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Chapter 06

Fluid Mechanics

6.0 Introduction

Fluid mechanics is a branch of applied mechanics concerned with the static and dynamics of fluid - both liquids and gases. The analysis of the behavior of fluids is based on the fundamental laws of mechanics, which relate continuity of mass and conservation of energy with force and momentum.

There are two aspects of fluid mechanics, which make it different to solid mechanics namely; the nature of a fluid is much different as compared with solid and in fluids it deals with continuous streams of fluid without a beginning or ending. Fluid is a substance, which deforms continuously, or flows, when subjected to shearing forces. If a fluid is at rest there is no shearing force acting.

6.1 Archimedes' Principle

Archimedes' principle states that a system submerged or floating in a fluid has buoyant force F_{buoy} acting on it whose magnitude is equal to that of the weight of the fluid displaced by the system. Buoyancy arises from the increase of fluid pressure with depth and the increase pressure exerted in all directions as stated by Pascal's law. Thus, there is an unbalanced upward force on the bottom of a submerged or floating object. An illustration is shown in Fig. 6.1. Since the "water ball" at left is exactly supported by the difference in pressure and the solid object at right experiences exactly the same pressure environment, it follows that the buoyancy force F_{buoy} on the solid object upward is equal to the weight of the water displaced according to Archimedes' principle.

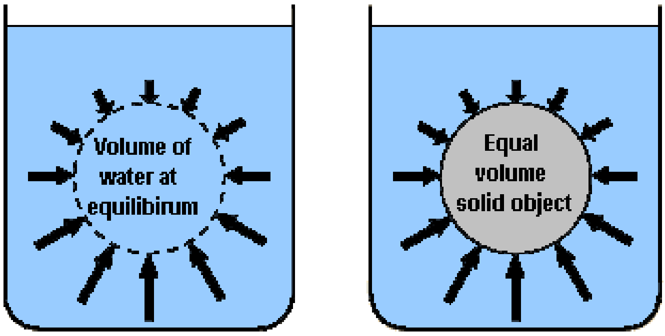


Figure 6.1: Illustration of buoyancy follows Archimedes' principle

Supposing that the volume of the water equivalent is V m^3, the buoyancy force F_{buoy} is equal to $V\rho_{water}g$. If the object has mass m, the apparent weight of the solid object shall be mg - $V\rho_{water}g$.

Let's consider the general case where an object has density ρ and volume V. Its weight W is equal to ρVg. When the object submerged in the fluid of density ρ_{fluid} and displaced a volume V' then the buoyant force F_{buoy} is $F_{buoy} = \rho_{fluid}V'g$. If the volume V is equal to V' and the object sinks to the bottom, this shall be mean $W > F_{buoy}$ and also implies that the density ρ of the object is larger than the density ρ_{fluid} of fluid. Likewise, the object floats. It implies that the density of object is smaller than the density of fluid.

If the object is neither sink nor float like a fish swimming in the sea, $\rho_{fluid}V'g = \rho Vg$. For ship that float on water, the condition $\rho_{fluid}V'g = \rho Vg$ is also satisfies. However, the volume is $V' < V$ because only a small portion of volume of ship is submerged in water. This implies that the density of ship is less than density of water.

Example 6.1
A rock is suspended from a spring scale in air and found to be weight of magnitude w. The rock is then submerged completely in water while attached to the scale. The new reading of scale is w_{sub}. Find the expression for the density ρ_{rock} of rock in terms of the scale readings and density of water ρ_{water}.

Solution
The weight of rock in air is $W = \rho_{rock}Vg$. The buoyant force is $F_{buoy} = \rho_{water}Vg$. The submerged weight is $w_{sub} = \rho_{rock}Vg - \rho_{water}Vg$. Thus, the ratio of submerged weight and weight in air is $\frac{w_{sub}}{w} = \frac{\rho_{rock}Vg - \rho_{water}Vg}{\rho_{rock}Vg}$. This implies that the density of rock $\rho_{rock} = \frac{w\rho_{water}}{w - w_{sub}}$.

6.1.1 Center of Buoyancy

The center of buoyancy for floating and submerged object would determine the stability of the system. For stability or equilibrium, everybody knows that the net force and net torque should be zero, which are $F_{net} = 0$ and $\tau_{net} = 0$. Therefore, for an equilibrium system, the center of mass COM and center of buoyancy COB should lie in same vertical line.

For a total submerged system such as a submarine, the center of mass COM should lie below center of buoyancy COB since the submarine is designed such that it is heavier at the bottom. If the submarine is tilted toward right, the COB is shifted toward right. The torque out of the page with respect to COM is restoring the tilt to equilibrium position. The illustration is shown in Fig. 6.2.

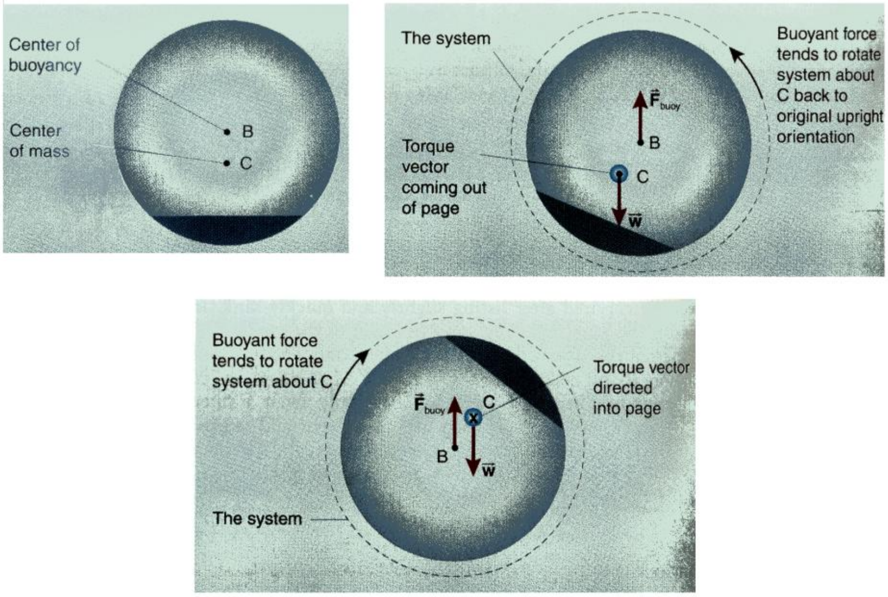


Figure 6.2: Center of buoyancy for a total submerged system

For the floating system such as the aircraft carrier, the COB is below the COM as shown in Fig. 6.3. If there is a tilt, the COB is move toward right align the metacenter. A restoring torque will move the aircraft carrier back to equilibrium.

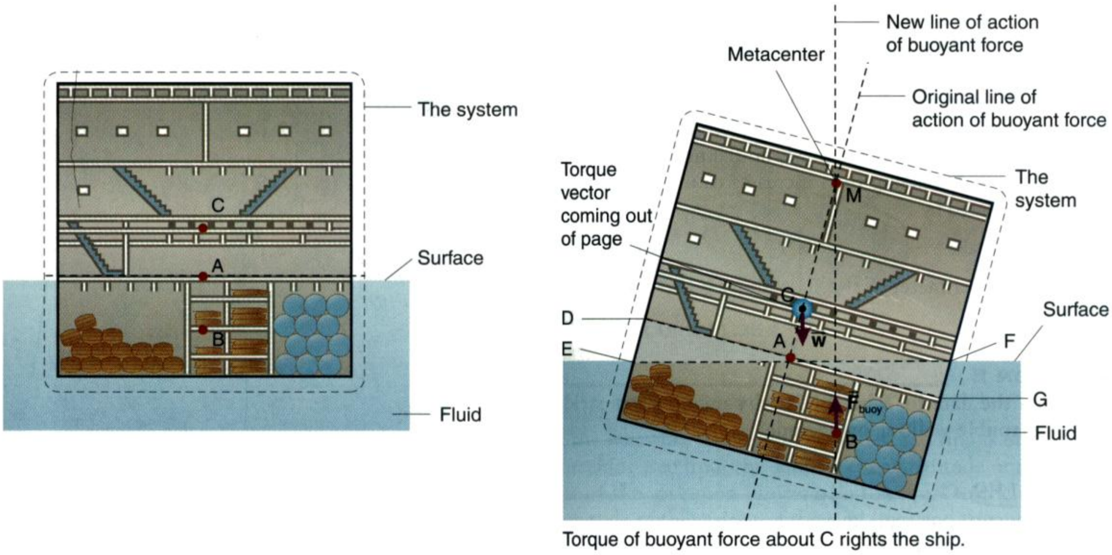


Figure 6.3: Center of buoyancy for a floating system

6.2 Newton's Law of Viscosity

Liquid and gas are both fluids cannot resist deformation force. As it flows, it is under the action of the force. Its shape will change continuously as long as the force is applied. The deformation is caused by shearing forces, which act tangentially to a surface as shown in Fig. 6.4. The force F acting tangentially on a rectangular (solid lined) element ABDC causes deformation that produces the dashed lined rhombus element a'b'c'd'.

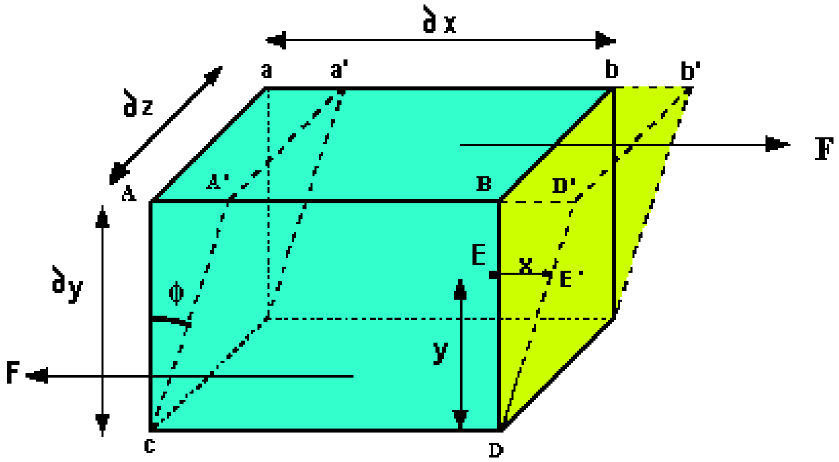


Figure 6.4: Shearing force acts on liquid

When a fluid is in motion shear stresses are developed if the particles of the fluid move relative to one another. When this happens adjacent particles have different velocities. If velocity of fluid is the same at every point then there is no shear stress produced and particles have zero *relative* velocity. An example is the flow of water in the pipe where at the wall of the pipe, the velocity of the water is zero. The velocity increases toward the centre of the pipe as its profile is shown in Fig. 6.5.

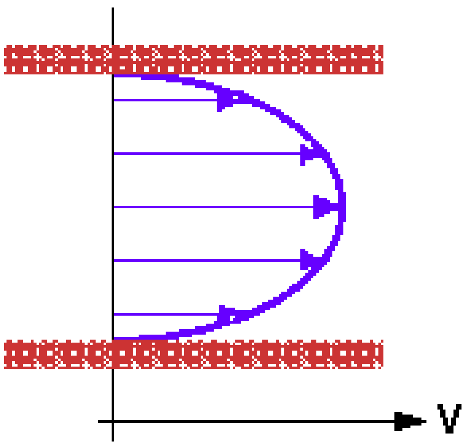


Figure 6.5: The velocity profile of water flow in the pipe

The shearing force F acts on the area on the top of the element. This area is given by A = δzδx. The shear stress σ is equal to force per unit area i.e. $\tau = \frac{F}{A}$.
The tan ϕ is the shear strain ε, which is defined as x/y. The rate of shear strain shall be equal to dε/dt, which is also equal to

$$\frac{d\varepsilon}{dt} = \frac{d(x)}{ydt} = \frac{u}{y} \tag{6.1}$$

where u is the velocity of the fluid particle at point E and u/y shall be the velocity gradient. In the differential for u/y shall be written as du/dy.

The result of experiment has shown that the shear stress τ is proportional to the rate of change of shear strain.

$$\tau = \mu \frac{du}{dy} \tag{6.2}$$

The constant of proportionality is known as coefficient of dynamic viscosity μ. This is also known as Newton's law of viscosity. The coefficient of dynamic viscosity μ is defined as the shear force per unit area or shear stress τ, required dragging one layer of fluid with unit velocity past another layer a unit distance away. For fluid that has constant viscosity shall be called Newtonian fluid, otherwise it is a non-Newtonian fluid. Figure 6.6 shows the typical viscosity of some fluids.

