PH 221-1D Spring 2013

Center of Mass and Linear Momentum

Lectures 22-23

Chapter 9

(Halliday/Resnick/Walker, Fundamentals of Physics 9th edition)

Chapter 9

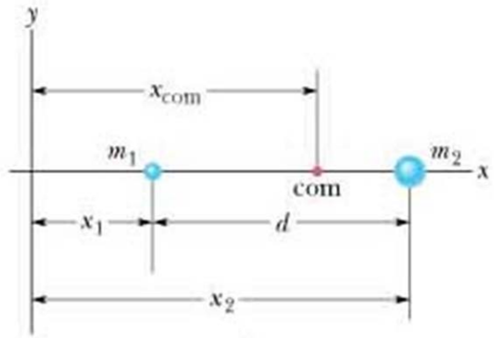
Center of Mass and Linear Momentum

In this chapter we will introduce the following new concepts:

-Center of mass (com) for a system of particles
-The velocity and acceleration of the center of mass
-Linear momentum for a single particle and a system of particles

We will derive the equation of motion for the center of mass, and discuss the principle of conservation of linear momentum

Finally we will use the conservation of linear momentum to study collisions in one and two dimensions and derive the equation of motion for rockets



(b)

The Center of Mass:

Consider a system of two particles of masses m_1 and m_2 at positions x_1 and x_2 , respectively. We define the position of the center of mass (com) as follows:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

We can generalize the above definition for a system of n particles as follows:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{m_1 + m_2 + m_3 + \ldots + m_n} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots + m_n x_n}{M} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i$$

Here M is the total mass of all the particles $M = m_1 + m_2 + m_3 + \ldots + m_n$

We can further generalize the definition for the center of mass of a system of particles in three dimensional space. We assume that the the i-th particle (mass m_i) has position vector $\vec{r}_i$

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

The position vector for the center of mass is given by the equation: $\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$

The position vector can be written as: $\vec{r}_{com} = x_{com}\hat{i} + y_{com}\hat{j} + z_{com}\hat{k}$

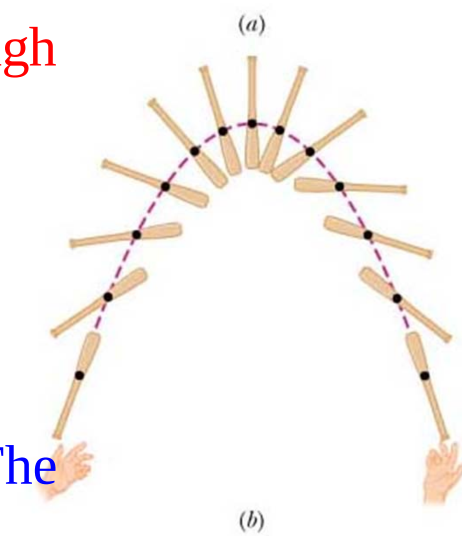
The components of $\vec{r}_{com}$ are given by the equations:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i \qquad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i \qquad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

The center of mass has been defined using the equations given above so that it has the following property:

The center of mass of a system of particles moves as though all the system's mass were concetrated there, and that the vector sum of all the external forces were applied there

The above statement will be proved later. An example is given in the figure. A baseball bat is flipped into the air and moves under the influence of the gravitation force. The center of mass is indicated by the black dot. It follows a parabolic path as discussed in Chapter 4 (projectile motion) All the other points of the bat follow more complicated paths



The Center of Mass for Solid Bodies

Solid bodies can be considered as systems with continuous distribution of matter

The sums that are used for the calculation of the center of mass of systems with discrete distribution of mass become integrals:

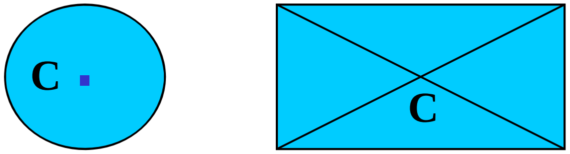
$$x_{com} = \frac{1}{M}\int x\,dm \qquad y_{com} = \frac{1}{M}\int y\,dm \qquad z_{com} = \frac{1}{M}\int z\,dm$$

The integrals above are rather complicated. A simpler special case is that of uniform objects in which the mass density $\rho = \frac{dm}{dV}$ is constant and equal to $\frac{M}{V}$

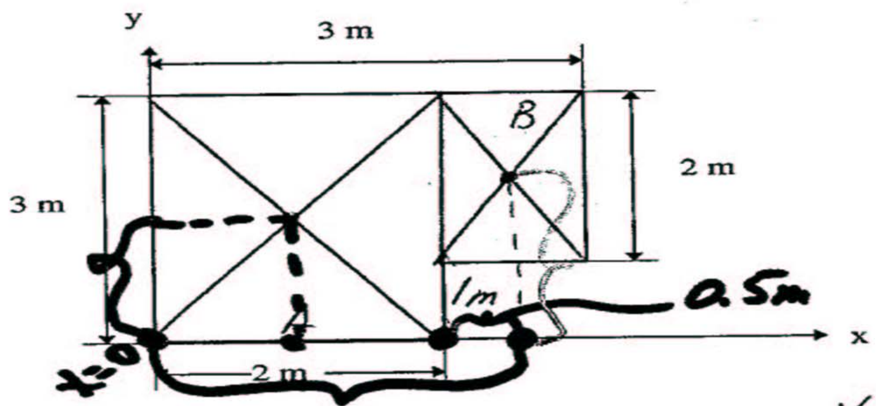
$$x_{com} = \frac{1}{V}\int x\,dV \qquad y_{com} = \frac{1}{V}\int y\,dV \qquad z_{com} = \frac{1}{V}\int z\,dV$$

In objects with symetry elements (symmetry point, symmetry line, symmetry plane) it is not necessary to eveluate the integrals. The center of mass lies on the symmetry element. For example the com of a uniform sphere coincides with the sphere center

In a uniform rectanglular object the com lies at the intersection of the diagonals



Locate the center of mass of the uniform plate shown in Fig.



1) $X_{CM_A} = 1m$

$Y_{CM_A} = 1.5m$

2) $X_{CM_B} = 2.5m$

$Y_{CM_B} = 2m$

3) $M_A = \rho S_A$; $M_B = \rho S_B$; $S_A = 3\times2 = 6m^2$ $S_A + S_B = 8m^2$

$S_B = 1\times2 = 2m^2$

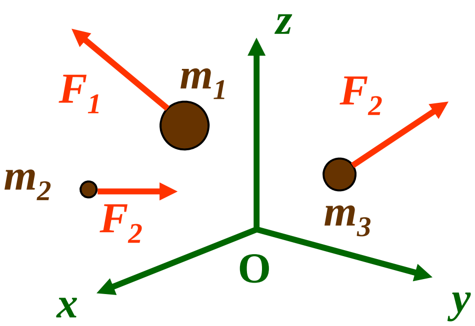
$M = \rho(S_A + S_B)$

4) $X_{CM} = \frac{M_A X_{CM_A} + M_B X_{CM_B}}{M} = \frac{\rho(S_A X_{CM_A} + S_B X_{CM_B})}{\rho(S_A + S_B)} =$

$= \frac{(6m^2)(1m) + (2m^2)(2.5m)}{(8m^2)} = 1.375m$

5) $Y_{CM} = \frac{M_A Y_{CM_A} + M_B Y_{CM_B}}{M} = \frac{\rho(S_A Y_{CM_A} + S_B Y_{CM_B})}{\rho(S_A + S_B)} =$

$= \frac{6\times1.5 + 2\times2}{8} = 1.625m$



Newton's Second Law for a System of Particles

Consider a system of n particles of masses $m_1, m_2, m_{3,} ..., m_n$

and position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, ..., \vec{r}_n$, respectively.

