Chapter 1. Introduction to Nonlinear Space Plasma Physics

The goal of this course, *Nonlinear Space Plasma Physics*, is to explore the formation, evolution, propagation, and characteristics of the large amplitude quasi-stationary nonlinear waves, structures, or turbulences commonly observed in the space plasmas and in numerical simulations of space plasmas. Familiar with theoretical solutions of these nonlinear plasma waves can help us to analyze and to properly explain these observed nonlinear phenomena.

Differences between linear plasma waves and nonlinear plasma waves are discussed below:

By definition, a function $f(x)$ is a linear function, if and only if, $f(ax+by) = af(x)+bf(y)$. Thus, if $f(ax+by) \neq af(x)+bf(y)$, then the function $f(x)$ is a nonlinear function, and $f(x)=0$ is a nonlinear equation.

A linear perturbation in space plasma can be considered as a linear combination of the eigen wave modes, which can be obtained from the *linear wave dispersion relations* of the space plasma. Under linear superimposition assumption, there is no interaction between these eigen wave modes. Thus, the coefficients of the linear combinations should not vary with time. We can predict the linear evolution of a linear perturbation based on the characteristics of these linear eigen wave modes.

The superimposition assumption is not applicable to nonlinear structures. Even if it is still possible to decompose a nonlinear wave into a linear combination of all the linear eigen wave modes of the background plasma, the coefficients of the linear combinations should vary with time. Thus, it is impossible to predict the evolution of the nonlinear wave based on the characteristics of these linear eigen modes.

Several methods have been introduced in literatures for the studying of the nonlinear waves in the space plasma. We shall discuss these methods in Section 1.1. Basic nonlinear equations of space plasmas are reviewed in Section 1.2. Generations of these nonlinear waves are discussed in Section 1.3.

Remarks on linear wave dispersion relations:

A plasma system can be described by a set of nonlinear partial differential equations (PDEs). The first step to study such a system is to linearize the set of nonlinear PDEs by Taylor expansion. Keeping only first order terms in the Taylor expansions, one can reduce the set of nonlinear PDEs into a set of linear PDEs. After Fourier transform and Laplace transform, the set of linear PDEs can be converted into a set of algebra equations. Linear dispersion relations can be obtained as a set of eigen states of these algebra equations.

Linear dispersion relations provide not only information on linear waves at different wavelength, they can also help us to classify nonlinear wave solutions obtained by other methods to be discussed in Section 1.1 and the rest chapters of this course.

1.1. Methods for Studying Nonlinear Waves

Quasi-linear Approximation

Quasi-linear approximation is a useful tool to study small but finite amplitude nonlinear waves. Quasi-linear approximation keeps first and second order terms in Taylor expansion of a nonlinear equation. Quasi-linear approximation is commonly used to study nonlinear phenomena due to wave-wave interactions. Quasi-linear approximation allows us to study nonlinear phenomena in multiple spatial and time scales. Solution of quasi-linear approximation may be a time-independent structure at a long timescale, but become a time-dependent structure at a short timescale.

Pseudo Potential Method

Pseudo potential method is commonly used to study one-dimensional steady-state nonlinear wave solutions. Pseudo potential method can help us to find analytical solutions of nonlinear equations with or without quasi-linear approximation. Unlike quasi-linear approximation, there is no standard procedure to determine pseudo potential of fully nonlinear equations. Fortunately, in nonlinear plasma physics, the pseudo potential can be obtained based on the conservation of energy flux.

Jump Conditions of Shocks and Discontinuities Obtained Based on Conservation of Fluxes

From conservation of mass flux, momentum flux, energy flux, and Maxwell's equations, one can obtain nonlinear jump conditions of shocks and discontinuities in collisionless plasma. Knowing solution space of jump conditions is the first step to study these nonlinear phenomena. Advanced studies of these nonlinear phenomena include, but not limit to, (1) study of generation mechanism of these nonlinear waves, (2) study of possible instabilities that might occur in the transition region, and (3) study of collisionless dissipation process in the transition region.

Numerical Simulation

Numerical simulation is a powerful tool to study evolutions of nonlinear waves in a self-consistent manner. Combination of numerical simulations and analytical solutions can help us to understand nonlinear wave behavior and underline physical processes in a complicated nonlinear system.

Probability Approach

Chaos, fractal, and turbulence are popular ways to describe different stages of nonlinear phenomena. Nonlinear wave solutions obtained analytically by pseudo-potential method can be considered as a chaos type of nonlinear phenomena. Waves found in shock transition region and instabilities occurred along the discontinuity surface often show turbulent nonlinear structures. These Chaos, fractal, and turbulence can also be studied based on a probability or statistic approach. The probability approach can be achieved by adding random noises onto a well-defined nonlinear structure or an analytic solution to model those small effects, which were neglected in the process of obtaining the simplified analytic solution. Various types of statistic tests provide another way to examine the characteristics of the observed turbulent structures.