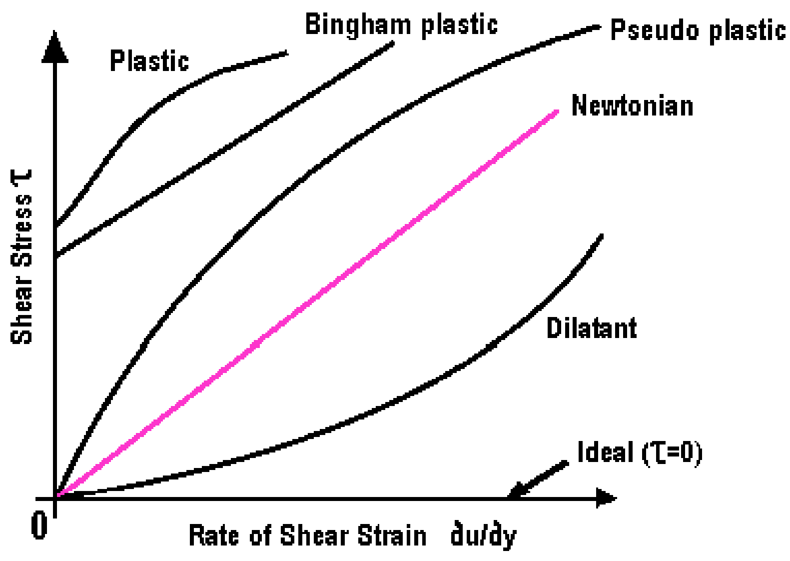


Figure 6.6: Viscosity of some fluids

From the results show in Fig. 6.6, the viscosity μ generally follows equation (6.3).

$$\tau = A + B\left(\frac{du}{dy}\right)^n \qquad (6.3)$$

where A, B and n are constants. For Newtonian fluids A = 0, B = m and n = 1.

The coefficient of dynamic viscosity of water is $1.14x10^{-3}kgm^{-1}s^{-1}$, air is $1.78x10^{-5}kgm^{-1}s^{-1}$, mercury is $1.55kgm^{-1}s^{-1}$, and paraffin oil is $1.9kgm\text{-}^{1}s^{-1}$.

6.2.1 Viscosity and Temperature

There is some molecules interchange between adjacent layers in liquids. But the molecules are much closer than in gas that their cohesive forces hold them in place much more rigidly. Thus, it reduces the molecules exchange. This cohesion plays an important role in the viscosity of liquid.

As the temperature of a fluid increases, it reduces the cohesive force and increases the molecular interchange. Reducing cohesive forces reduces shear stress, while increasing molecules interchange increases shear stress. Thus, one can see there is a complex relationship between molecules exchange and cohesive force on viscosity. In general the reduction of cohesive force is more than increase of molecules exchange. Thus, the viscosity of liquid is decreased as temperature increases.

High pressure can also change the viscosity of a liquid. As pressure increases the relative movement of molecules requires more energy hence viscosity increases.

The molecules of gas are only weakly bounded by cohesive force between molecules, as they are far apart. Between adjacent layers, there is a continuous exchange of molecules. Molecules of a slower layer move to faster layers causing a drag, while molecules moving the other way exert an acceleration force. Mathematical considerations of this momentum exchange can lead to Newton law of viscosity.

If temperature of a gas increases, the momentum exchange between layers will increase thus increasing viscosity. It can be viewed as the temperature increases, it reduces the cohesive force further increase more molecules exchange that increases the viscosity.

Viscosity will also change with pressure - but under normal condition this change is negligible in gasses.

Kinematics viscosity υ is defined as the ratio of dynamic viscosity to mass density, which is

$$\nu = \frac{\mu}{\rho} \tag{6.4}$$

The unit for kinematics viscosity is Stoke, whereby 1 stokes ST = $1.0x10^{-4}m^2s^{-1}$.

Example 6.2
The density of oil is $850kg/m^3$. Find its relative density and Kinematics viscosity if the dynamic viscosity is $5.0x10^{-3}kg/ms$.

Solution
The relative density of fluid is defined as the rate of its density to the density of water. Thus, the relative density of oil is 850/1000 = 0.85.

Kinematics viscosity is defined as the ratio of dynamic viscosity to mass density, which is $5.0x10^{-3}/0.85 = 5.88x10^{-3}\ m^2/s$ = 58.8 ST.

6.3 Pressure Measurement by Manometer

In this section, various types of manometers for pressure measurement shall be discussed and analyzed.

Pressure is the ratio of perpendicular force exerted to an area. Thus, pressure has dimension of Nm^{-2}, in which $1.0Nm^{-2}$ is also termed as one pascal. i.e. $1.0Nm^{-2}$ = 1.0Pa. One atmospheric pressure 1.00atm is equal to $1.013x10^5$Pa. Another commonly use scale for measuring the pressure is bar in which 1.0 bar is equal to $1.0x10^5Nm^{-2}$.

The simplest manometer is a tube, open at the top attached to the top of a vessel containing liquid at a pressure (higher than atmospheric pressure) to be measured. An example can be seen in Fig. 6.7. This simple device is known as a *piezometer tube*. As the tube is open to the atmosphere, the pressure measured is relative to atmospheric pressure is called *gauge pressure*.

The pressure at point A is $\rho g h_1 + P_0$, where P_0 is the atmospheric pressure. Similarly, the pressure at point B is $P_0 + \rho g h_2$. Pressure $\rho g h_1$ and $\rho g h_2$ are termed as *gauge pressure*.

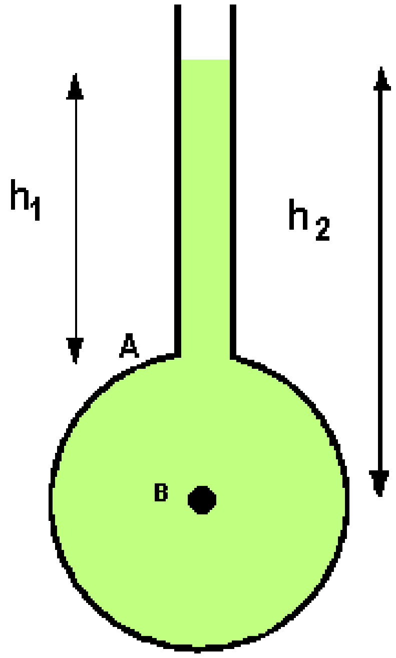


Figure 6.7: A piezometer tube manometer

"U"-tube manometer enables the pressure of both liquids and gases to be measured with the same instrument. The "U" tube manometer is shown in Fig. 6.8 filled with a fluid called the *manometric fluid*. The fluid whose pressure is being measured should have a density less than the density of the manometric fluid and the two fluids should be immiscible, which does not mix readily.

Pressure at point B and C are the same. Pressure P_B at point B is equal to pressure P_A at point A plus $\rho g h_1$ i.e. $P_B = P_A + \rho g h_1$. The pressure at point C is equal to atmospheric pressure P_{atm} plus $\rho_{man} g h_2$ i.e. $P_C = P_{atm} + \rho_{man} g h_2$. Equating the pressure at point A and C should yield expression $P_A + \rho g h_1 = P_{atm} + \rho_{man} g h_2$. If the density of fluid to be measured is much lesser than density of manometric fluid than pressure at point A is approximately equal to $P_A = P_{atm} + \rho_{man} g h_2$.

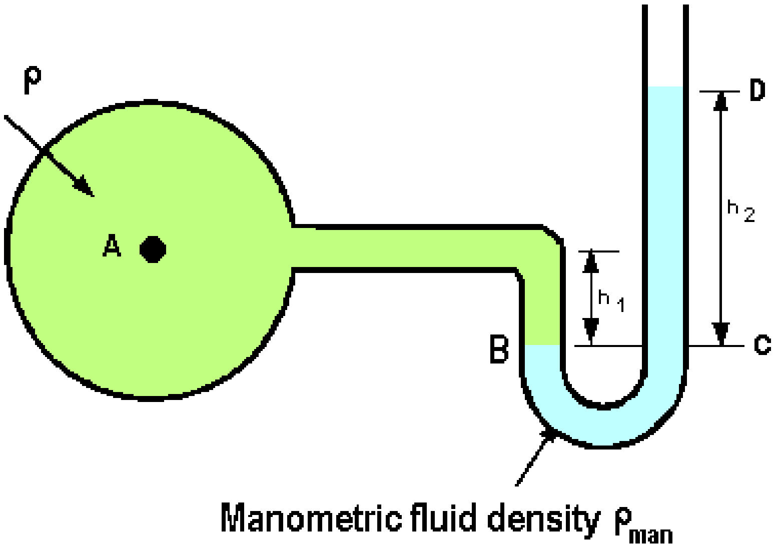


Figure 6.8: A "U" tube manometer

Pressure difference can be measured using a "U"-Tube manometer. The "U"-tube manometer is connected to a pressurized vessel at two points the pressure difference between these two points can be measured as shown in Fig. 6.9.

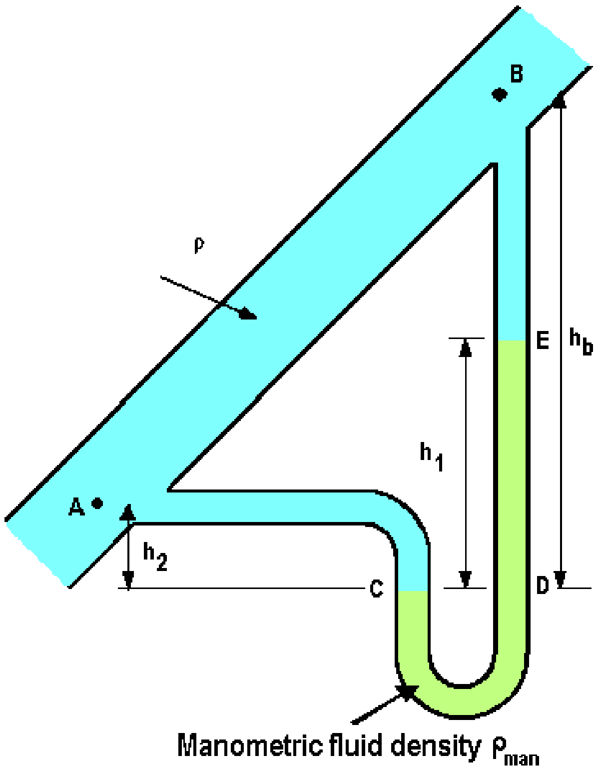


Figure 6.9: Pressure difference measurement by the "U"-Tube manometer

Point C and D have same pressure. The pressure at point A is $P_A = P_C - \rho h_2 g$. Pressure point B is $P_B = P_D - \rho_{man} h_1 g - \rho(h_b - h_1)g$. This shall mean that pressure difference between point A and point B is $P_A - P_B = P_C - \rho h_2 g - P_D + \rho_{man} h_1 g + \rho(h_b - h_1)g = \rho g(h_b - h_2) + h_1 g(\rho_{man} - \rho)$.

The advanced "U" tube manometer is used to measure the pressure difference ($P_1 - P_2$) that has the manometer shown in Fig. 6.10.

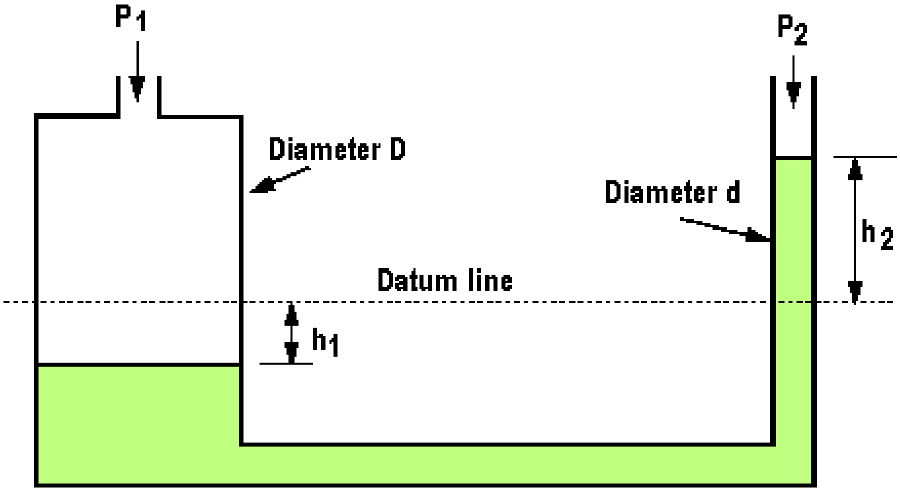


Figure 6.10: An advanced "U" tube manometer

When there is no pressure difference, the level of the manometric fluid shall be stayed at datum line. The volume of level decrease in the left hand side shall be equal to the volume of level raised in right hand-side. This implies that $\pi\left(\frac{D}{2}\right)^2 h_1 = \pi\left(\frac{d}{2}\right)^2 h_2$ and $h_1 = \left(\frac{d}{D}\right)^2 h_2$, Thus, the pressure difference ($P_1 - P_2$) shall be $h_1\rho_{man}g + h_2\rho_{man}g = \left(\frac{d}{D}\right)^2 h_2\rho_{man}g + h_2\rho_{man}g = h_2\rho_{man}g\left[1+\left(\frac{d}{D}\right)^2\right]$.

