Central Results from Newton's Principia Mathematica
–
I. The Body of Least Resistance

Josef Bemelmans

Institute for Mathematics, RWTH Aachen

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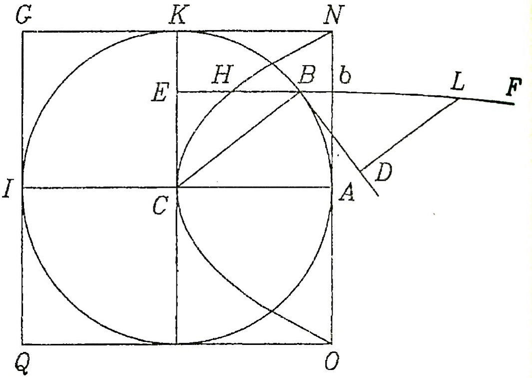
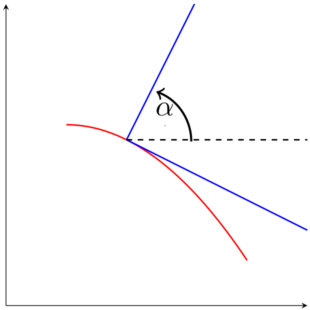
Outline

Proposition XXXIV

In a rare medium consisting of particles that are equal and arranged freely at equal distances from one another; let a sphere and a cylinder – described with equal diameters – move with equal velocity along the axis of the cylinder; then the resistance of the sphere will be half the resistance of the cylinder.



Formulation of the Problem, Proof by Standard Methods



Force of particle at point P: **e** with $\|\mathbf{e}\| = 1$.

Force at P in direction of **n**: $cos\,\alpha$.

Force at P in direction of **e**: $cos^2\,\alpha$.

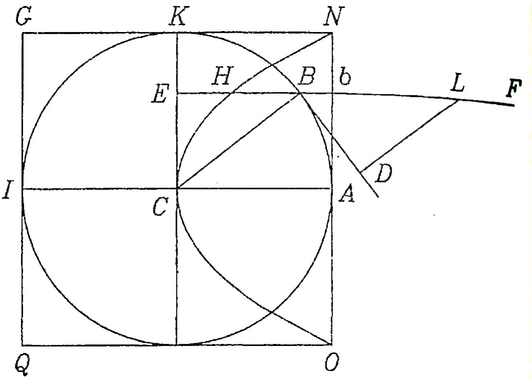
$R(u) = \int_0^{2\pi}\int_0^{r_0} cos^2\,\alpha \cdot cos\,\alpha \cdot \sqrt{1+|u'|^2}\, r\,dr\,d\phi = 2\pi \int_0^{r_0} \frac{1}{1+|u'|^2}\, r\,dr$

because of $cos\alpha = \frac{1}{\sqrt{1+|u'|^2}}$.

Resistance of sphere: $u_1(r) = \sqrt{r_0^2 - r^2}, R(u_1) = \frac{\pi}{2} \cdot r_0^2$.

Resistance of cylinder: $u_2 = 1, R(u_2) = \pi \cdot r_0^2$.

Newton's Proof



$F(B), F(b) :=$ force acting in B or b, resp., $|BC| = 1, |LB| = 1$.
$F(B) : F(b) = LD : LB = BE : BC$.
$F_e :=$ component of F acting in direction e;
therefore $F_e(B) : F_e(b) = BE^2 : BC^2$.

Summation of forces: Define H on the line EF by

$$bH = \frac{BE^2}{BC} = BC \cdot \frac{BE^2}{BC^2} = AC \cdot cos^2\alpha.$$

For b on ON, this defines a parabola with x= EH, y = EC:

$$bH = AC - x,$$

$$BE^2 = BC^2 - CE^2 = AC^2 - y^2,$$

$$AC - x = bH = \frac{BE^2}{BC} = \frac{1}{BC} \cdot (AC^2 - y^2).$$

Hence, $y^2 = AC \cdot x$.

