Chapter 1

Rotation of an Object About a Fixed Axis

1.1 The Important Stuff

1.1.1 Rigid Bodies; Rotation

So far in our study of physics we have (with few exceptions) dealt with *particles*, objects whose spatial dimensions were unimportant for the questions we were asking. We now deal with the (elementary!) aspects of the motion of *extended objects*, objects whose dimensions are important.

The objects that we deal with are those which maintain a rigid shape (the mass points maintain their relative positions) but which can change their *orientation* in space. They can have **translational motion**, in which their center of mass moves but also **rotational motion**, in which we can observe the changes in direction of a set of axes that is "glued to" the object. Such an object is known as a **rigid body**. We need only a small set of angles to describe the rotation of a rigid body. Still, the general motion of such an object can be quite complicated.

Since this *is* such a complicated subject, we specialize further to the case where *a line of points of the object is fixed* and the object spins about a **rotation axis** fixed in space. When this happens, every individual point of the object will have a circular path, although the *radius* of that circle will depend on which mass point we are talking about. And the orientation of the object is completely specified by *one variable*, an angle θ which we can take to be the angle between some reference line "painted" on the object and the x axis (measured counter-clockwise, as usual).

Because of the nice mathematical properties of expressing the measure of an angle in *radians*, we will usually express angles in radians all through our study of rotations; on occasion, though, we may have to convert to or from degrees or revolutions. Revolutions, degrees and radians are related by:

$$1\,\text{revolution} = 360\,\text{degrees} = 2\pi\,\text{radians}$$

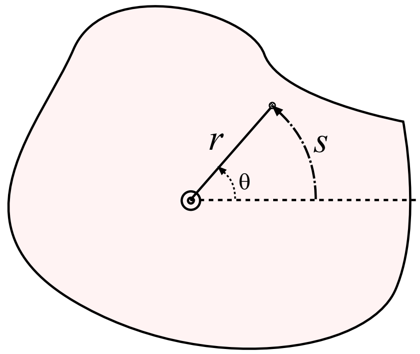

Figure 1.1: A point on the rotating object is located a distance r from the axis; as the object rotates through an angle θ it moves a distance s.

[Later, because of its importance, we will deal with the motion of a (round) object which rolls along a surface without slipping. This motion involves rotation *and* translation, but it is not much more complicated than rotation about a fixed axis.]

1.1.2 Angular Displacement

As a rotating object moves through an angle θ from the starting position, a mass point on the object at radius r will move a distance s; s length of arc of a circle of radius r, subtended by the angle θ.

When θ is in radians, these are related by

$$\theta = \frac{s}{r} \qquad \theta \text{ in radians} \tag{1.1}$$

If we think about the consistency of the *units* in this equation, we see that since s and r both have units of length, θ is really *dimensionless*; but since we are assuming radian measure, we will often write “rad” next to our angles to keep this in mind.

1.1.3 Angular Velocity

The angular position of a rotating changes with time; as with linear motion, we study the rate of change of θ with time t. If in a time period Δt the object has rotated through an angular displacement $\Delta\theta$ then we define the **average angular velocity** for that period as

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t} \tag{1.2}$$

A more interesting quantity is found as we let the time period Δt be vanishingly small. This gives us the **instantaneous angular velocity**, ω:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \tag{1.3}$$

Angular velocity has units of $\frac{\text{rad}}{\text{s}}$, or equivalently, $\frac{1}{\text{s}}$ or s^{-1}.

In more advanced studies of rotational motion, the angular velocity of a rotating object is defined in such a way that it is a *vector* quantity. For an object rotating counterclockwise about a fixed axis, this vector has magnitude ω and points outward along the axis of rotation. For our purposes, though, we will treat ω as a *number* which can be positive or negative, depending on the direction of rotation.

1.1.4 Angular Acceleration; Constant Angular Acceleration

The rate at which the angular velocity changes is the angular acceleration of the object. If the object's (instantaneous) angular velocity changes by $\Delta\omega$ within a time period Δt, then the **average angular acceleration** for this period is

$$\overline{\alpha} = \frac{\Delta\omega}{\Delta t} \tag{1.4}$$

But as you might expect, much more interesting is the **instantaneous angular acceleration**, defined as

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \tag{1.5}$$

We can derive simple equations for rotational motion if we know that α is constant. (Later we will see that this happens if the "torque" on the object is constant.) Then, if θ_0 is the initial angular displacement, ω_0 is the initial angular velocity and α is the *constant* angular acceleration, then we find:

$$\omega = \omega_0 + \alpha t \tag{1.6}$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \tag{1.7}$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{1.8}$$

$$\theta = \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \tag{1.9}$$

where θ and ω are the angular displacements and velocity at time t. θ_0 and ω_0 are the values of the angle and angular velocity at $t = 0$.

These equations have *exactly the same form* as the equations for one–dimensional *linear* motion given in Chapter 2 of Vol. 1. The correspondences of the variables are:

$$x \leftrightarrow \theta \qquad v \leftrightarrow \omega \qquad a \leftrightarrow \alpha \ .$$

It is almost always simplest to set $\theta_0 = 0$ in these equations, so you will often see Eqs. 1.6—1.9 written with this substitution already made.

1.1.5 Relationship Between Angular and Linear Quantities

As we wrote in Eq. 1.1, when a rotating object has an angular displacement $\Delta\theta$, then a point on the object at a radius r travels a distance $s = r\theta$. This is a relation between the angular motion of the point and the "linear" motion of the point (though here "linear" is a bit of a misnomer because the point has a circular path). The distance of the point from the axis does not change, so taking the time derivative of this relation give the instantaneous speed of the particle as:

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega \tag{1.10}$$

which we similarly call the point's **linear speed** (or, **tangential speed**), v_T) to distinguish it from the *angular* speed. Note, all points on the rotating object have the same *angular* speed but their linear speeds depend on their distances from the axis.

Similarly, the time derivative of the Eq. 1.10 gives the linear acceleration of the point:

$$a_T = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{1.11}$$

Here it is *essential* to distinguish the *tangential* acceleration from the *centripetal* acceleration that we recall from our study of uniform circular motion. It is *still* true that a point on the wheel at radius r will have a centripetal acceleration given by:

$$a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = r\omega^2 \tag{1.12}$$

These two components specify the acceleration vector of a point on a rotating object. (Of course, if α is zero, then $a_T = 0$ and there is *only* a centripetal component.)

1.1.6 Rotational Kinetic Energy

Because a rotating object is made of many mass points in motion, it has kinetic energy; but since each mass point has a *different* speed v our formula from *translational* particle motion, $K = \frac{1}{2}mv^2$ no longer applies. If we label the mass points of the rotating object as m_i, having individual (different!) linear speeds v_i, then the total kinetic energy of the rotating object is

$$K_{\rm rot} = \sum_i \tfrac{1}{2}m_i v_i^2 = \tfrac{1}{2}\sum_I m_i v_i^2$$

If r_i is the distance of the $i^{\rm th}$ mass point form the axis, then $v_i = r_i\omega$ and we then have:

$$K_{\rm rot} = \tfrac{1}{2}\sum_i m_i(r_i\omega)^2 = \tfrac{1}{2}\sum_i (m_i r_i^2)\omega^2 = \tfrac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 \ .$$

The sum $\sum_i m_i r_i^2$ is called the **moment of inertia** for the rotating object (which we discuss further in the next section), and usually denoted I. (It is also called the **rotational inertia** in some books.) It has units of $\mathrm{kg}\cdot\mathrm{m}^2$ in the SI system. With this simplification, our last equation becomes

$$K_{\rm rot} = \tfrac{1}{2}I\omega^2 \tag{1.13}$$

1.1.7 The Moment of Inertia; The Parallel Axis Theorem

For a rotating object composed of many *mass points*, the moment of inertia I is given by

$$I = \sum_i m_i r_i^2 \tag{1.14}$$

I has units of $\mathrm{kg \cdot m^2}$ in the SI system, and as we use it in elementary physics, it is a scalar [1] (i.e. a single *number* which in fact is always positive). More frequently we deal with a rotating object which is a *continuous* distribution of mass, and for this case we have the more general expression

$$I = \int r^2 dm \tag{1.15}$$

Here, the integral is performed over the volume of the object and at each point we evaluate r^2, where r is the distance measured perpendicularly from the rotation axis.

The evaluation of this integral for various of cases of interest is a common exercise in multi-variable calculus. In most of our problems we will only be using a few basic geometrical shapes, and the moments of inertia for these are given in Figure 1.2 and in Figure 1.3.

Suppose the moment of inertia for an object of mass M with the rotation axis *passing through the center of mass* is $I_{\rm CM}$. Now suppose we displace the axis *parallel to itself* by a distance D. This situation is shown in Fig. 1.4. The moment of inertia of the object about the new axis will have a new value I, given by

$$I = I_{\rm CM} + MD^2 \tag{1.16}$$

Eq. 1.16 is known as the **Parallel Axis Theorem** and is sometimes handy for computing moments of inertia if we already have a listing for a moment of inertia through the object's center of mass.

1.1.8 Torque

We can impart an acceleration to a rotating object by exerting a force on it at a particular point. But it turns out that the force is not the simplest quantity to use in studying rotations; rather it is the *torque* imparted by the force.

Suppose the force $\mathbf{F}$ (whose direction lies in the plane of rotation) is applied at a point $\mathbf{r}$ (relative to the rotation axis which is at the origin O). Suppose that the (smallest) angle between $\mathbf{r}$ and $\mathbf{F}$ is ϕ. Then the magnitude of the **torque** exerted on the object by this force is

$$|\tau| = rF \sin\phi \tag{1.17}$$

By some very simple regrouping, this equation can be written as

$$\tau = r(F \sin\phi) = rF_t$$

[1] In advanced mechanics it is treated as a *matrix*, but we don't need to make things that complicated!

