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Abstract

Malmquist and Hicks–Moorsteen productivity indexes are the two most widely used indexes for measuring productivity growth. The former, which has been proposed initially, has a nice feature of decomposing productivity growth into two important sources: efficiency change and technical change components. The technical change component is considered measuring the distances between the isoquants along a single direction. When technical change is not Hicks-neutral and is biased towards certain factor inputs or outputs, the distance between the isoquants is dependent on the direction selected. In this case, if we adopt a single direction for measuring the distances, we can only locally capture technical change, which is a global phenomenon by nature. To rectify this problem, we propose a more global index of technical change that measures the distance between the isoquants by utilizing two directions. Along with the existing measure of efficiency change, this allows us to define a corresponding productivity index. This index turns out to be the geometric mean of the Malmquist and the Hicks-Moorsteen productivity indexes under constant returns to scale technology. While there has been a long discussion on which index is more preferable between the two productivity indexes, we give a justification to using the geometric mean of these two indexes.

Key Words: Productivity, Biased technical change, Malmquist productivity index, Hicks–Moorsteen productivity index

JEL classification: C14, D24, O47, O51

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1. Introduction

Technical change is one of the important sources of productivity growth. It increases the productivity of factor inputs or lowers the production cost of outputs, thus increasing the productivity and profits of firms. There exist different types of technical change: While Hicks-neutral technical change affects all factor inputs or all outputs equally, biased technical change affects these variably. The latter favours certain inputs by increasing their productivity while maintaining or lowering the productivity of other inputs, or it favours particular outputs by decreasing their production costs while maintaining or increasing the production costs of other outputs.

As Acemoglu (2002, 2009) summarized, various types of biased technical change have been documented over the last 100 years. A recent notable example is skilled-biased technical change, which benefits the use of skilled workers.¹ It is considered as one of the main reasons for the rise in the college wage premium, which accelerated starting the 1970s in the US.² This issue continues to attract considerable attention globally.³ However, while substantial effort has been devoted to searching for the appropriate productivity indexes, enough attention has not been paid to the question whether technical change is Hicks-neutral or biased.

The Malmquist and the Hicks–Moorsteen productivity indexes are among those most widely used for measuring productivity growth. While the former is introduced by Caves et al. (1982), the latter is first formulated by Bjurek (1996). As first demonstrated by Färe et al. (1994), one of the features of the Malmquist productivity index is that it can be decomposed into two importance sources of productivity growth: efficiency change and technical change.

All feasible input–output vectors are located below the production frontier. A firm’s efficiency improves when its input–output vector gets closer to the frontier. The efficiency change component in the Malmquist productivity index is constructed to measure the change in the distance from the input–output vector to the frontier across two periods. On the other hand, technical change generally means the shift in the production frontier; larger shift in the production frontier indicates greater technical change. Thus, the technical change component in the Malmquist productivity index is constructed to measure the distance between the production frontiers of two periods.

Distance between two production frontiers is measured relative to the reference input–output vector. The input–output vectors which are in fact chosen in the two periods being compared are obvious candidates for the reference vector. Selection of each vector leads to a distinct measure of the distance between frontiers. Thus, there are two equally reasonable measures of the distance between frontiers. Following Fisher (1922), the technical change component in the Malmquist productivity index is defined by the geometric mean of these two measures of the distance to avoid arbitrariness in choice of reference vector. However, once we examine the distance between two production frontiers in terms of the isoquants, the technical change

¹ See Acemoglu and Autor (2011).

² Terms such as ‘directed technological change’ or ‘directed technical change’ are also widely used.

³ See Adermon and Gustavsson (2015) and Buera et al. (2015) as examples of recent studies on skill-biased technical change.

component is found to measure the distance between each pair of isoquants along a single direction only.⁴

When technical change is Hicks-neutral, the isoquant expands or contracts proportionally. Thus, the distance between the isoquants is invariant regardless of the choice of the direction. However, biased technical change moves the isoquant towards certain inputs or outputs. Under this condition, the distance between the isoquants differs depending on the choice of the direction. Thus, the amount of technical change documented under the current measure based on a single direction might be significantly depreciated or appreciated if the distance is measured along a different direction. Technical change means a shift in the production frontiers. Since the frontier is defined over the set of all feasible input vectors, technical change should be viewed as a global phenomenon. The current technical change component in the Malmquist productivity index is problematic because it only utilizes the local information based on a single direction for measuring such a global phenomenon.

To avoid this problem, we propose an alternative index of technical change that measures the distance between the isoquants along two directions. Increasing the number of directions for measuring the distance between two isoquants helps us to capture the shift in the production frontier more globally. Along with an existing measure of efficiency change, we can also define a corresponding productivity index, which turns out to be the geometric mean of the Malmquist and the Hicks–Moorsteen productivity indexes under the constant returns to scale technology. The relationship between the proposed productivity index and the Malmquist productivity index is explored in detail in the end.

In the literature, there is no consensus on which productivity index is better. Balk (2001) justifies the Malmquist productivity index based on the constant returns to scale distance function because it incorporates all the underlying sources of productivity change. On the other hand, O’Donnell (2012) advocates the Hicks–Moorsteen productivity index over the Malmquist productivity index because one cannot interpret the latter as an index measuring total factor productivity growth. The result of the present paper can be interpreted as providing a justification to using the geometric mean of these two popular indexes under biased technical change, rather than choosing one of them.⁵

Färe et al. (1997) is an important exception that examines the Malmquist productivity index by explicitly taking into account the case of biased technical change. Recognizing the possibility that bias in technical change can be effectively utilized for improving productivity growth, the technical change component in the Malmquist productivity index is further decomposed into a magnitude term and a bias term.⁶ Since the purpose of the present paper is to accurately measure the amount of technical change, it might be rephrased as searching for an appropriate magnitude term. However, the magnitude term in Färe et al. (1997) measures the distance

⁴ Coelli et al. (2005) clearly illustrate Hicks neutral technical change and biased technical change by using the isoquants.