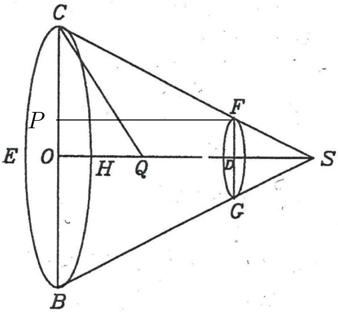
Archimedes:

Paraboloid has half the volume of its enclosing cylinder.

Scholium, Part I

By the same method other figures can be compared with one another with respect to resistance, and those that are more suitable for continuing their motion in resisting mediums can be found. For example, let it be required to construct a frustrum CBGF of a cone with the circular base CEBH (which is described with center O and radius OC) and with the height OD, which is resisted less than any other frustrum constructed with the same base and height and moving forward along the direction of the axis toward D; bisect the height OD in Q, and produce OQ to S so that QS is equal to QC, and S will be the vertex of the cone whose frustrum is required.

Scholium, Part I, Fig. 1



Proof by Standard Methods

$OC =: a; OD =: b;\ DF =: c; DS =: l.$

Force at an arbitrary point of the envelop:

$F = \sin^2\beta,\ \beta = \sphericalangle OSC,\ \beta = \frac{\pi}{2} - \alpha.$

Total resistance: $\pi c^2 + \pi \cdot (a^2 - c^2) \cdot \sin^2\beta.$

From $\sin\beta = \frac{OC}{CS}$ and

$CS^2 = CO^2 + OS^2:\ \sin^2\beta = \frac{a^2}{a^2 + (b+l)^2}.$

Intercept theorem: $\frac{c}{a} = \frac{l}{b+l}.$

$\Rightarrow\ R(l) = \pi \cdot a^2 \cdot \frac{a^2 + l^2}{a^2 + (b+l)^2}.$

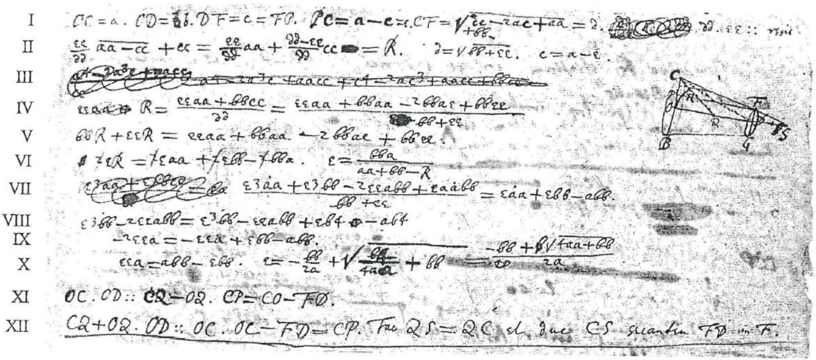
$$\frac{dR}{dl}(l) = 0 \Rightarrow l = -\frac{b}{2} + \sqrt{\frac{b^2}{4} + a^2}.$$

$$OS = b + l = \frac{b}{2} + \sqrt{\frac{b^2}{4} + a^2} = OQ + \sqrt{OQ^2 + OC^2}$$
$$= OQ + QC.$$

On the other hand:
$OS = QS + OQ$

$\Rightarrow QC = QS.$

Newton's Proof



(I): Definitions of quantities a,b,and c.

(I.c): Definition of P: OP = DF; FP is slip of the pen

(I.d): Definition of quantity e, parameter of the family of cones

(I.e): $CF^2 = PF^2 + CP^2 = OD^2 + CP^2 = b^2 + (a-c)^2$, hence $CF = \sqrt{b^2 + (a-c)^2} =: d$

(II): Definition of resistance, $sin^2\beta = \frac{e^2}{d^2}$ is (I.f), only partly visible. c^2 is area of the top surface, $a^2 - c^2$ weighted area of the envelop of the cone.

