CE1259 – STRENGTH OF MATERIALS

UNIT I STRESS, STRAIN DEFORMATION OF SOLIDS

Rigid and Deformable bodies – Strength, stiffness and stability – Stresses: Tensile, compressive and shear – Deformation of simple and compound bars under axial load – Thermal stress – Elastic constants – Strain energy and unit strain energy – Strain energy in uniaxial loads.

UNIT II BEAMS - LOADS AND STRESSES

Types of beams: Supports and loads – Shear force and bending moment in beams – Cantilever, simply supported and overhanging beams – Stresses in beams – Theory of simple bending – Stress variation along the length and in the beam section – Effect of shape of beam section on stress induced – Shear stresses in beams – Shear flow.

UNIT III TORSION

Analysis of torsion of circular bars – Shear stress distribution – Bars of solid and hollow circular section – Stepped shaft – Twist and torsion stiffness – Compound shafts – Fixed and simply supported shafts – Application to close-coiled helical springs – Maximum shear stress in spring section including Wahl Factor – Deflection of helical coil springs under axial loads – Design of helical coil springs – stresses in helical coil springs under torsion loads.

UNIT IV BEAM DEFLECTION

Elastic curve of Neutral axis of the beam under normal loads – Evaluation of beam deflection and slope: Double integration method, Macaulay method, and Moment-area method – Columns – End conditions – Equivalent length of a column – Euler equation – Slenderness ratio – Rankine formula for columns.

UNIT V ANALYSIS OF STRESSES IN TWO DIMENSIONS

Biaxial state of stresses – Thin cylindrical and spherical shells – Deformation in thin cylindrical and spherical shells – Biaxial stresses at a point – Stresses on inclined plane – Principal planes and stresses – Mohr's circle for biaxial stresses – Maximum shear stress – Strain energy in bending and torsion.

TEXT BOOKS

1. Popov, E.P., "Engineering Mechanics of Solids", Prentice Hall of India, 1997.
2. Beer, F.P. and Johnston, R.," Mechanics of Materials", 3rd Edition, McGraw-Hill Book Co, 2002.

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1. Nash, W.A., "Theory and Problems in Strength of Materials", Schaum Outline Series, McGraw-Hill Book Co, 1995.
2. Kazimi, S.M.A., "Solid Mechanics", Tata McGraw-Hill Publishing Co., 1981.
3. Timoshenko, S.P., "Elements of Strength of Materials", Tata McGraw-Hill, 1997.

UNIT I STRESS, STRAIN DEFORMATION OF SOLIDS

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INTRODUCTION

In materials science, the **strength** of a material is its ability to withstand an applied stress without failure. The applied stress may be tensile, compressive, or shear. It is a subject which deals with loads, elastic and forces acting on the material. For example, an external load applied to an elastic material or internal forces acting on the material. Deformation (e.g. bending) of the material is called **strain**, while the intensity of the internal resisting force is called **stress**. The strength of any material relies on three different type of analytical method: strength, stiffness and stability, where strength means load carrying capacity, stiffness means deformation or elongation, and stability means ability to maintain its initial configuration. Yield strength refers to the point on the engineering stress-strain curve (as opposed to true stress-strain curve) beyond which the material begins deformation that cannot be reversed upon removal of the loading. Ultimate strength refers to the point on the engineering stress-strain curve corresponding to the maximum stress.

A material's strength is dependent on its microstructure. The engineering processes to which a material is subjected can alter this microstructure. The variety of strengthening mechanisms that alter the strength of a material includes work hardening, solid solution strengthening, precipitation hardening and grain boundary strengthening and can be quantified and qualitatively explained. However, strengthening mechanisms are accompanied by the caveat that some mechanical properties of the material may degenerate in an attempt to make the material stronger. For example, in grain boundary strengthening, although yield strength is maximized with decreasing grain size, ultimately, very small grain sizes make the material brittle. In general, the yield strength of a material is an adequate indicator of the material's mechanical strength. Considered in tandem with the fact that the yield strength is the parameter that predicts plastic deformation in the material, one can make informed decisions on how to increase the strength of a material depending its micro structural properties and the desired end effect. Strength is considered in terms of compressive strength, tensile strength, and shear strength, namely the limit states of compressive stress, tensile stress and shear stress, respectively. The effects of dynamic loading are probably the most important practical part of the strength of materials, especially the problem of fatigue. Repeated loading often initiates brittle cracks, which grow slowly until failure occurs.

However, the term **strength of materials** most often refers to various methods of calculating stresses in structural members, such as beams, columns and shafts. The methods that can be employed to predict the response of a structure under loading and its susceptibility to various failure modes may take into account various properties of the materials other than material (yield or ultimate) strength. For example failure in buckling is dependent on material stiffness (Young's Modulus).

Engineering science is usually subdivided into number of topics such as

1. Solid Mechanics
2. Fluid Mechanics
3. Heat Transfer

The solid mechanics as a subject may be defined as a branch of applied mechanics that deals with behaviors of solid bodies subjected to various types of loadings. This is usually subdivided into further two streams i.e Mechanics of rigid bodies or simply Mechanics and Mechanics of deformable solids.

The mechanics of deformable solids which is branch of applied mechanics is known by several names i.e. strength of materials, mechanics of materials etc.

Mechanics of rigid bodies:

The mechanics of rigid bodies is primarily concerned with the static and dynamic behavior under external forces of engineering components and systems which are treated as infinitely strong and undeformable Primarily we deal here with the forces and motions associated with particles and rigid bodies.

Mechanics of deformable solids :

Mechanics of solids:

The mechanics of deformable solids is more concerned with the internal forces and associated changes in the geometry of the components involved. Of particular importance are the properties of the materials used, the strength of which will determine whether the components fail by breaking in service, and the stiffness of which will determine whether the amount of deformation they suffer is acceptable. Therefore, the subject of mechanics of materials or strength of materials is central to the whole activity of engineering design. Usually the objectives in analysis here will be the determination of the stresses, strains, and deflections produced by loads. Theoretical analyses and experimental results have an equal roles in this field.

Analysis of stress and strain :

Concept of stress : Let us introduce the concept of stress as we know that the main problem of engineering mechanics of material is the investigation of the internal resistance of the body, i.e. the nature of forces set up within a body to balance the effect of the externally applied forces.

The externally applied forces are termed as loads. These externally applied forces may be due to any one of the reason.

(i) due to service conditions

(ii) due to environment in which the component works

(iii) through contact with other members

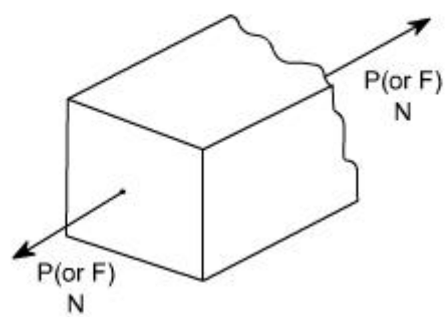
(iv) due to fluid pressures

(v) due to gravity or inertia forces.

As we know that in mechanics of deformable solids, externally applied forces acts on a body and body suffers a deformation. From equilibrium point of view, this action should be opposed or reacted by internal forces which are set up within the particles of material due to cohesion.

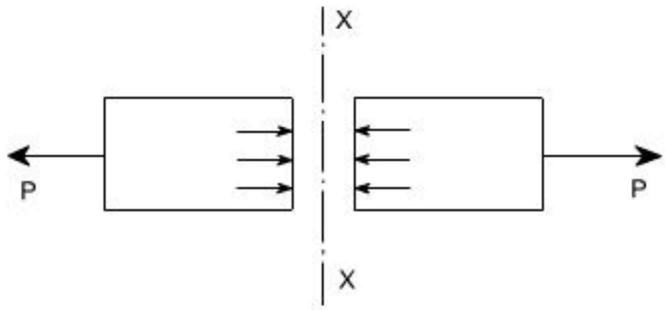
These internal forces give rise to a concept of stress. Therefore, let us define a stress Therefore, let us define a term stress

Stress:



Let us consider a rectangular bar of some cross – sectional area and subjected to some load or force (in Newtons)

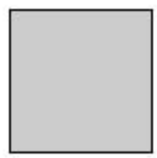
Let us imagine that the same rectangular bar is assumed to be cut into two halves at section XX. The each portion of this rectangular bar is in equilibrium under the action of load P and the internal forces acting at the section XX has been shown



Now stress is defined as the force intensity or force per unit area. Here we use a symbol s to represent the stress.

$$\sigma = \frac{P}{A}$$

Where A is the area of the X – section



Here we are using an assumption that the total force or total load carried by the rectangular bar is uniformly distributed over its cross – section.

But the stress distributions may be for from uniform, with local regions of high stress known as stress concentrations. If the force carried by a component is not uniformly distributed over its cross – sectional area, A, we must consider a small area, ‘dA' which carries a small load dP, of the total force ‘P', Then definition of stress is

$$\boxed{\sigma = \frac{\delta F}{\delta A}}$$

As a particular stress generally holds true only at a point, therefore it is defined mathematically as

$$\boxed{\sigma = \lim_{\delta A \to 0} \frac{\delta F}{\delta A}}$$

Units :

The basic units of stress in S.I units i.e. (International system) are N / m^2 (or Pa)

MPa = 10^6 Pa

GPa = 10^9 Pa

KPa = 10^3 Pa

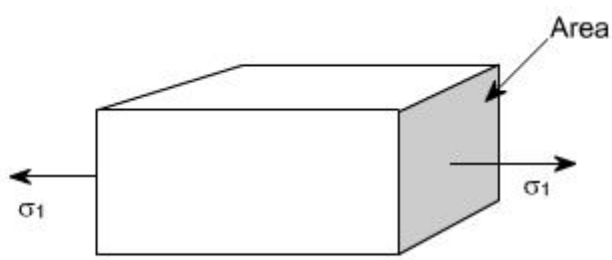
Some times N / mm^2 units are also used, because this is an equivalent to MPa. While US customary unit is pound per square inch psi.

TYPES OF STRESSES :

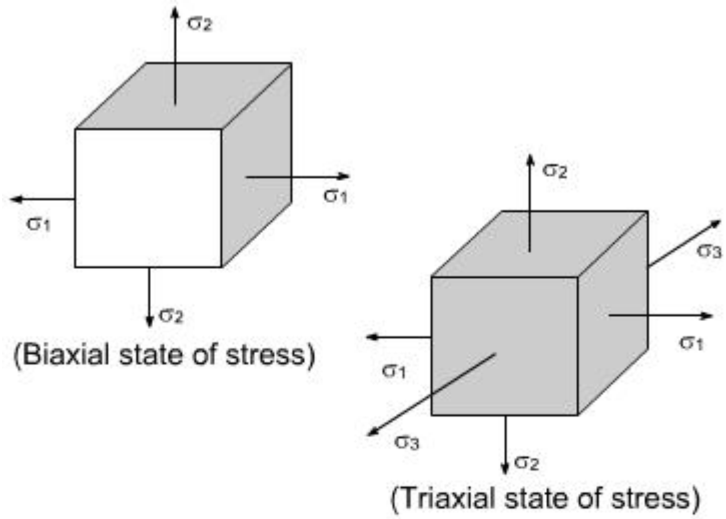
only two basic stresses exists : (1) normal stress and (2) shear shear stress. Other stresses either are similar to these basic stresses or are a combination of these e.g. bending stress is a combination tensile, compressive and shear stresses. Torsional stress, as encountered in twisting of a shaft is a shearing stress.

Let us define the normal stresses and shear stresses in the following sections.

Normal stresses : We have defined stress as force per unit area. If the stresses are normal to the areas concerned, then these are termed as normal stresses. The normal stresses are generally denoted by a Greek letter (s)

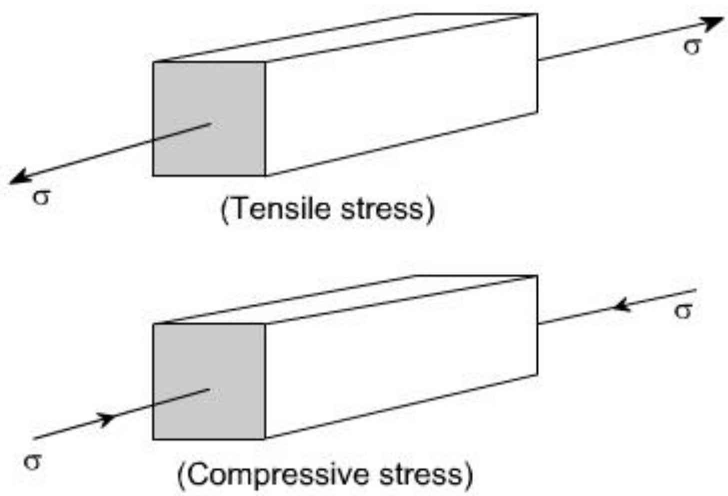


This is also known as uniaxial state of stress, because the stresses acts only in one direction however, such a state rarely exists, therefore we have biaxial and triaxial state of stresses where either the two mutually perpendicular normal stresses acts or three mutually perpendicular normal stresses acts as shown in the figures below :

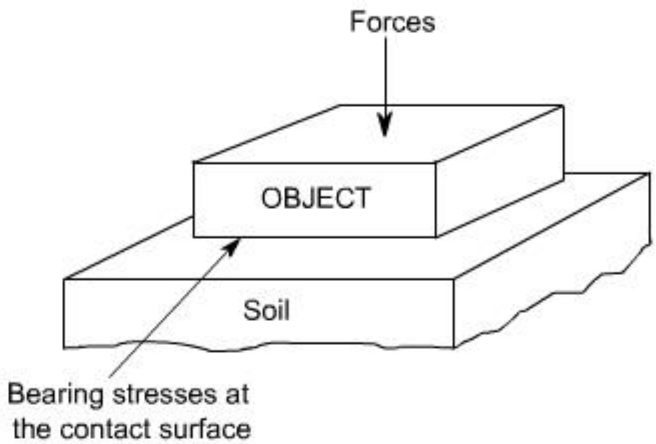


Tensile or compressive stresses :

The normal stresses can be either tensile or compressive whether the stresses acts out of the area or into the area

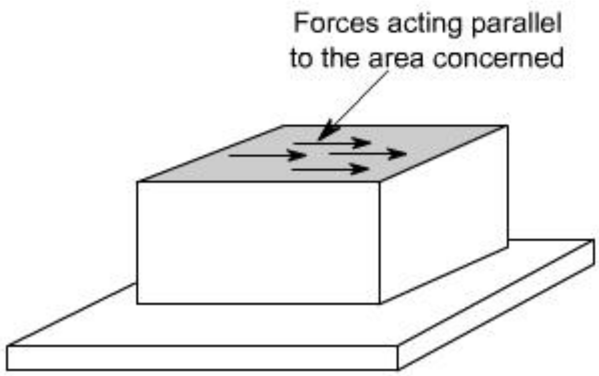


Bearing Stress : When one object presses against another, it is referred to a bearing stress (They are in fact the compressive stresses).



Shear stresses :

Let us consider now the situation, where the cross – sectional area of a block of material is subject to a distribution of forces which are parallel, rather than normal, to the area concerned. Such forces are associated with a shearing of the material, and are referred to as shear forces. The resulting force interistes are known as shear stresses.



The resulting force intensities are known as shear stresses, the mean shear stress being equal to

$$\tau = \frac{P}{A}$$

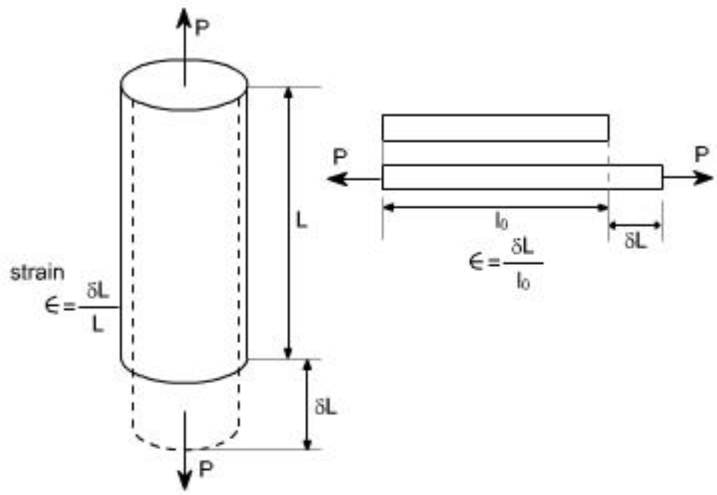
Where P is the total force and A the area over which it acts.

CONCEPT OF STRAIN

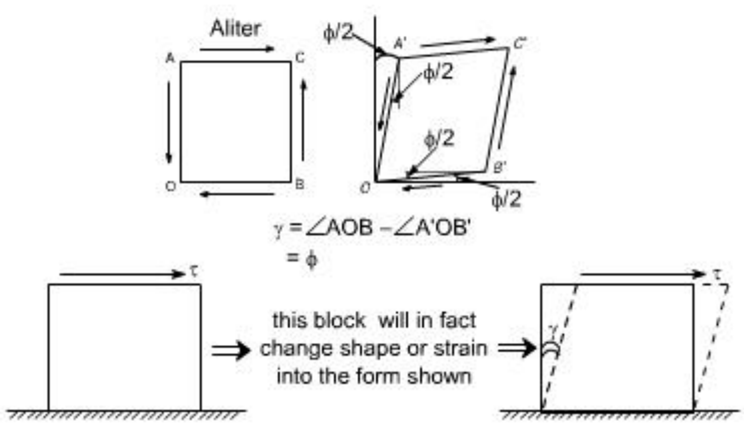
<u>Concept of strain :</u> if a bar is subjected to a direct load, and hence a stress the bar will change in length. If the bar has an original length L and changes by an amount dL, the strain produce is defined as follows:

$$\text{strain}(\varepsilon) = \frac{\text{change in length}}{\text{orginal length}} = \frac{\delta L}{L}$$

Strain is thus, a measure of the deformation of the material and is a nondimensional Quantity i.e. it has no units. It is simply a ratio of two quantities with the same unit.



Shear strain: As we know that the shear stresses acts along the surface. The action of the stresses is to produce or being about the deformation in the body consider the distortion produced b shear sheer stress on an element or rectangular block



This shear strain or slide is f and can be defined as the change in right angle. or The angle of deformation g is then termed as the shear strain. Shear strain is measured in radians & hence is non – dimensional i.e. it has no unit.So we have two types of strain i.e. normal stress & shear stresses.

Hook's Law :

A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.

Hook's law therefore states that

Stress (s) a strain(Î)

$$\boxed{\frac{\text{stress}}{\text{strain}} = \text{constant}}$$

Modulus of elasticity : Within the elastic limits of materials i.e. within the limits in which Hook's law applies, it has been shown that

Stress / strain = constant

This constant is given by the symbol E and is termed as the modulus of elasticity or Young's modulus of elasticity

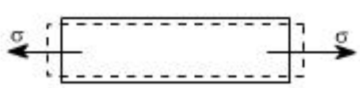
$$E = \frac{\text{strain}}{\text{stress}} = \frac{\sigma}{\epsilon}$$

$$= P/A \big/ \delta L/L$$

Thus $\boxed{E = \frac{PL}{A\delta L}}$

The value of Young's modulus E is generally assumed to be the same in tension or compression and for most engineering material has high, numerical value of the order of 200 GPa

Poisson's ratio: If a bar is subjected to a longitudinal stress there will be a strain in this direction equal to s / E . There will also be a strain in all directions at right angles to s . The final shape being shown by the dotted lines.



It has been observed that for an elastic materials, the lateral strain is proportional to the longitudinal strain. The ratio of the lateral strain to longitudinal strain is known as the poison's ratio .

Poison's ratio (m) = - lateral strain / longitudinal strain

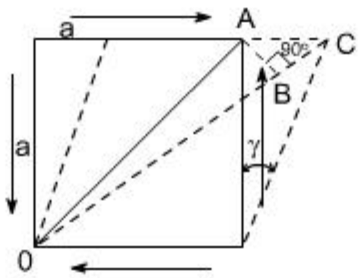
For most engineering materials the value of m his between 0.25 and 0.33.

RELATION AMONG ELASTIC CONSTANTS

Relation between E, G and u :

Let us establish a relation among the elastic constants E,G and u. Consider a cube of material of side ‘a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below.

Assuming that the strains are small and the angle A C B may be taken as 45^0.



Therefore strain on the diagonal OA

= Change in length / original length

Since angle between OA and OB is very small hence OA @ OB therefore BC, is the change in the length of the diagonal OA

$$\text{Thus, strain on diagonal OA} = \frac{BC}{OA}$$

$$= \frac{AC\cos 45^0}{OA}$$

$$OA = \frac{a}{\sin 45^0} = a.\sqrt{2}$$

$$\text{hence} \qquad \text{strain} = \frac{AC}{a\sqrt{2}}.\frac{1}{\sqrt{2}}$$

$$= \frac{AC}{2a}$$

but AC = $a\gamma$

where γ = shear strain

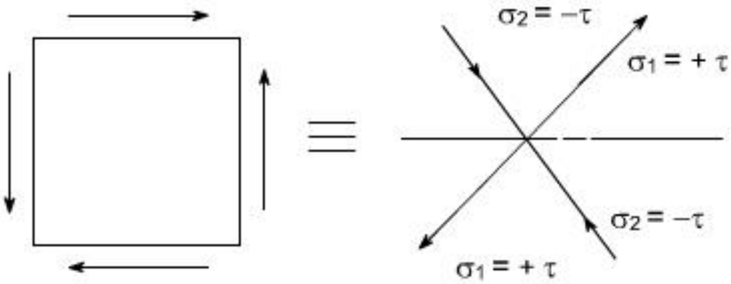
$$\text{Thus, the strain on diagonal} = \frac{a\gamma}{2a} = \frac{\gamma}{2}$$

From the definition

$$G = \frac{\tau}{\gamma} \text{ or } \gamma = \frac{\tau}{G}$$

$$\text{thus, the strain on diagonal} = \frac{\gamma}{2} = \frac{\tau}{2G}$$

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain.



Thus, for the direct state of stress system which applies along the diagonals:

$$\text{strain on diagonal} = \frac{\sigma_1}{E} - \mu\frac{\sigma_2}{E}$$
$$= \frac{\tau}{E} - \mu\frac{(-\tau)}{E}$$
$$= \frac{\tau}{E}(1+\mu)$$

equating the two strains one may get

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\mu)$$

or $\boxed{E = 2G(1+\mu)}$

We have introduced a total of four elastic constants, i.e E, G, K and g. It turns out that not all of these are independent of the others. Infact given any two of then, the other two can be found.

Again $E = 3K(1-2\gamma)$

$$\Rightarrow \frac{E}{3(1-2\gamma)} = K$$

if $\gamma = 0.5$ $K = \infty$

$$\epsilon_v = \frac{(1-2\gamma)}{E}(\epsilon_x + \epsilon_y + \epsilon_z) = 3\frac{\sigma}{E}(1-2\gamma)$$

(for $\epsilon_x = \epsilon_y = \epsilon_z$ hydrostatic state of stress)

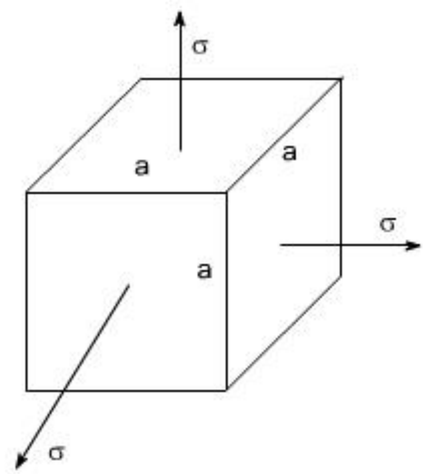
$\epsilon_v = 0$ if $\gamma = 0.5$

irrespective of the stresses i.e, the material is incompressible.

When g = 0.5 Value of k is infinite, rather than a zero value of E and volumetric strain is zero, or in other words, the material is incompressible.

Relation between E, K and u :

Consider a cube subjected to three equal stresses s as shown in the figure below



The total strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress s is given as

$$= \frac{\sigma}{E} - \gamma\frac{\sigma}{E} - \gamma\frac{\sigma}{E}$$

$$= \frac{\sigma}{E}(1-2\gamma)$$

volumetre strain = 3.linear strain

volumetre strain = $\epsilon_x + \epsilon_y + \epsilon_z$

or thus, $\epsilon_x = \epsilon_y = \epsilon_z$

volumetric strain = $3\frac{\sigma}{E}(1-2\gamma)$

By definition

Bulk Modulus of Elasticity (K) = $\frac{\text{Volumetric stress}(\sigma)}{\text{Volumetric strain}}$

or

Volumetric strain = $\frac{\sigma}{k}$

Equating the two strains we get

$$\frac{\sigma}{k} = 3.\frac{\sigma}{E}(1-2\gamma)$$

$$\boxed{E = 3K(1-2\gamma)}$$

<u>Relation between E, G and K :</u>

The relationship between E, G and K can be easily determained by eliminating u from the already derived relations

E = 2 G (1 + u) and E = 3 K (1 - u)

Thus, the following relationship may be obtained

$$\boxed{E = \frac{9\,GK}{(3K + G)}}$$

Relation between E, K and g :

From the already derived relations, E can be eliminated

$$E = 2G(1 + \gamma)$$
$$E = 3K(1 - 2\gamma)$$

Thus, we get

$$3k(1 - 2\gamma) = 2G(1 + \gamma)$$

therefore

$$\gamma = \frac{(3K - 2G)}{2(G + 3K)}$$

or

$$\boxed{\gamma = 0.5(3K - 2G)(G + 3K)}$$

Engineering Brief about the elastic constants :

We have introduced a total of four elastic constants i.e E, G, K and u. It may be seen that not all of these are independent of the others. Infact given any two of them, the other two can be determined. Further, it may be noted that

$$E = 3K(1 - 2\gamma)$$

or

$$K = \frac{E}{(1 - 2\gamma)}$$

if $\gamma = 0.5$; $K = \infty$

Also $\epsilon_v = \frac{(1 - 2\gamma)}{E}(\sigma_x + \sigma_y + \sigma_z)$

$= \frac{(1 - 2\gamma)}{E}.3\sigma$ (for hydrostatic state of stress i.e $\sigma_x = \sigma_y = \sigma_z = \sigma$)

hence if u = 0.5, the value of K becomes infinite, rather than a zero value of E and the volumetric strain is zero or in other words, the material becomes incompressible

Further, it may be noted that under condition of simple tension and simple shear, all real materials tend to experience displacements in the directions of the applied forces and Under hydrostatic loading they tend to increase in volume. In other words the value of the elastic constants E, G and K cannot be negative

Therefore, the relations

$$E = 2\,G\,(\,1 + u\,)$$

$$E = 3\,K\,(\,1 - u\,)$$

Yields $\boxed{-1 \leq \nu \leq 0.5}$

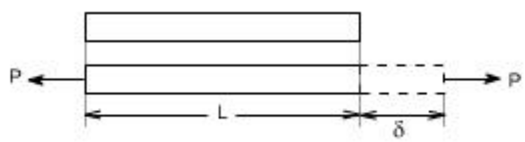
In actual practice no real material has value of Poisson's ratio negative . Thus, the value of u cannot be greater than 0.5, if however u > 0.5 than $\hat{I}_v$ = -ve, which is physically unlikely because when the material is stretched its volume would always increase.

Members Subjected to Uniaxial Stress

Members in Uni – axial state of stress

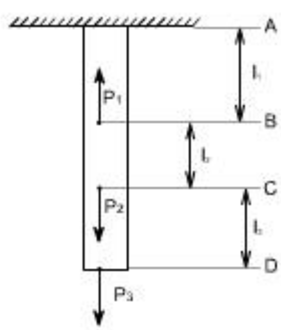
Introduction: [For members subjected to uniaxial state of stress]

For a prismatic bar loaded in tension by an axial force P, the elongation of the bar can be determined as



$$\delta = \frac{PL}{AE} \qquad \text{———} \quad (1)$$

Suppose the bar is loaded at one or more intermediate positions, then equation (1) can be readily adapted to handle this situation, i.e. we can determine the axial force in each part of the bar i.e. parts AB, BC, CD, and calculate the elongation or shortening of each part separately, finally, these changes in lengths can be added algebraically to obtain the total charge in length of the entire bar.



When either the axial force or the cross – sectional area varies continuosly along the axis of the bar, then equation (1) is no longer suitable. Instead, the elongation can be found by considering a deferential element of a bar and then the equation (1) becomes

$$d\delta = \frac{P_x dx}{E.A_x}$$

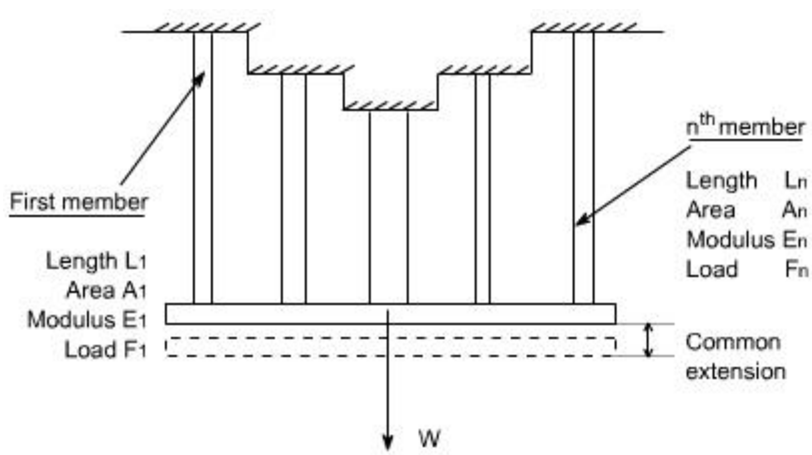
$$\delta = \int_0^l \frac{P_x dx}{E.A_x}$$

i.e. the axial force $P_x$and area of the cross – section A_x must be expressed as functions of x. If the expressions for $P_x$and A_x are not too complicated, the integral can be evaluated analytically, otherwise Numerical methods or techniques can be used to evaluate these integrals.

Thermal stresses, Bars subjected to tension and Compression

Compound bar: In certain application it is necessary to use a combination of elements or bars made from different materials, each material performing a different function. In over head electric cables or Transmission Lines for example it is often convenient to carry the current in a set of copper wires surrounding steel wires. The later being designed to support the weight of the cable over large spans. Such a combination of materials is generally termed compound bars.

Consider therefore, a compound bar consisting of n members, each having a different length and cross sectional area and each being of a different material. Let all member have a common extension 'x' i.e. the load is positioned to produce the same extension in each member.



<u>Energy Methods</u>

Strain Energy

Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy** :

Strain Energy in uniaxial Loading

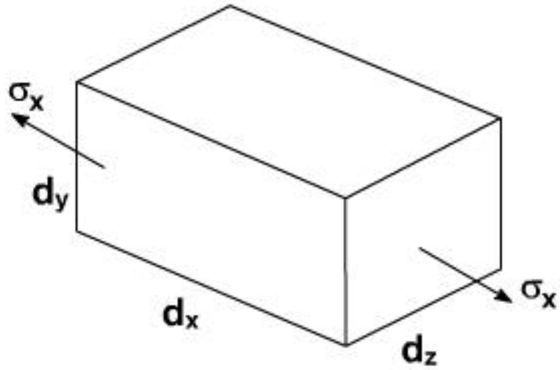


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress s_x.

The forces acting on the face of this element is s_x. dy. dz

where

dydz = Area of the element due to the application of forces, the element deforms to an amount = Î$_x$ dx

Î$_x$ = strain in the material in x – direction

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain as shown in Fig . 2.

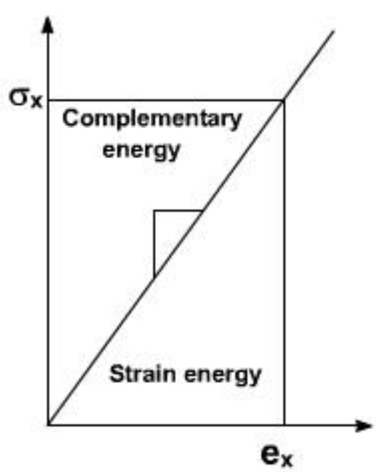


Fig .2

\ From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

Hence average force on the element is equal to ½ s_x . dy. dz.

\ Therefore the workdone by the above force

Force = average force x deformed length

= ½ s_x. dydz . $Î_x$. dx

For a perfectly elastic body the above work done is the internal strain energy “du”.

$$\boxed{du = \frac{1}{2}\sigma_x dydz\, \epsilon_x\, dx} \qquad .....(2)$$

$$= \frac{1}{2}\sigma_x \epsilon_x\, dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x\, \epsilon_x\, dv} \qquad ....(3)$$

where dv = dxdydz

= Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x\, \epsilon_x} \qquad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy – density ‘u_o' .

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\,\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^{\,2}}{2E} = \frac{E\,\epsilon_x^{\,2}}{2}} \qquad ......(5)$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^{\,2}}{2E} dv} \qquad ....(6)$$

In the case of a rod of uniform cross – section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

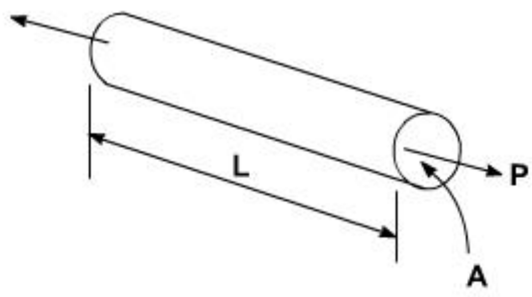


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^2}{2E} dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_0^L \frac{P^2}{2EA^2} A dx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.
L = Length of the bar

$$\boxed{U = \frac{P^2 L}{2AE}} \qquad .....(7)$$

Modulus of resilience :

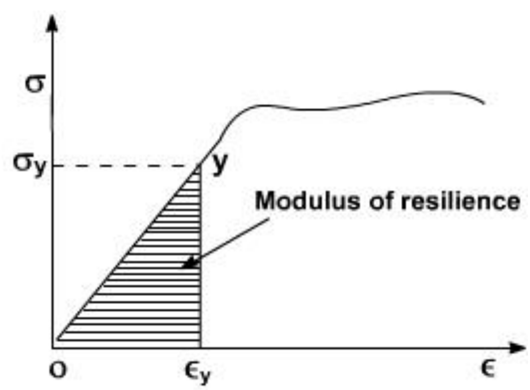


Fig .4

Suppose ' s_x' in strain energy equation is put equal to s_y i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

So $\boxed{U_y = \frac{\sigma_y^2}{2E}}$ .....(8)

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion 'OY' of the stress – strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

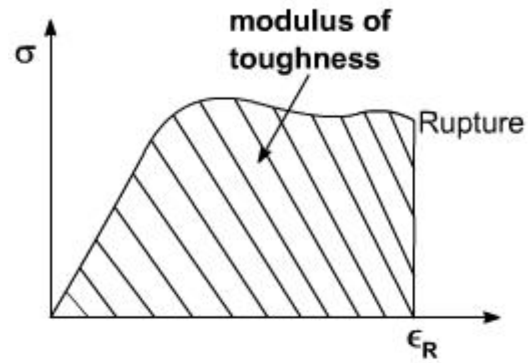


Fig .5

Suppose 'Î' [strain] in strain energy expression is replaced by $Î_R$ strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\epsilon_x \, dx = \frac{E\epsilon_R^2}{2} dv$$

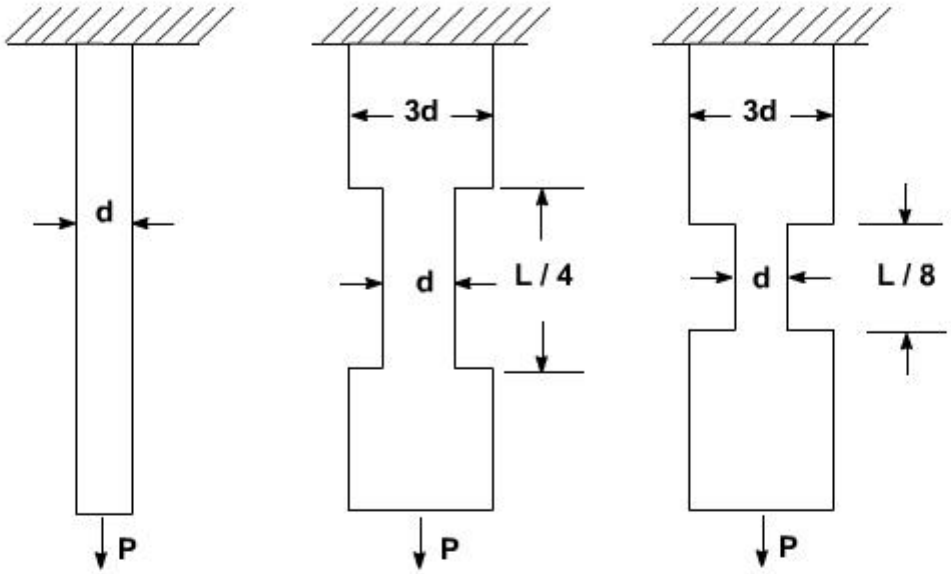
$$\boxed{U = \frac{E\epsilon_R^2}{2}} \qquad \ldots\ldots(9)$$

From the stress – strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length 'L' but different shapes are shown in fig below. The first bar has a diameter 'd' over its entire length, the second had this diameter over one – fourth of its length, and the third has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2 L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2 (L/4)}{2EA} + \frac{P^2 (3L/4)}{2E9A} = \frac{P^2 L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

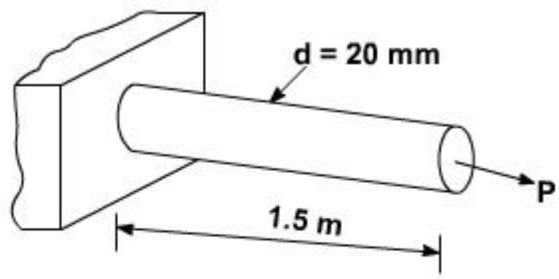
$$U_3 = \frac{P^2 (L/8)}{2EA} + \frac{P^2 (7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2 L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution :

Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4} d^2 L$$
$$= \frac{\pi}{4} 20 \times 1.5 \times 10^3$$
$$= 471 \times 10^3 \ mm^3$$

Yield Strength :

As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to s_x .

$$U = \frac{\sigma_y^2}{2E}$$
$$0.144 = \frac{\sigma_y^2}{2 \times (200 \times 10^3)}$$
$$\boxed{\sigma_y = 200 \text{ Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

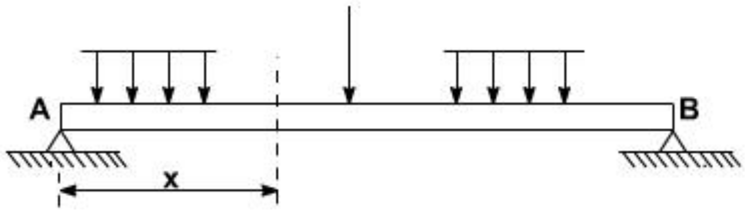


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\boxed{\sigma = \frac{MY}{I}}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$$= \int \frac{M^2 . y^2}{2EI^2} dv \qquad .....(10)$$

Substituting dv = dxdA

Where dA = elemental cross-sectional area

$\frac{M^2 . y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

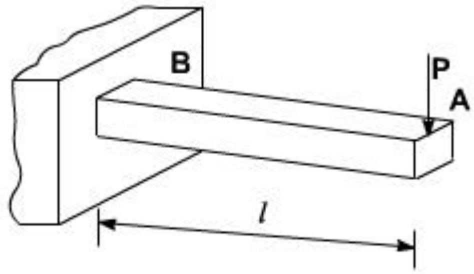
$$U = \int_0^L \frac{M^2}{2EI^2}\left(\int y^2 dA\right) dx \qquad .....(11)$$

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx} \qquad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.



Solution : The bending moment at a distance x from end A is defined as

$$\boxed{M = -Px}$$

Substituting the above value of M in the expression of strain energy we may write

$$U = \int_0^L \frac{P^2x^2}{2EI}\,dx$$

$$\boxed{U = \int_0^L \frac{P^2L^3}{EI}}$$

Problem 2 :

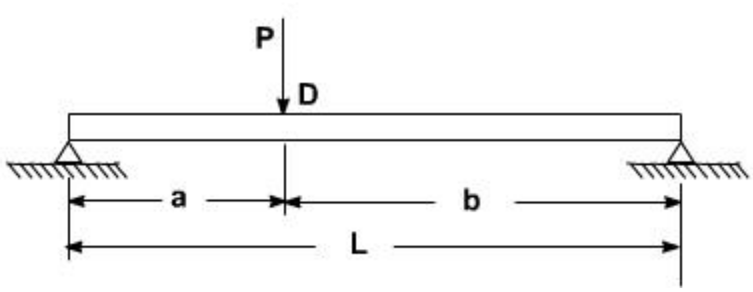
a. Determine the expression for strain energy of the prismatic beam AB for the loading as shown in figure below. Take into account only the effect of normal stresses due to bending.

b. Evaluate the strain energy for the following values of the beam

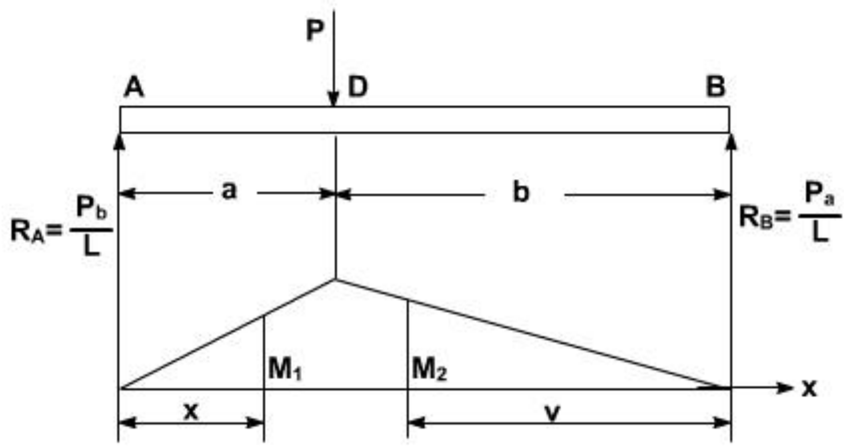
P = 208 KN ; L = 3.6 m = 3600 mm

A = 0.9 m = 90mm ; b = 2.7m = 2700 mm

E = 200 GPa ; I = 104 x 10^8 mm^4



Solution:

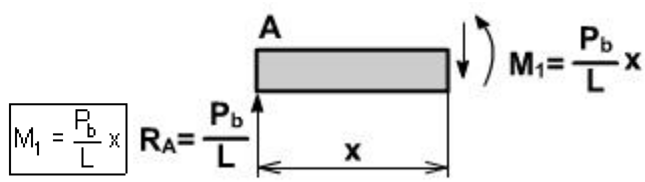


a.