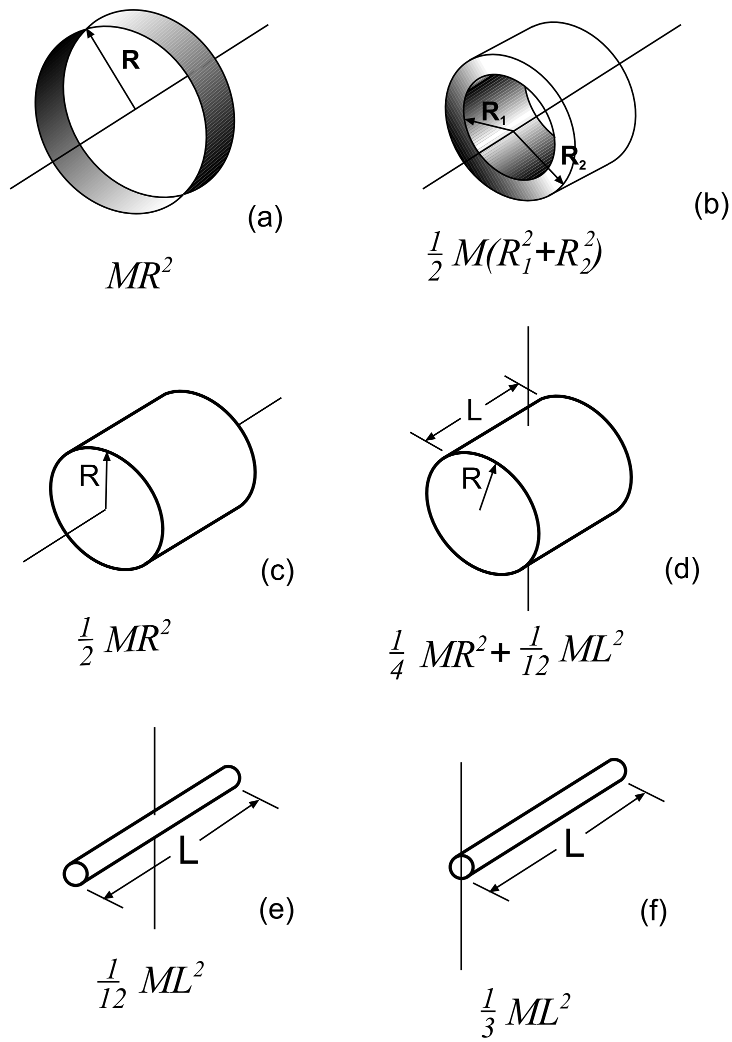

Figure 1.2: Formulae for the moment of inertia I for simple shapes about the axes, as shown. In each case, the mass of the object is M. (a) Hoop about symmetry axis. (b) Annular cylinder about symmetry axis. (c) Solid cylinder (disk) about symmetry axis. (d) Solid cylinder (disk) about axis through CM, perpendicular to symmetry axis. (e) Thin rod about axis through CM, perpendicular to length. (f) Thin rod about axis through one end, perpendicular to length.

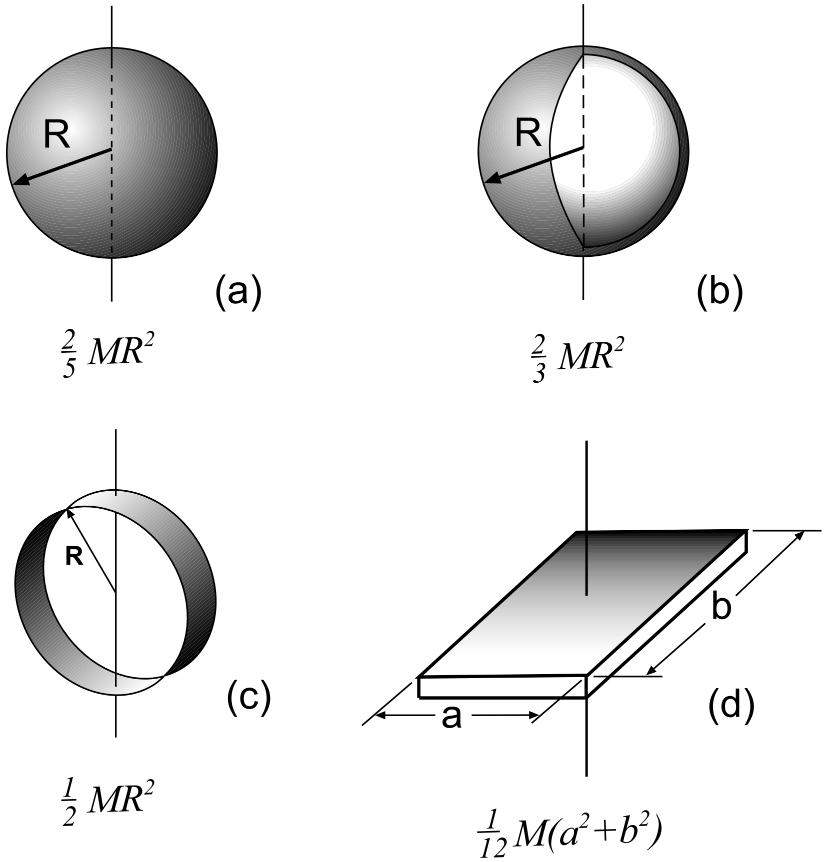

Figure 1.3: More formulae for moment of inertia I for simple shapes about the axes, as shown. (a) Solid sphere; axis is through a diameter. (b) Spherical shell; axis is through a diameter. (c) Hoop; axis is through a diameter. (d) Rectangular slab; axis is perpendicular, through the center.

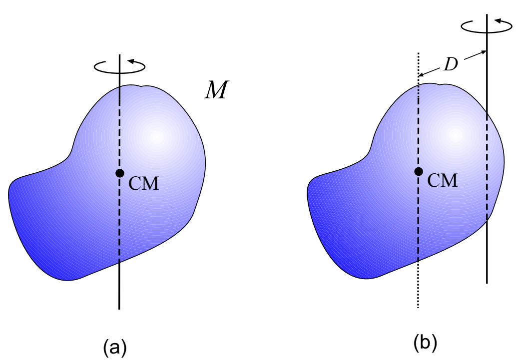

Figure 1.4: (a) Moment of inertia about an axis through the center of mass of an object is I_{CM}. (b) We displace the axis so that it is parallel and a distance D away. New moment of inertia is given by the Parallel Axis Theorem. This object looks like some kind of potato, but the theorem will work for any vegetable.

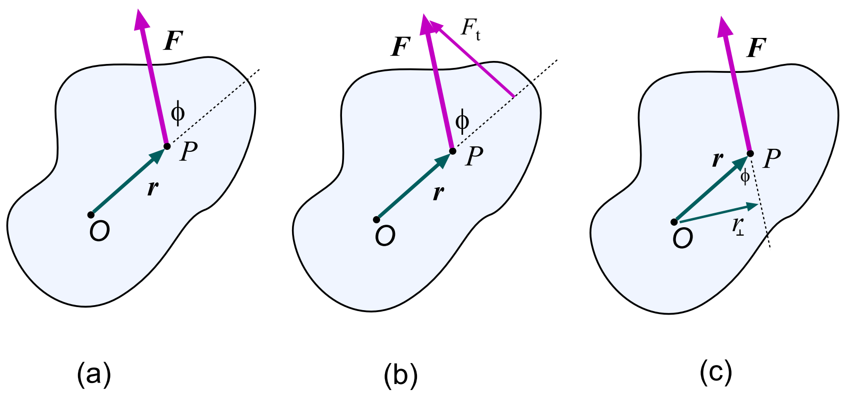

Figure 1.5: (a) Rigid body rotates about point O. A force $\mathbf{F}$ is exerted at point P. (b) Torque τ can be viewed as the product of the distance r and the tangential component of the force, F_t. (c) Torque τ can *also* be viewed as the product of the force F and the distance $r_\perp$ (often called the moment arm, or lever arm).

where $F_t = F\sin\phi$ is the component of the force perpendicular to $\mathbf{r}$, or as

$$\tau = (r\sin\phi)F = r_\perp F$$

where $r_\perp = r\sin\phi$ is the distance between the axis and the line which we get by "extending" the force vector into a line often called the **line of action**. The distance $r_\perp$ is called the **moment arm** of the force $\mathbf{F}$. These different ways of thinking about the terms in Eq. 1.17 are illustrated in Fig. 1.5.

Formula 1.17 gives us the magnitude of the torque; strictly speaking, torque is a *vector* and in more advanced studies of rotations, it must be treated as such. The (vector) torque due a force $\mathbf{F}$ is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

so we see that Eq. 1.17 gives the magnitude of this *vector*. For now we will treat torque as a *number* which is positive if the force gives a counterclockwise rotation and negative if the force gives a clockwise rotation. What we are calling τ here is really the component of the torque *vector* along the rotation axis.

To repeat, τ is positive or negative depending on whether the rotation which the force (acting alone) induces is counter-clockwise or clockwise. (However — on occasion we may want to choose the other convention.)

When a number of individual forces act on a rotating object, we can compute the net torque:

$$\tau_{\rm net} = \sum \tau_i$$

1.1.9 Torque and Angular Acceleration (Newton's Second Law for Rotations)

The angular acceleration of a rotating object is proportional to the net torque on the object; they are related by:

$$\tau_{\rm net} = I\alpha \qquad (1.18)$$

Linear Quantity	Angular Quantity
x	θ
v	ω
a	α
m	I
F	τ
p	L

Table 1.1: Correspondences between linear and angular quantities

where I is the moment of inertia of the object.

This equation looks a lot like Newton's Second Law for one–dimensional motion, $F_x = ma_x$.

1.1.10 Work, Energy and Power in Rotational Motion

As with the (abbreviated) formula for the work done by a force for a small displacement, linear motion, $W = F_x dx$, we have a formula for the work done by a *torque* for a small angular displacement $d\theta$:

$$W = \tau d\theta$$

For a finite angular displacement from θ_i to θ_f, the work done is

$$W = \int_{\theta_i}^{\theta_f} \tau \, d\theta$$

1.1.11 The New Equations Look Like the Old Equations

Although the *rotational* motion we have begun to study in this chapter is really quite a different thing from the *linear* motion we have studied up to now, it is of great help in using (or at least remembering) the equations by drawing correspondences between the new rotational quantities and the old linear quantities. These are summarized in Table 1.1. For completeness, we include the quantity *angular momentum*, L, which will be discussed in the next chapter.

The basic relations between the old linear quantities and the corresponding relations between the new angular quantities are summarized in Table 1.2. Of course, the first three pairs of equations deal with the special case of constant linear or angular acceleration.

A few pointers for solving problems which involve rotations:

• The objects which rotate are (obviously) all extended objects, that is, they have dimensions (width, height. . .) which can't be ignored. In some problems the center of mass of the rigid body *does* move and we may need to understand how the gravitational potential energy of the object changes. The basic answer is that we compute the gravitational potential energy by treating all the mass of the object as being located at its center of mass.

Linear Relation	Angular Relation
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
$F = ma$	$\tau = I\alpha$
$p = mv$	$L = I\omega$

Table 1.2: Correspondences between basic equations for linear and angular motion.

• Some problems involve a pulley which turns because a string is tightly wrapped around it or because a string passes over the pulley and does not slip against it. In those cases the string exerts a force on the pulley which is tangential, and we use this fact in computing the torque on the pulley. But see the next point...

