity on morphisms; and then we can *define*

$$f^\dagger = (f^*)_* = (f_*)^*.$$

Concretely, in terms of matrices $()^*$ is *transpose*, $()_*$ is *complex conjugation*, and the adjoint is the *conjugate transpose*. Each of these three operations can be expressed in terms of the other two. For example, $f^* = (f^\dagger)_*$. All three of these operations are important in articulating the foundational structure of quantum mechanics. All three can be presented at the abstract level as functors, as we shall now show.

4.3 Axiomatization of Strong Compact Closure

We shall adopt the most concise and elegant axiomatization of strongly compact closed categories, which takes the adjoint as primitive, following [Abramsky and Coecke, 2005].

It is convenient to build the definition up in several stages, as in [Selinger, 2007].

DEFINITION 10. A *dagger category* is a category \mathbf{C} equipped with an identity-on-objects, contravariant, strictly involutive functor $f \mapsto f^\dagger$:

$$1^\dagger = 1, \quad (g \circ f)^\dagger = f^\dagger \circ g^\dagger, \quad f^{\dagger\dagger} = f.$$

We define an arrow $f : A \rightarrow B$ in a dagger category to be *unitary* if it is an isomorphism such that $f^{-1} = f^\dagger$. An endomorphism $f : A \rightarrow A$ is *self-adjoint* if $f = f^\dagger$.

DEFINITION 11. A *dagger symmetric monoidal category* $(\mathbf{C}, \otimes, I, \lambda, \rho, \alpha, \sigma, \dagger)$ combines dagger and symmetric monoidal structure, with the requirement that the natural isomorphisms $\lambda, \rho, \alpha, \sigma$ are componentwise unitary, and moreover that \dagger is a strong monoidal functor:

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger.$$

Finally we come to the main definition.

DEFINITION 12. A *strongly compact closed category* is a dagger symmetric monoidal category which is compact closed, and such that the following diagram commutes:

$$\begin{array}{ccc} I & \xrightarrow{\eta_A} & A^* \otimes A \\ & \searrow \epsilon_A & \downarrow \sigma_{A^*, A} \\ & & A \otimes A^* \end{array}$$

This implies that the counit is *definable* from the unit and the adjoint:

$$\epsilon_A = \eta_A^\dagger \circ \sigma_{A, A^*}$$

and similarly the unit can be defined from the counit and the adjoint. Furthermore, it is in fact possible to replace the two commuting diagrams required in the definition of compact closure by one. We refer to [Abramsky and Coecke, 2005] for the details.

DEFINITION 13. In any strongly compact closed category \mathbf{C} , we can define a covariant monoidal functor

$$A \mapsto A^*, \quad f : A \rightarrow B \mapsto f_* = (f^\dagger)^* : A^* \rightarrow B^* .$$

Examples Our central example is of course **FdHilb**. Any compact closed category such as **Rel**, in which $(\)^*$ is the identity on objects, is trivially strongly compact closed (we just take $f^\dagger = f^*$). Note that in this case $f_* = f^{**} = f$. Thus in **Rel** the adjoint is relational converse. The category of finite-dimensional real inner product spaces and linear maps, with $A = A^*$, offers another example of this situation. A construction of free strongly compact closed categories over dagger categories is given in [Abramsky, 2005].

Scalars Self-adjoint scalars $s = s^\dagger$ in strongly compact closed categories are of special interest. In the case of **FdHilb**, these are the *positive reals* \mathbb{R}^+ . The passage from s to ss^\dagger , which is self-adjoint, will track the passage in quantum mechanics from *amplitudes* to *probabilities*.

4.4 Inner Products and Dirac Notation

With the adjoint available, it is straightforward to interpret Dirac notation — the indispensable everyday notation of quantum mechanics and quantum information. A *ket* is simply an arrow $\psi : I \rightarrow A$, which we can write as $|\psi\rangle$ for emphasis. We think of kets as *states*, of a given type of system A . The corresponding *bra* will then be $\psi^\dagger : A \rightarrow I$, which we can think of as a *costate*.

Example In **FdHilb**, a linear map $f : \mathbb{C} \rightarrow \mathcal{H}$ can be identified with the vector $f(1) = \psi \in \mathcal{H}$: by linearity, all other values of f are determined by ψ . Even better, we can identify f with its image, which is the *ray* or one-dimensional subspace of \mathcal{H} spanned by ψ — the proper notion of (pure) state of a quantum system.

DEFINITION 14. Given $\psi, \phi : I \rightarrow A$ we define their *abstract inner product* $\langle \psi | \phi \rangle$ as

$$\psi^\dagger \circ \phi : I \longrightarrow I .$$

Note that this is a scalar, as it should be. In **FdHilb**, this definition coincides with the usual inner product. In **Rel**, for $x, y \subseteq \{*\} \times X$:

$$\langle x \mid y \rangle = 1_{\mathbf{I}}, \quad x \cap y \neq \emptyset \quad \text{and} \quad \langle x \mid y \rangle = 0_{\mathbf{I}}, \quad x \cap y = \emptyset.$$

We now show that two of the basic properties of adjoints in **FdHilb** hold in generality in the abstract setting.

PROPOSITION 15. *For $\psi : \mathbf{I} \rightarrow A$, $\phi : \mathbf{I} \rightarrow B$ and $f : B \rightarrow A$ we have*

$$\langle f^\dagger \circ \psi \mid \phi \rangle_B = \langle \psi \mid f \circ \phi \rangle_A.$$

Proof: $\langle f^\dagger \circ \psi \mid \phi \rangle = (f^\dagger \circ \psi)^\dagger \circ \phi = \psi^\dagger \circ f \circ \phi = \langle \psi \mid f \circ \phi \rangle$. ■

PROPOSITION 16. *Unitary morphisms $U : A \rightarrow B$ preserve the inner product, that is for all $\psi, \phi : \mathbf{I} \rightarrow A$ we have*

$$\langle U \circ \psi \mid U \circ \phi \rangle_B = \langle \psi \mid \phi \rangle_A.$$

Proof: By Proposition 15, $\langle U \circ \psi \mid U \circ \phi \rangle_B = \langle U^\dagger \circ U \circ \psi \mid \phi \rangle_A = \langle \psi \mid \phi \rangle_A$. ■

Finally, we show how the inner product can be defined in terms of the ‘complex conjugate’ functor $(\)_*$.

PROPOSITION 17. *For $\psi, \phi : \mathbf{I} \rightarrow A$ we have:*

$$\langle \psi \mid \phi \rangle_A = \mathbf{I} \xrightarrow{\rho_{\mathbf{I}}} \mathbf{I} \otimes \mathbf{I} \xrightarrow{1_{\mathbf{I}} \otimes u_{\mathbf{I}}} \mathbf{I} \otimes \mathbf{I}^* \xrightarrow{\phi \otimes \psi_*} A \otimes A^* \xrightarrow{\epsilon_A} \mathbf{I}.$$

Proof: Since $u_{\mathbf{I}} = \rho_{\mathbf{I}^*}^{-1} \circ \eta_{\mathbf{I}}$ by naturality of ρ we have

$$\eta_{\mathbf{I}} = \rho_{\mathbf{I}^*} \circ \rho_{\mathbf{I}^*}^{-1} \circ \eta_{\mathbf{I}} = \rho_{\mathbf{I}^*} \circ u_{\mathbf{I}} = (u_{\mathbf{I}} \otimes 1_{\mathbf{I}}) \circ \rho_{\mathbf{I}}$$

where we use $\rho^{-1} = \rho^\dagger$ and similarly we obtain $\epsilon_{\mathbf{I}} = \rho_{\mathbf{I}}^\dagger \circ (1_{\mathbf{I}} \otimes u_{\mathbf{I}}^\dagger)$. Hence by $1_{\mathbf{I}} = u_{\mathbf{I}}^\dagger \circ u_{\mathbf{I}}$ and the analogues to Lemmas 6 and 9 for the counit we obtain

$$\begin{aligned} \psi^\dagger \circ \phi &= \rho_{\mathbf{I}}^\dagger \circ ((\psi^\dagger \circ \phi) \otimes 1_{\mathbf{I}}) \circ \rho_{\mathbf{I}} = \epsilon_{\mathbf{I}} \circ (\psi^\dagger \otimes 1_{\mathbf{I}^*}) \circ (\phi \otimes 1_{\mathbf{I}^*}) \circ \epsilon_{\mathbf{I}}^\dagger \\ &= \lrcorner \psi^\dagger \lrcorner \circ (\phi \otimes 1_{\mathbf{I}^*}) \circ \epsilon_{\mathbf{I}}^\dagger \\ &= \epsilon_{\mathbf{I}} \circ (1_{\mathbf{I}} \otimes \psi_*) \circ (\phi \otimes 1_{\mathbf{I}^*}) \circ \epsilon_{\mathbf{I}}^\dagger \end{aligned}$$

which is equal to $\epsilon_{\mathbf{I}} \circ (\phi \otimes \psi_*) \circ (1_{\mathbf{I}} \otimes u_{\mathbf{I}}) \circ \rho_{\mathbf{I}}$. ■

4.5 Dissection of the bipartite projector

Projectors are a basic building block in the von Neumann-style foundations of quantum mechanics, and in standard approaches to quantum logic. It is a notable feature of our approach that we are able, at the abstract level of strongly compact

closed categories, to delineate a *fine-structure* of bipartite projectors, which can be applied directly to the analysis of information flow in quantum protocols.

We define a *projector* on an object A in a strongly compact closed category to be an arrow $P : A \rightarrow A$ which is idempotent and self-adjoint:

$$P^2 = P, \quad P = P^\dagger.$$

PROPOSITION 18. *Suppose we have a state $\psi : I \rightarrow A$ which is normalized, meaning $\langle \psi | \psi \rangle = 1_I$. Then the ‘ket-bra’ $|\psi\rangle\langle\psi| = \psi \circ \psi^\dagger : A \rightarrow A$ is a projector.*

Proof: Self-adjointness is clear. For idempotence:

$$\psi \circ \psi^\dagger \circ \psi \circ \psi^\dagger = \langle \psi | \psi \rangle \bullet \psi \circ \psi^\dagger = \psi \circ \psi^\dagger.$$

■

We now want to apply this idea in a more refined form to a state $\psi : I \rightarrow A^* \otimes B$ of a compound system. Note that, by Map-State duality:

$$\mathbf{C}(I, A^* \otimes B) \equiv \mathbf{C}(A, B)$$

any such state ψ corresponds biuniquely to the name of a map $f : A \rightarrow B$, *i.e.* $\psi = \ulcorner f \urcorner$. This arrow witnesses an information flow from A to B , and we will use this to expose the information flow inherent in the corresponding projector.

Explicitly, we define

$$P_f := \ulcorner f \urcorner \circ (\ulcorner f \urcorner)^\dagger = \ulcorner f \urcorner \circ \lrcorner f_{* \lrcorner} : A^* \otimes B \rightarrow A^* \otimes B,$$

that is, we have an assignment

$$P_- : \mathbf{C}(I, A^* \otimes B) \longrightarrow \mathbf{C}(A^* \otimes B, A^* \otimes B) :: \psi \mapsto \psi \circ \psi^\dagger$$

from bipartite elements to bipartite projectors. Note that the strong compact closed structure is essential in order to define P_f as an endomorphism.

