Chain transitivity, attractivity and strong repellors for semidynamical systems

Morris W. Hirsch*
Department of Mathematics
University of California
Berkeley, CA, 94720

Hal L. Smith† and Xiao-Qiang Zhao‡
Department of Mathematics
Arizona State University
Tempe, AZ 85287–1804

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Abstract Some properties of internally chain transitive sets for continuous maps in metric spaces are presented. Applications are made to attractivity, convergence, strong repellors, uniform persistence and permanence. A result of Schreiber on robust permanence is improved.

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1 Introduction

The notion of chain recurrence, introduced by Conley [11], is a way of getting at the recurrence properties of a dynamical system. It has remarkable connections to the structure of attractors. These ideas have recently been extended to noncompact spaces by Hurley [21].

Chain recurrence is proving increasingly useful in a variety of fields; accordingly, it is of interest to identify chain recurrent sets and to analyze their structure. Chain recurrence plays a central role in the theory of exponential dichotomies for linear evolutionary systems developed by Sacker and Sell. See [26] and the references to their earlier work therein. Mischaikow, Smith and Thieme [23] show that omega limit sets of asymptotically autonomous semiflows are internally chain recurrent for the limiting vector field,

*Supported by NSF Grant DMS 9802182
†Supported by NSF Grant DMS 9700910
‡Present address: Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John's, NF A1C 5S7, Canada

and Benaïm and Hirsch [5] obtained similar results for asymptotic pseudotrajectories for continuous semiflows. Benaïm [3] has shown that limit sets of certain stochastic approximation processes are characterized as internally chain recurrent continua for the associated mean vector field; this has been applied to economics [18], epidemiology [6], game theory [8] and numerical analysis [19]. In [16] it is shown that compact internally chain transitive sets for strongly monotone dynamical systems either are unordered or else are contained in totally ordered, compact arcs of equilibria; and the latter alternative cannot occur if the dynamical system is real analytic and dissipative. This result is applied to stochastic approximation and game theory in [7]. Chain recurrence in surface flows is analyzed in [4].

Outside the subject of topological dynamics, chain recurrence has been used to characterize the property of uniform persistence (or permanence) for dynamical systems, an idea that arose out of population biology; see Garay [13], Hofbauer and So [20], Schreiber [27], and Smith and Zhao [31]. Looked at abstractly, uniform persistence is the notion that a closed subset of the state space (e.g., the set of extinction for one or more populations) is repelling for the dynamics on the complementary set. One of the principal tools in the theory of uniform persistence is the Butler-McGehee lemma [10, 12, 15, 29, 32, 33, 35]. This says that an omega limit set which intersects an isolated invariant set M, but is not contained in M, must contain positive and negative orbits outside M whose respective omega and alpha limit sets lie in M. In the present paper we show that the Butler-McGehee property of omega limit sets is shared by chain transitive sets for a dynamical system. This class of sets, we show, contains the omega limit sets of perturbations of true (precompact) orbits of dynamical systems, including asymptotic pseudo-orbits and orbits of asymptotically autonomous dynamical processes. By extending the Butler-McGehee result we are able to extend earlier results on attractivity, convergence and uniform persistence to perturbed dynamical systems. In particular, we show that uniform persistence is stable under a broad class of perturbations.

Schreiber [27] has recently proved a robust permanence result for Kolmogorov-type vector field on $\mathbb{R}^n_+$ generating a dissipative flow. More precisely, he gives sufficient conditions for uniform persistence to hold uniformly for the given system as well as for all small C^r perturbations of it ($r \geq 1$). As an application of our general result on the stability of uniform persistence, we improve this result by establishing that the uniform persistence is stable to perturbation by a C^0-small Lipschitz vector field.

In the first section of this paper basic definitions are given. In addition, we show that limit sets of precompact orbits of (not necessarily invertible) maps are internally chain transitive, and that omega limit sets of certain perturbed orbits also have this property. The Butler-McGehee lemma and some other properties of internal chain transitive sets are the focus of section 3. Our main results on strong repellors and uniform persistence are contained in section 4.

2 Chain transitive sets

Let Z be the set of integers and Z_+ the set of nonnegative integers. Let X be a metric space with metric d and $f : X \to X$ be a continuous map. A subset $A \subset X$ is said to be an **attractor** for f if A is nonempty, compact and invariant ($f(A) = A$), and there exists some open neighborhood U of A in X such that $\lim_{n\to\infty} \sup_{x\in U}\{d(f^n(x), A)\} = 0$. If $A \neq X$, then A is a *proper attractor*. A **global attractor** for $f : X \to X$ is an attractor which attracts every point in X. For a nonempty invariant set M, the set $W^s(M) := \{x \in X : \lim_{n\to\infty} d(f^n(x), M) = 0\}$ is called the **stable set** of M. The **omega limit set** of x is defined in the usual way as $\omega(x) = \{y \in X : f^{n_k}(x) \to y, \text{ for some } n_k \to \infty\}$. A **negative orbit** through $x = x_0$ is a sequence $\gamma^-(x) = \{x_k\}_{k=-\infty}^{0}$ such that $f(x_{k-1}) = x_k$ for integers $k \leq 0$. There may be no negative orbit through x and even if there is one, it may not be unique. Of course, a point of an invariant set always has at least one negative orbit contained in the invariant set. For a given negative orbit $\gamma^-(x)$ we define its **alpha limit set** as $\alpha(\gamma^-) = \{y \in X : x_{n_k} \to y \text{ for some } n_k \to -\infty\}$.

Definition 2.1 A point $x \in X$ is said to be **chain recurrent** if for any $\epsilon > 0$, there is a finite sequence of points $x_1, \cdots, x_m$ in X ($m > 1$) with $x_1 = x = x_m$ such that $d(f(x_i), x_{i+1}) < \epsilon$ for all $1 \leq i \leq m-1$. The set of all chain recurrent points for $f : X \to X$ is denoted by $R(X, f)$. Let $A \subset X$ be a nonempty invariant set. We call A **internally chain recurrent** if $R(A, f) = A$, and **internally chain transitive** if the following stronger condition holds: For any $a, b \in A$ and any $\epsilon > 0$, there is a finite sequence $x_1, \cdots, x_m$ in A with $x_1 = a, x_m = b$ such that $d(f(x_i), x_{i+1}) < \epsilon, 1 \leq i \leq m-1$. The sequence $\{x_1, \cdots, x_m\}$ is called an ϵ-chain in A connecting a and b.

Following LaSalle [22], we call a compact invariant set A **invariantly connected** if it cannot be decomposed into two disjoint closed nonempty invariant sets. An internally chain recurrent set need not have this property— e.g., a pair of fixed points. However, it is easy to see that every internally chain transitive set is invariantly connected.

We give some examples of internally chain transitive sets.

Lemma 2.1 Let $f : X \to X$ be a continuous map. Then the omega (alpha) limit set of any precompact positive (negative) orbit is internally chain transitive

Proof. Let $x \in X$ and set $x_n = f^n(x)$. Assume x has a precompact orbit $\gamma = \{x_n\}$, and denote its omega limit set by ω. Then ω is nonempty, compact, invariant and $\lim_{n\to\infty} d(x_n, \omega) = 0$. Let $\epsilon > 0$ be given. By the continuity of f and compactness of ω, there exists $\delta \in (0, \frac{\epsilon}{3})$ with the following property: If u, v are points in the open δ-neighborhood U of ω with $d(u, v) < \delta$, then $d(f(u), f(v)) < \frac{\epsilon}{3}$. Since x_n approaches ω as $n \to \infty$, there exists $N > 0$ such that $x_n \in U$ for all $n \geq N$.

Let $a, b \in \omega$ be arbitrary. There exist $k > m \geq N$ such that $d(x_m, f(a)) < \frac{\epsilon}{3}$ and

$d(x_k, b) < \frac{\epsilon}{3}$. The sequence

$$\{y_0 = a,\ y_1 = x_m, \cdots, y_{k-m} = x_{k-1},\ y_{k-m+1} = b\}$$

is an $\frac{\epsilon}{3}$-chain in X connecting a and b. Since for each $y_i \in U$ for $i = 1, \cdots, k-m$, we can choose $z_i \in \omega$ such that $d(z_i, y_i) < \delta$. Let $z_0 = a$ and $z_{k-m+1} = b$. Then for $i = 0, 1, \cdots, k-m$ we have:

$$\begin{aligned} d(f(z_i), z_{i+1}) &\leq d(f(z_i), f(y_i)) + d(f(y_i), y_{i+1}) + d(y_{i+1}, z_{i+1}) \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} \end{aligned}$$

Thus the sequence $z_0, z_1, \cdots, z_{k-m}, z_{k-m+1}$ is an ϵ-chain in ω connecting a and b. Therefore ω is internally chain transitive. By a similar argument, we can prove the internal chain transitivity of alpha limit sets of precompact negative orbits. ∎

Remark 2.1 Bowen [9] proved that omega limit sets of precompact orbits of continuous invertible maps are internally chain transitive. Robinson [24] proved that omega limit sets of precompact orbits of continuous maps are internally chain recurrent.