• When a string does pass over a *real* pulley (i.e. it has mass even though we might still ignore friction in the bearings) and exerts forces on the pulley, the string tension will *not* be the same on both sides of the place where it is in contact. This differs from the problems in the chapter on forces where pulleys were involved; there, we used the idealization of massless pulleys, and the tension *was* the same on both sides.

1.2 Worked Examples

1.2.1 Angular Displacement

1. (a) What angle in radians is subtended by an arc that has length $1.80\,\mathrm{m}$ and is part of a circle of radius $1.20\,\mathrm{m}$? (b) Express the same angle in degrees. (c) The angle between two radii of a circle is $0.620\,\mathrm{rad}$. What arc length is subtended if the radius is $2.40\,\mathrm{m}$? [HRW5 11-1]

(a) Eq. 1.1 relates arclength, radius and subtended angle. We find:

$$\theta = \frac{s}{r} = \frac{1.80\,\mathrm{m}}{1.20\,\mathrm{m}} = 1.50\,\mathrm{rad}$$

The subtended angle is 1.50 radians.

(b) To express this angle in degrees use the relation: $360\,\mathrm{deg} = 2\pi\,\mathrm{rad}$ (or, $180\,\mathrm{deg} = \pi\,\mathrm{rad}$). Then we have:

$$1.50\,\mathrm{rad} = (1.50\,\mathrm{rad})\left(\frac{180\,\mathrm{deg}}{\pi\,\mathrm{rad}}\right) = 85.9^\circ$$

(c) We can find the arc length subtended by an angle θ by the relation: $s = \theta r$. Then for an angle of $0.620\,\mathrm{rad}$ and radius $2.40\,\mathrm{m}$, the arclength is

$$s = \theta r = (0.620)(2.40\,\mathrm{m}) = 1.49\,\mathrm{m}\ .$$

1.2.2 Angular Velocity

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2. What is the angular speed in radians per second of (a) the Earth in its orbit about the Sun and (b) the Moon in its orbit about the Earth? [Ser4 10-3]

(a) The Earth goes around in a (nearly!) circular path with a period of one year. In seconds, this is:

$$1\,\mathrm{yr} = (1\,\mathrm{yr})\left(\frac{365.25\,\mathrm{day}}{1\,\mathrm{yr}}\right)\left(\frac{24\,\mathrm{hr}}{1\,\mathrm{day}}\right)\left(\frac{3600\,\mathrm{s}}{1\,\mathrm{hr}}\right) = 3.156\times 10^{7}\,\mathrm{s}$$

In one year its angular displacement is 2π radians (all the way around) so its angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{(3.156\times 10^{7}\,\mathrm{s})} = 1.99\times 10^{-7}\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

(b) How long does it take the moon to go around the earth? That's a number we need to look up. You *ought* to know that it is about a month, but any good reference source will tell you that it is 27.3 days, officially called the *sidereal period* of the moon.. (Things get confusing because of the motion of the earth; full moons occur every 29.5 days, a period which is called the *synodic period*.) Converting to seconds, we have:

$$P = 27.3\,\mathrm{days} = (27.3\,\mathrm{days})\left(\frac{24\,\mathrm{hr}}{1\,\mathrm{day}}\right)\left(\frac{3600\,\mathrm{s}}{1\,\mathrm{hr}}\right) = 2.36\times 10^{6}\,\mathrm{s}$$

In that length of time the angular displacement of the moon is 2π so its angular speed is

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{(2.36\times 10^{6}\,\mathrm{s})} = 2.66\times 10^{-6}\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

1.2.3 Angular Acceleration; Constant Angular Acceleration

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3. The angular position of a point on the rim of a rotating wheel is given by $\theta = 4.0t - 3.0t^2 + t^3$, where θ is in radians if t is given in seconds. (a) What are the angular velocities at $t = 2.0\,\mathrm{s}$ and $t = 4.0\,\mathrm{s}$? (b) What is the average angular acceleration for the time interval that begins at $t = 2.0\,\mathrm{s}$ and ends at $t = 4.0\,\mathrm{s}$? (c) What are the instantaneous angular accelerations at the beginning and end of this time interval? [HRW5 11-5]

(a) In the problem we are given the angular position θ as a function of time. To find the (instantaneous) angular velocity at any time, use Eq. 1.3 and find:

$$\begin{aligned}\omega(t) &= \frac{d\theta}{dt} = \frac{d}{dt}\left(4.0t - 3.0t^2 + t^3\right)\\ &= 4.0 - 6.0t + 3.0t^2\end{aligned}$$

where, if t is given in seconds, ω is given in $\frac{\mathrm{rad}}{\mathrm{s}}$.

The angular velocities at the given times are then

$$\omega(2.0\,\mathrm{s}) = 4.0 - 6.0(2.0) + 3.0(2.0)^2 = 4.0\,\frac{\mathrm{rad}}{\mathrm{s}}$$

$$\omega(4.0\,\mathrm{s}) = 4.0 - 6.0(4.0) + 3.0(4.0)^2 = 28.0\,\frac{\mathrm{rad}}{\mathrm{s}}$$

(b) Since we have the values of ω and $t = 2.0\,\mathrm{s}$ and $t = 4.0\,\mathrm{s}$, Eq. 1.4 gives the average angular acceleration for the interval:

$$\begin{aligned} \overline{\alpha} &= \frac{\Delta\omega}{\Delta t} = \frac{(28.0\,\frac{\mathrm{rad}}{\mathrm{s}} - 4.0\,\frac{\mathrm{rad}}{\mathrm{s}})}{(4.0\,\mathrm{s} - 2.0\,\mathrm{s})} \\ &= 12.0\,\frac{\mathrm{rad}}{\mathrm{s}^2} \end{aligned}$$

The average angular acceleration is $12.0\,\frac{\mathrm{rad}}{\mathrm{s}^2}$.

(c) We find the instantaneous angular acceleration from Eq. 1.5:

$$\begin{aligned} \alpha(t) &= \frac{d\omega}{dt} = \frac{d}{dt}\left(4.0 - 6.0t + 3.0t^2\right) \\ &= -6.0 + 6.0t \end{aligned}$$

where, if t is given in seconds, α is given in $\frac{\mathrm{rad}}{\mathrm{s}^2}$.

Then at the beginning and end of our time interval the angular accelerations are:

$$\alpha(2.0\,\mathrm{s}) = -6.0 + 6.0(2.0) = 6.0\,\frac{\mathrm{rad}}{\mathrm{s}^2}$$

$$\alpha(4.0\,\mathrm{s}) = -6.0 + 6.0(4.0) = 18.0\,\frac{\mathrm{rad}}{\mathrm{s}^2}$$

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4. An electric motor rotating a grinding wheel at $100\,\mathrm{rev/min}$ is switched off. Assuming constant negative angular acceleration of magnitude $2.00\,\frac{\mathrm{rad}}{\mathrm{s}^2}$, (a) how long does it take the wheel to stop? (b) Through how many radians does it turn during the time found in (a)? [Ser4 10-5]

(a) Convert the initial rotation rate to radians per second:

$$100\,\frac{\mathrm{rev}}{\mathrm{min}} = \left(100\,\frac{\mathrm{rev}}{\mathrm{min}}\right)\left(\frac{2\pi\,\mathrm{rad}}{1\,\mathrm{rev}}\right)\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) = 10.5\,\frac{\mathrm{rad}}{\mathrm{s}}$$

When the wheel has stopped then of course its angular velocity is zero. Since we know ω_0, ω and α we can use Eq. 1.6 to get the elapsed time:

$$\omega = \omega_0 + \alpha t \qquad \Longrightarrow \qquad t = \frac{(\omega - \omega_0)}{\alpha}$$

and we get:

$$t = \frac{(0 - 10.5\,\frac{\mathrm{rad}}{\mathrm{s}})}{(-2.00\,\frac{\mathrm{rad}}{\mathrm{s}^2})} = 5.24\,\mathrm{s}$$

The wheel takes 5.24 s to stop.

(b) We want to find the angular displacement θ during the time of stopping. Since we know that the angular acceleration is constant we can use Eq 1.9, and it might be simplest to do so. then we have:

$$\theta = \tfrac{1}{2}(\omega_0 + \omega)t = \tfrac{1}{2}(10.5\,\tfrac{\mathrm{rad}}{\mathrm{s}} + 0)(5.24\,\mathrm{s}) = 27.5\,\mathrm{rad} \quad .$$

The wheel turns through 27.5 radians in coming to a halt.

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5. A phonograph turntable rotating at $33\frac{1}{3}\,\mathrm{rev/min}$ slows down and stops in $30\,\mathrm{s}$ after the motor is turned off. (a) Find its (uniform) angular acceleration in units of $\mathrm{rev/min^2}$. (b) How many revolutions did it make in this time? [HRW5 11-12]

(a) Here we are given the initial angular velocity of the turntable and its final angular velocity (namely zero, when it stops) and the time interval between them. We can use Eq. 1.6 to find α, which we are told is constant. We have:

$$\alpha = \frac{\omega - \omega_0}{t}$$

We don't need to convert the units of the data to radians and seconds; if we watch our units, we *can* use revolutions and minutes. Noting that the time for the turntable to stop is $t = 30\,\mathrm{s} = 0.50\,\mathrm{min}$, and with $\omega_0 = 33.3\,\frac{\mathrm{rev}}{\mathrm{min}}$ and $\omega = 0$ we find:

$$\alpha = \frac{0 - 33.3\,\frac{\mathrm{rev}}{\mathrm{min}}}{0.50\,\mathrm{min}} = -66.7\,\frac{\mathrm{rev}}{\mathrm{min}^2}$$

The angular acceleration of the turntable during the time of stopping was $-66.7\,\frac{\mathrm{rev}}{\mathrm{min}^2}$. (The minus sign indicates a *deceleration*, that is, an angular acceleration opposite to the sense of the angular velocity.)

(b) Here we want to find the value of θ at $t = 0.50\,\mathrm{min}$. To get this, we can use either Eq. 1.7 or Eq. 1.9. With $\theta_0 = 0$, Eq. 1.9 gives us:

$$\begin{aligned} \theta &= \theta_0 + \tfrac{1}{2}(\omega_0 + \omega)t \\ &= \tfrac{1}{2}(33.3\,\tfrac{\mathrm{rev}}{\mathrm{min}} + 0)(0.50\,\mathrm{min}) \\ &= 8.33\,\mathrm{rev} \end{aligned}$$

The turntable makes 8.33 revolutions as it slows to a halt.