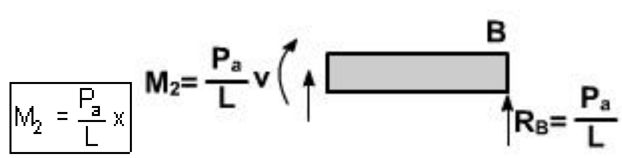
Bending Moment : Using the free – body diagram of the entire beam, we may determine the values of reactions as follows:

$R_A = P_b / L$ $R_B = P_a / L$

For Portion AD of the beam, the bending moment is



For Portion DB, the bending moment at a distance v from end B is



Strain Energy :

Since strain energy is a scalar quantity, we may add the strain energy of portion AD to that of DB to obtain the total strain energy of the beam.

$$U = U_{AD} + U_{DB}$$

$$= \int_0^a \frac{M_1^2}{2EI} dx + \int_0^b \frac{M_2^2}{2EI} dv$$

$$= \frac{1}{2EI} \int_0^a \left(\frac{P_b}{L} x \right)^2 dx + \frac{1}{2EI} \int_0^b \left(\frac{P_a}{L} v \right)^2 dx$$

$$= \frac{1}{2EI} \frac{P^2}{L^2} \left(\frac{b^2 a^3}{3} + \frac{a^2 b^3}{3} \right)$$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL^2} (a + b)}$$

Since$(a + b) = L$

$$\boxed{U = \frac{P^2 a^2 b^2}{6EIL}}$$

b. Substituting the values of P, a, b, E, I, and L in the expression above.

$$U = \frac{(200 \times 10^3)^2 \times (900)^2 \times (2700)^2}{6 (200 \times 10^3) \times (104 \times 10^6) \times (3600)} = 5.27 \times 10^7 \text{ KN.m}$$

Problem

3) Determine the modulus of resilience for each of the following materials.

a. Stainless steel . E = 190 GPa s_y = 260MPa

b. Malleable constantan E = 165GPa s_y = 230MPa

c. Titanium E = 115GPa s_y = 830MPa

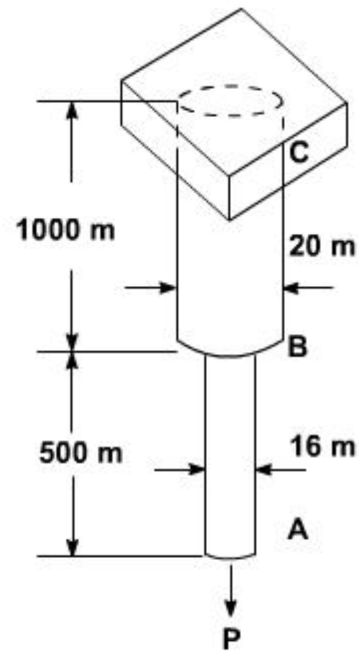
d. Magnesium E = 45GPa s_y = 200MPa

4) For the given Loading arrangement on the rod ABC determine

(a). The strain energy of the steel rod ABC when

P = 40 KN.

(b). The corresponding strain energy density in portions AB and BC of the rod.



UNIT I

STRESS STRAIN DEFORMATION OF SOLIDS

PART- A (2 Marks)

1. What is Hooke's Law?
2. What are the Elastic Constants?
3. Define Poisson's Ratio.
4. Define: Resilience, proof resilience and modulus of resilience.
5. Distinguish between rigid and deformable bodies.
6. Define stress and strain.
7. Define Shear stress and Shear strain.
8. Define elastic limit.
9. Define volumetric strain.
10. Define tensile stress and compressive stress.
11. Define young's Modulus.
12. Define modulus of rigidity.
13. Define thermal stress.

PART- B (16 Marks)

1. A rod of 150 cm long and diameter 2.0cm is subjected to an axial pull of 20 KN. If the modulus of elasticity of the material of the rod is 2x 105 N/mm2
Determine 1. Stress 2. Strain 3. the elongation of the rod

2. The extension in a rectangular steel bar of length 400mm and thickness 10mm is found to 0.21mm .The bar tapers uniformly in width from 100mm to 50mm. If E for the bar is 2x 105 N/mm2 ,Determine the axial load on the bar

UNIT II BEAMS - LOADS AND STRESSES

Types of beams: Supports and loads – Shear force and bending moment in beams – Cantilever, simply supported and overhanging beams – Stresses in beams – Theory of simple bending – Stress variation along the length and in the beam section – Effect of shape of beam section on stress induced – Shear stresses in beams – Shear flow.

Introduction:

In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These type of members are termed as beam.

There are various ways to define the beams such as

Definition I: A beam is a laterally loaded member, whose cross-sectional dimensions are small as compared to its length.

Definition II: A beam is nothing simply a bar which is subjected to forces or couples that lie in a plane containing the longitudnal axis of the bar. The forces are understood to act perpendicular to the longitudnal axis of the bar.

Definition III: A bar working under bending is generally termed as a beam.

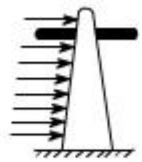
Materials for Beam:

The beams may be made from several usable engineering materials such commonly among them are as follows:

- Metal
- Wood
- Concrete
- Plastic

Examples of Beams:

Refer to the figures shown below that illustrates the beam



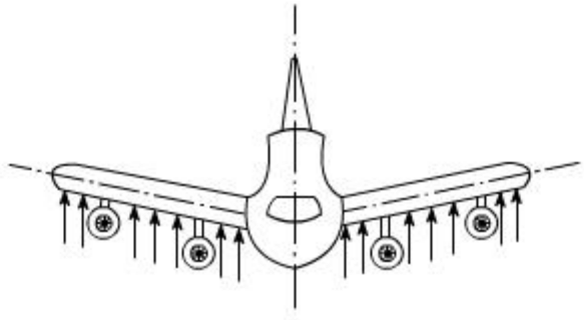


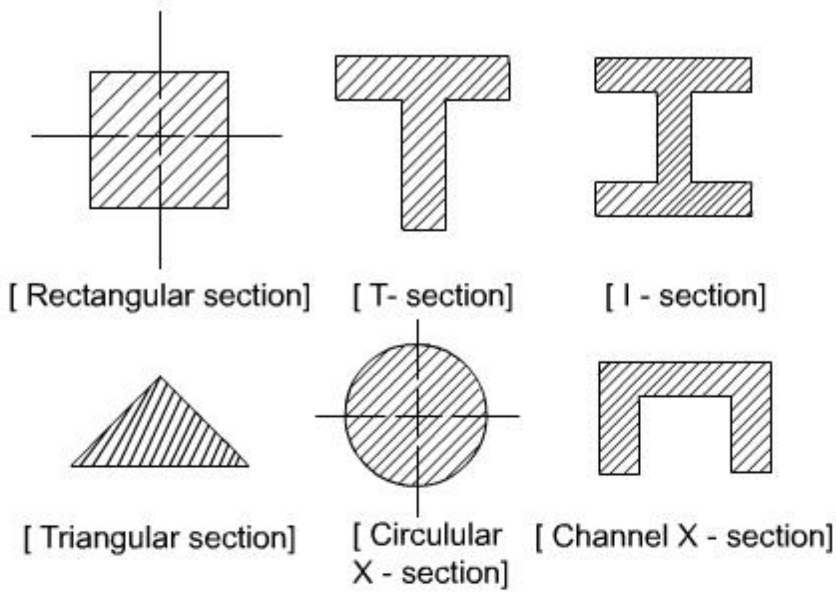
Fig 1 **Fig 2**

In the fig.1, an electric pole has been shown which is subject to forces occurring due to wind; hence it is an example of beam.

In the fig.2, the wings of an aeroplane may be regarded as a beam because here the aerodynamic action is responsible to provide lateral loading on the member.

Geometric forms of Beams:

The Area of X-section of the beam may take several forms some of them have been shown below:



Issues Regarding Beam:

Designer would be interested to know the answers to following issues while dealing with beams in practical engineering application

- At what load will it fail

- How much deflection occurs under the application of loads.

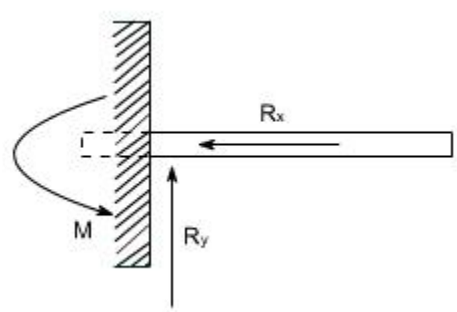
Classification of Beams:

Beams are classified on the basis of their geometry and the manner in which they are supported.

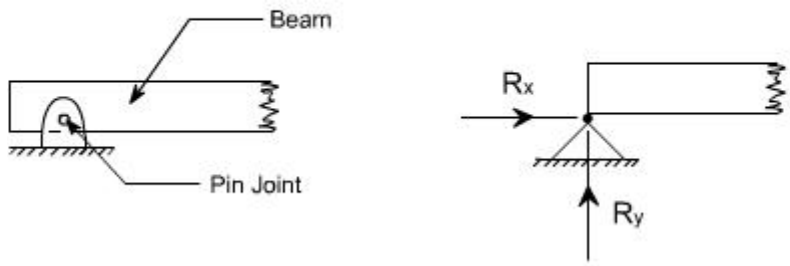
Classification I: The classification based on the basis of geometry normally includes features such as the shape of the X-section and whether the beam is straight or curved.

Classification II: Beams are classified into several groups, depending primarily on the kind of supports used. But it must be clearly understood why do we need supports. The supports are required to provide constrainment to the movement of the beams or simply the supports resists the movements either in particular direction or in rotational direction or both. As a consequence of this, the reaction comes into picture whereas to resist rotational movements the moment comes into picture. On the basis of the support, the beams may be classified as follows:

Cantilever Beam: A beam which is supported on the fixed support is termed as a cantilever beam: Now let us understand the meaning of a fixed support. Such a support is obtained by building a beam into a brick wall, casting it into concrete or welding the end of the beam. Such a support provides both the translational and rotational constrainment to the beam, therefore the reaction as well as the moments appears, as shown in the figure below

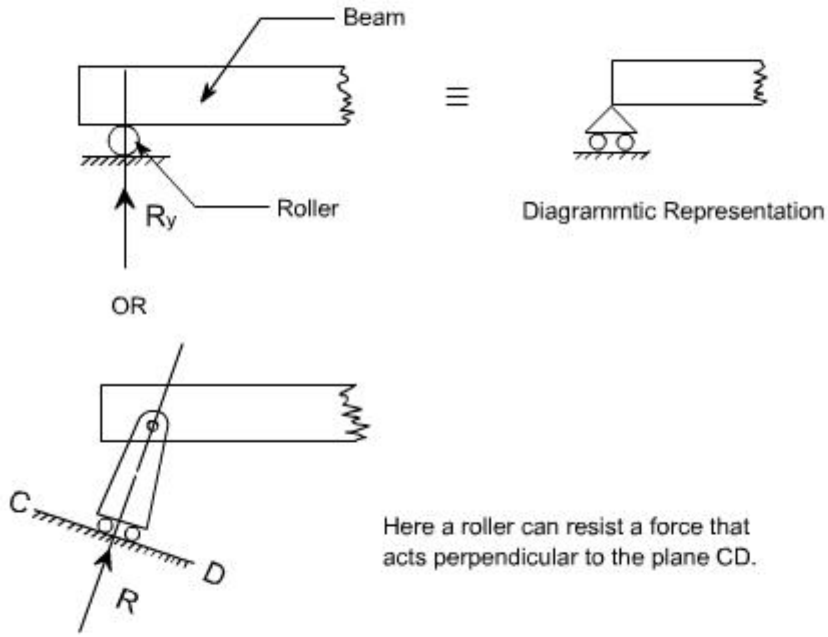


Simply Supported Beam: The beams are said to be simply supported if their supports creates only the translational constraints.



(a) Actual Representation (b) Diagrammatic Representation

Some times the translational movement may be allowed in one direction with the help of rollers and can be represented like this



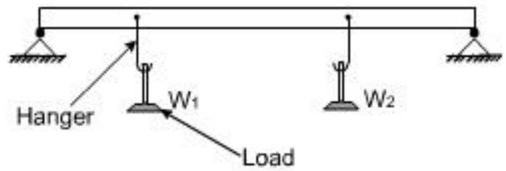
Statically Determinate or Statically Indeterminate Beams:

The beams can also be categorized as statically determinate or else it can be referred as statically indeterminate. If all the external forces and moments acting on it can be determined from the equilibrium conditions alone then. It would be referred as a statically determinate beam, whereas in the statically indeterminate beams one has to consider deformation i.e. deflections to solve the problem.

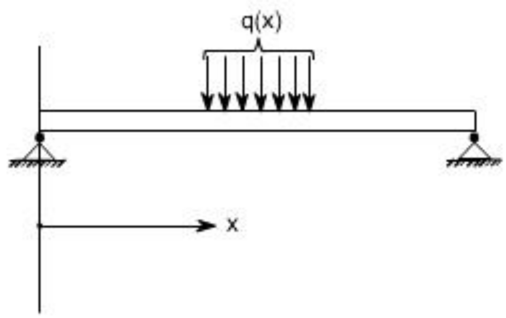
Types of loads acting on beams:

A beam is normally horizontal where as the external loads acting on the beams is generally in the vertical directions. In order to study the behaviors of beams under flexural loads. It becomes pertinent that one must be familiar with the various types of loads acting on the beams as well as their physical manifestations.

A. Concentrated Load: It is a kind of load which is considered to act at a point. By this we mean that the length of beam over which the force acts is so small in comparison to its total length that one can model the force as though applied at a point in two dimensional view of beam. Here in this case, force or load may be made to act on a beam by a hanger or though other means

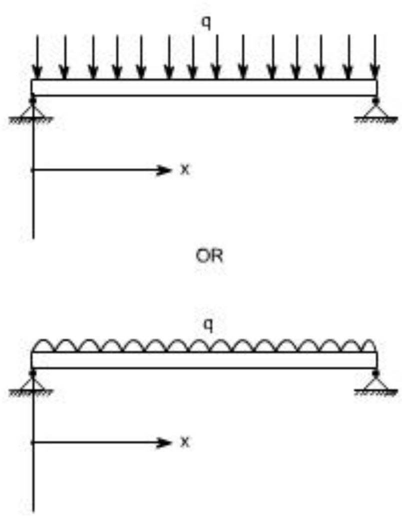


B. Distributed Load: The distributed load is a kind of load which is made to spread over a entire span of beam or over a particular portion of the beam in some specific manner

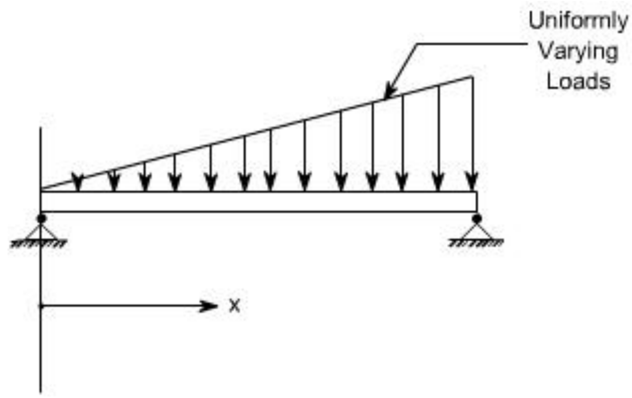


In the above figure, the rate of loading 'q' is a function of x i.e. span of the beam, hence this is a non uniformly distributed load.

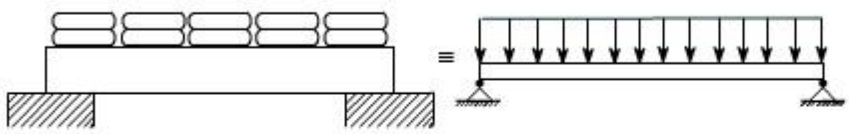
The rate of loading 'q' over the length of the beam may be uniform over the entire span of beam, then we cell this as a uniformly distributed load (U.D.L). The U.D.L may be represented in either of the way on the beams



some times the load acting on the beams may be the uniformly varying as in the case of dams or on inclind wall of a vessel containing liquid, then this may be represented on the beam as below:

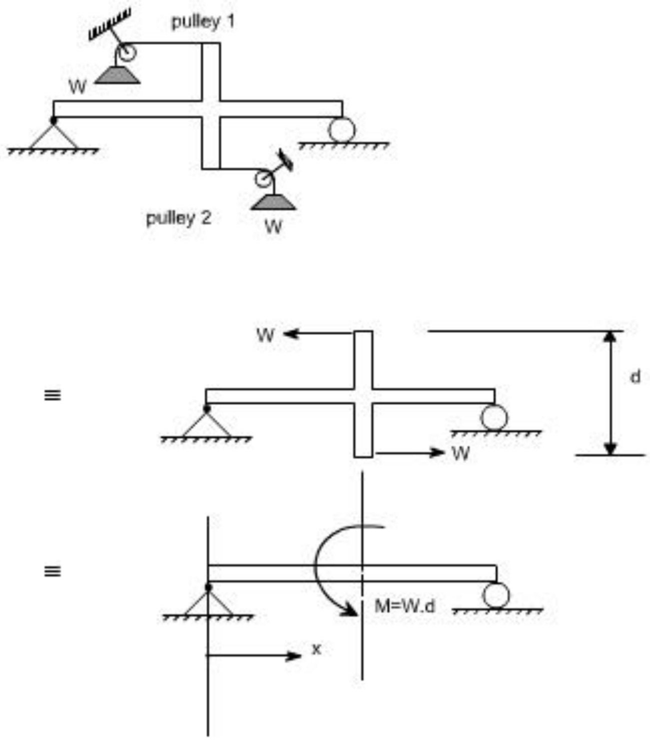


The U.D.L can be easily realized by making idealization of the ware house load, where the bags of grains are placed over a beam.



Concentrated Moment:

The beam may be subjected to a concentrated moment essentially at a point. One of the possible arrangement for applying the moment is being shown in the figure below:



Concept of Shear Force and Bending moment in beams:

When the beam is loaded in some arbitrarily manner, the internal forces and moments are developed and the terms shear force and bending moments come into pictures which are helpful to analyze the beams further. Let us define these terms

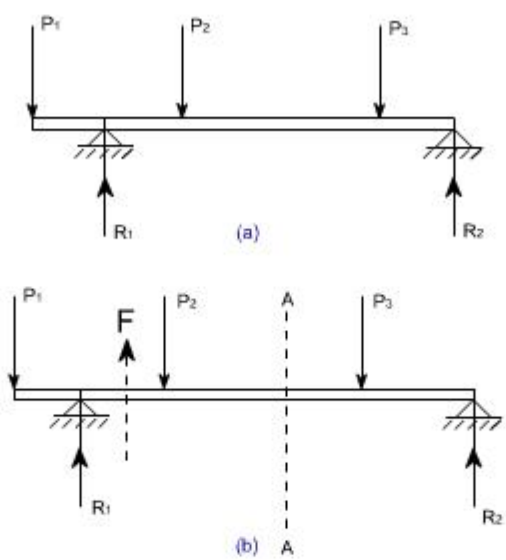


Fig 1

Now let us consider the beam as shown in fig 1(a) which is supporting the loads P_1, P_2, P_3 and is simply supported at two points creating the reactions R_1 and R_2 respectively. Now let us assume that the beam is to divided into or imagined to be cut into two portions at a section AA. Now let us assume that the resultant of loads and reactions to the left of AA is ‘F' vertically upwards, and since the entire beam is to remain in equilibrium, thus the resultant of forces to the right of AA must also be F, acting downwards. This forces ‘F' is as a shear force. The shearing force at any x-section of a beam represents the tendency for the portion of the beam to one side of the section to slide or shear laterally relative to the other portion.

Therefore, now we are in a position to define the shear force ‘F' to as follows:

At any x-section of a beam, the shear force ‘F' is the algebraic sum of all the lateral components of the forces acting on either side of the x-section.

Sign Convention for Shear Force:

The usual sign conventions to be followed for the shear forces have been illustrated in figures 2 and 3.

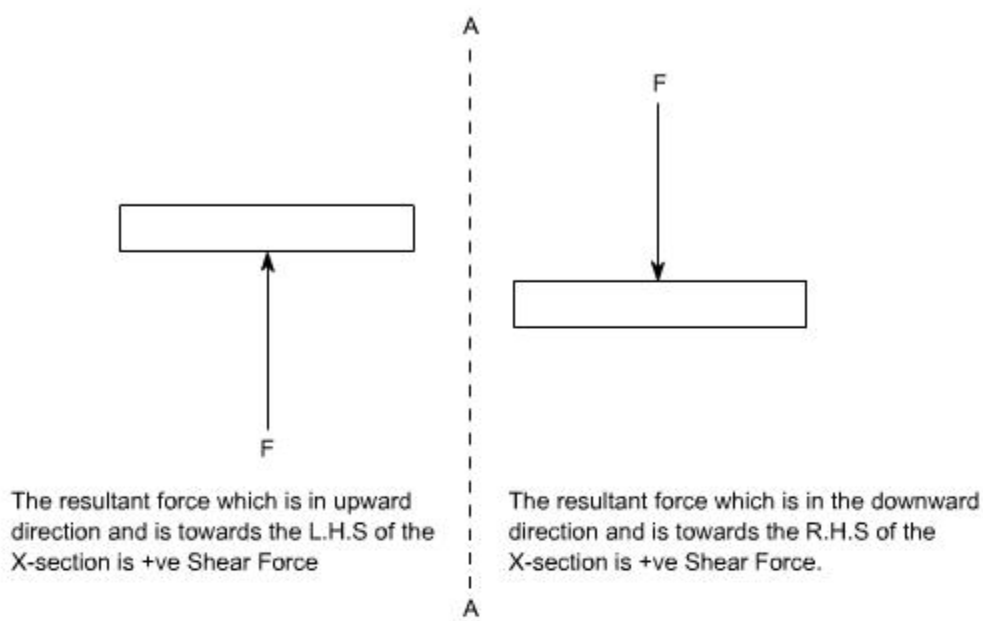


Fig 2: Positive Shear Force

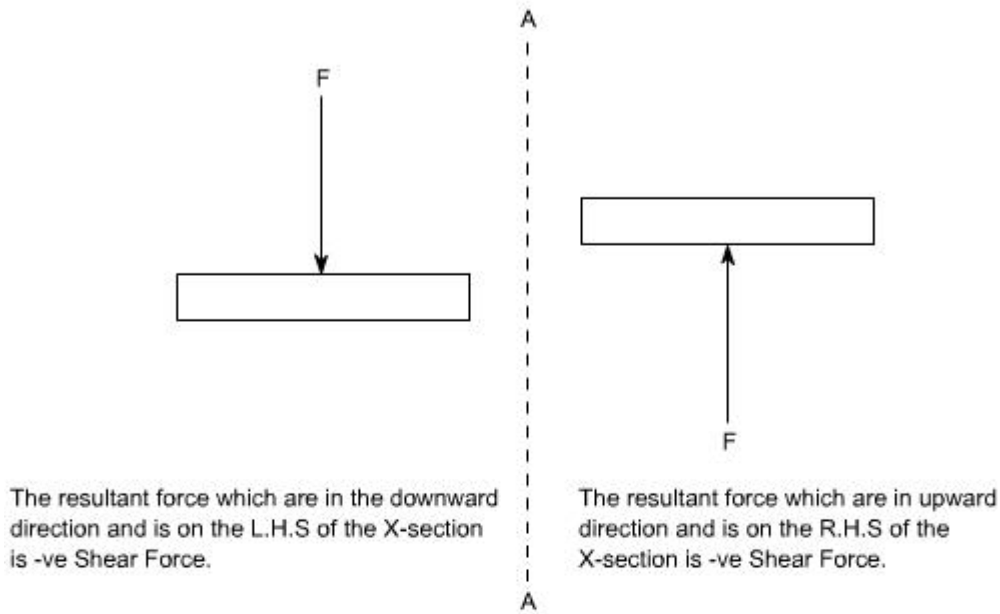


Fig 3: Negative Shear Force

Bending Moment:

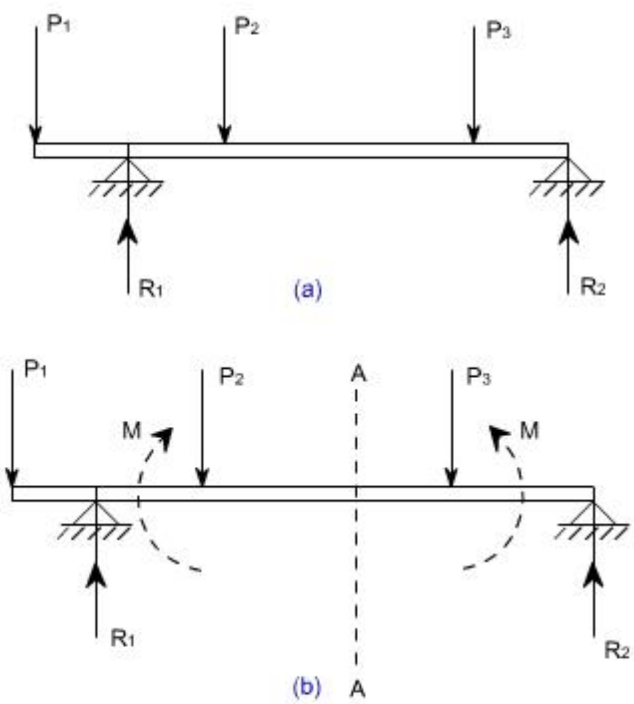


Fig 4

Let us again consider the beam which is simply supported at the two prints, carrying loads P_1, P_2 and P_3 and having the reactions R_1 and R_2 at the supports Fig 4. Now, let us imagine that the beam is cut into two potions at the x-section AA. In a similar manner, as done for the case of shear force, if we say that the resultant moment about the section AA of all the loads and reactions to the left of the x-section at AA is M in C.W direction, then moment of forces to the right of x-section AA must be 'M' in C.C.W. Then 'M' is called as the Bending moment and is abbreviated as B.M. Now one can define the bending moment to be simply as the algebraic sum of the moments about an x-section of all the forces acting on either side of the section

Sign Conventions for the Bending Moment:

For the bending moment, following sign conventions may be adopted as indicated in Fig 5 and Fig 6.

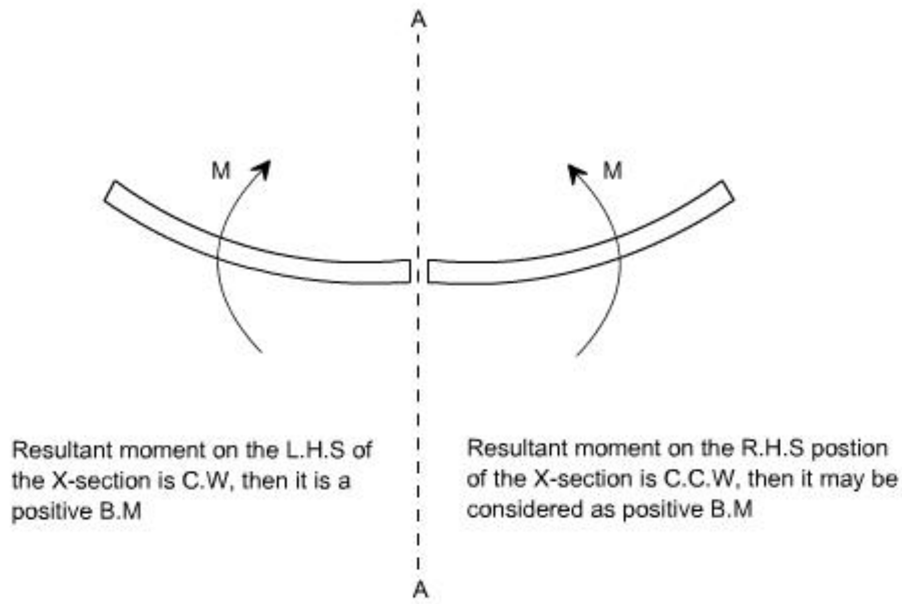


Fig 5: Positive Bending Moment

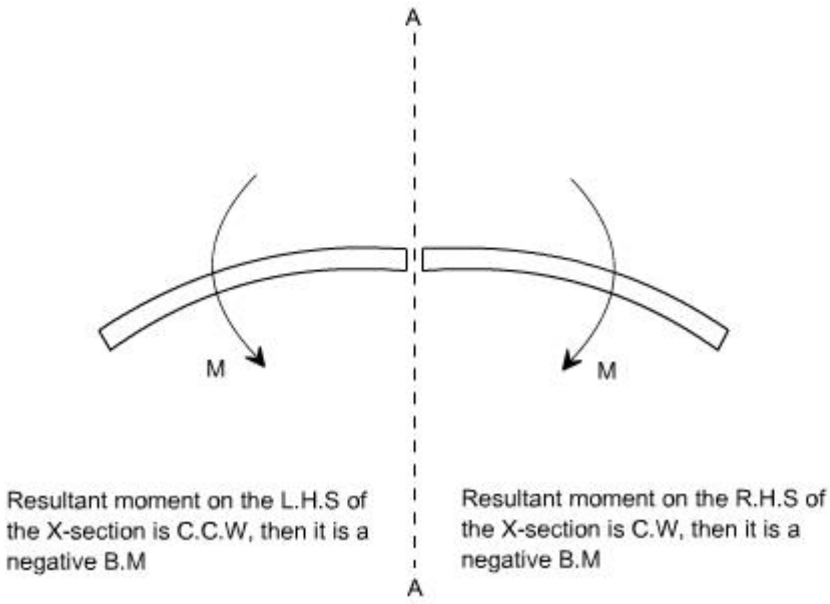


Fig 6: Negative Bending Moment

Some times, the terms ‘Sagging' and Hogging are generally used for the positive and negative bending moments respectively.

Bending Moment and Shear Force Diagrams:

The diagrams which illustrate the variations in B.M and S.F values along the length of the beam for any fixed loading conditions would be helpful to analyze the beam further.

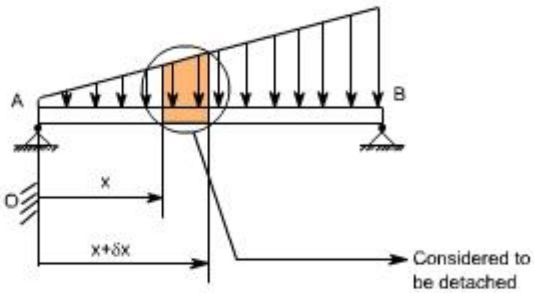
Thus, a shear force diagram is a graphical plot, which depicts how the internal shear force ‘F' varies along the length of beam. If x dentotes the length of the beam, then F is function x i.e. F(x).

Similarly a bending moment diagram is a graphical plot which depicts how the internal bending moment ‘M' varies along the length of the beam. Again M is a function x i.e. M(x).

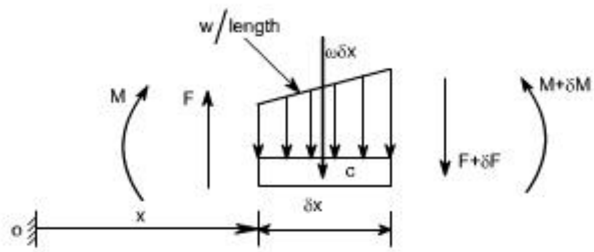
Basic Relationship Between The Rate of Loading, Shear Force and Bending Moment:

The construction of the shear force diagram and bending moment diagrams is greatly simplified if the relationship among load, shear force and bending moment is established.

Let us consider a simply supported beam AB carrying a uniformly distributed load w/length. Let us imagine to cut a short slice of length dx cut out from this loaded beam at distance ‘x' from the origin ‘0'.



Let us detach this portion of the beam and draw its free body diagram.



The forces acting on the free body diagram of the detached portion of this loaded beam are the following

• The shearing force F and F+ **d**F at the section x and x + **d**x respectively.

• The bending moment at the sections x and x + **d**x be M and M + dM respectively.

• Force due to external loading, if ‘w' is the mean rate of loading per unit length then the total loading on this slice of length **d**x is w. **d**x, which is approximately acting through the centre ‘c'. If the loading is assumed to be uniformly distributed then it would pass exactly through the centre ‘c'.

This small element must be in equilibrium under the action of these forces and couples.

Now let us take the moments at the point ‘c'. Such that

$$M + F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = M + \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + (F + \delta F).\frac{\delta x}{2} = \delta M$$

$$\Rightarrow F.\frac{\delta x}{2} + F.\frac{\delta x}{2} + \delta F.\frac{\delta x}{2} = \delta M \text{ [Neglecting the product of}$$

$$\delta F \text{ and } \delta x \text{ being small quantities]}$$

$$\Rightarrow F.\delta x = \delta M$$

$$\Rightarrow F = \frac{\delta M}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\boxed{F = \frac{dM}{dx}} \qquad \ldots\ldots\ldots\ldots \qquad (1)$$

Re solving the forces vertically we get

$$w.\delta x + (F + \delta F) = F$$

$$\Rightarrow w = -\frac{\delta F}{\delta x}$$

Under the limits $\delta x \rightarrow 0$

$$\Rightarrow w = -\frac{dF}{dx} \text{ or } -\frac{d}{dx}(\frac{dM}{dx})$$

$$\boxed{w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}} \qquad \ldots\ldots\ldots\ldots (2)$$

Conclusions: From the above relations,the following important conclusions may be drawn

• From Equation (1), the area of the shear force diagram between any two points, from the basic calculus is the bending moment diagram

$$M = \int F.dx$$

• The slope of bending moment diagram is the shear force,thus

$$F = \frac{dM}{dx}$$

Thus, if F=0; the slope of the bending moment diagram is zero and the bending moment is therefore constant.'

• The maximum or minimum Bending moment occurs where $\frac{dM}{dx} = 0.$

The slope of the shear force diagram is equal to the magnitude of the intensity of the distributed loading at any position along the beam. The –ve sign is as a consequence of our particular choice of sign conventions

Procedure for drawing shear force and bending moment diagram:

Preamble:

The advantage of plotting a variation of shear force F and bending moment M in a beam as a function of ‘x' measured from one end of the beam is that it becomes easier to determine the maximum absolute value of shear force and bending moment.

Further, the determination of value of M as a function of ‘x' becomes of paramount importance so as to determine the value of deflection of beam subjected to a given loading.

Construction of shear force and bending moment diagrams:

A shear force diagram can be constructed from the loading diagram of the beam. In order to draw this, first the reactions must be determined always. Then the vertical components of forces and reactions are successively summed from the left end of the beam to preserve the mathematical sign conventions adopted. The shear at a section is simply equal to the sum of all the vertical forces to the left of the section.

When the successive summation process is used, the shear force diagram should end up with the previously calculated shear (reaction at right end of the beam. No shear force acts through the beam just beyond the last vertical force or reaction. If the shear force diagram closes in this fashion, then it gives an important check on mathematical calculations.

The bending moment diagram is obtained by proceeding continuously along the length of beam from the left hand end and summing up the areas of shear force diagrams giving due regard to sign. The process of obtaining the moment diagram from the shear force diagram by summation is exactly the same as that for drawing shear force diagram from load diagram.

It may also be observed that a constant shear force produces a uniform change in the bending moment, resulting in straight line in the moment diagram. If no shear force exists along a certain portion of a beam, then it indicates that there is no change in moment takes place. It may also further observe that dm/dx= F therefore, from the fundamental theorem of calculus the maximum or minimum moment occurs where the shear is zero. In order to check the validity of the bending moment diagram, the terminal conditions for the moment must be satisfied. If the end is free or pinned, the computed sum must be equal to zero. If the end is built in, the moment computed by the summation must be equal to the one calculated initially for the reaction. These conditions must always be satisfied.

Illustrative problems:

In the following sections some illustrative problems have been discussed so as to illustrate the procedure for drawing the shear force and bending moment diagrams

1. A cantilever of length carries a concentrated load 'W' at its free end.

Draw shear force and bending moment.

Solution:

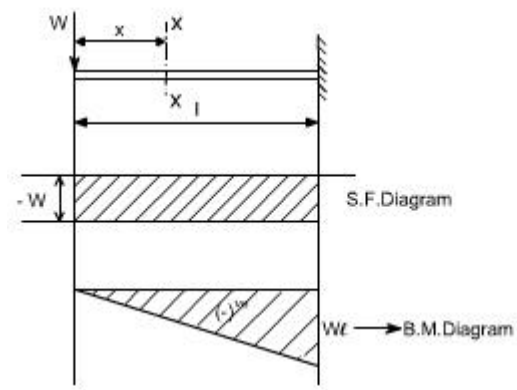
At a section a distance x from free end consider the forces to the left, then F = -W (for all values of x) -ve sign means the shear force to the left of the x-section are in downward direction and therefore negative

Taking moments about the section gives (obviously to the left of the section)

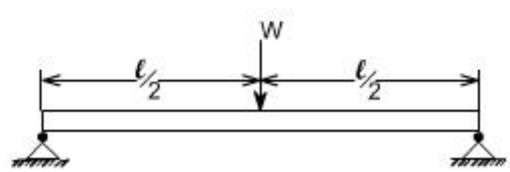
M = -Wx (-ve sign means that the moment on the left hand side of the portion is in the anticlockwise direction and is therefore taken as –ve according to the sign convention)

so that the maximum bending moment occurs at the fixed end i.e. M = -W l

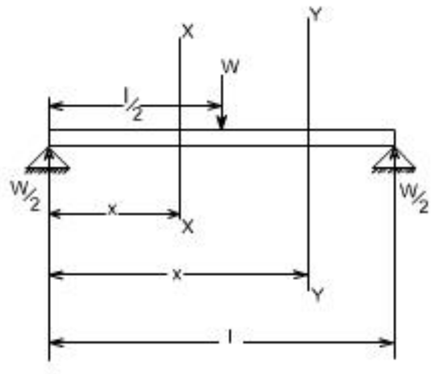
From equilibrium consideration, the fixing moment applied at the fixed end is Wl and the reaction is W. the shear force and bending moment are shown as,



2. Simply supported beam subjected to a central load (i.e. load acting at the mid-way)



By symmetry the reactions at the two supports would be W/2 and W/2. now consider any section X-X from the left end then, the beam is under the action of following forces.

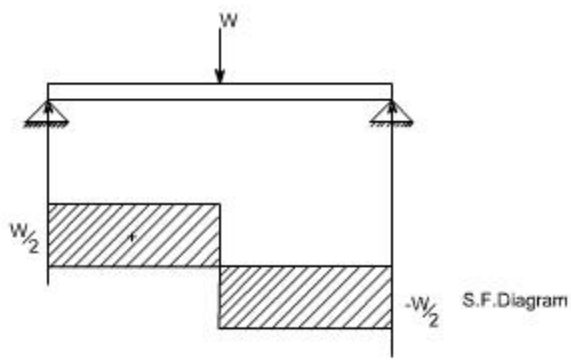


.So the shear force at any X-section would be = W/2 [Which is constant upto x < l/2]

If we consider another section Y-Y which is beyond l/2 then

$S.F_{Y-Y} = \frac{W}{2} - W = \frac{-W}{2}$ for all values greater = l/2

Hence S.F diagram can be plotted as,



.For B.M diagram:

If we just take the moments to the left of the cross-section,

$$B.M_{X-X} = \frac{W}{2}\, x \text{ for } x \text{ lies between } 0 \text{ and } l/2$$

$$B.M_{\text{at } x = \frac{l}{2}} = \frac{W}{2}\,\frac{l}{2} \text{ i.e } B.M \text{ at } x = 0$$

$$= \frac{Wl}{4}$$

$$B.M_{Y-Y} = \frac{W}{2}\,x - W\left(x - \frac{l}{2}\right)$$

Again

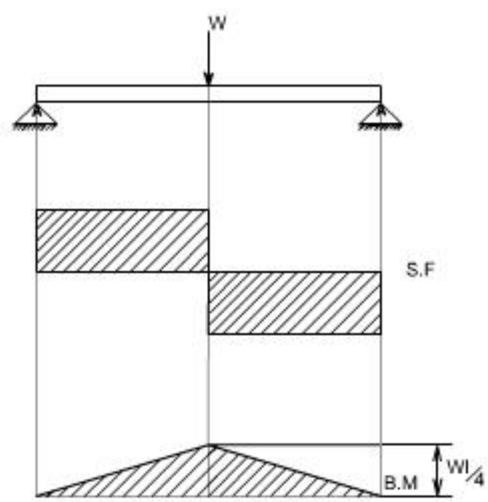
$$= \frac{W}{2}\,x - Wx + \frac{Wl}{2}$$

$$= -\frac{W}{2}\,x + \frac{Wl}{2}$$

$$B.M_{\text{at } x = l} = -\frac{Wl}{2} + \frac{Wl}{2}$$

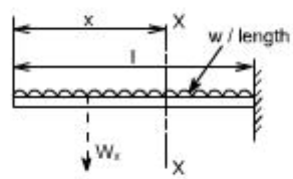
$$= 0$$

Which when plotted will give a straight relation i.e.



It may be observed that at the point of application of load there is an abrupt change in the shear force, at this point the B.M is maximum.

3. A cantilever beam subjected to U.d.L, draw S.F and B.M diagram.



Here the cantilever beam is subjected to a uniformly distributed load whose intensity is given w / length.

Consider any cross-section XX which is at a distance of x from the free end. If we just take the resultant of all the forces on the left of the X-section, then

$S.F_{xx} = -Wx$ for all values of 'x'. **---------- (1)**

$S.F_{xx} = 0$

$S.F_{xx \text{ at } x=1} = -Wl$

So if we just plot the equation No. (1), then it will give a straight line relation. Bending Moment at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting through the centre of gravity.

Therefore, the bending moment at any cross-section X-X is

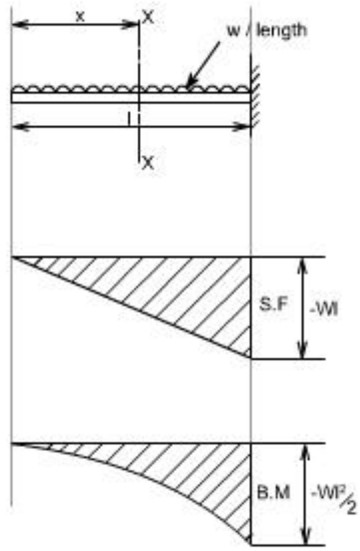
$$B.M_{X\text{-}X} = -Wx\,\frac{x}{2}$$
$$= -W\frac{x^2}{2}$$

The above equation is a quadratic in x, when B.M is plotted against x this will produces a parabolic variation.

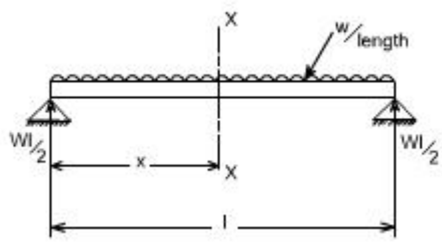
The extreme values of this would be at x = 0 and x = l

$$B.M_{\text{at } x=l} = -\frac{Wl^2}{2}$$
$$= \frac{Wl}{2} - Wx$$

Hence S.F and B.M diagram can be plotted as follows:



4. Simply supported beam subjected to a uniformly distributed load [U.D.L].



The total load carried by the span would be

= intensity of loading x length

= w x l

By symmetry the reactions at the end supports are each wl/2

If x is the distance of the section considered from the left hand end of the beam.