The position vector of the center of mass is given by:

$M\vec{r}_{com} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + ... + m_n\vec{r}_n$ We take the time derivative of both sides $\rightarrow$

$$M\frac{d}{dt}\vec{r}_{com} = m_1\frac{d}{dt}\vec{r}_1 + m_2\frac{d}{dt}\vec{r}_2 + m_3\frac{d}{dt}\vec{r}_3 + ... + m_n\frac{d}{dt}\vec{r}_n \rightarrow$$

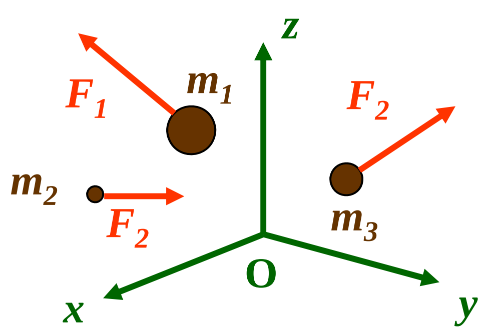
$M\vec{v}_{com} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ... + m_n\vec{v}_n$ Here $\vec{v}_{com}$ is the velocity of the com

and $\vec{v}_i$ is the velocity of the i-th particle. We take the time derivative once more $\rightarrow$

$$M\frac{d}{dt}\vec{v}_{com} = m_1\frac{d}{dt}\vec{v}_1 + m_2\frac{d}{dt}\vec{v}_2 + m_3\frac{d}{dt}\vec{v}_3 + ... + m_n\frac{d}{dt}\vec{v}_n \rightarrow$$

$M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ... + m_n\vec{a}_n$ Here $\vec{a}_{com}$ is the acceleration of the com

and $\vec{a}_i$ is the acceleration of the i-th particle



$$M\vec{a}_{com} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \ldots + m_n\vec{a}_n$$

We apply Newton's second law for the i-th particle:

$m_i\vec{a}_i = \vec{F}_i$ Here $\vec{F}_i$ is the net force on the i-th particle

$$M\vec{a}_{com} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_n$$

The force $\vec{F}_i$ can be decomposed into two components: applied and internal

$\vec{F}_i = \vec{F}_i^{app} + \vec{F}_i^{\text{int}}$ The above equation takes the form:

$$M\vec{a}_{com} = \left(\vec{F}_1^{app} + \vec{F}_1^{\text{int}}\right) + \left(\vec{F}_2^{app} + \vec{F}_2^{\text{int}}\right) + \left(\vec{F}_3^{app} + \vec{F}_3^{\text{int}}\right) + \ldots + \left(\vec{F}_n^{app} + \vec{F}_n^{\text{int}}\right) \rightarrow$$

$$M\vec{a}_{com} = \left(\vec{F}_1^{app} + \vec{F}_2^{app} + \vec{F}_3^{app} + \ldots + \vec{F}_n^{app}\right) + \left(\vec{F}_1^{\text{int}} + \vec{F}_2^{\text{int}} + \vec{F}_3^{\text{int}} + \ldots + \vec{F}_n^{\text{int}}\right)$$

The sum in the first parenthesis on the RHS of the equation above is just $\vec{F}_{net}$

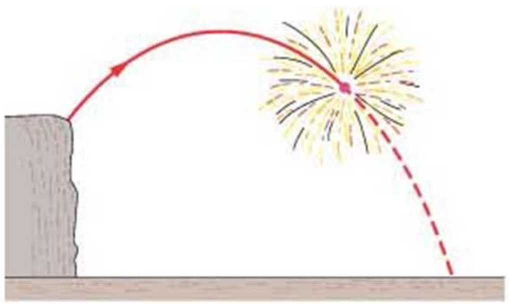
The sum in the second parethesis on the RHS vanishes

by virtue of Newton's third law.

The equation of motion for the center of mass becomes: $M\vec{a}_{com} = \vec{F}_{net}$

In terms of components we have:

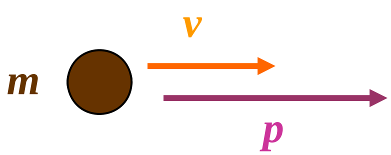
$F_{net,x} = Ma_{com,x}$ $F_{net,y} = Ma_{com,y}$ $F_{net,z} = Ma_{com,z}$



$$M\vec{a}_{com} = \vec{F}_{net}$$

$$F_{net,x} = Ma_{com,x}$$
$$F_{net,y} = Ma_{com,y}$$
$$F_{net,z} = Ma_{com,z}$$

The equations above show that the center of mass of a system of particles moves as though all the system's mass were concetrated there, and that the vector sum of all the external forces were applied there. A dramatic example is given in the figure. In a fireworks display a rocket is launched and moves under the influence of gravity on a parabolic path (projectile motion). At a certain point the rocket explodes into fragments. If the explosion had not occured, the rocket would have continued to move on the parabolic trajectory (dashed line). The forces of the explosion, even though large, are all internal and as such cancel out. The only external force is that of gravity and this remains the same before and after the explosion. This means that the center of mass of the fragments folows the same parabolic trajectory that the rocket would have followed had it not exploded



$$\vec{p} = m\vec{v}$$

Linear Momentum

Linear momentum $\vec{p}$ of a particle of mass m and velocity $\vec{v}$ is defined as: $\vec{p} = m\vec{v}$

The SI unit for lineal momentum is the kg.m/s

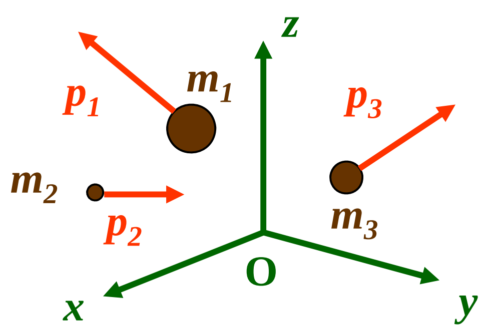
Below we will prove the following statement: The time rate of change of the linear momentum of a particle is equal to the magnitude of net force acting on the particle and has the direction of the force

In equation form: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$ We will prove this equation using Newton's second law

$$\vec{p} = m\vec{v} \rightarrow \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(m\vec{v}\right) = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}_{net}$$

This equation is stating that the linear momentum of a particle can be changed only by an external force. If the net external force is zero, the linear momentum cannot change

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$



The Linear Momentum of a System of Particles

In this section we will extedend the definition of linear momentum to a system of particles. The i-th particle has mass m_i, velocity $\vec{v}_i$, and linear momentum $\vec{p}_i$

We define the linear momentum of a system of n particles as follows:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_n = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots + m_n\vec{v}_n = M\vec{v}_{com}$$

The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity $\vec{v}_{com}$ of the center of mass