⁵ Peyrache (2013) characterizes a family of productivity index that is more general than these two productivity indexes. Note that the Malmquist and the Hicks–Moorsteen productivity indexes that he deals with are defined with respect to specific reference technology, rather than employing the geometric mean.

⁶ Chen and Yu (2014) documented significant bias terms accounting for productivity growth of many regions based on Färe et al. (1997).

between production frontiers by using a single reference input–output vector. It is worth noting that the information explored for measuring the shift in production frontier is even more limited than the technical change component in the Malmquist productivity index. Thus, one can regard the magnitude term as a more problematic measure of technical change.

Asmild and Tam (2007) propose the global Malmquist productivity index. The technical change component in the global Malmquist productivity index is considered as a global measure of technical change, since it is the geometric mean of the distances between production frontiers measured with respect to a variety of reference input–output vectors. However, only input–output combinations that have been chosen in each period are adopted as reference vectors. Thus, we cannot deny the possibility that the technical change component in the global Malmquist productivity index measures the distance between isoquants along a single direction. Therefore, the global Malmquist productivity index is still not immune from the problem encountered by the Malmquist productivity index under biased technical change.⁷

The paper is organised as follows. Section 2 illustrates the case where the technical change component in the Malmquist productivity index fails to appropriately capture the impact of technical change when it is not Hicks-neutral. Section 3 contains the main result of the present paper. It proposes an alternative measure of technical change, which more globally captures the shift in production frontiers. Along the current efficiency change component, we also present an alternative productivity index. Section 4 discusses the relationships between this index and other productivity indexes. Section 5 concludes the paper.

2. Malmquist Productivity Index and Biased Technical Change

Consider a firm transforming input vector $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}_+^N$ to output vector $\mathbf{y} = (y_1, \dots, y_M) \in \mathbb{R}_+^M$. The technology set T^t represents the production technology that a firm can access at period t . It is the set of all feasible combinations of input–output vectors and defined as follows:

$$T^t \equiv \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{N+M} : \mathbf{x} \text{ can produce } \mathbf{y} \text{ in period } t\}. \quad (1)$$

The production technology represented by T^t can be alternatively expressed by either output or input sets. The output set $P^t(\mathbf{x})$ is the set of all output vectors \mathbf{y} that is attainable from given \mathbf{x} at period t , which is defined by $P^t(\mathbf{x}) \equiv \{\mathbf{y} \in \mathbb{R}_+^M : (\mathbf{x}, \mathbf{y}) \in T^t\}$. On the other hand, the input set $L^t(\mathbf{y})$ is the set of all input vector \mathbf{x} that can produce \mathbf{y} at period t , which is defined by $L^t(\mathbf{y}) \equiv \{\mathbf{x} \in \mathbb{R}_+^N : (\mathbf{x}, \mathbf{y}) \in T^t\}$.

We assume that the technology satisfies the following regularity conditions: (T.1) no free lunch: $(\mathbf{0}_N, \mathbf{y}) \notin T^t$ for all $\mathbf{y} \in \mathbb{R}_+^M \setminus \mathbf{0}_M$; (T.2) no production is possible with given input: $(\mathbf{x}, \mathbf{0}_M) \notin T^t$ for all $\mathbf{x} \in \mathbb{R}_+^N$; (T.3) strong disposability of outputs: if $(\mathbf{x}, \mathbf{y}) \in T^t$ and $\mathbf{y}^* \leq \mathbf{y}$, then $(\mathbf{x}, \mathbf{y}^*) \in T^t$; (T.4) strong disposability of inputs: if

⁷ We need to employ the hypothetical input–output combinations as reference vectors in order to measure the distance between isoquants, which are production frontiers conditional on inputs or outputs of each period.

$(\mathbf{x}, \mathbf{y}) \in T^t$ and $\mathbf{x}^* \leq \mathbf{x}$, then $(\mathbf{x}^*, \mathbf{y}) \in P^t$; (T.5) T^t is closed and (T.6) $P^t(\mathbf{x})$ is bounded. These conventional axioms on the technology allow us to define the distance functions (Färe and Primont, 1995). The distance functions provide represent the underlying technology and are indispensable tools for measuring productivity growth in general condition.

The period t output distance function $D_o^t: \mathbb{R}_+^{N+M} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ measures the distance between an input–output vector (\mathbf{x}, \mathbf{y}) and the production frontier of period t by considering a minimal proportional contraction of the output vector \mathbf{y} , given an input vector \mathbf{x} . It is defined as follows:

$$D_o^t(\mathbf{x}, \mathbf{y}) \equiv \inf\{\theta > 0: (\mathbf{x}, \mathbf{y}/\theta) \in T^t\}. \quad (2)$$

The period t input distance function $D_i^t: \mathbb{R}_+^{N+M} \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ measures the distance from an input–output vector (\mathbf{x}, \mathbf{y}) to the production frontier of period t by considering a maximal proportional expansion of the input vector \mathbf{x} , given an output vector \mathbf{y} . It is defined as follows:

$$D_i^t(\mathbf{y}, \mathbf{x}) \equiv \sup\{\theta > 0: (\mathbf{x}/\theta, \mathbf{y}) \in T^t\}. \quad (3)$$

We consider the problem of measuring productivity growth that takes place from a period 0 to another period 1 by utilizing productivity indexes. Both the Malmquist and the Hicks–Moorsteen productivity indexes are defined by the distance functions.