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6. A disk, initially rotating at $120\,\frac{\mathrm{rad}}{\mathrm{s}}$, is slowed down with a constant angular acceleration of magnitude $4.0\,\frac{\mathrm{rad}}{\mathrm{s}^2}$. (a) How much time elapses before the disk stops? (b) Through what angle does the disk rotate in coming to rest? [HRW5 11-13]

(a) We are given the initial angular velocity of the disk, $\omega_0 = 120\,\frac{\text{rad}}{\text{s}}$. (We let the positive sense of rotation be the same as that of the initial motion.) We are given the *magnitude* of the disk's angular acceleration as it slows, but then we must write

$$\alpha = -4.0\,\frac{\text{rad}}{\text{s}^2}\ .$$

The final angular velocity (when the disk has stopped!) is $\omega = 0$. Then from Eq. 1.6 we can solve for the time t:

$$\omega = \omega_0 + \alpha t \qquad \Longrightarrow \qquad t = \frac{\omega - \omega_0}{\alpha}$$

and we get:

$$t = \frac{(0 - 120\,\frac{\text{rad}}{\text{s}})}{(-4.0\,\frac{\text{rad}}{\text{s}^2})} = 30.0\,\text{s}$$

(b) We'll let the initial angle be $\theta_0 = 0$. We can now use any of the constant–α equations containing θ to solve for it; let's choose Eq. 1.8, which gives us:

$$\omega^2 = \omega_0^2 + 2\alpha(\theta) \qquad \Longrightarrow \qquad \theta = \frac{(\omega^2 - \omega_0^2)}{2\alpha}$$

and we get:

$$\theta = \frac{(0^2 - (120\,\frac{\text{rad}}{\text{s}})^2)}{2(-4.0\,\frac{\text{rad}}{\text{s}^2})} = 1800\,\text{rad}$$

The disk turns through an angle of 1800 radians before coming to rest.

---

7. A wheel, starting from rest, rotates with a constant angular acceleration of $2.00\,\frac{\text{rad}}{\text{s}^2}$. During a certain $3.00\,\text{s}$ interval, it turns through $90.0\,\text{rad}$. (a) How long had the wheel been turning before the start of the $3.00\,\text{s}$ interval? (b) What was the angular velocity of the wheel at the start of the $3.00\,\text{s}$ interval? [HRW5 11-19]

(a) We are told that some time after the wheel starts from rest we measure the angular displacement for *some* $3.00\,\text{s}$ interval and it is $9.00\,\text{rad}$. Suppose we start measuring time at the beginning of this interval; since this time measurement isn't from the beginning of the wheel's motion, we'll call it t'. Now, with the usual choice $\theta_0 = 0$ we know that at $t' = 3.00\,\text{s}$ we have $\theta = 90.0\,\text{rad}$. Also $\alpha = 2.00\,\frac{\text{rad}}{\text{s}^2}$. Using Eq. 1.7 get:

$$90.0\,\text{rad} = \omega_0(3.00\,\text{s}) + \tfrac{1}{2}(2.00\,\tfrac{\text{rad}}{\text{s}^2})(3.00\,\text{s})^2$$

which we can use to solve for ω_0:

$$\omega_0(3.00\,\text{s}) = 90.0\,\text{rad} - \tfrac{1}{2}(2.00\,\tfrac{\text{rad}}{\text{s}^2})(3.00\,\text{s})^2 = 81.0\,\text{rad}$$

so that

$$\omega_0 = 27.0\,\frac{\text{rad}}{\text{s}}$$

(Looking ahead, we can see that we've already answered part (b)!)

Now suppose we measure time from the beginning of the wheel's motion with the variable t. We want to find the length of time required for ω to get up to the value $27.0\,\frac{\rm rad}{\rm s}$. For this period the initial angular velocity is $\omega_0 = 0$ and the final angular velocity is $27.0\,\frac{\rm rad}{\rm s}$. Since we have α we can use Eq. 1.6 to get t:

$$\omega = \omega_0 + \alpha t \qquad \Longrightarrow \qquad t = \frac{\omega - \omega_0}{\alpha}$$

which gives

$$t = \frac{(27.0\,\frac{\rm rad}{\rm s} - 0\,\frac{\rm rad}{\rm s})}{2.00\,\frac{\rm rad}{\rm s^2}} = 13.5\,{\rm s}$$

This tells us that the wheel had been turning for 13.5 s before the start of the 3.00 s interval.

(b) In part (a) we found that at the beginning of the 3.00 s interval the angular velocity was $27.0\,\frac{\rm rad}{\rm s}$.

1.2.4 Relationship Between Angular and Linear Quantities

---

8. What is the angular speed of a car travelling at $50\,\frac{\rm km}{\rm hr}$ and rounding a circular turn of radius $110\,{\rm m}$? [HRW5 11-27]

To work consistently in SI units, convert the speed of the car:

$$v = 50\,\frac{\rm km}{\rm hr} = (50\,\frac{\rm km}{\rm hr})\left(\frac{10^3\,{\rm m}}{1\,{\rm km}}\right)\left(\frac{1\,{\rm hr}}{3600\,{\rm s}}\right) = 13.9\,\frac{\rm m}{\rm s}$$

The relation between the car's "linear" speed v and its angular speed ω as it goes around the track is $v = r\omega$. This gives:

$$\omega = \frac{v}{r} = \frac{13.9\,\frac{\rm m}{\rm s}}{110\,{\rm m}} = 0.126\,\frac{\rm rad}{\rm s}$$

---

9. An astronaut is being tested in a centrifuge. The centrifuge has a radius of $10\,{\rm m}$ and, in starting, rotates according to $\theta = 0.30t^2$, where t in seconds gives θ in radians. When $t = 5.0\,{\rm s}$, what are the astronaut's (a) angular velocity, (b) linear speed, (c) tangential acceleration (magnitude only) and (d) radial acceleration (magnitude only)? [HRW5 11-32]

(a) We are given θ as a function of time. We get the angular velocity from its definition, Eq. 1.3,

$$\omega = \frac{d\theta}{dt} = \frac{d}{dt}(0.30t^2) = 0.60t$$

where we mean that when t is in seconds, ω is given in $\frac{\rm rad}{\rm s}$. When $t = 5.0\,{\rm s}$ this is

$$\omega = (0.60)(5.0)\,\frac{\rm m}{\rm s} = 3.0\,\frac{\rm rad}{\rm s}$$

(b) The linear speed of the astronaut is found from Eq. 1.10 (the linear or tangential speed of a mass point):

$$v = R\omega = (10\,\mathrm{m})(0.60t) = 6.0t$$

where we mean that when t is given in seconds, v is given in $\frac{\mathrm{m}}{\mathrm{s}}$. When $t = 5.0\,\mathrm{s}$ this is

$$v = (6.0)(5.0)\,\tfrac{\mathrm{m}}{\mathrm{s}} = 30.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

(c) The (magnitude of the) tangential acceleration of a mass point is given by Eq. 1.11. We will need the angular acceleration α, which is

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}(0.60t) = 0.60$$

so that

$$a_T = R\alpha = (10\,\mathrm{m})(0.60) = 6.0\,\tfrac{\mathrm{m}}{\mathrm{s}^2}\ .$$

Here, since a_T is constant we have written in the appropriate units, which are *mpss*. (Since a_T is constant, the answer is the same at $t = 5.0\,\mathrm{s}$ as at any other time.)

(d) The radial acceleration of the astronaut is our old friend (?) the centripetal acceleration. From Eq. 1.12 we can get the magnitude of a_c from:

$$a_c = R\omega^2 = (10\,\mathrm{m})(0.60t)^2 = 3.6t^2$$

where we mean that if t is given in seconds, a_c is given in *mpss*. When $t = 5.0\,\mathrm{s}$ this is

$$a_c = (3.6)(5.0)^2\,\tfrac{\mathrm{m}}{\mathrm{s}^2} = 90\,\tfrac{\mathrm{m}}{\mathrm{s}^2}$$

1.2.5 Rotational Kinetic Energy

---

10. Calculate the rotational inertia of a wheel that has a kinetic energy of $24,400\,\mathrm{J}$ **when rotating at** $602\,\mathrm{rev/min}$**.** [HRW5 11-45]

Find the angular speed ω of the wheel, in $\frac{\mathrm{rad}}{\mathrm{s}}$:

$$\omega = 602\,\tfrac{\mathrm{rev}}{\mathrm{min}}\left(\frac{2\pi\,\mathrm{rad}}{1\,\mathrm{rev}}\right)\left(\frac{1\,\mathrm{min}}{60\,\mathrm{s}}\right) = 63.0\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

We have ω and the kinetic energy of rotation, K_{rot}, so we can find the rotational kinetic energy from

$$K_{\mathrm{rot}} = \tfrac{1}{2}I\omega^2 \qquad \Longrightarrow \qquad I = \frac{2K_{\mathrm{rot}}}{\omega^2}$$

We get:

$$I = \frac{2(24,400\,\mathrm{J})}{(63.0\,\frac{\mathrm{rad}}{\mathrm{s}})^2} = 12.3\,\mathrm{kg}\cdot\mathrm{m}^2$$

The moment of inertia of the wheel is $12.3\,\mathrm{kg}\cdot\mathrm{m}^2$.

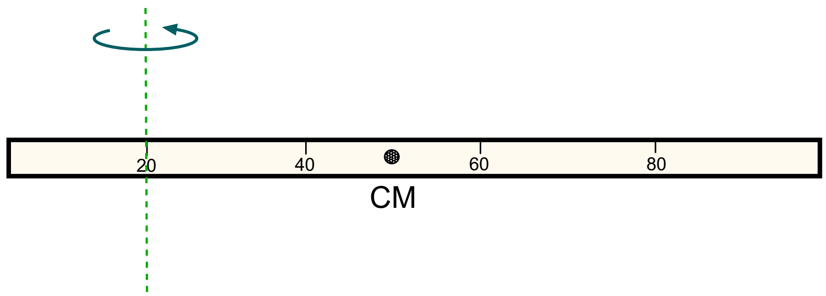

Figure 1.6: Rotating system for Example 11.

1.2.6 The Moment of Inertia (and More Rotational Kinetic Energy)

---

11. Calculate the rotational inertia of a meter stick with mass $0.56\,\mathrm{kg}$**, about an axis perpendicular to the stick and located at the** $20\,\mathrm{cm}$ **mark.** [HRW5 11-53]

A picture of this rotating system is given in Fig. 1.6. The stick is one meter long (being a meter stick and all that) and we take it to be uniform so that its center of mass is at the $50\,\mathrm{cm}$ mark. But the axis of rotation goes through the $20\,\mathrm{cm}$ mark.

Now if the axis *did* pass through the center of mass (perpendicular to the stick), we would know how to find the rotational inertia; from Figure 1.2 we see that it would be

$$I_{\mathrm{CM}} = \tfrac{1}{12}ML^2 = \tfrac{1}{12}(0.56\,\mathrm{kg})(1.00\,\mathrm{m})^2 = 4.7\times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$$

The rotational inertia about *our* axis will not be the same.