S.F at any X-section X-X is

$$= \frac{Wl}{2} - Wx$$
$$= W\left(\frac{l}{2} - x\right)$$

Giving a straight relation, having a slope equal to the rate of loading or intensity of the loading.

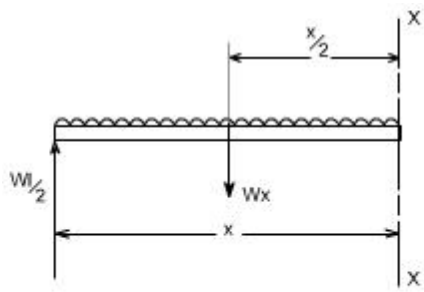
$$S.F_{\text{at } x=0} = \frac{wl}{2} - wx$$

so at

$$S.F_{\text{at } x=\frac{l}{2}} = 0 \text{ hence the S.F is zero at the centre}$$

$$S.F_{\text{at } x=l} = -\frac{Wl}{2}$$

The bending moment at the section x is found by treating the distributed load as acting at its centre of gravity, which at a distance of x/2 from the section



$$B.M_{x\text{-}x} = \frac{Wl}{2}x - Wx.\frac{x}{2}$$

so the

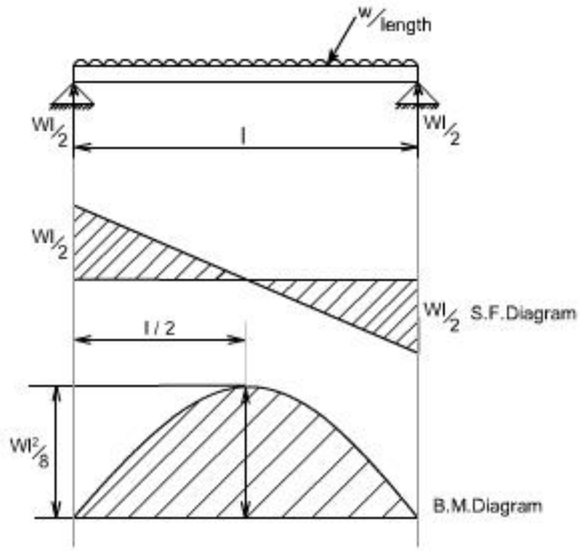
$$= W.\frac{x}{2}(l - 2) \quad \ldots\ldots(2)$$

$$B.M_{\text{at } x=0} = 0$$

$$B.M_{\text{at } x=l} = 0$$

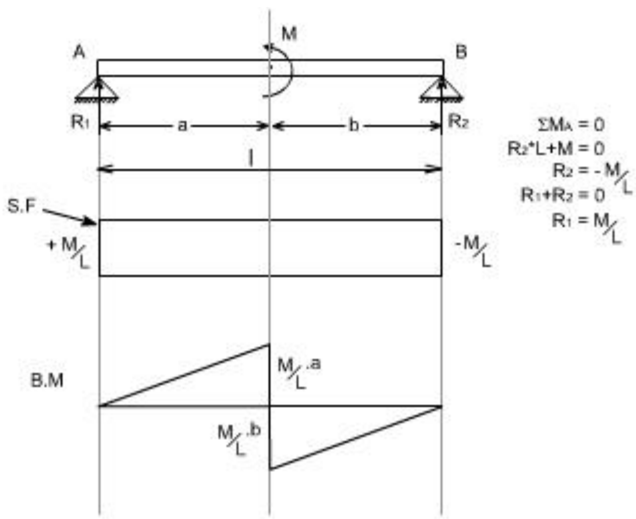
$$B.M\Big|_{\text{at } x=l} = -\frac{Wl^2}{8}$$

So the equation (2) when plotted against x gives rise to a parabolic curve and the shear force and bending moment can be drawn in the following way will appear as follows:



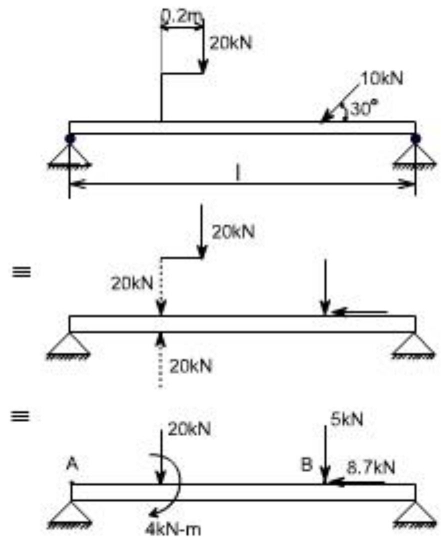
5. Couple.

When the beam is subjected to couple, the shear force and Bending moment diagrams may be drawn exactly in the same fashion as discussed earlier.



6. Eccentric loads.

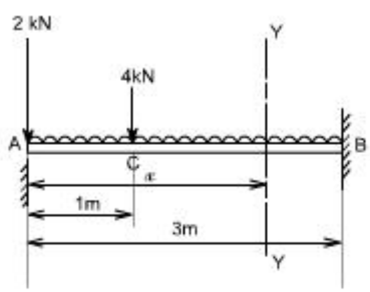
When the beam is subjected to an eccentric loads, the eccentric load are to be changed into a couple/ force as the case may be, In the illustrative example given below, the 20 kN load acting at a distance of 0.2m may be converted to an equivalent of 20 kN force and a couple of 2 kN.m. similarly a 10 kN force which is acting at an angle of 30^0 may be resolved into horizontal and vertical components.The rest of the procedure for drawing the shear force and Bending moment remains the same.



6. Loading changes or there is an abrupt change of loading:

When there is an aabrupt change of loading or loads changes, the problem may be tackled in a systematic way.consider a cantilever beam of 3 meters length. It carries a uniformly distributed load of 2 kN/m and a concentrated loads of 2kN at the free end and 4kN at 2 meters from fixed end.The shearing force and bending moment diagrams are required to be drawn and state the maximum values of the shearing force and bending moment.

Solution



Consider any cross section x-x, at a distance x from the free end

Shear Force at x-x = -2 -2x $\quad\quad 0 < x < 1$

S.F at x = 0 i.e. at A = -2 kN

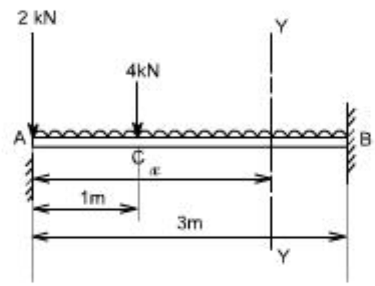
S.F at x = 1 = -2-2 = - 4kN

S.F at C (x = 1) = -2 -2x - 4 $\quad$ Concentrated load

= - 2 - 4 -2x1 kN

= - 8 kN

Again consider any cross-section YY, located at a distance x from the free end



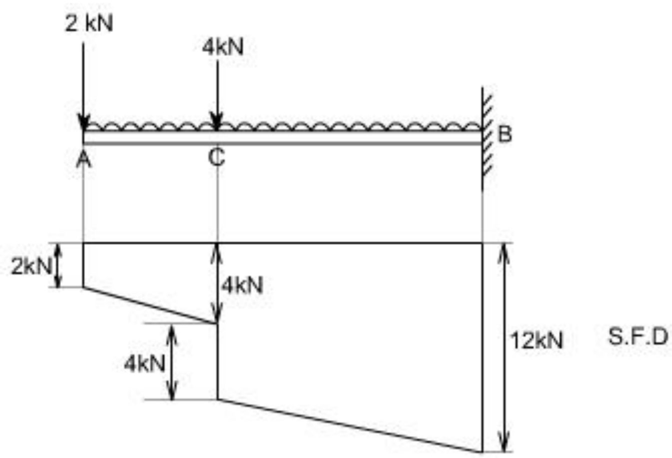
S.F at Y-Y = -2 - 2x - 4   1< x < 3

This equation again gives S.F at point C equal to -8kN

S.F at x = 3 m = -2 -4 -2x3

= -12 kN

Hence the shear force diagram can be drawn as below:



For bending moment diagrams – Again write down the equations for the respective cross sections, as consider above

Bending Moment at xx = -2x - 2x.x/2 valid upto AC

B.M at x = 0 = 0

B.M at x =1m = -3 kN.m

For the portion CB, the bending moment equation can be written for the x-section at Y-Y .

B.M at YY = -2x - 2x.x/2 - 4(x -1)

This equation again gives,

B.M at point C = - 2.1 - 1 - 0 i.e. at x = 1

= -3 kN.m

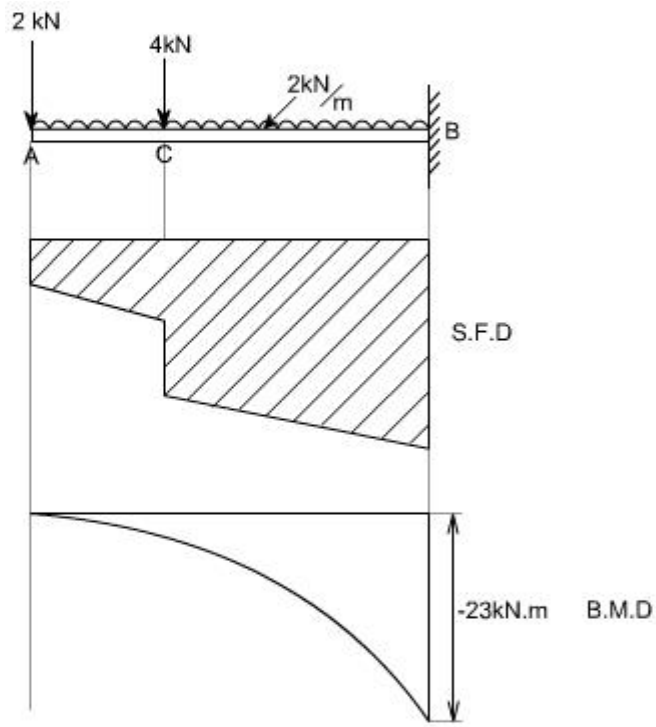
B.M at point B i.e. at x = 3 m

= - 6 - 9 - 8

= - 23 kN-m

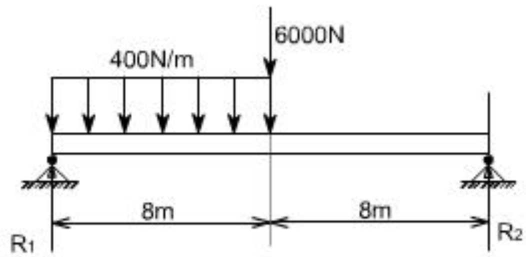
The variation of the bending moment diagrams would obviously be a parabolic curve

Hence the bending moment diagram would be



7. Illustrative Example :

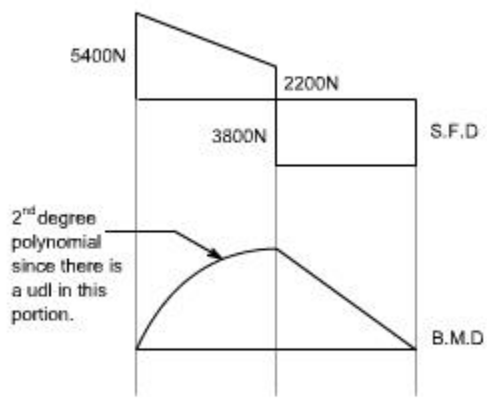
In this there is an abrupt change of loading beyond a certain point thus, we shall have to be careful at the jumps and the discontinuities.



For the given problem, the values of reactions can be determined as

R2 = 3800N and R1 = 5400N

The shear force and bending moment diagrams can be drawn by considering the X-sections at the suitable locations.



8. Illustrative Problem :

The simply supported beam shown below carries a vertical load that increases uniformly from zero at the one end to the maximum value of 6kN/m of length at the other end .Draw the shearing force and bending moment diagrams.

Solution

Determination of Reactions

For the purpose of determining the reactions R1 and R2 , the entire distributed load may be replaced by its resultant which will act through the centroid of the triangular loading diagram.

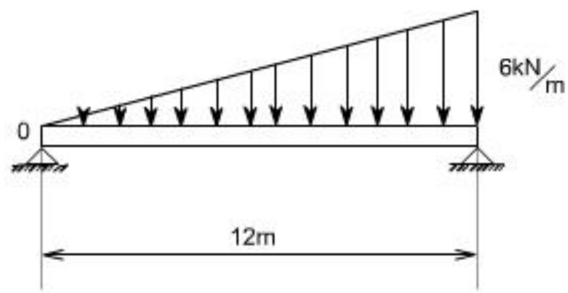
So the total resultant load can be found like this-

Average intensity of loading = (0 + 6)/2

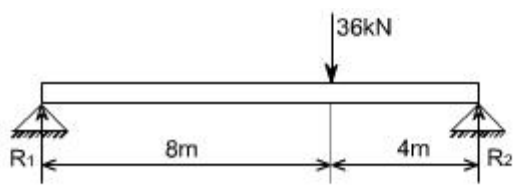
= 3 kN/m

Total Load = 3 x 12

= 36 kN



Since the centroid of the triangle is at a 2/3 distance from the one end, hence 2/3 x 3 = 8 m from the left end support.



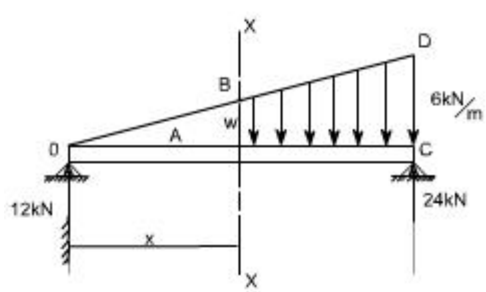
Now taking moments or applying conditions of equilibrium

36 x 8 = R2 x 12

R1 = 12 kN

R2 = 24 kN

Note: however, this resultant can not be used for the purpose of drawing the shear force and bending moment diagrams. We must consider the distributed load and determine the shear and moment at a section x from the left hand end.



Consider any X-section X-X at a distance x, as the intensity of loading at this X-section, is unknown let us find out the resultant load which is acting on the L.H.S of the X-section X-X, hence

So consider the similar triangles

OAB & OCD

$$\frac{w}{6} = \frac{x}{12}$$
$$w = \frac{x}{2} \ k\frac{N}{m}$$

In order to find out the total resultant load on the left hand side of the X-section

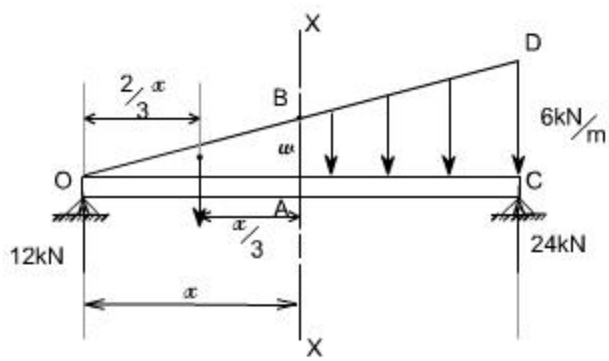
Find the average load intensity

$$= \frac{0 + \frac{x}{2}}{2}$$
$$= \frac{x}{4}\, k\frac{N}{m}$$

Therefore the total load over the length x would be

$$= \frac{x}{4}.x \text{ kN}$$
$$= \frac{x^2}{4} \text{ kN}$$

Now these loads will act through the centroid of the triangle OAB. i.e. at a distance 2/3 x from the left hand end. Therefore, the shear force and bending momemt equations may be written as



$$S.F_{at\ xx} = \left(12 - \frac{x^2}{4}\right) \text{ kN}$$

valid for all values of x ....(1)

$$B.M_{at\ xx} = 12\,x - \frac{x^2}{4}.\frac{x}{3}$$

$$B.M_{at\ xx} = 12\,x - \frac{x^3}{12} \text{ kN-m}$$

valid for all values of x ....(2)

$$S.F_{at\ x=0} = 12 \text{ kN}$$

$$S.F_{at\ x=12m} = 12 - \frac{12 \times 12}{4}$$

$$= -24 \text{ kN}$$

In order to find out the point where S.F is zero

$$\left(12 - \frac{x^2}{4}\right) = 0$$

x = 6.92 m (selecting the positive values)

Again

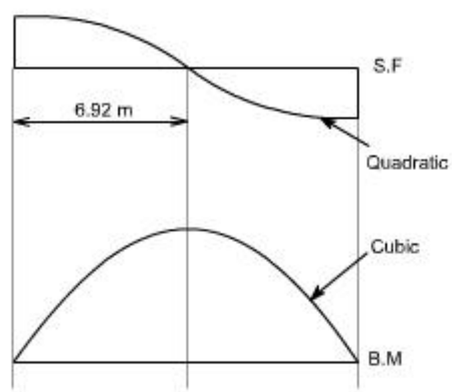
$$B.M_{at\ x=0} = 0$$

$$B.M_{at\ x=12} = 12 \times 12 - \frac{12^3}{12}$$

$$= 0$$

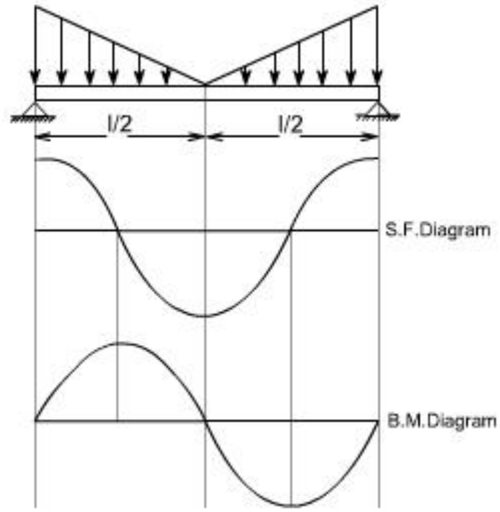
$$B.M_{at\ x=6.92} = 12 \times 6.92 - \frac{6.92^3}{12}$$

$$= 55.42 \text{ kN-m}$$



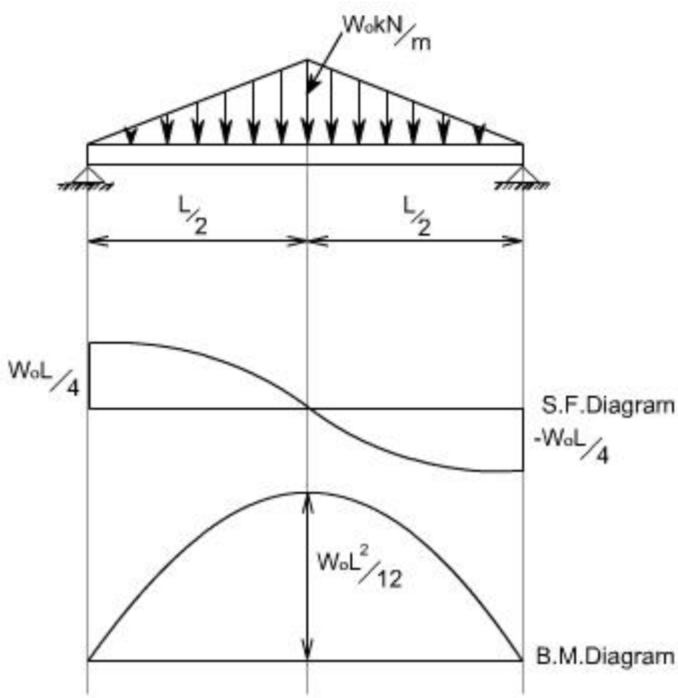
9. Illustrative problem :

In the same way, the shear force and bending moment diagrams may be attempted for the given problem



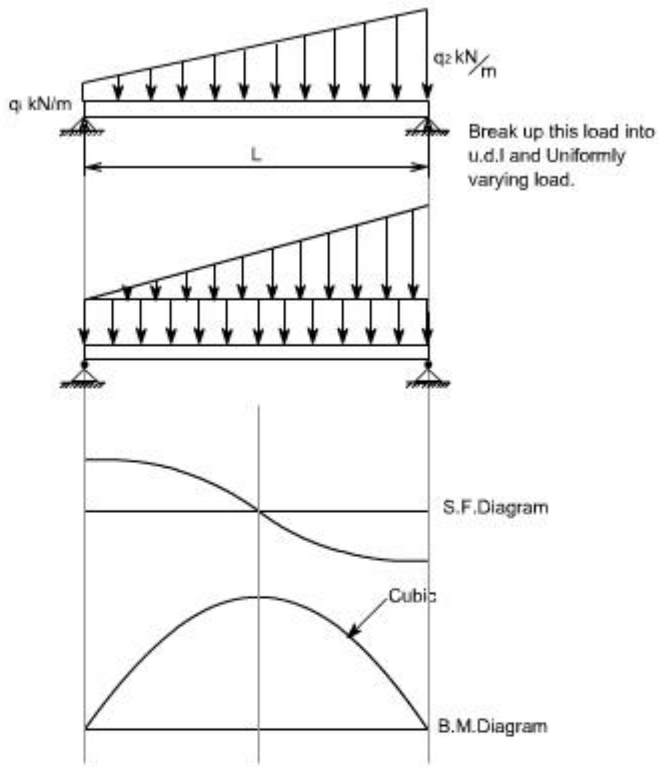
10. Illustrative problem :

For the uniformly varying loads, the problem may be framed in a variety of ways, observe the shear force and bending moment diagrams

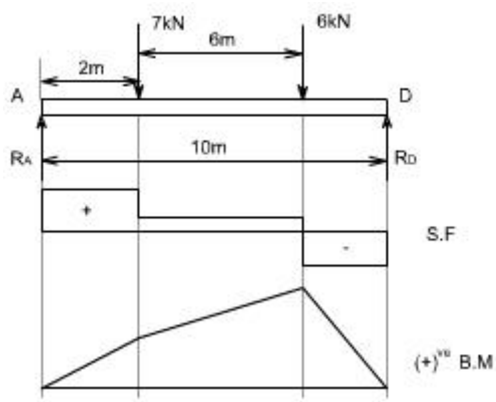


11. Illustrative problem :

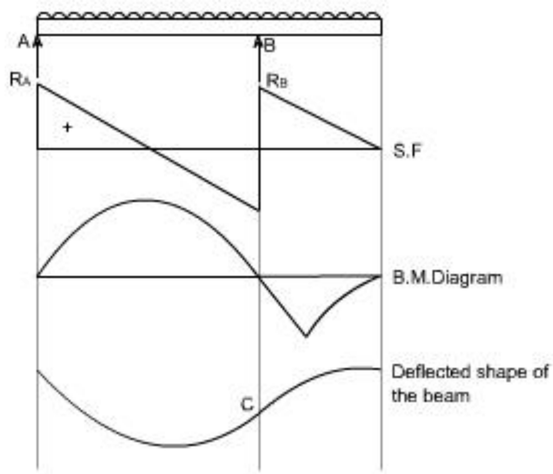
In the problem given below, the intensity of loading varies from q_1 kN/m at one end to the q_2 kN/m at the other end.This problem can be treated by considering a U.d.i of intensity q_1 kN/m over the entire span and a uniformly varying load of 0 to (q_2- q_1)kN/m over the entire span and then super impose teh two loadings.



Point of Contraflexure:

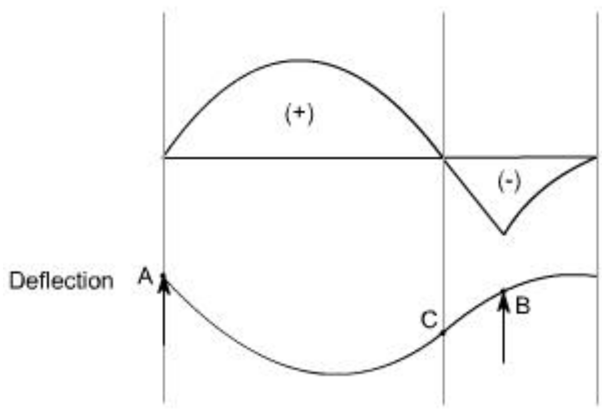


Consider the loaded beam a shown below along with the shear force and Bending moment diagrams for It may be observed that this case, the bending moment diagram is completely positive so that the curvature of the beam varies along its length, but it is always concave upwards or sagging.However if we consider a again a loaded beam as shown below along with the S.F and B.M diagrams, then



It may be noticed that for the beam loaded as in this case,

The bending moment diagram is partly positive and partly negative.If we plot the deflected shape of the beam just below the bending moment



This diagram shows that L.H.S of the beam 'sags' while the R.H.S of the beam 'hogs'

The point C on the beam where the curvature changes from sagging to hogging is a point of contraflexure.

OR

It corresponds to a point where the bending moment changes the sign, hence in order to find the point of contraflexures obviously the B.M would change its sign when it cuts the X-axis therefore to get the points of contraflexure equate the bending moment equation equal to zero.The fibre stress is zero at such sections

Note: there can be more than one point of contraflexure

Simple Bending Theory OR Theory of Flexure for Initially Straight Beams

(The normal stress due to bending are called flexure stresses)

Preamble:

When a beam having an arbitrary cross section is subjected to a transverse loads the beam will bend. In addition to bending the other effects such as twisting and buckling may occur, and to investigate a problem that includes all the combined effects of bending, twisting and buckling could become a complicated one. Thus we are interested to investigate the bending effects alone, in order to do so, we have to put certain constraints on the geometry of the beam and the manner of loading.

Assumptions:

The constraints put on the geometry would form the **assumptions:**

1. Beam is initially **straight** , and has a **constant cross-section.**

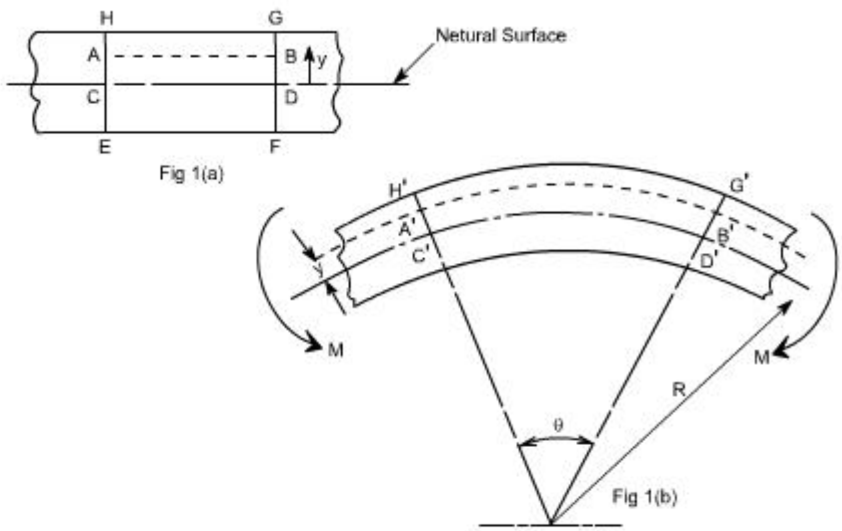
2. Beam is made of **homogeneous material** and the beam has a **longitudinal plane of symmetry.**

3. Resultant of the applied loads lies in the plane of symmetry.

4. The geometry of the overall member is such that bending not buckling is the primary cause of failure.

5. Elastic limit is nowhere exceeded and **‘E'** is same in tension and compression.

6. Plane cross - sections remains plane before and after bending.



Let us consider a beam initially unstressed as shown in fig 1(a). Now the beam is subjected to a constant bending moment (i.e. 'Zero Shearing Force') along its length as would be obtained by applying equal couples at each end. The beam will bend to the radius R as shown in Fig 1(b)

As a result of this bending, the top fibers of the beam will be subjected to tension and the bottom to compression it is reasonable to suppose, therefore, **that some where between the two there are points at which the stress is zero. The locus of all such points is known as neutral axis** . The radius of curvature R is then measured to this axis. For symmetrical sections the N. A. is the axis of symmetry but what ever the section N. A. will always pass through the centre of the area or centroid.

The above restrictions have been taken so as to eliminate the possibility of 'twisting' of the beam.

Concept of pure bending:

Loading restrictions:

As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since $\frac{dM}{dX} = F = 0$ or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

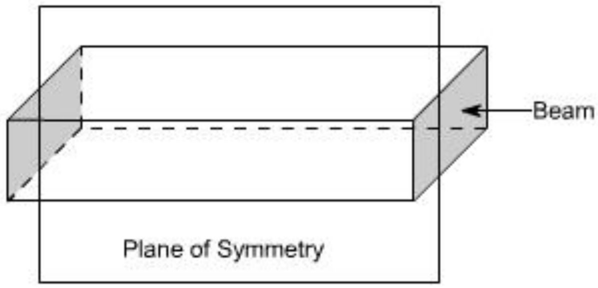


Fig (1)

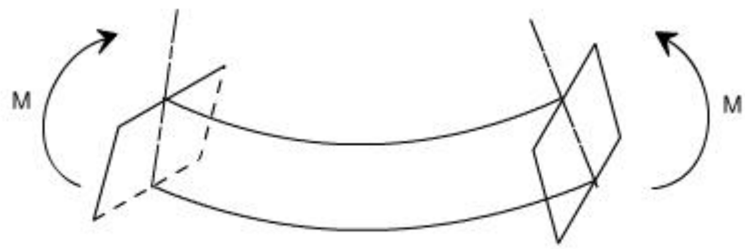
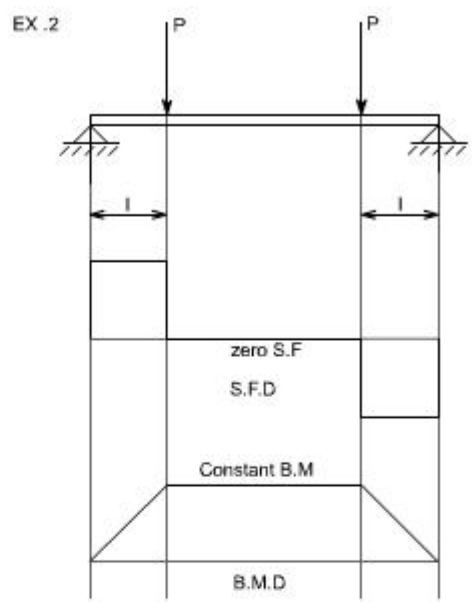
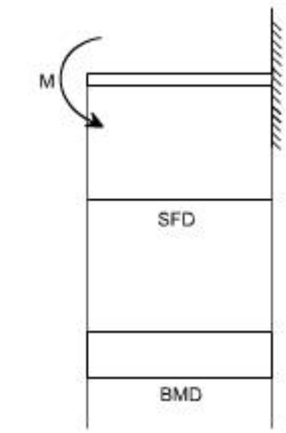


Fig (2)

When a member is loaded in such a fashion it is said to be in **pure bending.** The examples of pure bending have been indicated in EX 1and EX 2 as shown below :



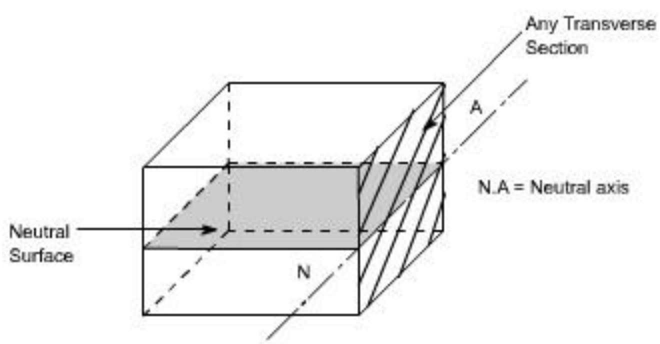
EX. 1



When a beam is subjected to pure bending are loaded by the couples at the ends, certain cross-section gets deformed and we shall have to make out the conclusion that,

1. Plane sections originally perpendicular to longitudinal axis of the beam remain plane and perpendicular to the longitudinal axis even after bending , i.e. the cross-section A'E', B'F' (refer Fig 1(a)) do not get warped or curved.

2. In the deformed section, the planes of this cross-section have a common intersection i.e. any time originally parallel to the longitudinal axis of the beam becomes an arc of circle.



We know that when a beam is under bending the fibres at the top will be lengthened while at the bottom will be shortened provided the bending moment M acts at the ends. In between these there are some fibres which remain unchanged in length that is they are not strained, that is they do not carry any stress. The plane containing such fibres is called neutral surface.

The line of intersection between the neutral surface and the transverse exploratory section is called the neutral axisNeutral axis **(N A)** .

<u>Bending Stresses in Beams or Derivation of Elastic Flexural formula :</u>

In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle q.

Consider now fiber AB in the material, at adistance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But AB = CD and CD = C'D'}$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

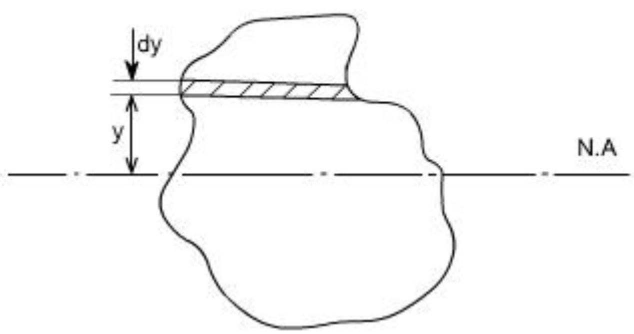
Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\text{However } \frac{\text{stress}}{\text{strain}} = E \qquad \text{where E = Young's Modulus of elasticity}$$

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$



Consider any arbitrary a cross-section of beam, as shown above now the strain on a fibre at a distance ‘y' from the N.A, is given by the expression

$$\sigma = \frac{E}{R}\,y$$

if the shaded strip is of area'dA'

then the force on the strip is

$$F = \sigma\,\delta A = \frac{E}{R}\,y\,\delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R}\,y^2\delta A$

The toatl moment for the whole

cross-section is therefore equal to

$$M = \sum \frac{E}{R}\,y^2\,\delta A = \frac{E}{R}\sum y^2\delta A$$

Now the term $\sum y^2\delta A$ is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.

Therefore

$$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

This equation is known as the Bending Theory Equation. The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Section Modulus:

From simple bending theory equation, the maximum stress obtained in any cross-section is given as

$$\sigma_{max}{}^{m} = \frac{M}{I}\,y_{max}{}^{m}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}{}^{m}}\,\sigma_{max}{}^{m}$$

For ready comparison of the strength of various beam cross-section this relationship is some times written in the form

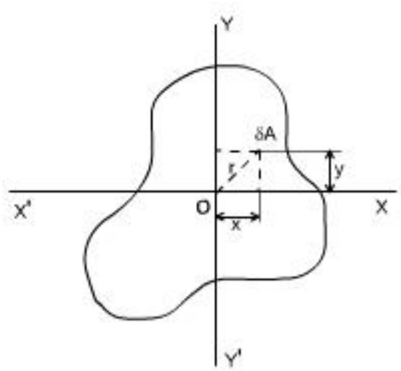
$M = Z\,\sigma_{max^m}$ where $Z = \dfrac{I}{y_{max^m}}$ **Is termed as section modulus**

The higher value of Z for a particular cross-section, the higher the bending moment which it can withstand for a given maximum stress.

<u>Theorems to determine second moment of area:</u> There are two theorems which are helpful to determine the value of second moment of area, which is required to be used while solving the simple bending theory equation.

Second Moment of Area :

Taking an analogy from the mass moment of inertia, the second moment of area is defined as the summation of areas times the distance squared from a fixed axis. (This property arised while we were driving bending theory equation). This is also known as the moment of inertia. An alternative name given to this is second moment of area, because the first moment being the sum of areas times their distance from a given axis and the second moment being the square of the distance or $\int y^2\,dA$.



Consider any cross-section having small element of area d A then by the definition

I_x(Mass Moment of Inertia about x-axis) = $\int y^2\,dA$ and I_y(Mass Moment of Inertia about y-axis) $= \int x^2\,dA$

Now the moment of inertia about an axis through ‘O' and perpendicular to the plane of figure is called the polar moment of inertia. (The polar moment of inertia is also the area moment of inertia).

i.e,

J = polar moment of inertia

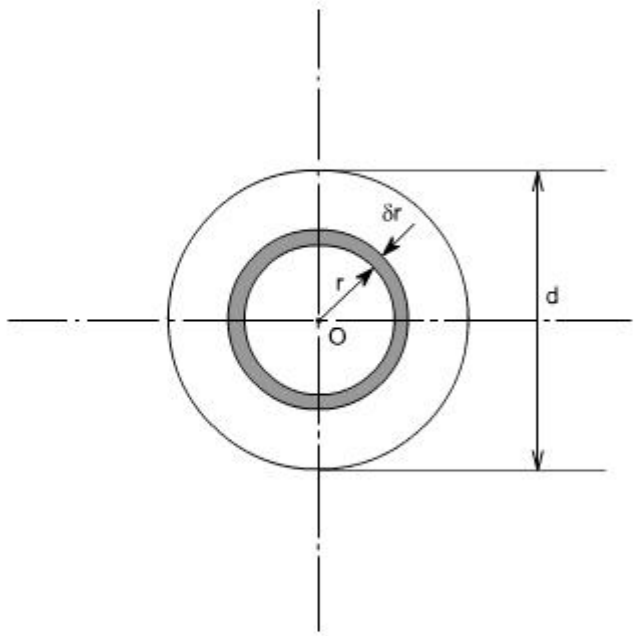
$$= \int r^2 dA$$
$$= \int (x^2 + y^2) dA$$
$$= \int x^2 dA + \int y^2 dA$$
$$= I_X + I_Y$$
$$\text{or } J = I_X + I_Y \qquad \text{..........} (1)$$

The relation (1) is known as the **perpendicular axis theorem** and may be stated as follows:

The sum of the Moment of Inertia about any two axes in the plane is equal to the moment of inertia about an axis perpendicular to the plane, the three axes being concurrent, i.e, the three axes exist together.

CIRCULAR SECTION :

For a circular x-section, the polar moment of inertia may be computed in the following manner



Consider any circular strip of thickness dr located at a radius 'r'.

Than the area of the circular strip would be dA = 2pr. dr

$$J = \int r^2 dA$$

Taking the limits of intergration from 0 to d/2

$$J = \int_0^{\frac{d}{2}} r^2 2\pi r \delta r$$

$$= 2\pi \int_0^{\frac{d}{2}} r^3 \delta r$$

$$J = 2\pi \left[\frac{r^4}{4}\right]_0^{\frac{d}{2}} = \frac{\pi d^4}{32}$$

however ,by perpendicular axistheorem

$$J = I_x + I_y$$

But for the circular cross-section ,the I_x and I_y are both equal being moment of inertia about a diameter

$$I_{dia} = \frac{1}{2} J$$

$$I_{dia} = \frac{\pi d^4}{64}$$

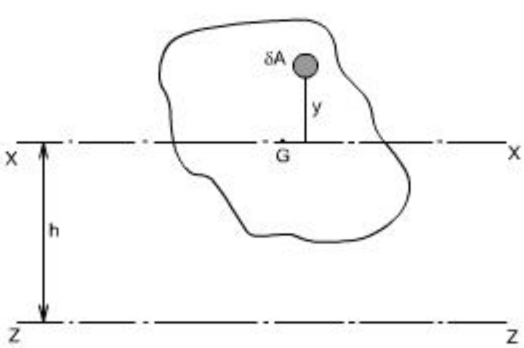
for a hollow circular sectionof diameter D and d,
thevaluesof J and I are defined as

$$J = \frac{\pi(D^4 - d^4)}{32}$$

Thus $$I = \frac{\pi(D^4 - d^4)}{64}$$

<u>Parallel Axis Theorem:</u>

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the centroid plus the area times the square of the distance between the axes.



If ‘ZZ' is any axis in the plane of cross-section and ‘XX' is a parallel axis through the centroid G, of the cross-section, then

$$I_z = \int (y+h)^2 \, dA \text{ by definition (moment of inertia about an axis ZZ)}$$

$$= \int \left(+ 2yh + h^2 \right) dA$$

$$= \int y^2 dA + h^2 \int dA + 2h \int y dA$$

$$\text{Since } \int y dA = 0$$

$$= \int y^2 dA + h^2 \int dA$$

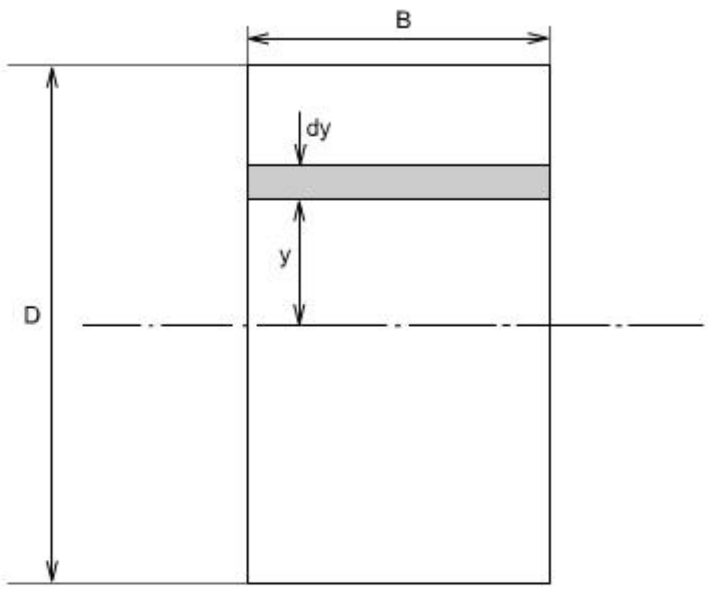
$$= \int y^2 dA + h^2 A$$

$$I_z = I_x + Ah^2 \quad I_x = I_G \text{ (since cross-section axes also pass through G)}$$

$$\text{Where } A = \text{Total area of the section}$$

Rectangular Section:

For a rectangular x-section of the beam, the second moment of area may be computed as below :



Consider the rectangular beam cross-section as shown above and an element of area **dA** , thickness **dy** , breadth **B** located at a distance **y** from the neutral axis, which by symmetry passes through the centre of section. The second moment of area **I** as defined earlier would be

$$I_{N.A} = \int y^2 dA$$

Thus, for the rectangular section the second moment of area about the neutral axis i.e., an axis through the centre is given by

$$I_{N.A} = \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 \left(B\,dy\right)$$

$$= B \int_{\frac{-D}{2}}^{\frac{D}{2}} y^2 dy$$

$$= B\left[\frac{y^3}{3}\right]_{\frac{D}{2}}^{\frac{-D}{2}}$$

$$= \frac{B}{3}\left[\frac{D^3}{8} - \left(\frac{-D^3}{8}\right)\right]$$

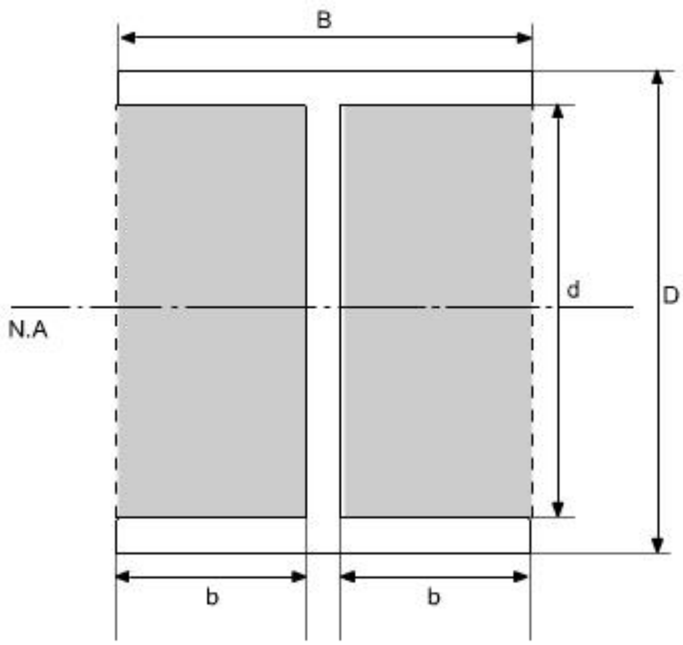
$$= \frac{B}{3}\left[\frac{D^3}{8} + \frac{D^3}{8}\right]$$

$$I_{N.A} = \frac{BD^3}{12}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of **0** to **D** .