(IV): Formula for the resistance

(VI): Differentiation with respect to e:

$b^2 \cdot \frac{dR}{de} + 2\,e\,R + e^2 \cdot \frac{dR}{de} = 2\,e\,a^2 - 2\,b\,a^2 + 2\,b^2\,e$

(VII): Terms with a dot on top cancel, cf.

$e\,a^2\,(b^2 + e^2) = e\,a^2\,b^2 + e^3\,a^2$

(X): Quadratic equation and its solution;

to see its geometric content, write solution in the form

(+) $\frac{a}{b} = \frac{\sqrt{\frac{1}{4}b^2 + a^2} - \frac{b}{2}}{e}$.

Quantity $b/2$ equals OQ and QD.

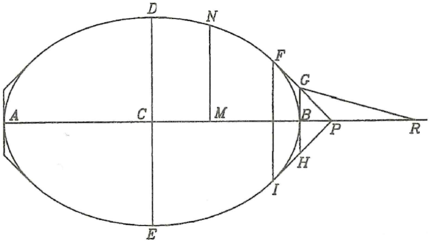
(XI): In geometrical terms (X.b) reads $\frac{OC}{OD} = \frac{CQ-OQ}{CP} = \frac{CQ-OQ}{CO-FD}$. This equation does not show the proposition $CQ = QS$; multiply (+) by $\left(\sqrt{\frac{b^2}{4} + a^2} + \frac{b}{2}\right)$ to get

$\frac{\sqrt{\frac{b^2}{4}+a^2}+\frac{b}{2}}{b} = \frac{a}{e}$, and this is (XII).

On the other hand, we have from the intercept theorem $\frac{CO}{CP} = \frac{OS}{PF} = \frac{OS}{OD} = \frac{OQ+QS}{OD}$, hence $CQ = QS$.

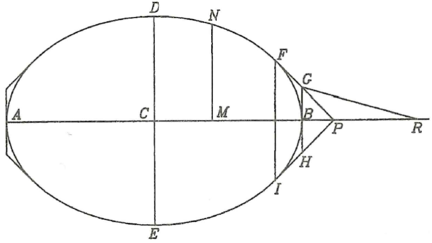
Newton concludes his notes with the sentence:
Fac QS = QC et duc CS secantem FD in F.

Scholium, Part II



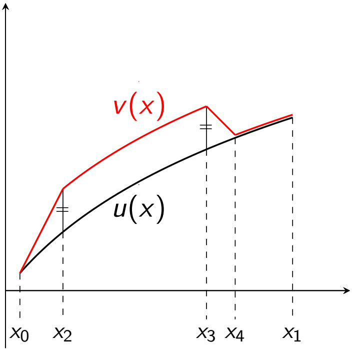
Note in passing that the angle CSB is always acute, it follows that if the solid ADBE is generated by a revolution of the elliptical or oval figure ADBE about the axis AB, and if the generating figure is touched by the three straight lines FG, GH, and HI in points F,B, and I, in such a way that GH is perpendicular to the axis in the point of contact B, and FG and HI meet in the said line GH at angles FGB and BHI of 135°, then the solid that is generated by the figure ADFGHIE about the same axis AB is less resisted than the former solid, provided that each of the two moves forward along the direction of its axis AB, and the end B of each one is in front. Indeed, I think that this proposition will be of some use for the construction of ships.

Scholium, Part III



But suppose the figure DNFG to be a curve of such a sort that if the perpendicular NM is dropped from any point N of that curve to the axis AB, and if from the given point G the straight line GR is drawn which is parallel to a straight line touching the figure in N and cuts the axis (produced) in R, then MN would be to GR as GR^3 to $4BR \times GB^2$. Then, in this case, the solid that is described by a revolution of this figure about the axis AB will, in moving in the aforesaid rare medium from A towards B, be resisted less than any other solid of revolution described with the same length and width.