6.4 Static Fluid

Fluid is said to be static if there is *no shearing force* acting on it. Any force between the fluid and the boundary must be acting at right angle to the boundary. Figure 6.11 shows the condition for fluid being static.

This definition is also true for curved surfaces as long as the force is acting perpendicular to the surface. In this case the force acting at any point is normal to the surface at that point as shown in Fig. 6.11. The definition is also true for any imaginary plane in a static fluid.

For any particle of fluid at rest, the particle will be in equilibrium - the sum of the components of forces in any direction will be zero i.e. net force = 0. The sum of the moments of forces on any particle about any point must also be zero. i.e. net torque = 0.

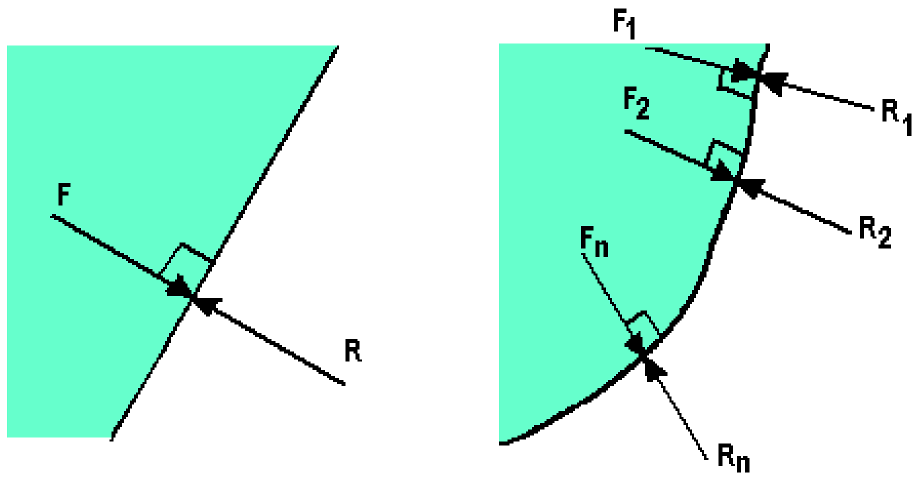


Figure 6.11: An illustration to show static fluid

Since at static condition, the force is acting perpendicular to the surface of contact, which can be different for different contact area, thus, it is convenient to use force per unit area, which is termed as *pressure*.

6.4.1 Pascal's Law for Pressure at a Point

Pascal's law of pressure states that at a particular point P, pressure acts on it equal in all directions. Let's take a point P in the fluid be denoted by a small element of fluid in the form of a triangular prism shown in Fig. 6.12.

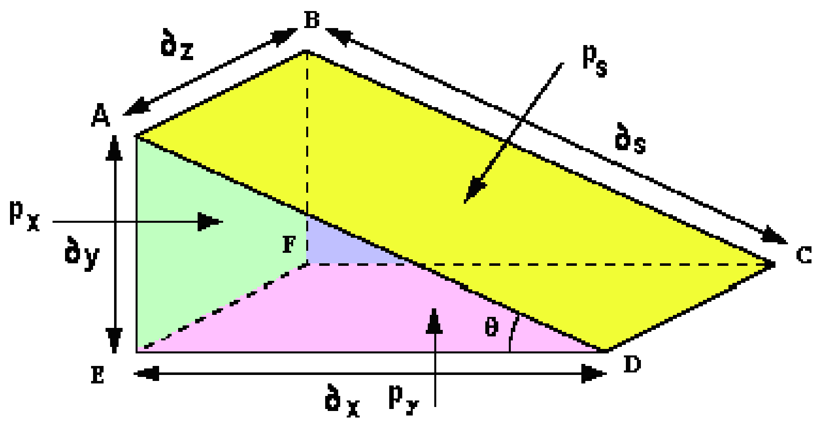


Figure 6.12: Pressure component acting on a point in the fluid

The pressures are pressure p_x in the x direction, p_y in the y direction, and p_s in the direction normal to the sloping face. Since the net force is equal to zero at static condition, the net force acting on both x and y directions should be zero.

For force acting in x-direction, $p_x\delta y\delta z = p_s\delta_s\delta_z\sin\theta = p_z\delta_s\delta_z\frac{\delta y}{\delta s} = p_z\delta_y\delta_z$. This result implies that $p_x = p_s$. For force acting on y-direction, the force relationship is $p_s\delta_z\delta_s\cos\theta + \frac{1}{2}\delta_z\delta_x\delta_y\rho g = p_y\delta_z\delta_x = p_s\delta_z\delta_s\frac{\delta x}{\delta s} + \frac{1}{2}\delta_z\delta_x\delta_y\rho g$, where $\frac{1}{2}\delta_z\delta_x\delta_y\rho g$ is the weight of prism. The pressure p_s is equal to p_y since $p_s = p_y$ since $\frac{1}{2}\delta_z\delta_x\delta_y\rho g$ is approximately equal to zero. Combining the result above, pressure acting on a point P is equal in all directions since $p_x = p_y = p_s$.

6.4.2 Variation of Pressure Vertically in Fluid under Gravity

There is pressure variation when fluid under gravity, which shall mean that the pressure at different height is different. Let's use Fig. 6.13 to derive the equation of pressure of different height of fluid under gravity.

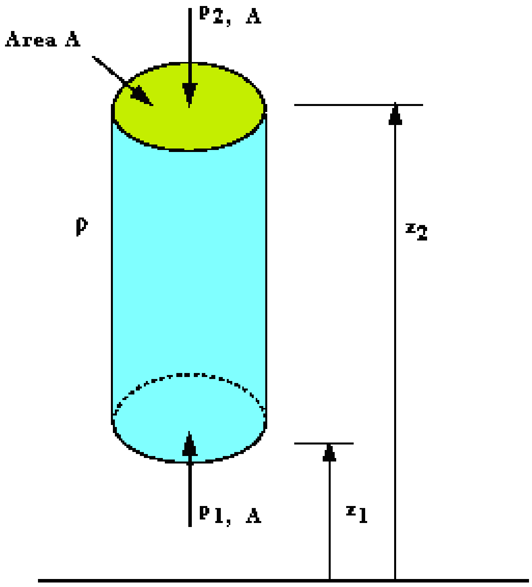


Figure 6.13: Pressure at different height in static fluid

The force F_1 acting upward at the bottom is $F_1 = p_1A$. The F_2 acting down from the top is $F_2 = p_2A$. The weight of cylindrical volume of fluid is also acting downward, which is $\rho gA(z_2 - z_1)$. At static equilibrium, the net force is equal to zero. Therefore $F_2 + \rho gA(z_2 - z_1) = F_1$, which shall mean

$$p_2 + \rho g\,(z_2 - z_1) = p_1 \quad (6.5)$$

6.4.3 Equality of Pressure at Same Level in Static Fluid

Pressure at the same level in static fluid is the same. Let's use Fig. 6.14 to prove the point.

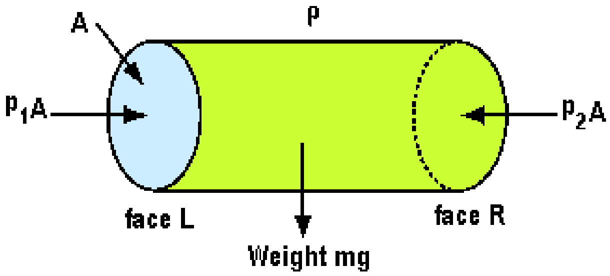


Figure 6.14: Pressure at same level in static fluid

The net horizontal force is equal to zero. This shall mean that $p_1A = p_2A$. This implies that pressure at same level is the same. This result is the same for any *continuous* fluid such as the case where two connected tanks, which appear not to have all directions connected.

6.4.4 General Equation for Variation of Pressure in Static Fluid

Based on the above two cases mentioned in Section 6.4.2 and 6.4.3, the variation of pressure in static fluid can be derived based the situation shown in Fig. 6.15.

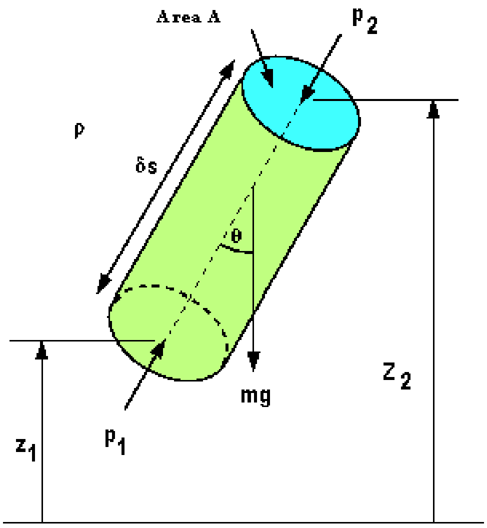


Figure 6.15: The variation of pressure in fluid

The weight of the cylindrical fluid along center axis is $\rho gA\delta s\cos\theta$. The force by pressure p_1 perpendicular to area A is p_1A and the force acting by pressure p_2 is p_2 perpendicular to area A is p_2A. At static equilibrium,

$$\rho g A \delta s \cos\theta = p_1 A - p_2 A \qquad (6.6)$$

where $\delta s = (z_2 - z_1)/\cos\theta$. For the same level case, $\theta = 90^0$, then $\rho g A \delta s \cos\theta = 0$, implying $p_1 = p_2$. For different level vertically, $\theta = 0^0$, $\cos\theta = 1$, $\rho g A \delta s \cos\theta = \rho g A(z_2 - z_1)$ implies that $\rho g A \delta s \cos\theta = p_1 = \rho g \delta(z_2 - z_1) + p_2$, the different level case.

Example 6.3
Find the height of column of water exerted by pressure of $500\text{x}10^3\text{Nm}^{-2}$ giving that the density of water is $1{,}000\text{kgm}^{-3}$.

Solution
The height of the column is $h = p/(\rho g) = 500\text{x}10^3/(1000\text{x}9.8) = 50.95\text{m}$.

6.5 Fluid Dynamics

There is motion in fluid, which shall mean the shearing force is not zero. The motion of fluid can be studied in the same way as the motion of solids using the fundamental laws of physics together with the physical properties of the fluid. In study of fluid dynamic, the term uniform, non-uniform, steady, and unsteady flows are used. Uniform flow shall mean the velocity is same at every point in the stream. Steady flow means the conditions such as pressure, velocity, and cross section area of flow may differ from point to point but do not change with time. Based on the definition, the flow of fluid can be classified into four categories, which are

1. *Steady uniform flow*. Conditions such as velocity, pressure, and cross-section of flow do not change with time. An example is the flow of water in a pipe of constant diameter at constant velocity.
2. *Steady non-uniform flow.* Conditions such as velocity, pressure, and cross section of flow change from point to point in the stream but do not change with time. An example is flow in a tapering pipe with constant velocity at the inlet - velocity will change as you move along the length of the pipe toward the exit.
3. *Unsteady uniform flow*. At a given instant in time the conditions at every point are the same, but will change with time. An example is a pipe of constant diameter connected to a pump pumping at a constant rate, which is then switched off.
4. *Unsteady non-uniform flow.* Every condition of the flow may change from point to point and with time at every point. For example waves in a channel.

In our study of fluid dynamic, we shall restrict ourselves for the steady uniform flow case.