The Malmquist productivity index is constructed by measuring the radial distances of the input–output vectors (\mathbf{x}, \mathbf{y}) to a reference technology.⁸ The technologies that are available at periods 0 and 1 are equally reasonable as the reference technology. Thus, the geometric mean of the two productivity indexes relative to the technology of each period defines the Malmquist productivity index. By adopting the input-oriented direction, the input-oriented Malmquist productivity index is defined as follows:

$$M \equiv \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^0)}{D_i^0(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2}. \quad (4)$$

Caves et al. (1982) first proposed the input-oriented and output-oriented Malmquist productivity indexes and simultaneously introduced the Malmquist quantity indexes of input and output. The Malmquist quantity indexes are constructed to capture changes in quantities of inputs \mathbf{x} and outputs \mathbf{y} . Their definitions also depend on the geometric mean of the two indexes, each of which is defined relative to the technology of the given period, such as the productivity index. The ratio of the

⁸ While the distances can be input- or output-oriented, we focus on the input-oriented Malmquist productivity index, which is defined by the input distance function here. All the reasoning we advance for this index is also applicable to the output-oriented Malmquist productivity index. The applicability of our result to the latter index is explicitly mentioned when we explain the propositions later.

Malmquist input quantity index to the Malmquist output quantity index defines the Hicks–Moorsteen productivity index as follows:⁹

$$HM \equiv \left(\frac{D_o^0(\mathbf{x}^0, \mathbf{y}^1) D_o^1(\mathbf{x}^1, \mathbf{y}^1)}{D_o^0(\mathbf{x}^0, \mathbf{y}^0) D_o^1(\mathbf{x}^1, \mathbf{y}^0)} \right)^{1/2} / \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^0)} \right)^{1/2}. \quad (5)$$

The present paper starts with the input-oriented Malmquist productivity index proposed by Caves et al. (1982) and suggests an alternative productivity index by rectifying its shortcomings. We later relate the productivity index introduced in this paper to the Hicks–Moorsteen productivity index. The Malmquist productivity index can be decomposed into efficiency change (EC) and technical change (TC) components as follows:

$$M = \underbrace{\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0)}{D_i^1(\mathbf{y}^1, \mathbf{x}^1)}}_{EC} \times \underbrace{\left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2}}_{TC} \quad (6)$$

Technical efficiency of a firm in period t is defined by the distance between the input–output vector $(\mathbf{x}^t, \mathbf{y}^t)$ and the production frontier of that period. The efficiency change component EC measures changes in technical efficiency over periods. On the other hand, technical change component TC measures the shift in the production frontiers over periods. It indicates the distance between the production frontiers of two periods. The measured distance depends on the reference input–output vector (\mathbf{x}, \mathbf{y}) . The input–output vectors chosen in two periods $(\mathbf{x}^0, \mathbf{y}^0)$ and $(\mathbf{x}^1, \mathbf{y}^1)$ are equally reasonable as the reference vector. Thus, the geometric mean of the two measures based on each vector defines the technical change component.

The present paper focuses on the technical change component TC while the efficiency change component EC is given. Figures 1 and 2 present the geometric intuition behind TC in case of two inputs $\mathbf{x} = (x_K, x_L)$ of capital x_K and labour x_L . Suppose points A and B indicate the input–output vectors of periods 0 and 1, respectively. The first component $D_i^1(\mathbf{y}^0, \mathbf{x}^0)/D_i^0(\mathbf{y}^0, \mathbf{x}^0)$ in TC coincides with OC/OC' . Thus, besides being the distance between the production frontiers of periods 0 and 1 while using the reference input–output vector $(\mathbf{y}^0, \mathbf{x}^0)$, it can also be regarded as the radial distance between the two input isoquants for \mathbf{y}^0 measured in the direction of the input vector \mathbf{x}^0 .¹⁰ The second component $D_i^1(\mathbf{y}^1, \mathbf{x}^1)/D_i^0(\mathbf{y}^1, \mathbf{x}^1)$ in TC coincides with OD/OD' . Thus, besides being the between the production frontiers of periods 0 and 1 while using the reference input–output vector $(\mathbf{y}^1, \mathbf{x}^1)$, it can also be regarded as the radial distance between the two input isoquants for \mathbf{y}^1 measured in the direction of the input vector \mathbf{x}^1 . Thus, computing the distance between the isoquants for \mathbf{y}^0 and \mathbf{y}^1 by using OC/OC' and OD/OD' , respectively, we can compute TC as follows:

⁹ Bjurek (1996) firstly formulated the Hicks–Moorsteen productivity index, originally named as ‘Malmquist total factor productivity index’. Diewert and Fox (2014) called it ‘Bjurek productivity index’.

¹⁰ In other words, the distance between two isoquants is measured along the direction \mathbf{y}^0 .

$$TC = \left(\frac{OC}{OC'} \frac{OD}{OD'} \right)^{1/2} \quad (7)$$

Thus, while technical change component TC is the geometric mean of the distance between two input isoquants for \mathbf{y}^0 and the distance between two input isoquants for \mathbf{y}^1 , only a single direction is adopted for measuring the distance between each pair of input isoquants: \mathbf{x}^0 for the distance between input isoquants for \mathbf{y}^0 , and \mathbf{x}^1 for the distance between input isoquants for \mathbf{y}^1 . Employing a single direction is harmless in the case of Hicks-neutral technical change as described in Figure 1, because the distance between two input isoquants is constant regardless of the choice of the direction. However, in the case of biased technical change, this distance depends on the direction. Thus, if we adopt only a single direction, we are likely to fail to capture the whole picture of technical change appropriately.