We note that our axis is displaced from the one through the CM by $30\,\mathrm{cm}$. Then the Parallel Axis Theorem (Eq. 1.16) tells us that the moment of inertia about our axis is given by

$$I = I_{\mathrm{CM}} + MD^2$$

where $I_{\mathrm{CM}} = 4.7\times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$, as we've already found, M is the mass of the rod and D is the distance the axis is displaced (parallel to itself), namely $30\,\mathrm{cm}$. We get:

$$I = 4.7\times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2 + (0.56\,\mathrm{kg})(0.30\,\mathrm{m})^2 = 9.7\times 10^{-2}\,\mathrm{kg}\cdot\mathrm{m}^2$$

So the rotational inertia of the stick *about the given axis* is $0.097\,\mathrm{kg}\cdot\mathrm{m}^2$.

---

12. Two masses M **and** m **are connected by a rigid rod of length** L **and negligible mass as in Fig. 1.7. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axis passes through the center of mass. Show that this moment of inertia is** $I = \mu L^2$**, where** $\mu = mM/(m+M)$**.** [Ser4 10-21]

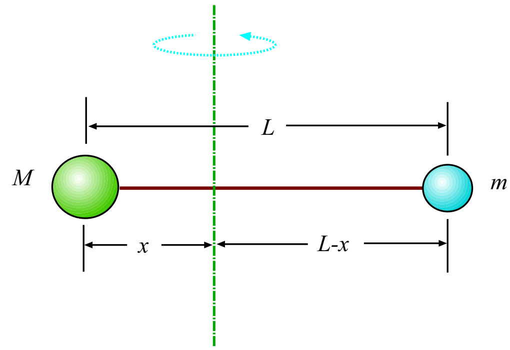

Figure 1.7: Rotating system for Example 12.

As noted in the figure, let the distance of the axis from the mass M be x; then its distance from mass m must be $L - x$. The definition of the moment of inertia tells us that for this system

$$
\begin{aligned}
I &= \sum_i m_i r_i^2 \\
&= Mx^2 + m(L-x)^2 = Mx^2 + m(L^2 - 2Lx + x^2) \\
&= (M+m)x^2 - 2mLx + mL^2
\end{aligned}
$$

The final expression shows that as a function of x, I is quadratic with a positive coefficient in front of the x^2 term. The graph of this function is a parabola which faces up, and it has a minimum at the value of x for which $dI/dx = 0$. Solving for this value, we get:

$$
\frac{dI}{dx} = 2(M+m)x - 2mL = 0 \qquad \Longrightarrow \qquad x = \frac{mL}{(M+m)}
$$

(Note, since $\frac{m}{(M+m)}$ is a number between zero and 1 this *is* a point between the two masses.)

Now, taking the origin to be at mass M, the coordinate of the center of mass of this system is located at

$$
x_{\rm CM} = \frac{(M \cdot 0 + m \cdot L)}{(m+M)} = \frac{mL}{(m+M)}
$$

so the axis of minimum I does indeed pass through the center of mass.

Substituting this value of x into the expression for I we find:

$$
\begin{aligned}
I_{\rm min} &= M\left(\frac{mL}{(M+m)}\right)^2 + m\left(L - \frac{mL}{(M+m)}\right)^2 \\
&= \frac{Mm^2L^2}{(M+m)^2} + m\left(\frac{L(M+m) - mL}{(M+m)}\right)^2 \\
&= \frac{Mm^2L^2}{(M+m)^2} + m\left(\frac{ML}{(M+m)}\right)^2 \\
&= \frac{Mm^2L^2 + mM^2L^2}{(M+m)^2} = \frac{mM(m+M)}{(M+m)^2}L^2
\end{aligned}
$$

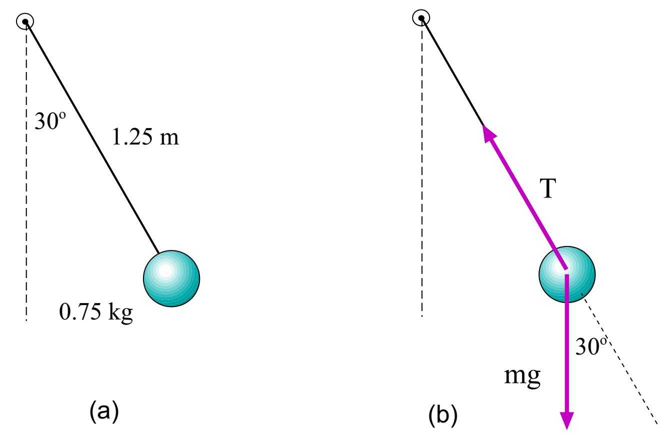

Figure 1.8: The pendulum of Example 13, at a position of 30° from the vertical. The second picture shows the forces acting on the ball at the end of the rod.

$$= \frac{mM}{(M+m)} L^2$$

Now if we let $\mu = mM/(M+m)$ we can write this as

$$I_{\min} = \mu L^2 \ .$$

1.2.7 Torque

---

13. A small $0.75\,\mathrm{kg}$ ball is attached to one end of a $1.25\,\mathrm{m}$ long massless rod, and the other end of the rod is hung from a pivot. When the resulting pendulum is 30° from the vertical, what is the magnitude of the torque about the pivot? [HRW5 11-61]

Draw a picture of the system! In Fig. 1.8 we show the basic geometry and also the forces which are acting on the ball at the end of the rod. (The rod is massless so no external forces act on it; we only need to worry about the torque produced by the forces acting on the ball.)

To calculate the torques due to each force, we need the magnitude of the force, the distance at which it acts from the pivot (here, it is $1.25\,\mathrm{m}$) and the angle between the force and the lever arm (that is, the line joining the pivot to the place where the force acts).

You might think we would want to find the tension T in the rod before calculating the torques, but it is not necessary. The force of the rod on the ball points *along* the vector $\mathbf{r}$. So $\phi = 0$ and there is no torque.

The force of gravity makes an angle of 30° from the vector $\mathbf{r}$ and since it has magnitude mg, the magnitude of its torque is

$$\tau = rF\sin\phi = (1.25\,\mathrm{m})(0.75\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})\sin 30^\circ = 4.6\,\mathrm{N}\cdot\mathrm{m}$$

The net torque has magnitude $4.6\,\mathrm{N}\cdot\mathrm{m}$.

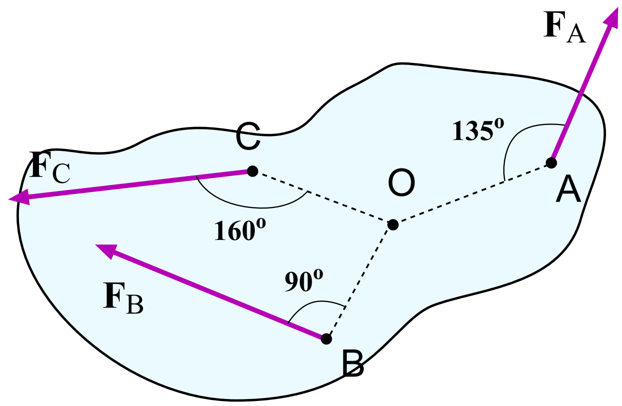

Figure 1.9: Forces acting on a rotating body in Example 14.

---

14. The body in Fig. 1.9 is pivoted at O. Three forces act on it in the directions shown on the figure: $F_A = 10\,\mathrm{N}$ at point A, $8.0\,\mathrm{m}$ from O; $F_B = 16\,\mathrm{N}$ at point B, $4.0\,\mathrm{m}$ from O; and $F_C = 19\,\mathrm{N}$ at point C, $3.0\,\mathrm{m}$ from O. What is the net torque about O? [HRW5 11-64]

Start with the force applied at point A. Here $r_A = 8.0\,\mathrm{m}$ and for our purposes ϕ_A can be taken as 135° or 45°. The *magnitude* of the torque imparted by $\mathbf{F}_A$ is

$$|\tau_A| = |r_A F_A \sin\phi_A| = |(8.0\,\mathrm{m})(10.0\,\mathrm{N})\sin(135^\circ)| = 56.6\,\mathrm{N}\cdot\mathrm{m}$$

Clearly $\mathbf{F}_A$ is a force which gives a torque in the counterclockwise (positive, usually) sense, so we write:

$$\tau_A = +56.6\,\mathrm{N}\cdot\mathrm{m}$$

Next, the force applied at B. With $r_B = 4.0\,\mathrm{m}$ and $\phi_B = 90^\circ$ we find:

$$|\tau_B| = |r_B F_B \sin\phi_B| = |(4.0\,\mathrm{m})(16.0\,\mathrm{N})\sin(90^\circ)| = 64.0\,\mathrm{N}\cdot\mathrm{m}$$

but $\mathbf{F}_B$ is clearly a force which gives a *clockwise* (i.e. negative) torque, and so

$$\tau_B = -64.0\,\mathrm{N}\cdot\mathrm{m}$$

And finally, the force applied at C. With $r_C = 3.0\,\mathrm{m}$ and $\phi_C = 160^\circ$, we have

$$|\tau_C| = |r_C F_C \sin\phi_C| = |(3.0\,\mathrm{m})(19.0\,\mathrm{N})\sin(160^\circ)| = 19.5\,\mathrm{N}\cdot\mathrm{m}\ .$$

One can see that $\mathbf{F}_C$ gives a counterclockwise torque and so

$$\tau_C = +19.5\,\mathrm{N}\cdot\mathrm{m}$$

Now find the total torque!

$$\tau_{\mathrm{net}} = \sum \tau_i = (56.6\,\mathrm{N}\cdot\mathrm{m}) + (-64.0\,\mathrm{N}\cdot\mathrm{m}) + (19.5\,\mathrm{N}\cdot\mathrm{m}) = 12.1\,\mathrm{N}\cdot\mathrm{m}$$

The net (counterclockwise) torque on the object is $12.1\,\mathrm{N}\cdot\mathrm{m}$.