6.5.1 Equation of Flow Continuity

Mass rate flow of the fluid is a measure of fluid out of the outlet per unit time. For example an empty bucket weighs 5.0kg. After 10 seconds of collecting water, the bucket weighs 9.5kg, the mass flow rate is $\left(\frac{9.5-5}{10}\right)$ = 0.45kgs^{-1}.

Volume flow rate Q is defined as the volume fluid discharge per unit time or discharge rate. Using the example above, the volume flow rate shall be $0.45/1{,}000 = 0.45\times10^{-3}\text{ms}^{-1}$. If the cross sectional area A of a pipe and the mean velocity u_m are known, then the volume rate flow Q is

$$Q = Au \tag{6.7}$$

The principle of conservation of mass shall be applied for non-compressible and compressible fluid. This shall mean the mass rate Q_1 enter into tube is equal to mass rate Q_2 out of the tube. i.e. $Q_1 = Q_2$. Applying this principle to the case of a streamline flow shown in Fig. 6.16, equation (6.8) is obtained

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 = \text{constant} \tag{6.8}$$

Equation (6.8) is also termed as continuity equation. For incompressible fluid, it has same density then $\rho_1 = \rho_2 = \rho$. Equation (6.8) becomes $A_1 u_1 = A_2 u_2$.

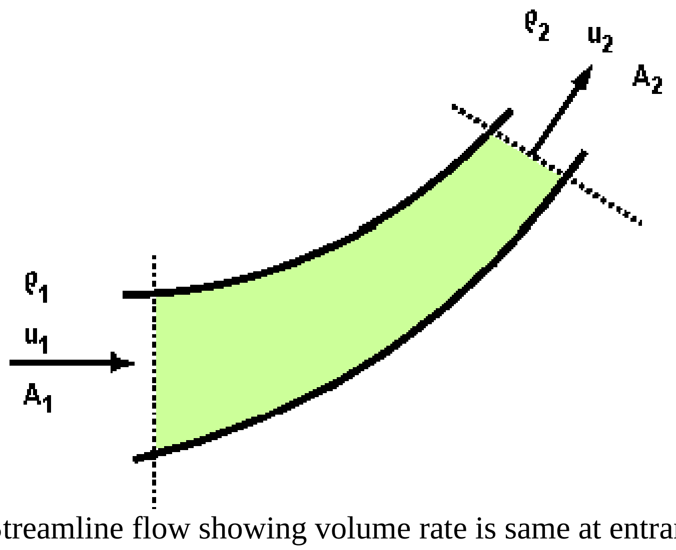
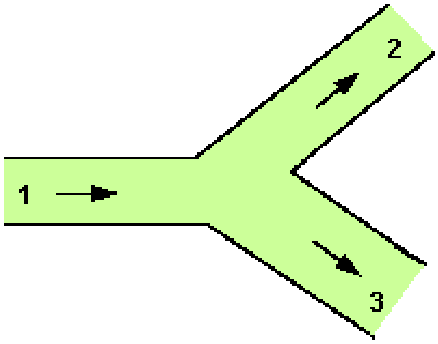


Figure 6.16: Streamline flow showing volume rate is same at entrance and outlet

Example 6.4
An uncompressible fluid flows into pipe 1 and distributes via pipe 2 and pipe 3 as shown in figure below. Pipe 1 has diameter 50mm and mean velocity 2.0m/s. Pipe 2 has diameter 40mm and it takes 30% of total discharge per sec. Pipe 3 has diameter 60mm. What are the values of discharge and mean velocity for pipe 2 and pipe 3?



Solution
Using the conservation of mass, the discharge rate Q_1 entering pipe 1 shall be equal to sum of mass rate in pipe 2 and pipe 3. i.e. $Q_1 = Q_2 + Q_3$. The discharge rate of pipe 1 shall be $\pi\left(\frac{d^2}{4}\right)u_1 = \pi\left(\frac{50x10^{-3}}{2}\right)^2 2 = 3.93x10^{-3}m^3/s$.

The discharge rate of pipe 2 shall be $1.18x10^{-3}$ m^3/s and discharge rate of pipe 3 shall be $2.75x10^{-3}m^3/s$.

The mean velocity of pipe 2 shall be $1.18x10^{-3}/\left(\pi\frac{(40x10^{-3})^2}{4}\right) = 0.939m/s$.

The mean velocity of pipe 3 shall be $2.75x10^{-3}/\left(\pi\frac{(60x10^{-3})^2}{4}\right) = 0.973m/s$.

6.5.2 Work Done and Energy

From the law of conservation of energy, it states that sum of kinetic energy KE and gravitational potential energy PE is constant. i.e. KE + PE = constant. If the fluid drop is falling from rest at the height h above the ground, its initial KE is zero and its PE is equal to mgh. The KE and PE when it touches the ground is $\frac{1}{2}mV^2$ and zero respectively. Thus, by conversation of energy mgh = $\frac{1}{2}mV^2$.
From Kinetic energy-work done theorem, $\Delta KE =$ net work done. This should mean that sum of the change of kinetic energy and net work done W_{net} is a

constant. i.e. $\Delta KE + W_{net}$ = constant. For the case of fluid, the net work done can be treated as volume multiplies by change of pressure ΔP, which is $W_{net} = V\Delta P$. This shall mean

$$\Delta P + \Delta KE/\text{Volume} = \text{constant} \qquad (6.9)$$

Equation (6.9) is known as Bernoulli's principle, which states that, an increase of pressure in the flowing fluid always resulting in decreasing of speed of fluid and vice versa. The principle has been demonstrated in our daily activity like the shower curtain get suck inwards when the water is first turned-on. Squeezing the bulb of a perfume bottle creating high speed of the perfume fluid reducing the pressure of the air subsequently draws the fluid-up. The window of the house tends to explode during the hurricane because the high-speed hurricane creates low pressure surrounding the house. The high pressure in the house pushes the window outward. The foil of the aircraft wing lifts the aircraft because the high speed airflow on top of the wing.

Let's derive the equation for water jet as shown in Fig. 6.17 using equation (6.8). The change in kinetic energy is

$$\frac{1}{2}u_2^2 - \frac{1}{2}u_1^2 + z_2 g - z_1 g = 0 \qquad (6.10)$$

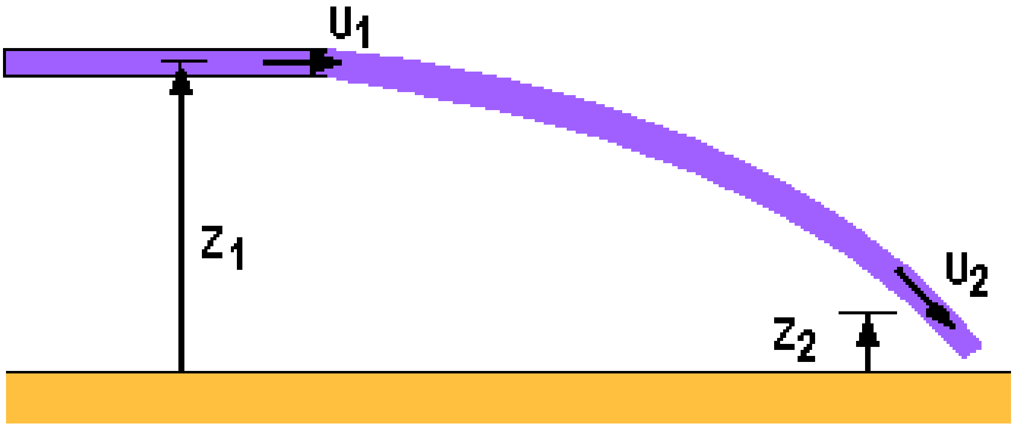


Figure 6.17: Water jet

The flow from reservoir as shown in Fig. 6.18, the initial kinetic energy is zero. Using the conversation energy, the final velocity of the water jet shall be

$$u_2 = \sqrt{2g(z_1 - z_2)} \qquad (6.11)$$

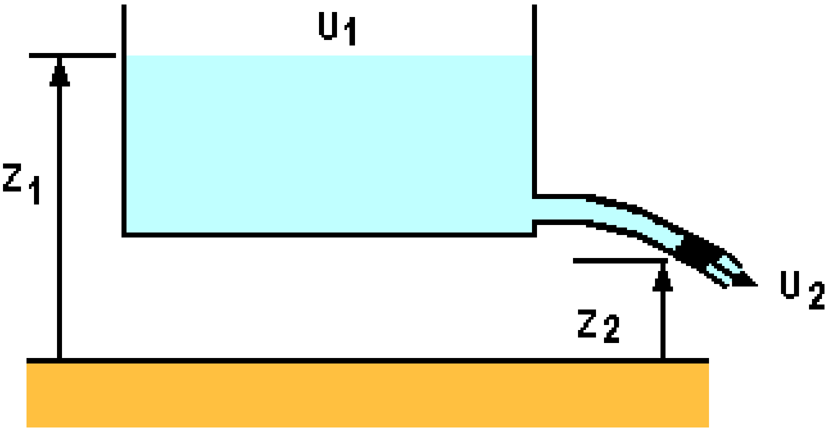


Figure 6.18: Flow from a reservoir

The examples considered above have condition of constant pressure with different velocity. Let's consider the case where there is variation of pressure and constant velocity such as the case shown in Fig. 6.19.

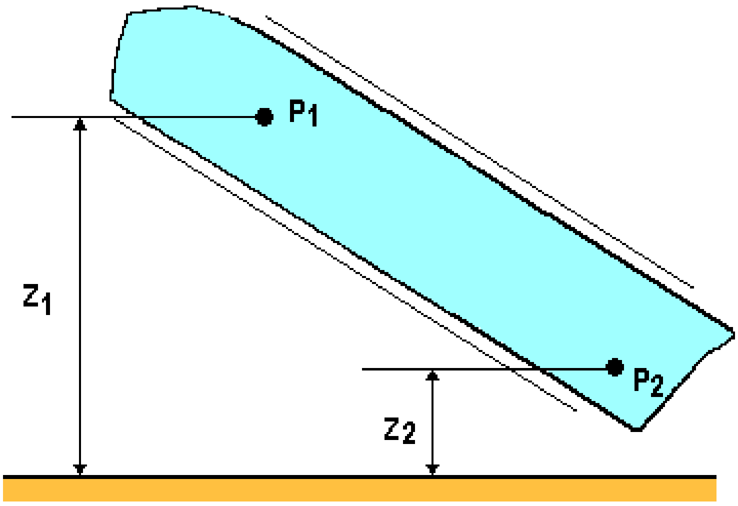


Figure 6.19: Fluid flow at different pressure

Pressure at point P_2 is equal to pressure at point P_1 plus the pressure difference which is $(z_1 - z_2)\rho g$. Therefore, the expression of pressure P_2 is $P_2 = P_1 + (z_1 - z_2)\rho g$. Rearrange this equation shall yield,

$$\frac{P_1}{\rho} + gz_1 = \frac{P_2}{\rho} + gz_2 \qquad (6.12)$$

For the case where there is variation of pressure and velocity, then combining equation (6.10) and (6.12) would yield the Bernoulli's equation (6.13).

$$\frac{P_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{P_2}{\rho g} + \frac{u_2^2}{2g} + z_2 \qquad (6.13)$$

6.6 Bernoulli's Equation

In this section, we shall begin with the derivation of Bernoulli's equation. Subsequently, the application using the Bernoulli's equation shall be discussed. The assumptions underlying the derivation of Bernoulli's equation are steady flow, density is constant, friction losses are negligible, and the streamline single type, which mean constant velocity.

In deriving the Bernoulli's equation, the principle of conservation of energy shall be used. This shall mean that at a point in fluid, the sum of work done by pressure P, kinetic energy KE of the fluid, and the potential energy PE shall be constant.

Let's consider a small element of the fluid of weight mg and cross sectional area *a* flows from section AB to section A'B' with velocity *u* and is situated at the height z from the reference line shown in Fig. 6.20.

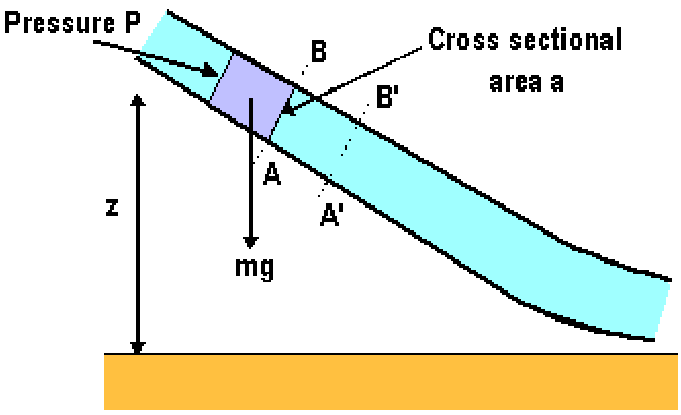


Figure 6.20: Derivation of Bernoulli's equation

The potential energy of the fluid element is mgz. The potential energy per unit weight shall be z, which is also named as *potential head*.