Figure 2 illustrates the case when technical change is biased. In this particular example, in Figures 1 and 2, the distance between C and C' and that between D and D' are set to be identical. Thus, the amount of technical change taking place in each case is evaluated as equal under TC . However, it is evident that the overall input isoquant shifts toward the origin greatly in Figure 2 compared to that in Figure 1. While the distance between two input isoquants for \mathbf{y}^1 measured along the direction \mathbf{x}^1 is OD/OD' , if we measure the distance along other ray steeper than \mathbf{x}^1 such as \mathbf{x}^0 , the measured distance OF/OF' becomes greater than OD/OD' . This suggests that using less capital and more labour, higher productivity growth would be attained.¹¹ Similarly, while the distance between two input isoquants for \mathbf{y}^0 measured along the direction \mathbf{x}^0 is OC/OC' , if we measure the distance along a ray flatter than \mathbf{x}^0 , such as \mathbf{x}^1 , the measured distance OE/OE' becomes greater than OC/OC' .

Therefore, it is reasonable to evaluate the amount of technical change illustrated in Figure 2 as higher than that in Figure 1. However, the current measure of technical change TC undervalues technical change in Figure 2 and evaluates the two cases equally. Thus, in the next section, we propose an alternative measure of technical change to avoid such a problem encountered in the case of biased technical change.

3. Two Directions for Measuring Technical Change

Technical change means a shift in the production frontier, which also induces the input isoquants for a given output to move. Since the frontier and the input isoquant are defined over all possible inputs, technical change should be regarded as a global phenomenon. The problem of the technical change component TC in the Malmquist productivity index comes from the fact that it intends to capture technical change only locally, utilizing a single direction. Therefore, we propose a more global measure of

¹¹ This is the situation when capital is not sufficiently substituted by labour under capital saving technical change. This might be attributed to the existence of inefficient subsidies and taxes or regulations. Färe et al. (2001) and Chen and Yu (2014) emphasized this point and empirically investigated whether technical change bias have been well exploited by substitution among factor inputs.

technical change, which adopts two directions for measuring the distances between each pair of input isoquants, as illustrated in Figure 2.

While TC uses a different direction for comparing each pair of input isoquants, our proposal is to adopt the same two directions \mathbf{y}^0 and \mathbf{y}^1 for comparing every pair of input isoquants. The distance between two input isoquants is captured by the geometric mean of one measure based on vector \mathbf{x}^0 and the other one based on vector \mathbf{x}^1 . Thus, in our example, the distance between the input isoquants for \mathbf{y}^0 is captured by the geometric mean of the distance between C and C' and that between E and E' , $((OC/OC') \times (OE/OE'))^{1/2}$. Similarly, the distance between the input isoquants for \mathbf{y}^1 is captured by the geometric mean of the distance between D and D' and that between F and F' , $((OD/OD') \times (OF/OF'))^{1/2}$. Finally, following the Malmquist productivity index, the geometric mean of the distance between input isoquants for \mathbf{y}^0 and that between input isoquants for \mathbf{y}^1 defines the *technical change component based on two directions*, TC_{MD} , as follows:

$$TC_{MD} = \left(\left(\frac{OC}{OC'} \frac{OE}{OE'} \right)^{1/2} \left(\frac{OD}{OD'} \frac{OF}{OF'} \right)^{1/2} \right)^{1/2} = \left(\frac{OC}{OC'} \frac{OE}{OE'} \frac{OD}{OD'} \frac{OF}{OF'} \right)^{1/4} \quad (8)$$

Now, we generalize it into the cases of more than two inputs by utilizing the input distance function. That is, the distance between the input isoquants for \mathbf{y}^0 measured along the direction of \mathbf{x}^0 is $D_i^1(\mathbf{y}^0, \mathbf{x}^0)/D_i^0(\mathbf{y}^0, \mathbf{x}^0)$ and that measured along the direction of \mathbf{x}^1 is $D_i^1(\mathbf{y}^0, \mathbf{x}^1)/D_i^0(\mathbf{y}^0, \mathbf{x}^1)$. Their geometric mean defines the distance between the input isoquants for \mathbf{y}^0 . Similarly, the distance between the input isoquants for \mathbf{y}^1 measured along the direction of \mathbf{x}^0 is $D_i^1(\mathbf{y}^1, \mathbf{x}^0)/D_i^0(\mathbf{y}^1, \mathbf{x}^0)$ and that measured along the direction of \mathbf{x}^1 is $D_i^1(\mathbf{y}^1, \mathbf{x}^1)/D_i^0(\mathbf{y}^1, \mathbf{x}^1)$. Their geometric mean defines the distance between the isoquants for \mathbf{y}^1 . Thus, we can formulate TC_{MD} as follows:

$$TC_{MD} = \left(\underbrace{\left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^1)} \right)^{1/2}}_{\text{distance between isoquants for } \mathbf{y}^0} \times \underbrace{\left(\frac{D_i^1(\mathbf{y}^1, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2}}_{\text{distance between isoquants for } \mathbf{y}^1} \right)^{1/2} \quad (9)$$

$$= \left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4}$$

By increasing the number of directions, TC_{MD} captures the shift in the production frontier or technical change more globally and is hence considered as a more

appropriate measure for the amount of technical change than the existing measure TC , especially in the presence of bias in technical change.¹²

Along with the current efficiency change component EC , TC_{MD} leads to an alternative productivity index, *productivity index based on two directions*, M_{MD} , as follows:¹³

$$\begin{aligned}
M_{MD} &= TE \times TC_{MD} \\
&= \frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0)}{D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \times \left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4} \\
&= \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^0)}{D_i^1(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4} \\
&\quad \times \left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4} \quad (10) \\
&= \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^0)}{D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right. \\
&\quad \left. \frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^0)}{D_i^1(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4}
\end{aligned}$$

Since TC_{MD} is considered as an appropriate technical change measure especially in the case of biased technical change, the corresponding productivity index M_{MD} can also be considered as an appropriate productivity index in the same situation. M_{MD} satisfies all the axioms that Malmquist productivity index does. However, it is worth noting that neither the current productivity index nor the Malmquist and the Hicks–Moorsteen productivity indexes satisfy transitivity.¹⁴

It is difficult to regard Eq. (10) itself as a measure of productivity growth. However, assuming constant returns to scale technology transforms this index to a more familiar form and makes it easier for us to recognize that this index measures productivity growth.