1.2.8 Torque and Angular Acceleration (Newton's Second Law for Rotations)

---

15. When a torque of $32.0\,\mathrm{N\cdot m}$ is applied to a certain wheel, the wheel acquires an angular acceleration of $25.0\,\frac{\mathrm{rad}}{\mathrm{s}^2}$. What is the rotational inertia of the wheel? [HRW5 11-65]

Torque, angular acceleration and the moment of inertia are related by $\tau = I\alpha$. Solving for I, we find:

$$I = \frac{\tau}{\alpha} = \frac{32.0\,\mathrm{N\cdot m}}{25.0\,\frac{\mathrm{rad}}{\mathrm{s}^2}} = 1.28\,\mathrm{kg\cdot m^2}$$

---

16. A thin spherical shell has a radius of $1.90\,\mathrm{m}$. An applied torque of $960\,\mathrm{N\cdot m}$ imparts to the shell an angular acceleration equal to $6.20\,\frac{\mathrm{rad}}{\mathrm{s}^2}$ about an axis through the center of the shell. (a) What is the rotational inertia of the shell about the axis of rotation? (b) Calculate the mass of the shell. [HRW5 11-69]

(a) From the relation between (net) torque, moment of inertia and angular acceleration, $\tau = I\alpha$ we have

$$I = \frac{\tau}{\alpha}$$

Using the given torque on the shell and its angular acceleration we find that the rotational inertia of the shell is

$$I = \frac{(960\,\mathrm{N\cdot m})}{(6.20\,\frac{\mathrm{rad}}{\mathrm{s}^2})} = 155\,\mathrm{kg\cdot m^2}\ .$$

(b) From Figure 1.3 the moment of inertia for a spherical shell of mass M and radius R rotating about any diameter is

$$I = \tfrac{2}{3}MR^2$$

We know I and R so we can solve for the mass of the shell:

$$M = \frac{3I}{2R^2} = \frac{3(155\,\mathrm{kg\cdot m^2})}{2(1.90\,\mathrm{m})^2} = 64.4\,\mathrm{kg}$$

---

17. A wheel of radius $0.20\,\mathrm{M}$ is mounted on a frictionless horizontal axis. A massless cord is wrapped around the wheel and attached to a $2.0\,\mathrm{kg}$ object that slides on a frictionless surface inclined at 20° with the horizontal, as shown in Fig. 1.10. The object accelerates down the incline at $2.0\,\frac{\mathrm{m}}{\mathrm{s}^2}$. What is the rotational inertia of the wheel about its axis of rotation? [HRW5 11-72]

As in most problems of this type, we will find a solution by diagramming the forces which act on each mass and then using Newton's laws to set up equations. Here there are *two* masses we need to isolate: The block and the pulley.

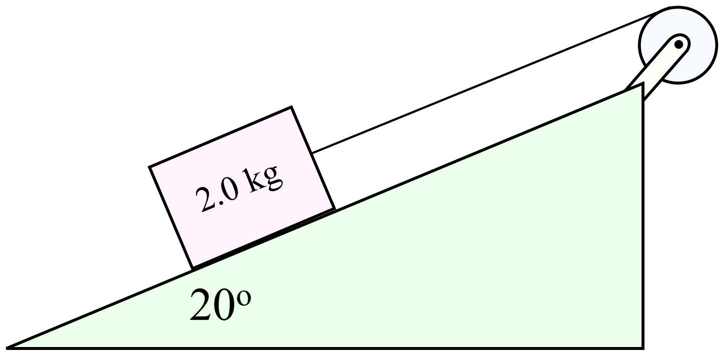

Figure 1.10: Sliding mass on string and pulley, as described in Example 17.

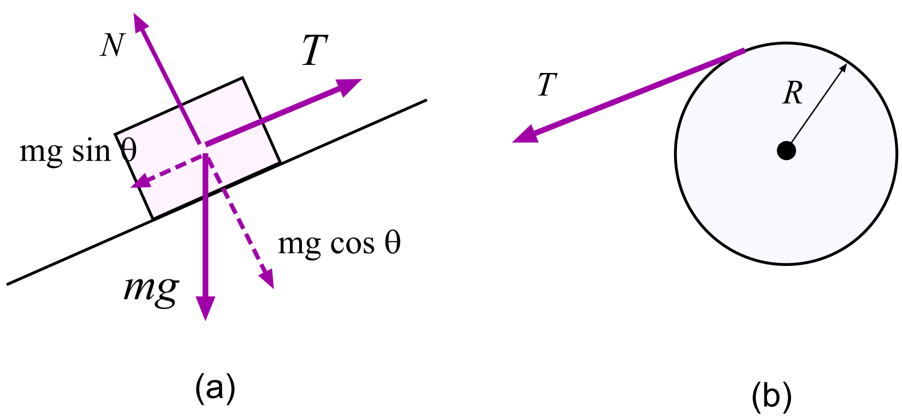

Figure 1.11: (a) Forces on the sliding mass in Example 17. (b) Force (and its position of application on the pulley) in Example 17.

We start with the sliding mass; the forces acting on it are shown in Fig. 1.11(a). There is the downward force of gravity mg on the mass, which we separate into its components: $mg\sin\theta$ down the slope and $mg\cos\theta$ into the plane. There is the normal force N of the surface (outward and perpendicular to the slope) and the tension of the string T which points up the slope. Since in this problem we take the surface to be frictionless, that's all.

The block can only move along the slope so the forces perpendicular to the slope must cancel. This gives $N = mg\cos\theta$, which we don't really need! If the acceleration down the slope is a, then adding up the "down–the–slope" forces and using Newton's 2$^{\rm nd}$ law gives:

$$mg\sin\theta - T = ma \tag{1.19}$$

Now consider the pulley. With the string wrapped around it at a radius R, the action of the string on the wheel is that of a tangential force of magnitude T applied at a distance R, shown in Fig. 1.11(b). The force is perpendicular to the line joining the axle and the point of application so that it gives a (counterclockwise) torque of magnitude $rF\sin\phi = RT$. Then the relation between torque and angular acceleration gives:

$$\tau = RT = I\alpha$$

where I is the moment of inertia of the wheel.

Now if the string is wrapped around the wheel then as it rolls off the edge of the wheel it must be true that the linear acceleration of any piece of the string is the same as the tangential acceleration of the wheel's edge, and this is also the same as the acceleration of the mass down the slope. Since the tangential acceleration of the edge of the wheel is $a_t = R\alpha$, we have $a = R\alpha$, or $\alpha = a/R$. Putting this into the last equation gives

$$RT = I\frac{a}{R} \qquad \text{or} \qquad T = \frac{Ia}{R^2}$$

and this gives an expression for T that we can put into Eq. 1.19. Doing this, we get

$$mg\sin\theta - \frac{Ia}{R^2} = ma$$

Solving for I we find

$$Ia/R^2 = m(g\sin\theta - a) \qquad \Longrightarrow \qquad I = mR^2\frac{(g\sin\theta - a)}{a}$$

Now we can plug in the numbers and we get:

$$I = (2.0\,\mathrm{kg})(0.20\,\mathrm{m})^2\frac{((9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})\sin 20^\circ - 2.0\,\frac{\mathrm{m}}{\mathrm{s}^2})}{(2.0\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 5.4\times 10^{-2}\,\mathrm{kg\cdot m^2}$$

The rotational inertia of the wheel is $0.054\,\mathrm{kg\cdot m^2}$.

---

18. A block of mass m_1 and one of mass m_2 are connected by a massless string over a pulley that is in the shape of a disk having radius R and mass M. In

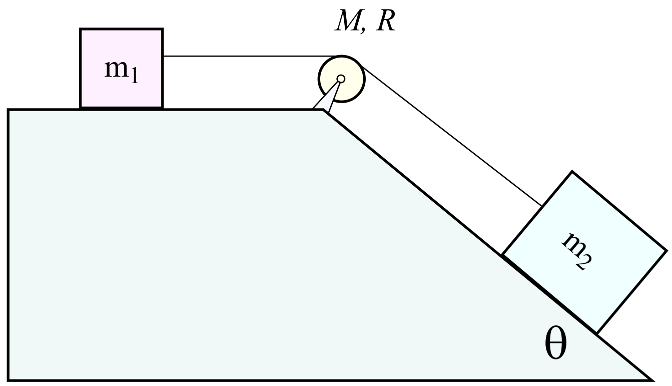

Figure 1.12: Masses joined by string and moving on wedge; pulley (a disk) has mass!

addition, the blocks are allowed to move on a fixed block–wedge of angle θ as in Fig. 1.12. The coefficient of kinetic friction for the motion of either mass on the wedge surface is μ_k. Determine the acceleration of the two blocks. [Ser4 10-29]

This problem is similar to ones you may have seen in the chapter on force problems. We will solve it in the same way, by drawing a free–body diagram for each mass, writing out Newton's $2^{\rm nd}$ Law for each mass and solving the equations. The difference comes from the fact that one of the mass elements in this problem (the pulley!) undergoes *rotational* motion because of the torques that are exerted on it. It will undergo an angular acceleration which will be related to the common *linear* acceleration of the two blocks.

We start by noting that for a problem where a string passes over a pulley *which has mass* there will be a *different* tension for each section of the string. (This differs from the case of the "ideal massless" pulley where the string tension was the *same* on both sides.) We will see better why this has to be true when we look at the torques acting on the pulley. The part of the string which is connected to m_1 will have a tension T_1 and the part of the string connected to m_2 will have a tension T_2.

Then we think about how the blocks are going to move in this situation. As they are connected by a string, the magnitudes of their accelerations will be the same; we can call it a. We *know* that regardless of the values of the masses, m_1 must move to the right and m_2 must move down the slope. (This assumes that friction is not so strong as to prohibit any motion!) And the pulley will be turning clockwise.

Now we can draw the force diagrams for the two blocks; we get the diagrams given in Fig. 1.13. On m_1 we have the downward force of gravity $m_1 g$ and the upward normal force N_1 of the surface; the string pulls to the right with a force of magnitude T_1 while there is a leftward friction force on the block which we denote $f_{k,1}$. We know that m_1 has an acceleration to right of magnitude a.

The vertical forces on m_1 must cancel out, giving $N_1 = m_1 g$. The kinetic friction force is then

$$f_{k,1} = \mu_k N_1 = \mu_k m_1 g$$

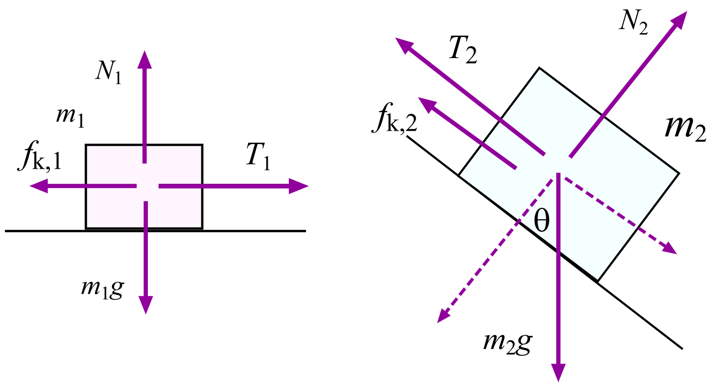

Figure 1.13: Free–body diagrams for the masses m_1 and m_2.