The kinetic energy of the element is $\frac{1}{2}mu^2$. The kinetic energy per unit weight shall be $\frac{u^2}{2g}$, which is also named as *velocity head*.

The force at section AB shall be Pa. When the element of weight mg moves from section AB to A'B', the volume that shall be $\frac{mg}{\rho g} = \frac{m}{\rho}$ and the distance traveled shall AA' or BB' which is equal to $\frac{m}{\rho a}$.

The work done by the element of fluid moves from AB to A'B' shall be the force multiplies by the distance AA'. This shall mean work done is equal to $Pa\frac{m}{\rho a} = \frac{Pm}{\rho}$. The work done per unit weight shall be $\frac{P}{\rho g}$, which is also named as *pressure head*.

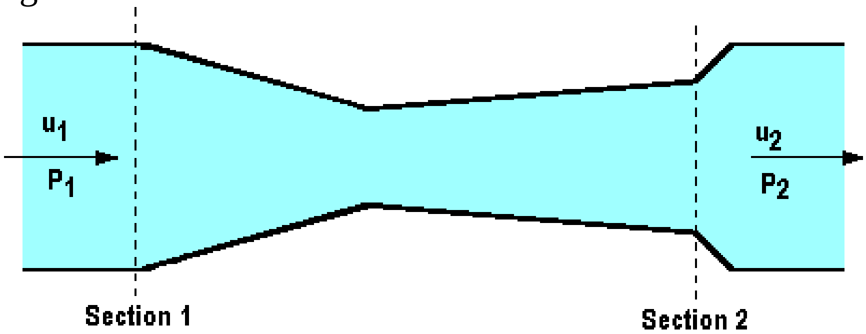
From conservation of energy, Bernoulli's equation shall be

$$\frac{P}{\rho g} + \frac{u^2}{2g} + z = \text{constant} = H \qquad (6.14)$$

where H is the total head.

Example 6.5

A fluid of constant density $960 kgm^{-3}$ is flowing steadily through a tube as shown in the figure. The diameters at the section 1 and section 2 are d_1 = 100mm and d_2 = 80mm respectively. The pressure gauge and velocity at section 1 are $200x10^3 Nm^{-2}$ and $5.0 ms^{-1}$ respectively. Determine the velocity and pressure gauge value at section 2.



Solution

Since the tube is horizontally placed $z_1 = z_2$ and Bernoulli's equation shall be $\frac{P_1}{\rho g} + \frac{u_1^2}{2g} = \frac{P_2}{\rho g} + \frac{u_2^2}{2g}$. To know the speed at section 2, the continuity shall be used to determine it, which is $u_1 A_1 = u_2 A_2$. This implies that $u_2 = \left(\frac{d_2}{d_1}\right)^2 u_1$ =

$5.0(80/100)^2 = 3.2ms^{-1}$. The pressure at section 2 shall be $P_2 = \rho\left(\frac{P_1}{\rho} + \frac{u_1^2}{2} - \frac{u_2^2}{2}\right)$. This shall mean that the pressure P_2 is $207.0x10^3Nm^{-2}$.

Let's do an analysis for various types of Bernoulli's heads by considering a reservoir that feeds water to the households through a pipe that has different diameter and rising over hill and going down the hill, and finally reaching the household level as shown in Fig. 6.21. The pressure at various point 1 to 4 has the relative magnitude order $P_4 > P_2 > P_3 > P_1$.

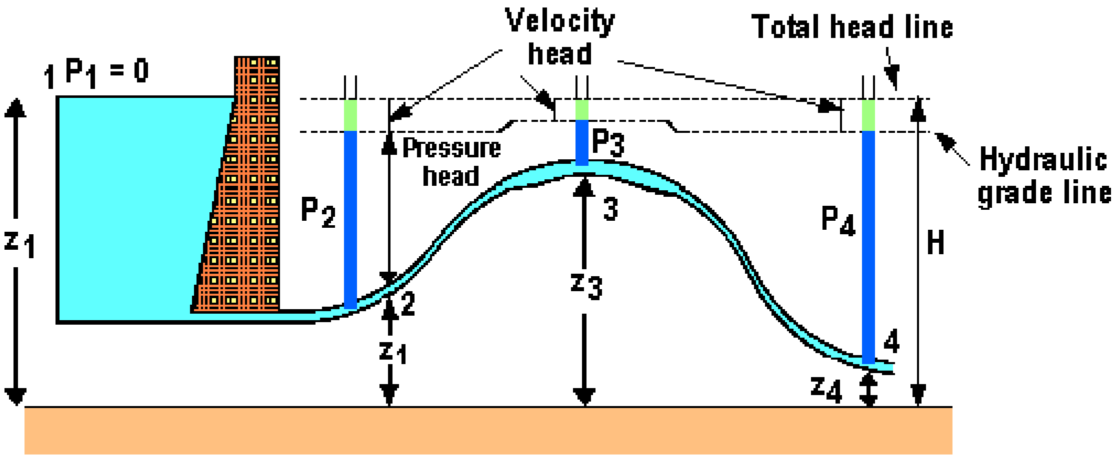


Figure 6.21: Analysis of Bernoulli's heads

The analysis shall be based on conservation of energy whereby the total head H is a constant, which is also Bernoulli's equation. At point 1, the total head H is consist of the potential head z_1, since the gauge pressure is zero and the velocity of water on the surface of dam is zero because the movement of water is practical at still.

If the tap at the household end is shut, then the velocity head at point 2, 3, and 4 shall be zero since the water in the pipe is in static condition. The total head H_2 at point 2 shall be equal to the sum of potential head z_2, pressure head $\frac{P_2'}{\rho g}$. i.e. $H_2 = z_2 + \frac{P_2'}{\rho g}$. The total head H_3 at point 3 shall be $H_3 = z_3 + \frac{P_3'}{\rho g}$ and the total head at point 4 shall be $H_4 = z_4 + \frac{P_4'}{\rho g}$. By Bernoulli's equation $H_1 = H_2 = H_3 = H_4$.

If the tap is open at the household, then the velocity will not be zero. The magnitude of pressure at point 2, 3, and 4 shall be lower based Bernoulli's principle. The total head at point 2 shall be $H_2 = z_2 + \frac{u_2^2}{2g} + \frac{P_2}{\rho g}$, at point 3 shall be $H_3 = z_3 + \frac{u_3^2}{2g} + \frac{P_3}{\rho g}$, and at point 4 shall be $H_4 = z_4 + \frac{u_4^2}{2g} + \frac{P_4}{\rho g}$.

Take note that according to continuity equation, velocity head should be equal if the diameter of the pipe is the same. The velocity head shall be smaller like the case at point 3 if the diameter of the pipe is large than that at other points.

If there is friction, which true in real case, the total head H shall not be the same at the reservoir point 1 and household end point. The total head at household end shall be $H_4 = H_1 - H_f$, where H_f is the head due to friction.

A number of applications of Bernoulli's equation and continuity equation such as pitot tube, venturi meter, flow through orifice, and etc. shall be discussed here.

6.6.1 Pitot Tube

A pitot tube has a streamline flow into a blunt body as shown in Fig. 6.22. Point 1 and point 2 has same level. This implies that the potential head of both points are the same.

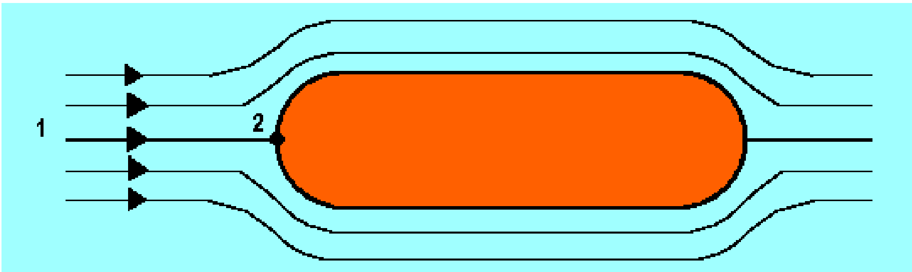


Figure 6.22: A pitot tube

The velocity head at point 1 shall be $\frac{u_1^2}{2g}$, whilst the velocity head at point 2 is zero since the velocity at point 2 is zero. The pressure head at point 1 and 2 shall be $\frac{P_1}{\rho g}$ and $\frac{P_2}{\rho g}$ respectively. From Bernoulli's equation, this shall mean $\frac{u_1^2}{2g} + \frac{P_1}{\rho g} = \frac{P_2}{\rho g}$. This implies that the pressure at point P_2 is equal to $P_1 + \frac{1}{2}\rho u_1^2$.

Note that the increase of pressure to bring the fluid to rest is termed *dynamic pressure*. The increase of pressure is $\frac{1}{2}\rho u_1^2$ is dynamic pressure. The total pressure P_2 is termed *stagnation pressure*.

6.6.2 Venturi Meter

Venturi meter is a device used for measuring discharge in a pipe. It consists of a rapidly converging section, which increases the velocity of flow and hence reduces the pressure. It then returns to the original dimension of the pipe by a gently diverging 'diffuser' section. By measuring the pressure difference, the discharge rate can be calculated. This method is a particularly accurate for flow measurement because the energy loss is very small. The meter is shown in Fig. 6.23.

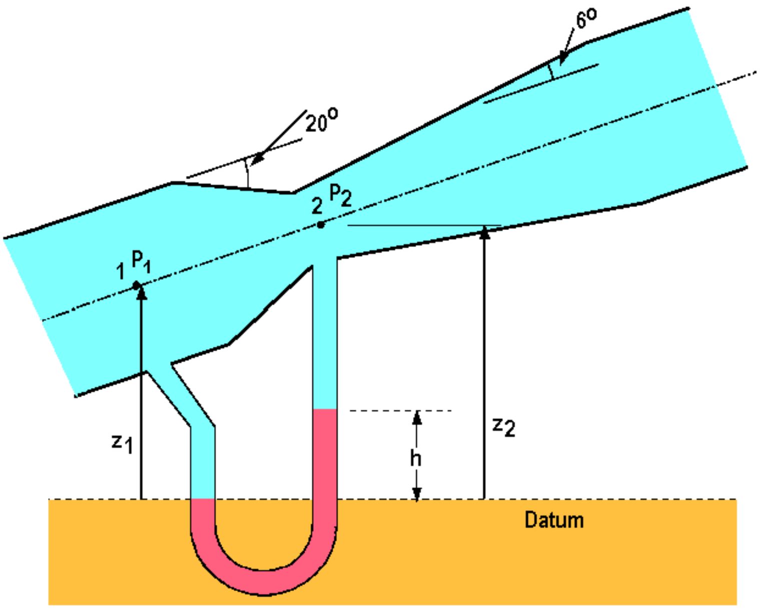


Figure 6.23: The Venturi meter

Applying Bernoulli's equation for point 1 and 2, it yields $z_1 + \frac{u_1^2}{2g} + \frac{P_1}{\rho g} = z_2 + \frac{u_2^2}{2g} + \frac{P_2}{\rho g}$. Using continuity equation (6.8), the volume rate Q = $u_1A_2 = u_2A_2$. This shall mean $u_2 = \frac{u_1 A_1}{A_2}$. Substituting this equation into the earlier equation shall yield $\frac{P_1 - P_2}{\rho g} + z_1 - z_2 = \frac{u_1^2}{2g}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$. Rearrange this equation for velocity u_1

shall be $u_1 = \sqrt{\dfrac{2g\left[\dfrac{P_1 - P_2}{\rho g} + z_1 - z_2\right]}{\left(\dfrac{A_1}{A_2}\right)^2 - 1}}$. From manometer reading at datum line, $P_1 + z_1\rho g = P_2 + (z_2 - h)\rho g + h\rho_{man}g$. This implies that $\dfrac{P_1 - P_2}{\rho g} + z_1 - z_2 = h\left(\dfrac{\rho_{man}}{\rho} - 1\right)$.