Therefore $I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3}$

These standards formulas prove very convenient in the determination of I_{NA} for build up sections which can be conveniently divided into rectangles. For instance if we just want to find out the Moment of Inertia of an I - section, then we can use the above relation.



$$I_{N.A} = I_{\text{of dotted rectangle}} - I_{\text{of shaded portion}}$$

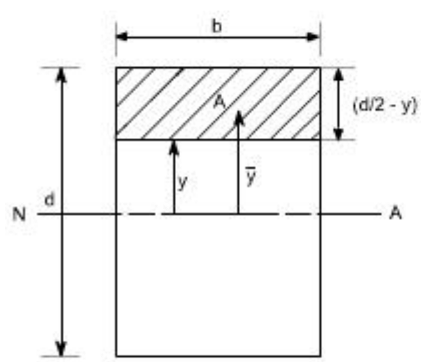
$$\therefore\ I_{N.A} = \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right)$$

$$I_{N.A} = \frac{BD^3}{12} - \frac{bd^3}{6}$$

Let us consider few examples to determaine the sheer stress distribution in a given X-sections

Rectangular x-section:

Consider a rectangular x-section of dimension b and d



A is the area of the x-section cut off by a line parallel to the neutral axis. $\bar{y}$ is the distance of the centroid of A from the neutral axis

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

for this case, $A = b\left(\frac{d}{2} - y\right)$

While $\bar{y} = \left[\frac{1}{2}\left(\frac{d}{2} - y\right) + y\right]$

i.e $\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$ and $z = b\,;\, I = \frac{b.d^3}{12}$

substituting all these values, in the formula

$$\tau = \frac{F.A.\bar{y}}{I.z}$$

$$= \frac{F.b.\left(\frac{d}{2} - y\right).\frac{1}{2}.\left(\frac{d}{2} + y\right)}{b.\frac{b.d^3}{12}}$$

$$= \frac{\frac{F}{2}.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{\frac{b.d^3}{12}}$$

$$= \frac{6.F.\left\{\left(\frac{d}{2}\right)^2 - y^2\right\}}{b.d^3}$$

This shows that there is a parabolic distribution of shear stress with y.

The maximum value of shear stress would obviously beat the location y = 0.

Such that $\tau_{max} = \frac{6.F}{b.d^3} \cdot \frac{d^2}{4}$

$$= \frac{3.F}{2.b.d}$$

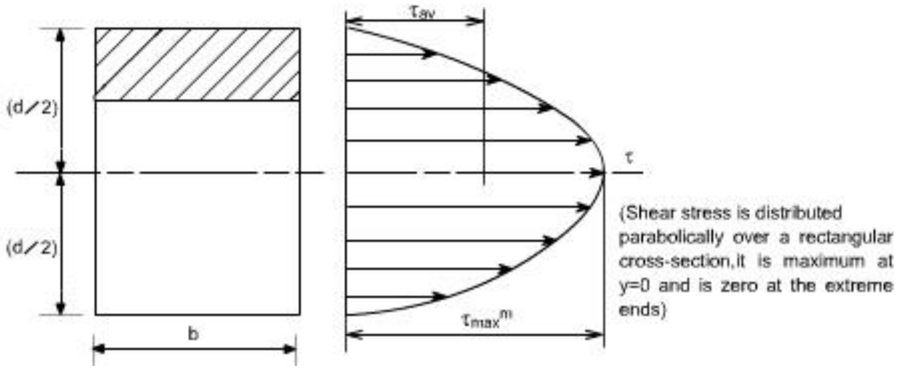
So $\boxed{\tau_{max} = \frac{3.F}{2.b.d}}$ The value of τ_{max} occurs at the neutral axis

The mean shear stress in the beam is defined as

τ_{mean} or $\tau_{avg} = F/A = F/b.d$

So $\tau_{max} = 1.5\ \tau_{mean} = 1.5\ \tau_{avg}$

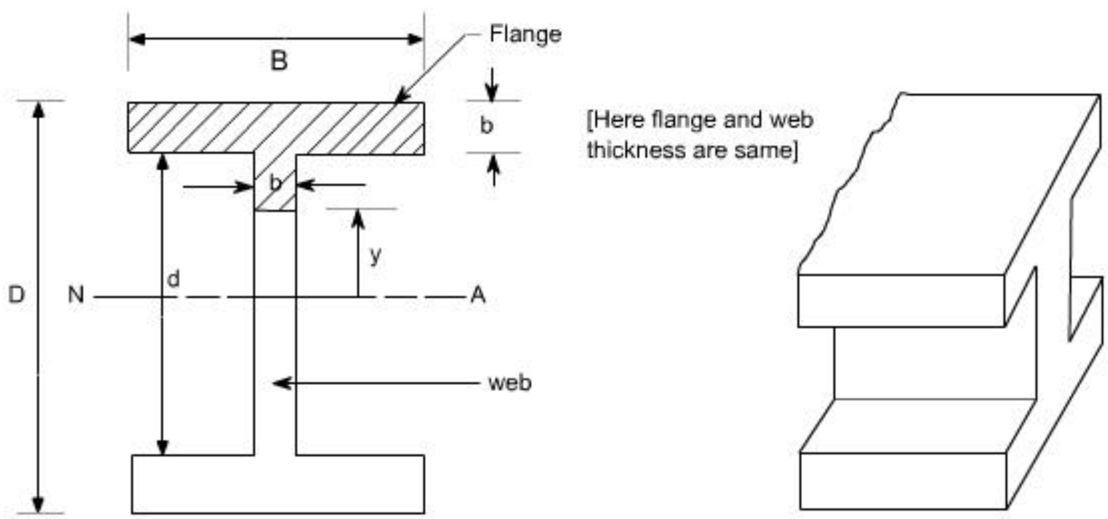
Therefore the shear stress distribution is shown as below.



It may be noted that the shear stress is distributed parabolically over a rectangular cross-section, it is maximum at y = 0 and is zero at the extreme ends.

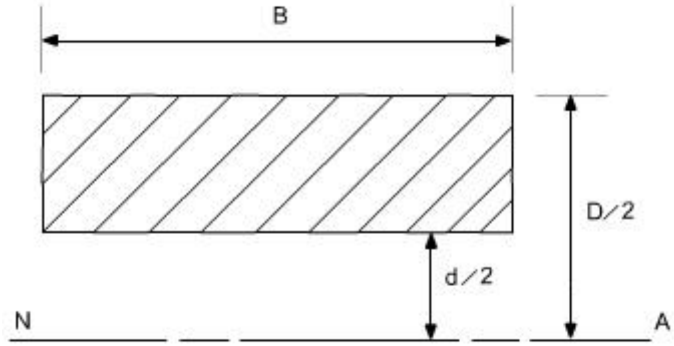
<u>I - section :</u>

Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \frac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

$$\text{Area of the flange} = B\left(\frac{D-d}{2}\right)$$

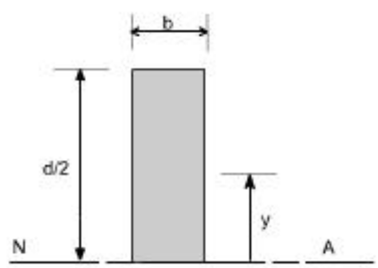
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\Big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

$$A\bar{y}\big|_{Total} = B\left(\frac{D - d}{2}\right)\left(\frac{D + d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

$$A\bar{y}\big|_{Total} = B\left(\frac{D^2 - d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2 - d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$

To get the maximum and minimum values of t substitute in the above relation.

y = 0 at N. A. And y = d/2 at the tip.

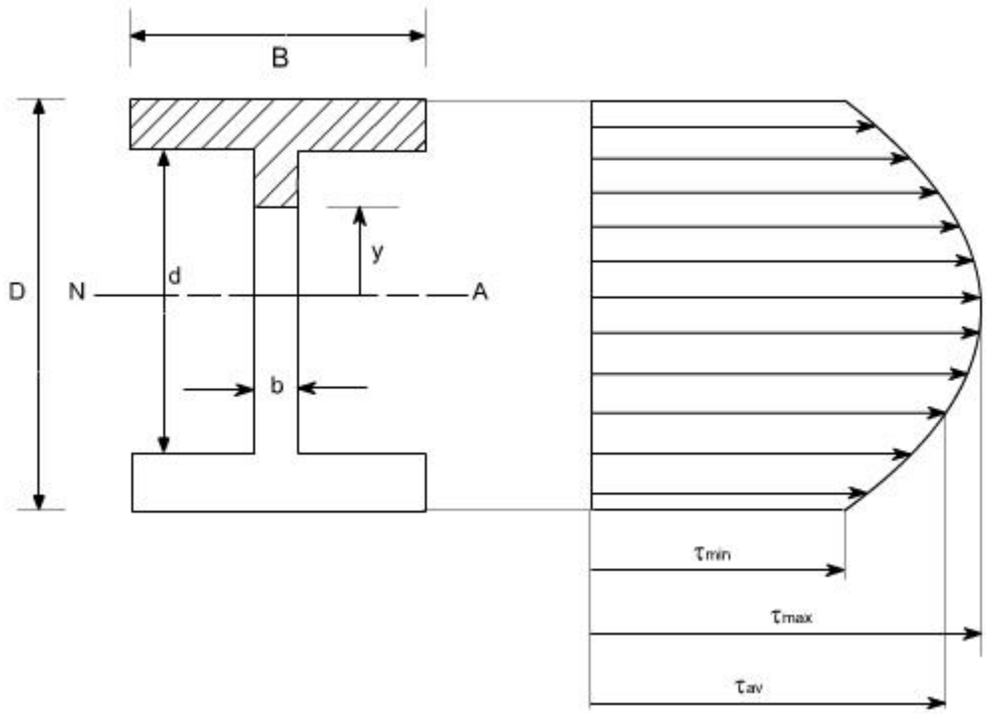
The maximum shear stress is at the neutral axis. i.e. for the condition y = 0 at N. A.

Hence, τ_{max} at $y = 0 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$ ....(2)

The minimum stress occur at the top of the web, the term bd 2 goes off and shear stress is given by the following expression

τ_{min} at $y = d/2 = \frac{F}{8bI}\left[B\left(D^2 - d^2\right)\right]$ ....(3)

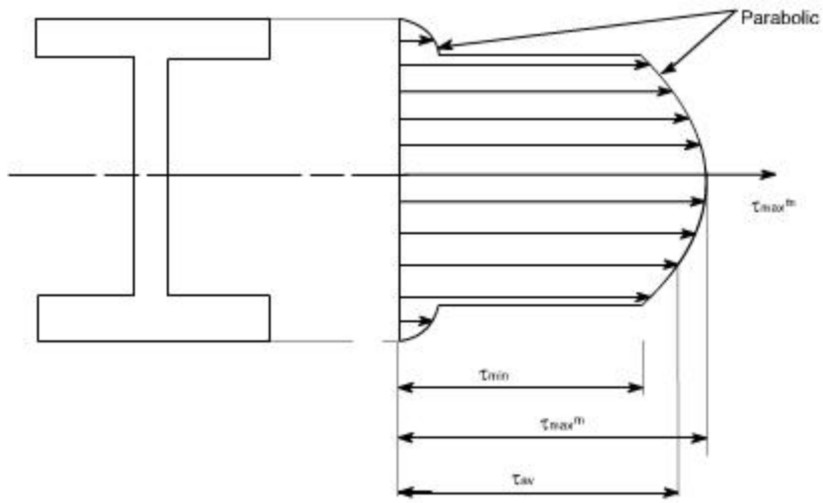
The distribution of shear stress may be drawn as below, which clearly indicates a parabolic distribution



$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$

Note: from the above distribution we can see that the shear stress at the flanges is not zero, but it has some value, this can be analyzed from equation (1). At the flange tip or flange or web interface y = d/2.Obviously than this will have some constant value and than onwards this will have parabolic distribution.

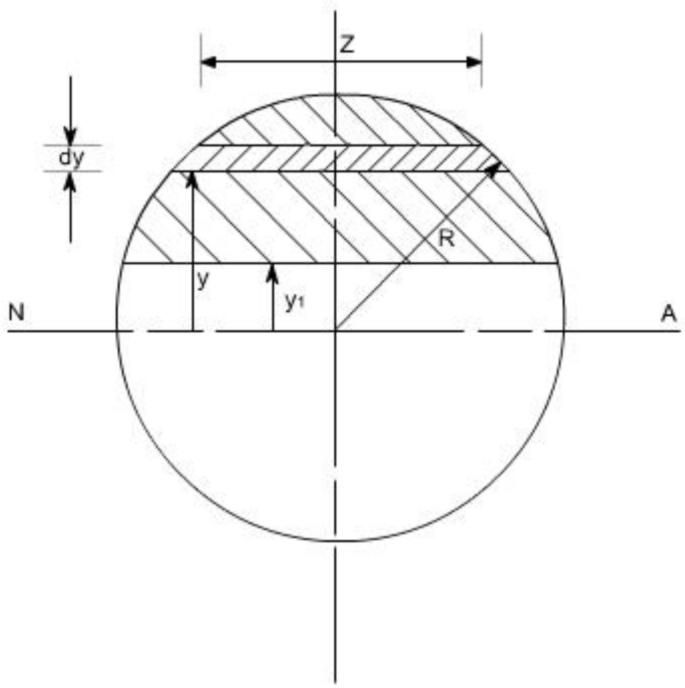
In practice it is usually found that most of shearing stress usually about 95% is carried by the web, and hence the shear stress in the flange is neglible however if we have the concrete analysis i.e. if we analyze the shearing stress in the flange i.e. writing down the expression for shear stress for flange and web separately, we will have this type of variation.



This distribution is known as the “top – hat” distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

Shear stress distribution in beams of circular cross-section:

Let us find the shear stress distribution in beams of circular cross-section. In a beam of circular cross-section, the value of Z width depends on y.



Using the expression for the determination of shear stresses for any arbitrary shape or a arbitrary section.

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{FA\int y\,dA}{Z\,I}$$

Where òy dA is the area moment of the shaded portion or the first moment of area.

Here in this case ‘dA' is to be found out using the Pythagoras theorem

$$\left(\frac{Z}{2}\right)^2 + y^2 = R^2$$

$$\left(\frac{Z}{2}\right)^2 = R^2 - y^2 \text{ or } \frac{Z}{2} = \sqrt{R^2 - y^2}$$

$$Z = 2\sqrt{R^2 - y^2}$$

$$dA = Z\,dy = 2.\sqrt{R^2 - y^2}\,.dy$$

$$I_{\text{N.A. for a circular cross-section}} = \frac{\pi R^4}{4}$$

Hence,

$$\tau = \frac{FA\,\bar{y}}{Z\,I} = \frac{F}{\frac{\pi R^4}{4}\,2\sqrt{R^2 - y^2}}\int_{y_1}^{R} 2\,y\sqrt{R^2 - y^2}\,dy$$

Where R = radius of the circle.

[The limits have been taken from y_1 to R because we have to find moment of area the shaded portion]

$$= \frac{4F}{\pi R^4\sqrt{R^2 - y^2}}\int_{y_1}^{R} y\sqrt{R^2 - y^2}\,dy$$

The integration yields the final result to be

$$\tau = \frac{4F\left(R^2 - y_1^2\right)}{3\pi R^4}$$

Again this is a parabolic distribution of shear stress, having a maximum value when $y_1 = 0$

$$\tau_{\max m}|y_1 = 0 = \frac{4F}{3\pi R^2}$$

Obviously at the ends of the diameter the value of $y_1 = \pm R$ thus $\tau = 0$ so this again a parabolic distribution; maximum at the neutral axis

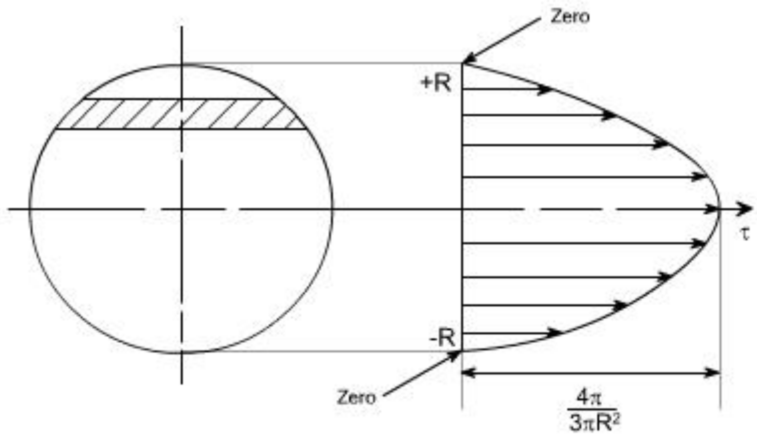
Also

$$\tau_{avg} \text{ or } \tau_{mean} = \frac{F}{A} = \frac{F}{\pi R^2}$$

Hence,

$$\boxed{\tau_{\max m} = \frac{4}{3}\,\tau_{avg}}$$

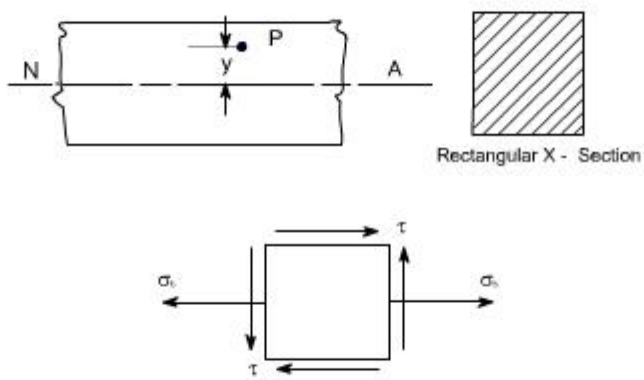
The distribution of shear stresses is shown below, which indicates a parabolic distribution



Principal Stresses in Beams

It becomes clear that the bending stress in beam s_x is not a principal stress, since at any distance y from the neutral axis; there is a shear stress t (or t_{xy} we are assuming a plane stress situation)

In general the state of stress at a distance y from the neutral axis will be as follows.



At some point ‘P' in the beam, the value of bending stresses is given as

$$\sigma_b = \frac{My}{I} \text{ for a beam of rectangular cross - section of dimensions b and d; } I = \frac{bd^3}{12}$$

$$\sigma_b = \frac{12\,My}{bd^3}$$

whereas the value shear stress in the rectangular cross - section is given as

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right]$$

Hence the values of principle stress can be determined from the relations,

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau_{xy}^2}$$

Letting $\sigma_y = 0$; $\sigma_x = \sigma_b$,the values of σ_1 and σ_2 can be computed as

$$\text{Hence } \sigma_1 / \sigma_2 = \frac{1}{2}\left(\frac{12My}{bd^3}\right) \pm \frac{1}{2}\sqrt{\left(\frac{12\,My}{bd^3}\right)^2 + 4\left(\frac{6\,F}{bd^3}\left(\frac{d^2}{4} - y^2\right)\right)^2}$$

$$\sigma_1, \sigma_2 = \frac{6}{bd^3}\left[My \pm \sqrt{\left\{M^2y^2 + F^2\left(\frac{d^2}{4} - y^2\right)^2\right\}}\right]$$

Also,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{putting } \sigma_y = 0$$

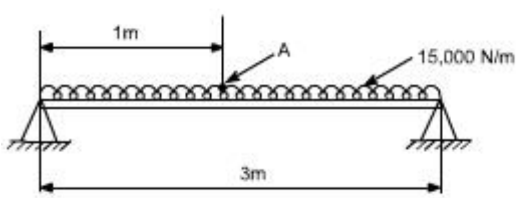
we get,

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x}$$

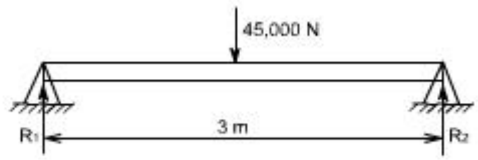
After substituting the appropriate values in the above expression we may get the inclination of the principal planes.

Illustrative examples: Let us study some illustrative examples,pertaining to determination of principal stresses in a beam

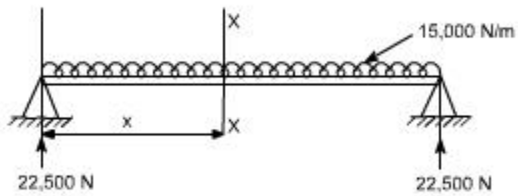
1. Find the principal stress at a point A in a uniform rectangular beam 200 mm deep and 100 mm wide, simply supported at each end over a span of 3 m and carrying a uniformly distributed load of 15,000 N/m.



Solution: The reaction can be determined by symmetry



$R_1 = R_2 = 22{,}500$ N



consider any cross-section X-X located at a distance x from the left end.

Hence,

$S.\ F_{\text{at XX}} = 22{,}500 - 15{,}000\ x$

$B.M_{\text{at XX}} = 22{,}500\ x - 15{,}000\ x\ (x/2) = 22{,}500\ x - 15{,}000\ .\ x^2 / 2$

Therefore,

$S.\ F_{\text{at X}} = 1\text{ m} = 7{,}500$ N

$B.\ M_{\text{at X}} = 1\text{ m} = 15{,}000$ N

$$S.F|_{x=1m} = 7{,}500\ \text{N}$$
$$B.M|_{x=1m} = 15{,}000\ \text{N.m}$$
$$\sigma_x = \frac{My}{I}$$
$$= \frac{15{,}000 \times 5 \times 10^{-2} \times 12}{10 \times 10^{-12} \times \left(20 \times 10^{-2}\right)^3}$$
$$\sigma_x = 11.25\ \text{MN/m}^2$$

For the compution of shear stresses

$$\tau = \frac{6F}{bd^3}\left[\frac{d^2}{4} - y^2\right] \qquad \text{putting } y = 50\text{ mm}, d = 200\text{ mm}$$
$$F = 7500\ \text{N}$$
$$\tau = 0.422\ \text{MN/m}^2$$

Now substituting these values in the principal stress equation,

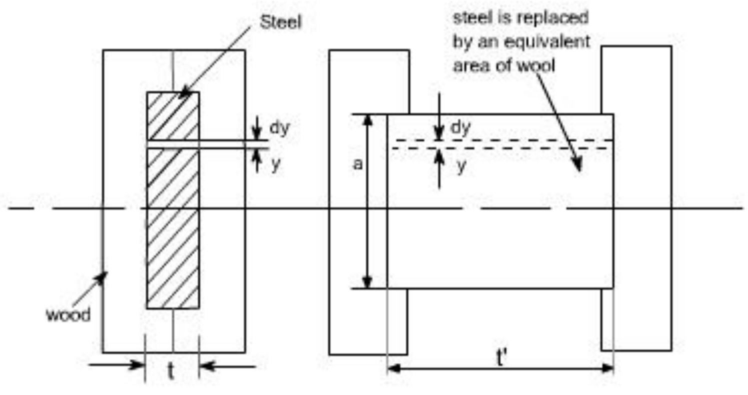
We get $s_1 = 11.27$ MN/m^2

$s_2 = -\ 0.025$ MN/m^2

Bending Of Composite or Flitched Beams

A composite beam is defined as the one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and a reinforcing steel plate, then it is termed as a flitched beam.

The bending theory is valid when a constant value of Young's modulus applies across a section it cannot be used directly to solve the composite-beam problems where two different materials, and therefore different values of E, exists. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.



Composite Section   Equivalent Section

Consider, a beam as shown in figure in which a steel plate is held centrally in an appropriate recess/pocket between two blocks of wood .Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength. i.e. the moment at any section must be the same in the equivalent section as in the original section so that the force at any given dy in the equivalent beam must be equal to that at the strip it replaces.

$$\sigma.t = \sigma'.t' \text{ or } \boxed{\frac{\sigma}{\sigma'} = \frac{t'}{t}}$$

recalling $\sigma = E.\varepsilon$

Thus

$$\varepsilon E t = \varepsilon' E' t'$$

Again, for true similarity the strains must be equal,

$$\varepsilon = \varepsilon' \text{ or } E t = E' t' \text{ or } \boxed{\frac{E}{E'} = \frac{t'}{t}}$$

Thus, $\boxed{t' = \frac{E}{E'}.t}$

Hence to replace a steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'.

The equivalent section is then one of the same materials throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows by utilizing the given relations.

$$\frac{\sigma}{\sigma} = \frac{t'}{t}$$

$$\frac{\sigma}{\sigma} = \frac{E}{E}$$

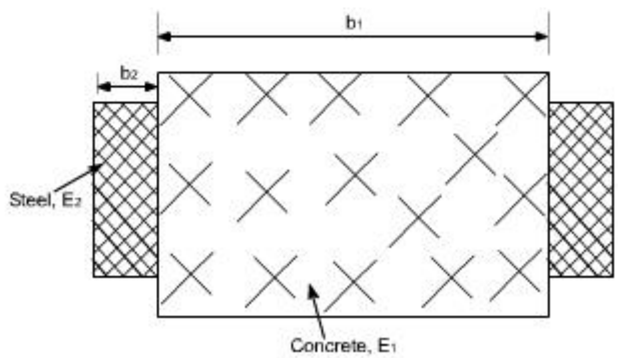
Stress in steel = modular ratio x stress in equivalent wood

The above procedure of course is not limited to the two materials treated above but applies well for any material combination. The wood and steel flitched beam was nearly chosen as a just for the sake of convenience.

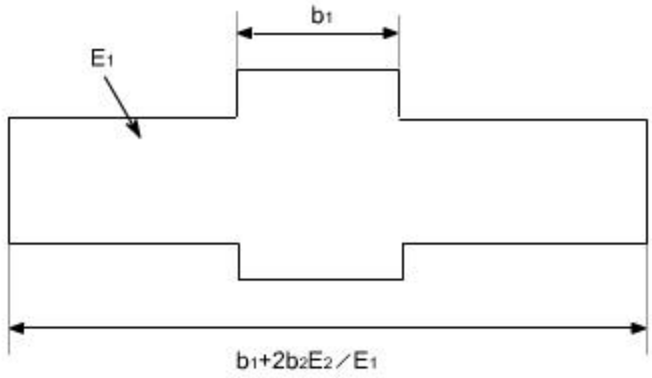
Assumption

In order to analyze the behavior of composite beams, we first make the assumption that the materials are bonded rigidly together so that there can be no relative axial movement between them. This means that all the assumptions, which were valid for homogenous beams are valid except the one assumption that is no longer valid is that the Young's Modulus is the same throughout the beam.

The composite beams need not be made up of horizontal layers of materials as in the earlier example. For instance, a beam might have stiffening plates as shown in the figure below.



Again, the equivalent beam of the main beam material can be formed by scaling the breadth of the plate material in proportion to modular ratio. Bearing in mind that the strain at any level is same in both materials, the bending stresses in them are in proportion to the Young's modulus.



BEAMS – LOADS AND STRESSES

PART- A (2 Marks)

1. State the different types of supports.
2. What is cantilever beam?
3. Write the equation for the simple bending theory.
4. What do you mean by the point of contraflexure?
5. Define beam.
6. Define shear force and bending moment.
7. What is Shear stress diagram?
8. What is Bending moment diagram?
9. What are the types of load?
10. Write the assumption in the theory of simple bending.
11. What are the types of beams?

PART- B (16 Marks)

1. Three planks of each 50 x200 mm timber are built up to a symmetrical I section for a beam. The maximum shear force over the beam is 4KN. Propose an alternate rectangular section of the same material so that the maximum shear stress developed is same in both sections. Assume then width of the section to be 2/3 of the depth.

2. A beam of uniform section 10 m long carries a udl of KN/m for the entire length and a concentrated load of 10 KN at right end. The beam is freely supported at the left end. Find the position of the second support so that the maximum bending moment in the beam is as minimum as possible. Also compute the maximum bending moment

3. A beam of size 150 mm wide, 250 mm deep carries a uniformly distributed load of w kN/m over entire span of 4 m. A concentrated load 1 kN is acting at a distance of 1.2 m from the left support. If the bending stress at a section 1.8 m from the left support is not to exceed 3.25 N/mm2 find the load w.

4. A cantilever of 2m length carries a point load of 20 KN at 0.8 m from the fixed end and

another point of 5 KN at the free end. In addition, a u.d.l. of 15 KN/m is spread over the entire length of the cantilever. Draw the S.F.D, and B.M.D.

5. A Simply supported beam of effective span 6 m carries three point loads of 30 KN, 25 KN and 40 KN at 1m, 3m and 4.5m respectively from the left support. Draw the SFD and BMD. Indicating values at salient points.

6. A Simply supported beam of length 6 metres carries a udl of 20KN/m throughout its length and a point of 30 KN at 2 metres from the right support. Draw the shear force and bending moment diagram. Also find the position and magnitude of maximum Bending moment.

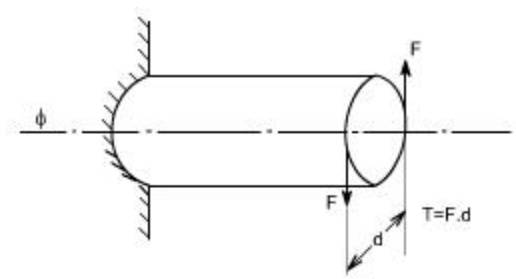
7. A Simply supported beam 6 metre span carries udl of 20 KN/m for left half of span and two point loads of 25 KN end 35 KN at 4 m and 5 m from left support. Find maximum SF and BM and their location drawing SF and BM diagrams.

UNIT III TORSION

Analysis of torsion of circular bars – Shear stress distribution – Bars of solid and hollow circular section – Stepped shaft – Twist and torsion stiffness – Compound shafts – Fixed and simply supported shafts – Application to close-coiled helical springs – Maximum shear stress in spring section including Wahl Factor – Deflection of helical coil springs under axial loads – Design of helical coil springs – stresses in helical coil springs under torsion loads.

Torsion of circular shafts

Definition of Torsion: Consider a shaft rigidly clamped at one end and twisted at the other end by a torque T = F.d applied in a plane perpendicular to the axis of the bar such a shaft is said to be in torsion.



Effects of Torsion: The effects of a torsional load applied to a bar are

(i) To impart an angular displacement of one end cross � section with respect to the other end.

(ii) To setup shear stresses on any cross section of the bar perpendicular to its axis.

GENERATION OF SHEAR STRESSES

The physical understanding of the phenomena of setting up of shear stresses in a shaft subjected to a torsion may be understood from the figure 1-3.

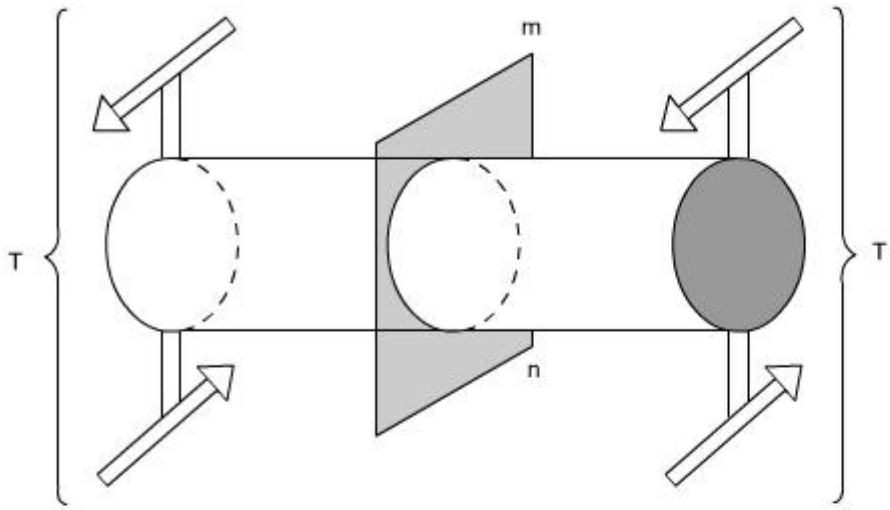


Fig 1: Here the cylindrical member or a shaft is in static equilibrium where T is the resultant external torque acting on the member. Let the member be imagined to be cut by some imaginary plane �mn'.

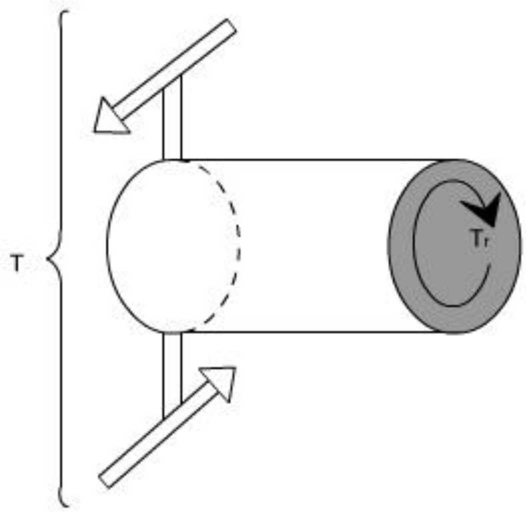


Fig 2: When the plane �mn' cuts remove the portion on R.H.S. and we get a fig 2. Now since the entire member is in equilibrium, therefore, each portion must be in equilibrium. Thus, the member is in equilibrium under the action of resultant external torque T and developed resisting Torque T_r .

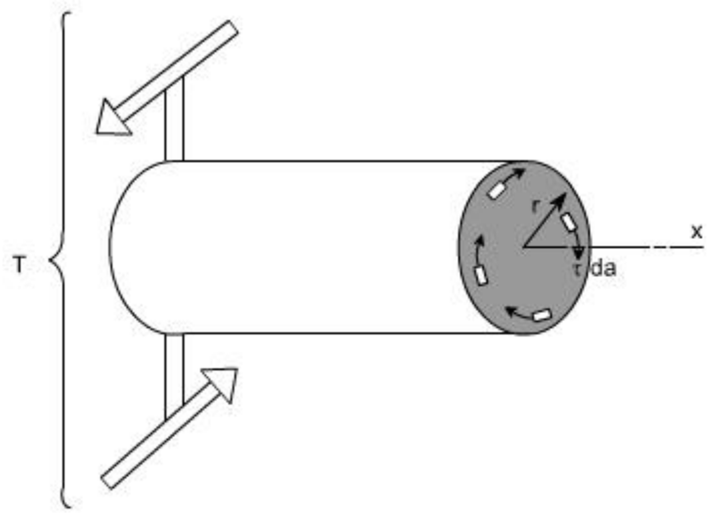


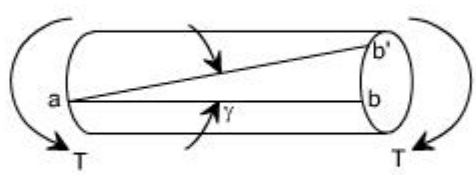
Fig 3: The Figure shows that how the resisting torque T_r is developed. The resisting torque T_r is produced by virtue of an infinites mal shear forces acting on the plane perpendicular to the axis of the shaft. Obviously such shear forces would be developed by virtue of sheer stresses.

Therefore we can say that when a particular member (say shaft in this case) is subjected to a torque, the result would be that on any element there will be shear stresses acting. While on other faces the complementary sheer forces come into picture. Thus, we can say that when a member is subjected to torque, an element of this member will be subjected to a state of pure shear.

Shaft: The shafts are the machine elements which are used to transmit power in machines.

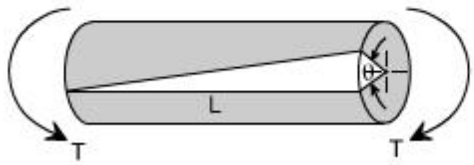
Twisting Moment: The twisting moment for any section along the bar / shaft is defined to be the algebraic sum of the moments of the applied couples that lie to one side of the section under consideration. The choice of the side in any case is of course arbitrary.

Shearing Strain: If a generator a � b is marked on the surface of the unloaded bar, then after the twisting moment 'T' has been applied this line moves to ab'. The angle �□ ' measured in radians, between the final and original positions of the generators is defined as the shearing strain at the surface of the bar or shaft. The same definition will hold at any interior point of the bar.



Modulus of Elasticity in shear: The ratio of the shear stress to the shear strain is called the modulus of elasticity in shear OR Modulus of Rigidity and in represented by the symbol $G = \frac{\tau}{r}$

Angle of Twist: If a shaft of length L is subjected to a constant twisting moment T along its length, than the angle □ through which one end of the bar will twist relative to the other is known is the angle of twist.

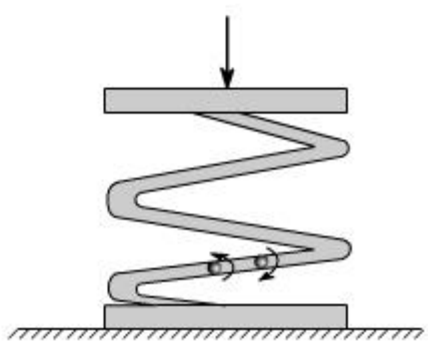


- Despite the differences in the forms of loading, we see that there are number of similarities between bending and torsion, including for example, a linear variation of stresses and strain with position.

  In torsion the members are subjected to moments (couples) in planes normal to their axes.

- For the purpose of desiging a circular shaft to withstand a given torque, we must develop an equation giving the relation between twisting moment, maximum shear stress produced, and a quantity representing the size and shape of the cross-sectional area of the shaft.

Not all torsion problems, involve rotating machinery, however, for example some types of vehicle suspension system employ torsional springs. Indeed, even coil springs are really curved members in torsion as shown in figure.



- Many torque carrying engineering members are cylindrical in shape. Examples are drive shafts, bolts and screw drivers.

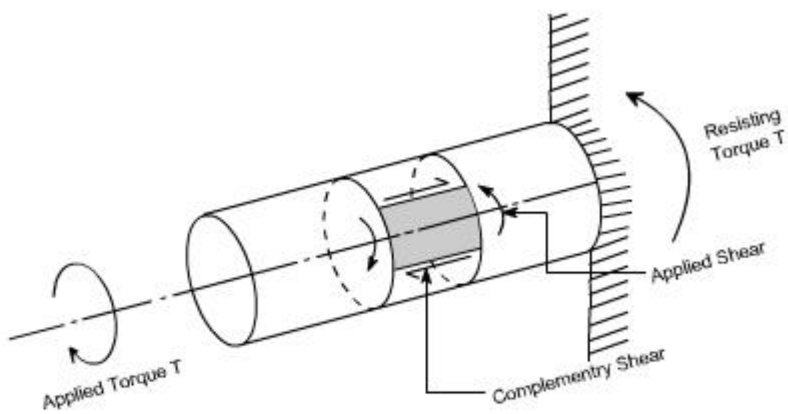
Simple Torsion Theory or Development of Torsion Formula : Here we are basically interested to derive an equation between the relevant parameters

Relationship in Torsion: $\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$

1 st Term: It refers to applied loading ad a property of section, which in the instance is the polar second moment of area.

2 nd Term: This refers to stress, and the stress increases as the distance from the axis increases.

3 rd Term: it refers to the deformation and contains the terms modulus of rigidity & combined term (□ l) which is equivalent to strain for the purpose of designing a circular shaft to with stand a given torque we must develop an equation giving the relation between Twisting moments max m shear stain produced and a quantity representing the size and shape of the cross � sectional area of the shaft.



Refer to the figure shown above where a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear, the moment of resistance developed by the shear stresses being every where equal to the magnitude, and opposite in sense, to the applied torque. For the purpose of deriving a simple theory to describe the behavior of shafts subjected to torque it is necessary make the following base assumptions.

Assumption:

(i) The materiel is homogenous i.e of uniform elastic properties exists throughout the material.

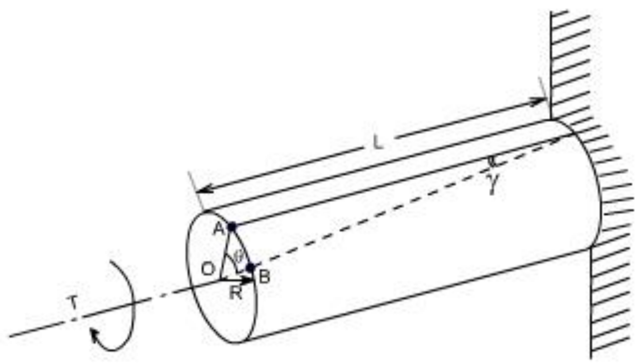
(ii) The material is elastic, follows Hook's law, with shear stress proportional to shear strain.

(iii) The stress does not exceed the elastic limit.

(iv) The circular section remains circular

(v) Cross section remain plane.

(vi) Cross section rotate as if rigid i.e. every diameter rotates through the same angle.



Consider now the solid circular shaft of radius R subjected to a torque T at one end, the other end being fixed Under the action of this torque a radial line at the free end of the shaft twists through an angle □ , point A moves to B, and AB subtends an angle � □ ' at the fixed end. This is then the angle of distortion of the shaft i.e the shear strain.

Since angle in radius = arc / Radius

arc AB = R□

= L □ [since L and □ also constitute the arc AB]

Thus, □ = R□ / L (1)

From the definition of Modulus of rigidity or Modulus of elasticity in shear

$$G = \frac{\text{shear stress}(\tau)}{\text{shear strain}(\gamma)}$$

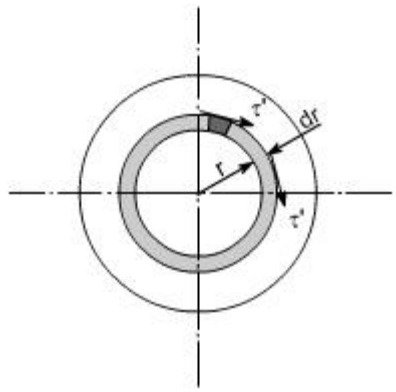
where γ is the shear stress set up at radius R.

Then $\frac{\tau}{G} = \gamma$

Equating the equations (1) and (2) we get $\frac{R\theta}{L} = \frac{\tau}{G}$

$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right)$ where τ' is the shear stress at any radius r.

Stresses: Let us consider a small strip of radius r and thickness dr which is subjected to shear stress τ'.



The force set up on each element

= stress x area

= τ' x $2\pi r$ dr (approximately)

This force will produce a moment or torque about the center axis of the shaft.