Let $\{S_n : X \to X\}_{n\geq 0}$, be a sequence of continuous maps. The **discrete dynamical process** (or *process* for short) generated by $\{S_n\}$ is the sequence $\{T_n : X \to X\}_{n\geq 0}$ defined by $T_0 = I =$ the identity map of X and

$$T_n = S_{n-1} \circ S_{n-2} \circ \cdots S_1 \circ S_0, \quad n \geq 1$$

The *orbit* of $x \in X$ under this process is the set $\gamma^+(x) = \{T_n(x) : n \geq 0\}$, and its omega limit set is

$$\omega(x) = \{y \in X : (\exists n_k \to \infty) \quad \lim_{k\to\infty} T_{n_k}(x) = y\}$$

If there is a continuous map S on X such that $S_n = S$, so that T_n is the nth iterate S^n, then $\{T_n\}$ is a special kind of process called the *discrete semiflow* generated by S. By an abuse of language we may refer to the map S as a discrete semiflow.

Definition 2.2 The process $\{T_n : X \to X\}$ is **asymptotically autonomous**, if there exists a continuous map $S : X \to X$ such that

$$n_j \to \infty,\ x_j \to x \Rightarrow \lim_{j\to\infty} S_{n_j}(x_j) = S(x)$$

We also say that $\{T_n\}$ is asymptotic to S.

It is easy to see from the triangle inequality that if $\lim_{n\to\infty} S_n = S$ uniformly on compact sets, then the process generated by $\{S_n\}$ is asymptotic to S.

By [35, Theorem 2.1] and [36, Theorem 1.2], the omega limit set of a precompact orbit of an asymptotically autonomous process $T_n : X \to X, n \geq 0$, with limit $S : X \to X$, is nonempty, compact, invariant and internally chain recurrent for S. Lemma 2.1 and the same embedding approach as in [35, 36], give the following additional result.

Lemma 2.2 Let $T_n : X \to X, n \geq 0$, be an asymptotically autonomous discrete process with limit $S : X \to X$. Then the omega limit set of any precompact orbit of $\{T_n\}$ is internally chain transitive for S.

Definition 2.3 Let $S : X \to X$ be a continuous map. A sequence $\{x_n\}$ in X is an **asymptotic pseudo-orbit** of S if

$$\lim_{n\to\infty} d(S(x_n), x_{n+1}) = 0.$$

The omega limit set of $\{x_n\}$ is the set of limits of subsequences.

Let $\{T_n\}$ be a discrete process in X generated by a sequence of continuous maps S_n that converges to a continuous map $S : X \to X$ uniformly on compact subsets of X. It is easy to see that every precompact orbit of $T_n : X \to X, n \geq 0$, is an asymptotic pseudo-orbit of S.

Remark 2.2 Consider the non-autonomous difference equation $x_{n+1} = f(n, x_n),\ n \geq 0$ on the metric space X. If we define $S_n = f(n, \cdot) : X \to X,\ n \geq 0$ and let $T_0 = I,\ T_n = S_{n-1} \circ \cdots \circ S_1 \circ S_0 : X \to X,\ n \geq 1$, then $x_n = T_n(x_0)$ and $\{x_n :\ n \geq 0\}$ is an orbit of the discrete process T_n. If $f(n, \cdot) \to \bar{f} : X \to X$ uniformly on compact subsets of X then T_n is asymptotically autonomous with limit $\bar{f}$. Furthermore, in this case any precompact orbit of the difference equation is an asymptotic pseudo-orbit of $\bar{f}$ since $d(\bar{f}(x_n), x_{n+1}) = d(\bar{f}(x_n), f(n, x_n)) \to 0$.

Lemma 2.3 The omega limit set of any precompact asymptotic pseudo-orbit of a continuous map $S : X \to X$ is nonempty, compact, invariant and internally chain transitive.

Proof. Let $\overline{Z}_+ = Z_+ \cup \{\infty\}$. For any given strictly increasing continuous function $\phi : [0, \infty) \to [0, 1)$ with $\phi(0) = 0$ and $\phi(\infty) = 1 (e.g., \phi(s) = \frac{s}{1+s})$, we can define a metric ρ on $\overline{Z}_+$ as $\rho(m_1, m_2) = |\phi(m_1) - \phi(m_2)|$, for any $m_1, m_2 \in \overline{Z}_+$, and then $\overline{Z}_+$ is compactified. Let $\{x_n :\ n \geq 0\}$ be a precompact asymptotic pseudo-orbit of $S : X \to X$, and denote its compact omega limit set by ω. Define a metric space

$$Y = (\{\infty\} \times X) \cup \{(n, x_n) :\ n \geq 0\}$$

and

$$g : Y \to Y, \quad g(n, x_n) = (n + 1, x_{n+1}),\ g(\infty, x) = (\infty, S(x)).$$

By Definition 2.3 and the fact that $d(x_{n+1}, S(x)) \leq d(x_{n+1}, S(x_n)) + d(S(x_n), S(x))$ for $x \in X, n \geq 0$, it easily follows that $g : Y \to Y$ is continuous. Let $\gamma^+(0, x_0) =$

$\{(n, x_n); n \geq 0\}$ be the positive orbit of $(0, x_0)$ for discrete semiflow $g^n : Y \to Y, n \geq 0$. Then $\gamma^+(0, x_0)$ is precompact in Y and its omega limit $\omega(0, x_0) = \{\infty\} \times \omega$, which by Lemma 2.1 is invariant and internally chain transitive for g. Applying the definition of g, we see that ω is invariant and internally chain transitive for S. ∎

Let A and B be two nonempty compact subsets of X. Recall that the Hausdorff distance between A and B is defined by

$$d_H(A, B) := \max\left(\sup\{d(x, B) : \ x \in A\}, \sup\{d(x, A) : \ x \in B\}\right)$$

We then have the following result.

Lemma 2.4 Let $S, S_n : X \to X$ for $n \geq 1$ be continuous. Let $\{D_n\}$ be a sequence of nonempty compact subsets of X with $\lim_{n\to\infty} d_H(D_n, D) = 0$ for some compact subset D of X. Assume that for each $n \geq 1$, D_n is invariant and internally chain transitive for S_n. If $S_n \to S$ uniformly on $D \cup (\cup_{n\geq1} D_n)$, then D is invariant and internally chain transitive for S.

Proof. Observe that the set $K = D \bigcup (\cup_{n\geq1} D_n)$ is compact; for an open cover of K also covers D, and hence a finite subcover provides a neighborhood of D which must also contain D_n for all large n. If $x \in D$ then there exist $x_n \in D_n$ such that $x_n \to x$. As $S_n(x_n) \in D_n$ and $S_n(x_n) \to S(x)$, we see that $S(x) \in D$. Thus $S(D) \subset D$. On the other hand, there exist $y_n \in D_n$ such that $S_n(y_n) = x_n$. We can assume that $y_{n_i} \to y \in D$ for some subsequence y_{n_i}, since $d_H(D_{n_i}, D) \to 0$. Then $x_{n_i} = S_{n_i}(y_{n_i}) \to S(y) = x$, showing that $S(D) = D$.

By uniform continuity and uniform convergence, for any $\epsilon > 0$ there exists $\delta \in (0, \epsilon/3)$ and a natural number N such that for $n \geq N$ and $u, v \in K$ with $d(u, v) < \delta$, we have

$$d(S_n(u), S(v)) \leq d(S_n(u), S(u)) + d(S(u), S(v)) < \epsilon/3.$$

Fix $n > N$ such that $d_H(D_n, D) < \delta$. For any $a, b \in D$, there are points $x, y \in D_n$ such that $d(x, a) < \delta$ and $d(y, b) < \delta$. As D_n is internally chain transitive for S_n, there is a δ-chain $\{z_1 = x, z_2, \cdots, z_{m+1} = y\}$ in D_n for S_n, connecting x to y. For each $i = 2, \cdots, m$ we can find $w_i \in D$ with $d(w_i, z_i) < \delta$ since D_n is contained in the δ-neighborhood of D. Let $w_1 = a, w_{m+1} = b$. We then have

$$\begin{aligned} d(S(w_i), w_{i+1}) &\leq d(S(w_i), S_n(z_i)) + d(S_n(z_i), z_{i+1}) + d(z_{i+1}, w_{i+1}) \\ &< \epsilon/3 + \delta + \delta < \epsilon \end{aligned}$$

for $i = 1, \cdots, m$. Thus $\{w_1 = a, w_2, \cdots, w_{m+1} = b\}$ is an ϵ-chain for S in D connecting a to b. ∎

Let $\Phi(t) : X \to X, t \in [0, \infty)$ be a continuous semiflow. That is, $(x, t) \to \Phi(t)x$ is continuous, $\Phi(0) = id_X$ and $\Phi(t) \circ \Phi(s) = \Phi(t + s)$ for $t, s \geq 0$. A nonempty invariant set $A \subset X$ for $\Phi(t)$ (i.e., $\Phi(t)A = A, t \geq 0$) is said to be internally chain transitive if for any $a, b \in A$ and any $\epsilon > 0, t_0 > 0$, there is a finite sequence $\{x_1 = a, x_2, \cdots, x_{m-1}, x_m = b; t_1, \cdots, t_{m-1}\}$ with $x_i \in A$ and $t_i \geq t_0, 1 \leq i \leq m-1$, such that $d(\Phi(t_i, x_i), x_{i+1}) < \epsilon$ for all $1 \leq i \leq m-1$. The sequence $\{x_1, \cdots, x_m; t_1, \cdots, t_{m-1}\}$ is called an (ϵ, t_0)-chain in A connecting a and b. We then have the following result.

Lemma 2.1′ Let $\Phi(t) : X \to X, t \geq 0$, be a continuous semiflow. Then the omega limit set of any precompact orbit is internally chain transitive.