Substituting this equation into u_1 velocity equation, $u_1 = \sqrt{\dfrac{2gh\left(\dfrac{\rho_{man}}{\rho} - 1\right)}{\left(\dfrac{A_1}{A_2}\right)^2 - 1}}$. Since the volume rate Q is equal to u_1A_1, therefore, $Q = A_1\sqrt{\dfrac{2gh\left(\dfrac{\rho_{man}}{\rho} - 1\right)}{\left(\dfrac{A_1}{A_2}\right)^2 - 1}}$. If there is loss due to friction, then the coefficient of volume rate C_d can be added. The volume rate, which is also the discharge rate Q shall be

$$Q = C_d A_1 A_2 \sqrt{\frac{2gh\left(\dfrac{\rho_{man}}{\rho} - 1\right)}{A_1^2 - A_2^2}} \qquad (6.15)$$

Note equation (6.15) is independent of height z_1 and z_2.

6.6.3 Flow through a Small Orifice

Let's now consider the flow through a small orifice, where the flow of fluid through a hole at the side closed to the base of the tank as shown in Fig. 6.24.

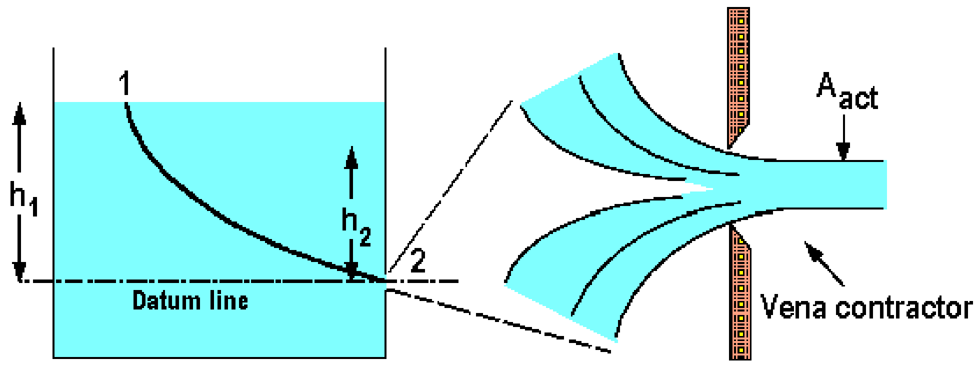


Figure 6.24: Flow through a small orifice

The shape of the hole edge is sharp to minimize the frictional loss. The fluid contracts after the orifice to a minimum value such that it becomes parallel, which streamline flow. At this point, the velocity and pressure are uniform across the jet. This convergence is called the *vena contracta* meaning contracted vein. In order to accurate calculate the flow, is necessary to know the amount of contraction.

Using Bernoulli's equation, at point 1 the velocity u_1 is zero, the pressure P_1 is equal to zero, whist the potential head is equal to h_1.

At the orifice, the jet is open to the air. Thus, the pressure P_2 is equal to zero. The potential head is equal to zero. Thus, equating the total head yields h $= \frac{u_2^2}{2g}$. This shall mean the velocity u_2 at orifice is $u_2 = \sqrt{2gh_1}$.

The volume discharge rate Q at point 2 shall be A_2u_2. To include the frictional force, the coefficient of velocity C_v shall be used such the $u_{act} = C_vu_2$. The cross sectional area A_{act} shall be $C_AA_{orifice}$ after taking *vena* contraction into consideration, where C_A is the coefficient of contraction. The actual volume rate Q shall then equal to

$$Q = C_vu_2\, C_AA_{orifice} \tag{6.16}$$

The time taken for the tank to drop from height h_1 to h_2 through the flow of fluid via the orifice can be calculated based on the continuity equation.

The volume rate is equal to Q = AV, where A is the cross sectional area of the tank and V is the velocity of the flow. The velocity V is also equal to dh/dt. Thus, the volume rate is $Q = -\int_{h_1}^{h_2} A\frac{dh}{dt}$. Negative sign denotes that the level is decreasing. Substituting equation (6.16) into this equation, the time t taken for the height of tank to fall from h_1 to h_2 shall be

$$\begin{aligned} t &= -\int_{h_2}^{h_1} \frac{A}{C_vC_AA_{orifice}} \cdot \frac{dh}{\sqrt{2gh}} \\ &= -\frac{2A}{C_vC_AA_{orifice}\sqrt{2g}} \cdot \left[\sqrt{h_2} - \sqrt{h_1}\right] \end{aligned} \tag{6.17}$$

6.6.4 Flow through Submerged Orifice

If there are two tanks next to each other and are connected by an orifice as shown in Fig. 6.25, then the orifice is considered as submerged orifice.

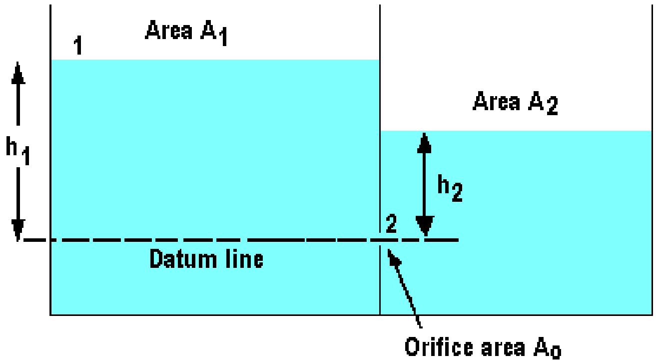


Figure 6.25: Two tanks joined by an orifice

At point one, the total head is h_1. At point 2, the total head is consist of pressure head $\frac{P_2}{\rho g}$, where P_2 is equal to $\rho g h_2$, and velocity head $\frac{u_2^2}{2g}$. This shall mean that the velocity u_2 at point 2 is

$$u_2 = \sqrt{2g(h_1 - h_2)} \tag{6.18}$$

The rate of discharge Q shall be $Q = C_A A_{orifice}\sqrt{2g(h_1 - h_2)}$.

The time taken for two tanks of different height to be equalized shall be calculated based on continuity equation. The volume rate is $Q = -A_1\frac{dh_1}{dt} = A_2\frac{dh_2}{dt}$. Rewrite this equation as $Qdt = -A_1dh_1 = A_2dh_2$. Letting $dh = dh_2 - dh_1$, $-A_1dh_1 = A_2d(h + h_1)$, this implies that $dh_1 = \frac{A_2dh}{A_1 + A_2}$. Since volume $Qdt = -A_1dh_1$ and discharge rate is $Q = C_A A_{orifice}\sqrt{2g(h_1 - h_2)}$, the volume is also equal to $C_A A_{oriffice}\sqrt{2g(h_1 - h_2)}\,dt = \frac{A_1A_2dh}{A_1 + A_2}$. For equalizing the tank, the height shall be from $(h_1 – h_2)$ to zero. The time taken for not equalizing the thank shall be

$$t = \int_{h_1 - h_2}^{h_3} \frac{A_1A_2dh}{(A_1 + A_2)C_d A_{orifice}\sqrt{2gh}} \tag{6.19}$$

$$= \frac{2A_1A_2dh}{(A_1 + A_2)C_dA_{orifice}\sqrt{2g}} \cdot \left[\sqrt{h_1 - h_2} - \sqrt{h_3}\right]$$

where h_3 is any value greater than 0 and less than $(h_1 - h_2)$. For equalization the time taken shall be $t = \frac{2A_1A_2dh}{(A_1 + A_2)C_dA_{orifice}\sqrt{2g}} \cdot \left[\sqrt{h_1 - h_2}\right]$ by setting h_3 equals to zero.

6.6.5 Flow through Weir

A notch, a device for measuring the discharge of fluid, is an opening in the side of a tank or reservoir, which extends above the surface of the liquid. A weir is a large version of a notch usually found in river. It can be a sharp crested type with a substantial width in the direction of flow, which used both as a flow measuring device and water level control.

In deriving the equation for weir, the velocity of the fluid approaching the weir is said to be small so that kinetic energy is assumed to be zero. However, for fast moving river, it is not true. The velocity through any elemental strip of fluid is dependent on the depth below the free surface. The assumptions are acceptable for tank with notch or reservoir with weir. Consider a horizontal strip of width *b* and depth *h* below the free surface, as shown in the Fig. 6.26.

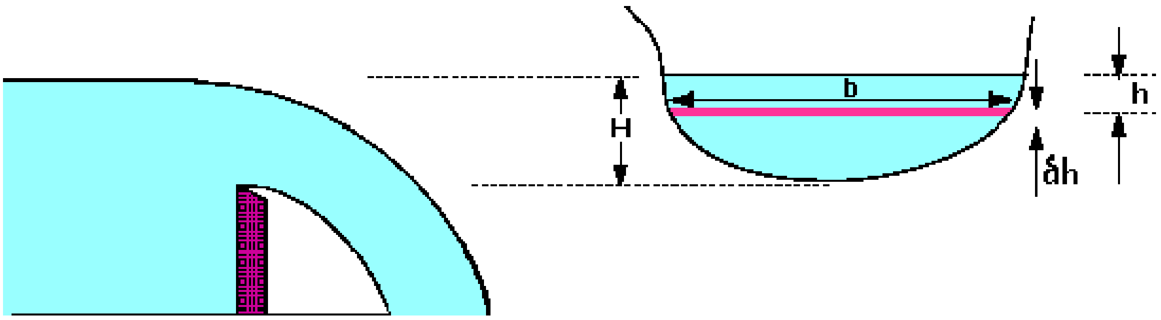


Figure 6.26: Elemental strip of fluid through the notch

The velocity *u* through the strip is $u = \sqrt{2gh}$ and the discharge rate through the strip is $dQ = uA = b\sqrt{2gh}\,dh$. The discharge rate Q shall be the integration of the equation for height limit from 0 to H. Thus,

$$Q = \int_0^H b\sqrt{2gh}\,dh \qquad (6.20)$$

Equation (6.20) shall be the general equation for the flow rate or discharge rate via the notch or weir.

For rectangular tank, the width of the strip shall be constant said B, the flow rate shall be equal to $Q = \int_0^H B\sqrt{2gh}\,dh = \frac{2}{3}B\sqrt{2g}H^{3/2}$.

For the "V" shaped notch, which is shown in Fig. 6.27, the width b of the strip is not a constant.

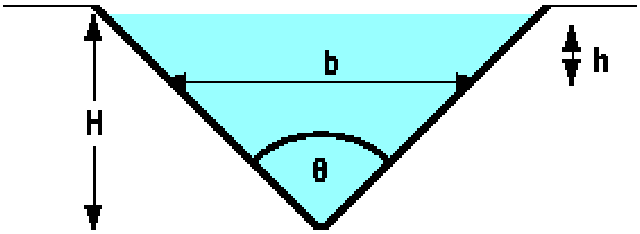


Figure 6.27: "V" shaped notch

From the figure, tan $(\theta/2) = \frac{b/2}{(H-h)}$, implying that b is equal to $b = 2(h-h)\tan\left(\frac{\theta}{2}\right)$. The discharge rate Q shall be $Q = \int_0^H 2(H-h)\tan\left(\frac{\theta}{2}\right)\sqrt{2gh}\,dh = \frac{8}{15}\tan\left(\frac{\theta}{2}\right)\sqrt{2g}H^{5/2}$.

6.7 The Momentum Equation and Its Applications

As it has been mentioned earlier, the analysis of fluid motion is performed in the same way as in solid mechanics by using Newton's laws of motion with account for the special properties of fluids when in motion.

In fluid mechanic, the mass of moving fluid is not clear like the case of solid. Thus, net force net $F_{net} = ma$ for Newton's second law may not be suitable to describe the motion of fluid. Instead, the rate change of momentum equals to the resultant force acting on the fluid in the direction of force is a more appropriate way to describe the motion of fluid.