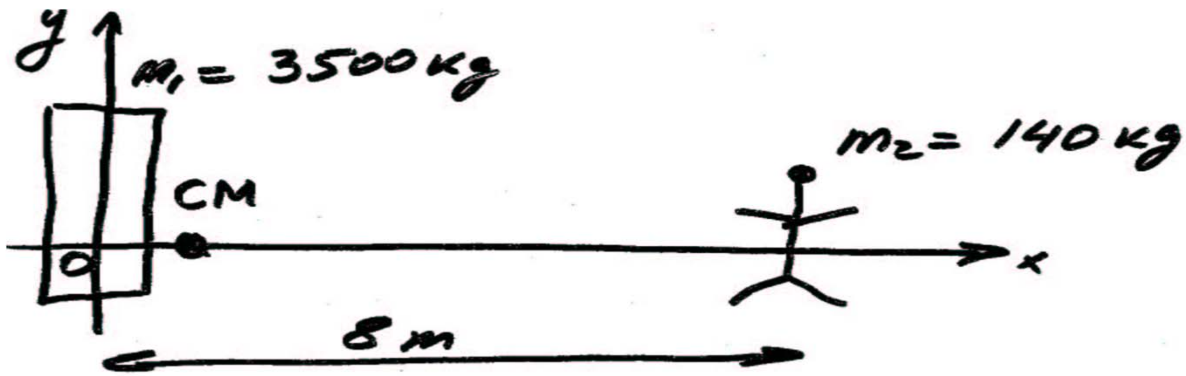
The time rate of change of $\vec{P}$ is: $\frac{d\vec{P}}{dt} = \frac{d}{dt}\left(M\vec{v}_{com}\right) = M\vec{a}_{com} = \vec{F}_{net}$

The linear momentum $\vec{P}$ of a system of particles can be changed only by a net external force $\vec{F}_{net}$. If the net external force $\vec{F}_{net}$ is zero $\vec{P}$ cannot change

$$\boxed{\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots + \vec{p}_n = M\vec{v}_{com}}$$

$$\boxed{\frac{d\vec{P}}{dt} = \vec{F}_{net}}$$

Example. Motion of the Center of Mass



1. Coordinate system with conv. origin point

2. Type of system

$F_{ext} = 0$ – isolated

$\vec{F}_{ext} = M\vec{a}_{CM}$, with $\vec{F}_{ext} = 0$ and $\vec{a}_{CM} = 0$

$\Rightarrow x_{CM} = \text{const} \Rightarrow$ They meet at the point of CM

3. $x_{CM} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \dfrac{0 + 140\,\text{kg} \times 8\,\text{m}}{3500\,\text{kg} + 140\,\text{kg}} =$

$= 0.31\ \text{m}$

During a "space walk" an astronaut floats in space 8.0 m from his Gemini spacecraft orbiting the Earth. He is tethered to the spacecraft by a long umbilical cord; to return he pulls himself in by this cord. How far does the spacecraft move toward him? The mass of the spacecraft, including is 3500 kg and the mass of the austronaut, including his space suit is 140 kg.

A fisherman in a boat catches a great white shark with a harpoon. The Shark struggles for a while and then becomes limp when at a distance of 300m from the boat. The fisherman pulls in the shark by the rope attached to the harpoon. During this operation, the boat (initially at rest) moves 50m in the direction of the shark. The mass of the boat is 5400kg. What is the mass of the shark? Pretend that the water exerts no friction.

Given:

$x_1 = 50m$

$x_1 + x_2 = 300m$

$m_1 = 5400\ kg$

Find m_2 -?

1. Isolated system $P = const$

$P_i = 0 \Rightarrow P_f = 0$; $P = MV_{CM} \Rightarrow V_{CM} = 0$

$x_{CM} = const \Rightarrow$ They meet at the point of CM

2. Choose system of coord. with origin at CM.

coordinate $m_1 : -x_1$

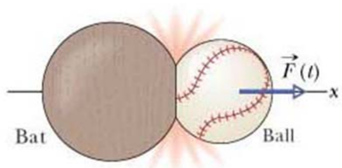
$m_2 : +x_2$

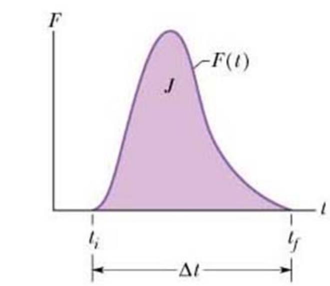
$$x_{CM} = \frac{m_1(-x_1) + m_2 x_2}{m_1 + m_2} = 0, \quad x_1 = 50, \quad x_1 + x_2 = 300 \Rightarrow -m_1 x_1 + m_2 x_2 = 0, \quad x_1 = 50, \quad x_2 = 300 - x_1 \Rightarrow$$

$$\Rightarrow m_2 = \frac{m_1 x_1}{x_2} = \frac{5400 \times 50}{250} = \boxed{1080 kg}$$

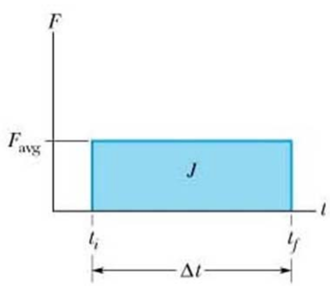
Collision and Impulse

We have seen in the previous discussion that the momentum of an object can change if there is a non-zero external force acting on the object. Such forces exist during the collision of two objects. These forces act for a brief time interval, they are large, and they are responsible for the changes in the linear momentum of the colliding objects.





(a)



(b)

Consider the collision of a baseball with a baseball bat

The collision starts at time t_i when the ball touches the bat and ends at t_f when the two objects separate

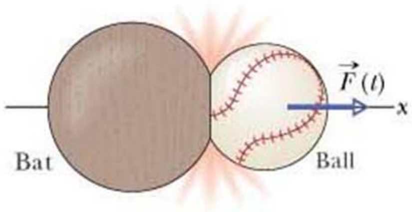
The ball is acted upon by a force $\vec{F}(t)$ during the collision

The magnitude $F(t)$ of the force is plotted versus t in fig.a

The force is non-zero only for the time interval $t_i < t < t_f$

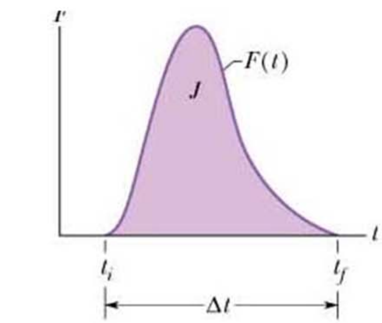
$$\vec{F}(t) = \frac{d\vec{p}}{dt}$$ Here $\vec{p}$ is the linear momentum of the ball

$$d\vec{p} = \vec{F}(t)dt \rightarrow \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$$



$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt \qquad \int_{t_i}^{t_f} d\vec{p} = \vec{p}_f - \vec{p}_i = \Delta\vec{p} \ = \text{change in momentum}$$

$\int_{t_i}^{t_f} \vec{F}(t)dt$ is known as the impulse $\vec{J}$ of the collision