Proof. Let $\omega = \omega(x)$ be the omega limit set of a precompact orbit $\gamma(x) = \{\Phi(t)x : t \geq 0\}$ in X. Then ω is nonempty, compact, invariant and $\lim_{t\to\infty} d(\Phi(t)x, \omega) = 0$. Let $\epsilon > 0$ and $t_0 > 0$ be given. By the uniform continuity of $\Phi(t)x$ for (t, x) in the compact set $[t_0, 2t_0] \times \omega$, there is a $\delta = \delta(\epsilon, t_0) \in (0, \frac{\epsilon}{3})$ such that for any $t \in [t_0, 2t_0]$ and u and v in the open δ-neighborhood U of ω with $d(u, v) < \delta$, there holds $d(\Phi(t)u, \Phi(t)v) < \frac{\epsilon}{3}$. It then follows that there exists a sufficiently large $T_0 = T_0(\delta) > 0$ such that $\Phi(t)x \in U$, for all $t \geq T_0$. For any $a, b \in \omega$, there exist $T_1 > T_0$ and $T_2 > T_0$ with $T_2 > T_1 + t_0$ such that $d(\Phi(T_1)x, \Phi(t_0)a) < \frac{\epsilon}{3}$ and $d(\Phi(T_2)x, b) < \frac{\epsilon}{3}$. Let m be the greatest integer which is not greater than $\frac{T_2 - T_1}{t_0}$, then $m \geq 1$. Set

$$y_1 = a, y_i = \Phi(T_1 + (i-2)t_0)x,\ i = 2, \cdots, m+1, y_{m+2} = b$$

and

$$t_i = t_0 \text{ for } i = 1, \cdots, m;\ t_{m+1} = T_2 - T_1 - (m-1)t_0$$

Then $t_{m+1} \in [t_0, 2t_0)$. It follows that $d(\Phi(t_i)y_i, y_{i+1}) < \frac{\epsilon}{3}$ for all $i = 1, \cdots, m+1$. Thus the sequence

$$\{y_1 = a, y_2, \cdots, y_{m+1}, y_{m+2} = b; t_1, t_2, \cdots, t_{m+1}\}$$

is an $(\frac{\epsilon}{3}, t_0)$-chain in X connecting a and b. Since $y_i \in U$ for $i = 2, \cdots, m+1$, we can choose $z_i \in \omega$ such that $d(z_i, y_i) < \delta$. Let $z_1 = a$ and $z_{m+2} = b$. It then follows that

$$\begin{aligned} d(\Phi(t_i)z_i, z_{i+1}) &\leq d(\Phi(t_i)z_i, \Phi(t_i)y_i) + d(\Phi(t_i)y_i, y_{i+1}) + d(y_{i+1}, z_{i+1}) \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3}, \quad i = 1, \cdots, m+1. \end{aligned}$$

This proves that the sequence $\{z_1 = a, z_2, \cdots, z_{m+1}, z_{m+2} = b; t_1, t_2, \cdots, t_{m+1}\}$ is an (ϵ, t_0)-chain in ω connecting a and b. Therefore ω is internally chain transitive. ∎

Remark 2.3 With Lemma 2.1′, it is easy to see there are analogues of Lemmas 2.2 and 2.3 for continuous semiflows. Moreover, the analogue of Lemma 2.4 for continuous semiflows follows by a similar argument.

3 Chain transitivity and attractivity

In this section, we discuss further properties of internally chain transitive sets, and give some applications. Throughout this section, X is a metric space with metric d and

$f : X \to X$ is a continuous map.

Lemma 3.1 A nonempty compact invariant set M is internally chain transitive if and only if M is the omega limit set of some asymptotic pseudo-orbit of f in M.

Proof. The sufficiency follows from Lemma 2.3. To prove the necessity, we can choose a point $x \in M$ since M is nonempty. For any $\epsilon > 0$, the compactness of M implies that there is a finite sequence of points $\{x_1 = x, x_2, \cdots, x_m, x_{m+1} = x\}$ in M such that its ϵ-net in X covers M, i.e., $M \subset \cup_{i=1}^m B(x_i, \epsilon)$, where $B(x_i, \epsilon) := \{y \in X : \ d(y, x_i) < \epsilon\}$. For each $1 \le i \le m$, since M is internally chain transitive, there is a finite ϵ-chain $\{y_1^i = x_i, y_2^i, \cdots, y_{n_i}^i, y_{n_i+1}^i = x_{i+1}\}$ in M connecting x_i and x_{i+1}. Then the sequence $\{x_1^1, \cdots, x_{n_1}^1, x_1^2, \cdots, x_{n_2}^2, \cdots, x_1^m, \cdots, x_{n_m}^m, x_{n_m+1}^m\}$ is a finite ϵ-chain in M connecting x and x, and its ϵ-net in X covers M.

For each integer k, letting $\epsilon = \frac{1}{k}$ in the above claim, we have a finite $\frac{1}{k}$-chain $\{z_1^k = x, z_2^k, \cdots, z_{l_k}^k, z_{l_k+1}^k = x\}$ in M whose $\frac{1}{k}$-net in X covers M. It then easily follows that the infinite sequence of points $\{z_1^1, \cdots, z_{l_1}^1, z_1^2, \cdots, z_{l_2}^2, \cdots, z_1^k, \cdots, z_{l_k}^k, \cdots\}$ is an asymptotic pseudo-orbit of f in M and its omega limit set is M. ∎

Block-Franke Lemma([2], Theorem A) Let K be a compact metric space and $f : K \to K$ be a continuous map. Then $x \notin R(K, f)$ if and only if there exists an attractor $A \subset K$ such that $x \in W^s(A) \setminus A$.

Lemma 3.2 A nonempty compact invariant set M is internally chain transitive if and only if $f|_M : M \to M$ has no proper attractor.

Proof. *Necessity.* Assume there is a proper attractor A for $f|_M : M \to M$. Then $A \neq \emptyset$ and $M \setminus A \neq \emptyset$. Since A is an attractor, there is an $\epsilon_0 > 0$ such that A attracts the open ϵ_0-neighborhood U of A in M. Choose $a \in M\setminus A$ and $b \in A$ and let $\{x_1 = a, x_2, \cdots, x_m = b\}$ be an ϵ_0-chain in M connecting a and b. Let $k = \min\{i : 1 \le i \le m,\ x_i \in A\}$. Since $b \in A$ and $a \notin A$, we have $2 \le k \le m$. Since $d(f(x_{k-1}), x_k) < \epsilon_0$, we have $f(x_{k-1}) \in U$ and hence $x_{k-1} \in W^s(A) \setminus A$. By Block-Franke Lemma, $x_{k-1} \notin R(M, f)$, which proves that M is not internally chain recurrent, and *a fortiori* not internally chain transitive.

Sufficiency. For any subset $B \subset X$ we define $\omega(B)$ to be the set of limits of sequences of the form $\{f^{n_k} x_k\}$ where $n_k \to \infty$ and $x_k \in B$. Since $f|_M : M \to M$ has no proper attractor, Block-Franke Lemma implies that M is internally chain recurrent. Given $a, b \in M$ and $\epsilon > 0$, let V be the set of all points x in M for which there is an ϵ-chain in M connecting a to x; this set contains a. For any $z \in V$, let

$$\{z_1 = a, z_2, \cdots, z_{m-1}, z_m = z\}$$

be an ϵ-chain in M connecting a to z. Since $\lim_{x \to z} d(f(z_{m-1}), x) = d(f(z_{m-1}), z) < \epsilon$, there is an open neighborhood U of z in M such that for any $x \in U$, $d(f(z_{m-1}), x) < \epsilon$.

Then $\{z_1 = a, z_2, \cdots, z_{m-1}, x\}$ is an ϵ-chain in M connecting a and x, and hence $U \subset V$. Thus V is an open set in M. We further claim that $f(\overline{V}) \subset V$. Indeed, for any $z \in \overline{V}$, by the continuity of f at z, we can choose a $y \in V$ such that $d(f(y), f(z)) < \epsilon$. Let $\{y_1 = a, y_2, \cdots, y_{m-1}, y_m = y\}$ be an ϵ-chain in M connecting a and y. It then follows that $\{y_1 = a, y_2, \cdots, y_{m-1}, y_m = y, y_{m+1} = f(z)\}$ is an ϵ-chain in M connecting a and $f(z)$, and hence $f(z) \in V$. By the compactness of M and [14, Lemma 2.1.2] applied to $f : M \to M$, it then follows that $\omega(\overline{V})$ is nonempty, compact, invariant and $\omega(\overline{V})$ attracts $\overline{V}$. Since $f(\overline{V}) \subset V$, we have $\omega(\overline{V}) \subset \overline{V}$ and hence $\omega(\overline{V}) = f(\omega(\overline{V})) \subset V$. Then $\omega(\overline{V})$ is an attractor in M. Now the nonexistence of proper attractor for $f : M \to M$ implies that $\omega(\overline{V}) = M$ and hence $V = M$. Clearly, $b \in M = V$, and hence, by the definition of V, there is an ϵ-chain in M connecting a and b. Therefore M is internally chain transitive. ∎

Recall that a nonempty invariant subset M of X is said to be **isolated** for $f : X \to X$ if it is the maximal invariant set in some neighborhood of itself.