Definition 1: The technology exhibits constant returns to scale¹⁵ if

¹² Färe et al. (1997) formulate the magnitude term in TC , capturing the amount of biased technical change by $D_i^1(\mathbf{y}^0, \mathbf{x}^0)/D_i^0(\mathbf{y}^0, \mathbf{x}^0)$, which becomes OC/OC' in Figure 2. Clearly, it adopts a smaller number of directions for measuring the isoquants than TC_{MD} and even TC do. It is also problematic that the amount of technical change solely depends on inputs and outputs vector of the previous period $(\mathbf{y}^0, \mathbf{x}^0)$.

¹³ Alternatively, it is rewritten such as $M_{MD} = \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0)}{D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{3/4} \left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^0)}{D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4}$. We adopt the formulation of (10) to easily relate M_{MD} to other productivity indexes.

¹⁴ It is worth noting that M_{MD} is homogeneous of degree minus one in input \mathbf{x} . If the underlying technology exhibits constant returns to scale, it is also homogeneous of degree one in output \mathbf{y} . While the input distance function is adopted in the present paper, we can also define based on the output distance function. In that case, M_{MD} is homogeneous of degree one in output \mathbf{y} , and degree minus one in input \mathbf{x} under constant returns to scale technology.

¹⁵ It can be alternatively defined by an output set, such as $L^t(\lambda \mathbf{y}) = \lambda L^t(\mathbf{y})$ for all $\lambda > 0$.

$$D_i^t(\mathbf{x}, \mathbf{y}) = 1/D_o^t(\mathbf{x}, \mathbf{y}). \quad (11)$$

Proposition 1: Assume that the technology exhibits constant returns to scale. Then, the productivity index based on two directions becomes the geometric mean of the input-oriented Malmquist and the Hicks–Moorsteen productivity indexes; $M_{MD} = \sqrt{M \times HM}$.¹⁶

The above proposition shows that the choice between the Malmquist and the Hicks–Moorsteen productivity indexes is not necessary and that the use of their mean is preferable especially in the case of biased technical change.¹⁷

4. Relationship among Productivity Indexes

Now we aim to sharpen our understanding of the proposed productivity index M_{MD} by relating it to the input-oriented Malmquist productivity index M . There are two cases when our amendment to the technical change component in the previous section is not necessary and the proposed productivity index M_{MD} becomes equal to the input-oriented Malmquist productivity index M .

The first case is when technical change is Hicks-neutral. It is certainly understandable that the adjustment correcting the measurement problems encountered under biased technical change is not required under Hicks-neutral technical change. However, understanding the productivity indexes in the case of Hicks-neutral technical change helps us to clarify the reason that the proposed productivity index M_{MD} coincides with the Malmquist productivity index M in other situations. Among the variants of Hicks-neutral technical change, the present study is concerned with the following condition¹⁸:

Definition 2: Technical change is Hicks input-neutral¹⁹ if

$$D_i^t(\mathbf{y}, \mathbf{x}) = A^t \tilde{D}_i(\mathbf{y}, \mathbf{x}) \quad (12)$$

where A^t is a scaling parameter and $\tilde{D}_i(\mathbf{y}, \mathbf{x})$ is the input distance function which does not depend on time t .

Eq. (12) indicates that the period t input distance function $D_i^t(\mathbf{y}, \mathbf{x})$ is decomposed into a time-variable parameter A^t and the input distance function $\tilde{D}_i(\mathbf{y}, \mathbf{x})$ which is

¹⁶ Since the input-oriented and the output-oriented Malmquist productivity indexes are equal under constant returns to scale technology, Proposition 1 also implies that M_{MD} coincides with the output-oriented Malmquist productivity index under constant returns to scale technology.

¹⁷ Kerstens and Van De Woestyne (2014) empirically compared the Malmquist and the Hicks–Moorsteen productivity indexes using unbalanced and balanced panel data and documented that they are close under constant returns to scale technology.

¹⁸ See Hicks (1932), Blackorby et al. (1976) and Färe and Grosskopf (1996).

¹⁹ Hicks output-neutral technical change is defined by $D_o^t(\mathbf{x}, \mathbf{y}) = B^t \tilde{D}_o(\mathbf{x}, \mathbf{y})$ with the output distance function which does not depend on time $\tilde{D}_o(\mathbf{x}, \mathbf{y})$ and a scaling parameter B^t .

constant in time. This is regarded as a separability condition with respect to time, which induces the radial expansion of the input isoquant over time.

The reason why two productivity indexes coincide under Hicks input neutral technical change can be intuitively explained in simple two-input cases depicted by Figures 1 and 2. Under Hicks input-neutral technical change in Figure 1, irrespective of the direction adopted, \mathbf{x}^0 or \mathbf{x}^1 , the distance between two isoquants for \mathbf{y}^0 is constant so that $OC/OC' = OE/OE'$ and that between two isoquants for \mathbf{y}^1 is constant so that $OD/OD' = OF/OF'$. Therefore, the two measures of technical change TC and TC_{MD} expressed in Eqs. (7) and (8) turn out to be the same, leading to the equality between the two corresponding productivity indexes M and M_{MD} . We show the equivalence result between M and M_{MD} in a more general case of more than three inputs as follows:

Proposition 2: Assume that technical change is Hicks input-neutral. Then, the input-oriented Malmquist productivity index and the productivity index based on two directions coincide; $M = M_{MD}$.²⁰