Applying Newton's 2nd law for the horizontal forces gives

$$T_1 - f_{k,1} = T_1 - \mu_k m_1 g = m_1 a$$

giving us our first "take–home" equation,

$$T_1 - \mu_k m_1 g = m_1 a \tag{1.20}$$

We move on to m_2. On this mass we have the downward force of gravity, $m_2 g$, (which we show as the sum of its components along the slope and perpendicular to the slope) the normal force from the surface, N_2, the force from the string (magnitude T_2, directed up the slope) and the force of kinetic friction, which has magnitude $f_{k,2}$ and is directed *up* the slope because we know that block m_2 is moving *down* the slope.

The forces (that is, their components) perpendicular to the slope must cancel out, and from our experience with similar problems we can see that this will give

$$N_2 = m_2 g \cos\theta \ .$$

This gives the magnitude of the force of kinetic friction on m_2,

$$f_{k,2} = \mu_k N_2 = \mu_k m_2 g \cos\theta$$

Now look at the force components on m_2 in the "down–the–slope" direction. By Newton's 2nd law they sum to give $m_2 a$, and so we write:

$$m_2 g \sin\theta - T_2 - f_{k,2} = m_2 a$$

We substitute for $f_{k,2}$ and get our next "take–home" equation,

$$m_2 g \sin\theta - T_2 - \mu_k m_2 g \cos\theta = m_2 a \tag{1.21}$$

The two equations we have found have *three* unknowns: T_1, T_2 and a. We will never solve the problem if we don't get a third equation, and it is the equation of (rotational) motion for the *pulley* that gives us the last equation.

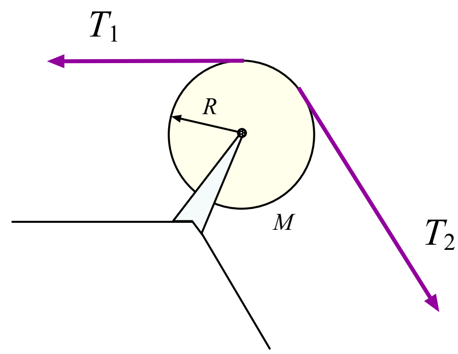

Figure 1.14: Forces acting on the pulley in Example 18.

The force diagram for the pulley is given in Fig. 1.14. When the string is wrapped over the pulley and does not slip, it acts to exert a tangential force T_2 at the rim of the disk in one direction and a tangential force T_1 in the other direction, as shown.

These forces act at a distance R from the pivot point (the pulley's axle) and each is perpendicular to the line joining the pivot and the point of application. If we take the *clockwise* torque as being positive, then the net torque on the disk is

$$\tau_{\rm net} = T_2 R - T_1 R = (T_2 - T_1)R \ .$$

From the rotational form of the second law, we then have

$$\tau_{\rm net} = (T_2 - T_1)R = I\alpha$$

where I is the moment of inertia of the pulley and α is its angular acceleration in the *clockwise* sense, since that is how we defined positive torque τ. We can use a couple other facts before finishing with this equation; we assume the pulley is a uniform disk, and so I is given by

$$I = \tfrac{1}{2}MR^2 \ .$$

Also, we know that the linear tangential acceleration of the rim of the disk must be the same as the linear acceleration of the string and also the masses m_1 and m_2. This gives us:

$$\alpha R = a \qquad \Longrightarrow \qquad \alpha = a/R$$

Putting both of these relations into the $\tau = I\alpha$ equation, we get

$$(T_2 - T_1)R = I\alpha = \tfrac{1}{2}MR^2\left(\frac{a}{R}\right) = \tfrac{1}{2}MRa$$

and so if we cancel a factor of R we have a third equation relating the three unknowns,

$$(T_2 - T_1) = \tfrac{1}{2}Ma \tag{1.22}$$

Interestingly enough, the radius R does not appear. Our final answers will not depend on it!

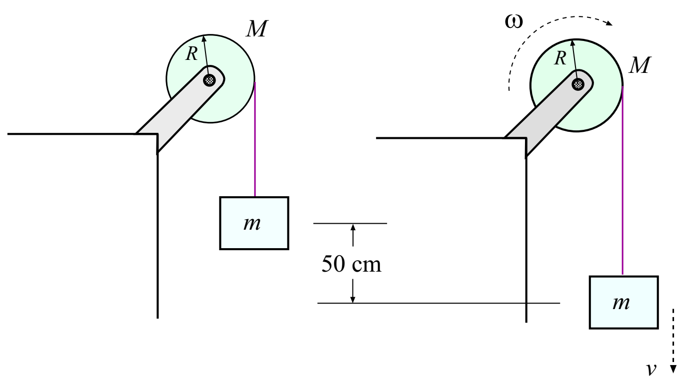

Figure 1.15: Before and after pictures for a mass–pulley system which starts from rest. Take the pulley to be a uniform disk.

At this point, we have three equations for three unknowns (Eqs. 1.20, 1.21 and 1.22.) *The physics part of the problem is done.* What remains is the *algebra* involved in solving for the unknowns T_1, T_2 and a. Eqs. 1.20 and 1.21 give

$$T_1 = m_1 a + \mu_k m_1 g \qquad \text{and} \qquad T_2 = -m_2 a + m_2 g \sin\theta - \mu_k m_2 g \cos\theta$$

When we put these into Eq. 1.22 we get:

$$-(m_2 + m_1)a - \mu_k g(m_2 \cos\theta + m_1) + m_2 g \sin\theta = \tfrac{1}{2}Ma$$

This contains *only* the acceleration a and so we can quickly solve for it. A little rearranging gives

$$(m_1 + m_2 + M/2)a = m_2 g \sin\theta - \mu_k g(m_1 + m_2 \cos\theta)$$

Isolating a and factoring out g gives

$$a = \frac{[m_2 \sin\theta - \mu_k(m_1 + m_2 \cos\theta)]}{(m_1 + m_2 + M/2)} g$$

for the acceleration of the blocks. This expression can only make sense if μ_k is not so big that it makes a *negative* in this expression.

1.2.9 Work, Energy and Power in Rotational Motion

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19. (a) If $R = 12\,\mathrm{cm}$, $M = 400\,\mathrm{g}$ and $m = 50\,\mathrm{g}$ in Fig. 1.15, find the speed of the block after it has descended $50\,\mathrm{cm}$ starting from rest. Solve the problem using energy conservation principles. (b) Repeat (a) with $R = 5.0\,\mathrm{cm}$. (Assume the pulley is a uniform disk.) [HRW5 11-77]

(a) If there is no friction in the axle of the pulley then mechanical energy will be conserved as the block descends. The change in total energy between the "before" and "after" pictures in Fig. 1.15 will be zero:

$$\Delta E = \Delta K + \Delta U = 0$$

Initially, the block is at rest and so is the pulley, so there is *no* kinetic energy in the system. In the final picture, the block has some speed v so it has kinetic energy (namely $\frac{1}{2}mv^2$) and the pulley is turning at some angular speed ω so that it too has kinetic energy. The kinetic energy of the pulley is $K_{\rm rot} = \frac{1}{2}I\omega^2$, so that the total kinetic energy in the "after" picture is

$$K_f = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2$$

We can simplify this expression by realizing that if the string does not slip on the pulley (and it is implied in the set-up that it is wrapped around it, so it doesn't) then the tangential speed of the edge of the disk is the same as that of the falling mass. This gives $\omega R = v$, or $\omega = v/R$. Also, assuming that the pulley is a uniform disk, we have $I = \frac{1}{2}MR^2$. Using these relations we get:

$$\begin{aligned} K_f &= \tfrac{1}{2}mv^2 + \tfrac{1}{2}\left(\tfrac{1}{2}MR^2\right)\left(\frac{v}{R}\right)^2 \\ &= \tfrac{1}{2}mv^2 + \tfrac{1}{4}Mv^2 = \left(\frac{m}{2} + \frac{M}{4}\right)v^2 \end{aligned}$$

The potential energy of the system changes only from the change in height of the suspended mass m. Its change in height is $\Delta y = -0.50\,\mathrm{m}$ so that the change in potential energy of the system is

$$\Delta U = mg\Delta y = mg(-0.50\,\mathrm{m})$$

Putting everything into the energy conservation equation we get

$$\Delta K + \Delta U = \left(\frac{m}{2} + \frac{M}{4}\right)v^2 + mg(-0.50\,\mathrm{m}) = 0$$

which we can use to solve for v since we know all the other quantities:

$$\left(\frac{m}{2} + \frac{M}{4}\right)v^2 = mg(0.50\,\mathrm{m}) = (0.050\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(0.50\,\mathrm{m}) = 0.245\,\mathrm{J}$$

$$\left(\frac{(0.050\,\mathrm{kg})}{2} + \frac{(0.400\,\mathrm{kg})}{4}\right)v^2 = (0.125\,\mathrm{kg})v^2 = 0.245\,\mathrm{J}$$

$$v^2 = \frac{(0.245\,\mathrm{J})}{(0.125\,\mathrm{kg})} = 1.96\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

$$v = 1.4\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The final speed of the block is $v = 1.4\,\frac{\mathrm{m}}{\mathrm{s}}$.

(b) In this part we are asked to find the final speed v if R has a different value. But if we look back at our solution for part (a) we see that we never used the given value of R! (It cancelled out when we wrote out the energy conservation condition in terms of m, M and v.) So we get the same answer: $v = 1.4\,\frac{\mathrm{m}}{\mathrm{s}}$.

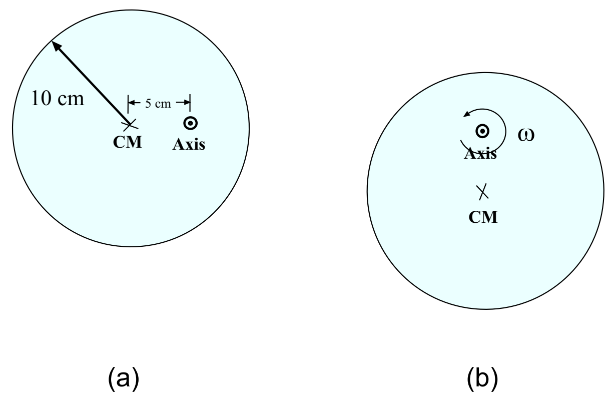

Figure 1.16: Cylinder rotating about an off–center axis, in Example 20. (a) is the initial position of the cylinder; (b) shows the cylinder as it moves through its lowest position.

20. A uniform cylinder of radius $10\,\mathrm{cm}$ and mass $20\,\mathrm{kg}$ is mounted so as to rotate freely about a horizontal axis that is parallel to and $5.0\,\mathrm{cm}$ from the central longitudinal axis of the cylinder. (a) What is the rotational inertia of the cylinder about the axis of rotation? (b) If the cylinder is released from rest with its central longitudinal axis at the same height as the axis about which the cylinder rotates, what is the angular speed of the cylinder as it passes through its lower position? (*Hint:* Use the principle of conservation of energy.) [HRW5 11-84]

(a) First, draw a picture of this system; Fig. 1.16 (a) shows an end–on view of the rotating cylinder. Its symmetry axis is labelled "CM" but its *rotational* axis is marked "Axis". The cylinder does not turn about its center! (If it did, its moment of inertia would be $I = \frac{1}{2}MR^2$, but that will not be the case here.) We need some other way of getting I.