Lemma 3.3 (Butler-McGehee type lemma) Let M be an isolated invariant set and L be a compact internally chain transitive set for $f : X \to X$. Assume that $L \cap M \neq \emptyset$ and $L \not\subset M$. Then:

(a) there exists a $u \in L \setminus M$ such that $\omega(u) \subset M$.

(b) there exists a $w \in L \setminus M$ and a negative orbit $\gamma^-(w) \subset L$ such that its α-limit set $\alpha(w) \subset M$.

Proof. By the assumption, we can choose $a \in L \cap M$ and $b \in L \setminus M$. For any integer $k \geq 1$, by the internal chain transitivity of L, there exist a $\frac{1}{k}$-chain $\{y_1^k = a, \cdots, y_{l_k+1}^k = b\}$ in L connecting a and b, and a $\frac{1}{k}$-chain $\{z_1^k = b, \cdots, z_{m_k+1}^k = a\}$ in L connecting b and a. Define a sequence of points by

$$\{x_n : n \geq 0\} := \{y_1^1, \cdots, y_{l_1}^1, z_1^1, \cdots, z_{m_1}^1, \cdots, y_1^k, \cdots, y_{l_k}^k, z_1^k, \cdots, z_{m_k}^k, \cdots, \}.$$

Then for any $k > 0$ and for all $n \geq N(k) := \sum_{j=1}^{k}(l_j+m_j)$, we have $d(f(x_n), x_{n+1}) < \frac{1}{k+1}$, and hence $\lim_{n\to\infty} d(f(x_n), x_{n+1}) = 0$. Thus $\{x_n\}_{n\geq 0} \subset L$ is a precompact asymptotic pseudo-orbit of $f : X \to X$. Let ω be its omega limit set.

As in the proof of Lemma 2.3, we define a metric space

$$Y = (\{\infty\} \times X) \cup \{(n, x_n); n \geq 0\}$$

and a continuous map

$$g : Y \to Y, \quad g(n, x_n) = (n + 1, x_{n+1}),\ g(\infty, x) = (\infty, f(x))$$

By the invariance and isolatedness of M for $f : X \to X$, it is easy to see that $\tilde{M} := \{\infty\} \times M$ is invariant and isolated for g. Note that $a, b \in \omega$ and $\omega(0, x_0) = \{\infty\} \times \omega$.

Clearly, $(\infty, a) \in \omega(0, x_0) \cap \tilde{M}$ and $(\infty, b) \in \omega(0, x_0) \setminus \tilde{M}$. By the Butler-McGehee lemma for omega limit sets ([12, Theorem 3.1]), it then follows that there exists $(\infty, u) \in \omega(0, x_0) \setminus \tilde{M}$ such that $\omega(\infty, u) \subset \tilde{M}$; and there exists $(\infty, w) \in \omega(0, x_0) \setminus \tilde{M}$ and a negative orbit $\gamma^-(\infty, w)$ such that its α-limit set $\alpha(\infty, w) \subset \tilde{M}$. Now the definition of g shows that u and w satisfy (a) and (b), respectively. ∎

In [31], a generalized Markus' theorem for maps is proved by using Lemma 3.2. For the completeness, we include this result here and also give an alternative, although somewhat longer, proof using Lemma 3.3.

Theorem 3.1(Strong Attractivity) ([31], Lemma 4.1) Let A be an attractor and C be a compact internally chain transitive set for $f : X \to X$. If $C \cap W^s(A) \neq \emptyset$, then $C \subset A$.

Proof. Clearly, A is isolated for $f : X \to X$. Let $x \in C \cap W^s(A)$. By the compactness and invariance of C, $\omega(x) \subset C$ and hence $\omega(x) \subset C \cap A$. Then $C \cap A \neq \emptyset$. Assume that, by contradiction, $C \not\subset A$, then, by Lemma 3.3, there exists $w \in C \setminus A$ with a full orbit $\gamma(w) = \{w_n : n \in Z\} \subseteq C$ and $\alpha(w) \subset A$. Since $w \notin A$, there exists an open neighborhood V of A such that $w \notin V$. Then, by the attractivity of A, there exist an open neighborhood U of A and an integer $n_0 > 0$ such that $S^n U \subset V$ for all $n \geq n_0$. Since $\alpha(w) \subset A$, there exists an integer $n_1 > n_0$ such that $w_{-n_1} \in U$, and hence $w = w_0 = S^{n_1}(w_{-n_1}) \in V$, which contradicts $w \notin V$. ∎

Let A and B be two isolated invariant sets. A is said to be **chained** to B, written $A \to B$, if there exists a full orbit through some $x \notin A \cup B$ such that $\omega(x) \subset B$ and $\alpha(x) \subset A$. A finite sequence $\{M_1, \cdots, M_k\}$ of invariant sets is called a **chain** if $M_1 \to M_2 \to \cdots \to M_k$. The chain is called a **cycle** if $M_k = M_1$.

Theorem 3.2 (Convergence) Assume that each fixed point of f is an isolated invariant set, that there is no cycle of fixed points, and that every precompact orbit converges to some fixed point of f. Then any compact internally chain transitive set is a fixed point of f.

Proof. Let C be a compact internally chain transitive set for $f : X \to X$. Then for any $x \in C$, we have $\gamma^+(x) \subset C$ and $\omega(x) \subset C$. Thus the convergence of $\gamma^+(x)$ implies that C contains some fixed point of f. Let $E = \{e \in C : f(e) = e\}$, then $E \neq \emptyset$ and, by the compactness of C and the isolatedness of each fixed point of f, $E = \{e_1, e_2, \cdots, e_m\}$ for some integer $m > 0$. Assume by way of contradiction that C is not singleton. Since $E \neq \emptyset$, there exists some i_1 $(1 \leq i_1 \leq m)$ such that $e_{i_1} \in C$, i.e., $C \cap \{e_{i_1}\} \neq \emptyset$. Since $C \not\subset \{e_{i_1}\}$, by Lemma 3.3, there exist $w_1 \in C \setminus \{e_{i_1}\}$ and a full orbit $\gamma(w_1) \subset C$ such that $\alpha(w_1) = e_{i_1}$. Since $\gamma^+(w_1) \subset C$, there exists some i_2 $(1 \leq i_2 \leq m)$ such that $\omega(w_1) = e_{i_2}$. Therefore, e_{i_1} is chained to e_{i_2}, i.e., $e_{i_1} \to e_{e_2}$. Since $C \cap \{e_{i_2}\} \neq \emptyset$ and

$C \not\subset \{e_{i_2}\}$, again by Lemma 3.3 , there exist $w_2 \in C \setminus \{e_{i_2}\}$ and a full orbit $\gamma(w_2) \subset C$ such that and $\alpha(w_2) = e_{i_2}$. We can repeat the above argument to get an i_3 $(1 \le i_3 \le m)$ such that $e_{i_2} \to e_{i_3}$. Since there is only a finite number of e_i's, we will eventually arrive at a cyclic chain of some fixed points of f, which contradicts our assumption. ∎

Let S be a compact metric space and $f : S \to S$ be a continuous map with $f(S) = S$. An ordered collection $\{M_1, \cdots, M_k\}$ of pairwise disjoint, compact and invariant subsets of S is called a **Morse decomposition** of S if for each $x \in S \setminus \cup_{i=1}^k M_i$ there is an i with $\omega(x) \subset M_i$ and for any negative orbit γ^- through x there is a $j > i$ with $\alpha(\gamma^-) \subset M_j$. By [25, Theorems 3.1.7 and 3.1.8] and their discrete-time versions, the current definition for Morse decomposition is equivalent to that in terms of Conley's repeller-attractor pairs(see, e.g., [25, Definition 3.1.5] for semiflows and [31, Definition 4.2] for maps).

A collection $\{M_1, \cdots, M_k\}$ of pairwise disjoint, compact and invariant subsets of S is called an **acyclic covering** of $\Omega(S) := \cup_{x \in S}\omega(x)$ if each M_i is isolated in S, $\Omega(S) \subset \cup_{i=1}^k M_i$, and no subset of M_i's forms a cycle in S. This concept is very important in the persistence theory (see, e.g., [10, 15]). The equivalence between acyclic coverings and Morse decompositions was first observed by Garay for (two sided) continuous flow on the boundary (see [13, Lemma]). In the following lemma, we formulate it in a general setting and give a complete proof which also provides an algorithm how to re-order an acyclic covering into an ordered Morse decomposition.

Lemma 3.4 A finite sequence $\{M_1, \cdots, M_k\}$ of pairwise disjoint, compact and invariant sets of f in S is an acyclic covering of $\Omega(S)$ if and only if it (after re-ordering) is a Morse decomposition of S.

Proof. *Necessity.* We first claim that for any subcollection $\mathcal{M}$ of M_i's, there exists an element $D \in \mathcal{M}$ such that D cannot be chained to any element in $\mathcal{M}$. Indeed, by contradiction, the nonexistence of such D would imply that some subset of M_i's from this finite collection $\mathcal{M}$ forms a cycle, which contradicts the acyclic condition. By this claim, we can re-order the total collection $\mathcal{M}_0 := \{M_1, \cdots, M_k\}$ by induction. First we choose an element, denoted by D_1, from the collection $\mathcal{M}_0$ such that D_1 cannot be chained to any element in $\mathcal{M}_0$. Suppose we have chosen $D_1, \cdots, D_m$, we further choose an element, denoted by D_{m+1}, from the collection $\mathcal{M}_m := \mathcal{M}_0 \setminus \{D_1, \cdots, D_m\}$ such that D_{m+1} cannot be chained to any element in $\mathcal{M}_m$. After k steps, we then get a re-ordered collection $\mathcal{D} := \{D_1, \cdots, D_k\}$. Moreover, for any $1 \le i < j \le k$, clearly we have $D_i, D_j \in \mathcal{M}_{i-1}$. Therefore, by the choice of D_i, D_i cannot be chained to any element in $\mathcal{M}_{i-1}$, and hence D_i cannot be chained to D_j.