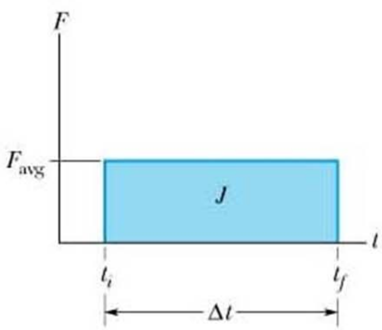


(a)

$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$ The magnitude of $\vec{J}$ is equal to the area under the F versus t plot of fig.a $\rightarrow \Delta\vec{p} = \vec{J}$

In many situations we do not know how the force changes with time but we know the average magnitude F_{ave} of the collision force. The magnitude of the impulse is given by:

$J = F_{ave}\Delta t$ where $\Delta t = t_f - t_i$



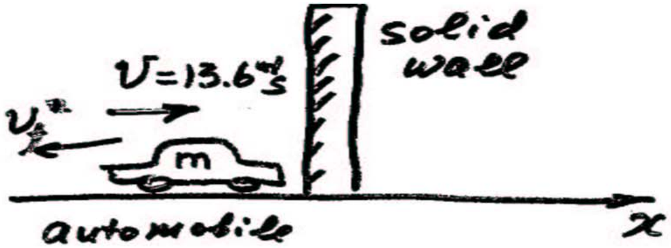
(b)

Geometrically this means that the the area under the F versus t plot (fig.a) is equal to the area under the F_{ave} versus t plot (fig.b)

$$\Delta p = J$$

$$J = F_{ave}\Delta t$$

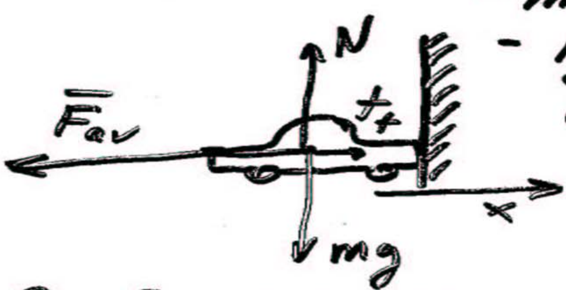
Collisions. Impulse and Momentum

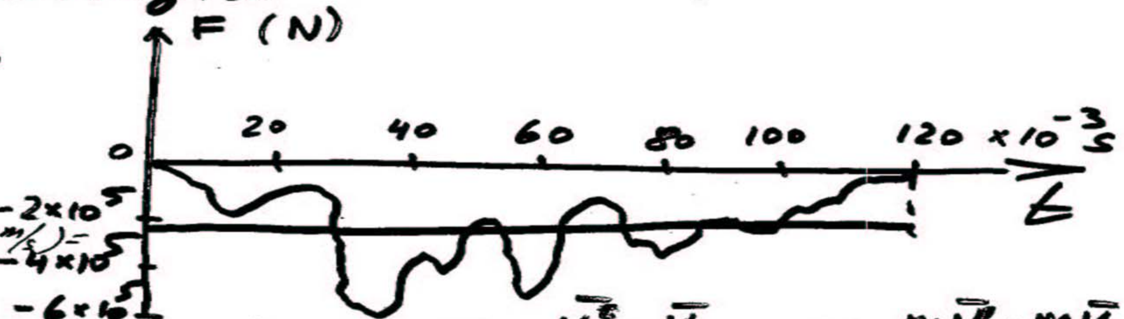


$v = 13.6\ m/s$
$v' = -1.3\ m/s$
$m = 1700\ kg$
$\Delta t = 0.120\ s$

Find $F_{x,ave} = ?$

- Collision involves a violent change of the motion.
- very strong forces (impulsive force)
  - act suddenly (contact)
  - last a short time
  - cease suddenly (separate)
  - vary in a complicated way, theoretic description is impossible.
  - much stronger than any other forces
  - produce large change in the motion. other forces produce insignificant changes.





$P_{fx} - P_{0x} = -mv_f - mv_0 =$
$= -(1700\,kg)(1.3\ m/s) - (1700\,kg)(13.6\ m/s) =$
$= -2.53\times10^4\ kg\cdot m/s$

$F_{av,x} = \frac{P_{fx} - P_{0x}}{\Delta t} =$
$= -\frac{2.53\times10^4}{120\times10^{-3}\,s} =$
$= \boxed{-2.11\times10^5\ N}$

$\bar{a} = \frac{\bar{v}_f - \bar{v}_0}{\Delta t};\ m\bar{a} = \frac{m\bar{v}_f - m\bar{v}_0}{\Delta t}$

$\bar{F}_{av} = \frac{\bar{P}_f - \bar{P}_0}{\Delta t}$ average force average over time

The Impulse-Momentum Theorem

A woman, driving a golf ball off a tee, gives the ball a velocity of +28 m/s. The mass of the ball is 0.045 kg, and the duration of the impact with the golf club is 6.0×10^{-3} s.

(a) What is the change in momentum of the ball?

(b) Determine the average force applied to the ball by the club.

(a) The change in momentum of the golf ball is

$$p_f - p_o = mv_f - mv_o = (0.045\ \text{kg})(28\ \text{m/s}) =$$

$$= \boxed{1.3\ \text{kg}\cdot\text{m/s, parallel to the ball velocity}}$$

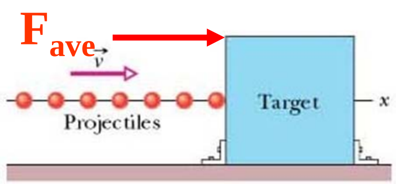
before collision, $v_o = 0$, ball at rest, $p_o = mv_o = 0$

after collision, v_f, $p_f = mv_f$ → x

(b) The average force applied by the club is

$$F_{av} = \frac{p_f - p_o}{\Delta t} = \frac{(1.3\ \text{kg}\cdot\text{m/s})}{(6.0\times10^{-3}\ \text{s})} =$$

$$= \boxed{220\ \text{N, parallel to the ball's velocity}}$$



Series of Collisions

Consider a target which collides with a steady stream of identical particles of mass m and velocity $\vec{v}$ along the x-axis

A number n of the particles collides with the target during a time interval Δt. Each particle undergoes a change Δp in momentum due to the collision with the target. During each collision a momentum change $-\Delta p$ is imparted on the target. The Impulse on the target during the time interval Δt is:

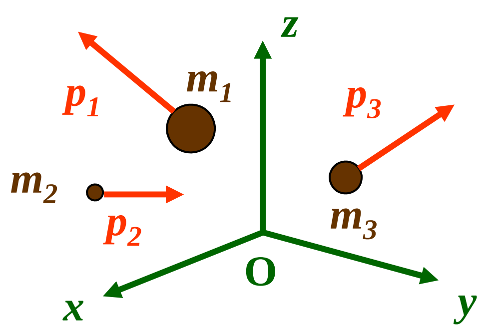
$J = -n\Delta p$ The average force on the target is:

$F_{ave} = \dfrac{J}{\Delta t} = \dfrac{-n\Delta p}{\Delta t} = -\dfrac{n}{\Delta t} m\Delta v$ Here Δv is the change in the velocity of each particle along the x-axis due to the collision with the target $\rightarrow$

$F_{ave} = -\dfrac{\Delta m}{\Delta t}\Delta v$ Here $\dfrac{\Delta m}{\Delta t}$ is the rate at which mass collides with the target

