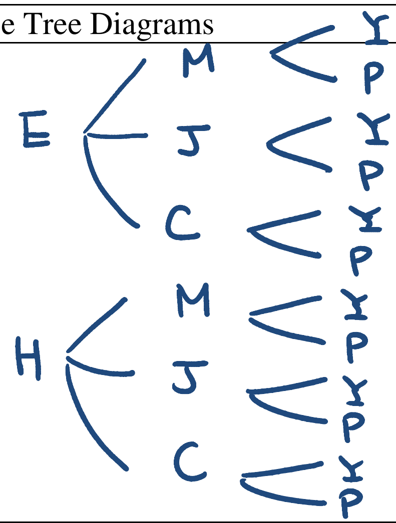
First Name: ________________________ Last Name: ________________________ Block: ______

Ch. 11 – Permutations, Combinations, and the Binomial Theorem

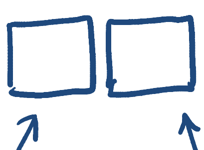
11.1 – Permutations

The Fundamental Counting Principle (FCP): If one item can be selected in m ways, and for each way a second item can be selected in n ways, then the two items can be selected in $m \cdot n$ ways.

Example 1:
A café has a lunch special consisting of an egg, or a ham sandwich (E or H); milk, juice, or coffee (M, J, or C); and yogurt or pie for dessert (Y or P). One item is chosen from each category. How many possible meals are there? How can you determine the number of possible meals without listing all of them?

Use Tree Diagrams
 12 possible meals.
Use Fundamental Counting Principle
$2 \times 3 \times 2 = 12$ possible meals

Example 2:
How many even 2-digit whole numbers are there?



1,2,3,4,5,
6,7,8,9

(9 ways to pick 1st digit)

0,2,4,6,8

(5 ways to pick 2nd digit)

using F.C.P:

$9 \times 5 = 45$ ways to select 2 digit numbers that are even.

Example 3:
In how many ways can a teacher seat three girls and two boys in a row of five seats if a boy must be seated at each end of the row?

2 ways	3	2	1	1 way

$= 2 \times 3 \times 2 \times 1 \times 1 = 12$ ways

B1	G1	G2	G3	B2
B1	G1	G3	G2	B2
B1	G2	G1	G3	B2
B1	G2	G3	G1	B2
B1	G3	G1	G2	B2
B1	G3	G2	G1	B2

Factorial Notation: For any positive integer n, the product of all of the positive integers up to and including n can be described using a factorial notation, $n!$

Ex: $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

In general: $n! = (n)(n-1)(n-2)\cdots(3)(2)(1)$

Note: $0! = 1$

To calculate $10!$ using a graphing calculator: 10 math $\rightarrow$ $\rightarrow$ $\rightarrow$ 4

Example 4:
How many three-digit numbers can you make using the digits 1, 2, 3, 4, and 5,

a) if repetition is allowed?

5 ways	5	5

$5 \times 5 \times 5 = 125$ ways

b) if repetition of digits is not allowed?

Using the Fundamental Counting Principle,

5 ways	4	3

$5 \times 4 \times 3 = 60$ ways

Using the Factorial Notation,

$$\frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 60 \text{ ways}$$

Permutation Involving Different (Distinct) Objects:
An ordered arrangement or sequence of all or part of a set.

The notation ${}_nP_r$ is used to represent the number of permutations, or arrangements in a definite order, of r items taken from a set of n distinct items. A formula for ${}_nP_r$ is ${}_nP_r = \frac{n!}{(n-r)!}$, $n \in N$.

Example 1:
How many permutations can be formed using all the letters of the word MUSIC? (how many ways to permute - alter/change the word MUSIC?)

5 ×	4 ×	3 ×	2 ×	1

= 5! 120

OR $_5P_5 = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 5! = 120$

Example 2:
How many 3-letter permutations can be formed from the letters of the word CLARINET?

8	7	6

$_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8\cdot7\cdot6 = 336$

On a graphing calculator: 8 MATH → → → 2 3

Permutation Involving Identical (Repeating) Objects:

Consider the permutations of the 4 letter in the word FUEL.

FUEL	FULE	FEUL	FELU	FLUE	FLEU
UFEL	UFLE	UEFL	UELF	ULFE	ULEF
EFUL	EFLU	EUFL	EULF	ELFU	ELUF
LFUE	LFEU	LUFE	LUEF	LEFU	LEUF

There are 24 permutations.

If we change the E in FUEL to L, we get the word FULL. If we change each E to L in the list of permutations above, we obtain:

FULL	FULL	FLUL	FLLU	FLUL	FLLU
UFLL	UFLL	ULFL	ULLF	ULFL	ULLF
LFUL	LFLU	LUFL	LULF	LLFU	LLUF
LFUL	LFLU	LUFL	LULF	LLFU	LLUF

The number of different permutations has now been reduced! There are now 12 different ways to arrange the letters.