= τ' . 2π r dr . r

= $2\pi . \tau' . r^2 . dr$

The total torque T on the section, will be the sum of all the contributions. $T = \int_0^R 2\pi\tau' r^2 \, dr$

Since τ' is a function of r, because it varies with radius so writing down τ' in terms of r from the equation (1).

i.e $\tau' = \frac{G\theta.r}{L}$

we get $T = \int_0^R 2\pi \frac{G\theta}{L}.r^3 dr$

$T = \frac{2\pi G\theta}{L}\int_0^R r^3 dr$

$= \frac{2\pi G\theta}{L}.\left[\frac{R^4}{4}\right]_0^R$

$= \frac{G\theta}{L}.\frac{2\pi R^4}{4}$

$= \frac{G\theta}{L}.\frac{\pi R^4}{2}$

$= \frac{G\theta}{L}.\left[\frac{\pi d^4}{32}\right]$ now substituting R = d/2

$= \frac{G\theta}{L}.J$

since $\frac{\pi d^4}{32}$ = J the polar moment of inertia

or $\frac{T}{J} = \frac{G\theta}{L}$ ......(2)

if we combine the equation no.(1) and (2) we get $\boxed{\frac{T}{J} = \frac{\tau'}{r} = \frac{G.\theta}{L}}$

Where

T = applied external Torque, which is constant over Length L;

J = Polar moment of Inertia

$= \frac{\pi d^4}{32}$ for solid shaft

$= \frac{\pi(D^4 - d^4)}{32}$ for a hollow shaft. [D = Outside diameter ; d = inside diameter]

G = Modules of rigidity (or Modulus of elasticity in shear)

 = It is the angle of twist in radians on a length L.

Tensional Stiffness: The tensional stiffness k is defined as the torque per radius twist

i.e, k = T /   = GJ / L

Power Transmitted by a shaft : If T is the applied Torque and is the angular velocity of the shaft, then the power transmitted by the shaft is

$$P = T.\omega = \frac{2\pi NT}{60} = \frac{2\pi NT}{60.10^3}\,kw$$

$$\text{where } N = rpm$$

Distribution of shear stresses in circular Shafts subjected to torsion :

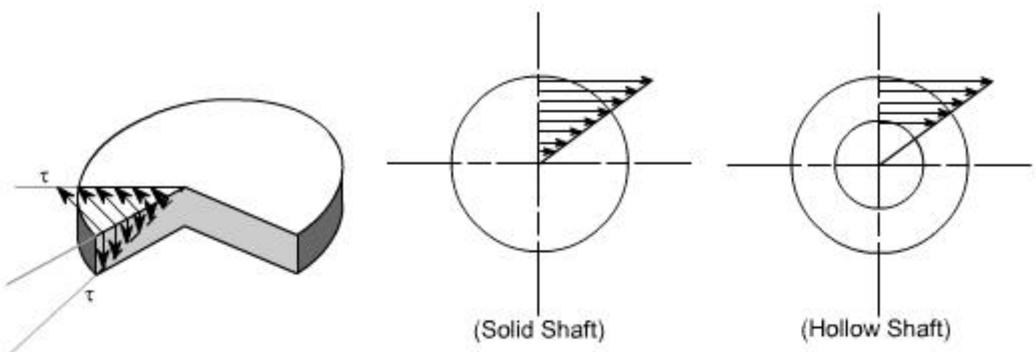
The simple torsion equation is written as

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

$$\text{or}$$

$$\tau = \frac{G\theta.r}{L}$$

This states that the shearing stress varies directly as the distance �r' from the axis of the shaft and the following is the stress distribution in the plane of cross section and also the complementary shearing stresses in an axial plane.



Hence the maximum strear stress occurs on the outer surface of the shaft where r = R

The value of maximum shearing stress in the solid circular shaft can be determined as

$$\frac{\tau}{r} = \frac{T}{J}$$

$$\tau_{max}\Big|_{r=d/2} = \frac{T.R}{J} = \frac{T}{\frac{\pi d^4}{32}} . d/2$$

where d=diameter of solid shaft

$$\text{or} \quad \tau_{max^m} = \frac{16T}{\pi d^3}$$

From the above relation, following conclusion can be drawn

(i) $\square_{max}{}^{m} \square$ T

(ii) $\square_{max}{}^{m} \square$ 1/d ^{3}

Power Transmitted by a shaft:

In practical application, the diameter of the shaft must sometimes be calculated from the power which it is required to transmit.

Given the power required to be transmitted, speed in rpm �N' Torque T, the formula connecting

These quantities can be derived as follows

$$P = T.\omega$$
$$= \frac{T.2\pi N}{60} \text{watts}$$
$$= \frac{2\pi NT}{60 \times 10^3} (\text{kw})$$

Torsional stiffness: The torsional stiffness k is defined as the torque per radian twist .

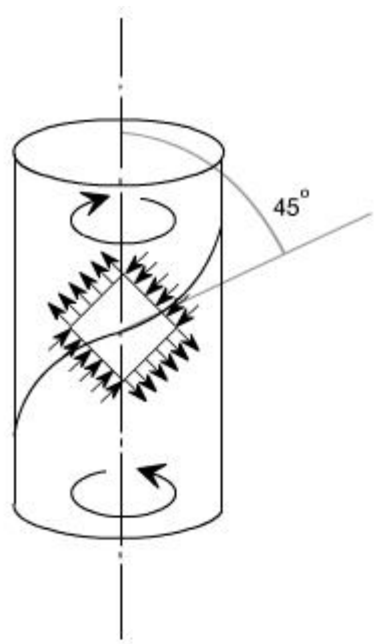
$$k = \frac{T}{\theta}$$
$$\text{i.e} \quad = \frac{GJ}{L}$$
$$\text{or} \quad k = \frac{G.J}{L}$$

For a ductile material, the plastic flow begins first in the outer surface. For a material which is weaker in shear longitudinally than transversely � for instance a wooden shaft, with the fibres parallel to axis the first cracks will be produced by the shearing stresses acting in the axial section and they will upper on the surface of the shaft in the longitudinal direction.

In the case of a material which is weaker in tension than in shear. For instance a, circular shaft of cast iron or a cylindrical piece of chalk a crack along a helix inclined at 45^0 to the axis of shaft often occurs.

Explanation: This is because of the fact that the state of pure shear is equivalent to a state of stress tension in one direction and equal compression in perpendicular direction.

A rectangular element cut from the outer layer of a twisted shaft with sides at 45^0 to the axis will be subjected to such stresses, the tensile stresses shown will produce a helical crack mentioned.



<u>TORSION OF HOLLOW SHAFTS:</u>

From the torsion of solid shafts of circular x � section , it is seen that only the material at the outer surface of the shaft can be stressed to the limit assigned as an allowable working stresses. All of the material within the shaft will work at a lower stress and is not being used to full capacity. Thus, in these cases where the weight reduction is important, it is advantageous to use hollow shafts. In discussing the torsion of hollow shafts the same assumptions will be made as in the case of a solid shaft. The general torsion equation as we have applied in the case of torsion of solid shaft will hold good

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

For the hollow shaft

$$J = \frac{\pi(D_0{}^4 - d_i{}^4)}{32} \qquad \text{where } D_0 = \text{Outside diameter}$$

$$d_i = \text{Inside diameter}$$

$$\text{Let } d_i = \frac{1}{2}.D_0$$

$$\tau_{max^m}\Big|_{solid} = \frac{16T}{\pi D_0{}^3} \qquad (1)$$

$$\tau_{max^m}\Big|_{hollow} = \frac{T.D_0/2}{\frac{\pi}{32}(D_0{}^4 - d_i{}^4)}$$

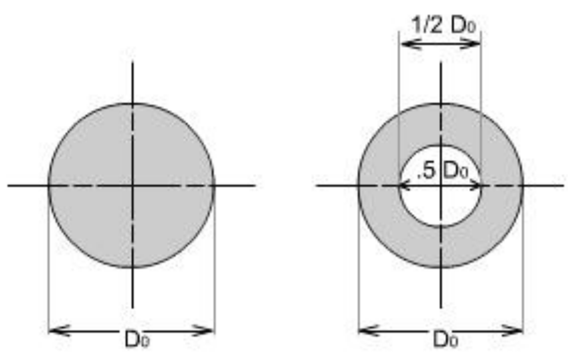
$$= \frac{16T.D_0}{\pi D_0{}^4\left[1-(d_i/D_0)^4\right]}$$

$$= \frac{16T}{\pi D_0{}^3\left[1-(1/2)^4\right]} = 1.066.\frac{16T}{\pi D_0{}^3} \qquad (2)$$

Hence by examining the equation (1) and (2) it may be seen that the □ $_{max}{}^{m}$ in the case of hollow shaft is 6.6% larger then in the case of a solid shaft having the same outside diameter.

Reduction in weight:

Considering a solid and hollow shafts of the same length 'l' and density '□' with d_i = 1/2 D_o



Weight of hollow shaft

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi (D_0/2)^2}{4}\right] l \times \rho$$

$$= \left[\frac{\pi D_0^2}{4} - \frac{\pi D_0^2}{16}\right] l \times \rho$$

$$= \frac{\pi D_0^2}{4}\left[1 - 1/4\right] l \times \rho$$

$$= 0.75 \frac{\pi D_0^2}{4} l \times \rho$$

$$\text{Weight of solid shaft} = \frac{\pi D_0^2}{4} . l . \rho$$

$$\text{Reduction in weight} = (1 - 0.75)\frac{\pi D_0^2}{4} l \times \rho$$

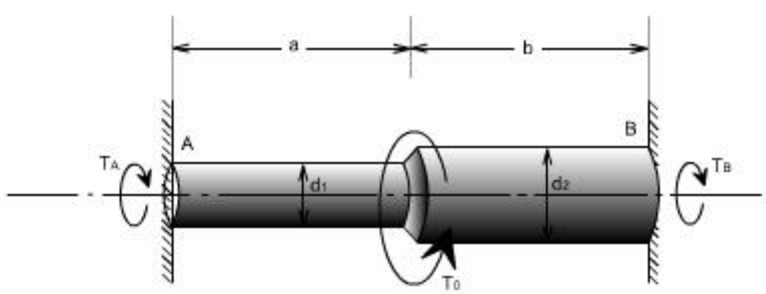
$$= 0.25 \frac{\pi D_0^2}{4} l \times \rho$$

Hence the reduction in weight would be just 25%.

<u>Illustrative Examples :</u>

<u>Problem 1</u>

A stepped solid circular shaft is built in at its ends and subjected to an externally applied torque. T_0 at the shoulder as shown in the figure. Determine the angle of rotation $\square_0$ of the shoulder section where T_0 is applied ?



Solution: This is a statically indeterminate system because the shaft is built in at both ends. All that we can find from the statics is that the sum of two reactive torque T_A and T_B at the built � in ends of the shafts must be equal to the applied torque T_0

Thus $T_A + T_B = T_0$ ------ (1)

[from static principles]

Where T_A ,T_B are the reactive torque at the built in ends A and B. wheeras T_0 is the applied torque

From consideration of consistent deformation, we see that the angle of twist in each portion of the shaft must be same.

i.e $\theta_a = \theta_b = \theta_0$

using the relation for angle of twist

$$\frac{T}{J} = \frac{G.\theta}{l}$$

$$\text{or } \theta_A = \frac{T_A a}{J_A G}$$

$$\theta_B = \frac{T_B a}{J_B G}$$

$$\Rightarrow \frac{T_A a}{J_A G} = \frac{T_B b}{J_B G} = \theta_0 \quad \text{or} \quad \frac{T_A}{T_B} = \frac{J_A}{J_B}.\frac{b}{a} \quad (2)$$

N.B: Assuming modulus of rigidity G to be same for the two portions

So the defines the ratio of T_A and T_B

So by solving (1) & (2) we get

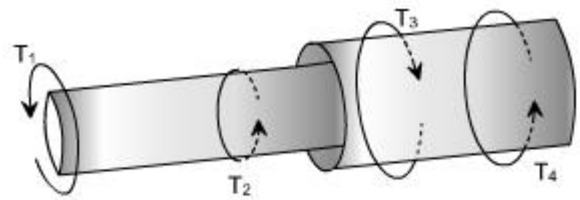
$$T_A = \frac{T_0}{1 + \frac{J_B a}{J_A b}}$$

$$T_b = \frac{T_0}{1 + \frac{J_a b}{J_b a}}$$

Using either of these values in (2) we have the angle of rotation θ_0 at the junction

$$\theta_0 = \frac{T_0.a.b}{[J_A.b + J_B.a]G}$$

Non Uniform Torsion: The pure torsion refers to a torsion of a prismatic bar subjected to torques acting only at the ends. While the non uniform torsion differs from pure torsion in a sense that the bar / shaft need not to be prismatic and the applied torques may vary along the length.



Here the shaft is made up of two different segments of different diameters and having torques applied at several cross sections. Each region of the bar between the applied loads between changes in cross section is in pure torsion, hence the formula's derived earlier may be applied. Then form the internal torque, maximum shear stress and angle of rotation for each region can be calculated from the relation

$$\frac{T}{J} = \frac{\tau}{r} \text{ and } \frac{T}{J} = \frac{G\theta}{L}$$

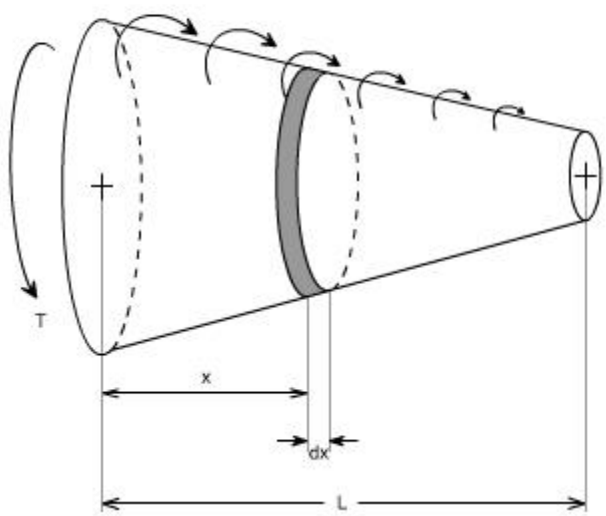
The total angle to twist of one end of the bar with respect to the other is obtained by summation using the formula

$$\theta = \sum_{i=1}^{n} \frac{T_i L_i}{G_i J_i}$$

i = index for no. of parts

n = total number of parts

If either the torque or the cross section changes continuously along the axis of the bar, then the □ (summation can be replaced by an integral sign ($\int$). i.e We will have to consider a differential element.



After considering the differential element, we can write $d\theta = \frac{T_x dx}{GI_x}$

Substituting the expressions for T_x and J_x at a distance x from the end of the bar, and then integrating between the limits 0 to L, find the value of angle of twist may be determined.

$$\theta = \int_0^L d\theta = \int_0^L \frac{T_x dx}{GI_x}$$

Closed Coiled helical springs subjected to axial loads:

Definition: A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when load is released.

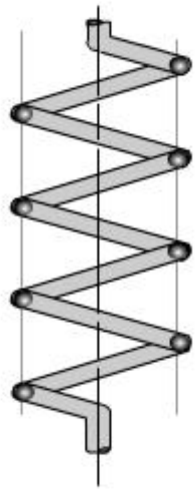
or

Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

Important types of springs are:

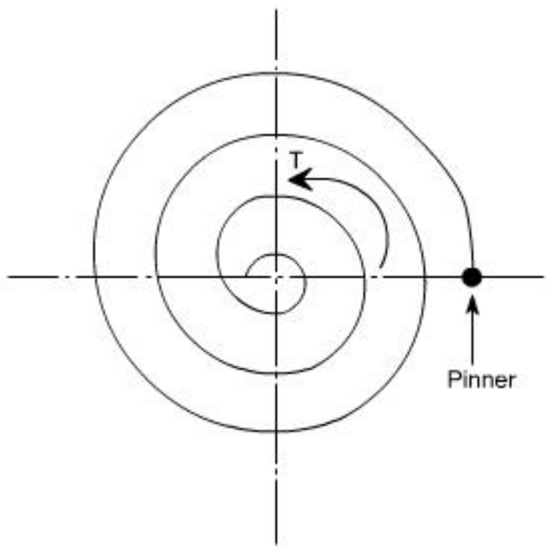
There are various types of springs such as

(i) helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix. In these type of springs the major stresses is torsional shear stress due to twisting. They are both used in tension and compression.

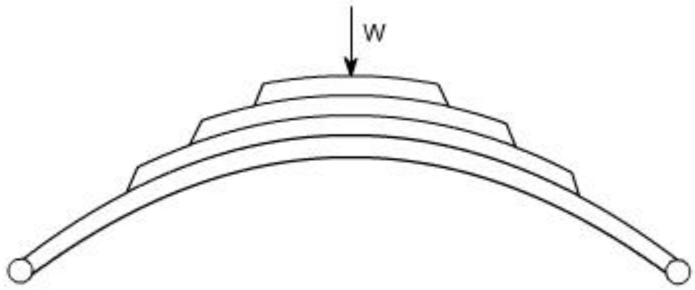


(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.

In this the major stresses are tensile and compression due to bending.



(iv) Leaf springs: They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency . Leaf springs may be full elliptic, semi elliptic or cantilever types, In these type of springs the major stresses which come into picture are tensile & compressive.



These type of springs are used in the automobile suspension system.

Uses of springs :

(a) To apply forces and to control motions as in brakes and clutches.

(b) To measure forces as in spring balance.

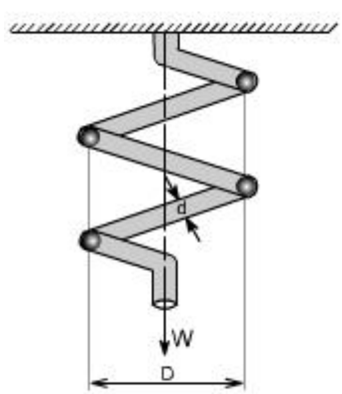
(c) To store energy as in clock springs.

(d) To reduce the effect of shock or impact loading as in carriage springs.

(e) To change the vibrating characteristics of a member as inflexible mounting of motors.

Derivation of the Formula :

In order to derive a necessary formula which governs the behaviour of springs, consider a closed coiled spring subjected to an axial load W.



Let

W = axial load

D = mean coil diameter

d = diameter of spring wire

n = number of active coils

C = spring index = D / d For circular wires

l = length of spring wire

G = modulus of rigidity

x = deflection of spring

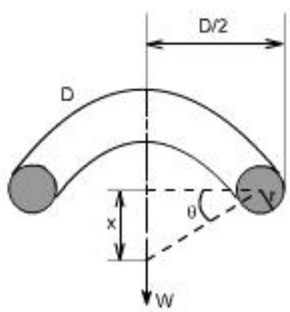
q = Angle of twist

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.

If q is the total angle of twist along the wire and x is the deflection of spring under the action of load W along the axis of the coil, so that

x = D / 2 . □

again l = □ D n [consider ,one half turn of a close coiled helical spring]



Assumptions: (1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a such a spring will be assumed to lie in a plane which is nearly □^{r} to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical. Hence to maintain equilibrium of a segment of the spring, only a shearing force V = F and Torque T = F. r are required at any X � section. In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is uniformly distributed and is negligible

so applying the torsion formula.

Using the torsion formula i.e

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G.\theta}{l}$$

and substitituting $J = \frac{\pi d^4}{32}; T = w.\frac{d}{2}$

$$\theta = \frac{2.x}{D}; l = \pi D.x$$

SPRING DEFLECTION

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{G.2x/D}{\pi D.n}$$

Thus,

$$x = \frac{8w.D^3.n}{G.d^4}$$

Spring striffness: The stiffness is defined as the load per unit deflection therefore

$$k = \frac{w}{x} = \frac{w}{\frac{8w.D^3.n}{G.d^4}}$$

Therefore

$$k = \frac{G.d^4}{8.D^3.n}$$

Shear stress

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max^m}}{d/2}$$

or $\tau_{max^m} = \frac{8wD}{\pi d^3}$

WAHL'S FACTOR :

In order to take into account the effect of direct shear and change in coil curvature a stress factor is defined, which is known as Wahl's factor

K = Wahl' s factor and is defined as $K = \frac{4c-1}{4c-4} + \frac{0.615}{c}$

Where C = spring index

= D/d

if we take into account the Wahl's factor than the formula for the shear stress becomes $\tau_{max^m} = \frac{16.T.k}{\pi d^3}$

Strain Energy : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

In the case of a spring the strain energy would be due to bending and the strain energy due to bending is given by the expansion

$$U = \frac{T^2 L}{2EI}$$

$$L = \pi D n$$

$$I = \frac{\pi d^4}{64}$$

so after substitution we get

$$U = \frac{32T^2 Dn}{E.d^4}$$

Example: A close coiled helical spring is to carry a load of 5000N with a deflection of 50 mm and a maximum shearing stress of 400 N/mm^2 .if the number of active turns or active coils is 8.Estimate the following:

(i) wire diameter

(ii) mean coil diameter

(iii) weight of the spring.

Assume G = 83,000 N/mm^2 ; □ = 7700 kg/m^3

solution :

(i) for wire diametre if W is the axial load, then

$$\frac{w.d/2}{\frac{\pi d^4}{32}} = \frac{\tau_{max}m}{d/2}$$

$$D = \frac{400}{d/2}.\frac{\pi d^4}{32}.\frac{2}{W}$$

$$D = \frac{400.\pi d^3.2}{5000.16}$$

$$D = 0.0314\,d^3$$

Futher, deflection is given as

$$x = \frac{8wD^3.n}{G.d^4}$$

on substituting the relevant parameters we get

$$50 = \frac{8.5000.(0.0314\,d^3)^3.8}{83{,}000.d^4}$$

$$d = 13.32\,mm$$

Therefore,

$D = .0314 \text{ x } (13.317)^3$mm

=74.15mm

D = 74.15 mm

Weight

mass or weight = volume. density

= area.length of the spring.density of spring material

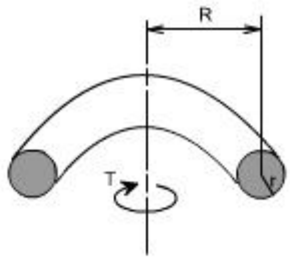
$$= \frac{\pi d^2}{4}.\pi D n.\rho$$

On substituting the relevant parameters we get

Weight = 1.996 kg

= 2.0kg

Close � coiled helical spring subjected to axial torque T or axial couple.



In this case the material of the spring is subjected to pure bending which tends to reduce Radius R of the coils. In this case the bending moment is constant through out the spring and is equal to the applied axial Torque T. The stresses i.e. maximum bending stress may thus be determined from the bending theory.

$$\sigma_{max} = \frac{M.y}{I} = \frac{T.d/2}{\frac{\pi d^4}{64}}$$

$$\sigma_{max} = \frac{32T}{\pi d^3}$$

Deflection or wind � up angle:

Under the action of an axial torque the deflection of the spring becomes the �wind � up� angle of the spring which is the angle through which one end turns relative to the other. This will be equal to the total change of slope along the wire, according to area � moment theorem

$$\theta = \int_0^L \frac{MdL}{EI} \quad \text{but } M = T$$

$$= \int_0^L \frac{T.dL}{EI} = \frac{T}{EI}\int_0^L dL$$

Thus, as 'T ' remains constant

$$\theta = \frac{T.L}{EI}$$

Futher

$$L = \pi D.n$$

$$I = \frac{\pi d^4}{64}$$

Therefore, on substitution, the value of θ obtained is

$$\boxed{\theta = \frac{64T.D.n}{E.d^4}}$$

Springs in Series: If two springs of different stiffness are joined endon and carry a common load W, they are said to be connected in series and the combined stiffness and deflection are given by the following equation.

$$\frac{W}{k} = x_1 + x_2 = \frac{W}{k_1} + \frac{W}{k_2}$$

or

$$\boxed{\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}}$$

Springs in parallel: If the two spring are joined in such a way that they have a common deflection �x' ; then they are said to be connected in parallel.In this care the load carried is shared between the two springs and total load $W = W_1 + W_2$

$$x = \frac{W}{k} = \frac{W_1}{k_1} = \frac{W_2}{k_2}$$

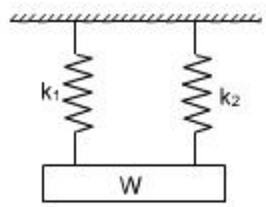
Thus $W_1 = \frac{Wk_1}{k}$

$W_2 = \frac{Wk_2}{k}$

Futher

$W=W_1+W_2$

thus $\boxed{k = k_1 + k_2}$



UNIT III

PART-A (2 Marks)

1. Define torsional rigidity of the solid circular shaft.
2. Distinguish between closed coil helical spring and open coil helical spring
3. What is meant by composite shaft?
4. What is called Twisting moment?
5. What is Polar Modulus ?
6. Define: Torsional rigidity of a shaft.
7. What do mean by strength of a shaft?
8. Write down the equation for Wahl factor.
9. Define: Torsional stiffness.
10. What are springs? Name the two important types.

PART- B (16 Marks)

1. Determine the diameter of a solid shaft which will transmit 300 KN at 250 rpm. The maximum shear stress should not exceed 30 N/mm2 and twist should not be more than 10 in a shaft length 2m. Take modulus of rigidity = 1x 105N/mm2.

2. The stiffness of the closed coil helical spring at mean diameter 20 cm is made of 3 cm diameter rod and has 16 turns. A weight of 3 KN is dropped on this spring. Find the height by which the weight should be dropped before striking the spring so that the spring may be compressed by 18 cm. Take C= 8x104 N/mm2.

3. It is required to design a closed coiled helical spring which shall deflect 1mm under an axial load of 100 N at a shear stress of 90 Mpa. The spring is to be made of round wire having shear modulus of 0.8 x 105 Mpa. The mean diameter of the coil is 10 times that of the coil wire. Find the diameter and length of the wire.

4. A steel shaft ABCD having a total length of 2400 mm is contributed by three different sections as follows. The portion AB is hollow having outside and inside diameters 80 mm and 50 mm respectively, BC is solid and 80 mm diameter. CD is also solid and 70 mm diameter. If the angle of twist is same for each section, determine the length of each portion and the total angle of twist. Maximum permissible shear stress is 50 Mpa and shear modulus 0.82 x 105 MPa

5. The stiffness of close coiled helical spring is 1.5 N/mm of compression under a maximum load of 60 N. The maximum shear stress in the wire of the spring is 125 N/mm2. The solid length of the spring (when the coils are touching) is 50 mm. Find the diameter of coil, diameter of wire and number of coils. C = 4.5

UNIT IV BEAM DEFLECTION

Elastic curve of Neutral axis of the beam under normal loads – Evaluation of beam deflection and slope: Double integration method, Macaulay method, and Moment-area method – Columns – End conditions – Equivalent length of a column – Euler equation – Slenderness ratio – Rankine formula for columns.

Deflection of Beams

Introduction:

In all practical engineering applications, when we use the different components, normally we have to operate them within the certain limits i.e. the constraints are placed on the performance and behavior of the components. For instance we say that the particular component is supposed to operate within this value of stress and the deflection of the component should not exceed beyond a particular value.

In some problems the maximum stress however, may not be a strict or severe condition but there may be the deflection which is the more rigid condition under operation. It is obvious therefore to study the methods by which we can predict the deflection of members under lateral loads or transverse loads, since it is this form of loading which will generally produce the greatest deflection of beams.

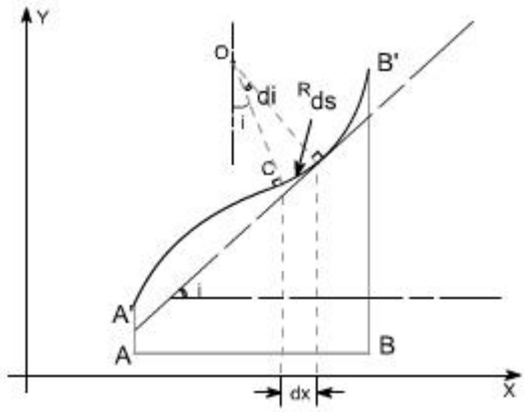
Assumption: The following assumptions are undertaken in order to derive a differential equation of elastic curve for the loaded beam

1. Stress is proportional to strain i.e. hooks law applies. Thus, the equation is valid only for beams that are not stressed beyond the elastic limit.

2. The curvature is always small.

3. Any deflection resulting from the shear deformation of the material or shear stresses is neglected.

It can be shown that the deflections due to shear deformations are usually small and hence can be ignored.



Consider a beam AB which is initially straight and horizontal when unloaded. If under the action of loads the beam deflect to a position A'B' under load or infact we say that the axis of the beam bends to a shape A'B'. It is customary to call A'B' the curved axis of the beam as the elastic line or deflection curve.

In the case of a beam bent by transverse loads acting in a plane of symmetry, the bending moment M varies along the length of the beam and we represent the variation of bending moment in B.M diagram. Futher, it is assumed that the simple bending theory equation holds good.

$$\boxed{\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}}$$

If we look at the elastic line or the deflection curve, this is obvious that the curvature at every point is different; hence the slope is different at different points.

To express the deflected shape of the beam in rectangular co-ordinates let us take two axes x and y, x-axis coincide with the original straight axis of the beam and the y – axis shows the deflection.

Futher,let us consider an element ds of the deflected beam. At the ends of this element let us construct the normal which intersect at point O denoting the angle between these two normal be di

But for the deflected shape of the beam the slope i at any point C is defined,

$\tan i = \frac{dy}{dx}$ ......(1) or $i = \frac{dy}{dx}$ Assuming $\tan i = i$

Futher

$ds = Rdi$

however,

$ds = dx$ [usually for small curvature]

Hence

$ds = dx = Rdi$

or $\boxed{\frac{di}{dx} = \frac{1}{R}}$

substituting the value of i, one get

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{R}$ or $\frac{d^2y}{dx^2} = \frac{1}{R}$

From the simple bending theory

$\frac{M}{I} = \frac{E}{R}$ or $M = \frac{EI}{R}$

so the basic differential equation governing the deflection of beams is

$$\boxed{M = EI\frac{d^2y}{dx^2}}$$

This is the differential equation of the elastic line for a beam subjected to bending in the plane of symmetry. Its solution y = f(x) defines the shape of the elastic line or the deflection curve as it is frequently called.

Relationship between shear force, bending moment and deflection: The relationship among shear force,bending moment and deflection of the beam may be obtained as

Differentiating the equation as derived

$\frac{dM}{dx} = EI\frac{d^3y}{dx^3}$ Recalling $\frac{dM}{dx} = F$

Thus,

$$F = EI\frac{d^3y}{dx^3}$$

Therefore, the above expression represents the shear force whereas rate of intensity of loading can also be found out by differentiating the expression for shear force

$$\text{i.e } w = -\frac{dF}{dx}$$

$$w = -EI\frac{d^4y}{dx^4}$$

Therefore if 'y' is the deflection of the loaded beam, then the following important relations can be arrived at

$$\boxed{\text{slope} = \frac{dy}{dx}}$$

$$\boxed{B.M = EI\frac{d^2y}{dx^2}}$$

$$\boxed{\text{Shear force} = EI\frac{d^3y}{dx^3}}$$

$$\boxed{\text{load distribution} = EI\frac{d^4y}{dx^4}}$$

Methods for finding the deflection: The deflection of the loaded beam can be obtained various methods.The one of the method for finding the deflection of the beam is the direct integration method, i.e. the method using the differential equation which we have derived.

Direct integration method: The governing differential equation is defined as

$$M = EI\frac{d^2y}{dx^2} \quad \text{or} \quad \frac{M}{EI} = \frac{d^2y}{dx^2}$$

on integrating one get,

$$\frac{dy}{dx} = \int\frac{M}{EI}dx + A \text{ ---- this equation gives the slope of the loaded beam.}$$

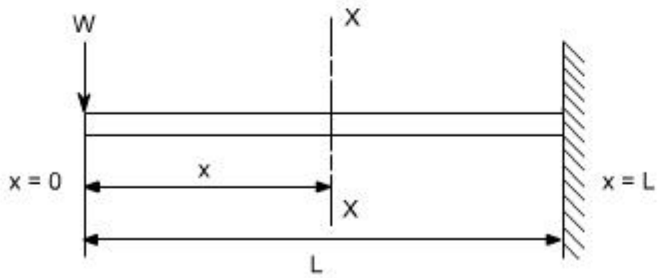
Integrate once again to get the deflection.

$$y = \int\int\frac{M}{EI}dx + Ax + B$$

Where A and B are constants of integration to be evaluated from the known conditions of slope and deflections for the particular value of x.

Illustrative examples : let us consider few illustrative examples to have a familiarty with the direct integration method

Case 1: Cantilever Beam with Concentrated Load at the end:- A cantilever beam is subjected to a concentrated load W at the free end, it is required to determine the deflection of the beam



In order to solve this problem, consider any X-section X-X located at a distance x from the left end or the reference, and write down the expressions for the shear force abd the bending moment

$$S.F|_{x-x} = -W$$

$$B.M|_{x-x} = -W.x$$

Therefore $M|_{x-x} = -W.x$

the governing equation $\frac{M}{EI} = \frac{d^2y}{dx^2}$

substituting the value of M interms of x then integrating the equation one get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{Wx}{EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{Wx}{EI}dx$$

$$\frac{dy}{dx} = -\frac{Wx^2}{2EI} + A$$

Integrating once more,

$$\int \frac{dy}{dx} = \int -\frac{Wx^2}{2EI}dx + \int A\,dx$$

$$y = -\frac{Wx^3}{6EI} + Ax + B$$

The constants A and B are required to be found out by utilizing the boundary conditions as defined below

i.e at x= L ; y= 0 --------------------- (1)

at x = L ; dy/dx = 0 -------------------- (2)

Utilizing the second condition, the value of constant A is obtained as

$$A = \frac{Wl^2}{2EI}$$

While employing the first condition yields

$$y = -\frac{WL^3}{6EI} + AL + B$$

$$B = \frac{WL^3}{6EI} - AL$$

$$= \frac{WL^3}{6EI} - \frac{WL^3}{2EI}$$

$$= \frac{WL^3 - 3WL^3}{6EI} = -\frac{2WL^3}{6EI}$$

$$B = -\frac{WL^3}{3EI}$$

Substituting the values of A and B we get

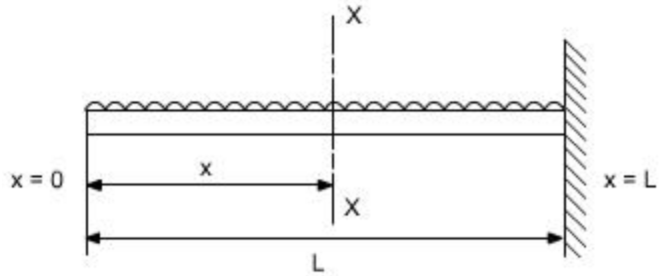
$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6EI} + \frac{WL^2x}{2EI} - \frac{WL^3}{3EI}\right]$$

The slope as well as the deflection would be maximum at the free end hence putting x=0 we get,

$$\boxed{y_{max} = -\frac{WL^3}{3EI}}$$

$$\boxed{(\text{Slope})_{max}m = +\frac{WL^2}{2EI}}$$

Case 2: A Cantilever with Uniformly distributed Loads:- In this case the cantilever beam is subjected to U.d.l with rate of intensity varying w / length.The same procedure can also be adopted in this case



$$S.F|_{x-x} = -w$$

$$B.M|_{x-x} = -w.x.\frac{x}{2} = w\left(\frac{x^2}{2}\right)$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -\frac{wx^2}{2EI}$$

$$\int \frac{d^2y}{dx^2} = \int -\frac{wx^2}{2EI}\,dx$$

$$\frac{dy}{dx} = -\frac{wx^3}{6EI} + A$$

$$\int \frac{dy}{dx} = \int -\frac{wx^3}{6EI}\,dx + \int A\,dx$$

$$y = -\frac{wx^4}{24EI} + Ax + B$$

Boundary conditions relevant to the problem are as follows:

1. At x = L; y = 0

2. At x= L; dy/dx = 0

The second boundary conditions yields

$$A = +\frac{wx^3}{6EI}$$

whereas the first boundary conditions yields

$$B = \frac{wL^4}{24EI} - \frac{wL^4}{6EI}$$

$$B = -\frac{wL^4}{8EI}$$

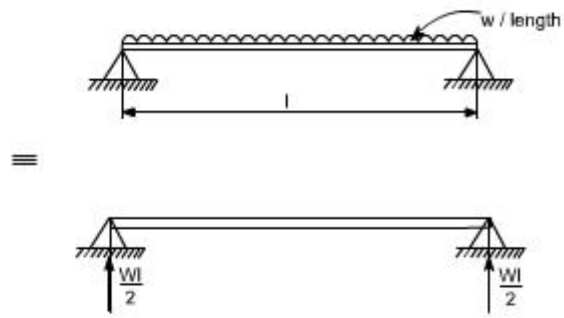
$$\text{Thus, } y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right]$$

So $y_{max}m$ will be at $x = 0$

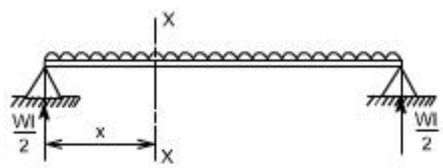
$$\boxed{y_{max}m = -\frac{wL^4}{8EI}}$$

$$\boxed{\left(\frac{dy}{dx}\right)_{max}m = \frac{wL^3}{6EI}}$$

Case 3: Simply Supported beam with uniformly distributed Loads:- In this case a simply supported beam is subjected to a uniformly distributed load whose rate of intensity varies as w / length.



In order to write down the expression for bending moment consider any cross-section at distance of x metre from left end support.



$$S.F|_{x-x} = w\left(\frac{l}{2}\right) - w.x$$

$$B.M|_{x-x} = w.\left(\frac{l}{2}\right).x - w.x.\left(\frac{x}{2}\right)$$

$$= \frac{wl.x}{2} - \frac{wx^2}{2}$$

The differential equation which gives the elastic curve for the deflected beam is

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{1}{EI}\left[\frac{wl.x}{2} - \frac{wx^2}{2}\right]$$

$$\frac{dy}{dx} = \int \frac{wlx}{2EI}dx - \int \frac{wx^2}{2EI}dx + A$$

$$= \frac{wlx^2}{4EI} - \frac{wx^3}{6EI} + A$$

Integrating, once more one gets

$$y = \frac{wlx^3}{12EI} - \frac{wx^4}{24EI} + A.x + B \qquad \text{-----(1)}$$

Boundary conditions which are relevant in this case are that the deflection at each support must be zero.

i.e. at x = 0; y = 0 : at x = l; y = 0

let us apply these two boundary conditions on equation (1) because the boundary conditions are on y, This yields B = 0.

$$0 = \frac{wl^4}{12EI} - \frac{wl^4}{24EI} + A.l$$

$$A = -\frac{wl^3}{24EI}$$

So the equation which gives the deflection curve is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

Futher

In this case the maximum deflection will occur at the centre of the beam where x = L/2 [i.e. at the position where the load is being applied].So if we substitute the value of x = L/2

Then $$y_{max^m} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$\boxed{y_{max^m} = -\frac{5wL^4}{384EI}}$$

Conclusions

(i) The value of the slope at the position where the deflection is maximum would be zero.

(ii) Thevalue of maximum deflection would be at the centre i.e. at x = L/2.

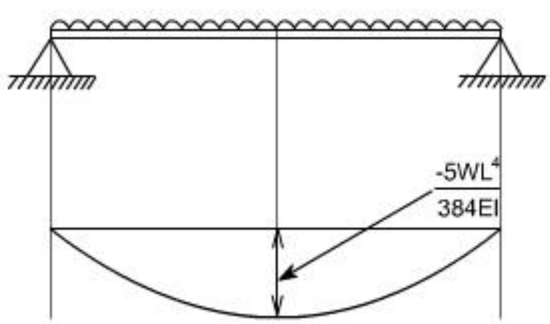
The final equation which is governs the deflection of the loaded beam in this case is

$$y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$

By successive differentiation one can find the relations for slope, bending moment, shear force and rate of loading.

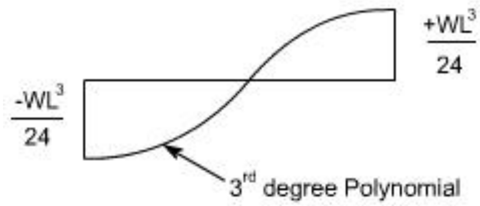
Deflection (y)

$$yEI = \left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right]$$



Slope (dy/dx)

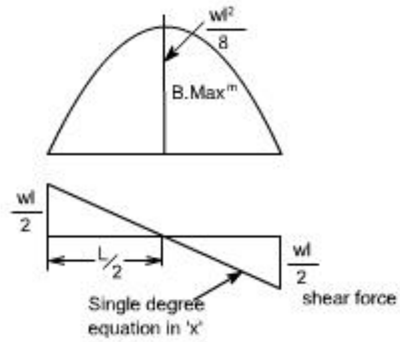
$$EI.\frac{dy}{dx} = \left[\frac{3wLx^2}{12} - \frac{4wx^3}{24} - \frac{wL^3}{24}\right]$$



So the bending moment diagram would be

Bending Moment

$$\frac{d^2y}{dx^2} = \frac{1}{EI}\left[\frac{wLx}{2} - \frac{wx^2}{2}\right]$$



Shear Force

Shear force is obtained by taking

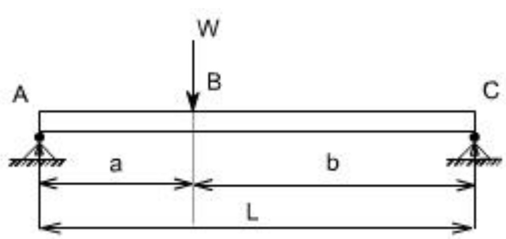
third derivative.

$$EI\frac{d^3y}{dx^3} = \frac{wL}{2} - w.x$$

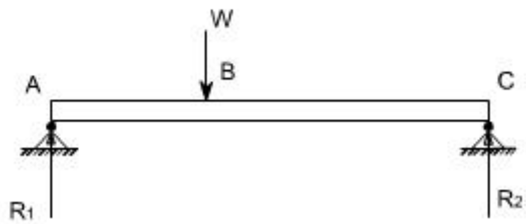
Rate of intensity of loading

$$EI\frac{d^4y}{dx^4} = -w$$

Case 4: The direct integration method may become more involved if the expression for entire beam is not valid for the entire beam.Let us consider a deflection of a simply supported beam which is subjected to a concentrated load W acting at a distance 'a' from the left end.



Let R_1 & R_2 be the reactions then,



B.M for the portion AB

$M|_{AB} = R_1.x \quad 0 \le x \le a$

B.M for the portion BC

$M|_{BC} = R_1.x - W(x - a) \quad a \le x \le l$

so the differential equation for the two cases would be,

$$EI\frac{d^2y}{dx^2} = R_1\,x$$

$$EI\frac{d^2y}{dx^2} = R_1\,x - W\,(x - a)$$

These two equations can be integrated in the usual way to find ‘y' but this will result in four constants of integration two for each equation. To evaluate the four constants of integration, four independent boundary conditions will be needed since the deflection of each support must be zero, hence the boundary conditions (a) and (b) can be realized.