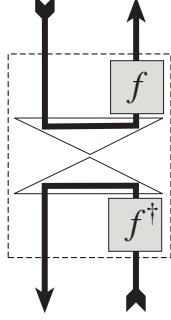
We can *normalize* these projectors P_f by considering $s_f \bullet P_f$ for $s_f := (\lrcorner f_{* \lrcorner} \circ \ulcorner f \urcorner)^{-1}$ (provided this inverse exists in $\mathbf{C}(I, I)$), yielding

$$(s_f \bullet P_f) \circ (s_f \bullet P_f) = s_f \bullet (\ulcorner f \urcorner \circ (s_f \bullet (\lrcorner f_{* \lrcorner} \circ \ulcorner f \urcorner))) \circ \lrcorner f_{* \lrcorner} = s_f \bullet P_f,$$

and also

$$(s_f \bullet P_f) \circ \ulcorner f \urcorner = \ulcorner f \urcorner \quad \text{and} \quad \lrcorner f_{* \lrcorner} \circ (s_f \bullet P_f) = \lrcorner f_{* \lrcorner}.$$

A picture corresponding to this decomposed bipartite projector is:



4.6 Trace

Another essential mathematical instrument in quantum mechanics is the *trace* of a linear map. In quantum information, extensive use is made of the more general notion of *partial trace*, which is used to trace out a subsystem of a compound system.

A general categorical axiomatization of the notion of partial trace has been given by Joyal, Street and Verity [Joyal, Street and Verity, 1996]. A trace in a symmetric monoidal category \mathbf{C} is a family of functions

$$\mathrm{Tr}_{A,B}^U : \mathbf{C}(A \otimes U, B \otimes U) \longrightarrow \mathbf{C}(A, B)$$

for objects A, B, U of \mathbf{C} , satisfying a number of axioms, for which we refer to [Joyal, Street and Verity, 1996]. This specializes to yield the total trace for endomorphisms by taking $A = B = \mathbf{I}$. In this case, $\mathrm{Tr}(f) = \mathrm{Tr}_{\mathbf{I},\mathbf{I}}^U(f) : \mathbf{I} \rightarrow \mathbf{I}$ is a scalar. Expected properties such as the invariance of the trace under cyclic permutations

$$\mathrm{Tr}(g \circ f) = \mathrm{Tr}(f \circ g)$$

follow from the general axioms.

Any compact closed category carries a canonical (in fact, a unique) trace. The definition can be given slightly more elegantly in the strongly compact closed case. For an endomorphism $f : A \rightarrow A$, the total trace is defined by

$$\mathrm{Tr}(f) = \epsilon_A \circ (f \otimes 1_{A^*}) \circ \epsilon_A^\dagger.$$

More generally, if $f : A \otimes C \rightarrow B \otimes C$, $\mathrm{Tr}_{A,B}^C(f) : A \rightarrow B$ is defined to be:

$$A \xrightarrow{\rho_A} A \otimes \mathbf{I} \xrightarrow{1 \otimes \epsilon_C^\dagger} A \otimes C \otimes C^* \xrightarrow{f \otimes 1_{C^*}} B \otimes C \otimes C^* \xrightarrow{1 \otimes \epsilon_C} B \otimes \mathbf{I} \xrightarrow{\rho_B^{-1}} B.$$

These definitions give rise to the standard notions of trace and partial trace in \mathbf{FdHilb} .

5 BIPRODUCTS, BRANCHING AND MEASUREMENTS

As we have seen, many of the basic ingredients of quantum mechanics are present in strongly compact closed categories. What is lacking is the ability to express the probabilistic branching arising from measurements, and the information flows from quantum to classical and back. We shall find this final piece of expressive power in a rather standard piece of additional categorical structure, namely *biproducts*.

5.1 Biproducts

Biproducts have been studied as part of the structure of Abelian categories. For further details, and proofs of the general results we shall cite in this sub-section, see e.g. [Mitchell, 1965; MacLane, 1998].

Recall that a *zero object* in a category is one which is both initial and terminal. If $\mathbf{0}$ is a zero object, there is an arrow

$$0_{A,B} : A \longrightarrow \mathbf{0} \longrightarrow B$$

between any pair of objects A and B . Let \mathbf{C} be a category with a zero object and binary products and coproducts. Any arrow

$$A_1 \amalg A_2 \xrightarrow{f} A_1 \amalg A_2$$

with injections $q_i : A_i \rightarrow A_1 \amalg A_2$ and projections $p_j : A_1 \amalg A_2 \rightarrow A_j$ can be written uniquely as a matrix

$$\begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}$$

where $f_{ij} := p_j \circ f \circ q_i : A_i \rightarrow A_j$. If the arrow

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is an isomorphism for all A_1, A_2 , then we say that \mathbf{C} has *biproducts*, and write $A \oplus B$ for the biproduct of A and B .

PROPOSITION 19 Semi-additivity. *If \mathbf{C} has biproducts, then we can define an operation of addition on each hom-set $\mathbf{C}(A, B)$ by*

$$\begin{array}{ccc} A & \xrightarrow{f+g} & B \\ \Delta \downarrow & & \uparrow \nabla \\ A \oplus A & \xrightarrow{f \oplus g} & B \oplus B \end{array}$$

for $f, g : A \rightarrow B$, where $\Delta = \langle 1_A, 1_A \rangle$ and $\nabla = [1_B, 1_B]$ are respectively the diagonal and codiagonal. This operation is associative and commutative, with 0_{AB} as an identity. Moreover, composition is bilinear with respect to this additive structure. Thus \mathbf{C} is enriched over abelian monoids.

Because of this automatic enrichment of categories with biproducts over abelian monoids, we say that such a category is *semi-additive*.

PROPOSITION 20. *If \mathbf{C} has biproducts, we can choose projections p_1, \dots, p_n and injections q_1, \dots, q_n for each $\bigoplus_{k=1}^{k=n} A_k$ satisfying*

$$p_j \circ q_i = \delta_{ij} \quad \text{and} \quad \sum_{k=1}^{k=n} q_k \circ p_k = 1_{\bigoplus_k A_k}$$

where $\delta_{ii} = 1_{A_i}$, and $\delta_{ij} = 0_{A_i, A_j}$, $i \neq j$.

5.2 Strongly compact closed categories with biproducts

We now come to the full mathematical structure we shall use as a setting for finitary quantum mechanics: namely *strongly compact closed categories with biproducts*.

A first point is that, because of the strongly self-dual nature of compact closed categories, weaker assumptions suffice in order to guarantee the presence of biproducts. The following elegant result is due to Robin Houston [2006], and was in fact directly motivated by [Abramsky and Coecke, 2004], the precursor to the present article.

THEOREM 21. *Let \mathbf{C} be a monoidal category with finite products and coproducts, and suppose that for every object A of \mathbf{C} , the functor $A \otimes -$ preserves products and the functor $- \otimes A$ preserves coproducts. Then \mathbf{C} has finite biproducts.*

Because a compact closed category is closed and self-dual, the existence of products implies that of coproducts, and vice versa, and the functor $- \otimes A$ is a left adjoint and hence preserves coproducts. Moreover, since $A^* \multimap B \simeq A^{**} \otimes B \simeq A \otimes B$, the functor $A \otimes -$ is a right adjoint and preserves products. Hence this result specializes to the following:

PROPOSITION 22. *If \mathbf{C} is a compact closed category with either products or coproducts, then it has biproducts, and hence is semiadditive.*

Examples There are many examples of compact closed categories with biproducts: the category of relations for a regular category with stable disjoint coproducts; the category of finitely generated projective modules over a commutative ring; the category of finitely generated free semimodules over a commutative semiring; and the category of free semimodules over a complete commutative semiring are all semi-additive compact closed categories. Examples have also arisen in a Computer Science context in the first author's work on Interaction Categories [Abramsky,

Gay and Nagarajan, 1995]. Compact closed categories with biproducts with additional assumptions, in particular that the category is abelian, have been studied in the mathematical literature on *Tannakian categories* [Deligne, 1990].

In the case of strongly compact closed categories, we need a coherence condition between the dagger and the biproduct structure. We say that a category is *strongly compact closed with biproducts* if we can choose biproduct structures p_i, q_i as in Proposition 20 such that $p_i^\dagger = q_i$ for $i = 1, \dots, n$.

PROPOSITION 23. *If \mathbf{C} is strongly compact closed with biproducts, then*

$$\sum_{k=1}^{k=n} p_k^\dagger \circ p_k = \sum_{k=1}^{k=n} q_k \circ q_k^\dagger = 1_{\bigoplus_k A_k}.$$

Moreover, there are natural isomorphisms

$$\nu_{A,B} : (A \oplus B)^* \simeq A^* \oplus B^* \quad \text{and} \quad \nu_1 : \mathbf{0}^* \simeq \mathbf{0},$$

and $(\)^\dagger$ preserves biproducts and hence is additive:

$$(f \oplus g)^\dagger = f^\dagger \oplus g^\dagger, \quad (f + g)^\dagger = f^\dagger + g^\dagger \quad \text{and} \quad 0_{A,B}^\dagger = 0_{B,A}.$$

Examples Examples of semi-additive strongly compact closed categories are the category $(\mathbf{Rel}, \times, +)$, where the biproduct is the disjoint union, and the category $(\mathbf{FdHilb}, \otimes, \oplus)$, where the biproduct is the direct sum.

Distributivity and classical information flow As we have already seen, in a compact closed category with biproducts, tensor distributes over the biproduct. This abstract-seeming observation in fact plays a crucial rôle in the representation of *classical information flow*. To understand this, consider a quantum system $A \otimes B$, composed from subsystems A(lice) and B(ob). Now suppose that Alice performs a local measurement, which we will represent as resolving her part of the system into say $A_1 \oplus A_2$. Here the biproduct is used to represent the different *branches* of the measurement. At this point, by the functorial properties of \oplus , Alice can perform actions $f_1 \oplus f_2$, which *depend on which branch of the measurement has been taken*. The global state of the system is $(A_1 \oplus A_2) \otimes B$, and as things stand Bob has no access to this measurement outcome. However, under distributivity we have

$$(A_1 \oplus A_2) \otimes B \simeq (A_1 \otimes B) \oplus (A_2 \otimes B)$$

which corresponds to propagating the classical information as to the measurement outcome ‘outwards’, so that it is now accessible to Bob, who can perform an action depending on this outcome, of the form $1_A \otimes (g_1 \oplus g_2)$.

We shall record distributivity in an explicit form for future use.

PROPOSITION 24 Distributivity of \otimes over \oplus . *In any monoidal closed category there is a right distributivity natural isomorphism $\tau_{A,B,C} : A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$, which is explicitly defined as*

$$\tau_{A,\cdot,\cdot} := \langle 1_A \otimes p_1, 1_A \otimes p_2 \rangle \quad \text{and} \quad \tau_{A,\cdot,\cdot}^{-1} := [1_A \otimes q_1, 1_A \otimes q_2].$$

A left distributivity isomorphism $\nu_{A,B,C} : (A \oplus B) \otimes C \simeq (A \otimes C) \oplus (B \otimes C)$ can be defined similarly.