A steady and non-uniform flow of fluid flowed in the same direction is shown in Fig. 6.28. The entrance inlet and exit have the parameters as shown in the figure. In time δt the volume of fluid that enters the entrance is $u_1 A_1 \delta t$. The mass of fluid in this time interval is $u_1 A_1 \rho_1 \delta t$. The momentum of fluid at this inlet shall be $u_1^2 A_1 \rho_1 \delta t$. Similarly, the momentum of fluid leaving the exit is $u_2^2 A_2 \rho_2 \delta t$. From Newton's second law, the net force F_{net} is equal to

$$F_{net} = \frac{u_2^2 A_2 \rho_2 \delta t - u_1^2 A_1 \rho_1 \delta t}{\delta t} = Q(\rho_2 u_2 - \rho_1 u_1) \qquad (6.21)$$

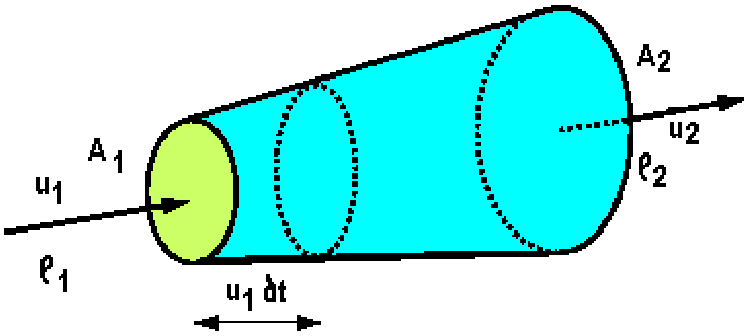


Figure 6.28: A steady and non-uniform one-direction flow of fluid

For fluid that has uniform density ρ, then $\rho_1 = \rho_2$. Equation (6.21) will also be equal to $F_{net} = \frac{u_2^2 A_2 \rho_2 \delta t - u_1^2 A_1 \rho_1 \delta t}{\delta t} = \frac{dm}{dt}(u_2 - u_1)$.

Let's extend the analysis to the case where the direction of the flow is not the same for the entrance and exit points as shown in Fig. 6.29.

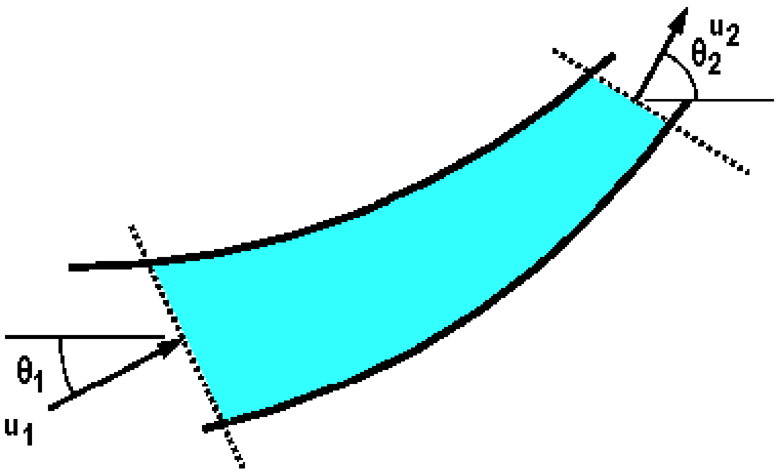


Figure 6.29: A steady and non-uniform two directions flow of fluid

Let's begin the analysis by resolving the force in x direction. The x-direction velocity component at entrance is $u_1\cos\theta_1$, whilst at exit point is $u_2\cos\theta_2$. The net force in x-direction shall be

$$F_{net\text{-}x} = Q\rho(u_2\cos\theta_2 - u_1\cos\theta_1) \qquad (6.22)$$

Similar analysis goes for the net force acting on y-direction is

$$F_{net\text{-}y} = Q\rho(u_2\sin\theta_2 - u_1\sin\theta_1) \qquad (6.23)$$

The resultant net force $F_{net\text{-}resultant}$ shall be equal to $F_{net\text{-}resultant} = \sqrt{F_{net-y}^2 + F_{net-x}^2}$. The force at a bend is equal to $-F_{net\text{-}resultant}$. The angle ϕ that the force acts with respect to x-axis is $\phi = \tan^{-1}\left(\frac{F_{net-y}}{F_{net-x}}\right)$.

From conservation of energy standpoint, the total force $F_{Total} = F_{net\text{-}resultant}$ of the fluid system is made of the force F_R exerted by the fluid at the bend, the force exerted by weight of fluid F_B, and the force exerted by the pressure outside the control volume F_P, which is

$$F_{Total} = F_R + F_B + F_P \qquad (6.23)$$

6.7.1 Force around the Pipe Bend

Let's study the force acting on a bend as it is shown in Fig. 6.30, when fluid changes its direction of flow. If the bend is not fixed, eventually it breaks due to large force. Owing to this one need to know how much force a support or thrust block would withstand.

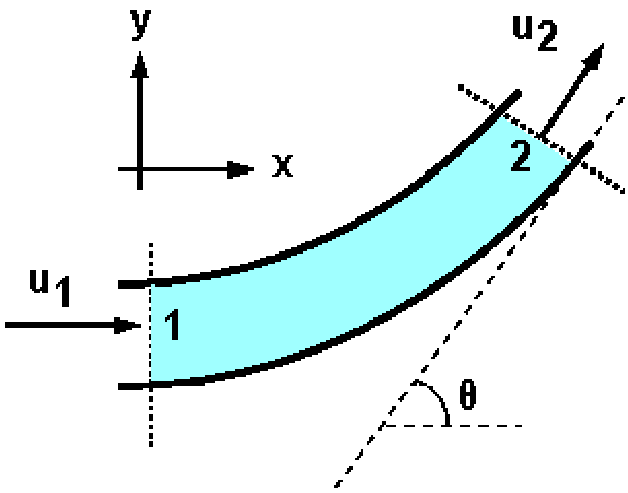


Figure 6.30: Flow around a pipe bend of constant cross-section

Along the x-direction, the total force $F_{T\text{-}x}$ is $F_{T\text{-}x} = Q\rho(u_2\cos\theta - u_1\cos 0) = Q\rho(u_2\cos\theta - u_1)$ and the force $F_{T\text{-}y}$ at y-direction shall be $F_{T\text{-}y} = Q\rho(u_2\sin\theta - u_1\sin 0) = Q\rho u_2\sin\theta$.

Along x-direction, the force due to pressure at different at inlet 1 and exit 2 is $F_{p\text{-}x} = A_1P_1\cos 0 - A_2P_2\cos\theta = A_1P_1 - A_2P_2\cos\theta$ and along y-direction is $F_{p\text{-}y} = A_1P_1\sin 0 - A_2p_2\sin\theta = - A_2P_2\sin\theta$.

There are no body forces in the x or y directions. The only body force is that exerted by gravity is acted perpendicular to the page.

Based on the above results, the x-direction force acts the bend is $F_{R\text{-}x} = Q\rho(u_2\cos\theta - u_1) - (A_1P_1 - A_2P_2\cos\theta)$, whilst the force acting in y-direction is $F_{R\text{-}y} = Q\rho u_2\sin\theta + A_2P_2\cos\theta$.

The resultant force shall be $F_R = \sqrt{F_{R-x}^2 + F_{R-y}^2}$ and the force acting on the bend shall be $- F_R$. The angle of acting shall be $\phi = \tan^{-1}\left(\frac{F_{R-y}}{F_{R-x}}\right)$.

6.7.2 Force on a Pipe Nozzle

Owing to the fluid is contracted at the nozzle; forces are induced in the nozzle. Anything holding the nozzle like a fireman must be strong enough to withstand these forces. Let' analyze these forces from the Fig. 6.31.

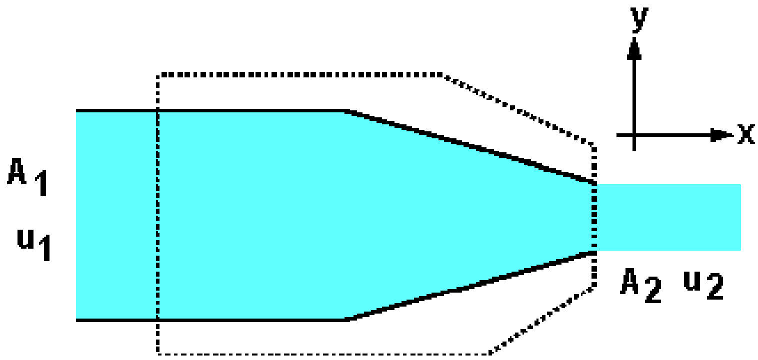


Figure 6.31: Force on pipe nozzle

The total force F_T is acting along *x*-direction is equal to $F_T = F_{T\text{-}x} = Q\rho(u_2 - u_1)$. After considering the continuity equation, the total force is $F_T = F_{T\text{-}x} = Q^2\rho\left(\frac{1}{A_2} - \frac{1}{A_1}\right)$.

The force due to pressure F_P is $P_1A_1 - P_2A_2$. The opening of the nozzle is at atmospheric pressure. Thus, the gauge pressure is zero. The force due to pressure shall be P_1A_1. The pressure P_1 can be calculated based on Bernoulli's equation $z_1 + \frac{u_1^2}{2g} + \frac{P_1}{\rho g} = z_2 + \frac{u_2^2}{2g} + \frac{P_2}{\rho g}$ for condition $z_1 = z_2$, $P_2 = 0$. The pressure P_1 shall be $P_1 = \frac{\rho}{2}\left(u_2^2 - u_1^2\right) = \frac{\rho Q^2}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)$. The force due pressure $F_P = F_{P\text{-}x} = \frac{\rho Q^2 A_1}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_a^2}\right)$. The force due to weight is zero because it is acting in y-direction. The resultant for $F_R = F_{R\text{-}x}$ shall be $F_{R\text{-}x} = Q^2\rho\left(\frac{1}{A_2} - \frac{1}{A_1}\right) - \frac{\rho Q^2 A_1}{2}\left(\frac{1}{A_2^2} - \frac{1}{A_a^2}\right)$.

6.7.3 Impact of Fluid Jet on a Plane

Consider the case where fluid is ejected horizontally on a vertical plane as shown in Fig, 6.32.

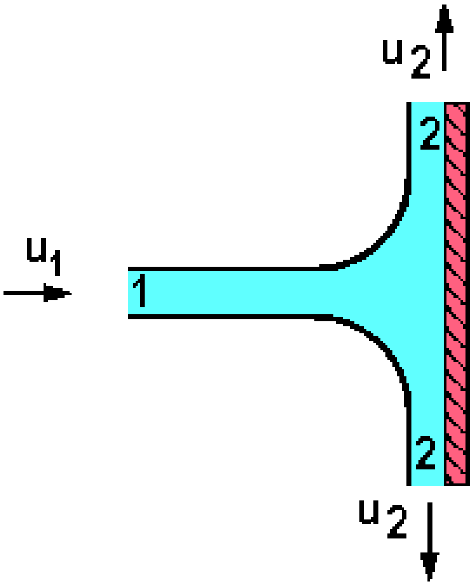


Figure 6.32: Jet impact on vertical plane

The total force acting in x-direction is $F_T = F_{T\text{-}x} = \rho Q(u_{2x} - u_1) = -\rho Q u_1$. The total force in y- direction $F_{T\text{-}y} = 0$ since there are two streams of same velocity on opposite direction.

The force due to pressure is zero since at point 1 and point 2 are at atmospheric pressure.

The weight is considered negligible. Thus, it is zero. The resultant $F_R = F_{R\text{-}x}$ is then equal to $-\rho Q u_1$.

6.8 Real Fluid

The flow of real fluids exhibits viscous effect due to shearing force that follows equation (6.8), $\tau = \mu \frac{du}{dy}$. The flow of fluid can be classified into laminar and turbulent flow. In laminar flow, the motion of the particles of fluid flows orderly in straight lines parallel to the pipe wall. For turbulent flow, the flow of particles is not in straight line. In 1880, Osborne Reynolds did many experiments with the set-up shown in Fig. 6.33 to determine the type of flow. He found an expression named after $R_e = \frac{\rho u d}{\mu}$, where u is the mean velocity, d is the diameter that determine the type of flow.

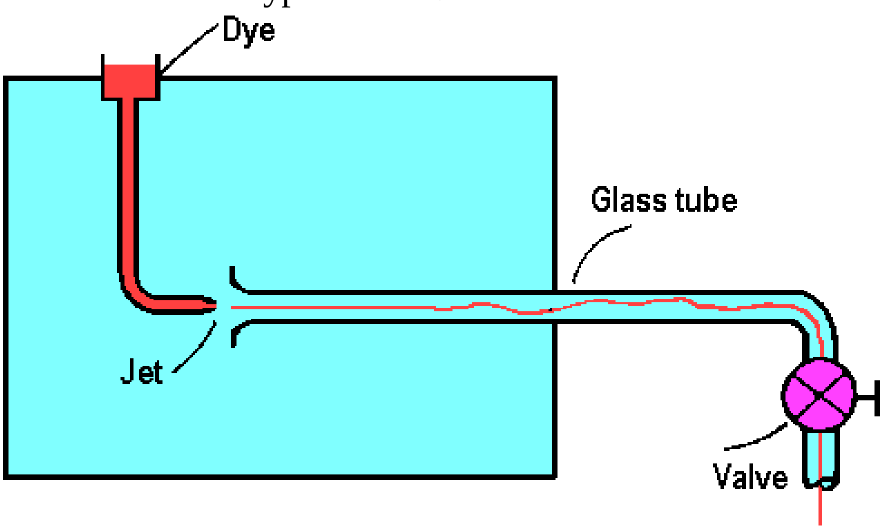


Figure 6.33: Experiment of Osbourne Reynold

He found that Reynolds number $R_e < 2000$ the flow is laminar. Re number in between 2,000 and 4,000 is transitional flow, and $R_e > 4,000$ is turbulent flow.