The first method, *quasi-linear approximation*, and the last method, *probability approach*, will not be addressed in this course. Results obtained from *numerical simulations* will be served as an example to demonstrate the powerfulness of combining the *jump conditions,* the *pseudo potential method,* and the *numerical simulations* to study nonlinear waves in space plasma.

1.2. Basic Equations of Space Plasmas

In this study, we consider a simplified plasma system, which is collisionless and non-relativistic. Gravitational force is considered to be much smaller than the Lorentz force and the observational frame is considered to be an inertial frame. Under these assumptions, we can use Vlasov plasma model, two-fluid plasma model, or one-fluid MHD or quasi-MHD plasma model to describe the variations of plasmas and fields at different spatial and temporal scales. Basic nonlinear equations of these plasma models are listed below.

To study kinetic plasma phenomena, the basic equations are Vlasov-Maxwell equations:

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{e_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0 \qquad (1.1)$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \sum_\alpha \int e_\alpha f_\alpha d\mathbf{v} \qquad (1.2)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (1.3)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.4}$$

$$\nabla \times \mathbf{B} = \mu_0 \sum_\alpha \int e_\alpha \mathbf{v} f_\alpha d\mathbf{v} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.5}$$

where subscript α denotes the αth species.

To study fluid plasma phenomena, the basic equations are fluid-Maxwell equations:

$$\frac{\partial}{\partial t}(n_\alpha) + \nabla \cdot (n_\alpha \mathbf{V}_\alpha) = 0 \tag{1.6$_\alpha$}$$

$$\frac{\partial}{\partial t}(m_\alpha n_\alpha \mathbf{V}_\alpha) + \nabla \cdot (m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathbf{P}_\alpha) = e_\alpha n_\alpha (\mathbf{E} + \mathbf{V}_\alpha \times \mathbf{B}) \tag{1.7$_\alpha$}$$

$$\frac{\partial}{\partial t}(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha) + \nabla \cdot [(\frac{1}{2} m_\alpha n_\alpha V_\alpha^2 + \frac{3}{2} p_\alpha)\mathbf{V}_\alpha + \mathbf{P}_\alpha \cdot \mathbf{V}_\alpha + \mathbf{q}_\alpha] = e n_\alpha \mathbf{V}_\alpha \cdot \mathbf{E} \tag{1.8$_\alpha$}$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \sum_\alpha e_\alpha n_\alpha \tag{1.9}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.10}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.11}$$

$$\nabla \times \mathbf{B} = \mu_0 \sum_\alpha e_\alpha n_\alpha \mathbf{V}_\alpha + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.12}$$

Exercise 1.1

(a) Derive Eqs. (1.6_α), (1.7_α), and (1.8_α) from Eq. (1.1).

(b) Define $n_\alpha, \mathbf{V}_\alpha, \mathbf{P}_\alpha, p_\alpha$, and $\mathbf{q}_\alpha$ based on distribution f_α.

Eqs. (1.6_α)~(1.12) are basic equations of a two-fluid or multiple-fluid system.

We can obtain one-fluid mass continuity equation, momentum equation, and energy equation from $\sum_\alpha m_\alpha (1.6_\alpha)$, $\sum_\alpha (1.7_\alpha)$, and $\sum_\alpha (1.8_\alpha)$, respectively, which yield

$$\frac{\partial}{\partial t}(\sum_\alpha m_\alpha n_\alpha) + \nabla \cdot (\sum_\alpha m_\alpha n_\alpha \mathbf{V}_\alpha) = 0 \tag{1.13}$$

$$\frac{\partial}{\partial t}(\sum_\alpha m_\alpha n_\alpha \mathbf{V}_\alpha) + \nabla \cdot [\sum_\alpha (m_\alpha n_\alpha \mathbf{V}_\alpha \mathbf{V}_\alpha + \mathbf{P}_\alpha) = \rho_c \mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{1.14}$$

$$\frac{\partial}{\partial t}\sum_\alpha(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2+\frac{3}{2}p_\alpha)+\nabla\cdot\sum_\alpha[(\frac{1}{2}m_\alpha n_\alpha V_\alpha^2+\frac{3}{2}p_\alpha)\mathbf{V}_\alpha+\mathbf{P}_\alpha\cdot\mathbf{V}_\alpha+\mathbf{q}_\alpha]=\mathbf{J}\cdot\mathbf{E} \quad (1.15)$$

Making use of Maxwell's equations, we can rewrite the above equations (1.13)~(1.15) into the following conservation forms:

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\mathbf{V})=0 \quad (1.13')$$

$$\frac{\partial}{\partial t}[\rho\mathbf{V}+\frac{1}{c^2}(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0})]+\nabla\cdot[\rho\mathbf{V}\mathbf{V}+\mathbf{P}+\mathbf{1}(\frac{\varepsilon_0E^2}{2}+\frac{B^2}{2\mu_0})-\varepsilon_0\mathbf{E}\mathbf{E}-\frac{\mathbf{B}\mathbf{B}}{\mu_0}]=0 \quad (1.14')$$

$$\frac{\partial}{\partial t}[\frac{1}{2}\rho V^2+\frac{3}{2}p+\frac{\varepsilon_0E^2}{2}+\frac{B^2}{2\mu_0}]+\nabla\cdot[(\frac{1}{2}\rho V^2+\frac{3}{2}p)\mathbf{V}+\mathbf{P}\cdot\mathbf{V}+\mathbf{q}+\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}]=0 \quad (1.15')$$