For any $x \in S \setminus \cup_{i=1}^k D_i$, By the assumption, we have $\omega(x) \subset \cup_{i=1}^k D_i$, and hence the invariant connectedness of $\omega(x)$ implies that $\omega(x) \subset D_i$ for some i. Let γ^- be any given negative orbit of f through x and let $\alpha = \alpha(\gamma^-)$. By Lemma 2.1, α is internally chain transitive for f. We further claim that $\alpha \subset D_j$ for some j. Indeed, assume that, by contradiction, $\alpha \not\subset D_m$ for all $1 \le m \le k$. Since $\alpha \subset S$ is compact and invariant,

$\alpha \cap (\cup_{i=1}^k M_i) \neq \emptyset$, and hence there exists some M_{i_1} $(1 \le i_1 \le k)$ such that $\alpha \cap D_{i_1} \neq \emptyset$. By Lemma 3.3, there exist $w_1 \in \alpha \backslash D_{i_1}$ and a full orbit $\gamma(w_1) \subset \alpha$ such that $\alpha(w_1) \subset D_{i_1}$. Since $w_1 \in \alpha \subset S$, $\omega(w_1) \subset \cup_{i=1}^k D_i$, and hence, by the invariant connectedness of $\omega(w_1)$, there exists some D_{i_2} $(1 \le i_2 \le k)$ such that $\omega(w_1) \subset D_{i_2}$. Therefore D_{i_1} is chained to D_{i_2}, i.e., $D_{i_1} \to D_{i_2}$. Clearly, $\omega(w_1) \subset \alpha$. Then $\alpha \cap D_{i_2} \neq \emptyset$. Again by Lemma 3.3, there exists $w_2 \in \alpha \setminus D_{i_2}$ and a full orbit $\gamma(w_2) \subset \alpha$ and $\alpha(w_2) \subset D_{i_2}$. We can repeat the above argument to get an i_3 $(1 \le i_3 \le k)$ such that $D_{i_2} \to D_{i_3}$. Since there is only a finite number of D_m's, we will eventually arrive at a cyclic chain of some D_m for f in S, which contradicts the acyclicity condition. It then follows that that $D_j \to D_i$, and hence, by the property of $\{D_1, \cdots, D_k\}$, we have $j > i$. Therefore $\{D_1, \cdots, D_k\}$ is a Morse decomposition of S.

Sufficiency. Since the $M_i, 1 \le i \le k$, are pairwise disjoint and compact, there exist k pairwise disjoint and closed subsets N_i of S such that M_i is contained in the interior of N_i, $1 \le i \neq j \le k$. In order to see that M_m is isolated in S, suppose that there exists an invariant set $M \subset \mathrm{Int}N_m$ but $M \not\subset M_m$. It follows that there is a $x \in M \cap (S \setminus \cup_{i=1}^k M_i)$. Let $\gamma \subset M$ be a full orbit through x. Clearly, $\omega(x) \subset \overline{M}$ and $\alpha(x) \subset \overline{M}$. Since $\{M_1, \cdots, M_k\}$ is a Morse decomposition of S, there exist $j > i$ such that $\omega(x) \subset M_i$ and $\alpha(x) \subset M_j$. Then $M_i \cap N_m \neq \emptyset$ and $M_j \cap N_m \neq \emptyset$, and hence $i = m = j$, which contradicts $j > i$. Thus each M_i is isolated in S. Clearly, the definition of Morse decompositions implies that $\Omega(S) \subset \cup_{i=1}^k M_i$. We further claim that if $M_{i_1} \to M_{i_2}$ then $i_1 > i_2$. Indeed, let $\gamma(x)$ be a full orbit through some $x \notin M_{i_1} \cup M_{i_2}$ such that $\omega(x) \subset M_{i_2}$ and $\alpha(x) \subset M_{i_1}$. If $x \in M_l$ for some l, we have $\omega(x) \subset M_l \cap M_{i_2}$ and $\alpha(x) \subset M_l \cap M_{i_1}$, and hence $i_1 = l = i_2$, contradicting that $x \notin M_{i_1} \cup M_{i_2}$. It follows that $x \in S \setminus \cup_{i=1}^k M_i$. Since $\{M_1, \cdots, M_k\}$ is a Morse decomposition of S, there exist $j > i$ such that $\omega(x) \in M_i$ and $\alpha(x) \in M_j$. Then we have $i_1 = j > i = i_2$. By this claim, it is easy to see that no subset of M_i's forms a cycle in S. Therefore $\{M_1, \cdots, M_k\}$ is an acyclic covering of $\Omega(S)$. ∎

4 Strong repellors and uniform persistence

Throughout this section, X is a metric space and $f : X \to X$ is a continuous map. Let $X_0 \subset X$ be an open set with $f(X_0) \subset X_0$. Define $\partial X_0 = X \setminus X_0$, and $M_\partial = \{x \in \partial X_0 : \ f^n(x) \in \partial X_0, n \ge 0\}$, which may be empty. Note that ∂X_0 need not be the boundary of X_0 as the notation suggests. This peculiar notation has become standard in persistence theory (see, e.g., [33]). We assume hereafter that every positive orbit of f is precompact. If $S \subset X$, define $\Omega(S) := \cup_{x \in S} \omega(x)$.

There are two traditional approaches in persistence theory, one using Morse decompositions and the other using acyclic coverings. The next lemma, together with Lemma 3.4, shows that the two approaches are identical.

Lemma 4.1 Suppose that there exists a maximal compact invariant set A_∂ of f in ∂X_0, that is, A_∂ is compact, invariant, possibly empty, and contains every compact invariant subset of ∂X_0. Then a finite sequence $\{M_1, \cdots, M_k\}$ of pairwise disjoint, compact and invariant subsets of ∂X_0, each of which is isolated in ∂X_0, is an acyclic covering of $\Omega(M_\partial)$ in ∂X_0 if and only if it (after re-ordering) is a Morse decomposition of A_∂.

Proof. Let $S = A_\partial$. Then we have $S \subset M_\partial$ and hence $\Omega(S) \subset \Omega(M_\partial)$. It follows that $\{M_1, \cdots, M_k\}$ is also an acyclic covering of $\Omega(S)$. So the necessity follows from Lemma 3.4. To prove the sufficiency, assume that $\{M_1, \cdots, M_k\}$ is a Morse decomposition of S. By Lemma 3.4, $\{M_1, \cdots, M_k\}$ is an acyclic covering of $\Omega(S)$ in S. Since S is the maximal compact invariant set in ∂X_0, any compact invariant set in ∂X_0 is a subset of S. Consequently, no subset of M_i's forms a cycle in ∂X_0 because such a cycle is compact and invariant so necessarily belongs to S, violating that $\{M_1, \cdots, M_k\}$ is a Morse decomposition of S. We further claim that $\Omega(M_\partial) \subset \cup_{i=1}^{k} M_i$. Indeed, for any $x \in M_\partial$, $\omega(x)$ is a compact, invariant, internally chain transitive set in ∂X_0(by Lemma 2.1). Then $\omega(x) \subset S$, and hence [31, Lemma 4.3] implies that $\omega(x) \subset R(S, f) \subset \cup_{i=1}^{k} M_i$. Therefore $\{M_1, .., M_k\}$ is an acyclic covering of $\Omega(M_\partial)$ in ∂X_0. ∎

Theorem 4.3 (Strong Repellors) Assume that

(C1) f has a global attractor A.

(C2) the maximal compact invariant set $A_\partial = A \cap M_\partial$ of f in ∂X_0, possibly empty, admits a Morse decomposition $\{M_1, \cdots, M_k\}$ with the following properties:

- M_i is isolated in X,
- $W^s(M_i) \cap X_0 = \emptyset$ for each $1 \le i \le k$.

Then there exists $\delta > 0$ such that for any compact internally chain transitive set L with $L \not\subset M_i$ for all $1 \le i \le k$, there holds $\inf_{x \in L} d(x, \partial X_0) > \delta$.

Proof. We first prove the following weaker conclusion:

Claim There is an $\epsilon > 0$ such that if L is a compact internally chain transitive set not contained in any M_i, then $\sup_{x \in L} d(x, \partial X_0) > \epsilon$.