Elementary Proof (with Modern Methods)



For a minimizer u, we get

$$\begin{cases} \int_{x_0}^{x_2} f(x, v'(x))\, dx \\ + \int_{x_3}^{x_4} f(x, v'(x))\, dx \\ - \int_{x_0}^{x_2} f(x, u'(x))\, dx \\ - \int_{x_3}^{x_4} f(x, u'(x))\, dx \geq 0. \end{cases}$$

By the mean value theorem one gets:
$\int_{x_0}^{x_2} f(x, v'(x))\, dx = (x_2 - x_0) \cdot f(x', \frac{(u(x_2)+k)-u(x_0)}{x_2-x_0})$ and
$\int_{x_3}^{x_4} f(x, v'(x))\, dx = (x_4 - x_3) \cdot f(x'', \frac{u(x_4)-(u(x_3)+k)}{x_4-x_3})$ with
$x' \in [x_0, x_2], x'' \in [x_3, x_4]$.

$$\begin{cases} \int_{x_0}^{x_2} f(x, v'(x))\, dx \\ + \int_{x_3}^{x_4} f(x, v'(x))\, dx \\ - \int_{x_0}^{x_2} f(x, u'(x))\, dx \\ - \int_{x_3}^{x_4} f(x, u'(x))\, dx \geq 0 \end{cases}$$

is equivalent to: $\int_{x_0}^{x_2} f(x, v'(x))\, dx + \int_{x_3}^{x_4} f(x, v'(x))\, dx$ is minimal for $k = 0$, hence, after approximating $u(x)$ by $u(x_0) + u'(x_0) \cdot (x_2 - x)$ in $[x_0, x_2]$ and, analogously, approximating in $[x_3, x_4]$, we get, with $h = x_2 - x_0 = x_4 - x_3$:
$h \cdot [f(x', \frac{u(x_2)-u(x_0)+k}{h}) + f(x'', \frac{u(x_4)-u(x_3)-k}{h})]$ is minimal in $k = 0$.
Then, after $h \to 0$: $f_p(x_0, u'(x_0)) = f_p(x_4, u'(x_4))$, and x_4 arbitrary.

Newton's Statement in Modern Notation

(*) $MN : GR = GR^3 : (4BR \, x \, GB^2)$.

Coordinates $r = MN, u(r) = CM$.

With $GB = c = const.$, set $GR = \frac{c}{cos\theta}$, then (*) becomes

$$\frac{r \cdot cos\theta}{c} = \frac{1}{4}\frac{1}{sin\theta} \cdot \frac{1}{cos^2\theta}.$$

$$\frac{r \cdot sin\theta}{cos\theta} = \frac{c}{4} \cdot (\frac{sin^2\theta + cos^2\theta}{cos^2\theta})^2 = \frac{c}{4} \cdot (1 + \frac{sin^2\theta}{cos^2\theta})^2.$$

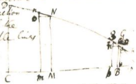
With $u'(r) = \frac{sin\theta}{cos\theta}$, this is $\frac{r \cdot u'(r)}{(1 + |u'(r)|^2)^2} = \frac{c}{4}$.

To ... Gregory

Sr

I now thank you heartily both for the very kind visit you made me, & for your care of Mr Pagets business. & the errata you gave me notice of in my book. The Lem. 1 in ye third book I could not recover as tis there stated, but I have don't another way wth a Demonstration, & altered very much the Proposition wch follows upon it concerning ye precession of ye Equinox. The whole is too long to set down. The figure wch feels ye least resistance in ye Schol. of Prop. XXXV Lib. II. is demonstrable by these steps

1. If upon BM be erected infinitely narrow parallelograms BGgb & MNom, & their distance Mb & altitudes MN, BG be given & the semisum of their bases $\frac{Mm + Bb}{2}$ be also given & called s & their semidifference $\frac{Mm - Bb}{2}$ be called x: And if the lines BG, bh, MN, MN, mo. cutt upon ye curve nNgG in ye points n, N, g & G, & the infinitely little lines on & hg be equal to one another & called c, & the figure mnNgGB be turned about its axis BM to generate a solid, & this solid move uniformly in water from M to B according to ye direction of its axis BM: the summ of the resistances of the two surfaces generated by the infinitely little lines Gg, Nn shall be least when gG qq is to nN qq as BG × Bb to MN × Mm.