$\frac{4!}{2!} = 4\times3 = 12$

The formula to deal with such permutation where there is a set of n objects with a of one kind that are identical, b of a second kind that are identical, and c of a third kind that are identical, and so on, can be arranged in $\frac{n!}{a!b!c!\ldots}$ different ways.

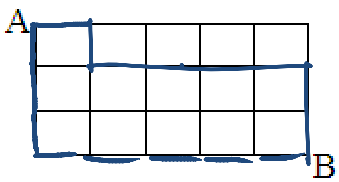
Example 1:

How many different 5-digit numbers can you make by arranging all of the digits of 1 7 1 7 1?

$$\frac{5!}{3!\,2!} = \frac{5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot2\cdot1\cdot\cancel{2}\cdot\cancel{1}} = 10$$ different 5-digit #s

Example 2:

In how many different ways can you walk from A to B in a three by five rectangular grid if you must move only down or to the right?



D D D R R R R R

R D R R R R D D ... so on

$$\frac{8!}{3!\,5!} = \frac{8\cdot7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} = 56$$

HW: p.524 #1 – 8, 10 – 11, 15

11.2 – Combinations

Scenario 1:
From a group of four students, three are to be elected to an executive committee **with a specific position.** The positions are as follows:

1st position: President
2nd position: Vice President
3rd position: Treasurer

1st 2nd 3rd
$\boxed{4 \mid 3 \mid 2} \rightarrow 24$

In how many ways can the positions be filled from this group?

1st position	2nd position	3rd position
1 S1	S2	S3
S1	S2	S4
2 S1	S3	S2
S1	S3	S4
S1	S4	S2
S1	S4	S3
3 S2	S1	S3
S2	S1	S4
4 S2	S3	S1
S2	S3	S4
S2	S4	S1
S2	S4	S3
5 S3	S1	S2
S3	S1	S4
6 S3	S2	S1
S3	S2	S4
S3	S4	S1
S3	S4	S2
S4	S1	S2
S4	S1	S3
S4	S2	S1
S4	S2	S3
S4	S3	S1
S4	S3	S2

$${}_4P_3 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = 4 \cdot 3 \cdot 2 = 24$$
ways the positions can be filled.

* Order does matter! This is an example of permutation.

Scenario 2:
Suppose that the three students are to be selected to serve on a committee (with no specific position). How many committees from the group of four students are possible?

* order does not matter.

S1 S2 S3 ← committee 1
S1 S2 S4 ← " 2
S1 S3 S4 ← " 3
S2 S3 S4 ← " 4

$$\frac{{}_4P_3}{\text{\# of ways selected students can be arranged}} = \frac{24}{3!} = \frac{24}{6} = 4$$

∴ 4 possible ways to form a committee

$_4C_3 = \frac{4!}{3!(4-3)!} = 4$

Combination: A selection of objects where order does not matter.

The number of combination of n different objects taken r at a time is denoted with a notation, $_nC_r$ and can be calculated using the following formula:

$$\boxed{{}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!}}$$

→ # of ways you can rearrange the selected items.

Example 1: Suppose that the four students are to be selected to serve on a committee (with no specific position).

a) How many committees from the group of 3 boys and 3 girls are possible?

$_6C_4 = \frac{6!}{4!(6-4)!} = \frac{6!}{4!(2!)} = \frac{6\cdot 5}{2} = \frac{30}{2} = 15$ ways to form a committee.

b) How many committees from the group of 3 boys and 3 girls are possible if 2 girls and 2 boys have to be selected.

① B_1 B_2 G_1 G_2
" G_1 G_3
" G_2 G_3

② B_1 B_3 G_1 G_2
" G_1 G_3
" G_2 G_3

③ B_2 B_3 G_1 G_2
" G_1 G_3
" G_2 G_3

Boys can be chosen in $_3C_2$ ways ($_3C_2 = \frac{3!}{2!(3-2)!} = \frac{3!}{2!} = 3$)

For each of these ways, girls can be chosen in $_3C_2$ ways.

$\therefore {}_3C_2 \times {}_3C_2 = 3 \times 3 = 9$ ways!

Example 2:
A standard deck of 52 playing cards consists of 4 suits (spades, hearts, diamonds, and clubs) of 13 cards each.

a) How many different 5-card hands can be formed?

$_{52}C_5 = \frac{52!}{5!(52-5)!} = \frac{52!}{5!(47!)} = 2,598,960$

b) How many different 5-card red hands can be formed? There are 26 red cards.