Indeed, assume that, by contradiction, there exists a sequence of compact internally chain transitive sets $\{D_n : n \ge 1\}$ with $D_n \not\subset M_i, 1 \le i \le k$, such that

$$\lim_{n\to\infty} \sup_{x \in D_n} d(x, \partial X_0) = 0.$$

Since $W^s(A) = X$, by Theorem 3.1, we have $D_n \subset A$ for all $n \ge 1$. In the compact metric space of compact nonempty subsets of A with Hausdorff distance d_H, the sequence $\{D_n : n \ge 1\}$ has a convergent subsequence. Without loss of generality, we assume that for some nonempty compact set $D \subset A$, $\lim_{n\to\infty} d_H(D_n, D) = 0$. Then for any $x \in D$, there exists $x_n \in D_n$ such that $\lim_{n\to\infty} x_n = x$. Clearly, $\lim_{n\to\infty} d(x_n, \partial X_0) = 0$, and hence there exists $y_n \in \partial X_0$ such that $\lim_{n\to\infty} d(x_n, y_n) = 0$. It then follows that $\lim_{n\to\infty} y_n = x$, and hence $x \in \overline{\partial X_0} = \partial X_0$. Thus $D \subset \partial X_0$. By Lemma 2.4 with $S_n = f$, D is internally chain transitive for f. It then follows that $D \subset A_\partial$ and [31, Lemma 4.3] implies that $D \subset R(A_\partial, f) \subset \cup_{i=1}^k M_i$. Then the invariant connectedness of D implies that $D \subset M_i$ for some i. Since $D_n \to M_i$ as $n \to \infty$, the isolatedness of M_i in X implies that $D_n \subset M_i$ for all large n, contradicting to our assumption. This proves the claim.

We now prove the theorem by contradiction. Assume there exists a sequence of compact, internally chain transitive sets $\{L_n : n \geq 1\}$ with $L_n \not\subset M_i, 1 \leq i \leq k, n \geq 1$, such that $\lim_{n\to\infty} \inf_{x\in L_n} d(x, \partial X_0) = 0$. As in the proof of the claim, we can assume that $\lim_{n\to\infty} d_H(L_n, L) = 0$, where L is a compact internally chain transitive set for $f : X \to X$ and $L \not\subset M_i$ for each $1 \leq i \leq k$. Clearly, there exist $x_n \in L_n, n \geq 1$, such that $\lim_{n\to\infty} d(x_n, \partial X_0) = 0$, and hence $L \cap \partial X_0 \neq \emptyset$. By the above claim, we can choose $a \in L \cap \partial X_0$ and $b \in L$ with $d(b, \partial X_0) > \epsilon$. As in the proof of Lemma 3.3, let $\{x_n : n \geq 0\}$ be the asymptotic pseudo-orbit determined by a and b in L. Then there are two subsequences x_{m_j} and x_{r_j} such that $x_{m_j} = a$ and $x_{r_j} = b$ for all $j \geq 1$. Note that $d(x_{s_j+1}, f(x)) \leq d(x_{s_j+1}, f(x_{s_j})) + d(f(x_{s_j}), f(x))$. By induction, it then follows that for any convergent subsequence $x_{s_j} \to x \in X, j \to \infty$, there holds $\lim_{j\to\infty} x_{s_j+n} = f^n(x)$ for any integer $n \geq 0$. We can further choose two sequences l_j and n_j with $l_j < m_j < n_j$ and $\lim_{j\to\infty} l_j = \infty$ such that $d(x_{l_j}, \partial X_0) > \epsilon$, $d(x_{n_j}, \partial X_0) > \epsilon$, and $d(x_k, \partial X_0) \leq \epsilon$ for any integer $k \in (l_j, n_j), j \geq 1$. Since $\{x_n : n \geq 0\}$ is a subset of the compact set L, we can assume that, after taking a convergent subsequence, $x_{l_j} \to x \in L$ as $j \to \infty$. Clearly, $d(x, \partial X_0) \geq \epsilon$ and hence $x \in X_0$. We further claim that the sequence $n_j - l_j$ is unbounded. Assume that, by contradiction, $n_j - l_j$ is bounded. Then $m_j - l_j$ is also bounded and hence we can assume that, after choosing a subsequence, $m_j - l_j = m$, where m is an integer. Since $f(X_0) \subset X_0$, we have $a = \lim_{j\to\infty} x_{m_j} = \lim_{j\to\infty} x_{l_j+m} = f^m(x) \in X_0$, which contradicts $a \in \partial X_0$. Thus we can assume that, by taking a subsequence, $n_j - l_j \to \infty$ as $j \to \infty$. Then for any integer $n \geq 1$, there is an integer $J = J(n) \geq 1$ such that $n_j - l_j > n$ for all $j \geq J$. Then we have $l_j < l_j + n < n_j$ and hence $d(x_{l_j+n}, \partial X_0) \leq \epsilon, j \geq J(n)$. Thus $f^n(x) = \lim_{j\to\infty} x_{l_j+n}$ satisfies $d(f^n(x), \partial X_0) \leq \epsilon, n \geq 1$. Since $x \in L$, we have $f^n(x) \in L, n \geq 0$. Thus, by Lemma 2.1, $\omega(x)$ is a compact, internally chain transitive set for $f : X \to X$. Moreover, $\sup_{y\in\omega(x)} d(y, \partial X_0) \leq \epsilon$. Appealing again to the claim, we conclude that $\omega(x) \subset M_i$ for some $1 \leq i \leq k$, and hence $x \in W^s(M_i) \cap X_0$. But this contradicts assumption (C2). ∎

Remark 4.1 By an argument similar to the last part of the proof of Theorem 4.3, with ∂X_0 replaced by an isolated invariant set M, we can prove Lemma 3.3 without appealing to the Butler-McGehee lemma for omega limit sets.

Remark 4.2 Recall that $f : X \to X$ is said to be **uniformly persistent** with respect to $(X_0, \partial X_0)$ if there exists $\eta > 0$ such that $\liminf_{n\to\infty} d(f^n(x), \partial X_0) \geq \eta$ for all

$x \in X_0$. If "inf" in this inequality is replaced with "sup", f is said to be **weakly uniformly persistent** with respect to $(X_0, \partial X_0)$. It then follows that Theorem 4.3, with $L = \omega(x), x \in X_0$, implies [20, Theorems 4.1 and 4.2]. Note that A_∂ is also a Morse decomposition of $f : A_\partial \to A_\partial$.

Remark 4.3 In view of Lemma 4.1, Theorem 4.3 holds if condition (C2) is replaced by

(C2′) There exists a finite sequence $M = \{M_1, \cdots, M_k\}$ of pairwise disjoint, compact and isolated invariant sets in ∂X_0 with the following properties:

- $\Omega(M_\partial) := \cup_{x \in M_\partial} \omega(x) \subset \cup_{i=1}^k M_i$,
- no subset of M forms a cycle in ∂X_0,
- M_i is isolated in X,
- $W^s(M_i) \cap X_0 = \emptyset$ for each $1 \le i \le k$.

Moreover, Theorem 4.3 with (C2) replaced by (C2′), applied to the omega limit sets of precompact orbits of continuous maps and asymptotically autonomous discrete processes, implies [34, Theorem 2.2]) and [35, Theorem 2.5], respectively.

Remark 4.4 If we restrict attention to omega limit sets L in Theorem 4.3, the assumption (C1) can be replaced by some weaker compactness assumptions near ∂X_0, see, e.g., [33] for a detailed discussion in the context of continuous semiflows. In particular, by the proof of Theorem 4.3, it follows that the weak uniform persistence implies uniform persistence for maps.

Let $S_m : X \to X,\ m \ge 0$, be a sequence of continuous maps such that every positive orbit for S_m has compact closure, and $S_m(X_0) \subset X_0$. Let $\omega_m(x)$ denote the omega limit of x for discrete semiflow S_m, and set $W = \overline{\cup_{m \ge 0, x \in X} \omega_m(x)}$.

Theorem 4.4 (STABILITY OF UNIFORM PERSISTENCE) Assume W is compact and $S_m \to S_0$ uniformly on W. In addition, assume:

(A1) S_0 satisfies (C1) and (C2) of Theorem 4.3 or (C1) and (C2′) of Remark 4.3.

(A2) there exist $\eta_0 > 0$ and a positive integer N_0 such that for $m \ge N_0$ and $x \in X_0$, $\limsup_{n\to\infty} d(S_m^n x, M_i) \ge \eta_0,\ 1 \le i \le k$.

Then there exist $\eta > 0$ and a positive integer N such that $\liminf_{n\to\infty}\ d(S_m^n x, \partial X_0) \ge \eta$ for $m \ge N$ and $x \in X_0$.

Proof. Assume that, by contradiction, there exists a sequence $\{x_k\}$ in X_0 and positive integers $m_k \to \infty$ satisfying $\liminf_{n\to\infty} d(S_{m_k}^n x_k, \partial X_0) \to 0$ as $k \to \infty$. By Lemma 2.1,

$\omega_{m_k}(x_k)$ is a compact internally chain transitive set for S_{m_k}. In the compact metric space of all compact subsets of W with Hausdorff distance d_H, the sequence $\{\omega_{m_k}(x_k)\}$ has a convergent subsequence. Without loss of generality, we assume that for some nonempty compact $L \subset W$, $\lim_{k\to\infty} d_H(\omega_{m_k}(x_k), L) = 0$. Clearly, there exist $y_k \in \omega_{m_k}(x_k)$ such that $\lim_{k\to\infty} d(y_k, \partial X_0) = 0$, and hence $L \cap \partial X_0 \neq \emptyset$. By Lemma 2.4, L is internally chain transitive for S_0. Since $L \cap \partial X_0 \neq \emptyset$, Theorem 4.3, applied to S_0, implies $L \subset M_i$ for some i. Therefore $\lim_{k\to\infty} \sup\{d(x, M_i) : x \in \omega_{m_k}(x_k)\} = 0$ and hence there exists a $k_0 > 0$ such that $m_{k_0} > N_0$ and $\omega_{m_{k_0}}(x_{k_0}) \subset \{x : d(x, M_i) < \frac{\eta_0}{2}\}$. Since $S^n_{m_{k_0}}(x_{k_0}) \to \omega_{m_{k_0}}(x_{k_0})$ as $n \to \infty$, we have $\limsup_{n\to\infty} d(S^n_{m_{k_0}}(x_{k_0}), M_i) \leq \frac{\eta_0}{2}$, which is a contradiction to assumption (A2). ∎

Corollary 4.5 (Uniform persistence uniform in parameters) Let Λ be a metric space with metric ρ. For each $\lambda \in \Lambda$, let $S_\lambda : X \to X$ be a continuous map that takesX_0 into itself, and such that $S_\lambda(x)$ is continuous in (λ, x). Assume that every positive orbit for S_λ has compact closure in X, and that the set $\bigcup_{\lambda\in\Lambda, x\in X} \omega_\lambda(x)$ has compact closure, where $\omega_\lambda(x)$ denotes the omega limit of x for discrete semiflow $\{S^n_\lambda\}$. Let $\lambda_0 \in \Lambda$ be fixed, and assume further that

(B1) $S_{\lambda_0} : X \to X$ has a global attractor, and *either* the maximal compact invariant set A_∂ of S_{λ_0} in ∂X_0 admits a Morse decomposition $\{M_1, \cdots, M_k\}$, *or* there exists an acyclic covering $\{M_1, \cdots, M_k\}$ of $\Omega(M_\partial)$ for S_{λ_0} in ∂X_0.