Semiring of scalars. In a strongly compact closed category with biproducts, the scalars form a commutative semiring. Moreover, scalar multiplication satisfies the usual additive properties

$$(s_1 + s_2) \bullet f = s_1 \bullet f + s_2 \bullet f, \quad 0 \bullet f = 0$$

as well as the multiplicative ones. For Hilbert spaces, this commutative semiring is the field of complex numbers. In **Rel** the commutative semiring of scalars is the Boolean semiring $\{0, 1\}$, with disjunction as sum.

Matrix calculus. We can write any arrow of the form $f : A \oplus B \rightarrow C \oplus D$ as a matrix

$$M_f := \begin{pmatrix} p_1^{C,D} \circ f \circ q_1^{A,B} & p_1^{C,D} \circ f \circ q_2^{A,B} \\ p_2^{C,D} \circ f \circ q_1^{A,B} & p_2^{C,D} \circ f \circ q_2^{A,B} \end{pmatrix}.$$

The sum $f + g$ of such morphisms corresponds to the matrix sum $M_f + M_g$ and composition $g \circ f$ corresponds to matrix multiplication $M_g \cdot M_f$. Hence categories with biproducts admit a matrix calculus.

5.3 Spectral Decompositions

We define a *spectral decomposition* of an object A to be a unitary isomorphism

$$U : A \rightarrow \bigoplus_{i=1}^{i=n} A_i.$$

(Here the ‘spectrum’ is just the set of indices $1, \dots, n$). Given a spectral decomposition U , we define morphisms

$$\psi_j := U^\dagger \circ q_j : A_j \rightarrow A \quad \text{and} \quad \pi_j := \psi_j^\dagger = p_j \circ U : A \rightarrow A_j,$$

diagrammatically

$$\begin{array}{ccc} A_j & \xrightarrow{\psi_j} & A \\ q_j \downarrow & \swarrow U & \downarrow \pi_j \\ \bigoplus_{i=1}^{i=n} A_i & \xrightarrow{p_j} & A_j \end{array}$$

and finally *projectors*

$$P_j := \psi_j \circ \pi_j : A \rightarrow A.$$

These projectors are *self-adjoint*

$$P_j^\dagger = (\psi_j \circ \pi_j)^\dagger = \pi_j^\dagger \circ \psi_j^\dagger = \psi_j \circ \pi_j = P_j$$

idempotent and *orthogonal*

$$P_i \circ P_j = \psi_i \circ \pi_i \circ \psi_j \circ \pi_j = \psi_i \circ \delta_{ij} \circ \pi_j = \delta_{ij}^A \circ P_i.$$

Moreover, they yield a *resolution of the identity*:

$$\begin{aligned} \sum_{i=1}^{i=n} P_i &= \sum_{i=1}^{i=n} \psi_i \circ \pi_i = \sum_{i=1}^{i=n} U^\dagger \circ q_i \circ p_i \circ U \\ &= U^\dagger \circ \left(\sum_{i=1}^{i=n} q_i \circ p_i \right) \circ U = U^{-1} \circ 1_{\bigoplus_i A_i} \circ U = 1_A. \end{aligned}$$

5.4 Bases and dimension

Writing $n \cdot X$ for types of the shape $\bigoplus_{i=1}^{i=n} X$ it follows by self-duality of the tensor unit I that

$$\nu_{I, \dots, I}^{-1} \circ (n \cdot u_I) : n \cdot I \simeq (n \cdot I)^*.$$

A *basis* for an object A is a unitary isomorphism

$$\text{base} : n \cdot I \rightarrow A.$$

Given bases base_A and base_B for objects A and B respectively we can define the matrix (m_{ij}) of any morphism $f : A \rightarrow B$ in those two bases as the matrix of

$$\text{base}_B^\dagger \circ f \circ \text{base}_A : n_A \cdot I \rightarrow n_B \cdot I.$$

PROPOSITION 25. *Given $f : A \rightarrow B$, $\text{base}_A : n_A \cdot I \rightarrow A$ and $\text{base}_B : n_B \cdot I \rightarrow B$ the matrix (m'_{ij}) of f^\dagger in these bases is the conjugate transpose of the matrix (m_{ij}) of f .*

Proof: $m'_{ij} = p_i \circ \text{base}_A^\dagger \circ f^\dagger \circ \text{base}_B \circ q_j = (p_j \circ \text{base}_B^\dagger \circ f \circ \text{base}_A \circ q_i)^\dagger = m_{ji}^\dagger$. ■

If in addition to the assumptions of Proposition 15 and Proposition 16 there exist bases for A and B , we can prove converses to both of them.

PROPOSITION 26. *If there exist bases for A and B then $f : A \rightarrow B$ is the adjoint to $g : B \rightarrow A$ if and only if*

$$\langle f \circ \psi \mid \phi \rangle_B = \langle \psi \mid g \circ \phi \rangle_A$$

for all $\psi : I \rightarrow A$ and $\phi : I \rightarrow B$.

Proof: Let (m_{ij}) be the matrix of f^\dagger and (m'_{ij}) the matrix of g in the given bases.

$$\begin{aligned}
 m_{ij} &= p_i \circ \text{base}_A^\dagger \circ f^\dagger \circ \text{base}_B \circ q_j \\
 &= \langle f \circ \text{base}_A \circ q_i \mid \text{base}_B \circ q_j \rangle_B \\
 &= \langle f \circ \psi \mid \phi \rangle_B \\
 &= \langle \psi \mid g \circ \phi \rangle_A \\
 &= \langle \text{base}_A \circ q_i \mid g \circ \text{base}_B \circ q_j \rangle_A \\
 &= p_i \circ \text{base}_A^\dagger \circ g \circ \text{base}_B \circ q_j \\
 &= m'_{ij}.
 \end{aligned}$$

Hence the matrix elements of g and f^\dagger coincide so g and f^\dagger are equal. The converse is Proposition 15. \blacksquare

PROPOSITION 27. *If there exist bases for A and B then a morphism $U : A \rightarrow B$ is unitary if and only if it preserves the inner product, that is for all $\psi, \phi : I \rightarrow A$ we have*

$$\langle U \circ \psi \mid U \circ \phi \rangle_B = \langle \psi \mid \phi \rangle_A.$$

Proof: We have $\langle U^{-1} \circ \psi \mid \phi \rangle_A = \langle U \circ U^{-1} \circ \psi \mid U \circ \phi \rangle_B = \langle \psi \mid U \circ \phi \rangle_B$ and hence by Proposition 26, $U^\dagger = U^{-1}$. The converse is given by Proposition 16. \blacksquare

Note also that when a basis is available we can assign to $\psi^\dagger : A \rightarrow I$ and $\phi : I \rightarrow A$ matrices

$$\left(\psi_1^\dagger \quad \cdots \quad \psi_n^\dagger \right) \quad \text{and} \quad \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix}$$

respectively, and the inner product becomes

$$\langle \psi \mid \phi \rangle = \left(\psi_1^\dagger \quad \cdots \quad \psi_n^\dagger \right) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} = \sum_{i=1}^n \psi_i^\dagger \circ \phi_i.$$

Dimension Interestingly, two different notions of dimension arise in our setting. We assign an *integer dimension* $\dim(A) \in \mathbb{N}$ to an object A provided there exists a base

$$\text{base} : \dim(A) \cdot I \rightarrow A.$$

Alternatively, we introduce the *scalar dimension* as

$$\dim_s(A) := \text{Tr}(1_A) = \epsilon_A \circ \epsilon_A^\dagger \in \mathbf{C}(I, I).$$

We also have:

$$\dim_s(I) = 1_I \quad \dim_s(A^*) = \dim_s(A) \quad \dim_s(A \otimes B) = \dim_s(A)\dim_s(B)$$

In $\mathbf{FdVec}_{\mathbb{K}}$ these notions of dimension coincide, in the sense that $\dim_s(V)$ is multiplication with the scalar $\dim(V)$. In \mathbf{Rel} the integer dimension corresponds to the cardinality of the set, and is only well-defined for finite sets, while $\dim_s(X)$ always exists; however, $\dim_s(X)$ can only take two values, $0_{\mathbb{I}}$ and $1_{\mathbb{I}}$, and the two notions of dimension diverge for sets of cardinality greater than 1.

5.5 Towards a representation theorem

As the results in this section have shown, under the assumption of biproducts we can replicate many of the familiar linear-algebraic calculations in Hilbert spaces. One may wonder how far we really are from Hilbert spaces.

The deep results by Deligne [1990] and Doplicher-Roberts [1989] on Tannakian categories, the latter directly motivated by algebraic quantum field theory, show that under additional assumptions, in particular that the category is abelian as well as compact closed, we obtain a representation into finite-dimensional modules over the ring of scalars. One would like to see a similar result in the case of strongly compact closed categories with biproducts, with the conclusion being a representation into inner-product spaces.

6 ABSTRACT QUANTUM MECHANICS: AXIOMATICS AND QUANTUM PROTOCOLS

We can identify the basic ingredients of finitary quantum mechanics in any semi-additive strongly compact closed category.

1. A *state space* is represented by an object A .
2. A *basic variable* ('type of qubits') is a state space Q with a given unitary isomorphism

$$\mathbf{base}_Q : \mathbb{I} \oplus \mathbb{I} \rightarrow Q$$

which we call the *computational basis* of Q . By using the isomorphism $n \cdot \mathbb{I} \simeq (n \cdot \mathbb{I})^*$ described in Section 5, we also obtain a computational basis for Q^* .

3. A *compound system* for which the subsystems are described by A and B respectively is described by $A \otimes B$. If we have computational bases \mathbf{base}_A and \mathbf{base}_B , then we define

$$\mathbf{base}_{A \otimes B} := (\mathbf{base}_A \otimes \mathbf{base}_B) \circ d_{nm}^{-1}$$

where

$$d_{nm} : n \cdot \mathbb{I} \otimes m \cdot \mathbb{I} \simeq (nm) \cdot \mathbb{I}$$

is the canonical isomorphism constructed using first the left distributivity isomorphism ν , and then the right distributivity isomorphism τ , to give the usual lexicographically-ordered computational basis for the tensor product.

4. Basic data transformations are unitary isomorphisms.

5a. A *preparation* in a state space A is a morphism $\psi : I \rightarrow A$ for which there exists a unitary $U : I \oplus B \rightarrow A$ such that

$$\begin{array}{ccc}
 I & \xrightarrow{\psi} & A \\
 q_1 \downarrow & \nearrow U & \\
 I \oplus B & &
 \end{array}$$

commutes.

5b. Consider a spectral decomposition

$$U : A \rightarrow \bigoplus_{i=1}^{i=n} A_i$$

with associated projectors P_j . This gives rise to the *non-destructive measurement*

$$\langle P_i \rangle_{i=1}^{i=n} : A \rightarrow n \cdot A.$$

The projectors $P_i : A \rightarrow A$ for $i = 1, \dots, n$ are called the *measurement branches*. This measurement is *non-degenerate* if $A_i = I$ for all $i = 1, \dots, n$. In this case we refer to U itself as a *destructive measurement* or *observation*. The morphisms $\pi_i = p_i \circ U : A \rightarrow I$ for $i = 1, \dots, n$ are called *observation branches*.