$_{26}C_5 = \frac{26!}{5!(26-5)!} = \frac{26!}{5!\,21!} = 65\,780$

c) How many different 5-card hands can be formed containing at least 3 black cards?
Case 1: 3 black cards and 2 red cards
(Note: The black cards can be chosen in ${}_{26}C_3$ ways, and for each of these ways, the red cards can be chosen in ${}_{26}C_2$ ways)

$${}_{26}C_3 \times {}_{26}C_2 = \frac{26!}{3!\,23!} \times \frac{26!}{2!\,24!} = 2600 \times 325 = 845000$$

Case 2: 4 black cards and 1 red card

$${}_{26}C_4 \times {}_{26}C_1 = \frac{26!}{4! \cdot 22!} \times \frac{26!}{1!\,25!} = 388\,700$$

Case 3: 5 black cards and 0 red card

$${}_{26}C_5 = 65\,780$$

Therefore, total number of combination is:

$$845\,000 + 388\,700 + 65\,780 = 1\,299\,480$$

Example 3: Express as factorials and simplify $\dfrac{{}_nC_5}{{}_{n-1}C_3}$.

$$\frac{n!}{5!(n-5)!} \div \frac{(n-1)!}{3!(n-1-3)!} = \frac{n!}{5!(n-5)!} \times \frac{3!(n-4)!}{(n-1)!}$$

$$= \frac{n \cdot \cancel{(n-1)!}}{5 \cdot 4 \cdot \cancel{3!}\,\cancel{(n-5)!}} \times \frac{\cancel{3!}(n-4)\cancel{(n-5)!}}{\cancel{(n-1)!}} = \frac{n(n-4)}{20}$$

HW: p.534 # 1 – 6, 10, 11, 13, 17, 18, 19

11.3 – Binomial Theorem

Pascal's Triangle: The triangle, Pascal's triangle, is named after the great French mathematician Blaise Pascal (1623 – 1662) because of his work with the properties of the triangle.

1st row												1											
2nd row											1		1										
3rd row										1		2		1									
4th row									1		3		3		1								
5th row								1		4		6		4		1							
6th row							1		5		10		10		5		1						
7th row						1		6		15		20		15		6		1					
8th row					1		7		21		35		35		21		7		1				
9th row				1		8		28		56		70		56		28		8		1			
10th row			1		9		36		84		126		126		84		36		9		1		
11th row		1		10		45		120		210		252		210		120		45		10		1	
12th row	1		11		55		165		330		462		462		330		165		55		11		1

1 12 66

Investigate:

1) Examine Pascal's triangle. Write the next few rows in the space provided.

2) Determine the sum of the numbers in each horizontal row. What pattern did you find?

3) Each number in Pascal's triangle can be written as a combination using the notation ${}_nC_r$, where n is the number of objects in the set and r is the number selected. Express the 4th row using combination notation. Check whether your combinations have the same values as the numbers in the 4th row of Pascal's triangle.

1st row				${}_0C_0$			
2nd row			${}_1C_0$		${}_1C_1$		
3rd row		${}_2C_0$		${}_2C_1$		${}_2C_2$	
4th row	${}_3C_0$		${}_3C_1$		${}_3C_2$		${}_3C_3$

Does ${}_3C_1 = {}_2C_0 + {}_2C_1$?   Does ${}_3C_2 = {}_2C_1 + {}_2C_2$?

Other than the first and last number in each row, can you say: ${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$?

What would be the third value in the 13th row of Pascal's triangle?

$${}_{12}C_2 = \frac{12!}{2!(10!)} = \frac{12 \cdot 11}{2} = 66$$

4) Expand the following binomials by multiplying. How does the coefficients relate to the numbers in Pascal's triangle?

$(x+y)^0$	1
$(x+y)^1$	$x + y$
$(x+y)^2$	$1x^2 + 2xy + 1y^2$
$(x+y)^3$	$1x^3 + 3x^2y + 3xy^2 + 1y^3$
$(x+y)^4$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$
6th row $\Rightarrow$ $(x+y)^5$	$1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$

Note the expansion of $(x+y)^5$ *as an example: The powers of* x *decrease from 5 to 0 in successive terms of the expansion. The powers of* y *increase from 0 to 5.*

Binomial Theorem:

You can use the binomial theorem to expand any power of a binomial expression.

$$(x+y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \quad \ldots \quad + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

where $n \in N$.

Examples:

1) Use the binomial theorem to expand $(a+b)^8$. $\leftarrow$ row 9th

$a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$

2) Use the binomial theorem to expand $(2p-3q)^4$. $\leftarrow$ row 5

$(2p)^4 + 4(2p)^3(-3q) + 6(2p)^2(-3q)^2 + 4(2p)(-3q)^3 + (-3q)^4$

$16p^4 - 96p^3q + 216p^2q^2 - 216pq^3 + 81q^4$

HW: p. 542 #1 – 7 (odd letters), 10, 11