If the particles stop after the collision then $\Delta v = 0 - v = -v$

If the particles bounce backwards then $\Delta v = -v - v = -2v$



Conservation of Linear Momentum

Consider a system of particles for which $\vec{F}_{net} = 0$

$$\frac{d\vec{P}}{dt} = \vec{F}_{net} = 0 \rightarrow \vec{P} = \text{ Constant}$$

If no net external force acts on a system of particles the total linear momentum $\vec{P}$ cannot change

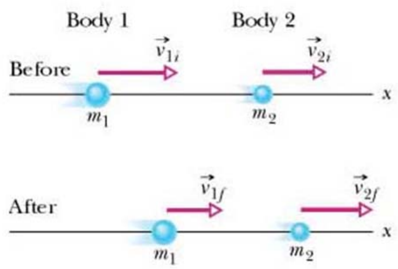
$$\begin{bmatrix}\text{total linear momentum} \\ \text{at some initial time } t_i\end{bmatrix} = \begin{bmatrix}\text{total linear momentum} \\ \text{at some later time } t_f\end{bmatrix}$$

The conservation of linear momentum is an importan principle in physics.

It also provides a powerful rule we can use to solve problems in mechanics such as collisions.

Note 1: In systems in which $\vec{F}_{net} = 0$ we can always apply conservation of linear momentum even when the internal forces are very large as in the case of colliding objects

Note 2: We will encounter problems (e.g. inelastic collisions) in which the energy is not conserved but the linear momentum is



Momentum and Kinetic Energy in Collisions

Consider two colliding objects with masses m_1 and m_2, initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ and final velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$, respectively

If the system is isolated i.e. the net force $\vec{F}_{net} = 0$ linear momentum is conserved

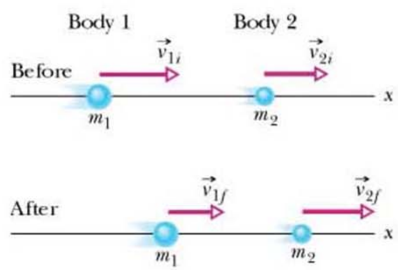
The conervation of linear momentum is true regardless of the the collision type

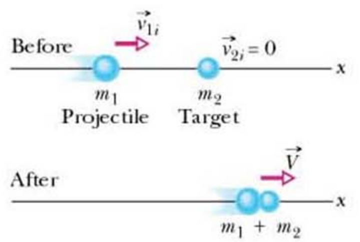
This is a powerful rule that allows us to determine the results of a collision without knowing the details. Collisions are divided into two broad classes: elastic and inelastic.

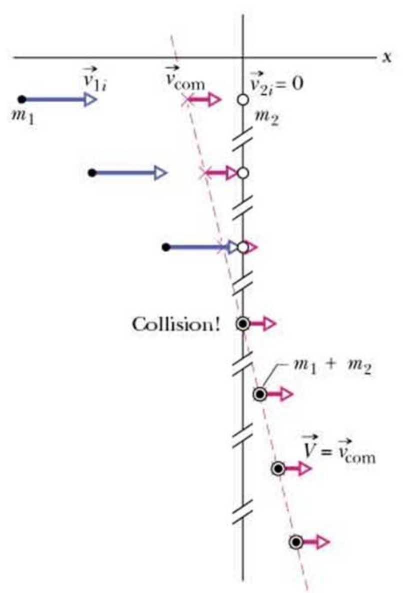
A collision is elastic if there is no loss of kinetic energy i.e. $K_i = K_f$

A collision is inelastic if kinetic energy is lost during the collision due to conversion into other forms of energy. In this case we have: $K_f < K_i$

A special case of inelastic collisions are known as completely inlelastic. In these collisions the two colliding objects stick together and they move as a single body. In these collisions the loss of kinetic energy is maximum







One Dimensional Inelastic Collisions

In these collisions the linear momentium of the colliding objects is conserved $\rightarrow$ $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$

$$m_1\vec{v}_{1i} + m_1\vec{v}_{2i} = m_1\vec{v}_{1f} + m_1\vec{v}_{2f}$$

One Dimensional Completely Inelastic Collisions

In these collisions the two colliding objects stick together and move as a single body. In the figure to the left we show a special case in which $\vec{v}_{2i} = 0. \rightarrow m_1 v_{1i} = m_1 V + m_2 V \rightarrow$

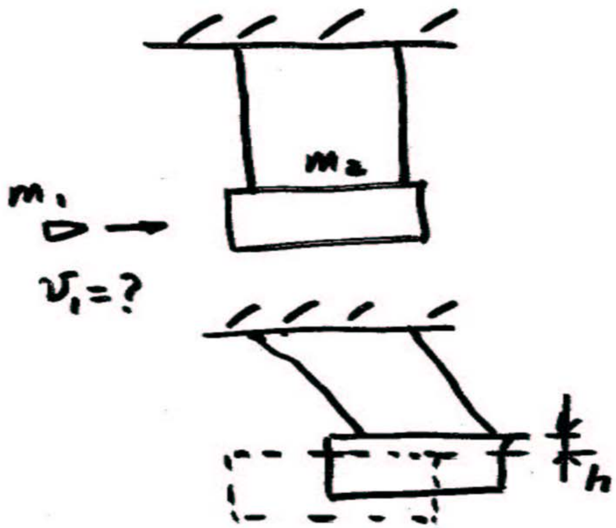
$$V = \frac{m_1}{m_1 + m_2} v_{1i}$$

The velocity of the center of mass in this collision

is $\vec{v}_{com} = \dfrac{\vec{P}}{m_1 + m_2} = \dfrac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2} = \dfrac{m_1\vec{v}_{1i}}{m_1 + m_2}$

In the picture to the left we show some freeze-frames of a totally inelastic collision

Inelastic Collision



Given $m_1 = 9.7 g$
$m_2 = 4.0 kg$
$h = 19 cm$

$v_1 = ?$

Block-bullet - isolated system at the instant of collision
- is not isolated after

1) $\bar{P}_f - \bar{P}_i = 0$

Collision
$\bar{P}$ - cons
E - not conserv.

$$(m_1+m_2)v - m_1 v_1 = 0$$

$$v = \frac{m_1 v_1}{m_1 + m_2}$$

2) after collision

$E_i = E_f$ conserv. of energy
$\bar{P}$ is not conserved

$$\frac{1}{2}(m_1+m_2)v^2 = (m_1+m_2)gh$$

$$v = \sqrt{2gh}$$

$$\frac{m_1 v_1}{m_1+m_2} = \sqrt{2gh}\,; \quad v_1 = \frac{m_1+m_2}{m_1}\sqrt{2gh} =$$