Note that the type of a non-destructive measurement makes it explicit that it is an operation which involves a non-deterministic transition (by contrast with the standard Hilbert space quantum mechanical formalism).

6a. Explicit biproducts represent the *branching* arising from the indeterminacy of measurement outcomes.

Hence an operation f acting on an explicit biproduct $A \oplus B$ should itself be an explicit biproduct, *i.e.* we want

$$f = f_1 \oplus f_2 : A \oplus B \rightarrow C \oplus D,$$

for $f_1 : A \rightarrow C$ and $f_2 : B \rightarrow D$. The dependency of f_i on the branch it is in captures *local* classical communication. The full force of non-local classical communication is enabled by Proposition 24.

6b. Distributivity isomorphisms represent *non-local classical communication*.

To see this, suppose e.g. that we have a compound system $Q \otimes A$, and we (non-destructively) measure the qubit in the first component, obtaining a new system state described by $(Q \oplus Q) \otimes A$. At this point, we know ‘locally’, *i.e.* at the site of the first component, what the measurement outcome is, but we have not propagated this information to the rest of the system A . However, after applying the distributivity isomorphism

$$(Q \oplus Q) \otimes A \simeq (Q \otimes A) \oplus (Q \otimes A)$$

the information about the outcome of the measurement on the first qubit has been propagated globally throughout the system, and we can perform operations on A depending on the measurement outcome, e.g. $(1_Q \otimes U_0) \oplus (1_Q \otimes U_1)$ where U_0, U_1 are the operations we wish to perform on A in the event that the outcome of the measurement we performed on Q was 0 or 1 respectively.

6.1 The Born rule

We now show how the *Born rule*, which is the key quantitative feature of quantum mechanics, emerges automatically from our abstract setting.

For a preparation $\psi : I \rightarrow A$ and spectral decomposition $U : A \rightarrow \bigoplus_{i=1}^{i=n} A_i$, with corresponding non-destructive measurement

$$\langle P_i \rangle_{i=1}^{i=n} : A \rightarrow n \cdot A,$$

we can consider the protocol

$$I \xrightarrow{\psi} A \xrightarrow{\langle P_i \rangle_{i=1}^{i=n}} n \cdot A.$$

We define scalars

$$\text{Prob}(P_i, \psi) := \langle \psi | P_i | \psi \rangle = \psi^\dagger \circ P_i \circ \psi.$$

PROPOSITION 28. *With notation as above,*

$$\text{Prob}(P_i, \psi) = (\text{Prob}(P_i, \psi))^\dagger$$

and

$$\sum_{i=1}^{i=n} \text{Prob}(P_i, \psi) = 1.$$

Hence we think of the scalar $\text{Prob}(P_j, \psi)$ as ‘the probability of obtaining the j ’th outcome of the measurement $\langle P_i \rangle_{i=1}^{i=n}$ on the state ψ ’.

Proof: From the definitions of preparation and the projectors, there are unitaries U, V such that

$$\text{Prob}(P_i, \psi) = (V \circ q_1)^\dagger \circ U^\dagger \circ q_i \circ p_i \circ U \circ V \circ q_1$$

for each i . Hence

$$\begin{aligned}
 \sum_{i=1}^{i=n} \text{Prob}(P_i, \psi) &= \sum_{i=1}^{i=n} p_1 \circ V^\dagger \circ U^\dagger \circ q_i \circ p_i \circ U \circ V \circ q_1 \\
 &= p_1 \circ V^\dagger \circ U^\dagger \circ \left(\sum_{i=1}^n q_i \circ p_i \right) \circ U \circ V \circ q_1 \\
 &= p_1 \circ V^{-1} \circ U^{-1} \circ 1_{n \cdot 1} \circ U \circ V \circ q_1 = p_1 \circ q_1 = 1_1.
 \end{aligned}$$

■

Moreover, since by definition $P_j = \pi_j^\dagger \circ \pi_j$, we can rewrite the Born rule expression as

$$\text{Prob}(P_j, \psi) = \psi^\dagger \circ P_j \circ \psi = \psi^\dagger \circ \pi_j^\dagger \circ \pi_j \circ \psi = (\pi_j \circ \psi)^\dagger \circ \pi_j \circ \psi = s_j^\dagger \circ s_j$$

for some scalar $s_j \in \mathbf{C}(\mathbf{I}, \mathbf{I})$. Thus s_j can be thought of as the ‘probability amplitude’ giving rise to the probability $s_j^\dagger \circ s_j$, which is of course self-adjoint. If we consider the protocol

$$\mathbf{I} \xrightarrow{\psi} A \xrightarrow{\langle \pi_i \rangle_{i=1}^{i=n}} n \cdot \mathbf{I}.$$

which involves an observation $\langle \pi_i \rangle_{i=1}^{i=n}$, then these scalars s_j correspond to the branches

$$\mathbf{I} \xrightarrow{\psi} A \xrightarrow{\pi_j} \mathbf{I}.$$

We now turn to the description of the quantum protocols previously discussed in Section 2 within our framework. In each case, we shall give a complete description of the protocol, including the quantum-to-classical information flows arising from measurements, and the subsequent classical-to-quantum flows corresponding to the classical communications and the actions depending on these performed as steps in the protocols. We shall in each case verify the correctness of the protocol, by proving that a certain diagram commutes. Thus these case studies provide evidence for the expressiveness and effectiveness of the framework.

Our general axiomatic development allows for considerable generality. The standard von Neumann axiomatization fits Quantum Mechanics perfectly, with room to spare. Our basic setting of strongly compact closed categories with biproducts is general enough to allow very different models such as **Rel**, the category of sets and relations. When we consider specific protocols such as teleportation, a kind of ‘Reverse Arithmetic’ (by analogy with Reverse Mathematics [Simpson, 1999]) arises. That is, we can characterize what requirements are placed on the semiring of scalars $\mathbf{C}(\mathbf{I}, \mathbf{I})$ (where \mathbf{I} is the tensor unit) in order for the protocol to be realized. This is often much less than requiring that this be the field of complex numbers, but in the specific cases which we shall consider, the requirements are sufficient to exclude **Rel**.

6.2 Quantum teleportation

DEFINITION 29. A teleportation base is a scalar s together with a morphism

$$\text{prebase}_T : 4 \cdot \mathbf{I} \rightarrow Q^* \otimes Q$$

such that:

- $\text{base}_T := s \bullet \text{prebase}_T$ is unitary.
- the four maps $\beta_j : Q \rightarrow Q$, where β_j is defined by $\lceil \beta_j \rceil := \text{prebase}_T \circ q_j$, are unitary.
- $2s^\dagger s = 1$.

The morphisms $s \bullet \lceil \beta_j \rceil$ are the base vectors of the teleportation base. A teleportation base is a Bell base when the Bell base maps $\beta_1, \beta_2, \beta_3, \beta_4 : Q \rightarrow Q$ satisfy²

$$\beta_1 = 1_Q \quad \beta_2 = \sigma_Q^\oplus \quad \beta_3 = \beta_3^\dagger \quad \beta_4 = \sigma_Q^\oplus \circ \beta_3$$

where

$$\sigma_Q^\oplus := \text{base}_Q \circ \sigma_{\mathbf{I}, \mathbf{I}}^\oplus \circ \text{base}_Q^{-1}.$$

A teleportation base defines a teleportation observation

$$\langle s^\dagger \bullet \lfloor \beta_i \rfloor \rangle_{i=1}^{i=4} : Q \otimes Q^* \rightarrow 4 \cdot \mathbf{I}.$$

To emphasize the identity of the individual qubits we label the three copies of Q we shall consider as Q_a, Q_b, Q_c . We also use labelled identities, e.g. $1_{bc} : Q_b \rightarrow Q_c$, and labelled Bell bases. Finally, we introduce

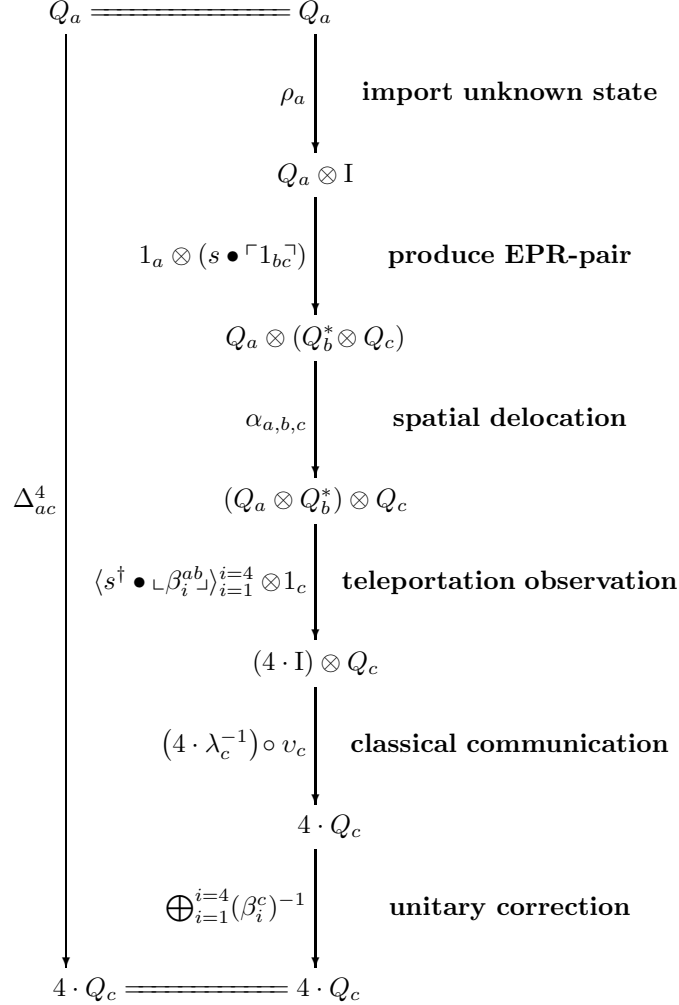
$$\Delta_{ac}^4 := \langle s^\dagger s \bullet 1_{ac} \rangle_{i=1}^{i=4} : Q_a \rightarrow 4 \cdot Q_c$$

as the *labelled, weighted diagonal*. This expresses the intended behaviour of teleportation, namely that the input qubit is propagated to the output along each branch of the protocol, with ‘weight’ $s^\dagger s$, corresponding to the probability amplitude for that branch. Note that the sum of the corresponding probabilities is

$$4(s^\dagger s)^\dagger s^\dagger s = (2s^\dagger s)(2s^\dagger s) = 1.$$

²This choice of axioms is sufficient for our purposes. One might prefer to axiomatize a notion of Bell base such that the corresponding Bell base maps are exactly the Pauli matrices — note that this would introduce a coefficient i in β_4 .