Further, since the deflection curve is smooth, the deflection equations for the same slope and deflection at the point of application of load i.e. at x = a. Therefore four conditions required to evaluate these constants may be defined as follows:

(a) at x = 0; y = 0 in the portion AB i.e. $0 \le x \le a$

(b) at x = l; y = 0 in the portion BC i.e. $a \le x \le l$

(c) at x = a; dy/dx, the slope is same for both portion

(d) at x = a; y, the deflection is same for both portion

By symmetry, the reaction R_1 is obtained as

$$R_1 = \frac{Wb}{a+b}$$

Hence,

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x \qquad 0 \le x \le a \text{ --------(1)}$$

$$EI\frac{d^2y}{dx^2} = \frac{Wb}{(a+b)}x - W(x-a) \qquad a \le x \le l \text{ ---------(2)}$$

integrating (1) and (2) we get,

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k_1 \qquad 0 \le x \le a \text{ --------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k_2 \qquad a \le x \le l \text{ --------(4)}$$

Using condition (c) in equation (3) and (4) shows that these constants should be equal, hence letting

$$K_1 = K_2 = K$$

Hence

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 + k \qquad 0 \le x \le a \text{ ------(3)}$$

$$EI\frac{dy}{dx} = \frac{Wb}{2(a+b)}x^2 - \frac{W(x-a)^2}{2} + k \qquad a \le x \le l \text{ ------- (4)}$$

Integrating agian equation (3) and (4) we get

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3 \qquad 0 \le x \le a \text{ ------(5)}$$

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4 \qquad a \le x \le l \text{ ------(6)}$$

Utilizing condition (a) in equation (5) yields

$$k_3 = 0$$

Utilizing condition (b) in equation (6) yields

$$0 = \frac{Wb}{6(a+b)}l^3 - \frac{W(l-a)^3}{6} + kl + k_4$$

$$k_4 = -\frac{Wb}{6(a+b)}l^3 + \frac{W(l-a)^3}{6} - kl$$

But $a + b = l$,

Thus,

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b)$$

Now lastly k_3 is found out using condition (d) in equation (5) and equation (6), the condition (d) is that,

At x = a; y; the deflection is the same for both portion

Therefore $y|_{\text{from equation 5}} = y|_{\text{from equation 6}}$

or

$$\frac{Wb}{6(a+b)}x^3 + kx + k_3 = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

$$\frac{Wb}{6(a+b)}a^3 + ka + k_3 = \frac{Wb}{6(a+b)}a^3 - \frac{W(a-a)^3}{6} + ka + k_4$$

Thus, $k_4 = 0;$

OR

$$k_4 = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6} - k(a+b) = o$$

$$k(a+b) = -\frac{Wb(a+b)^2}{6} + \frac{Wb^3}{6}$$

$$k = -\frac{Wb(a+b)}{6} + \frac{Wb^3}{6(a+b)}$$

so the deflection equations for each portion of the beam are

$$EIy = \frac{Wb}{6(a+b)}x^3 + kx + k_3$$

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{- - - -}\mathbf{for\ 0 \le x \le a}\text{ - - - - - (7)}$$

and for other portion

$$EIy = \frac{Wb}{6(a+b)}x^3 - \frac{W(x-a)^3}{6} + kx + k_4$$

Substituting the value of 'k' in the above equation

$$EIy = \frac{Wbx^3}{6(a+b)} - \frac{W(x-a)^3}{6} - \frac{Wb(a+b)x}{6} + \frac{Wb^3x}{6(a+b)} \qquad \text{For }\mathbf{for\ a \le x \le l}\text{ - - - - - (8)}$$

so either of the equation (7) or (8) may be used to find the deflection at x = a

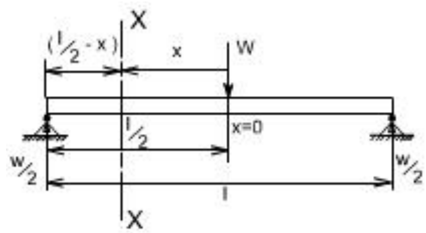
hence substituting x = a in either of the equation we get

$$Y|_{x=a} = -\frac{Wa^2b^2}{3EI(a+b)}$$

OR if a = b = l/2

$$\boxed{Y_{max^m} = -\frac{WL^3}{48EI}}$$

ALTERNATE METHOD: There is also an alternative way to attempt this problem in a more simpler way. Let us considering the origin at the point of application of the load,



$$S.F|_{xx} = \frac{W}{2}$$

$$B.M|_{xx} = \frac{W}{2}\left(\frac{l}{2} - x\right)$$

substituting the value of M in the governing equation for the deflection

$$\frac{d^2y}{dx^2} = \frac{\frac{W}{2}\left(\frac{l}{2} - x\right)}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WLx}{4} - \frac{Wx^2}{4}\right] + A$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^2}{12}\right] + Ax + B$$

Boundary conditions relevant for this case are as follows

(i) at x = 0; dy/dx= 0

hence, A = 0

(ii) at x = l/2; y = 0 (because now l / 2 is on the left end or right end support since we have taken the origin at the centre)

Thus,

$$0 = \left[\frac{WL^3}{32} - \frac{WL^3}{96} + B\right]$$

$$B = -\frac{WL^3}{48}$$

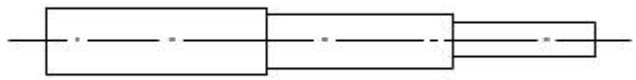
Hence he equation which governs the deflection would be

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right]$$

Hence

$$\boxed{\begin{array}{lll} Y_{max^m}\Big|_{at\,x=0} = -\dfrac{WL^3}{48EI} & & \text{At the centre} \\ \left(\dfrac{dy}{dx}\right)_{max^m}\Bigg|_{at\,x=\pm\frac{L}{2}} = \pm\dfrac{WL^2}{16EI} & & \text{At the ends} \end{array}}$$

Hence the integration method may be bit cumbersome in some of the case. Another limitation of the method would be that if the beam is of non uniform cross section,



i.e. it is having different cross-section then this method also fails.

So there are other methods by which we find the deflection like

1. Macaulay's method in which we can write the different equation for bending moment for different sections.

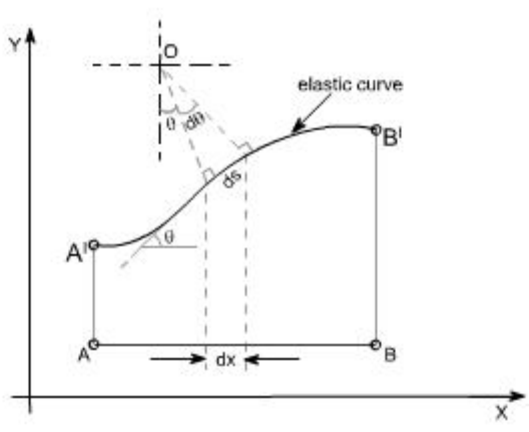
2. Area moment methods

3. Energy principle methods

THE AREA-MOMENT / MOMENT-AREA METHODS:

The area moment method is a semi graphical method of dealing with problems of deflection of beams subjected to bending. The method is based on a geometrical interpretation of definite integrals. This is applied to cases where the equation for bending moment to be written is cumbersome and the loading is relatively simple.

Let us recall the figure, which we referred while deriving the differential equation governing the beams.



It may be noted that d**q** is an angle subtended by an arc element ds and M is the bending moment to which this element is subjected.

We can assume,

ds = dx [since the curvature is small]

hence, R d**q** = ds

$$\frac{d\theta}{ds} = \frac{1}{R} = \frac{M}{EI}$$

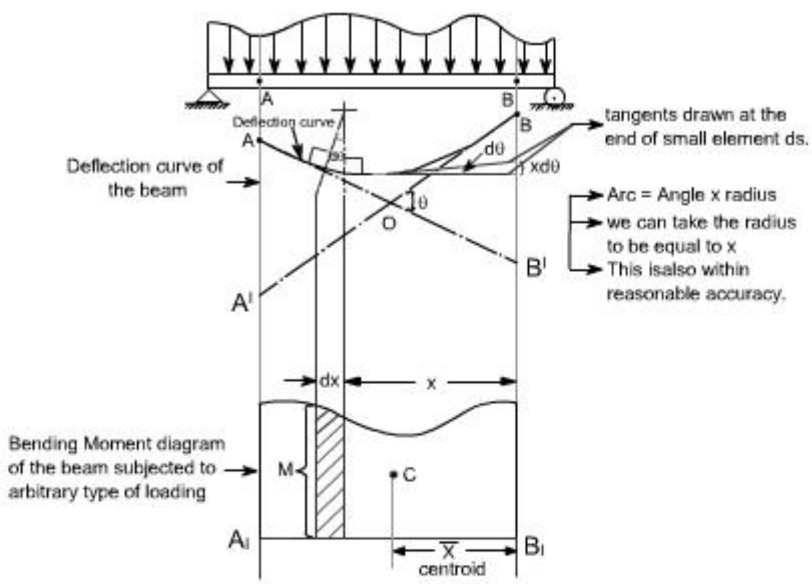
$$\frac{d\theta}{ds} = \frac{M}{EI}$$

But for small curvature[but θ is the angle ,slope is $\tan\theta = \frac{dy}{dx}$ for small angles $\tan\theta \approx \theta$,hence $\theta \cong \frac{dy}{dx}$ so we get $\frac{d^2y}{dx^2} = \frac{M}{EI}$ by putting $ds \approx dx$]

Hence,

$$\frac{d\theta}{dx} = \frac{M}{EI} \text{ or } \boxed{d\theta = \frac{M.dx}{EI}} \quad - - - - - (1)$$

The relationship as described in equation (1) can be given a very simple graphical interpretation with reference to the elastic plane of the beam and its bending moment diagram



Refer to the figure shown above consider AB to be any portion of the elastic line of the loaded beam and $A_1B_1$is its corresponding bending moment diagram.

Let AO = Tangent drawn at A

BO = Tangent drawn at B

Tangents at A and B intersects at the point O.

Futher, AA ' is the deflection of A away from the tangent at B while the vertical distance B'B is the deflection of point B away from the tangent at A. All these quantities are futher understood to be very small.

Let ds ≈ dx be any element of the elastic line at a distance x from B and an angle between at its tangents be d**q**. Then, as derived earlier

$$d\theta = \frac{M.dx}{EI}$$

This relationship may be interpreted as that this angle is nothing but the area M.dx of the shaded bending moment diagram divided by EI.

From the above relationship the total angle **q** between the tangents A and B may be determined as

$$\boxed{\theta = \int_A^B \frac{Mdx}{EI} = \frac{1}{EI}\int_A^B Mdx}$$

Since this integral represents the total area of the bending moment diagram, hence we may conclude this result in the following theorem

<u>Theorem I:</u>

$$\left\{\begin{matrix}\text{slope or }\theta \\ \text{between any two points}\end{matrix}\right\} = \left\{\begin{matrix}\frac{1}{EI}\times\text{area of B.M diagram between} \\ \text{corresponding portion of B.M diagram}\end{matrix}\right\}$$

Now let us consider the deflection of point B relative to tangent at A, this is nothing but the vertical distance BB'. It may be note from the bending diagram that bending of the element ds contributes to this deflection by an amount equal to x d**q** [each of this intercept may be considered as the arc of a circle of radius x subtended by the angle **q**]

Hence the total distance B'B becomes $\delta = \int_A^B x\,d\theta$

The limits from A to B have been taken because A and B are the two points on the elastic curve, under consideration]. Let us substitute the value of d**q** = M dx / EI as derived earlier

$$\delta = \int_A^B x\frac{Mdx}{EI} = \int_A^B \frac{Mdx}{EI}.x$$ [This is infact the moment of area of the bending moment diagram]

Since M dx is the area of the shaded strip of the bending moment diagram and x is its distance from B, we therefore conclude that right hand side of the above equation represents first moment area with respect to B of the total bending moment area between A and B divided by EI.

Therefore,we are in a position to state the above conclusion in the form of theorem as follows:

Theorem II:

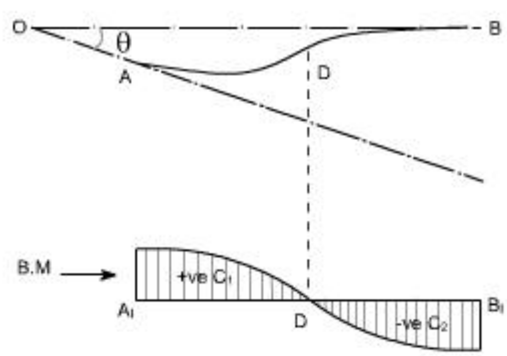
Deflection of point 'B' relative to point A $= \frac{1}{EI} \times \left\{ \begin{array}{l} \text{first moment of area with respect} \\ \text{to point B, of the total B.M diagram} \end{array} \right\}$

Futher, the first moment of area, according to the definition of centroid may be written as $A\bar{x}$, where $\bar{x}$ is equal to distance of centroid and a is the total area of bending moment

Thus, $\boxed{\delta_A = \frac{1}{EI} A\bar{X}}$

Therefore,the first moment of area may be obtained simply as a product of the total area of the B.M diagram betweenthe points A and B multiplied by the distance $\bar{x}$ to its centroid C.

If there exists an inflection point or point of contreflexure for the elastic line of the loaded beam between the points A and B, as shown below,



Then, adequate precaution must be exercised in using the above theorem. In such a case B. M diagram gets divide into two portions +ve and –ve portions with centroids $C_1$and C_2. Then to find an angle **q** between the tangentsat the points A and B

$$\theta = \int_A^D \frac{Mdx}{EI} - \int_D^B \frac{Mdx}{EI}$$

And similarly for the deflection of B away from the tangent at A becomes

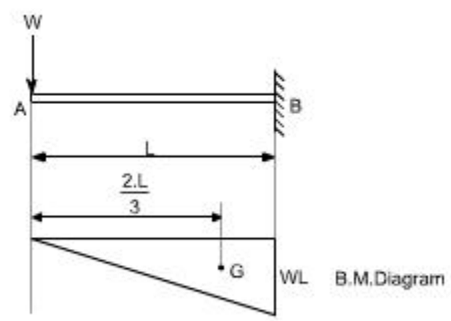
$$\delta = \int_A^D \frac{M.dx}{EI}.x - \int_B^D \frac{M.dx}{EI}.x$$

Illustrative Examples: Let us study few illustrative examples, pertaining to the use of these theorems

Example 1:

1. A cantilever is subjected to a concentrated load at the free end.It is required to find out the deflection at the free end.

Fpr a cantilever beam, the bending moment diagram may be drawn as shown below



Let us workout this problem from the zero slope condition and apply the first area - moment theorem

$$\text{slope at A} = \frac{1}{EI}\left[\text{Area of B.M diagram between the points A and B}\right]$$
$$= \frac{1}{EI}\left[\frac{1}{2}L.WL\right]$$
$$= \frac{WL^2}{2EI}$$

The deflection at A (relative to B) may be obtained by applying the second area - moment theorem

NOTE: In this case the point B is at zero slope.

Thus,

$$\delta = \frac{1}{EI}[\text{first moment of area of B.M diagram between A and B about A}]$$

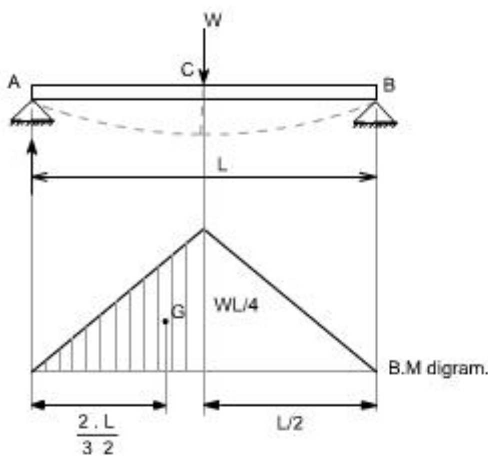
$$= \frac{1}{EI}[A\bar{y}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}L.WL\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{3EI}$$

Example 2: Simply supported beam is subjected to a concentrated load at the mid span determine the value of deflection.

A simply supported beam is subjected to a concentrated load W at point C. The bending moment diagram is drawn below the loaded beam.



Again working relative to the zero slope at the centre C.

$$\text{slope at } A = \frac{1}{EI}[\text{Area of B.M diagram between A and C}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\right]$$ we are taking half area of the B.M because we have to work out this relative to a zero slope

$$= \frac{WL^2}{16EI}$$

Deflection of A relative to C = central deflection of C

or

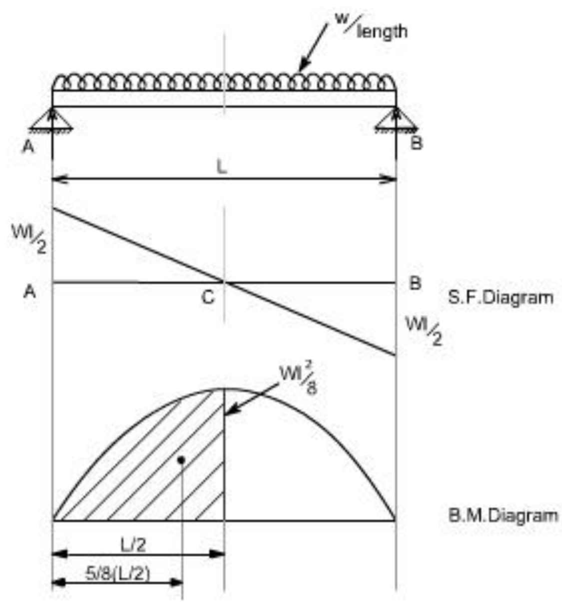
$$\delta_C = \frac{1}{EI}[\text{Moment of B.M diagram between points A and C about A}]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{2}\right)\left(\frac{L}{2}\right)\left(\frac{WL}{4}\right)\frac{2}{3}L\right]$$

$$= \frac{WL^3}{48EI}$$

Example 3: A simply supported beam is subjected to a uniformly distributed load, with a intensity of loading W / length. It is required to determine the deflection.

The bending moment diagram is drawn, below the loaded beam, the value of maximum B.M is equal to $Wl^2 / 8$



So by area moment method,

$$\text{Slope at point C w.r.t point A} = \frac{1}{EI}\left[\text{Area of B.M diagram between point A and C}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\right)\left(\frac{WL^2}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{WL^3}{24EI}$$

Deflection at point C relative to A

$$= \frac{1}{EI}\left[A\,\bar{y}\right]$$

$$= \frac{1}{EI}\left[\left(\frac{WL^3}{24}\right)\left(\frac{5}{8}\right)\left(\frac{L}{2}\right)\right]$$

$$= \frac{5}{384EI}.WL^4$$

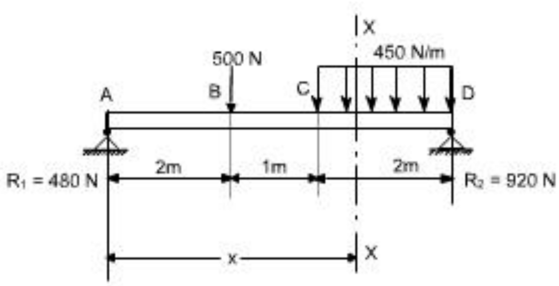
Macaulay's Methods

If the loading conditions change along the span of beam, there is corresponding change in moment equation. This requires that a separate moment equation be written between each change of load point and that two integration be made for each such moment equation. Evaluation of the constants introduced by each integration can become very involved. Fortunately, these complications can be avoided by writing single moment equation in such a way that it becomes continuous for entire length of the beam in spite of the discontinuity of loading.

Note : In Macaulay's method some author's take the help of unit function approximation (i.e. Laplace transform) in order to illustrate this method, however both are essentially the same.

For example consider the beam shown in fig below:

Let us write the general moment equation using the definition $M = (\sum M)_L$, Which means that we consider the effects of loads lying on the left of an exploratory section. The moment equations for the portions AB,BC and CD are written as follows

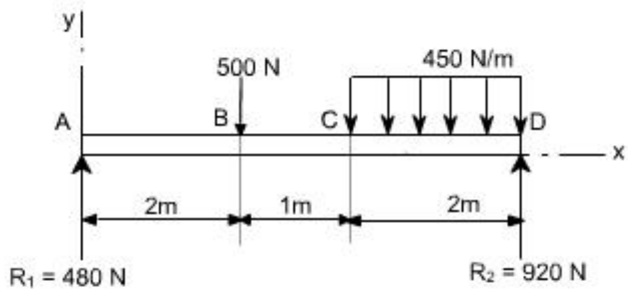


$$M_{AB} = 480\ x\ \text{N.m}$$

$$M_{BC} = \left[480\ x - 500\,(x-2)\right]\text{N.m}$$

$$M_{CD} = \left[480\ x - 500\,(x-2) - \frac{450}{2}(x-3)^2\right]\text{N.m}$$

It may be observed that the equation for M_{CD} will also be valid for both M_{AB} and M_{BC} provided that the terms (x - 2) and $(x - 3)^2$ are neglected for values of x less than 2 m and 3 m, respectively. In other words, the terms (x - 2) and $(x - 3)^2$ are nonexistent for values of x for which the terms in parentheses are negative.

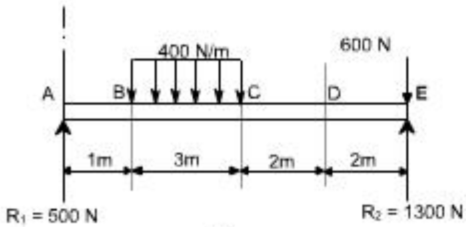


As an clear indication of these restrictions,one may use a nomenclature in which the usual form of parentheses is replaced by pointed brackets, namely, ‹ ›. With this change in nomenclature, we obtain a single moment equation

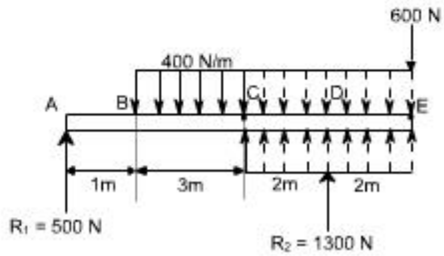
$$M = \left(480x - 500\langle x - 2 \rangle - \frac{450}{2}\langle x - 3 \rangle^2 \right) \mathrm{N.m}$$

Which is valid for the entire beam if we postulate that the terms between the pointed brackets do not exists for negative values; otherwise the term is to be treated like any ordinary expression.

As an another example, consider the beam as shown in the fig below. Here the distributed load extends only over the segment BC. We can create continuity, however, by assuming that the distributed load extends beyond C and adding an equal upward-distributed load to cancel its effect beyond C, as shown in the adjacent fig below. The general moment equation, written for the last segment DE in the new nomenclature may be written as:



(a)



$$M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right) \text{N.m}$$

It may be noted that in this equation effect of load 600 N won't appear since it is just at the last end of the beam so if we assume the exploratary just at section at just the point of application of 600 N than x = 0 or else we will here take the X - section beyond 600 N which is invalid.

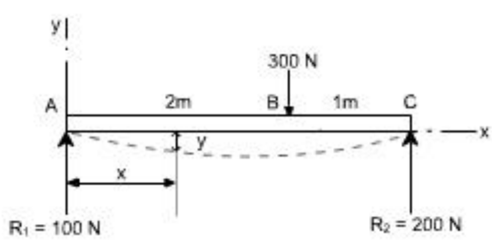
Procedure to solve the problems

(i). After writing down the moment equation which is valid for all values of ‘x' i.e. containing pointed brackets, integrate the moment equation like an ordinary equation.

(ii). While applying the B.C's keep in mind the necessary changes to be made regarding the pointed brackets.

llustrative Examples :

1. A concentrated load of 300 N is applied to the simply supported beam as shown in Fig.Determine the equations of the elastic curve between each change of load point and the maximum deflection in the beam.



Solution : writing the general moment equation for the last portion BC of the loaded beam,

$$EI\frac{d^2y}{dx^2} = M = \left(100x - 300\langle x-2\rangle\right)N.m \qquad \ldots\ldots.(1)$$

Integrating twice the above equation to obtain slope and the deflection

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 + C_1\right)N.m^2 \qquad \ldots\ldots.(2)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 + C_1x + C_2\right)N.m^3 \quad \ldots\ldots.(3)$$

To evaluate the two constants of integration. Let us apply the following boundary conditions:

1. At point A where x = 0, the value of deflection y = 0. Substituting these values in Eq. (3) we find C_2 = 0.keep in mind that $< x -2 >^3$ is to be neglected for negative values.

2. At the other support where x = 3m, the value of deflection y is also zero.

substituting these values in the deflection Eq. (3), we obtain

$$0 = \left(\frac{50}{3}3^3 - 50\langle 3-2\rangle^3 + 3.C_1\right) \text{or } C_1 = -133N.m^2$$

Having determined the constants of integration, let us make use of Eqs. (2) and (3) to rewrite the slope and deflection equations in the conventional form for the two portions.

segment AB ($0 \le x \le 2m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 133\right)N.m^2 \qquad \ldots\ldots.(4)$$

$$EIy = \left(\frac{50}{3}x^3 - 133x\right)N.m^3 \qquad \ldots\ldots.(5)$$

segment BC ($2m \le x \le 3m$)

$$EI\frac{dy}{dx} = \left(50x^2 - 150\langle x-2\rangle^2 - 133x\right)N.m^2 \ldots\ldots.(6)$$

$$EIy = \left(\frac{50}{3}x^3 - 50\langle x-2\rangle^3 - 133x\right)N.m^3 \ldots\ldots.(7)$$

Continuing the solution, we assume that the maximum deflection will occur in the segment AB. Its location may be found by differentiating Eq. (5) with respect to x and setting the derivative to be equal to zero, or, what amounts to the same thing, setting the slope equation (4) equal to zero and solving for the point of zero slope.

We obtain

$50x^2 – 133 = 0$ or x = 1.63 m (It may be kept in mind that if the solution of the equation does not yield a value < 2 m then we have to try the other equations which are valid for segment BC)

Since this value of x is valid for segment AB, our assumption that the maximum deflection occurs in this region is correct. Hence, to determine the maximum deflection, we substitute x = 1.63 m in Eq (5), which yields

$$EIy\big|_{max} m = -145 N.m^3 \qquad ....(8)$$

The negative value obtained indicates that the deflection y is downward from the x axis.quite usually only the magnitude of the deflection, without regard to sign, is desired; this is denoted by d, the use of y may be reserved to indicate a directed value of deflection.

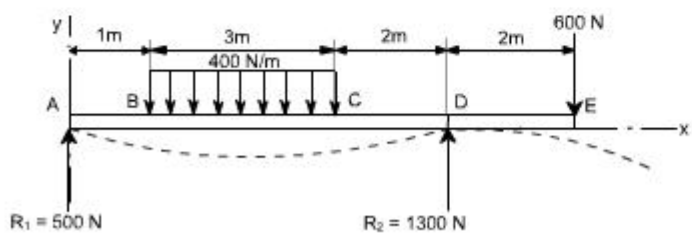
if E = 30 Gpa and I = 1.9 x 10^6 mm^4 = 1.9 x 10^{-6} m^4 , Eq. (h) becomes

$$y\big|_{max} m = \left(30\times10^9\right)\left(1.9\times10^{-6}\right)$$

Then $= -2.54 mm$

Example 2:

It is required to determine the value of EIy at the position midway between the supports and at the overhanging end for the beam shown in figure below.



Solution:

Writing down the moment equation which is valid for the entire span of the beam and applying the differential equation of the elastic curve, and integrating it twice, we obtain

$$EI\frac{d^2y}{dx^2} = M = \left(500x - \frac{400}{2}(x-1)^2 + \frac{400}{2}(x-4)^2 + 1300(x-6)\right)N.m$$

$$EI\frac{dy}{dx} = \left(250x^2 - \frac{200}{3}(x-1)^3 + \frac{200}{3}(x-4)^3 + 650(x-6)^2 + C_1\right)N.m$$

$$EIy = \left(\frac{250}{3}x^3 - \frac{50}{3}(x-1)^4 + \frac{50}{3}(x-4)^4 + \frac{650}{3}(x-6)^3 + C_1x + C_2\right)N.m^3$$

To determine the value of C_2, It may be noted that EIy = 0 at x = 0,which gives C_2 = 0.Note that the negative terms in the pointed brackets are to be ignored Next,let us use the condition that EIy = 0 at the right support where x = 6m.This gives

$$0 = \frac{250}{3}(6)^3 - \frac{50}{3}(5)^4 + \frac{50}{3}(2)^4 + 6C_1 \text{ or } C_1 = -1308 \text{N.m}^2$$

Finally, to obtain the midspan deflection, let us substitute the value of x = 3m in the deflection equation for the segment BC obtained by ignoring negative values of the bracketed terms á x - 4 ñ^4 and á x - 6 ñ^3. We obtain

$$EIy = \frac{250}{3}(3)^3 - \frac{50}{3}(2)^4 - 1308(3) = -1941 \text{ N.m}^3$$

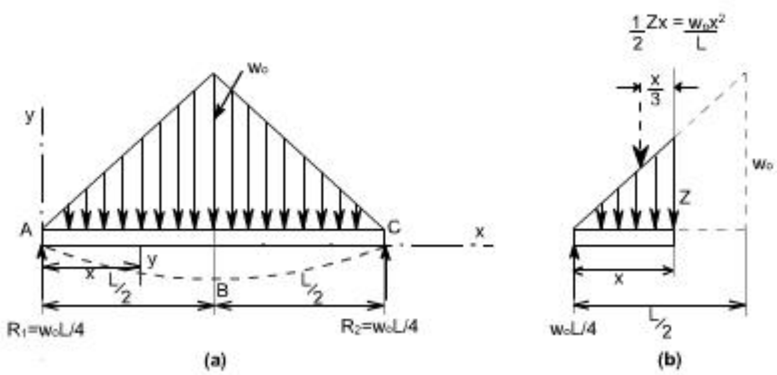
For the overhanging end where x=8 m,we have

$$EIy = \left(\frac{250}{3}(8)^3 - \frac{50}{3}(7)^4 + \frac{50}{3}(4)^4 + \frac{650}{3}(2)^3 - 1308(8)\right)$$

$$= -1814 \text{N.m}^3$$

Example 3:

A simply supported beam carries the triangularly distributed load as shown in figure. Determine the deflection equation and the value of the maximum deflection.



Solution:

Due to symmetry, the reactionsis one half the total load of 1/2w_0L, or $R_1 = R_2 = 1/4w_0L$.Due to the advantage of symmetry to the deflection curve from A to B is the mirror image of that from C to B. The condition of zero deflection at A and of zero slope at B do not require the use of a general moment equation. Only the moment equation for segment AB is needed, and this may be easily written with the aid of figure(b).

Taking into account the differential equation of the elastic curve for the segment AB and integrating twice, one can obtain

$$EI\frac{d^2y}{dx^2} = M_{AB} = \frac{w_0L}{4}x - \frac{w_0x^2}{L}.\frac{x}{3} \quad ......(1)$$

$$EI\frac{dy}{dx} = \frac{w_0Lx^2}{8} - \frac{w_0x^4}{12L} + C_1 \quad ......(2)$$

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} + C_1x + C_2 ......(3)$$

In order to evaluate the constants of integration,let us apply the B.C'swe note that at the support A, y = 0 at x = 0.Hence from equation (3), we get $C_2 = 0$. Also,because of symmetry, the slope dy/dx = 0 at midspan where x = L/2.Substituting these conditions in equation (2) we get

$$0 = \frac{w_0L}{8}\left(\frac{L}{2}\right)^2 - \frac{w_0}{12L}\left(\frac{L}{2}\right)^4 + C_1 C_1 = -\frac{5w_0L^3}{192}$$

Hence the deflection equation from A to B (and also from C to B because of symmetry) becomes

$$EIy = \frac{w_0Lx^3}{24} - \frac{w_0x^5}{60L} - \frac{5w_0L^3x}{192}$$

Whichreducesto

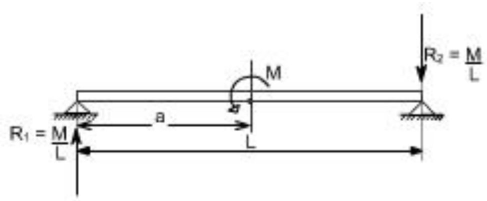
$$EIy = -\frac{w_0x}{960L}\left(25L^4 - 40L^2x^2 + 16x^4\right)$$

The maximum deflection at midspan,where x = L/2 is then found to be

$$\boxed{EIy = -\frac{w_0L^4}{120}}$$

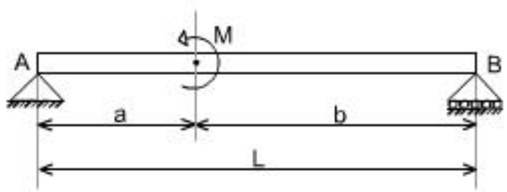
Example 4: couple acting

Consider a simply supported beam which is subjected to a couple M at adistance 'a' from the left end. It is required to determine using the Macauley's method.



To deal with couples, only thing to remember is that within the pointed brackets we have to take some quantity and this should be raised to the power zero.i.e. M á x - a ñ^0 . We have taken the power 0 (zero) ' because ultimately the term M á x - a ñ^0 Should have the moment units.Thus with integration the quantity á x - a ñ becomes either á x - a ñ^{1}or á x - a ñ^2

Or



Therefore, writing the general moment equation we get

$$M = R_1 x - M\langle x - a\rangle \text{ or } EI\frac{d^2y}{dx^2} = M$$

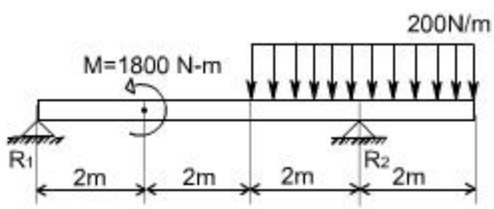
Integrating twice we get

$$EI\frac{dy}{dx} = R_1.\frac{x^2}{2} - M\langle x - a\rangle^1 + C_1$$

$$EI.y = R_1.\frac{x^3}{6} - \frac{M}{2}\langle x - a\rangle^2 + C_1 x + C_2$$

Example 5:

A simply supported beam is subjected to U.d.l in combination with couple M. It is required to determine the deflection.



This problem may be attemped in the some way. The general moment equation my be written as

$$\begin{aligned} M(x) &= R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle\langle x - 4\rangle}{2} + R_2\langle x - 6\rangle \\ &= R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle \end{aligned}$$

Thus,

$$EI\frac{d^2y}{dx^2} = R_1 x - 1800\langle x - 2\rangle^0 - \frac{200\langle x - 4\rangle^2}{2} + R_2\langle x - 6\rangle$$

Integrate twice to get the deflection of the loaded beam.

Elastic Stability Of Columns

Introduction:

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions.

Columns:

Short, thick members are generally termed columns and these usually fail by crushing when the yield stress of the material in compression is exceeded.

Struts:

Long, slender columns are generally termed as struts, they fail by buckling some time before the yield stress in compression is reached. The buckling occurs owing to one the following reasons.

(a). the strut may not be perfectly straight initially.

(b). the load may not be applied exactly along the axis of the Strut.

(c). one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties through out the strut.

In all the problems considered so far we have assumed that the deformation to be both progressive with increasing load and simple in form i.e. we assumed that a member in simple tension or compression becomes progressively longer or shorter but remains straight. Under some circumstances however, our assumptions of progressive and simple deformation may no longer hold good and the member become unstable. The term strut and column are widely used, often interchangeably in the context of buckling of slender members.]

At values of load below the buckling load a strut will be in stable equilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should than be possible to gently deflect the strut into a simple sine wave provided that the amplitude of wave is kept small.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling, this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached, owing to the reasons stated earlier.

The resistance of any member to bending is determined by its flexural rigidity EI and is The quantity I may be written as $I = Ak^2$,

Where I = area of moment of inertia

A = area of the cross-section

k = radius of gyration.

The load per unit area which the member can withstand is therefore related to k. There will be two principal moments of inertia, if the least of these is taken then the ratio

$$\frac{l}{k} \quad \text{i.e.} \quad \frac{\text{length of member}}{\text{least radius of gyration}}$$

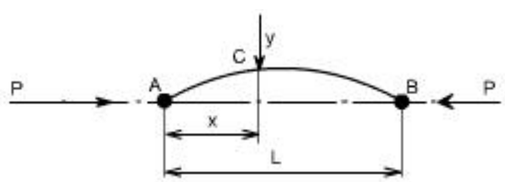
Is called the slenderness ratio. It's numerical value indicates whether the member falls into the class of columns or struts.

Euler's Theory : The struts which fail by buckling can be analyzed by Euler's theory. In the following sections, different cases of the struts have been analyzed.

Case A: Strut with pinned ends:

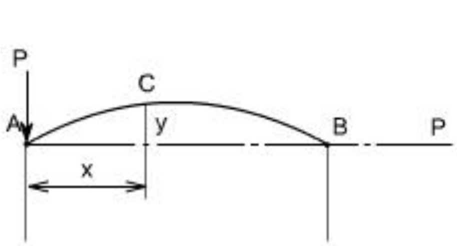
Consider an axially loaded strut, shown below, and is subjected to an axial load ‘P' this load ‘P' produces a deflection ‘y' at a distance ‘x' from one end.

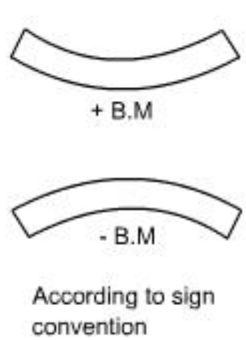
Assume that the ends are either pin jointed or rounded so that there is no moment at either end.



Assumption:

The strut is assumed to be initially straight, the end load being applied axially through centroid.





$B.M|_C = -Py$

Futher,we know that

$$EI\,\frac{d^2y}{dx^2} = M$$

$$EI\,\frac{d^2y}{dx^2} = -P.y = M$$

In this equation ‘M' is not a function ‘x'. Therefore this equation can not be integrated directly as has been done in the case of deflection of beams by integration method.

Thus,

$$EI\,\frac{d^2y}{dx^2} + Py = 0$$

Though this equation is in ‘y' but we can't say at this stage where the deflection would be maximum or minimum.

So the above differential equation can be arranged in the following form $\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$

Let us define a operator

D = d/dx

$(D^2 + n^2)$ y =0 where n^2 = P/EI

This is a second order differential equation which has a solution of the form consisting of complimentary function and particular integral but for the time being we are interested in the complementary solution only[in this P.I = 0; since the R.H.S of Diff. equation = 0]

Thus y = A cos (nx) + B sin (nx)

Where A and B are some constants.

Therefore $y = A\cos\sqrt{\frac{P}{EI}}\,x + B\sin\sqrt{\frac{P}{EI}}\,x$

In order to evaluate the constants A and B let us apply the boundary conditions,

(i) at x = 0; y = 0

(ii) at x = L ; y = 0

Applying the first boundary condition yields A = 0.

Applying the second boundary condition gives

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$$

Thus either $B = 0$, or $\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0$

if B=0, that y0 for all values of x hence the strut has not buckled yet. Therefore, the solution required is

$$\sin\left(L\sqrt{\frac{P}{EI}}\right) = 0 \quad \text{or} \quad \left(L\sqrt{\frac{P}{EI}}\right) = \pi \text{ or } nL = \pi$$

$$\text{or} \quad \sqrt{\frac{P}{EI}} = \frac{\pi}{L} \quad \text{or} \quad P = \frac{\pi^2 EI}{L^2}$$

From the above relationship the least value of P which will cause the strut to buckle, and it is called the " **Euler Crippling Load** " P_e from which w obtain.

$$\boxed{P_e = \frac{\pi^2 EI}{L^2}}$$

It may be noted that the value of I used in this expression is the least moment of inertia

It should be noted that the other solutions exists for the equation

$$\sin\left(l\sqrt{\frac{P}{EI}}\right) = 0 \qquad \text{i.e. } \sin nL = 0$$

The interpretation of the above analysis is that for all the values of the load P, other than those which make sin nL = 0; the strut will remain perfectly straight since

y = B sin nL = 0

For the particular value of

$$P_e = \frac{\pi^2 EI}{L^2}$$

$$\sin nL = 0 \quad \text{or } nL = \pi$$

$$\text{Therefore} \quad n = \frac{\pi}{L}$$

$$\text{Hence } y = B\sin nx = B \sin\frac{\pi x}{L}$$

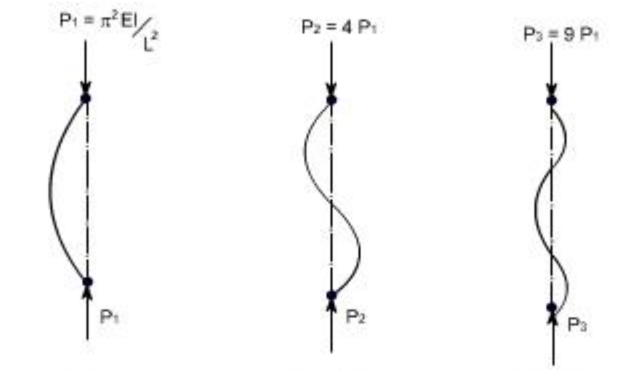
Then we say that the strut is in a state of neutral equilibrium, and theoretically any deflection which it suffers will be maintained. This is subjected to the limitation that 'L' remains sensibly constant and in practice slight increase in load at the critical value will cause the deflection to increase appreciably until the material fails by yielding.

Further it should be noted that the deflection is not proportional to load, and this applies to all strut problems; like wise it will be found that the maximum stress is not proportional to load.

The solution chosen of nL = p is just one particular solution; the solutions nL= 2p, 3p, 5p etc are equally valid mathematically and they do, infact, produce values of ‘Pe' which are equally valid for modes of buckling of strut different from that of a simple bow. Theoretically therefore, there are an infinite number of values of Pe , each corresponding with a different mode of buckling.

The value selected above is so called the fundamental mode value and is the lowest critical load producing the single bow buckling condition.

The solution nL = 2p produces buckling in two half – waves, 3p in three half-waves etc.



$nL = \pi$
Fundamental Mode
(First harmonic)

$nL = 2\pi$
Second harmonic
(mid point bracing)

$nL = 3\pi$
Third harmonic
(Third point bracing)

$$L\sqrt{\frac{P}{EI}} = \pi \quad \text{or} \quad P_1 = \frac{\pi^2 EI}{L^2}$$

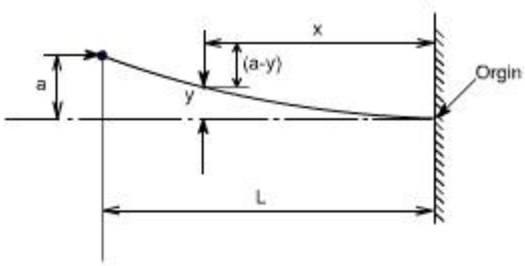
$$\text{If } L\sqrt{\frac{P}{EI}} = 2\pi \text{ or } P_2 = \frac{4\pi^2 EI}{L^2} = 4P_1$$

$$\text{If } L\sqrt{\frac{P}{EI}} = 3\pi \text{ or } P_3 = \frac{9\pi^2 EI}{L^2} = 9P_1$$

If load is applied sufficiently quickly to the strut, then it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse.