$$= \frac{0.0097 + 4.0}{0.0097}\sqrt{2 \times 9.81 \times 0.19} = 800\ m/s$$

The Principle of Conservation of Linear Momentum

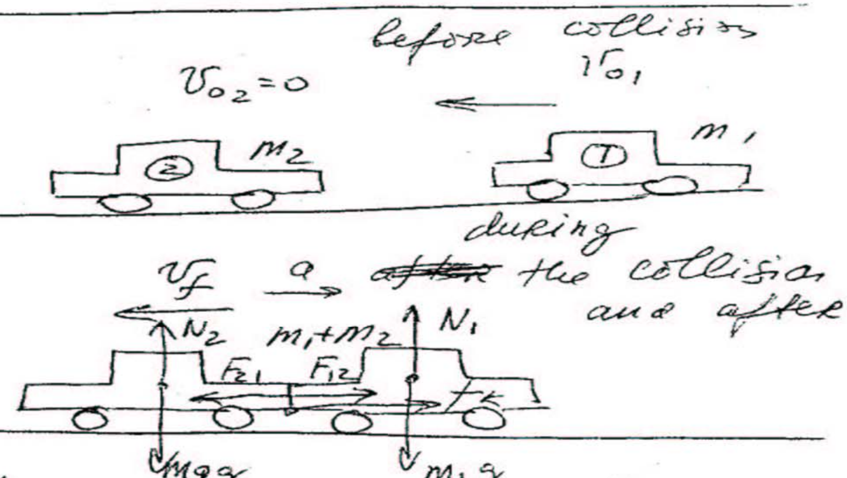
An automobile has a mass of 2300 kg and a velocity of $+16\ m/s$. It makes a rear-end collision with a stationary car whose mass is 1800 kg. The cars lock bumpers and skid off together with the wheels locked.

a) What is the velocity of the two cars just after the collision?

b) Find the impulse (magnitude and direction) that acts on the skidding cars from just after the collision?

c) If the coeff. of kinetic friction between the wheels of the cars and the pavement is $\mu_K = 0.8$, determine how far the cars skid before coming to rest.

(+)



• The total linear momentum of the two-car system is conserved because no net external force acts on the system during the collision (ignore friction during the collision) – isolated system

$$\underbrace{(m_1 + m_2) v_f}_{\text{total momentum after collision}} = \underbrace{m_1 v_{01} + m_2 v_{02}}_{\text{total momentum before collision}} \quad (v_{02} \to 0)$$

a) $$v_f = \frac{m_1 v_{01} + m_2 v_{02}}{m_1 + m_2} = \frac{(2300\ kg)(+16\ m/s) + 0}{(2300\ kg) + (1800\ kg)} = \boxed{+9.0\ m/s}$$

b) According to the impulse-momentum theorem

$$\underbrace{F_{av}\,\Delta t}_{\text{impulse due to friction}} = \underbrace{(m_1+m_2)\,v_{final}}_{\text{final momentum}} - \underbrace{(m_1+m_2)\,v_{after}}_{\text{Total momentum just after collision}}$$

$v_{final} = 0$ since the cars come to a halt

$v_{after} = v_f = +\,9.0\ m/s$

$$F_{av}\,\Delta t = 0 - (m_1+m_2)\,v_{after} = -(2300\,kg + 1800\,kg)(9.0\ m/s) = \boxed{-3.7\times10^{4}\ N\cdot s}$$

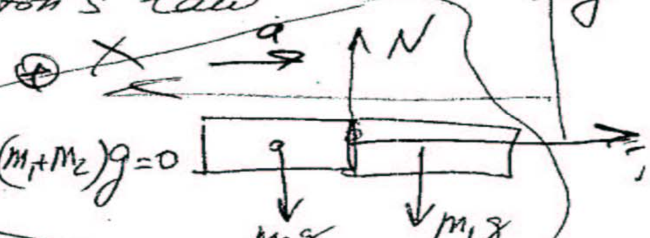
(−) – impulse due to friction acts opposite to the direction of motion of the 2 car system.

c) $v_{final}^{2}=0 = v_{after}^{2} - 2ax\ ;\quad 0 = v_f^{2} - 2\left(\dfrac{f_k}{m_1+m_2}\right)x$

$-(m_1+m_2)\,a = -f_k$ – second Newton's law

$$a = \frac{f_k}{m_1+m_2}$$

$$x = \frac{(m_1+m_2)\,v_f^{2}}{2 f_k}$$

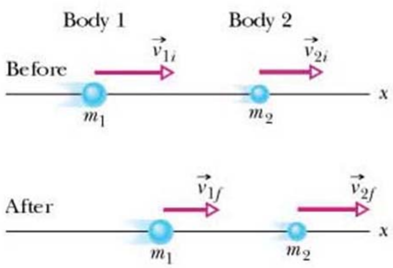


$N-(m_1+m_2)g=0$

$f_k = \mu_k N = \mu_k (m_1+m_2) g$

$$x = \frac{\cancel{(m_1+m_2)}\,v_f^{2}}{2\mu_k\,\cancel{(m_1+m_2)}\,g} = \frac{v_f^{2}}{2\mu_k g} = \frac{(9.0\ m/s)^{2}}{2\,(0.80)(9.80\ m/s^{2})} =$$

$$= \boxed{5.2\ m}$$



One-Dimensional Elastic Collisons

Consider two colliding objects with masses m_1 and m_2, initial velocities $\vec{v}_{1i}$ and $\vec{v}_{2i}$ and final velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$, respectively

Both linear momentum and kinetic energy are conserved.

Linear momentum conservation: $m_1 v_{1i} + m_1 v_{2i} = m_1 v_{1f} + m_1 v_{2f}$ (eqs.1)

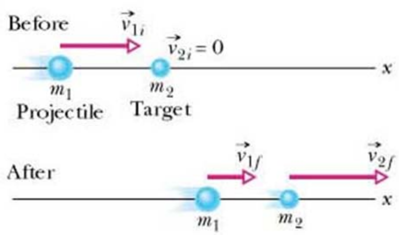
Kinetic energy conservation: $\frac{m_1 v_{1i}^2}{2} + \frac{m_1 v_{2i}^2}{2} = \frac{m_1 v_{1f}^2}{2} + \frac{m_2 v_{2f}^2}{2}$ (eqs.2)

We have two equations and two unknowns, v_{1f} and v_{2f}

If we solve equations 1 and 2 for v_{1f} and v_{2f} we get the following solutions:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i}$$



Special Case of elastic Collisions-Stationary Target $(v_{2i} = 0)$

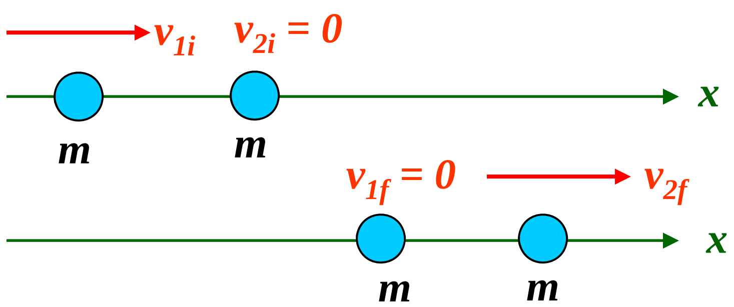
The substitute $v_{2i} = 0$ in the two solutions for v_{1f} and v_{2f}

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i} \rightarrow v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} v_{2i} \rightarrow v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Below we examine several special cases for which we know the outcome of the collision from experience