One-fluid charge continuity equation, and generalized Ohm's Law can be obtained from $\sum_\alpha e_\alpha(1.6_\alpha)$, and $\sum_\alpha\frac{e_\alpha}{m_\alpha}(1.7_\alpha)$, respectively. They are

$$\frac{\partial}{\partial t}\rho_c+\nabla\cdot\mathbf{J}=0 \quad (1.16)$$

$$\frac{\partial}{\partial t}\mathbf{J}+\nabla\cdot[\sum_\alpha(e_\alpha n_\alpha\mathbf{V}_\alpha\mathbf{V}_\alpha+\frac{e_\alpha}{m_\alpha}\mathbf{P}_\alpha)=\sum_\alpha(\frac{e^2n_\alpha}{m_\alpha})\mathbf{E}+\sum_\alpha(\frac{e^2n_\alpha\mathbf{V}_\alpha}{m_\alpha})\times\mathbf{B} \quad (1.17)$$

Eqs. (1.9)~(1.17) are useful equations for studying low frequency waves in one-fluid plasma.

Exercise 1.2

(a) Derive Eqs. (1.13'), (1.14'), and (1.15') from Eqs. (1.13), (1.14), (1.15), and Eqs. (1.9)~(1.12).

(b) Define ρ_c, $\mathbf{J}$, ρ, $\mathbf{V}$, $\mathbf{P}$, p, and $\mathbf{q}$ in terms of m_α, n_α, $\mathbf{V}_\alpha$, $\mathbf{P}_\alpha$, p_α, and $\mathbf{q}_\alpha$.

1.3. Generation of Nonlinear Waves

For a given equilibrium state, one can linearize the above nonlinear equations to obtain linear dispersion relation in a Vlasov-Maxwell system or in a fluid-Maxwell system. If there is a linear wave mode with positive growth rate $\omega_i>0$, then the linear disturbance will grow into finite or large amplitude waves. The linearized equations are no longer applicable to the nonlinear large amplitude waves. We have to use the original nonlinear equations to describe these nonlinear waves' behavior. Saturation of wave amplitude in the nonlinear stage is an important research topic in study nonlinear plasma physics.

A system with linear stability, i.e., $\omega_i = 0$, for all wave modes, may still be unstable by an external nonlinear disturbance. Again, we have to use the original nonlinear equations to describe these nonlinear waves' behavior. To find out generation mechanism of such nonlinear waves is another interesting subject in space research.

Figure 1.1 illustrates different types of equilibrium states. Case (a) is an unstable equilibrium state. Case (b) is a stable equilibrium state. Case (c) is an equilibrium state, which is stable under small amplitude perturbation, but unstable if the perturbation amplitude is large enough. Case (d) is an equilibrium state, which is unstable under linear approximation, but the wave amplitude will be saturated in the nonlinear stage. Case (e) shows a typical example of global coupling between a linear-stable equilibrium state and a linear-unstable equilibrium state. In this case, a linear-stable equilibrium state A is likely to be disturbed nonlinearly by a near-by linear-unstable equilibrium state B. But both states A and B will be confined under the dashed line and to fulfill the nonlinear saturation conditions.

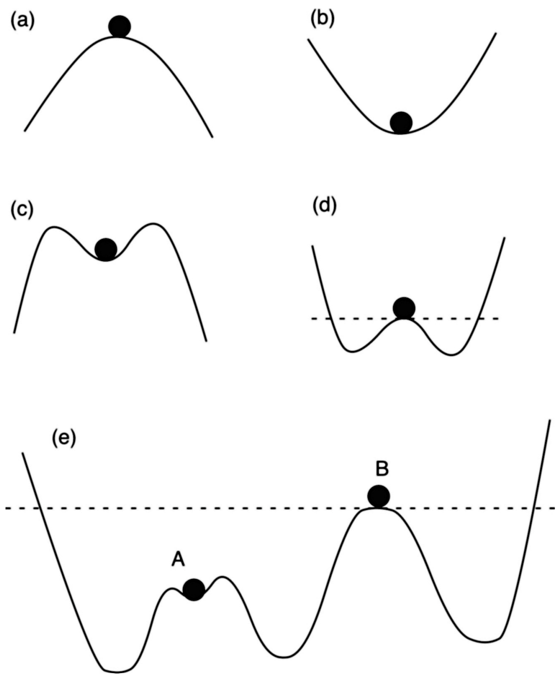

Figure 1.1. Different types of equilibrium states. See text for detail discussion.