THEOREM 30. *The following diagram commutes.*

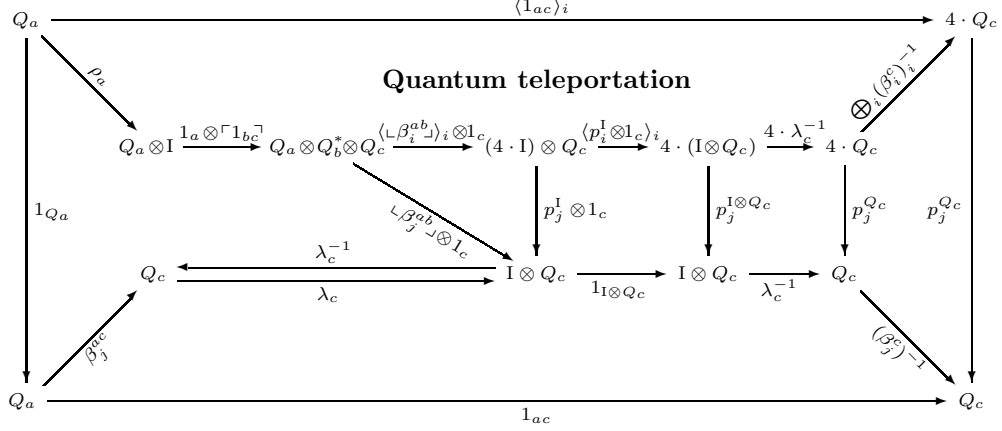


The right-hand-side of the above diagram is our formal description of the teleportation protocol; the commutativity of the diagram expresses the correctness of the protocol. Hence any strongly compact closed category with biproducts admits quantum teleportation provided it contains a teleportation base. If we do a Bell-base observation then the corresponding unitary corrections are

$$\beta_i^{-1} = \beta_i \quad \text{for } i \in \{1, 2, 3\} \quad \text{and} \quad \beta_4^{-1} = \beta_3 \circ \sigma_Q^\oplus.$$

Proof: For each $j \in \{1, 2, 3, 4\}$ we have a commutative diagram of the form below. The top trapezoid is the statement of the Theorem. We ignore the scalars – which

cancel out against each other – in this proof.



We use the universal property of the product, naturality of λ and the explicit form of the natural isomorphism $v_c := \langle p_i^I \otimes 1_c \rangle_{i=1}^4$. In the specific case of a Bell-base observation we use $1_Q^\dagger = 1_Q$, $(\sigma_Q^\oplus)^\dagger = \sigma_Q^\oplus$ and $(\sigma_Q^\oplus \circ \beta_3)^\dagger = \beta_3^\dagger \circ (\sigma_Q^\oplus)^\dagger = \beta_3 \circ \sigma_Q^\oplus$. ■

Although in **Rel** teleportation works for ‘individual observational branches’ it fails to admit the full teleportation protocol since there are only two automorphisms of Q (which is just a two-element set, *i.e.* the type of ‘classical bits’), and hence there is no teleportation base.

We now consider sufficient conditions on the ambient category \mathbf{C} for a teleportation base to exist. We remark firstly that if $\mathbf{C}(\mathbf{I}, \mathbf{I})$ contains an additive inverse for 1, then it is a ring, and moreover all additive inverses exist in each hom-set $\mathbf{C}(A, B)$, so \mathbf{C} is enriched over Abelian groups. Suppose then that $\mathbf{C}(\mathbf{I}, \mathbf{I})$ is a ring with $1 \neq -1$. We can define a morphism

$$\text{prebase}_T = \text{base}_{Q^* \otimes Q} \circ M : 4 \cdot \mathbf{I} \rightarrow Q^* \otimes Q$$

where M is the endomorphism of $4 \cdot \mathbf{I}$ determined by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

The corresponding morphisms β_j will have 2×2 matrices determined by the columns of this 4×4 matrix, and will be unitary. If $\mathbf{C}(\mathbf{I}, \mathbf{I})$ furthermore contains a scalar s satisfying $2s^\dagger s = 1$, then $s \bullet \text{prebase}_T$ is unitary, and the conditions for a teleportation base are fulfilled. Suppose we start with a ring R containing an element s satisfying $2s^2 = 1$. (Examples are plentiful, *e.g.* any subring of \mathbb{C} ,

or of $\mathbb{Q}(\sqrt{2})$, containing $\frac{1}{\sqrt{2}}$). The category of finitely generated free R -modules and R -linear maps is strongly compact closed with biproducts, and admits a teleportation base (in which s will appear as a scalar with $s = s^\dagger$), hence realizes teleportation.

6.3 Logic-gate teleportation

Logic gate teleportation of qubits requires only a minor modification as compared to the teleportation protocol.

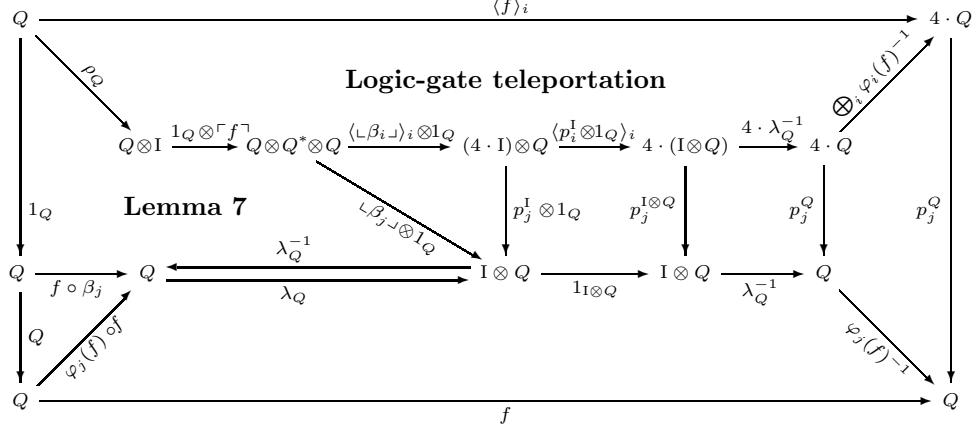
THEOREM 31. *Let unitary morphism $f : Q \rightarrow Q$ be such that for each $i \in \{1, 2, 3, 4\}$ a morphism $\varphi_i(f) : Q \rightarrow Q$ satisfying $f \circ \beta_i = \varphi_i(f) \circ f$ exists. The diagram of Theorem 30 with the modifications made below commutes.*

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 & & \downarrow \\
 & & Q_a \otimes I \\
 & & \downarrow \\
 & & 1_a \otimes (s \bullet \ulcorner f \urcorner) \quad \text{produce } f\text{-state} \\
 & & \downarrow \\
 & & Q_a \otimes (Q_b^* \otimes Q_c) \\
 & & \vdots \\
 & & \downarrow \\
 & & 4 \cdot Q_c \\
 & & \downarrow \\
 & & \bigoplus_{i=1}^4 (\varphi_i(f))^{-1} \quad \text{unitary correction} \\
 & & \downarrow \\
 \Delta_{ac}^4 \circ f & & 4 \cdot Q_c \\
 \downarrow & & \text{=====} \\
 & & 4 \cdot Q_c
 \end{array}$$

The right-hand-side of the diagram is our formal description of logic-gate teleportation of $f : Q \rightarrow Q$; the commutativity of the diagram under the stated conditions expresses the correctness of logic-gate teleportation for qubits.

Proof: The top trapezoid is the statement of the Theorem. The a , b and c -labels are the same as in the proof of teleportation. For each $j \in \{1, 2, 3, 4\}$ we have a

diagram of the form below. Again we ignore the scalars in this proof.



■

This two-dimensional case does not yet provide a universal computational primitive, which requires teleportation of $Q \otimes Q$ -gates [Gottesman and Chuang, 1999]. We present the example of teleportation of a CNOT gate [Gottesman and Chuang, 1999] (see also [Coecke, 2003] Section 3.3).

Given a Bell base we define a CNOT gate as one which acts as follows on tensors of the Bell base maps:

$$\text{CNOT} \circ (\sigma_Q^{\oplus} \otimes 1_Q) = (\sigma_Q^{\oplus} \otimes \sigma_Q^{\oplus}) \circ \text{CNOT} \quad \text{CNOT} \circ (1_Q \otimes \sigma_Q^{\oplus}) = (1_Q \otimes \sigma_Q^{\oplus}) \circ \text{CNOT}$$

$$\text{CNOT} \circ (\beta_3 \otimes 1_Q) = (\beta_3 \otimes 1_Q) \circ \text{CNOT} \quad \text{CNOT} \circ (1_Q \otimes \beta_3) = (\beta_3 \otimes \beta_3) \circ \text{CNOT}$$

It follows from this that

$$\text{CNOT} \circ (\beta_4 \otimes 1_Q) = (\beta_4 \otimes \sigma_Q^{\oplus}) \circ \text{CNOT} \quad \text{CNOT} \circ (1_Q \otimes \beta_4) = (\beta_3 \otimes \beta_4) \circ \text{CNOT}$$

from which in turn it follows by bifactoriality of the tensor that the required unitary corrections factor into single qubit actions, for which we introduce a notation by setting

$$\text{CNOT} \circ (\beta_i \otimes 1_Q) = \varphi_1(\beta_i) \circ \text{CNOT} \quad \text{CNOT} \circ (1_Q \otimes \beta_i) = \varphi_2(\beta_i) \circ \text{CNOT}$$

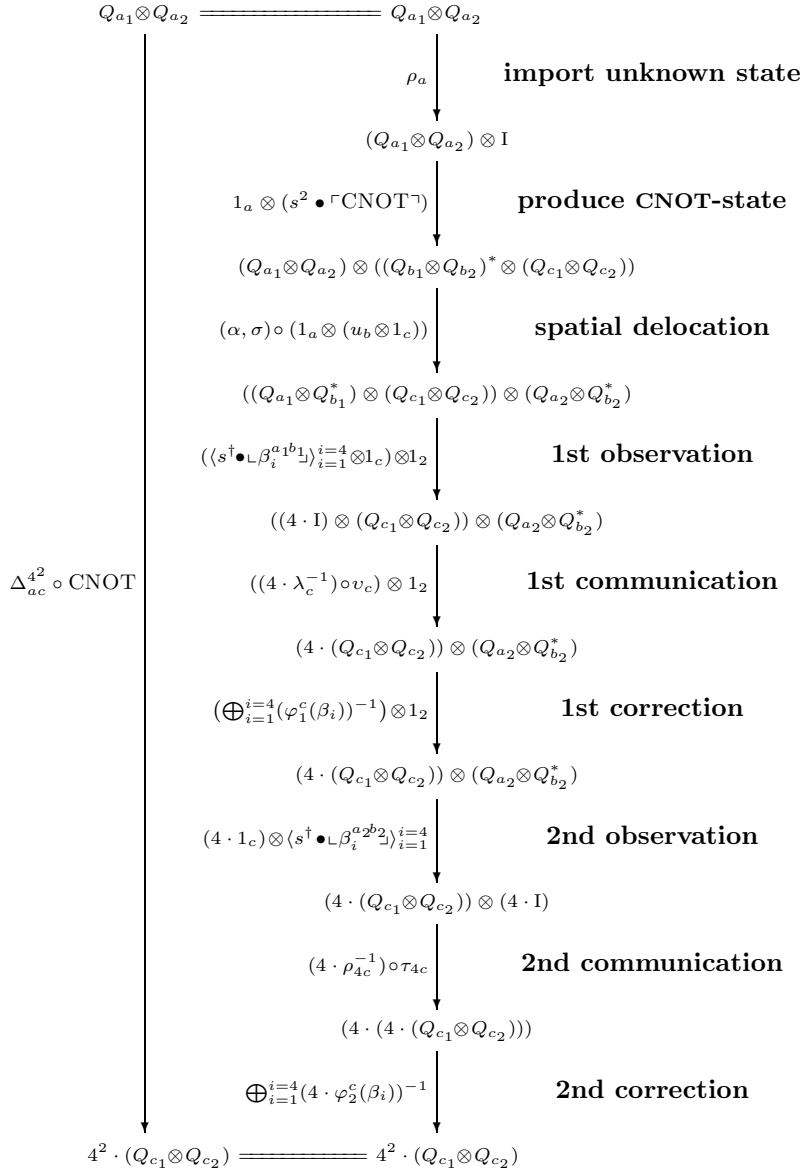
The reader can verify that for

$$4^2 \cdot (Q_{c_1} \otimes Q_{c_2}) := 4 \cdot (4 \cdot (Q_{c_1} \otimes Q_{c_2}))$$

and

$$\Delta_{ac}^{4^2} := \langle s^\dagger s \bullet \langle s^\dagger s \bullet 1_{ac} \rangle_{i=1}^{i=4} \rangle_{i=1}^{i=4} : Q_{a_1} \otimes Q_{a_2} \rightarrow 4^2 \cdot (Q_{c_1} \otimes Q_{c_2})$$

the following diagram commutes.