1. Equal masses $m_1 = m_2 = m$

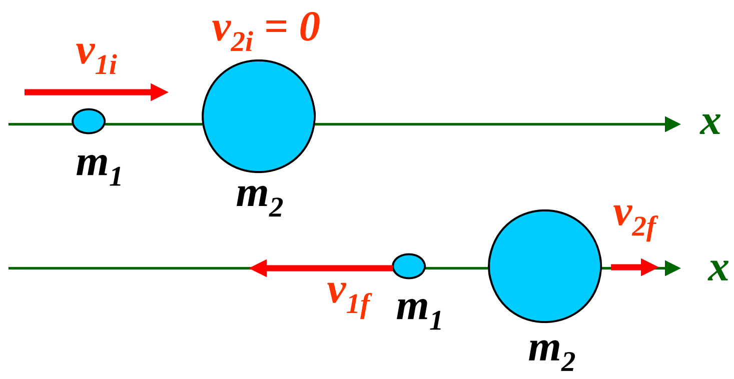


$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{m - m}{m + m} v_{1i} = 0$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2m}{m + m} v_{1i} = v_{1i}$$

The two colliding objects have exchanged velocities

2. A massive target $m_2 > m_1 \rightarrow \frac{m_1}{m_2} < 1$

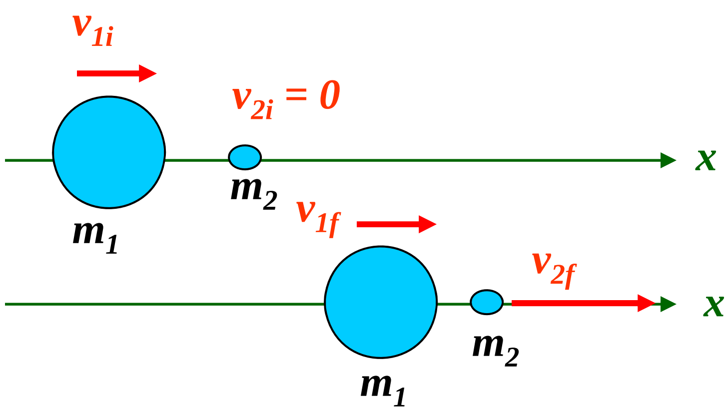


$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{\frac{m_1}{m_2} - 1}{\frac{m_1}{m_2} + 1} v_{1i} \approx -v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2\left(\frac{m_1}{m_2}\right)}{\frac{m_1}{m_2} + 1} v_{1i} \approx 2\left(\frac{m_1}{m_2}\right) v_{1i}$$

Body 1 (small mass) bounces back along the incoming path with its speed practically unchanged.

Body 2 (large mass) moves forward with a very small speed because $\frac{m_1}{m_2} < 1$



2. A massive projectile $m_1 > m_2 \rightarrow \frac{m_2}{m_1} < 1$

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} = \frac{1 - \frac{m_2}{m_1}}{1 + \frac{m_2}{m_1}} v_{1i} \approx v_{1i}$$

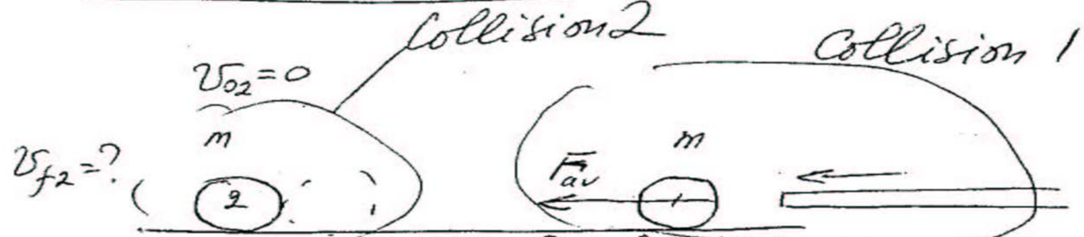
$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} = \frac{2}{1 + \frac{m_2}{m_1}} v_{1i} \approx 2v_{1i}$$

Body 1 (large mass) keeps on going scarcely slowed by the collision .

Body 2 (small mass) charges ahead at twice the speed of body 1

Elastic Collision

A cue ball (mass=0.165 kg) is at rest on a frictionless pool table. The ball is hit dead center by a pool stick which applies an impulse of +1.50 N·s to the ball. The ball then slides along the table and makes an elastic head-on collision with a second ball of equal mass that is initially at rest. Find the velocity of the second ball just after it is struck.



$F_{av}\Delta t = 1.50\ N \cdot s$

$m = 0.165\ kg;\ v_{02}=0$

$v_{f2} = ?$

- There are 2 collisions
- For the 2nd collision: The velocity of the second ball just after the collision v_{f2} can be found from

Equation $v_{f2} = \left(\frac{2m_1}{m_1+m_2}\right) v_{01}$ which is the solution of the system of two equations describing elastic collision

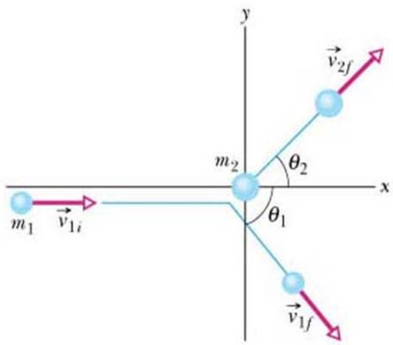
$$\left.\begin{aligned} \underbrace{m_1 v_{f1} + m_2 v_{f2}}_{P_{after}} &= \underbrace{m_1 v_{01} + m_2 v_{02}}_{P_{before}} \\ \underbrace{\tfrac{1}{2} m_1 v_{f1}^2 + \tfrac{1}{2} m_2 v_{f2}^2}_{KE_{after}} &= \underbrace{\tfrac{1}{2} m_1 v_{01}^2 + \tfrac{1}{2} m v_{02}^2}_{KE_{before}} \end{aligned}\right\}$$

($v_{02} = 0$)

- We must know v_{01}. From the 1st coll. $F_{av}\Delta t = (m v_{01})_{\text{after 1 collision}} - (m v_{01})_{\text{before 1 collision}}$ (= 0)

$$v_{01} = \frac{F\Delta t}{m} = \frac{+1.50\ N \cdot s}{0.165\ kg} = 9.09\ m/s$$

- $v_{f2} = \left(\frac{2m_1}{m_1+m_2}\right) v_{01} = \left(\frac{2m}{m+m}\right) v_{01} = v_{01} = +9.09\ m/s$



Collisions in Two Dimensions

In this section we will remove the restriction that the colliding objects move along one axis. Instead we assume that the two bodies that participate in the collision move in the *xy*-plane. Their masses are m_1 and m_2

The linear momentum of the sytem is conserved: $\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$

If the system is elastic the kinetic energy is also conserved: $K_{1i} + K_{2i} = K_{1f} + K_{2f}$

We assume that m_2 is stationary and that after the collision particle 1 and particle 2 move at angles θ_1 and θ_2 with the initial direction of motion of m_1

In this case the conservation of momentum and kinetic energy take the form:

$x-$axis: $m_1 v_{1i} = m_1 v_{1f} \cos\theta_1 + m_2 v_{2f} \cos\theta_2$ (eqs.1)

$y-$axis: $0 = -m_1 v_{1f} \sin\theta_1 + m_2 v_{2f} \sin\theta_2$ (eqs.2)

$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{2f}^2 + \frac{1}{2} m_2 v_{2f}^2$ (eqs.3) We have three equations and seven variables:

Two masses: m_1, m_2 three speeds: v_{1i}, v_{1f}, v_{2f} and two angles: θ_1, θ_2. If we know the values of four of these parameters we can calculate the remaining three

Problem 72. Two 2.0 kg bodies, A and B collide. The velocities before the collision are $\vec{v}_A = (15\hat{i} + 30\,\hat{j})$ m/s and $\vec{v}_B = (-10\hat{i} + 5.0\,\hat{j})$ m/s. After the collision, $\vec{v}_A' = (-5.0\hat{i} + 20\,\hat{j})$ m/s. What are (a) the final velocity of B and (b) the change in the total kinetic energy (including sign)?