Hicks-neutral technical change is a condition restricting the types of technical change over time. On the other hand, homotheticity is a characteristic of production technology. It is another condition under which two productivity indexes coincide regardless of the types of technical change. Among the variants of homotheticity condition,²¹ the present study is concerned with the following condition:

Definition 3: The technology is input homothetic²² if

$$D_i^t(\mathbf{y}, \mathbf{x}) = D_i^t(\mathbf{1}, \mathbf{x})/H^t(\mathbf{y}), \quad (13)$$

where H^t is a non-decreasing function and $\mathbf{1} = (1, \dots, 1)$ is a constant input vector.²³

Eq. (13) indicates that the period t input distance function $D_i^t(\mathbf{y}, \mathbf{x})$ is decomposed into $H^t(\mathbf{y})$, a function of the output vector and the input distance function $D_i^t(\mathbf{1}, \mathbf{x})$, which is independent of the output vector. This is regarded as a separability condition between input vector \mathbf{x} and output vector \mathbf{y} , which induces the radial expansion of the input isoquant when output vector \mathbf{y} changes.

The reason why two productivity indexes coincide under input homotheticity can also be intuitively explained in simple two input cases depicted by Figures 1 and 2.²⁴ Under input homothetic technology, irrespective of the direction adopted, \mathbf{x}^0 or \mathbf{x}^1 , the distance between two isoquants of period 0 is constant so that $OF/OC = OD/OE$

²⁰ Similarly, we can show that when technical change is Hicks output-neutral, the output-oriented Malmquist productivity index and the productivity index based on two directions coincide.

²¹ Primont and Primont (1994) discusses a variety of homotheticity conditions, including input homotheticity.

²² It can be alternatively defined by an input set such as $L^t(\mathbf{x}) = H^t(\mathbf{x})L^t(\mathbf{1})$. Input distance function $D_i^t(\mathbf{1}, \mathbf{x})$ can be $D_i^t(\bar{\mathbf{y}}, \mathbf{x})$ for arbitrary input vector $\bar{\mathbf{y}}$.

²³ Output homotheticity is defined by $D_o^t(\mathbf{x}, \mathbf{y}) = D_o^t(\mathbf{1}, \mathbf{y})/G^t(\mathbf{x})$, where G^t is a non-increasing function.

²⁴ Although Figure 1 might be considered to depict the input homothetic technology, examining the equality between two productivity indexes in Figure 2 gives better insight about the role of condition of input homotheticity.

and that between two isoquants of period 1 is constant so that $0F'/0C' = 0D'/0E'$, leading to $(0E/0E') \times (0F/0F') = (0C/0C') \times (0D/0D')$. Therefore, the two measures of technical change, TC and TC_{MD} , expressed in Eqs. (7) and (8) turn out to be the same, thus leading to the equality between the two corresponding productivity indexes, M and M_{MD} . We show the equivalence result between M and M_{MD} in a more general case of more than three inputs as follows:

Proposition 3: Assume that technology exhibits input homotheticity. Then, the input-oriented Malmquist productivity index and the productivity index based on two directions coincide; $M = M_{MD}$.²⁵

As emphasized above, while input homotheticity restricts the current production technology, it does not impose any restrictions on technical change. Thus, Proposition 3 indicates that the productivity index based on two directions, M_{MD} , is equal to the input-oriented Malmquist productivity index, M , even under the biased technical change. However, it is worthwhile noting that it does not mean that M_{MD} fails to surmount the problem associated with M . Instead, it means that the problem that is supposed to be encountered under biased technical change never shows up when technology exhibits input homotheticity.

The distances among four isoquants measured along different directions constitute technical change components TC and TC_{MD} of productivity indexes M and M_{MD} as shown in Figures 1 and 2. Propositions 2 and 3 state that a certain separability condition imposed on a family of production technology leads to equality between the two productivity indexes, M and M_{MD} . Since each separability condition induces some isoquants to be the radial expansion of other isoquants, the distances between these isoquants become constant regardless of the direction selected. Thus, some elements of the two technical change components TC and TC_{MD} that are originally considered to be different because different directions are adopted, become equal. This leads to the equality between the two corresponding productivity indexes M and M_{MD} . This is the logic behind the proofs of Propositions 2 and 3.

While Proposition 1 states that M_{MD} is equal to the geometric mean of M and HM under constant returns to scale, Hicks input neutral technical change or input homotheticity implies the equality between M and M_{MD} without assuming constant returns to scale. Thus, we can characterize the conditions leading to the equality between M and HM as corollaries of the above propositions as follows:²⁶

Corollary 1: Assume that the technology exhibits constant returns to scale and that technical change is Hicks input-neutral. Then, the input-oriented Malmquist and the Hicks–Moorsteen productivity indexes coincide; $M = HM$.

²⁵ Similarly, we can show that when technology exhibits constant returns to scale and input homotheticity, the output-oriented Malmquist productivity index and the productivity index based on two directions coincide.

²⁶ The two corollaries derive two sufficient conditions for the equality between the Malmquist and the Hicks–Moorsteen productivity indexes, which is recently found by Mizobuchi (2015). It is also pointed out that the constant returns to scale and inverse homothetic technology given by Färe et al. (1996) is a sufficient (rather than a necessary and sufficient) condition for two productivity indexes to coincide, as long as their definitions are based on the geometric mean.

Corollary 2: Assume that technology exhibits constant returns to scale and input homotheticity. Then, the input-oriented Malmquist and the Hicks–Moorsteen productivity indexes coincide; $M = HM$.