6.4 Entanglement swapping

THEOREM 32. *Setting*

$$\begin{aligned}
\gamma_i &:= (\beta_i)_* \\
P_i &:= s^\dagger s \bullet (\ulcorner \gamma_i \urcorner \circ \llcorner \beta_i \llcorner) \\
\zeta_i^{ac} &:= \bigoplus_{i=1}^{i=4} ((1_b^* \otimes \gamma_i^{-1}) \otimes (1_d^* \otimes \beta_i^{-1})) \\
\Theta_{ab} &:= 1_d^* \otimes \langle P_i \rangle_{i=1}^{i=4} \otimes 1_c \\
\Omega_{ab} &:= \langle s^\dagger s^3 \bullet (\ulcorner 1_{ba} \urcorner \otimes \ulcorner 1_{dc} \urcorner) \rangle_{i=1}^{i=4}
\end{aligned}$$

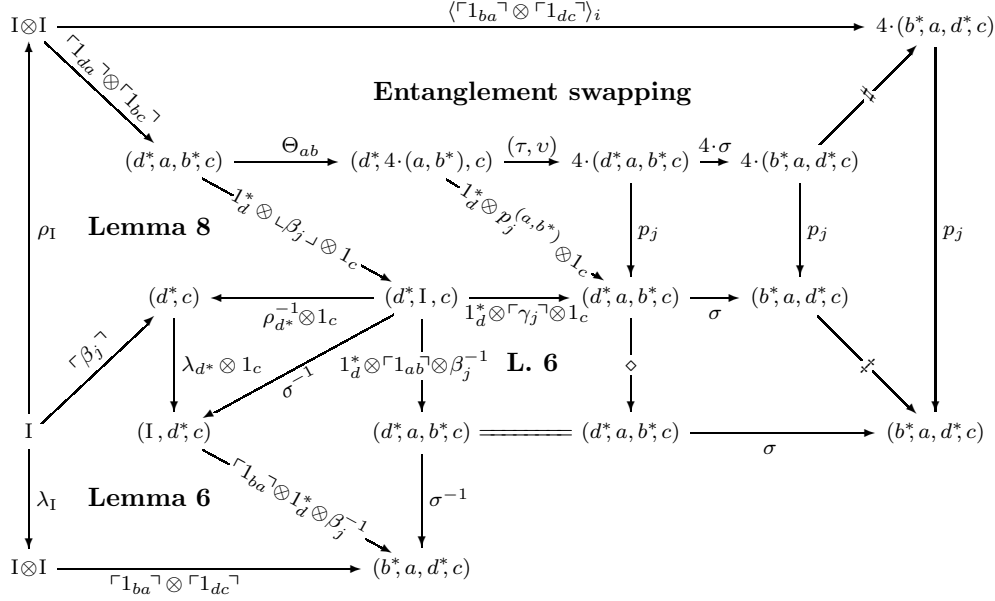
the following diagram commutes.

$$\begin{array}{ccc}
I \otimes I & \xlongequal{\hspace{2cm}} & I \otimes I \\
\downarrow \Omega_{ab} & & \downarrow \text{produce EPR-pairs} \\
& & s^2 \bullet (\ulcorner 1_{da} \urcorner \otimes \ulcorner 1_{bc} \urcorner) \\
& & \downarrow \\
& & (Q_d^* \otimes Q_a) \otimes (Q_b^* \otimes Q_c) \\
& & \downarrow \alpha \quad \text{spatial delocation} \\
& & Q_d^* \otimes (Q_a \otimes Q_b^*) \otimes Q_c \\
& & \downarrow \Theta_{ab} \quad \text{Bell-base measurement} \\
& & Q_d^* \otimes (4 \cdot (Q_a \otimes Q_b^*)) \otimes Q_c \\
& & \downarrow (4 \cdot (\alpha, \sigma)) \circ (\tau, \nu) \quad \text{classical communication} \\
& & 4 \cdot ((Q_b^* \otimes Q_a) \otimes (Q_d^* \otimes Q_c)) \\
& & \downarrow \zeta_i^{ac} \quad \text{unitary correction} \\
& & 4 \cdot ((Q_b^* \otimes Q_a) \otimes (Q_d^* \otimes Q_c))
\end{array}$$

The right-hand-side of the above diagram is our formal description of the entanglement swapping protocol.

Proof: The top trapezoid is the statement of the Theorem. We have a diagram of the form below for each $j \in \{1, 2, 3, 4\}$. To simplify the notation of the types

we set (a^*, b, c^*, d) for $Q_a^* \otimes Q_b \otimes Q_c^* \otimes Q_d$ etc. Again we ignore the scalars in this proof.



where

$$\ddagger := \bigoplus_i (1_b^* \otimes \gamma_i^{-1} \otimes 1_d^* \otimes \beta_i^{-1}) \quad , \quad \dagger := 1_b^* \otimes \gamma_j^{-1} \otimes 1_d^* \otimes \beta_j^{-1} \quad \text{and} \quad \diamond := 1_d^* \otimes \gamma_j^{-1} \otimes 1_b^* \otimes \beta_j^{-1}$$

■

We use $\gamma_i = (\beta_i)_*$ rather than β_i to make P_i an endomorphism and hence a projector. The general definition of a ‘bipartite entanglement projector’ is

$$P_f := \lceil f^\square \circ \lfloor f_* \rceil = \lceil f^\square \circ \lfloor f^\dagger \rceil \circ \sigma_{A^*, B} : A^* \otimes B \rightarrow A^* \otimes B$$

for $f : A \rightarrow B$, so in fact $P_i = P_{(\beta_i)_*}$.

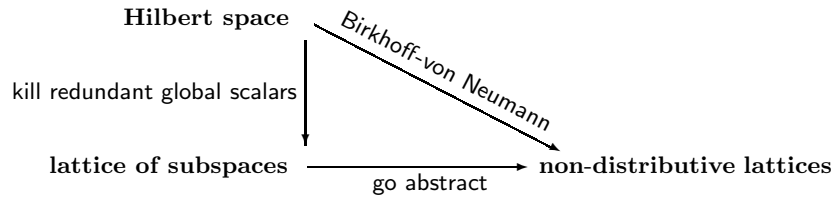
7 EXTENSIONS AND FURTHER DEVELOPMENTS

Since its first publication in 2004, a number of elaborations on the categorical quantum axiomatics described above have been proposed, by ourselves in collaboration with members of our group, Ross Duncan, Dusko Pavlovic, Eric Oliver Paquette, Simon Perdrix and Bill Edwards, and also by others elsewhere, most notably Peter Selinger and Jamie Vicary. We shall present some of the main developments.

7.1 Projective structure

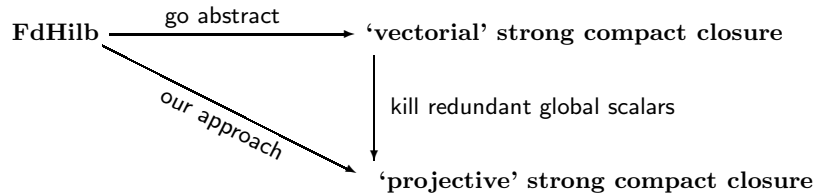
We shall discuss our first topic at considerably greater length than the others we shall cover in this survey. The main reason for this is that it concerns the passage to a projective point of view, which makes for an evident comparison with the standard approaches to quantum logic going back to [Birkhoff and von Neumann, 1936]. Thus it seems appropriate to go into some detail in our coverage of this topic, in the context of the Handbook in which this article will appear.

The axiomatics we have given corresponds to the pure state picture of quantum mechanics. The very fact that we can faithfully carry out linear-algebraic calculations using the semi-additive structure provided by biproducts means that states will typically carry redundant global phases, as is the case for vectors in Hilbert spaces. Eliminating these means ‘going projective’. The *quantum logic* tradition provides one way of doing so [Birkhoff and von Neumann, 1936]. Given a Hilbert space one eliminates global scalars by passing to the projection lattice. The non-Boolean nature of the resulting lattice is then taken to be characteristic for quantum behaviour. This leads one then to consider certain classes of non-distributive lattices as ‘quantum structures’.



It is well-known that there is no obvious counterpart for the Hilbert space tensor product when passing to these more general classes of lattices. This is one reason why Birkhoff-von Neumann style quantum logic never penetrated the mainstream physics community, and is particularly unfortunate in the light of the important role that the tensor product plays in quantum information and computation.

But one can also start from the whole category of finite dimensional Hilbert spaces and linear maps \mathbf{FdHilb} . Then we can consider ‘strongly compact closed categories + some additive structure’ as its appropriate abstraction, and hope to find some abstractly valid counterpart to ‘elimination of redundant global scalars’.



The major advantage which such a construction has is that the tensor product is now part of the mathematical object under consideration, and hence will not be lost in the passage from vectorial spaces to projective ones.

This passage was realised in [Coecke, 2007a] as follows. For morphisms in \mathbf{FdHilb} , i.e. linear maps, if we have that $f = e^{i\theta} \cdot g$ with $\theta \in [0, 2\pi[$ for $f, g : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, then

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger.$$

Now in abstract generality, given a strongly compact closed category \mathbf{C} , we can define a new category $\mathbf{WProj}(\mathbf{C})$ with the same objects as those of \mathbf{C} , but with

$$\mathbf{WProj}(\mathbf{C})(A, B) := \{f \otimes f^\dagger \mid f \in \mathbf{C}(A, B)\}$$

as hom-sets and in which composition is given by

$$(f \otimes f^\dagger) \circ (g \otimes g^\dagger) := (f \circ g) \otimes (f \circ g)^\dagger.$$

One easily shows that $\mathbf{WProj}(\mathbf{C})$ is again a strongly compact closed category. The abstract counterpart to elimination of global phases is expressed by the following propositions.