(B2) There exists $\delta_0 > 0$ such that for any $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) < \delta_0$ and any $x \in X_0$, $\limsup_{n\to\infty} d(S^n_\lambda x, M_i) \geq \delta_0,\ 1 \leq i \leq k$.

Then there exists $\delta > 0$ such that $\liminf_{n\to\infty} d(S^n_\lambda x, \partial X_0) \geq \delta$ for any $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) < \delta$ and any $x \in X_0$.

Proof. Clearly, (B1) and (B2) imply that (A1) holds for $S_0 := S_{\lambda_0} : X \to X$. If the conclusion were false we could find sequences $x_k \in X_0$ and λ_k with $\lambda_k \to \lambda_0$ such that $\liminf_{n\to\infty} d(S^n_k x_k, \partial X_0) \to 0$ as $k \to \infty$, where $S_k := S_{\lambda_k} \to S_0$ uniformly on W. But this contradicts Theorem 4.4. ∎

Remark 4.5 Corollary 4.5 is very similar to [31, Theorem 4.3]. The difference lies in that the existence of a global attractor $A_0 \subset X_0$ for $S_{\lambda_0} : X_0 \to X_0$ is assumed in [31, Theorem 4.3].

Remark 4.6 By using similar arguments, we can prove the analogues of Lemmas 3.1-3.4, Theorems 3.1-3.2 and 4.3-4.4 and Corollary 4.5 for continuous semiflows.

As an application of Corollary 4.5, consider the Kolmogorov-type ordinary differential equation:

$$x_i' = x_i f_i(x) \equiv F_i(x) \tag{1}$$

on $P \equiv \mathbb{R}^n_+$ where f is a C^1 vector field on P. Let $P^0 = \{x \in P : x_i > 0, 1 \le i \le n\}$ and for $M > 0$ let $P_M = \{x \in P : \ x_i \le M, 1 \le i \le n\}$ and $P_M^0 = P_M \cap P^0$. Denote by ϕ_t^f the semiflow generated by (1). Let $C_L = C_{Lip}(P_M, \mathbb{R}^n)$ be the space of Lipschitz vector fields on P_M. Below, $\|x\|$ denotes a norm of vector $x \in \mathbb{R}^n$.

According to [27], a compact invariant K of ϕ_t^f is said to be **unsaturated** if

$$\min_{\mu \in \mathcal{M}(f,K)} \max_{1 \le i \le n} \int f_i d\mu > 0,$$

where $\mathcal{M}(f, K)$ is the set of ϕ_t^f-invariant Borel probability measures with support contained in K. In particular, an equilibrium e of ϕ_t^f is unsaturated if and only if $f_i(e) > 0$ for some $1 \le i \le n$, and a periodic orbit $\gamma = \{u(t) : t \in [0, T]\}$ of ϕ_t^f, with minimal period $T > 0$, is unsaturated if and only if $\int_0^T f_i(u(s))ds > 0$ for some $1 \le i \le n$.

Corollary 4.6 (Robust permanence) Assume that

(D1) there exists $M > 0$ such that $x \in P_M$ and $x_i = M$ implies $f_i(x) \le 0$;

(D2) the maximal compact invariant set of ϕ_t^f on $P_M \backslash P_M^0$ admits a Morse decomposition $\{M_1, \cdots, M_k\}$ such that each M_i is unsaturated for ϕ_t^f.

Then there exist $\epsilon, \eta > 0$ such that for $g \in C_L$ satisfying (D1) and

$$\sup_{x \in P_M} \|f(x) - g(x)\| < \epsilon \tag{2}$$

and for $x \in P_M^0$, it follows that

$$\eta \le y_i \le M, \ 1 \le i \le n, \text{ for all } y \in \omega_g(x). \tag{3}$$

Here, $\omega_g(x)$ denotes the omega limit set of x for the system $x_i' = x_i g_i(x)$.

Proof. Let $\Lambda = \{g \in C_L : (D1) \text{ holds for } g\}$ (endowed with the uniform metric), and consider the family of semiflows ϕ_t^g on $X = P_M$ with $X_0 = P_M^0$. Here, ϕ_t^g denotes the semiflow generated by $x_i' = x_i g_i(x) \equiv G_i(x)$. The continuity of the map $(g, x, t) \to \phi_t^g(x)$ is well-known. The closure of $\bigcup_{g \in \Lambda, x \in P_M} \omega_g(x)$ is compact in P_M. Clearly, $\phi_t^f : X \to X$ has a global attractor. By (the continuous-time version of) Corollary 4.5, it suffices to prove that condition (B2) holds, which is implied by the following lemma. ∎

Lemma 4.2 Let $\lambda_0 = f \in \Lambda$. If $K \subset P_M$ is an unsaturated compact invariant set for ϕ_t^f, then condition (B2) holds for K.

Proof. Assume that, by contradiction, (B2) is not true for K. We will use a similar idea as in [27] to construct a ϕ_t^f-invariant Borel measure $\mu \in \mathcal{M}(f, K)$ such that μ is saturated for ϕ_t^f. It then follows that there exist two sequences $g^m \in \Lambda$ and $y^m \in X_0$ such that $\rho(g^m, f) := \sup\limits_{x \in P_M} \|g^m(x) - f(x)\| < \frac{1}{m}$ and

$$\limsup_{t\to\infty} d(\phi_t^{g^m}(y^m), K) < \frac{1}{m}, \ \forall m \geq 1, \tag{4}$$

and hence there is a sequence of s_m such that

$$d(\phi_t^{g^m}(y^m), K) < \frac{1}{m}, \ \forall t \geq s_m, m \geq 1.$$

Let $x^m = \phi^{g^m}(s_m, y^m)$. Then $x^m \in X_0$ and the flow property of $\phi_t^{g^m}$ implies that

$$d(\phi_t^{g^m}(x^m), K) < \frac{1}{m}, \ \forall t \geq 0, \ m \geq 1, \tag{5}$$

Let $f = (f_1, \cdots, f_n)$. Since

$$\ln\left(\frac{[\phi_t^{g^m}(x^m)]_i}{x_i^m}\right) = \int_0^t g_i^m(\phi_s^{g^m}(x^m))ds, \ \forall t \in \mathbb{R}, \ 1 \leq \mathrm{i} \leq \mathrm{n}, \ \mathrm{m} \geq 1. \tag{6}$$

By inequality (5), it easily follows that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t g_i^m(\phi_s^{g^m}(x^m))ds \leq 0, \ \ 1 \leq i \leq n, \ m \geq 1. \tag{7}$$

Then we can choose a sequence t_m such that $t_m \geq m$ and

$$\frac{1}{t_m}\int_0^{t_m} g_i^m(\phi_s^{g^m}(x^m))ds < \frac{1}{m}, \ \ 1 \leq i \leq n, \ m \geq 1. \tag{8}$$