(a) Conservation of linear momentum implies

$$m_A\vec{v}_A + m_B\vec{v}_B = m_A\vec{v}'_A + m_B\vec{v}'_B\,.$$

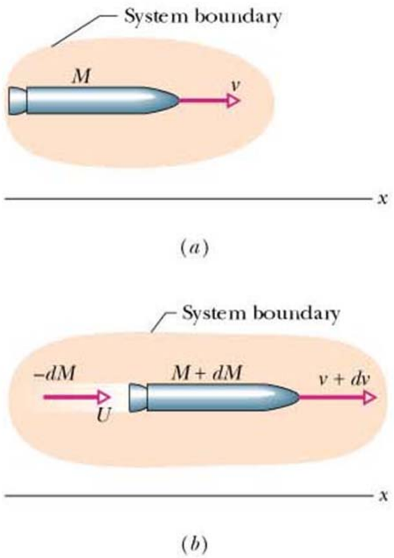
Since $m_A = m_B = m = 2.0$ kg, the masses divide out and we obtain

$$\begin{aligned}\vec{v}'_B &= \vec{v}_A + \vec{v}_B - \vec{v}'_A = (15\hat{\mathrm{i}} + 30\hat{\mathrm{j}})\ \mathrm{m/s} + (-10\,\hat{\mathrm{i}} + 5\hat{\mathrm{j}})\ \mathrm{m/s} - (-5\,\hat{\mathrm{i}} + 20\,\hat{\mathrm{j}})\ \mathrm{m/s}\\ &= (10\,\hat{\mathrm{i}} + 15\,\hat{\mathrm{j}})\ \mathrm{m/s}\,.\end{aligned}$$

(b) The final and initial kinetic energies are

$$\begin{aligned}K_f &= \frac{1}{2}mv'^2_A + \frac{1}{2}mv'^2_B = \frac{1}{2}(2.0)\left((-5)^2 + 20^2 + 10^2 + 15^2\right) = 8.0\times10^2\ \mathrm{J}\\ K_i &= \frac{1}{2}mv^2_A + \frac{1}{2}mv^2_B = \frac{1}{2}(2.0)\left(15^2 + 30^2 + (-10)^2 + 5^2\right) = 1.3\times10^3\ \mathrm{J}\,.\end{aligned}$$

The change kinetic energy is then $\Delta K = -5.0 \times 10^2$ J (that is, 500 J of the initial kinetic energy is lost).



Systems with Varying Mass: The Rocket

A rocket of mass M and speed v ejects mass backwards at a constant rate $\frac{dM}{dt}$. The ejected material is expelled at a constant speed v_{rel} relative to the rocket. Thus the rocket loses mass and accelerates forward. We will use the conservation of linear momentum to determine the speed v of the rocket

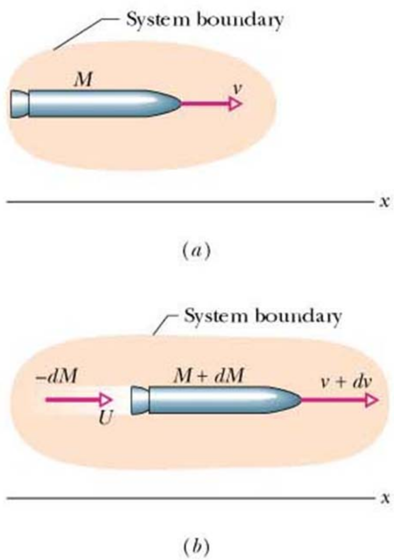
In figures (a) and (b) we show the rocket at times t and $t + dt$. If we assume that there are no external forces acting on the rocket, linear momentum is conserved

$$p(t) = p\left(t + dt\right) \rightarrow Mv = -dMU + \left(M + dM\right)\left(v + dv\right) \quad \text{(eqs.1)}$$

Here dM is a negative number because the rocket's mass decreases with time t

U is the velocity of the ejected gases with respect to the inertial reference frame in which we measure the rocket's speed v. We use the transformation equation for velocities (Chapter 4) to express U in terms of v_{rel} which is measured with respect to the rocket. $U = v + dv - v_{rel}$ We substitute U in equation 1 and we get:

$$Mdv = -dMv_{rel}$$



Using the conservation of linear momentum we derived the equation of motion for the rocket

$Mdv = -dMv_{rel}$ (eqs.2) We assume that material is ejected from the rocret's nozzle at a constant rate

$\frac{dM}{dt} = -R$ (eqs.3) Here R is a constant positive number, the positive mass rate of fuel consumprion.

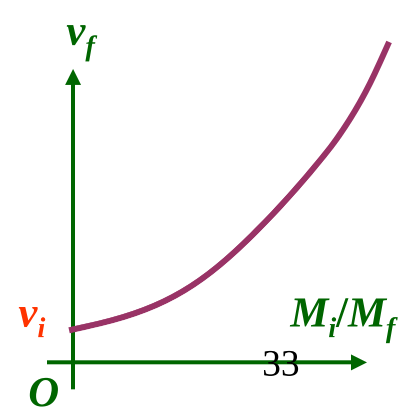
We devide both sides of eqs.(2) by dt $\rightarrow M\frac{dv}{dt} = -\frac{dM}{dt}v_{rel} = Rv_{rel} \rightarrow Ma = Rv_{rel}$

(First rocket equation) Here a is the rocket's acceleration, Rv_{rel} the thrust of the rocket engine. We use equation 2 to determine the rocket's speed as function of time

$dv = -v_{rel}\frac{dM}{M}$ We integrate both sides $\rightarrow \int_{v_i}^{v_f} dv = -v_{rel}\int_{M_i}^{M_f}\frac{dM}{M} \rightarrow$

$$v_f - v_i = -v_{rel}\left[\ln M\right]_{M_i}^{M_f} = v_{rel}\left[\ln M\right]_{M_f}^{M_i} = v_{rel}\ln\frac{M_i}{M_f}$$

$v_f - v_i = v_{rel}\ln\frac{M_i}{M_f}$ (Second rocket equation)



Problem 78. A 6090 kg space probe moving nose-first toward Jupiter at 105 m/s relative to the Sun fires its rocket engine, ejecting 80.0 kg of exhaust at a speed of 253 m/s relative to the space probe. What is the final velocity of the probe?

$$v_f = v_i + v_{\text{rel}} \ln\left(\frac{M_i}{M_f}\right) = 105\ \text{m/s} + (253\ \text{m/s})\ \ln\left(\frac{6090\ \text{kg}}{6010\ \text{kg}}\right) = 108\ \text{m/s}.$$