PROPOSITION 33. [Coecke, 2007a] *For morphisms f, g and scalars s, t in a strongly compact closed category, we have*

$$s \bullet f = t \bullet g \wedge s \circ s^\dagger = t \circ t^\dagger = 1_I \implies f \otimes f^\dagger = g \otimes g^\dagger.$$

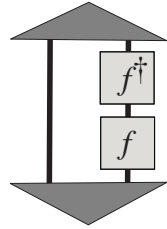
PROPOSITION 34. [Coecke, 2007a] *For morphisms f and g in a strongly compact closed category with scalars S we have*

$$f \otimes f^\dagger = g \otimes g^\dagger \implies \exists s, t \in S. s \bullet f = t \bullet g \wedge s \circ s^\dagger = t \circ t^\dagger.$$

In particular we can set

$$s := (\ulcorner f^\urcorner)^\dagger \circ \ulcorner f^\urcorner \quad \text{and} \quad t := (\ulcorner g^\urcorner)^\dagger \circ \ulcorner g^\urcorner.$$

While the first proposition is straightforward, the second one is somewhat more surprising. It admits a simple graphical proof. We represent units by dark triangles and their adjoints by the same triangle but depicted upside down. Other morphisms are depicted by square boxes as before, with the exception of scalars which are depicted by ‘diamonds’. The scalar $s := (\ulcorner f^\urcorner)^\dagger \circ \ulcorner f^\urcorner$ is depicted as



Bifactoriality means that we can move these boxes upward and downward, and naturality provides additional modes of movement, e.g. scalars admit arbitrary movements. Now, given that $f \otimes f^\dagger = g \otimes g^\dagger$, that is, in a picture,

$$\begin{array}{c} \uparrow \\ \boxed{f} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{f^\dagger} \\ \downarrow \end{array} = \begin{array}{c} \uparrow \\ \boxed{g} \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{g^\dagger} \\ \downarrow \end{array}$$

we need to show that $s \bullet f = t \bullet g$ and $s \circ s^\dagger = t \circ t^\dagger$ for some choice of scalars s and t , that is, in a picture,

$$\begin{array}{c} \diamond s \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{f} \\ \downarrow \end{array} = \begin{array}{c} \diamond t \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{g} \\ \downarrow \end{array} \qquad \begin{array}{c} \diamond s \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{s^\dagger} \\ \downarrow \end{array} = \begin{array}{c} \diamond t \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \boxed{t^\dagger} \\ \downarrow \end{array}$$

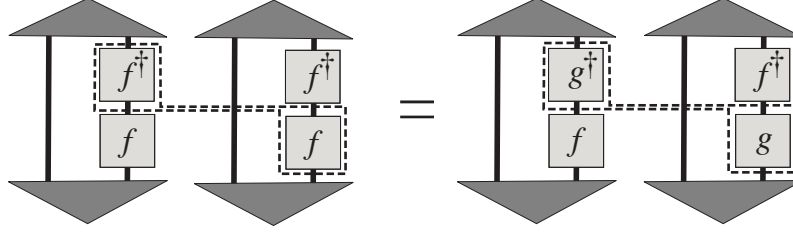
The choice that we will make for s and t is

$$\begin{array}{c} \diamond s \\ \downarrow \end{array} := \begin{array}{c} \text{---} \\ \uparrow \\ \boxed{f^\dagger} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \text{---} \end{array} \qquad \begin{array}{c} \diamond t \\ \downarrow \end{array} := \begin{array}{c} \text{---} \\ \uparrow \\ \boxed{g^\dagger} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \text{---} \end{array}$$

Then we indeed have $s \bullet f = t \bullet g$ since in

$$\begin{array}{c} \text{---} \\ \uparrow \\ \boxed{f^\dagger} \quad \boxed{f} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \text{---} \\ \uparrow \\ \boxed{g^\dagger} \quad \boxed{g} \\ \downarrow \\ \boxed{f} \\ \downarrow \\ \text{---} \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

the areas within the dotted line are equal by assumption. We also have that $s \circ s^\dagger = t \circ t^\dagger$ since



which completes the proof.

As expected, biproducts do not survive the passage from \mathbf{C} to $\mathbf{WProj}(\mathbf{C})$ but the weaker structure which results still suffices for a comprehensive description of the protocols we have discussed in this article. In particular, the distributivity natural isomorphisms

$$\text{dist}_{0,l} : A \otimes 0 \simeq 0$$

$$\text{dist}_l : A \otimes (B \oplus C) \simeq (A \otimes B) \oplus (A \otimes C)$$

$$\text{dist}_{0,r} : 0 \otimes A \simeq 0$$

$$\text{dist}_r : (B \oplus C) \otimes A \simeq (B \otimes A) \oplus (C \otimes A).$$

carry over to $\mathbf{WProj}(\mathbf{C})$. Details can be found in [Coecke, 2007a].

Our framework also allows a precise general statement of the incompatibility of biproducts with projective structure.

Call a strongly compact closed category *projective* iff equality of projections implies equality of the corresponding states, that is,

$$(1) \quad \forall \psi, \phi : I \rightarrow A. \quad \psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi.$$

PROPOSITION 35. [Coecke, 2007a] *If a strongly compact closed category with biproducts is projective and the semiring of scalars admits negatives, i.e. is a ring, then we have $1 = -1$, that is, there are no non-trivial negatives.*

Having no non-trivial negatives of course obstructs the description of interference.

7.2 Mixed states and Completely Positive Maps

The categorical axiomatics set out in this article primarily refers to the pure-state picture of quantum mechanics. However, for many purposes, in particular those of quantum information, it is *mixed states*, acted on by *completely positive maps*, which provide the most appropriate setting. Peter Selinger [2007] proposed a general categorical construction, directly in the framework of the categorical axiomatics of [Abramsky and Coecke, 2004] which has been described in this article, to capture the passage from the pure states to the mixed states picture.

The construction proceeds as follows. Given any strongly compact closed category \mathbf{C} we define a new category $\mathbf{CPM}(\mathbf{C})$ with the same objects as \mathbf{C} but with morphisms given by

$$\mathbf{CPM}(\mathbf{C})(A, B) :=$$

$$\{(1_B \otimes \epsilon_C \otimes 1_{B^*}) \circ (1_{B \otimes C} \otimes \sigma_{B^*, C^*}) \circ (f \otimes f_*) \mid f \in \mathbf{C}(A, B \otimes C)\}$$

where for simplicity we assume that the monoidal structure is strict. Composition in $\text{CPM}(\mathbf{C})$ is inherited pointwise from \mathbf{C} . The morphisms of the category $\text{CPM}(\mathbf{FdHilb})$ are exactly the *completely positive maps*, and the morphisms in the hom-set $\text{CPM}(\mathbf{FdHilb})(\mathbf{C}, \mathcal{H})$ are exactly the self-adjoint operators with positive trace on \mathcal{H} . The category $\text{WProj}(\mathbf{C})$ faithfully embeds in $\text{CPM}(\mathbf{C})$ by setting

$$f \otimes f^\dagger \mapsto f \otimes f_*.$$

Metaphorically, we have

$$\frac{\text{CPM}(\mathbf{C})}{\text{WProj}(\mathbf{C})} = \frac{\text{density operators}}{\text{projectors}}.$$

For more details on the CPM-construction we refer the reader to [Selinger, 2007].

Recently it was shown that the CPM-construction does not require strong compact closure, but only dagger symmetric monoidal structure. Details are in [Coecke, 2007]. An axiomatic presentation of categories of completely positive maps is given in [Coecke, 2008].

7.3 Generalised No-Cloning and No-Deleting theorems

The No-Cloning theorem [Dieks, 1982; Wootters and Zurek, 1982] is a basic limitative result for quantum mechanics, with particular significance for quantum information. It says that there is no unitary operation which makes perfect copies of an unknown (pure) quantum state. A stronger form of this result is the No-Broadcasting theorem [Barnum *et al.*, 1996], which applies to mixed states. There is also a No-Deleting theorem [Pati and Braunstein, 2000].

The categorical and logical framework which we have described provides new possibilities for exploring the structure, scope and limits of quantum information processing, and the features which distinguish it from its classical counterpart. One area where some striking progress has already been made is the axiomatics of No-Cloning and No-Deleting. It is possible to delimit the classical-quantum boundary here in quite a subtle way. On the one hand, we have the strongly compact closed structure which is present in the usual Hilbert space setting for QIC, and which we have shown accounts in generality for the phenomena of entanglement. Suppose we were to assume that *either* copying *or* deleting were available in a strongly compact closed category as *uniform operations*. Mathematically, a uniform copying operation means a *natural diagonal*

$$\Delta_A : A \rightarrow A \otimes A$$

i.e. a monoidal natural transformation, which moreover is co-associative and co-

commutative:

$$\begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \xrightarrow{1 \otimes \Delta} A \otimes (A \otimes A) \\
 \parallel & & \downarrow \alpha_{A,A,A} \\
 A & \xrightarrow{\Delta} & A \otimes A \xrightarrow{\Delta \otimes 1} (A \otimes A) \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\Delta} & A \otimes A \\
 \searrow \Delta & & \downarrow \sigma_{A,A} \\
 & & A \otimes A
 \end{array}$$

Thinking of the diagonal associated with the usual cartesian product, one sees immediately that co-commutativity and co-associativity are basic requirements for a copying operation: if I have two copies of the same thing, it does not matter which order they come in, and if I produce three copies by iterating the copying operation, which copy I choose to perform the second copying operation on is immaterial. Naturality, on the other hand, corresponds essentially to *basis-independence* in the Hilbert space setting; it says that the operation exists ‘for logical reasons’, in a representation-independent form.

We have shown recently that under these assumptions *the category trivializes*; in other words, that this combination of quantum and classical features is *inconsistent*, leading to a collapse of the structure. The precise form of the result is that under these hypotheses every endomorphism in the category is a scalar multiple of the identity.

Similar generalizations of the No-Deleting theorem [Pati and Braunstein, 2000] and the No-Broadcasting theorem [Barnum *et al.*, 1996] also hold. Papers on these results are in preparation.

One striking feature of these results is that they are visibly in the same genre as a well-known result by Joyal in categorical logic [Lambek and Scott, 1986] showing that a ‘Boolean cartesian closed category’ trivializes, which provides a major roadblock to the computational interpretation of classical logic. In fact, they strengthen Joyal’s result, insofar as the assumption of a full categorical product (diagonals *and* projections) in the presence of a classical duality is weakened. This shows a heretofore unsuspected connection between limitative results in proof theory and No-Go theorems in quantum mechanics.

Another interesting point is the way that this result is delicately poised. The basis structures to be discussed in the next sub-section do assume commutative comonoid structures existing in strongly compact closed categories—indeed with considerable additional properties, such as the Frobenius identity. Not only is this consistent, such structures correspond to a major feature of Hilbert spaces, namely orthonormal bases. The point is that there are many such bases for a given Hilbert space, and none are canonical. Indeed, the *choice* of basis corresponds to the *choice* of measurement set-up, to be made by a ‘classical observer’. The key ingredient which leads to inconsistency, and which basis structures lack, is *naturality*, which, as we have already suggested, stands as an abstract proxy for basis-independence.