For the resistances of the surfaces generated by the revolution of Gg & Nn are as $\frac{BG}{Gg^{quad}}$ & $\frac{MN}{Nn^{quad}}$, that is, if Gg quad & Nn quad be called p & q, as $\frac{BG}{p}$ & $\frac{MN}{q}$ & their summ $\frac{BG}{p} + \frac{MN}{q}$ is least when the fluxion thereof $-\frac{BG \times \dot{p}}{pp} - \frac{MN \times \dot{q}}{qq}$ is nothing, or $-\frac{BG \times \dot{p}}{pp} = +\frac{MN \times \dot{q}}{qq}$.

Now $p = Gg^{quad} = Bb^{quad} + gh^{quad} = ss - 2sx + xx + c$... & by ye same argument $q = 2s\dot{x} + 2$... & therefore $\dot{p} = -2s\dot{x} + 2x\dot{x}$ & therefore $\frac{BG \times 2s\dot{x} - 2x\dot{x}}{pp} = \frac{MN \times 2s\dot{x} + 2x\dot{x}}{qq}$, or $\frac{BG \times s - x}{pp} = \frac{MN \times}{qq}$... that is, BG^{qq} to nN^{qq} as BG × Bb ... & therefore pp is to qq as BG × s − x to MN × s + x that is MN × Mm.

2. If the curve line DnNgG be such that the surface of the solid generated by its revolution feels ye least resistance of any solid with the same top & bottom BG & CD ... than the resistance of the two narrow annular surfaces generated by ye revolution of the ... intermediate solid bgNM be removed ... less than if the ... as to be congruence it is the least that can be ... & therefore gG^{qq} is to nN^{qq} as BG ... to BG ...

... be equal to hG so that ye angle ... will ... 6.

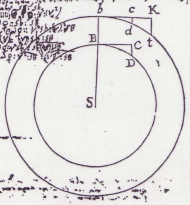
Reactions of Other Scientists

C.Huygens: works out the details, see his notes from April, 22nd and 25th, 1691, in Oeuvres, XII, pp. 325-332, 335-341.
G.W.Leibniz: notes in his copy of Newton's Principia:
"investigandum ex isoperimetris facillime progrediens"
"isoperimetris" crossed out and replaced by "isolabis".

Prop. IV. Theor. IV.

Corporum quæ diversos circulos æquabili motu describunt, vires centripetas ad centra eorundem circulorum tendere, & esse inter se ut arcuum simul descriptorum quadrata applicata ad circulorum radios.

Corpora *B*, *b* in circumferentiis circulorum *BD*, *bd* gyrantia, simul describant arcus *BD*, *bd*. Quoniam sola vi insita describerent tangentes *BC*, *bc* his arcubus æquales, manifestum est quod vires centripetæ sunt quæ perpetuo retrahunt corpora de tangentibus ad circumferentias circulorum, atq; adeo hæ sunt ad invicem in ratione prima spatiorum nascentium *CD*, *cd*: tendunt vero ad centra circulorum per Theor. II, propterea quod areæ radiis descriptæ ponuntur temporibus proportionales. Fiat figura *tkb* figuræ *DCB* similis, & per Lemma V, lineola *CD* erit ad lineolam *kt* ut arcus *BD* ad arcum *bt*: nec non, per Lemma XI, lineola nascens *tk* ad lineolam nascentem *dc* ut *bt quad.* ad *bd quad.* & ex æquo lineola nascens *DC* ad lineolam nascentem *dc* ut $BD \times bt$ ad *bd quad.* seu quod perinde est, ut $\frac{BD \times bt}{Sb}$ ad $\frac{bd\ quad.}{Sb}$, adeoq; (ob æquales rationes $\frac{bt}{Sb}$ & $\frac{BD}{SB}$) ut $\frac{BD\ quad.}{SB}$ ad $\frac{bd\ quad.}{Sb}$. Q. E. D.