Define a sequence of Borel probability measures μ_m on $\mathbb{R}_+^{\mathrm{n}}$ by

$$\int h d\mu_m = \frac{1}{t_m}\int_0^{t_m} h(\phi_s^{g^m}(x^m))ds, \ \ m \geq 1, \tag{9}$$

for any continuous function $h \in C(\mathbb{R}_+^{\mathrm{n}}, \mathbb{R})$. By inequality (5), it then follows that μ_m lies in space $\mathcal{M}(V)$ of Borel probability measures with support in the compact set $V = \{x \in \mathbb{R}_+^{\mathrm{n}} : \mathrm{d(x,K)} \leq 1\}$. By the weak* compactness of $\mathcal{M}(V)$, we can assume that μ_m converges in the weak* topology to some $\mu \in \mathcal{M}(V)$ as $m \to \infty$. We claim that μ is invariant under ϕ_t^f, i.e., $\mu(\phi_t^f(B)) = \mu(B)$ for any $t \in \mathbb{R}$ and any Borel set $B \subseteq \mathbb{R}_+^{\mathrm{n}}$. It suffices to verify that $\int h \circ \phi_t^f d\mu = \int h d\mu$ for any $h \in C(\mathbb{R}_+^{\mathrm{n}}, \mathbb{R})$ and $t \in \mathbb{R}$. For any fixed $t > 0$, since

$$\begin{aligned}&\int_0^{t_m} \left(h \circ \phi_t^{g^m}(\phi_s^{g^m}(x^m)) - h(\phi_s^{g^m}(x^m))\right) ds \\ &= \int_0^{t_m} h \circ \phi_{t+s}^{g^m}(x^m)ds - \int_0^{t_m} h(\phi_s^{g^m}(x^m))ds\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^{t_m-t} h\circ\phi^{g^m}_{t+s}(x^m)ds + \int_{t_m-t}^{t_m} h\circ\phi^{g^m}_{t+s}(x^m)ds\right)\\
&\quad - \left(\int_0^{t} h(\phi^{g^m}_{s}(x^m))ds + \int_{t}^{t_m} h(\phi^{g^m}_{s}(x^m))ds\right)\\
&= \left(\int_0^{t_m-t} h\circ\phi^{g^m}_{t+s}(x^m)ds + \int_0^{t} h\circ\phi^{g^m}_{t_m+u}(x^m)du\right)\\
&\quad - \left(\int_0^{t} h(\phi^{g^m}_{s}(x^m))ds + \int_0^{t_m-t} h(\phi^{g^m}_{t+v}(x^m))dv\right)\\
&= \int_0^t \left(h\left(\phi^{g^m}_{t_m+s}(x^m)\right) - h(\phi^{g^m}_{s}(x^m))\right)ds,
\end{aligned}
$$

we get

$$
\begin{aligned}
\left|\int (h\circ\phi^f_t - h)d\mu\right| &= \lim_{m\to\infty}\left|\int (h\circ\phi^f_t - h)d\mu_m\right|\\
&= \lim_{m\to\infty}\left|\frac{1}{t_m}\int_0^{t_m}\left(h\circ\phi^f_t(\phi^{g^m}_s(x^m)) - h(\phi^{g^m}_s(x^m))\right)ds\right|\\
&\le \limsup_{m\to\infty}\frac{1}{t_m}\left(\left|\int_0^{t_m}\left(h\circ\phi^{g^m}_t(\phi^{g^m}_s(x^m)) - h(\phi^{g^m}_s(x^m))\right)ds\right|\right.\\
&\quad \left. + \left|\int_0^{t_m}\left(h\circ\phi^f_t(\phi^{g^m}_s(x^m)) - h\circ\phi^{g^m}_t(\phi^{g^m}_s(x^m))\right)ds\right|\right)\\
&= \limsup_{m\to\infty}\frac{1}{t_m}\left(\left|\int_0^{t}\left(h(\phi^{g^m}_{t_m+s}(x^m)) - h(\phi^{g^m}_s(x^m))\right)ds\right|\right.\\
&\quad \left. + \left|\int_0^{t_m}\left(h\circ\phi^f_t(\phi^{g^m}_s(x^m)) - h\circ\phi^{g^m}_t(\phi^{g^m}_s(x^m))\right)ds\right|\right).
\end{aligned}
$$

Therefore, using inequality (5), boundedness of $h(\cdot)$ on V and uniform convergence of $g^m \to f$ (and hence of $\phi^{g^m}_t \to \phi^f_t$) on P_M, we have $\int (h\circ\phi^f_t - h)d\mu = 0$ for any $h \in C(\mathbb{R}^n_+, \mathbb{R})$ and $t > 0$. For any $t > 0$ and $p \in C(\mathbb{R}^n_+, \mathrm{R})$, letting $h = p\circ\phi^f_{-t}$, we then get $\int (p - p\circ\phi^f_{-t})d\mu = \int (p\circ\phi^f_{-t}\circ\phi^f_t - p\circ\phi^f_{-t})d\mu = 0$. Then μ is invariant for $\phi^f_s, s \in \mathbb{R}$. By inequality (5) and weak* convergence, it follows that $\mu \in \mathcal{M}(f, K)$. For any $1 \le i \le n$, using the uniform convergence of g^m to f on V and inequality (8), we further have

$$
\begin{aligned}
\int f_i du &= \lim_{m\to\infty}\int f_i d\mu_m\\
&\le \lim_{m\to\infty}\frac{1}{t_m}\int_0^{t_m}(f_i - g^m_i)(\phi^{g^m}_s(x^m))ds + \limsup_{m\to\infty}\frac{1}{t_m}\int_0^{t_m} g^m_i(\phi^{g^m}_s(x^m))ds \le 0.
\end{aligned}
$$

But this contradicts the unsaturatedness of K for ϕ^f_t. ∎

Remark 4.7 Corollary 4.6 may be compared to Schreiber's main result on robust permanence result ([27, Theorem 4.3]). His result guarantees permanence under C^r-small perturbation, $r \ge 1$, while ours guarantees permanence under the wider class of C^0-small perturbation by locally Lipschitz vector fields.

For many systems, the boundary dynamics are simple: every bounded orbit on the boundary converges to an equilibrium or a nontrivial periodic orbit. By a **critical element** of (1) we mean an equilibrium point or a nontrivial periodic orbit. Our usual notation for a critical element is $\gamma = \{u(t) : 0 \leq t \leq T\}$ where $u(t)$ is a T-periodic solution of (1) and T is the minimal period which may be zero for an equilibrium. Let $W^s(\gamma)$ denote the stable manifold of γ.

Corollary 4.7 (Robust permanence) Let **(D1)** hold, and assume that

(D3) there exist hyperbolic critical elements $\gamma^i \in P_{M'} \backslash P^0_{M'}$ for some $M' < M$, $1 \leq i \leq k$, satisfying:

- (i) $P \setminus P^0 \subset \cup_{i=1}^k W^s(\gamma^i)$;
- (ii) for each γ^i, there exists k such that $x_k = 0$ on γ^i and $\int_0^{T_i} f_k(u^i(s))ds > 0$;
- (iii) no subset of $\{\gamma^1, \gamma^2, \cdots, \gamma^k\}$ forms a cycle in $P \setminus P^0$.

Then the conclusion of Corollary 4.6 holds.

Proof. By assumptions (D1) and (D3) and Lemma 4.1, $\{\gamma^1, \gamma^2, \cdots, \gamma^k\}$ is a Morse decomposition of the maximal compact invariant set for ϕ^f_t on $P_M \setminus P^0_M$. Since (D3) (ii) implies that each γ^i is unsaturated for ϕ^f_t (see [27, Section 3]), the conclusion follows from Corollary 4.6. Here we give an alternative and more elementary proof without using the concept of invariant measures. As in the proof of Corollary 4.6, clearly (B1) holds, and then it suffices to prove that condition (B2) holds, which is implied by the following claim.

Claim For each γ^i, there is $\epsilon > 0$ such that for $g \in C_L$ satisfying (2) and $x \in P^0_M$ with $d(x, \gamma^i) < \epsilon$ there exists $t > 0$ such that $d(\phi^g_t(x), \gamma^i) \geq \epsilon$.

Proof of Claim: Without loss of generality suppose that $u(t) = u(t+T) = u^i(t) = (0, \cdots, 0, u_l(t), \cdots, u_n(t))$ with $u_j(t) > 0$ for all t. We will argue the case when γ is a nontrivial periodic orbit ($T > 0$) as the case for an equilibrium is simpler. Set $\lambda = T^{-1} \int_0^T f_k(u(s))ds > 0$ where $k < l$ is an index as in (D3)(ii) above. Let K be a common Lipschitz constant for f and F on P_M. Choose $\epsilon > 0$ such that

$$\epsilon[1 + K(1 + MT)\exp(KT)] < \lambda/2.$$

A standard Gronwall argument shows that if $d(x, \gamma) < \epsilon$, so $\|x - u(s)\| < \epsilon$ for some $s \in [0, T)$, and (2) holds, then

$$\|x(t) - u(t+s)\| \leq \epsilon(1 + MT)\exp(KT), \ 0 \leq t \leq T.$$

Here, we have simplified notation by setting $x(t) = \phi^g_t(x)$ and we use that $\sup_{x \in P_M} \|F(x) - G(x)\| < \epsilon M$. Now, suppose by way of contradiction that $d(x(t), \gamma) < \epsilon$ for all $t \geq 0$.

The inequality

$$\begin{aligned} g_k(x(t)) &\geq f_k(u(s+t)) - |g_k(x(t)) - f_k(x(t))| - |f_k(x(t)) - f_k(u(t+s))| \\ &\geq f_k(u(t+s)) - \epsilon - \epsilon K(1+MT)\exp(KT) \\ &\geq f_k(u(t+s)) - \lambda/2, \end{aligned}$$

which holds for $0 \leq t \leq T$, implies that $x_k(t)$ satisfies

$$x_k'(t) \geq x_k(t)[f_k(u(t+s)) - \lambda/2].$$

Integrating, we have,

$$x_k(T) \geq x_k(0)\exp(\lambda T/2).$$

By assumption, $d(x(T), \gamma) < \epsilon$ so we may apply the previous argument again to get $x_k(2T) \geq x_k(0)\exp(2\lambda T/2)$, and by induction, we have that $x_k(nT) \geq x_k(0)\exp(n\lambda T/2)$. As the right hand side increases without bound as n increases we contradict that $d(x(t), \gamma) < \epsilon$ for $t \geq 0$. This proves the claim. Because (B2) holds, our result follows from (the continuous-time version of) Corollary 4.5. ∎

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