There is one special feature about *this* rotation which will give us I. The rotation takes place about an axis which is *parallel* to an axis for which we *do* know I, namely the symmetry axis. For that axis, we have $I_{\mathrm{CM}} = \frac{1}{2}MR^2$ (where M and R are the mass and radius of the cylinder). Our axis is displaced from that one by a distance $D = 5.0\,\mathrm{cm} = 0.050\,\mathrm{m}$. Then the Parallel Axis Theorem, Eq. 1.16, gives us:

$$\begin{aligned} I &= I_{\mathrm{CM}} + MD^2 \\ &= \tfrac{1}{2}MR^2 + MD^2 \\ &= \tfrac{1}{2}(20\,\mathrm{kg})(0.10\,\mathrm{m})^2 + (20\,\mathrm{kg})(0.050\,\mathrm{m})^2 \\ &= 0.15\,\mathrm{kg}\cdot\mathrm{m}^2 \end{aligned}$$

So the rotational (moment of) inertia of the cylinder about the given axis is $0.15\,\mathrm{kg}\cdot\mathrm{m}^2$.

(b) The cylinder starts in the position shown Fig. 1.16 (a), that is, with the symmetry axis at the same height as the rotational axis. We know that the force of gravity acts on the cylinder to give it an angular velocity which increases as the cylinder swings downward. We want to know the angular velocity when the center of mass of the cylinder is at its lowest point.

It would be very hard to find the final ω using torques, because t he force of gravity does not act on the cylinder in the same way through the swing. As with similar problems in particle (point–mass) motion, it is easier to use *conservation of evergy*. If there is no friction in the axis, then total mechanical energy is conserved, which we can write as:

$$\Delta E = \Delta K + \Delta U = 0$$

Now, the cylinder starts from rest when it is at the position shown in (a) so that $K_i = 0$. In position (b), the cylinder has angular speed ω about the axis so that the final kinetic energy is

$$K_f = \tfrac{1}{2} I\omega^2$$

where I is the moment of inertia we found in part (a) of this problem.

There is a change in potential energy of the cylinder because there is a change in height of its center of mass. For an extended object we find the gravitational potential energy by imagining that all the mass is concentrated at the center of mass. Then for the cylinder, since its center of mass has fallen by a distance 5.0 cm (compare picture (a) and picture (b)) we see that the change in U is:

$$\Delta U = Mg\Delta y = (20\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(-0.050\,\mathrm{m}) = -9.80\,\mathrm{J}$$

Plugging everything into our energy conservation equation, we find

$$\Delta K + \Delta U = \tfrac{1}{2} I\omega^2 - 9.80\,\mathrm{J} = 0$$

for which we solve for ω:

$$\begin{aligned} \omega^2 &= \frac{2(9.80\,\mathrm{J})}{I} = \frac{2(9.80\,\mathrm{J})}{(0.15\,\mathrm{kg}\cdot\mathrm{m}^2)} \\ &= 131\,\mathrm{s}^{-2} \end{aligned}$$

which gives us

$$\omega = 11.4\,\tfrac{\mathrm{rad}}{\mathrm{s}}$$

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21. A tall, cylinder–shaped chimney falls over when its base is ruptured. Treating the chimney as a thin rod with height h, express the (a) radial and (b) tangential components of the linear acceleration of the top of the chimney as functions of the angle θ made by the chimney with the vertical. (c) At what angle θ does the linear acceleration equal g? [HRW5 11-87]

(a) As suggested in the problem we *model* the falling chimney as a thin rod of height h and mass M which turns about a pivot which is fixed to the ground at one place. (A real falling chimney may behave differently…it might break as it falls and the base may move along the ground. Real life is lots more complicated.) The model is shown in Fig. 1.17 (a). The angle θ measures the (instantaneous) angle of the chimney with the vertical. We know that

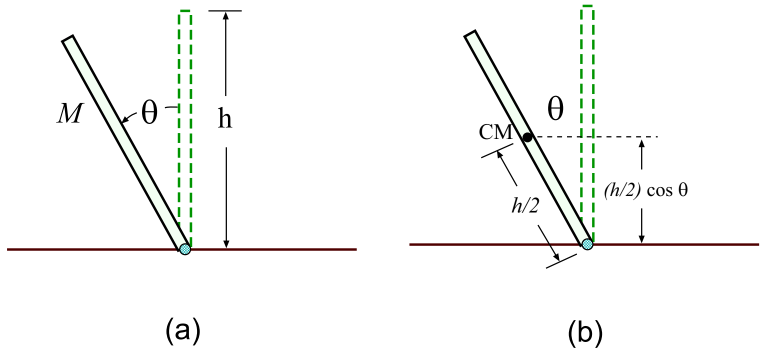

Figure 1.17: (a) The falling chimney described in Example 21. (b) Position of the center of mass of the chimney.

as the chimney falls over (starting from rest) it will lose gravitational potential energy and pick up rotational kinetic energy from its rotation about the pivot. Its angular speed will increase as it falls.

In part (a) we are asked for the radial acceleration of the end of the chimney when the chimney has rotated through an angle θ. That's just the centripetal acceleration a_c of the top of the chimney, which is given by Eq. 1.12 and since the radius of the circle in which the top moves is h, this is given by

$$a_{\rm c} = \frac{v^2}{h} = h\omega^2 \ .$$

All that we need to know to compute $a_{\rm c}$ is the angular speed of the rod, and for this we can use energy conservation for the rod as it falls. (Energy conservation is good for finding final speeds!) When the rod is in the initial (vertical) position, it is at rest so it has no kinetic energy. Suppose we measure height from the ground level; then the center of mass of the rod is at a height $h/2$ and so the initial gravitational potential energy is $Mg\frac{h}{2}$. So the initial total energy is

$$E_i = \frac{Mgh}{2}$$

Now suppose the rod has fallen through an angle θ, as shown in Fig. 1.17 (b). If it has angular speed ω at that time, then its kinetic energy is $\frac{1}{2}I\omega^2$, where I is the moment of inertia of the rotating chimney. For a uniform rod of length h rotating about one end, this is $I = \frac{1}{3}Mh^2$, so we have:

$$K_f = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\left(\tfrac{1}{3}Mh^2\right)\omega^2 = \tfrac{1}{6}Mh^2\omega^2$$

As for the potential energy in this position, we can see from geometry that the center of mass of the rod is at a height of $\frac{h}{2}\cos\theta$. So the rod's final potential energy is

$$U_f = Mg\frac{h}{2}\cos\theta \ .$$

and its final total mechanical energy is

$$E_f = K_f + U_f = \tfrac{1}{6}Mh^2\omega^2 + Mg\frac{h}{2}\cos\theta$$

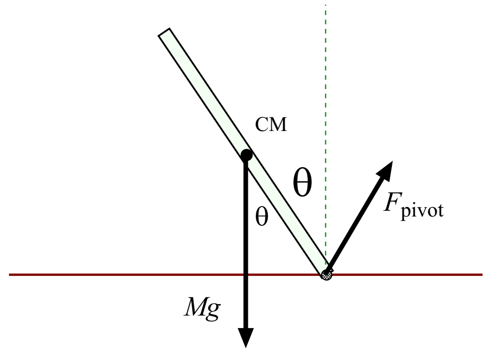

Figure 1.18: Forces acting on falling rod (chimney)

Energy conservation (we assume that this pivot is frictionless...), $E_i = E_f$ gives

$$\frac{Mgh}{2} = \tfrac{1}{6}Mh^2\omega^2 + Mg\frac{h}{2}\cos\theta$$

which can solve for ω^2. Rearranging, we get:

$$\tfrac{1}{6}Mh^2\omega^2 = \frac{Mgh}{2}(1-\cos\theta)$$

$$\omega^2 = \frac{3g}{h}(1-\cos\theta)$$

which we can use in expression for $a_{\rm c}$:

$$a_{\rm c} = h\omega^2 = h\frac{3g}{h}(1-\cos\theta) = 3g(1-\cos\theta) \ .$$

The radial part of the acceleration of the tip of the chimney has this magnitude and is of course directed inward toward the pivot.

(b) From Eq. 1.11, the tangential component of the acceleration for a point on a rotating object a distance h away from the pivot is $a_T = h\alpha$ where α is the object's angular acceleration. So we need to find the angular acceleration α of the chimney when it has fallen through an angle θ. We can get α by finding the net torque acting on the rod and then using $\tau = I\alpha$.

In Fig. 1.18 we show the forces that act on the rod as it falls, and their points of application. Gravity (effectively) pulls downward at the center of the rod with a force Mg, and the pivot also exerts a force on the rod. It's not so clear which direction the latter force points, but in the end it does not matter because we only want the *torques* that these forces exert about the pivot point and force of the pivot itself will give no torque.

The force of gravity is applied at a distance $h/2$ from the pivot. It makes an angle θ with the line which joins the pivot and application point. Here, it gives a torque in the counter-clockwise sense. So the torque due to gravity is

$$\tau = \frac{h}{2}Mg\sin\theta$$

and this is also the *total* (counter-clockwise) torque because the force of the pivot gives none. Then $\tau_{\rm net} = I\alpha$ gives us

$$\frac{h}{2}Mg\sin\theta = I\alpha$$

where I is the moment of inertia of the rod. For a uniform rod of length h rotating about one end, it is $I = \frac{1}{3}Mh^2$ so we get:

$$\frac{h}{2}Mg\sin\theta = \tfrac{1}{3}Mh^2\alpha$$

Some algebra gives us:

$$\alpha = \frac{3g}{2h}\sin\theta$$

And at last we find the tangential acceleration of the end of the chimney:

$$a_T = h\alpha = h\frac{3g}{2h}\sin\theta = \tfrac{3}{2}g\sin\theta \ .$$

The answers to (a) and (b) do not depend on the mass M of the chimney.

(c) We want to know at what angle the linear (i.e. the *tangential*) acceleration of the chimney–top is equal to g. Using our answer from part (b), we have the condition:

$$\tfrac{3}{2}g\sin\theta = g$$

Solve for θ:

$$\sin\theta = \tfrac{2}{3} \qquad \Longrightarrow \qquad \theta = \sin^{-1}\left(\tfrac{2}{3}\right) = 41.8^\circ$$

The chimney will have fallen by 41.8° when the linear acceleration of its top is equal to g.