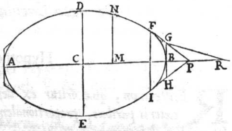


Corol. 1. Hinc vires centripetæ sunt ut velocitatum quadrata applicata ad radios circulorum.

Corol. 2. Et reciproce ut quadrata temporum periodicorum ap-

[327]

em eundem CB generatur, minus reſiſtitur quam ſolidum prius; ſi modo utrumque ſecundum plagam axis ſui AB progrediatur, & utriuſque terminus B præcedat. Quam quidem propoſitionem in conſtruendis Navibus non inutilem futuram eſſe cenſeo.



Quod ſi figura $DNFB$ ejuſmodi ſit ut, ſi ab ejus puncto quovis N ad axem AB demittatur perpendiculum NM, & a puncto dato G ducatur recta GR quæ parallela ſit rectæ figuram tangenti in N, & axem productum ſecet in R, fuerit MN ad GR ut GR *cub.* ad $4 BR \times GBq$: Solidum quod figuræ hujus revolutione circa axem AB facta deſcribitur, in Medio raro & Elaſtico ab A verſus B velociſſime movendo, minus reſiſtetur quam aliud quodvis eadem longitudine & latitudine deſcriptum Solidum circulare.

Prop. XXXVI. Prob. VIII.

Invenire reſiſtentiam corporis Sphærici in Fluido raro & Elaſtico velociſſime progredientis. (Vide Fig. Pag. 325.)

Deſignet $ABKI$ corpus Sphæricum centro C ſemidiametro CA deſcriptum. Producatur CA primo ad S deinde ad R, ut ſit AS pars tertia ipſius CA, & CR ſit ad CS ut denſitas corporis Sphærici ad denſitatem Medii. Ad CR erigantur perpendicula PC, RX, centroque R & Aſymptotis CR, RX deſcribatur Hyperbola quævis PVY. In CR capiatur CT longitudinis cujuſvis, & erigatur perpendiculum TV abſcindens aream Hyperbolicam $PCTV$, & ſit CZ latus hujus areæ applicatæ ad rectam PC. Dico quod motus quem globus, deſcribendo ſpatium CZ, ex reſiſtentia Medii amittet, erit ad ejus motum totum ſub initio ut longitudo CT ad longitudinem CR quamproxime. Nam

Note on title page: “p.253”.

This note refers to Scholium about the new infinitesimal methods.

Letter to Leibniz in 1676, and answer from Leibniz: has developed method “a mea vix abludentem praeterquam in verborum et notarum formulis”, and has stated that Newton had invented the method earlier. Leibniz is called “geometres peritissimus”, “vir clarissimus”.

Scholium in third edition: Newton refers to letter to J. Collins from December 10, 1672. There, he compares his methods to R. de Sluze’s (whose construction of tangents has not been published then) and J. Hudde’s (who developed such methods for a restricted class of curves).

PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA

Autore *JS. NEWTON*, *Trin. Coll. Cantab. Soc.* Matheseos Professore *Lucasiano*, & Societatis Regalis Sodali.

IMPRIMATUR·
S. PEPYS, *Reg. Soc.* PRÆSES.
Julii 5. 1686.

LONDINI,
Jussu *Societatis Regiæ* ac Typis *Josephi Streater*. Prostant Venales apud *Sam. Smith* ad insignia Principis *Walliæ* in Cœmiterio D. *Pauli*, aliosq; nonnullos Bibliopolas. *Anno* MDCLXXXVII.