<u>struts and columns with other end conditions:</u> Let us consider the struts and columns having different end conditions

Case b: One end fixed and the other free:



writing down the value of bending moment at the point C

$$B.M|_c = P(a - y)$$

Hence, the differential equation becomes,

$$EI \frac{d^2y}{dx^2} = P(a - y)$$

On rearranging we get

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

Let $\frac{P}{EI} = n^2$

Hence in operator form, the differential equation reduces to $(D^2 + n^2) y = n^2a$

The solution of the above equation would consist of complementary solution and particular solution, therefore

y_{gen} = A cos(nx) + sin(nx) + P. I

where

P.I = the P.I is a particular value of y which satisfies the differential equation

Hence $y_{P.I}$ = a

Therefore the complete solution becomes

Y = A cos(nx) + B sin(nx) + a

Now imposing the boundary conditions to evaluate the constants A and B

(i) at x = 0; y = 0

This yields A = -a

(ii) at x = 0; dy/dx = 0

This yields B = 0

Hence

y = -a cos(nx) + a

Futher, at x = L; y = a

Therefore a = - a cos(nx) + a   or 0 = cos(nL)

Now the fundamental mode of buckling in this case would be

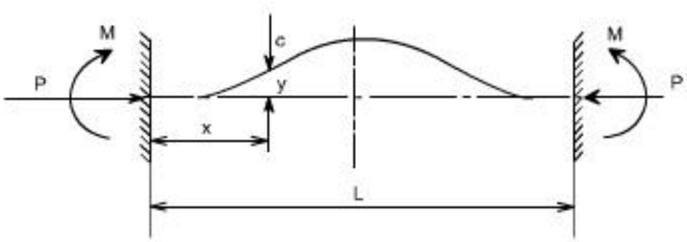
$$nL = \frac{\pi}{2}$$

$\sqrt{\frac{P}{EI}}\, L = \frac{\pi}{2}$, Therefore, the Euler's crippling load is given as

$$P_e = \frac{\pi^2 EI}{4L^2}$$

Case 3

Strut with fixed ends:



Due to the fixed end supports bending moment would also appears at the supports, since this is the property of the support.

Bending Moment at point C = M – P.y

$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\text{or } \frac{d^2y}{dx^2} + \frac{P}{EI} = \frac{M}{EI}$$

$n^2 = \frac{P}{EI}$, Therefore in the operator from, the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{M}{EI}$$

$$y_{general} = y_{complementary} + y_{particular\,integral}$$

$$y|_{P.I} = \frac{M}{n^2EI} = \frac{M}{P}$$

Hence the general solution would be

$$y = B\,Cosnx + A\,Sinnx + \frac{M}{P}$$

Boundry conditions relevant to this case are at x=0;y=0

$$B = -\frac{M}{P}$$

Also at $x = 0; \frac{dy}{dx} = 0$ hence

A=0

Therefore,

$$y = -\frac{M}{P}\,Cos\,nx + \frac{M}{P}$$

$$y = \frac{M}{P}\left(1 - Cos\,nx\right)$$

Futher, it may be noted that at $x = L; y = 0$

$$\text{Then } 0 = \frac{M}{P}\left(1 - Cos\,nL\right)$$

Thus, either $\frac{M}{P} = 0$ or $\left(1 - Cos\,nL\right) = 0$

obviously, $\left(1 - Cos\,nL\right) = 0$

$\cos nL = 1$

Hence the least solution would be

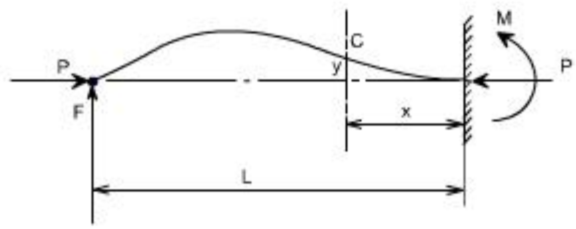
$nL = 2\pi$

$\sqrt{\frac{P}{EI}}\,L = 2\pi$, Thus, the buckling load or crippling load is

Thus, $$P_e = \frac{4\pi^2.EI}{L^2}$$

Case 4

<u>One end fixed, the other pinned</u>



In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin. The moment of F about the built in end then balances the fixing moment.

With the origin at the built in end, the B,M at C is given as

$$EI\frac{d^2y}{dx^2} = -Py + F(L - x)$$

$$EI\frac{d^2y}{dx^2} + Py = F(L - x)$$

Hence

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L - x)$$

In the operator form the equation reduces to

$$\left(D^2 + n^2\right)y = \frac{F}{EI}(L - x)$$

$$y_{particular} = \frac{F}{n^2EI}(L - x) \text{ or } y = \frac{F}{P}(L - x)$$

The full solution is therefore

$$y = A\,Cos\,mx + B\,Sin\,nx + \frac{F}{P}(L - x)$$

The boundry conditions relevants to the problem are at $x=0; y=0$

$$\text{Hence } A = -\frac{FL}{P}$$

$$\text{Also at } x = 0; \frac{dy}{dx} = 0$$

$$\text{Hence } B = \frac{F}{nP}$$

$$\text{or } y = -\frac{FL}{P}Cos\,nx + \frac{F}{nP}Sin\,nx + \frac{F}{P}(L - x)$$

$$y = \frac{F}{nP}\left[Sin\,nx - nL\,Cos\,nx + n(L - x)\right]$$

Also when x = L ; y = 0

Therefore

nL Cos nL = Sin nL   or tan nL = nL

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is nL = 4.49radian

$$\text{or} \sqrt{\frac{P}{EI}}\, L = 4.49$$

$$\frac{P_e}{EI} L^2 = 20.2$$

$$P_e = \frac{2.05\pi^2\ EI}{L^2}$$

Equivalent Strut Length:

Having derived the results for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form.

$$\text{i.e. } P_e = \frac{\pi^2 EI}{L^2}$$

Where L is the equivalent length of the strut and can be related to the actual length of the strut depending on the end conditions.

The equivalent length is found to be the length of a simple bow(half sine wave) in each of the strut deflection curves shown. The buckling load for each end condition shown is then readily obtained. The use of equivalent length is not restricted to the Euler's theory and it will be used in other derivations later.

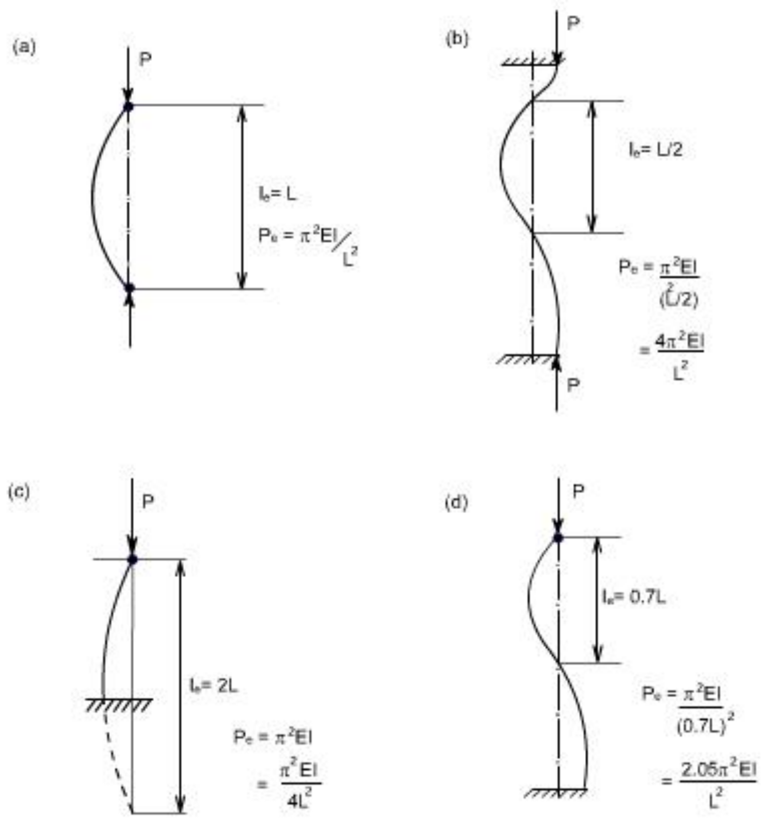
The critical load for columns with other end conditions can be expressed in terms of the critical load for a hinged column, which is taken as a fundamental case.

For case(c) see the figure, the column or strut has inflection points at quarter points of its unsupported length. Since the bending moment is zero at a point of inflection, the freebody diagram would indicates that the middle half of the fixed ended is equivalent to a hinged column having an effective length $L_e = L / 2$.

The four different cases which we have considered so far are:

(a) Both ends pinned (c) One end fixed, other free

(b) Both ends fixed (d) One end fixed and other pinned



Comparison of Euler Theory with Experiment results

Limitations of Euler's Theory :

In practice the ideal conditions are never [i.e. the strut is initially straight and the end load being applied axially through centroid] reached. There is always some eccentricity and initial curvature present. These factors needs to be accommodated in the required formula's.

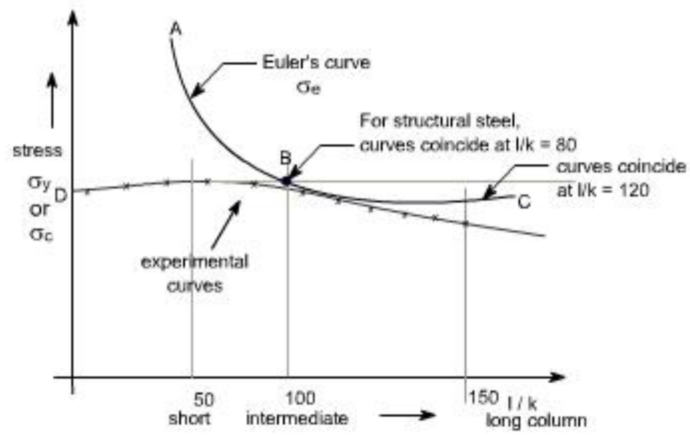
It is realized that, due to the above mentioned imperfections the strut will suffer a deflection which increases with load and consequently a bending moment is introduced which causes failure before the Euler's load is reached. Infact failure is by stress rather than by buckling and the deviation from the Euler value is more marked as the slenderness-ratio l/k is reduced. For values of l/k < 120 approx, the error in applying the Euler theory is too great to allow of its use. The stress to cause buckling from the Euler formula for the pin ended strut is

$$\text{Euler's stress, } \sigma_e = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

$$\text{But, } I = Ak^2$$

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

A plot of s_e versus l / k ratio is shown by the curve ABC.



Allowing for the imperfections of loading and strut, actual values at failure must lie within and below line CBD.

Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio i.e.l/k=40 to l/k=100.

(a) Straight – line formulae :

The permissible load is given by the formulae

$P = \sigma_y A\left[1 - n\left(\frac{l}{k}\right)\right]$ Where the value of index ‘n' depends on the material used and the end conditions.

(b) Johnson parabolic formulae : The Johnson parabolic formulae is defined as

$P = \sigma_y A\left[1 - b\left(\frac{l}{k}\right)^2\right]$ where the value of index ‘b' depends on the end conditions.

(c) Rankine Gordon Formulae :

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

Where P_e = Euler crippling load

P_c = Crushing load or Yield point load in Compression

P_R = Actual load to cause failure or Rankine load

Since the Rankine formulae is a combination of the Euler and crushing load for a strut.

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c}$$

For a very short strut P_e is very large hence 1/ P_e would be large so that 1/ P_e can be neglected.

Thus $P_R = P_c$, for very large struts, P_e is very small so 1/ P_e would be large and 1/ P_c can be neglected ,hence $P_R = P_e$

The Rankine formulae is therefore valid for extreme values of 1/k.It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus rewriting the formula in terms of stresses, we have

$$\frac{1}{\sigma A} = \frac{1}{\sigma_e A} + \frac{1}{\sigma_y A}$$

$$\frac{1}{\sigma} = \frac{1}{\sigma_e} + \frac{1}{\sigma_y}$$

$$\frac{1}{\sigma} = \frac{\sigma_e + \sigma_y}{\sigma_e . \sigma_y}$$

$$\sigma = \frac{\sigma_e . \sigma_y}{\sigma_e + \sigma_y} = \frac{\sigma_y}{1 + \frac{\sigma_y}{\sigma_e}}$$

For struts with both ends pinned

$$\sigma_e = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + \frac{\sigma_y}{\pi^2 E}\left(\frac{l}{k}\right)^2}$$

$$\sigma = \frac{\sigma_y}{1 + a\left(\frac{l}{k}\right)^2}$$

Where $a = \frac{\sigma_y}{\pi^2 EI}$ and the value of 'a' is found by conducting experiments on various materials. Theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end conditions.

Therefore $\boxed{\text{Rankine load} = \frac{\sigma_y . A}{1 + a\left(\frac{l}{k}\right)^2}}$

Typical values of 'a' for use in Rankine formulae are given below in table.

Material	**s_y or s_c MN/m^2**	**Value of a**	
		Pinned ends	**Fixed ends**
Low carbon steel	315	1/7500	1/30000
Cast Iron	540	1/1600	1/64000
Timber	35	1/3000	1/12000

note a = 4 x (a for fixed ends)

Since the above values of ‘a' are not exactly equal to the theoretical values , the Rankine loads for long struts will not be identical to those estimated by the Euler theory as estimated.

Strut with initial Curvature :

As we know that the true conditions are never realized , but there are always some imperfections. Let us say that the strut is having some initial curvature. i.e., it is not perfectly straight before loading. The situation will influence the stability. Let us analyze this effect.

by a differential calculus

$$R_0 \approx \frac{1}{d^2y_0/dx^2} \text{(Approximately)}$$

$$\text{Futher } \frac{E}{R} = \frac{M}{I} \text{ and } \frac{EI}{R} = M$$

$$\text{But for this case } EI\left[\frac{1}{R} - \frac{1}{R_0}\right] = M$$

since strut is having some initial curvature

Now putting

$$\frac{1}{R} = \frac{d^2y}{dx^2} \text{ and } \frac{1}{R_0} = \frac{d^2y_0}{dx^2}$$

Where ‘ y_0' is the value of deflection before the load is applied to the strut when the load is applied to the strut the deflection increases to a value ‘y'. Hence

$$EI\left[\frac{d^2y}{dx^2}-\frac{d^2y_0}{dx^2}\right]=M$$

$$EI\frac{d^2y}{dx^2}-EI\frac{d^2y_0}{dx^2}=M$$

$$EI\frac{d^2y}{dx^2}=M+EI\frac{d^2y_0}{dx^2}$$

$$EI\frac{d^2y}{dx^2}=-Py+EI\frac{d^2y_0}{dx^2}$$

If the pinended strut is under the action of a load P then obviously the BM would be ' $-py$'

Hence

$$EI\frac{d^2y}{dx^2}+Py=EI\frac{d^2y_0}{dx^2}$$

$$\frac{d^2y}{dx^2}+\frac{Py}{EI}=\frac{d^2y_0}{dx^2}$$

Again letting

$$\frac{P}{EI}=n^2$$

$$\frac{d^2y}{dx^2}+n^2y=\frac{d^2y_0}{dx^2}$$

The initial shape of the strut y_0 may be assumed circular, parabolic or sinusoidal without making much difference to the final results, but the most convenient form is

$y_0=C.\sin\frac{\pi x}{l}$ where C is some constant or here it is amplitude

Which satisfies the end conditions and corresponds to a maximum deviation 'C'. Any other shape could be analyzed into a Fourier series of sine terms. Then

$$\frac{d^2y}{dx^2}+n^2y=\frac{d^2y_0}{dx^2}=\frac{d^2}{dx^2}\left[C.\sin\frac{\pi x}{l}\right]=\left(-C.\frac{\pi^2}{l^2}\right)\sin\left(\frac{\pi x}{l}\right)$$

The computer solution would be therefore be

$$y_{general}=y_{complementry}+y_{PI}$$

$$y=A\cos nx+B\sin nx+\frac{C.\frac{\pi^2}{l^2}}{\left(\frac{\pi^2}{l^2}\right)-n^2}\sin\left(\frac{\pi x}{l}\right)$$

Boundary conditions which are relevant to the problem are

at x = 0 ; y = 0 thus B = 0

Again

when x = l ; y = 0 or x = l / 2 ; dy/dx = 0

the above condition gives B = 0

Therefore the complete solution would be

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

Again the above solution can be slightly rearranged.since

$$P_e = \frac{\pi^2 EI}{l^2}$$

hence the term $\frac{\frac{\pi^2}{l^2}}{\frac{\pi^2}{l^2} - n^2}$ after multiplying the denominator & numerator by EI is equal to

$$\frac{\frac{\pi^2 EI}{l^2}}{\frac{\pi^2 EI}{l^2} - n^2 EI} = \left[\frac{P_e}{P_e - P}\right]$$

Since $n^2 = \frac{P}{EI}$

where P_e = Euler's load P=applied load

Thus

$$y = \frac{C.\frac{\pi^2}{l^2}}{\left\{\left(\frac{\pi^2}{l^2}\right) - n^2\right\}} \sin\left(\frac{\pi x}{l}\right)$$

$$y = \left\{\frac{C.P_e}{P_e - P}\right\} \sin\left(\frac{\pi x}{l}\right)$$

The crippling load is again

$$P = P_e = \frac{\pi^2 EI}{l^2}$$

Since the BM for a pin ended strut at any point is given as

M = -Py and

Max BM = P y_{max}

Now in order to define the absolute value in terms of maximum amplitude let us use the symbol as '^'.

$$\hat{M} = P.\hat{y}$$
$$= C.\frac{P.P_e}{(P_e - p)}$$

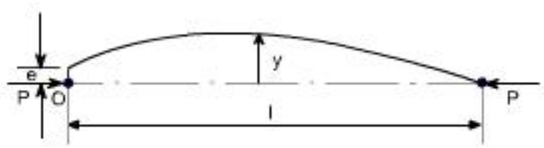
Therefore $\hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$ since $y_{max^m} = \frac{P_e}{[P_e - p]}$

$\sin\frac{\pi x}{l} = 1$ when $\frac{\pi x}{l} = \frac{\pi}{2}$

Hence $\hat{M} = \frac{C\ P\ P_e}{[P_e - p]}$

Strut with eccentric load

Let 'e' be the eccentricity of the applied end load, and measuring y from the line of action of the load.



Then $E\,I\,\frac{d^2y}{dx^2} = -\,P\,y$

or $(D^2 + n^2)\,y = 0$ where $n^2 = P / EI$

Therefore $y_{general} = y_{complementary}$

$= A\sin nx + B\cos nx$

applying the boundary conditions then we can determine the constants i.e.

at $x = 0$; $y = e$ thus $B = e$

at $x = l / 2$; $dy / dx = 0$

Therefore

$$A\cos\frac{nl}{2} - B\sin\frac{nl}{2} = 0$$

$$A\cos\frac{nl}{2} = B\sin\frac{nl}{2}$$

$$A = B\tan\frac{nl}{2}$$

$$A = e\tan\frac{nl}{2}$$

Hence the complete solution becomes

y = A sin(nx) + B cos(nx)

substituting the values of A and B we get

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

Note that with an eccentric load, the strut deflects for all values of P, and not only for the critical value as was the case with an axially applied load. The deflection becomes infinite for tan (nl)/2 = ∞ i.e. nl = p giving the same crippling load $P_e = \frac{\pi^2 EI}{l^2}$. However, due to additional bending moment set up by deflection, the strut will always fail by compressive stress before Euler load is reached.

Since

$$y = e\left[\tan\frac{nl}{2}\sin nx + \cos nx\right]$$

$$y_{max^m}\Big|_{at\ x = \frac{l}{2}} = e\left[\tan\left(\frac{nl}{2}\right)\sin\frac{nl}{2} + \cos\frac{nl}{2}\right]$$

$$= e\left[\frac{\sin^2\frac{nl}{2} + \cos^2\frac{nl}{2}}{\cos\frac{nl}{2}}\right]$$

$$= e\left[\frac{1}{\cos\frac{nl}{2}}\right] = e\ \sec\frac{nl}{2}$$

Hence maximum bending moment would be

$$M_{max^m} = P\ y_{max^m}$$

$$= P\,e\ \sec\frac{nl}{2}$$

Now the maximum stress is obtained by combined and direct strain

$$\sigma = \frac{P}{A} + \frac{M}{Z} \text{ stress due to bending } \frac{\sigma}{y} = \frac{M}{I};$$

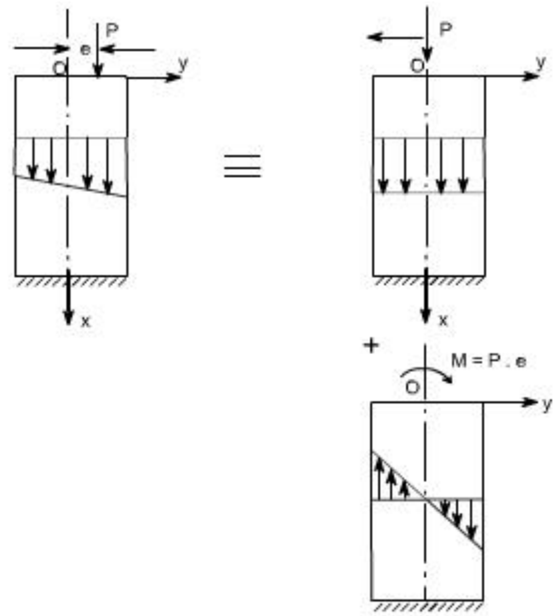
$$M = \sigma\frac{I}{y};\ \sigma_{max} = \frac{M}{Z} \text{ Wher } Z = I/y \text{ is section modulus}$$

The second term is obviously due the bending action.

Consider a short strut subjected to an eccentrically applied compressive force P at its upper end. If such a strut is comparatively short and stiff, the deflection due to bending action of the eccentric load will be neglible compared with eccentricity ‘e' and the principal of super-imposition applies.

If the strut is assumed to have a plane of symmetry (the xy - plane) and the load P lies in this plane at the distance ‘e' from the centroidal axis ox.

Then such a loading may be replaced by its statically equivalent of a centrally applied compressive force ‘P' and a couple of moment P.e



1. The centrally applied load P produces a uniform compressive $\sigma_1 = \frac{P}{A}$ stress over each cross-section as shown by the stress diagram.

2. The end moment 'M' produces a linearly varying bending stress $\sigma_2 = \frac{My}{I}$ as shown in the figure.

Then by super-impostion, the total compressive stress in any fibre due to combined bending and compression becomes,

$$\sigma = \frac{P}{A} + \frac{My}{I}$$

$$\sigma = \frac{P}{A} + \frac{M}{I/y}$$

$$\boxed{\sigma = \frac{P}{A} + \frac{M}{Z}}$$

UNIT IV

BEAM DEFLECTION

PART-A (2 Marks)

1. What are the advantages of Macaulay method over the double integration method, for finding the slope and deflections of beams?
2. State the limitations of Euler's formula.
3. Define crippling load.
4. State Mohr's theorem.
5. State any three assumption made in Euler's column theory.
6. What are the different modes of failures of a column?
7. Write down the Rankine formula for columns.
8. What is effective or equivalent length of column?
9. Define Slenderness Ratio.
10. Define the terms column and strut.

PART- B (16 Marks)

1. A simply supported beam of 10 m span carries a uniformly distributed load of 1 kN/m over the entire span. Using Castigliano's theorem, find the slope at the ends. EI = 30,000 kN/m2.

2. A 2m long cantilever made of steel tube of section 150 mm external diameter and10mm thick is loaded. If E=200 GN/m2 calculate (1) The value of W so that the maximum bending stress is 150 MN/m (2) The maximum deflection for the loading

3. A beam of length of 10 m is simply supported at its ends and carries two point loads of 100 KN and 60 KN at a distance of 2 m and 5 m respectively from the left support. Calculate the deflections under each load. Find also the maximum deflection.
Take I = 18 X 108 mm4 and E = 2 X 105.

4. i) A column of solid circular section, 12 cm diameter, 3.6 m long is hinged at both ends. Rankine's constant is 1 / 1600 and _c= 54 KN/cm2. Find the buckling load.
ii) If another column of the same length, end conditions and rankine constant but of 12 cm X 12 cm square cross-section, and different material, has the same buckling load, find the value of _c of its material.

5. A beam of length of 6 m is simply supported at its ends. It carries a uniformly distributed load of 10 KN/m as shown in figure. Determine the deflection of the beam at its mid-point and also the position and the maximum deflection. Take EI=4.5 X 108 N/mm2.

6. An overhanging beam ABC is loaded as shown is figure. Determine the deflection of the beam at point C. Take I = 5 X 108 mm4 and E = 2 X 105 N/mm2.

7. A cantilever of length 2 m carries a uniformly distributed load of 2.5 KN/m run for a length

of 1.25 m from the fixed end and a point load of 1 KN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and E=1 X 104 N/mm2

8. A cantilever of length 2m carries a uniformly distributed load 2 KN/m over a length of 1m from the free end, and a point load of 1 KN at the free end. Find the slope and deflection at the free end if E = 2.1 X 105 N/mm2 and I = 6.667 X 107 mm4 .

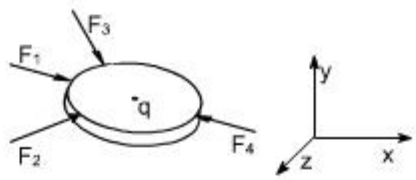
9. Determine the section of a hollow C.I. cylindrical column 5 m long with ends firmly built in. The column has to carry an axial compressive load of 588.6 KN. The internal diameter of the column is 0.75 times the external diameter. Use Rankine's constants.
a = 1 / 1600, _c = 57.58 KN/cm2 and F.O.S = 6.

UNIT V ANALYSIS OF STRESSES IN TWO DIMENSIONS

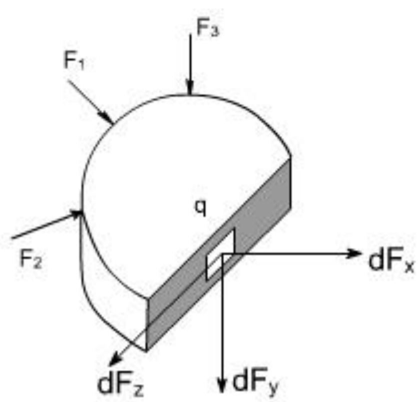
Biaxial state of stresses – Thin cylindrical and spherical shells – Deformation in thin cylindrical and spherical shells – Biaxial stresses at a point – Stresses on inclined plane – Principal planes and stresses – Mohr’s circle for biaxial stresses – Maximum shear stress – Strain energy in bending and torsion.

General State of stress at a point :

Stress at a point in a material body has been defined as a force per unit area. But this definition is some what ambiguous since it depends upon what area we consider at that point. Let us, consider a point �q' in the interior of the body



Let us pass a cutting plane through a pont 'q' perpendicular to the x - axis as shown below



The corresponding force components can be shown like this

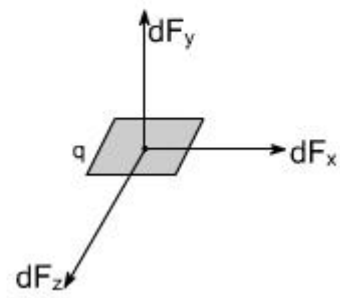
$dF_x = \square_{xx}. da_x$

$dF_y = \square_{xy}. da_x$

$dF_z = \square_{xz}. da_x$

where da_x is the area surrounding the point 'q' when the cutting plane □ ^{r} is to x - axis.

In a similar way it can be assummed that the cutting plane is passed through the point 'q' perpendicular to the y - axis. The corresponding force components are shown below



The corresponding force components may be written as

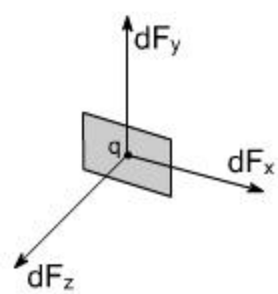
$dF_x = \square_{yx}. da_y$

$dF_y = \square_{yy}. da_y$

$dF_z = \square_{yz}. da_y$

where da_y is the area surrounding the point 'q' when the cutting plane □ ^{r} is to y - axis.

In the last it can be considered that the cutting plane is passed through the point 'q' perpendicular to the z - axis.



The corresponding force components may be written as

$dF_x = \square_{zx}. da_z$

$dF_y = \square_{zy}. da_z$

$dF_z = \square_{zz}. da_z$

where da_z is the area surrounding the point 'q' when the cutting plane □ ^{r} is to z - axis.

Thus, from the foregoing discussion it is amply clear that there is nothing like stress at a point 'q' rather we have a situation where it is a combination of state of stress at a point q. Thus, it becomes imperative to understand the term state of stress at a point 'q'. Therefore, it becomes easy to express astate of stress by the scheme as discussed earlier, where the stresses on the three mutually perpendiclar planes are labelled in the manner as shown earlier. the state of stress as depicted earlier is called the general or a triaxial state of stress that can exist at any interior point of a loaded body.

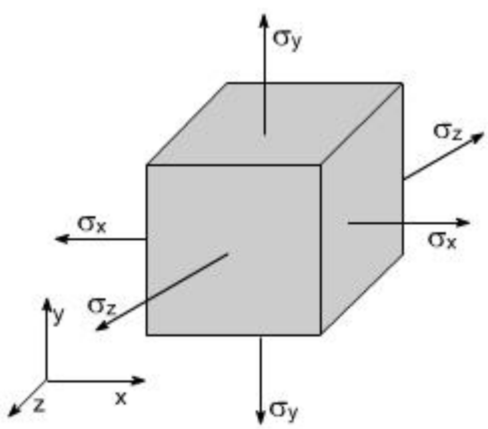
Before defining the general state of stress at a point. Let us make overselves conversant with the notations for the stresses.

We have already chosen to distinguish between normal and shear stress with the help of symbols □ and □ .

Cartesian - co-ordinate system

In the Cartesian co-ordinates system, we make use of the axes, X, Y and Z

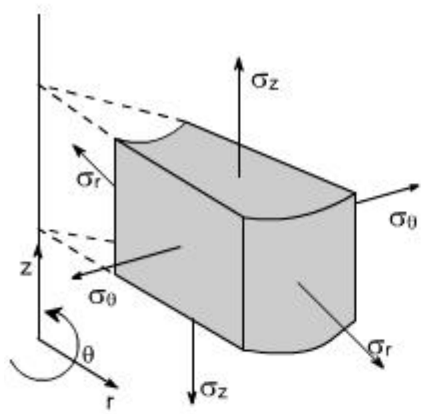
Let us consider the small element of the material and show the various normal stresses acting the faces



Thus, in the Cartesian co-ordinates system the normal stresses have been represented by □$_x$, □$_y$and □$_z$.

Cylindrical - co-ordinate system

In the Cylindrical - co-ordinate system we make use of co-ordinates r, □ and Z.



Thus, in the Cylindrical co-ordinates system, the normal stresses i.e components acting over a element is being denoted by $\square_r$, $\square_{\square}$and $\square_z$.

Sign convention : The tensile forces are termed as (+ve) while the compressive forces are termed as negative (-ve).

First sub � script : it indicates the direction of the normal to the surface.

Second subscript : it indicates the direction of the stress.

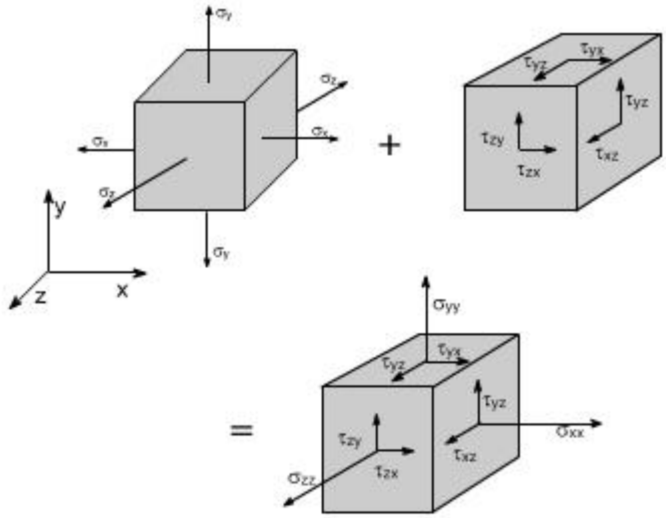
It may be noted that in the case of normal stresses the double script notation may be dispensed with as the direction of the normal stress and the direction of normal to the surface of the element on which it acts is the same. Therefore, a single subscript notation as used is sufficient to define the normal stresses.

Shear Stresses : With shear stress components, the single subscript notation is not practical, because such stresses are in direction parallel to the surfaces on which they act. We therefore have two directions to specify, that of normal to the surface and the stress itself. To do this, we stress itself. To do this, we attach two subscripts to the symbol ' □ ' , for shear stresses.

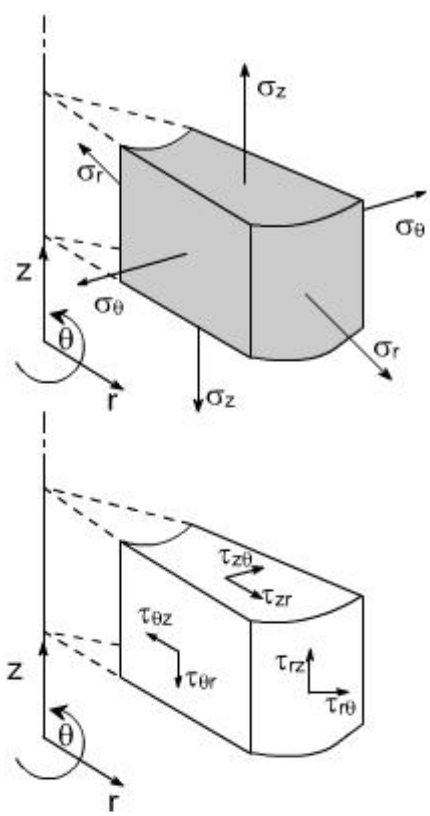
In cartesian and polar co-ordinates, we have the stress components as shown in the figures.

$\square_{xy}$, $\square_{yx}$, $\square_{yz}$, $\square_{zy}$, $\square_{zx}$, $\square_{xz}$

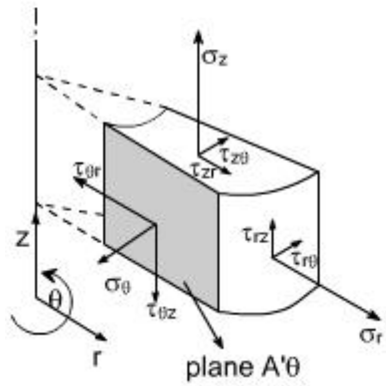
$\square_{r\square}$, $\square_{\square r}$, $\square_{\square z}$, $\square_{z\square}$,$\square_{zr}$, $\square_{rz}$



So as shown above, the normal stresses and shear stress components indicated on a small element of material seperately has been combined and depicted on a single element. Similarly for a cylindrical co-ordinate system let us shown the normal and shear stresses components separately.



Now let us combine the normal and shear stress components as shown below :



Now let us define the state of stress at a point formally.

State of stress at a point :

By state of stress at a point, we mean an information which is required at that point such that it remains under equilibrium. or simply a general state of stress at a point involves all the normal stress components, together with all the shear stress components as shown in earlier figures.

Therefore, we need nine components, to define the state of stress at a point

$\square_{x}$ $\square_{xy}$ $\square_{xz}$

$\square_{y}$ $\square_{yx}$ $\square_{yz}$

$\square_{z}$ $\square_{zx}$ $\square_{zy}$

If we apply the conditions of equilibrium which are as follows:

$\square\ F_x = 0$; $\square\ M_x = 0$

$\square\ F_y = 0$; $\square\ M_y = 0$

$\square\ F_z = 0$; $\square\ M_z = 0$

Then we get

$\square_{xy} = \square_{yx}$

□$_{yz}$ = □$_{zy}$

□$_{zx}$ = □$_{xy}$

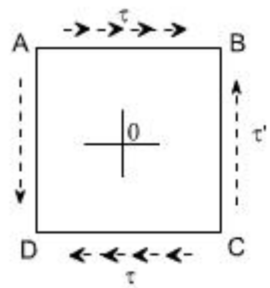
Then we will need only six components to specify the state of stress at a point i.e

□$_{x}$, □$_{y}$, □$_{z}$, □$_{xy}$, □$_{yz}$, □$_{zx}$

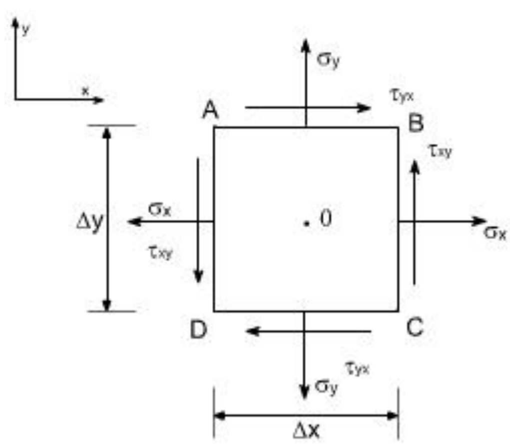
Now let us define the concept of complementary shear stresses.

Complementary shear stresses:

The existence of shear stresses on any two sides of the element induces complementary shear stresses on the other two sides of the element to maintain equilibrium.



on planes AB and CD, the shear stress □ acts. To maintain the static equilibrium of this element, on planes AD and BC, □ ' should act, we shall see that□ ' which is known as the complementary shear stress would come out to equal and opposite to the□ □ □ . Let us prove this thing for a general case as discussed below:



The figure shows a small rectangular element with sides of length □ x,□ y parallel to x and y directions. Its thickness normal to the plane of paper is □ z in z� direction. All nine normal and shear stress components may act on the element, only those in x and y directions are shown.

Sign convections for shear stresses:

Direct stresses or normal stresses

- tensile +ve

- compressive �ve

Shear stresses:

- tending to turn the element C.W +ve.

- tending to turn the element C.C.W � ve.

The resulting forces applied to the element are in equilibrium in x and y direction. (Although other normal and shear stress components are not shown, their presence does not affect the final conclusion).

Assumption : The weight of the element is neglected.

Since the element is a static piece of solid body, the moments applied to it must also be in equilibrium. Let �O' be the centre of the element. Let us consider the axis through the point �O'. the resultant force associated with normal stresses $\square_x$ and $\square_y$ acting on the sides of the element each pass through this axis, and therefore, have no moment.

Now forces on top and bottom surfaces produce a couple which must be balanced by the forces on left and right hand faces

Thus,

$\square_{yx} . \square x . \square z . \square y = \square_{xy} . \square x . \square z . \square y$

$\boxed{\tau_{yx} = \tau_{xy}}$

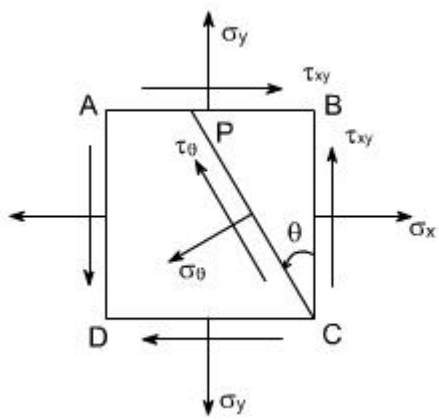
In other word, the complementary shear stresses are equal in magnitude. The same form of relationship can be obtained for the other two pair of shear stress components to arrive at the relations

$$\boxed{\begin{aligned}\tau_{zy} &= \tau_{zy}\\ \tau_{zx} &= \tau_{xz}\end{aligned}}$$

GRAPHICAL SOLUTION � MOHR'S STRESS CIRCLE

The transformation equations for plane stress can be represented in a graphical form known as Mohr's circle. This grapical representation is very useful in depending the relationships between normal and shear stresses acting on any inclined plane at a point in a stresses body.

To draw a Mohr's stress circle consider a complex stress system as shown in the figure



The above system represents a complete stress system for any condition of applied load in two dimensions

The Mohr's stress circle is used to find out graphically the direct stress □ and sheer stress□□ on any plane inclined at □ to the plane on which □$_x$ acts.The direction of □ here is taken in anticlockwise direction from the BC.

STEPS:

In order to do achieve the desired objective we proceed in the following manner

(i) Label the Block ABCD.

(ii) Set up axes for the direct stress (as abscissa) and shear stress (as ordinate)

(iii) Plot the stresses on two adjacent faces e.g. AB and BC, using the following sign convention.

Direct stresses tensile positive; compressive, negative

Shear stresses � tending to turn block clockwise, positive

� tending to turn block counter clockwise, negative

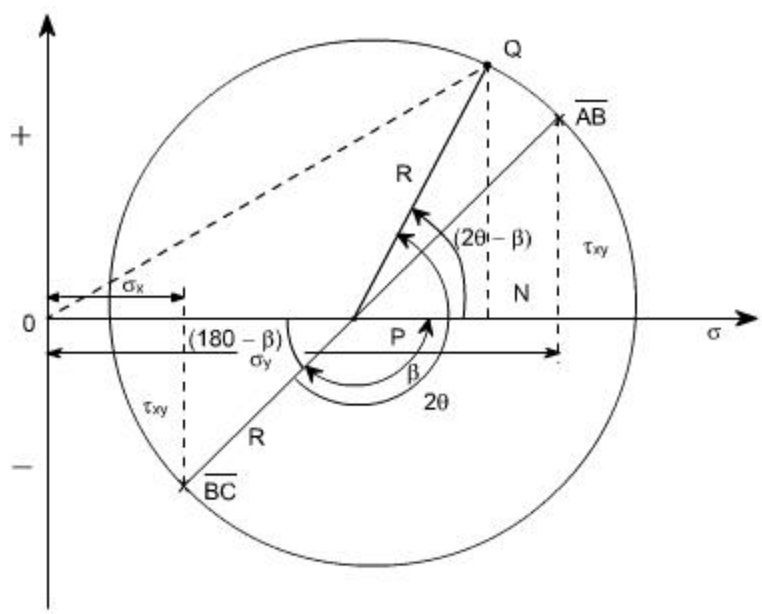
[i.e shearing stresses are +ve when its movement about the centre of the element is clockwise]

This gives two points on the graph which may than be labeled as $\overline{AB}$ and $\overline{BC}$ respectively to denote stresses on these planes.

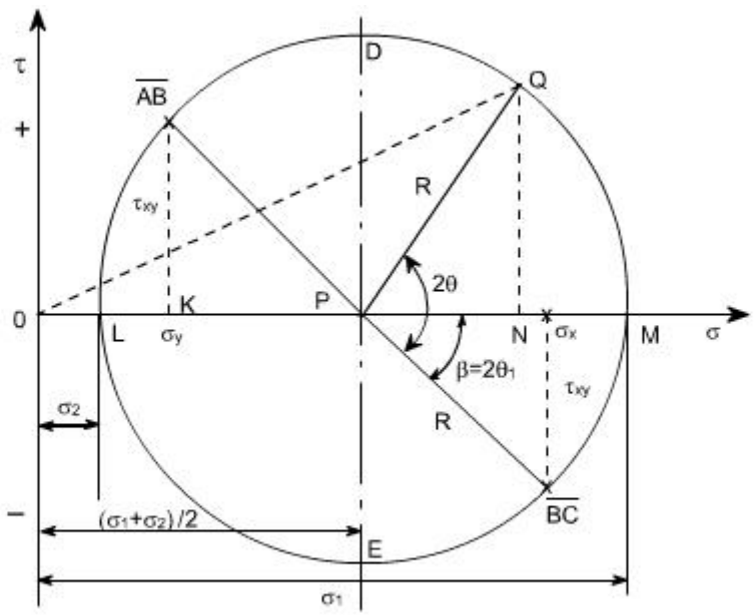
(iv) Join $\overline{AB}$ and $\overline{BC}$.