In the real fluid there is friction. Thus, the pressure upstream is usually higher than downstream even though it may be placed parallel as shown in Fig. 6.34. If the pressure at the upstream end is *P*, and at the downstream end the pressure has fallen by ΔP to (p - ΔP), then the driving force due to pressure, which is F = Pressure x Area, can then be written as driving force is equal to pressure force at upstream minus pressure force at downstream.

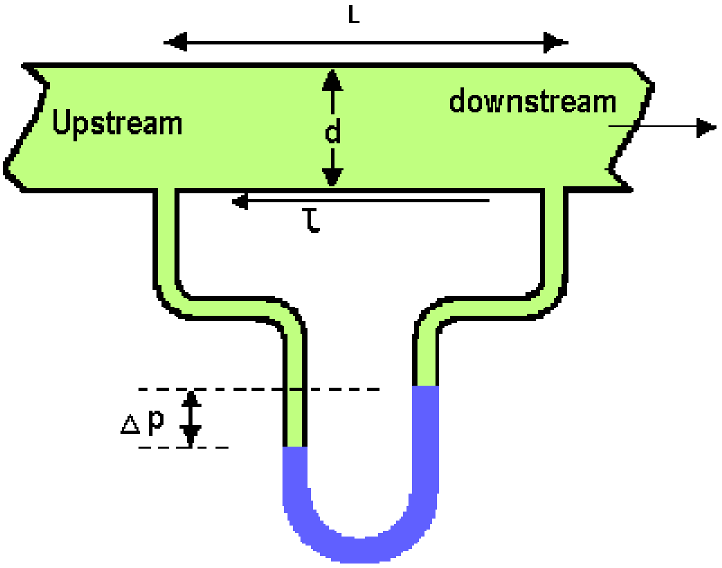


Figure 6.34: Pressure difference between upstream and downstream

Thus, the force is $\Delta PA = \Delta P\,\pi\frac{d^2}{4}$. This force is balanced by the shearing force at the wall of the pipe, which is $2\pi\frac{d}{2}L\tau = \pi dL\tau$. ΔP is also equal to $\Delta P = h_f\rho g$, where h_f is the lost height due to friction. From the above equations, the shearing force shall be $\tau = \frac{\Delta Pd}{4L}$. Since the flow is laminar type, therefore, the shearing force can be generalized as equal to $\tau = \frac{\Delta Pr}{2L}$ for any cylindrical flow of radius r as it is shown in Fig. 6.35.

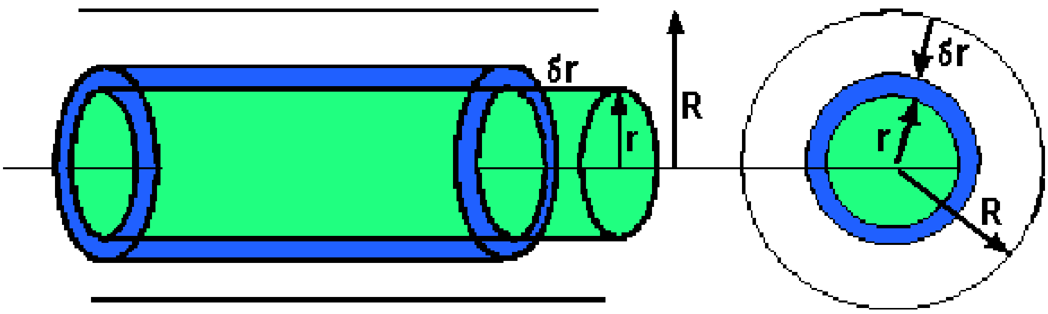


Figure 6.35: Cylindrical laminar flow of radius r

Shearing force $\tau = \frac{\Delta \mathrm{Pr}}{2\mathrm{L}}$ is also equal to $-\mu \frac{du}{dr}$. Negative shall mean the reference is from the center of pipe instead of at the wall of the pipe. From the equation $\tau = \frac{\Delta \mathrm{Pr}}{2\mathrm{L}} = -\mu \frac{du}{dr}$, the distribution of velocity of laminar flow across the pipe can be calculated by integrating for $u = u_{max}$ when r = 0 and u = 0 for r = R. This shall mean that $\frac{\Delta \mathrm{Pr}}{2\mathrm{L}} \int dr = -\mu \int du$. The result of the integration shall be $u = \frac{\Delta \mathrm{Pr}^2}{4\mathrm{L}\mu}$ + C, where C = $\frac{\Delta \mathrm{PR}^2}{4\mathrm{L}\mu}$. Thus, the distribution of velocity of laminar flow across the pipe of radius R is

$$u = \frac{\Delta \mathrm{P}(\mathrm{R}^2 - \mathrm{r}^2)}{4\mathrm{L}\mu} \tag{6.25}$$

The equation has the parabolic result, which is what have been shown in Fig. 6.5.

The mean velocity u_m is $u_m = \int_0^R \frac{\Delta \mathrm{Pr}}{4\mathrm{L}\mu} dr = \frac{\Delta \mathrm{PR}^2}{8\mathrm{L}\mu}$. The flow rate Q shall be equal to

$$\mathrm{Q} = \mathrm{A}u_\mathrm{m} = \pi \frac{\Delta \mathrm{PR}^4}{8\mathrm{L}\mu} = \pi \frac{\Delta \mathrm{Pd}^4}{128\mathrm{L}\mu} \tag{6.26}$$

Equation (6.25) is also called *Hagen-Poiseuille equation* for laminar flow in a pipe.

As it has been mentioned earlier, ΔP is also equal to $\Delta P = h_f \rho g$, where h_f is the loss of pressure head caused by friction, therefore the discharge rate is Q $= \pi \frac{\rho g h_f d^4}{128\mathrm{L}\mu}$ and the mean velocity u_m = $\frac{h_f \rho g R^2}{8\mathrm{L}\mu}$.

Tutorials

6.1. Explain why the viscosity of a liquid decreases while for a gas increases with a temperature rise.

6.2. The following is a table of measurement for a fluid at constant temperature. Determine the dynamic viscosity of the fluid.

du/dy (s^{-1})	0.00	0.20	0.40	0.60	0.80
τ (N m^{-2})	0.00	0.01	1.90	3.10	4.00

du/dy (s^{-1})	0.00	0.20	0.40	0.60	0.80
τ (N m^{-2})	0.00	0.01	1.90	3.10	4.00
Gradient	0.00	0.05	4.75	5.17	5.00

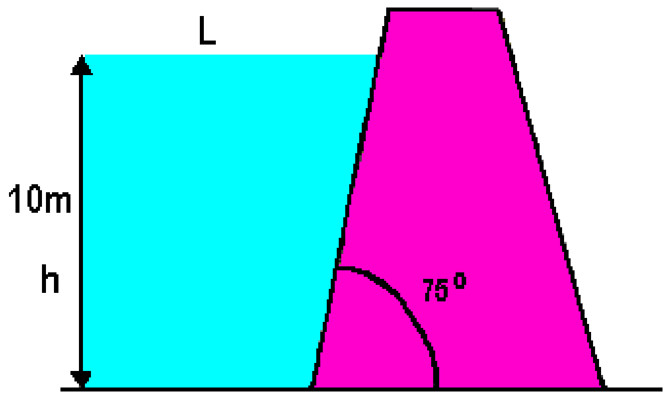
6.3. The velocity distribution of a viscous liquid (dynamic viscosity μ = 0.9Ns/m^2) flowing over a fixed plate is given by $u = 0.68y - y^2$ (u is velocity in m/s and y is the distance from the plate in m). What are the shear stresses at the plate surface and at y = 0.34m?

6.4. Derive an expression for the total force acting on the wall of dam that has water height H and width W.

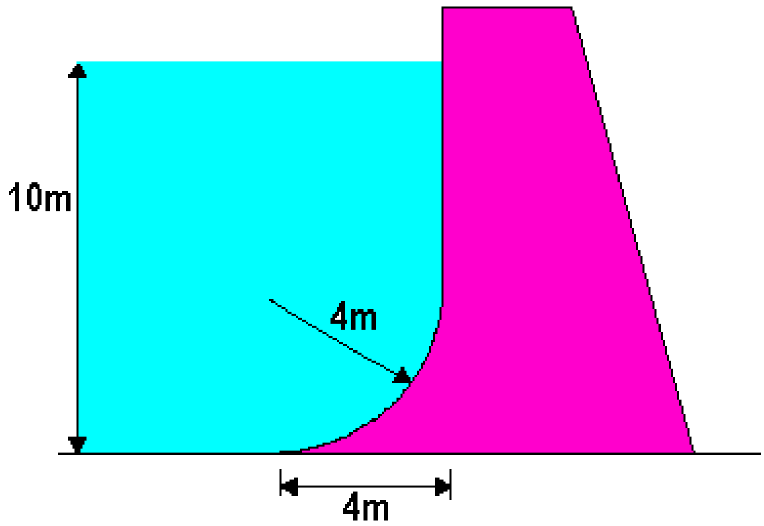
6.5. Air is a compressible fluid and assuming that the pressure p is direction proportional to the density ρ of air. Derive an expression for pressure of air at altitude H above sea level.

6.6. In a fluid the velocity measured at a distance of 75mm from the boundary is 1.125m/s. The fluid has absolute viscosity 0.048 Pa-s. What is the velocity gradient and shear stress at the boundary assuming a linear velocity distribution?

6.7. A concrete dam has the cross-sectional profile and width b as shown in the figure. Calculate the magnitude, direction and position of action of the resultant force exerted by the water per unit width of dam?



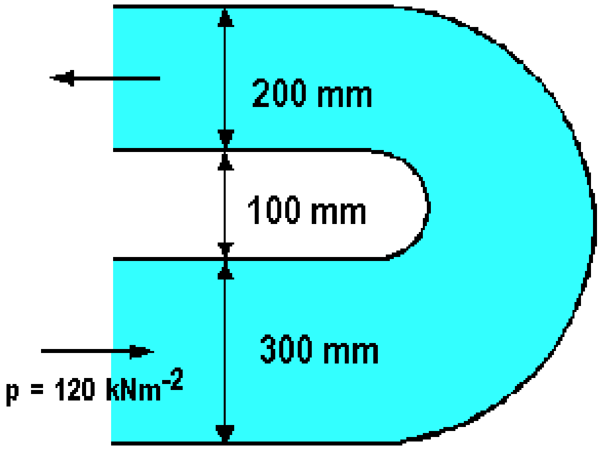
6.8. A design for a dam has the cross-sectional profile composed of a vertical face with a circular curved section at the base as shown in figure. Calculate the resultant force and its direction of application per unit width of this dam.



6.9. At the end of a channel is a sharp edged rectangular weir with a width of 400mm and a coefficient of discharge of 0.65. The water is flowing at a depth 0.16m above the base of the weir. If this weir is replaced by a 90^o V-notch weir with the same coefficient of discharge, what will be the necessary upstream depth of water to achieve the same discharge as the rectangular weir?

6.10. A venturi meter with an entrance diameter of 0.3m and a throat diameter of 0.2m is used to measure the volume of gas flowing through a pipe. The discharge coefficient of the meter is 0.96. Assuming the specific weight of the gas to be constant at $19.62 N/m^3$, calculate the volume flowing when the pressure difference between the entrance and the throat is measured as 0.06m on a water U-tube manometer.

6.11. Water flows along a circular pipe and is turned vertically through 180 °by a bend as shown in the figure. The rate of flow in the pipe is 20 litres/s, the pressure measured at the entrance to the bend is $120 kN/m^2$ and the volume of fluid in the bend is $0.1 m^3$. What is the magnitude and direction of the force exerted by the fluid on the bend? Ignore any friction losses.


200 mm
100 mm
300 mm
p = 120 kNm-2