While the former is derived from Propositions 1 and 2, the latter is derived from Propositions 1 and 3. These corollaries assure that constant returns to scale is indispensable for equating M and HM . It plays a key role for transforming the output distance function to the input distance function or vice versa.²⁷

5. Conclusions

The present paper starts with examining how to appropriately measure technical change when it is not Hicks-neutral and is biased towards certain inputs. The current measure captures the amount of technical change by the distance between two isoquants measured along a single direction only. We propose a more global index of technical change adopting two directions for measuring the distance between the isoquants. This, along with the existing measure of efficiency change, leads to an alternative productivity index that is valuable especially in the case of biased technical change. When the technology exhibits constant returns to scale, the proposed productivity index coincides with the geometric mean of two popular productivity indexes: the Malmquist and the Hicks–Moorsteen productivity indexes.

While we consider the biased technical change that favours the use of certain factor inputs in this paper, there exists a technical change that favours the production of certain outputs. The most notable example is investment-specific technical change, which has been documented elsewhere (Hulten, 1992; Greenwood et al., 1997; and Cummins and Violante, 2002). We can alternatively formulate the technical change component based on two directions by using the output distance function. The resulting index is a variant of the output-oriented Malmquist productivity index, which is a suitable measure for technical change that is biased towards certain outputs. However, we are unable to recommend which index should be used when technical change is known to be biased towards certain inputs as well as certain outputs simultaneously. Fortunately, under constant returns to scale, a variant of the input-oriented and of the output-oriented Malmquist productivity indexes coincide, since the input-oriented and the output-oriented Malmquist productivity indexes coincide. Thus, this means the proposed productivity index is considered to be applicable to all types of biased technical change under this condition.²⁸

Moreover, Proposition 2 helps us to find the indication of bias in technical change by comparing the two productivity indexes. However, no clue is provided as to which factors the technical change is biased toward. Thus, despite the knowledge of the technical change being Hicks-neutral and the properly computed amount of such biased technical change, it remains unclear whether the biased technical change

²⁷ Note that since the input-oriented and the output-oriented Malmquist productivity indexes are equal under constant returns to scale technology, these corollaries also imply the equality between the output-oriented Malmquist and the Hicks–Moorsteen productivity indexes.

²⁸ However, we face the problem of choosing between the input-oriented and output-oriented measures, when technology does not exhibit constant returns to scale.

favours capital or labour or more specific input such as skilled worker. We leave the question of detecting the direction of biased technical change for future research.²⁹

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²⁹ Klump et al. (2007) empirically locate the direction of bias in technical change in the US by assuming CES production function in a simple one-output case. Our general approach that captures biased technical change by the isoquants might be applicable to their analysis.

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Appendix A

Proof of Proposition 1

$$\begin{aligned}
M_{MD} &= \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^0)}{D_i^0(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right) \\
&\quad \times \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^1) D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^0)}{D_i^0(\mathbf{y}^1, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1) D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{\frac{1}{4}} \\
&\hspace{15em} \text{from (10),} \\
&= \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^0)}{D_i^0(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right) \\
&\quad \times \left(\frac{D_o^0(\mathbf{x}^0, \mathbf{y}^1) D_o^1(\mathbf{x}^1, \mathbf{y}^1) D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^0)}{D_o^0(\mathbf{x}^0, \mathbf{y}^0) D_o^1(\mathbf{x}^1, \mathbf{y}^0) D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{\frac{1}{4}} \\
&\hspace{15em} \text{from (11),} \\
&= \left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^0, \mathbf{x}^0)}{D_i^0(\mathbf{y}^1, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2} \frac{\left(\frac{D_o^0(\mathbf{x}^0, \mathbf{y}^1) D_o^1(\mathbf{x}^1, \mathbf{y}^1)}{D_o^0(\mathbf{x}^0, \mathbf{y}^0) D_o^1(\mathbf{x}^1, \mathbf{y}^0)} \right)^{1/2}}{\left(\frac{D_i^0(\mathbf{y}^0, \mathbf{x}^1) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^0)} \right)^{1/2}} \\
&\hspace{15em} \text{from (4) and (5).} \\
&= \sqrt{M \times HM}
\end{aligned}$$

Proof of Proposition 2

$$\begin{aligned}
TC_{MD} &= \left(\frac{A(1)\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^0) A(1)\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^1) A(1)\tilde{D}_i(\mathbf{y}^1, \mathbf{x}^0) A(1)\tilde{D}_i(\mathbf{y}^1, \mathbf{x}^1)}{A(0)\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^0) A(0)\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^1) A(0)\tilde{D}_i(\mathbf{y}^1, \mathbf{x}^0) A(0)\tilde{D}_i(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/4} \\
&\hspace{15em} \text{from (10) and (12),} \\
&= \frac{A(1)}{A(0)} \left(\frac{\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^0) \tilde{D}_i(\mathbf{y}^1, \mathbf{x}^1)}{\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^0) \tilde{D}_i(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2} \\
&= \left(\frac{A(1)\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^0) A(1)\tilde{D}_i(\mathbf{y}^1, \mathbf{x}^1)}{A(0)\tilde{D}_i(\mathbf{y}^0, \mathbf{x}^0) A(0)\tilde{D}_i(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2} \\
&\hspace{15em} \text{from (12),} \\
&= \left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2} = TC
\end{aligned}$$

from (10).

Since $M = EC \times TC$ and $M_{MD} = EC \times TC_{MD}$, it implies that $M = M_{MD}$.