(v) The point P where this line cuts the s axis is than the centre of Mohr's stress circle and the line joining $\overline{AB}$ and $\overline{BC}$ is diameter. Therefore the circle can now be drawn.

Now every point on the circle then represents a state of stress on some plane through C.



Proof:



Consider any point Q on the circumference of the circle, such that PQ makes an angle 2θ with BC, and drop a perpendicular from Q to meet the s axis at N.Then OQ represents the resultant stress on the plane an angle θ to BC. Here we have assumed that $\sigma_x > \sigma_y$

Now let us find out the coordinates of point Q. These are ON and QN.

From the figure drawn earlier

$$ON = OP + PN$$

$$OP = OK + KP$$

$$OP = \sigma_y + 1/2\ (\sigma_x - \sigma_y)$$

$$= \sigma_y / 2 + \sigma_y / 2 + \sigma_x / 2 + \sigma_y / 2$$

$$= (\sigma_x + \sigma_y) / 2$$

$PN = R\cos(2\theta - \beta)$

hence ON = OP + PN

$$= (\sigma_x + \sigma_y) / 2 + R\cos(2\theta - \beta)$$

$$= (\sigma_x + \sigma_y) / 2 + R\cos 2\theta \cos\beta + R\sin 2\theta \sin\beta$$

now make the substitutions for $R\cos\beta$ and $R\sin\beta$.

$$R\cos\beta = \frac{(\sigma_x - \sigma_y)}{2};\ R\sin\beta = \tau_{xy}$$

Thus,

$ON = 1/2\,(\sigma_x + \sigma_y) + 1/2\,(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy}\sin 2\theta$ (1)

Similarly $QM = R\sin(2\theta - \beta)$

$= R\sin 2\theta\cos\beta - R\cos 2\theta\sin\beta$

Thus, substituting the values of $R\cos\beta$ and $R\sin\beta$, we get

$QM = 1/2\,(\sigma_x - \sigma_y)\sin 2\theta - \tau_{xy}\cos 2\theta$ (2)

If we examine the equation (1) and (2), we see that this is the same equation which we have already derived analytically

Thus the co-ordinates of Q are the normal and shear stresses on the plane inclined at θ to BC in the original stress system.

N.B: Since angle $\overline{BC}$ PQ is 2θ on Mohr's circle and not θ it becomes obvious that angles are doubled on Mohr's circle. This is the only difference, however, as They are measured in the same direction and from the same plane in both figures.

Further points to be noted are :

(1) The direct stress is maximum when Q is at M and at this point obviously the sheer stress is zero, hence by definition OM is the length representing the maximum principal stresses σ_1 and $2\theta_1$ gives the angle of the plane θ_1 from BC. Similar OL is the other principal stress and is represented by σ_2

(2) The maximum shear stress is given by the highest point on the circle and is represented by the radius of the circle.

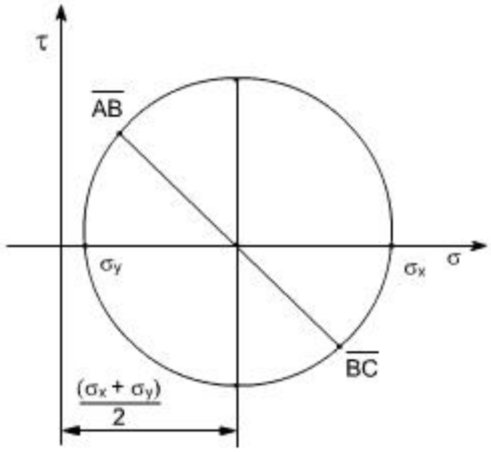
This follows that since shear stresses and complimentary sheer stresses have the same value; therefore the centre of the circle will always lie on the s axis midway between σ_x and σ_y . [since $+\tau_{xy}$ & $-\tau_{xy}$ are shear stress & complimentary shear stress so they are same in magnitude but different in sign.]

(3) From the above point the maximum sheer stress i.e. the Radius of the Mohr's stress circle would be

$$\frac{(\sigma_x - \sigma_y)}{2}$$

While the direct stress on the plane of maximum shear must be mid � may between $\square_x$ and $\square_y$ i.e

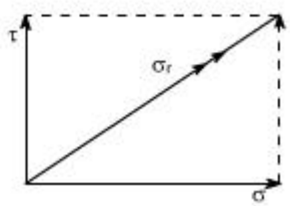
$$\frac{(\sigma_x + \sigma_y)}{2}$$



(4) As already defined the principal planes are the planes on which the shear components are zero.

Therefore are conclude that on principal plane the sheer stress is zero.

(5) Since the resultant of two stress at 90^0 can be found from the parallogram of vectors as shown in the diagram.Thus, the resultant stress on the plane at q to BC is given by OQ on Mohr's Circle.



(6) The graphical method of solution for a complex stress problems using Mohr's circle is a very powerful technique, since all the information relating to any plane within the stressed element is contained in the single construction. It thus, provides a convenient and rapid means of solution. Which is less prone to arithmetical errors and is highly recommended.

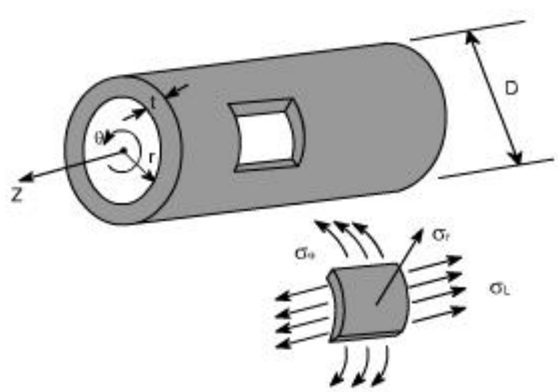
Pressurized thin walled cylinder:

Preamble : Pressure vessels are exceedingly important in industry. Normally two types of pressure vessel are used in common practice such as cylindrical pressure vessel and spherical pressure vessel.

In the analysis of this walled cylinders subjected to internal pressures it is assumed that the radial plans remains radial and the wall thickness dose not change due to internal pressure. Although the internal pressure acting on the wall causes a local compressive stresses (equal to pressure) but its value is neglibly small as compared to other stresses & hence the sate of stress of an element of a thin walled pressure is considered a biaxial one.

Further in the analysis of them walled cylinders, the weight of the fluid is considered neglible.

Let us consider a long cylinder of circular cross - section with an internal radius of R_2 and a constant wall thickness�t' as showing fig.



This cylinder is subjected to a difference of hydrostatic pressure of �p' between its inner and outer surfaces. In many cases, �p' between gage pressure within the cylinder, taking outside pressure to be ambient.

By thin walled cylinder we mean that the thickness�t' is very much smaller than the radius R_i and we may quantify this by stating than the ratio t / R_i of thickness of radius should be less than 0.1.

An appropriate co-ordinate system to be used to describe such a system is the cylindrical polar one r, □ , z shown, where z axis lies along the axis of the cylinder, r is radial to it and □ □ is the angular co-ordinate about the axis.

The small piece of the cylinder wall is shown in isolation, and stresses in respective direction have also been shown.

Type of failure:

Such a component fails in since when subjected to an excessively high internal pressure. While it might fail by bursting along a path following the circumference of the cylinder. Under normal circumstance it fails by circumstances it fails by bursting along a path parallel to the axis. This suggests that the hoop stress is significantly higher than the axial stress.

In order to derive the expressions for various stresses we make following

Applications :

Liquid storage tanks and containers, water pipes, boilers, submarine hulls, and certain air plane components are common examples of thin walled cylinders and spheres, roof domes.

ANALYSIS : In order to analyse the thin walled cylinders, let us make the following assumptions :

- There are no shear stresses acting in the wall.
- The longitudinal and hoop stresses do not vary through the wall.
- Radial stresses □$_r$ which acts normal to the curved plane of the isolated element are neglibly small as compared to other two stresses especially when $\left[\frac{t}{R_i} < \frac{1}{20}\right]$

The state of tress for an element of a thin walled pressure vessel is considered to be biaxial, although the internal pressure acting normal to the wall causes a local compressive stress equal to the internal pressure, Actually a state of tri-axial stress exists on the inside of the vessel. However, for then walled pressure vessel the third stress is much smaller than the other two stresses and for this reason in can be neglected.

Thin Cylinders Subjected to Internal Pressure:

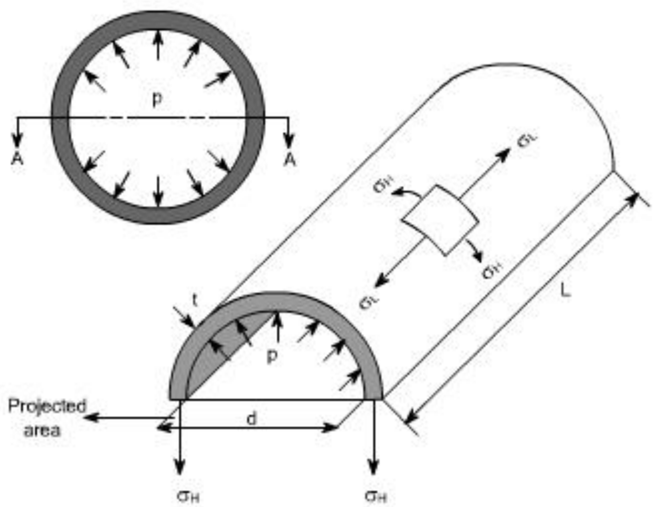
When a thin � walled cylinder is subjected to internal pressure, three mutually perpendicular principal stresses will be set up in the cylinder materials, namely

- Circumferential or hoop stress
- The radial stress
- Longitudinal stress

now let us define these stresses and determine the expressions for them

Hoop or circumferential stress:

This is the stress which is set up in resisting the bursting effect of the applied pressure and can be most conveniently treated by considering the equilibrium of the cylinder.



In the figure we have shown a one half of the cylinder. This cylinder is subjected to an internal pressure p.

i.e. p = internal pressure

d = inside diametre

L = Length of the cylinder

t = thickness of the wall

Total force on one half of the cylinder owing to the internal pressure 'p'

= p x Projected Area

= p x d x L

= **p .d. L** ------- (1)

The total resisting force owing to hoop stresses σ_H set up in the cylinder walls

= **2 .σ_H .L.t** ---------(2)

Because σ_H.L.t. is the force in the one wall of the half cylinder.

the equations (1) & (2) we get

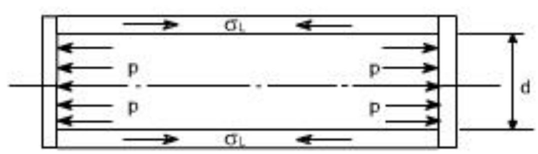
2 . σ_H . L . t = p . d . L

$$\sigma_H = (p \cdot d) / 2t$$

Circumferential or hoop Stress (σ_H) = (p .d)/ 2t

Longitudinal Stress:

Consider now again the same figure and the vessel could be considered to have closed ends and contains a fluid under a gage pressure p.Then the walls of the cylinder will have a longitudinal stress as well as a ciccumferential stress.



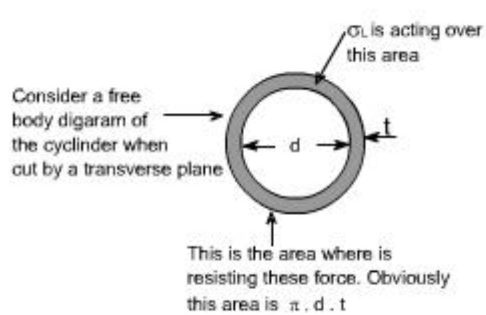
Total force on the end of the cylinder owing to internal pressure

= pressure x area

$= p \times \pi d^2/4$

Area of metal resisting this force = π d.t. (approximately)

because π d is the circumference and this is multiplied by the wall thickness



Hence the longitudnal stresses

$$\text{Set up} = \frac{\text{force}}{\text{area}} = \frac{[p \times \pi d^2/4]}{\pi d t}$$

$$= \frac{pd}{4t} \quad \text{or} \quad \sigma_L = \frac{pd}{4t}$$

or alternatively from equilibrium conditions

$$\sigma_L.(\pi d t) = p.\frac{\pi d^2}{4}$$

$$\text{Thus} \ \boxed{\sigma_L = \frac{pd}{4t}}$$

Energy Methods

Strain Energy

Strain Energy of the member is defined as the internal work done in defoming the body by the action of externally applied forces. This energy in elastic bodies is known as **elastic strain energy** :

Strain Energy in uniaxial Loading

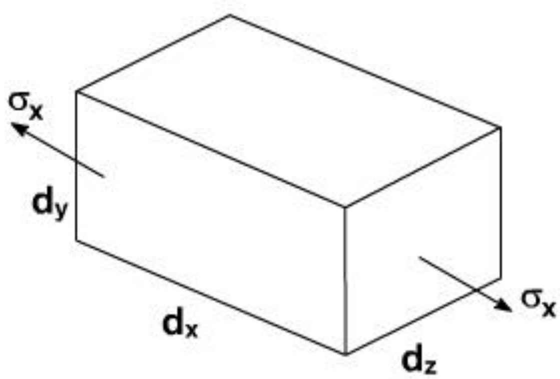


Fig .1

Let as consider an infinitesimal element of dimensions as shown in Fig .1. Let the element be subjected to normal stress σ_x.

The forces acting on the face of this element is σ_x. dy. dz

where

dydz = Area of the element due to the application of forces, the element deforms to an amount = ε_x dx

□ $\square_x$ = strain in the material in x � direction

$$= \frac{\text{Change in length}}{\text{Orginal in length}}$$

Assuming the element material to be as linearly elastic the stress is directly proportional to strain as shown in Fig . 2.

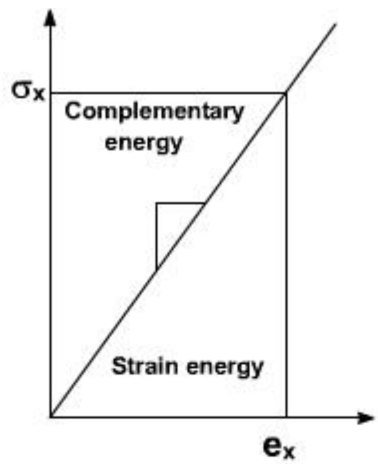


Fig .2

□ □ From Fig .2 the force that acts on the element increases linearly from zero until it attains its full value.

Hence average force on the element is equal to ½ $\square_x$. dy. dz.

□ Therefore the workdone by the above force

Force = average force x deformed length

= ½ $\square_x$. dydz . $\square_x$. dx

For a perfectly elastic body the above work done is the internal strain energy �du�.

$$\boxed{du = \frac{1}{2}\sigma_x dydz\, \epsilon_x\, dx} \qquad .....(2)$$

$$= \frac{1}{2}\sigma_x \epsilon_x\, dxdydz$$

$$\boxed{du = \frac{1}{2}\sigma_x\, \epsilon_x\, dv} \qquad ......(3)$$

where dv = dxdydz

= Volume of the element

By rearranging the above equation we can write

$$U_o = \boxed{\frac{du}{dv} = \frac{1}{2}\sigma_x\,\epsilon_x} \qquad .....(4)$$

The equation (4) represents the strain energy in elastic body per unit volume of the material its strain energy � density �u_o' .

From Hook's Law for elastic bodies, it may be recalled that

$$\boxed{\sigma = E\,\epsilon}$$

$$\boxed{U_o = \frac{du}{dv} = \frac{\sigma_x^{\,2}}{2E} = \frac{E\,\epsilon_x^{\,2}}{2}} \qquad ......(5)$$

$$\boxed{U = \int_{Vol} \frac{\sigma_x^{\,2}}{2E}\,dv} \qquad ....(6)$$

In the case of a rod of uniform cross � section subjected at its ends an equal and opposite forces of magnitude P as shown in the Fig .3.

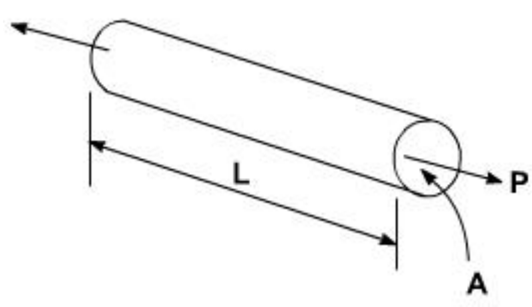


Fig .3

$$U = \int_{Vol}^{L} \frac{\sigma_x^{\,2}}{2E}\,dv \qquad \sigma_x = \frac{P}{A}$$

$$U = \int_{0}^{L} \frac{P^2}{2EA^2}\,A\,dx \qquad dv = Adx = \text{Element volume}$$

A = Area of the bar.
L = Length of the bar

$$\boxed{U = \frac{P^2 L}{2AE}} \qquad .....(7)$$

Modulus of resilience :

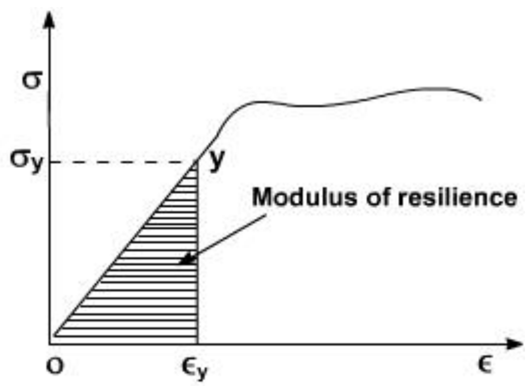


Fig .4

Suppose � $□_x$� in strain energy equation is put equal to $□_y$ i.e. the stress at proportional limit or yield point. The resulting strain energy gives an index of the materials ability to store or absorb energy without permanent deformation

So $$\boxed{U_y = \frac{\sigma_y^{\,2}}{2E}} \qquad .....(8)$$

The quantity resulting from the above equation is called the Modulus of resilience

The modulus of resilience is equal to the area under the straight line portion �OY' of the stress � strain diagram as shown in Fig .4 and represents the energy per unit volume that the material can absorb without yielding. Hence this is used to differentiate materials for applications where energy must be absorbed by members.

Modulus of Toughness :

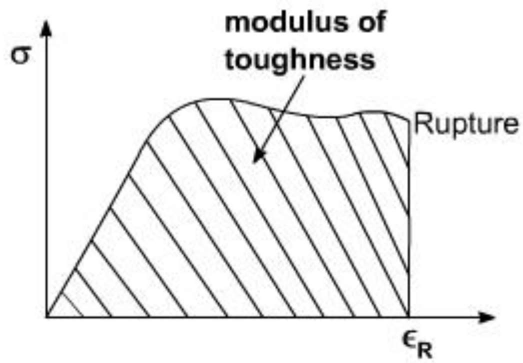


Fig .5

Suppose �□ ' [strain] in strain energy expression is replaced by□$_R$ strain at rupture, the resulting strain energy density is called modulus of toughness

$$U = \int_0^{\epsilon} E\,\epsilon_x \, dx = \frac{E\,\epsilon_R^2}{2} dv$$

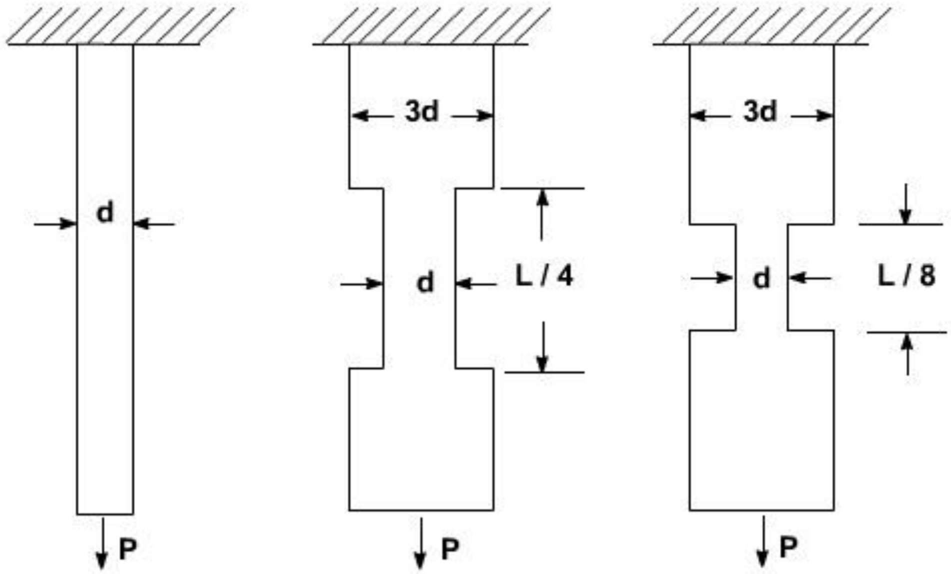
$$\boxed{U = \frac{E\,\epsilon_R^2}{2}} \qquad \ldots\ldots(9)$$

From the stress � strain diagram, the area under the complete curve gives the measure of modules of toughness. It is the materials.

Ability to absorb energy upto fracture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact Load depends upon the toughness of the material used.

ILLUSTRATIVE PROBLEMS

1. Three round bars having the same length �L' but different shapes are shown in fig below. The first bar has a diameter �d' over its entire length, the second had this diameter over one � fourth of its length, and the third has this diameter over one eighth of its length. All three bars are subjected to the same load P. Compare the amounts of strain energy stored in the bars, assuming the linear elastic behavior.



Solution :

1.The strain Energy of the first bar is expressed as

$$U_1 = \frac{P^2L}{2EA}$$

2.The strain Energy of the second bar is expressed as

$$U_2 = \frac{P^2(L/4)}{2EA} + \frac{P^2(3L/4)}{2E9A} = \frac{P^2L}{6EA}$$

$$\boxed{U_2 = \frac{U_1}{3}}$$

3.The strain Energy of the third bar is expressed as

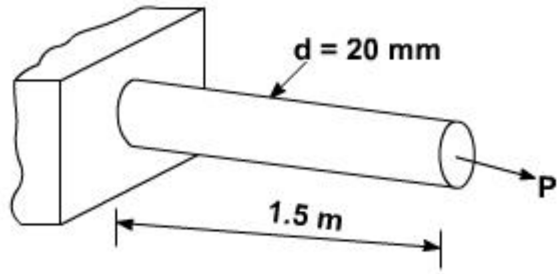
$$U_3 = \frac{P^2(L/8)}{2EA} + \frac{P^2(7L/8)}{2E(9A)}$$

$$U_3 = \frac{P^2L}{9EA}$$

$$\boxed{U_3 = \frac{2U_1}{9}}$$

From the above results it may be observed that the strain energy decreases as the volume of the bar increases.

2. Suppose a rod AB must acquire an elastic strain energy of 13.6 N.m using E = 200 GPa. Determine the required yield strength of steel. If the factor of safety w.r.t. permanent deformation is equal to 5.



Solution :

Factor of safety = 5

Therefore, the strain energy of the rod should be u = 5 [13.6] = 68 N.m

Strain Energy density

The volume of the rod is

$$V = AL = \frac{\pi}{4}d^2L$$
$$= \frac{\pi}{4}\,20 \times 1.5 \times 10^3$$
$$= 471 \times 10^3\ mm^3$$

Yield Strength :

As we know that the modulus of resilience is equal to the strain energy density when maximum stress is equal to σ_x .

$$U = \frac{\sigma_y^{\,2}}{2E}$$
$$0.144 = \frac{\sigma_y^{\,2}}{2 \times (200 \times 10^3)}$$
$$\boxed{\sigma_y = 200\,\mathrm{Mpa}}$$

It is important to note that, since energy loads are not linearly related to the stress they produce, factor of safety associated with energy loads should be applied to the energy loads and not to the stresses.

Strain Energy in Bending :

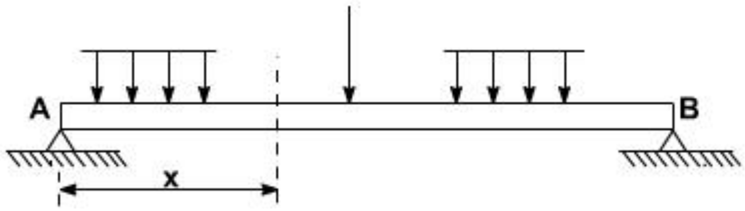


Fig .6

Consider a beam AB subjected to a given loading as shown in figure.

Let

M = The value of bending Moment at a distance x from end A.

From the simple bending theory, the normal stress due to bending alone is expressed as.

$$\boxed{\sigma = \frac{MY}{I}}$$

Substituting the above relation in the expression of strain energy

i.e. $U = \int \frac{\sigma^2}{2E} dv$

$$= \int \frac{M^2 . y^2}{2EI^2} dv \qquad .....(10)$$

Substituting dv = dxdA

Where dA = elemental cross-sectional area

$\frac{M^2 . y^2}{2EI^2} \rightarrow$ is a function of x alone

Now substitiuting for dy in the expression of U.

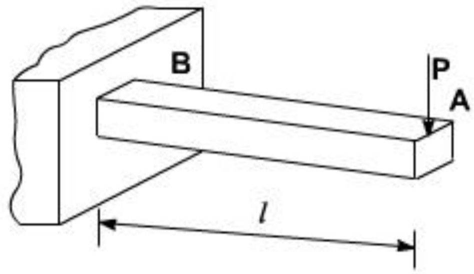
$$U = \int_0^L \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx \qquad .....(11)$$

We know $\int y^2 dA$ represents the moment of inertia 'I' of the cross-section about its neutral axis.

$$\boxed{U = \int_0^L \frac{M^2}{2EI} dx} \qquad .....(12)$$

ILLUSTRATIVE PROBLEMS

1. Determine the strain energy of a prismatic cantilever beam as shown in the figure by taking into account only the effect of the normal stresses.



Solution : The bending moment at a distance x from end
A is defined as

$\boxed{M = -Px}$

Substituting the above value of M in the expression of strain energy we may write

$$U = \int_0^L \frac{P^2 x^2}{2EI}\, dx$$

$$\boxed{U = \int_0^L \frac{P^2 L^3}{EI}}$$

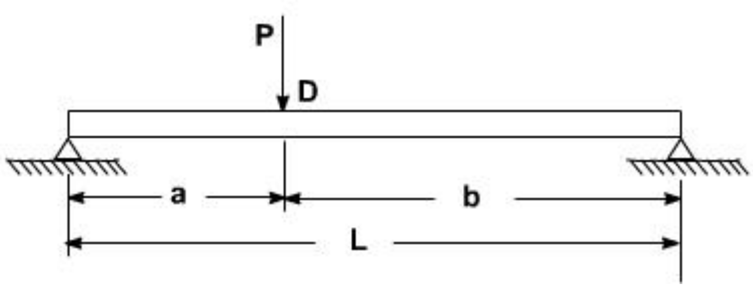
Problem 2 :

a. Determine the expression for strain energy of the prismatic beam AB for the loading as shown in figure below. Take into account only the effect of normal stresses due to bending.
b. Evaluate the strain energy for the following values of the beam

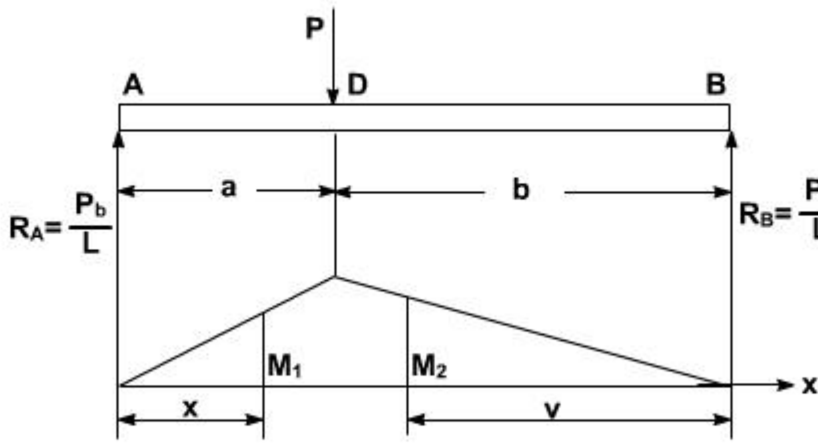
P = 208 KN ; L = 3.6 m = 3600 mm

A = 0.9 m = 90mm ; b = 2.7m = 2700 mm

E = 200 GPa ; I = 104 x 10^8 mm^4



Solution:



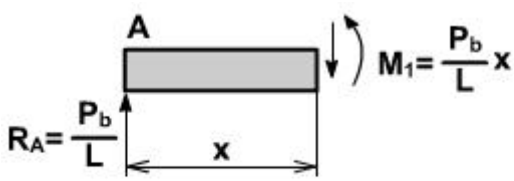
a.

Bending Moment : Using the free � body diagram of the entire beam, we may determine the values of reactions as follows:

$R_A = P_b / L$ $R_B = P_a / L$

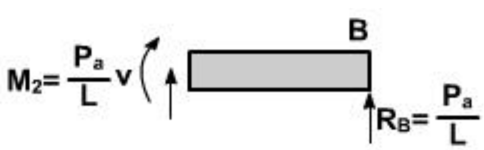
For Portion AD of the beam, the bending moment is

$\boxed{M_1 = \frac{P_b}{L} x}$



For Portion DB, the bending moment at a distance v from end B is

$\boxed{M_2 = \frac{P_a}{L} x}$



Strain Energy :

Since strain energy is a scalar quantity, we may add the strain energy of portion AD to that of DB to obtain the total strain energy of the beam.

$$U = U_{AD} + U_{DB}$$
$$= \int_0^a \frac{M_1^2}{2EI}dx + \int_0^b \frac{M_2^2}{2EI}dv$$
$$= \frac{1}{2EI}\int_0^a \left(\frac{P_b}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{P_a}{L}v\right)^2 dx$$
$$= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right)$$
$$\boxed{U = \frac{P^2a^2b^2}{6EIL^2}(a+b)}$$
Since$(a + b) = L$
$$\boxed{U = \frac{P^2a^2b^2}{6EIL}}$$

b. Substituting the values of P, a, b, E, I, and L in the expression above.

$$U = \frac{(200 \times 10^3)^2 \times (900)^2 \times (2700)^2}{6\,(200 \times 10^3) \times (104 \times 10^6) \times (3600)} = 5.27 \times 10^7 \text{ KN.m}$$

Problem

3) Determine the modulus of resilience for each of the following materials.

a. Stainless steel . E = 190 GPa □ □ □ □$_y$= 260MPa

b. Malleable constantan E = 165GPa □ □ □ □$_y$= 230MPa

c. Titanium E = 115GPa □ □ □ □$_y$= 830MPa

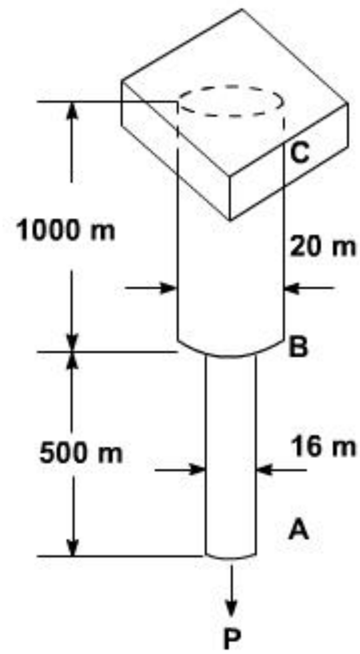
d. Magnesium E = 45GPa □ □ □ □$_y$ = 200MPa

4) For the given Loading arrangement on the rod ABC determine

(a). The strain energy of the steel rod ABC when

P = 40 KN.

(b). The corresponding strain energy density in portions AB and BC of the rod.



UNIT V

ANALYSIS OF STRESSES IN TWO DIMENSIONS

PART-A (2 Marks)

1. Distinguish between thick and thin cylinders.
2. Define Principal planes and principal stress.
3. Define: Thin cylinders. Name the stresses set up in a thin cylinder subjected to internal fluid pressure.
4. What is Mohr's circle & name any the situations where it is used?
5. Define principal planes and principal stresses.
6. Draw Mohr's Circle for given shear stress q.
7. What is the necessary condition for maximum shear stress?
8. Define Obliquity.
9. Define Strain energy and resilience.
10. Define proof resilience and modulus of resilience.

PART- B (16 Marks)

1. A Thin cylindrical shell 3 m long has 1m internal diameter and 15 mm metal thickness. Calculate the circumferential and longitudinal stresses induced and also the change in the dimensions of the shell, if it is subjected to an internal pressure of1.5 N/mm2 Take E = 2x105 N/mm2 and poison's ratio =0.3. Also calculate change in volume.

2. A closed cylindrical vessel made of steel plates 4 mm thick with plane ends, carries fluid under pressure of 3 N/mm2 The diameter of the cylinder is 25cm and length is 75 cm.

Calculate the longitudinal and hoop stresses in the cylinder wall and determine the change in diameter, length and Volume of the cylinder. Take E =2.1x105 N/mm2 and 1/m = 0.286.

3. A rectangular block of material is subjected to a tensile stress of 110 N/mm2 on one plane and a tensile stress of 47 N/mm2 on the plane at right angle to the former plane and a tensile stress of 47 N/mm2 on the plane at right angle to the former. Each of the above stress is accompanied by a shear stress of 63 N/mm2 Find (i) The direction and magnitude of each of the principal stress (ii) Magnitude of greatest shear stress

4. At a point in a strained material, the principal stresses are100 N/mm2 (T) and 40 N/mm2 (C). Determine the resultant stress in magnitude and direction in a plane inclined at 600 to the axis of major principal stress. What is the maximum intensity of shear stress in the material at the point?

5. A rectangular block of material is subjected to a tensile stress of 210 N/mm2 on one plane and a tensile stress of 28 N/mm2 on the plane at right angle to the former plane and a tensile stress of 28 N/mm2 on the plane at right angle to the former. Each of the above stress is accompanied by a shear stress of 53 N/mm2 Find (i) The direction and magnitude of each of the principal stress (ii) Magnitude of greatest shear stress

6 A closed cylindrical vessel made of steel plates 5 mm thick with plane ends, carries fluid under pressure of 6 N/mm2 The diameter of the cylinder is 35cm and length is 85 cm. Calculate the longitudinal and hoop stresses in the cylinder wall and determine the change in diameter, length and Volume of the cylinder. Take E =2.1x105 N/mm2 and 1/m = 0.286.

7. At a point in a strained material, the principal stresses are 200 N/mm2 (T) and 60 N/mm2 (C) Determine the direction and magnitude in a plane inclined at 600 to the axis of major principal stress. What is the maximum intensity of shear stress in the material at the point

8. At a point in a strained material, the principal stresses are 100 N/mm2 (T) and 40 N/mm2 (C) Determine the direction and magnitude in a plane inclined at 600 to the axis of major principal stress. What is the maximum intensity of shear stress in the material at the point

SUB CODE/NAME: CE1259 STRENGTH OF MATERIALS

QUESTION BANK

UNIT I

STRESS STRAIN DEFORMATION OF SOLIDS

PART- A (2 Marks)

1. What is Hooke's Law?
2. What are the Elastic Constants?
3. Define Poisson's Ratio.
4. Define: Resilience, proof resilience and modulus of resilience.
5. Distinguish between rigid and deformable bodies.
6. Define stress and strain.
7. Define Shear stress and Shear strain.
8. Define elastic limit.
9. Define volumetric strain.
10. Define tensile stress and compressive stress.
11. Define young's Modulus.
12. Define modulus of rigidity.
13. Define thermal stress.

PART- B (16 Marks)

1. A rod of 150 cm long and diameter 2.0cm is subjected to an axial pull of 20 KN. If the modulus of elasticity of the material of the rod is 2x 105 N/mm2
Determine 1. Stress 2. Strain 3. the elongation of the rod

2. The extension in a rectangular steel bar of length 400mm and thickness 10mm is found to 0.21mm .The bar tapers uniformly in width from 100mm to 50mm. If E for the bar is 2x 105 N/mm2 ,Determine the axial load on the bar

UNIT II

BEAMS – LOADS AND STRESSES

PART- A (2 Marks)

1. State the different types of supports.
2. What is cantilever beam?
3. Write the equation for the simple bending theory.
4. What do you mean by the point of contraflexure?
5. Define beam.
6. Define shear force and bending moment.

7. What is Shear stress diagram?
8. What is Bending moment diagram?
9. What are the types of load?
10. Write the assumption in the theory of simple bending.
11. What are the types of beams?

PART- B (16 Marks)

1. Three planks of each 50 x200 mm timber are built up to a symmetrical I section for a beam. The maximum shear force over the beam is 4KN. Propose an alternate rectangular section of the same material so that the maximum shear stress developed is same in both sections. Assume then width of the section to be 2/3 of the depth.

2. A beam of uniform section 10 m long carries a udl of KN/m for the entire length and a concentrated load of 10 KN at right end. The beam is freely supported at the left end. Find the position of the second support so that the maximum bending moment in the beam is as minimum as possible. Also compute the maximum bending moment

3. A beam of size 150 mm wide, 250 mm deep carries a uniformly distributed load of w kN/m over entire span of 4 m. A concentrated load 1 kN is acting at a distance of 1.2 m from the left support. If the bending stress at a section 1.8 m from the left support is not to exceed 3.25 N/mm2 find the load w.

4. A cantilever of 2m length carries a point load of 20 KN at 0.8 m from the fixed end and another point of 5 KN at the free end. In addition, a u.d.l. of 15 KN/m is spread over the entire length of the cantilever. Draw the S.F.D, and B.M.D.

5. A Simply supported beam of effective span 6 m carries three point loads of 30 KN, 25 KN and 40 KN at 1m, 3m and 4.5m respectively from the left support. Draw the SFD and BMD. Indicating values at salient points.

6. A Simply supported beam of length 6 metres carries a udl of 20KN/m throughout its length and a point of 30 KN at 2 metres from the right support. Draw the shear force and bending moment diagram. Also find the position and magnitude of maximum Bending moment.

7. A Simply supported beam 6 metre span carries udl of 20 KN/m for left half of span and two point loads of 25 KN end 35 KN at 4 m and 5 m from left support. Find maximum SF and BM and their location drawing SF and BM diagrams.

UNIT III

TORSION

PART-A (2 Marks)

1. Define torsional rigidity of the solid circular shaft.
2. Distinguish between closed coil helical spring and open coil helical spring

3. What is meant by composite shaft?
4. What is called Twisting moment?
5. What is Polar Modulus ?
6. Define: Torsional rigidity of a shaft.
7. What do mean by strength of a shaft?
8. Write down the equation for Wahl factor.
9. Define: Torsional stiffness.
10. What are springs? Name the two important types.

PART- B (16 Marks)

1. Determine the diameter of a solid shaft which will transmit 300 KN at 250 rpm. The maximum shear stress should not exceed 30 N/mm2 and twist should not be more than 10 in a shaft length 2m. Take modulus of rigidity = 1x 105N/mm2.

2. The stiffness of the closed coil helical spring at mean diameter 20 cm is made of 3 cm diameter rod and has 16 turns. A weight of 3 KN is dropped on this spring. Find the height by which the weight should be dropped before striking the spring so that the spring may be compressed by 18 cm. Take C= 8x104 N/mm2.

3. It is required to design a closed coiled helical spring which shall deflect 1mm under an axial load of 100 N at a shear stress of 90 Mpa. The spring is to be made of round wire having shear modulus of 0.8 x 105 Mpa. The mean diameter of the coil is 10 times that of the coil wire. Find the diameter and length of the wire.

4. A steel shaft ABCD having a total length of 2400 mm is contributed by three different sections as follows. The portion AB is hollow having outside and inside diameters 80 mm and 50 mm respectively, BC is solid and 80 mm diameter. CD is also solid and 70 mm diameter. If the angle of twist is same for each section, determine the length of each portion and the total angle of twist. Maximum permissible shear stress is 50 Mpa and shear modulus 0.82 x 105 MPa

5. The stiffness of close coiled helical spring is 1.5 N/mm of compression under a maximum load of 60 N. The maximum shear stress in the wire of the spring is 125 N/mm2. The solid length of the spring (when the coils are touching) is 50 mm. Find the diameter of coil, diameter of wire and number of coils. C = 4.5

UNIT IV

BEAM DEFLECTION

PART-A (2 Marks)
1. What are the advantages of Macaulay method over the double integration method, for finding the slope and deflections of beams?
2. State the limitations of Euler’s formula.

3. Define crippling load.
4. State Mohr’s theorem.
5. State any three assumption made in Euler’s column theory.
6. What are the different modes of failures of a column?
7. Write down the Rankine formula for columns.
8. What is effective or equivalent length of column?
9. Define Slenderness Ratio.
10. Define the terms column and strut.

PART- B (16 Marks)

1. A simply supported beam of 10 m span carries a uniformly distributed load of 1 kN/m over the entire span. Using Castigliano’s theorem, find the slope at the ends. EI = 30,000 kN/m2.

2. A 2m long cantilever made of steel tube of section 150 mm external diameter and10mm thick is loaded. If E=200 GN/m2 calculate (1) The value of W so that the maximum bending stress is 150 MN/m (2) The maximum deflection for the loading

3. A beam of length of 10 m is simply supported at its ends and carries two point loads of 100 KN and 60 KN at a distance of 2 m and 5 m respectively from the left support. Calculate the deflections under each load. Find also the maximum deflection.
Take I = 18 X 108 mm4 and E = 2 X 105.

4. i) A column of solid circular section, 12 cm diameter, 3.6 m long is hinged at both ends. Rankine’s constant is 1 / 1600 and _c= 54 KN/cm2. Find the buckling load.
ii) If another column of the same length, end conditions and rankine constant but of 12 cm X 12 cm square cross-section, and different material, has the same buckling load, find the value of _c of its material.

5. A beam of length of 6 m is simply supported at its ends. It carries a uniformly distributed load of 10 KN/m as shown in figure. Determine the deflection of the beam at its mid-point and also the position and the maximum deflection. Take EI=4.5 X 108 N/mm2.

6. An overhanging beam ABC is loaded as shown is figure. Determine the deflection of the beam at point C. Take I = 5 X 108 mm4 and E = 2 X 105 N/mm2.

7. A cantilever of length 2 m carries a uniformly distributed load of 2.5 KN/m run for a length of 1.25 m from the fixed end and a point load of 1 KN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and E=1 X 104 N/mm2

8. A cantilever of length 2m carries a uniformly distributed load 2 KN/m over a length of 1m from the free end, and a point load of 1 KN at the free end. Find the slope and deflection at the free end if E = 2.1 X 105 N/mm2 and I = 6.667 X 107 mm4 .

9. Determine the section of a hollow C.I. cylindrical column 5 m long with ends firmly built in. The column has to carry an axial compressive load of 588.6 KN. The internal diameter of the column is 0.75 times the external diameter. Use Rankine's constants.
a = 1 / 1600, _c = 57.58 KN/cm2 and F.O.S = 6.

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