7.4 *Basis Structures and Classical Information*

In this article, an approach to measurements and classical information has been developed based on biproducts. This emphasizes the *branching structure* of measurements due to their probabilistic outcomes.

One may distinguish the ‘multiplicative’ from the ‘additive’ levels of our axiomatization (using the terminology of Linear logic [Girard, 1987]). The multiplicative, purely tensorial level of strongly compact closed categories shows, among other things, how a remarkable amount of multilinear algebra, encompassing much of the structure needed for quantum mechanics and quantum information, can be done without any substrate of linear algebra. Moreover, this level of the axiomatization carries a very nice diagrammatic calculus, which we have sampled informally. In general, the return on structural insights gained from the axiomatization seems very good. The additive level of biproducts reinstates a linear (or ‘semilinear’) level of structure, albeit with fairly weak assumptions, and there is more of a sense of recapitulating familiar definitions and calculations. While a diagrammatic calculus is still available here (see [Abramsky and Duncan, 2006]), it is subject to a combinatorial unwieldiness familiar from process algebra in Computer Science [Milner, 1989] (cf. the ‘Expansion Theorem’).

An alternative approach to measurements and classical information has been developed in a series of papers [Coecke and Pavlovic, 2007; Coecke and Paquette, 2006; Coecke, Pavlovic and Vicary, 2008a; Coecke, Paquette and Perdrix, 2008] under various names, the best of which is probably ‘basis structure’. Starting from the standard idea that a measurement set-up corresponds to a choice of orthonormal basis, the aim is to achieve an axiomatization of the notion of basis as an additional structure. Of course, the notion of basis developed in Section 5 has all the right properties, but it is defined in terms of biproducts, while the aim here is to achieve an axiomatization purely at the multiplicative level.

This is done in an interesting way, bringing the informatic perspective to the fore. One can see the choice of a basis as determining a notion of ‘classical data’, namely the basis vectors. These vectors are subject to the classical operations of *copying* and *deleting*, so in a sense classical data, defined with respect to a particular choice of basis, stands as a contrapositive to the No-Cloning and No-Deleting theorems. Concretely, having chosen a basis $\{|i\rangle\}$ on a Hilbert space \mathcal{H} , we can define linear maps

$$\mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H} :: |i\rangle \mapsto |ii\rangle, \quad \mathcal{H} \longrightarrow \mathbb{C} :: |i\rangle \mapsto 1$$

which do correctly copy and delete the basis vectors (the ‘classical data’), although not of course the other vectors.

These considerations lead to the following definition. A *basis structure* on an object A in a strongly compact closed category is a commutative comonoid structure on A

$$\text{Copy} : A \rightarrow A \otimes A, \quad \text{Delete} : A \rightarrow I$$

subject to a number of additional axioms, the most notable of which is the *Frobenius identity* [Carboni and Walters, 1987]. In **FdHilb** these structures exactly correspond to orthonormal bases [Coecke, Pavlovic and Vicary, 2008a], which justifies their name and interpretation.

Quantum measurements can be defined relative to these structures, as self-adjoint Eilenberg-Moore coalgebras for the comonads induced by the above comonoids [Coecke and Pavlovic, 2007]. In **FdHilb** these indeed correspond exactly to projective spectra. The Eilenberg-Moore coalgebra square

$$\begin{array}{ccc}
 A & \xrightarrow{\text{Measure}} & X \otimes A \\
 \text{Measure} \downarrow & & \downarrow 1_X \otimes \text{Measure} \\
 X \otimes A & \xrightarrow{\text{Copy} \otimes 1_A} & X \otimes X \otimes A
 \end{array}$$

can be seen as an operational expression of von Neumann’s projection postulate in a resource-sensitive setting: measuring twice is the same as measuring once and then copying the measurement outcome. This abstract notion of measurement admits generalisation to POVMs and PMVMs, for which a generalised Naimark dilatation theorem can be proved at the abstract level [Coecke and Paquette, 2006].

Within **CPM(C)** the *decoherence* aspect of quantum measurement, which, concretely in **FdHilb**, is the completely positive map which eliminates non-diagonal elements relative to the measurement basis, arises as

$$\text{Copy} \circ \text{Copy}^\dagger : X \otimes X \rightarrow X \otimes X$$

where X is now taken to be self-dual, that is, $X = X^*$.

These basis structures not only allow for classical data, measurement and control operations to be described but also provide useful expressiveness when discussing multipartite states and unitaries. For example, they capture GHZ-states in a canonical fashion [Coecke and Pavlovic, 2007], and enable an elegant description of the state-transfer protocol [Coecke, Paquette and Perdrix, 2008].

Vicary showed that if one drops the co-commutativity requirement of basis structures in **FdHilb**, then, rather than all orthonormal bases, one finds exactly all finite dimensional C^* -algebras [Vicary, 2008].

The fact that these multiplicative basis structures allow measurements to be expressed without any explicit account of branching may be compared to the way that the pure λ -calculus can be used to encode booleans and conditionals [Barendregt, 1984]. From this perspective, explicit branching can be seen to have its merits, while *model-checking* [Clarke *et al.*, 1999] has done much to ameliorate the combinatorial unwieldiness mentioned above. It is likely that further insights will be gained by a deeper understanding of the relationships between the additive and multiplicative levels.

7.5 Complementary observables and phases

A further step is taken in [Coecke and Duncan, 2008], where abstract counterparts to (relative) phases are defined. Given a basis structure on X and a point $\psi : I \rightarrow X$ its action on X is defined to be the morphism

$$\Delta(\psi) := \text{Copy}^\dagger \circ (\psi \otimes 1_X) : X \rightarrow X.$$

From the axioms of basis structures it follows that the set of all these actions on the hom-set $\mathbf{C}(I, X)$ is a commutative monoid. Defining *unbiased points* as those $\psi \in \mathbf{C}(I, X)$ for which $\Delta(\psi)$ is unitary, the corresponding set of *unbiased actions* is always an abelian group, which we call the *phase group*. In the case of the qubit in **FdHilb** the phase group corresponds to the equator of the Bloch sphere, that is, indeed, to relative phase data.

Also in [Coecke and Duncan, 2008] an axiomatics is proposed for *complementary observables*. It is shown that for all known constructions of complementary quantum observables, the corresponding basis structures obey a ‘scaled’ variant of the bialgebra laws. This scaled bialgebra structure together with the phase group is sufficiently expressive to describe all linear maps, hence all multipartite states and unitary operators, in **FdHilb**. It provides an abstract means to reason about quantum circuits and to translate between quantum computational models, such as the circuit model and the measurement-based model.

As an application, a description is given of the quantum Fourier transform, the key ingredient of Shor’s factoring algorithm [Shor, 1994], the best-known example of a quantum algorithm.

7.6 The quantum harmonic oscillator

Jamie Vicary [2007] gave a purely categorical treatment of the quantum harmonic oscillator, directly in the setting described in this article, of strongly compact closed categories with biproducts. In Linear logic terminology, he introduced an ‘exponential level’ of structure, corresponding to Fock space. This provides a monoidal adjunction that encodes the raising and lowering operators into a co-commutative comonoid. Generalised coherent states arise through the hom-set isomorphisms defining the adjunction, and it is shown that they are eigenstates of the lowering operators. Similar results were independently obtained in [Fiore, 2007] in an abstract ‘formal power series’ context, with a motivation stemming from Joyal’s theory of species.

7.7 Automated quantum reasoning

The structures uncovered by the research programme we have described provide a basis for the design of software tools for automated reasoning about quantum phenomena, protocols and algorithms. Several MSc students at Oxford University

Computing Laboratory have designed and implemented such tools for their Masters Thesis projects. An ongoing high-level comprehensive approach has recently be initiated by Lucas Dixon and Ross Duncan [2008].

7.8 *Diagrammatic reasoning*

We have used a diagrammatic notation for tensor categories in an informal fashion. In fact, this diagrammatic notation, which can be traced back at least to Penrose [1971], was made fully formal by Joyal and Street [1991]; topological applications can be found in [Turaev, 1994].

The various structures which have arisen in the above discussion, such as strong compact closure, biproducts, dagger Frobenius comonoids, phase groups, scaled bialgebras, and the exponential structures used in the description of the quantum harmonic oscillator, all admit intuitive diagrammatic presentations in this tradition. References on these include [Abramsky and Duncan, 2006; Coecke and Paquette, 2006; Coecke and Duncan, 2008; Vicary, 2007]. Tutorial introductions to these diagrammatic calculi are given in [Coecke and Paquette, 2008; Selinger, 2008a]

These diagrammatic calculi provide very effective tools for the communication of the structural ideas. The software tools mentioned in the previous sub-section all support the presentation and manipulation of such diagrams as their interface to the user.

7.9 *Free constructions*

In [Abramsky, 2005] a number of free constructions are described in a simple, synthetic and conceptual manner, including the free strongly compact closed category over a dagger category, and the free traced monoidal category. The Kelly-Lapalaza [1980] construction of the free compact closed category is recovered in a structured and conceptual fashion.

These descriptions of free categories in simple combinatorial terms provide a basis for the use of diagrammatic calculi as discussed in the previous sub-section.

7.10 *Temperley-Lieb algebra and connections to knot theory and topological quantum field theory*

Our basic categorical setting has been that of symmetric monoidal categories. If we weaken the assumption of symmetry, to *braided* or *pivotal* categories, we come into immediate contact with a wide swathe of developments relating to knot theory, topology, topological quantum field theories, quantum groups, etc. We refer to [Freyd and Yetter, 1989; Kaufmann, 1991; Turaev, 1994; Yetter, 2001; Kock, 2003; Street, 2007] for a panorama of some of the related literature.

In [Abramsky, 2007], connections are made between the categorical axiomatics for quantum mechanics developed in this article, and the Temperley-Lieb algebra,

7.11 Logical syntax

In [Abramsky and Duncan, 2006] a strongly normalising proof-net calculus corresponding to the logic of strongly compact closed categories with biproducts is presented. The calculus is a full and faithful representation of the free strongly compact closed category with biproducts on a given category with an involution. This syntax can be used to represent and reason about quantum processes.

In [Duncan, 2006] this is extended to a description of the free strongly compact category generated by a monoidal category. This is applied to the description of the measurement calculus of [Danos *et al.*, 2007].

7.12 Completeness

In [Selinger, 2008] Selinger showed that finite-dimensional Hilbert spaces are equationally complete for strongly compact closed categories. This result shows that if we want to verify an equation expressed purely in the language of strongly compact closed categories, then it suffices to verify that it holds for Hilbert spaces.

7.13 Toy quantum categories

In [Coecke and Edwards, 2008] it is shown that Spekkens' well-known 'toy model' of quantum mechanics described in [Spekkens, 2007] can be regarded as an instance of the categorical quantum axiomatics. The category **Spek** is defined to be the dagger symmetric monoidal subcategory of **Rel** generated by those objects whose cardinality is a power of 4, the symmetry group on 4 elements, and a well-chosen copying-deleting pair for the 4 element set.

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