Proof of Proposition 3

$$TC_{MD} = \left(\frac{D_i^1(\mathbf{1}, \mathbf{x}^0)/H^1(\mathbf{y}^0) D_i^1(\mathbf{1}, \mathbf{x}^1)/H^1(\mathbf{y}^0) D_i^1(\mathbf{1}, \mathbf{x}^0)/H^1(\mathbf{y}^1) D_i^1(\mathbf{1}, \mathbf{x}^1)/H^1(\mathbf{y}^1)}{D_i^0(\mathbf{1}, \mathbf{x}^0)/H^0(\mathbf{y}^0) D_i^0(\mathbf{1}, \mathbf{x}^1)/H^0(\mathbf{y}^0) D_i^0(\mathbf{1}, \mathbf{x}^0)/H^0(\mathbf{y}^1) D_i^0(\mathbf{1}, \mathbf{x}^1)/H^0(\mathbf{y}^1)} \right)^{1/4}$$

from (10) and (13),

$$= \left(\frac{\left(D_i^1(\mathbf{1}, \mathbf{x}^0) D_i^1(\mathbf{1}, \mathbf{x}^1) \right) / \left(H^1(\mathbf{y}^0) H^1(\mathbf{y}^1) \right)}{\left(D_i^0(\mathbf{1}, \mathbf{x}^0) D_i^0(\mathbf{1}, \mathbf{x}^1) \right) / \left(H^0(\mathbf{y}^0) H^0(\mathbf{y}^1) \right)} \right)^{1/2}$$

$$= \left(\frac{D_i^1(\mathbf{1}, \mathbf{x}^0)/H^1(\mathbf{y}^0) D_i^1(\mathbf{1}, \mathbf{x}^1)/H^1(\mathbf{y}^1)}{D_i^0(\mathbf{1}, \mathbf{x}^0)/H^0(\mathbf{y}^0) D_i^0(\mathbf{1}, \mathbf{x}^1)/H^0(\mathbf{y}^1)} \right)^{1/2}$$

from (13),

$$= \left(\frac{D_i^1(\mathbf{y}^0, \mathbf{x}^0) D_i^1(\mathbf{y}^1, \mathbf{x}^1)}{D_i^0(\mathbf{y}^0, \mathbf{x}^0) D_i^0(\mathbf{y}^1, \mathbf{x}^1)} \right)^{1/2} = TC$$

from (10).

Since $M = EC \times TC$ and $M_{MD} = EC \times TC_{MD}$, it implies that $M = M_{MD}$.

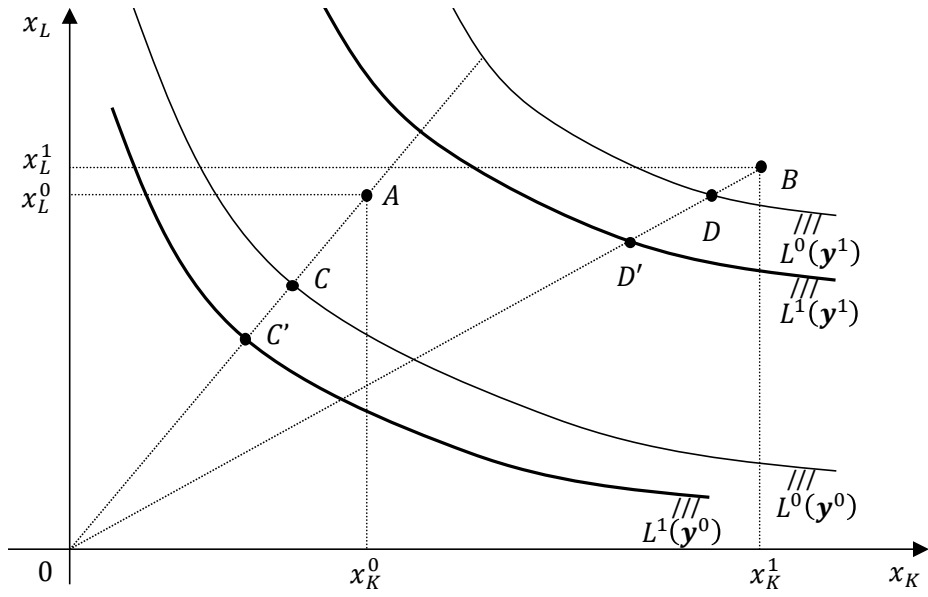


Figure 1: Case of Hicks Neutral Technical Change

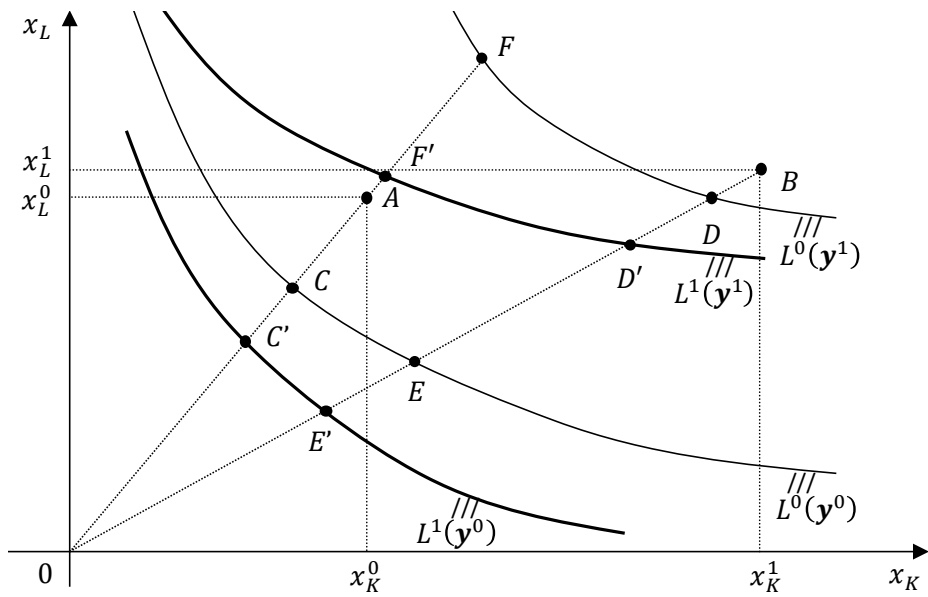


Figure 2: Case of Biased Technical Change