Chapter 1

The geometry of nature: fractals

Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line.

B.B.Mandelbrot [1]

1.1 Measure and dimension

THE ACT OF MEASUREMENT IS THE FUNDAMENTAL ACT OF SCIENTIFIC ENQUIRE. This includes observation, counting and actual measurement. ACT OF MEASURE

When, for example, we measure the length L of an object, we first need to choose a unit of length, say l, then we count how many times our unit fits into it. Since L will almost never be an exact multiple of l, our estimate will almost never be completely precise. To obtain better and better estimates we need to use smaller and smaller rods (i.e. smaller units l). These new units will progressively fit the length L more and more finely, ultimately giving a very precise estimate. We see here one fundamental fact about measurements: THE ESTIMATE WE MAKE DEPENDS ON THE UNIT WE ARE EMPLOYING, i.e. the length is a function of the unit $L(l)$. MEASUREMENTS DEPEND ON THE UNIT USED

When we try to measure the area of an object we construct small squares of side l and we count how many of these squares are needed to cover the entire object. When instead we measure the volume of an object we construct small MEASURING AREAS, VOLUMES AND TIME LAPSES cubes of side l and we fill the volume keeping track of how many cubes we have used. In both cases the precision of the estimate will depend on the unit scale l we chose. The same happens if we want to measure a time lapse; we choose a unit of time and we count how may times this fits the time lapse.

In all cases measurement is made by counting. This motivates the definition of the d–dimensional measure at resolution l, of a physical or mathematical object, as:

DEFINITION OF MEASURE

$$M_d(l) = N(l)l^d \,, \tag{1.1}$$

where $N(l)$ is the number of d–dimensional cubes of volume l^d we used to fill the object under study.

We used the term "dimension" but we have not defined it. As we will see there are many different ways to define the concept of dimension. The dimension we introduced in (1.1), and that we indicate by d, is the *topological dimension.* For us this is the dimension of the unit used.

TOPOLOGICAL DIMENSION

Only for few regular geometrical objects like a line, a square or a cube, the outcome of the measurement can be made independent of the resolution: $M_d(l) \equiv M_d$ for all l. This objects that we call *regular objects* are purely mathematical.

REGULAR OBJECTS

But already the diagonal of a square escapes from this class of objects. In most cases our measurement at scale l is only an approximate one: the unit of measure never exactly fits the entire object. To do a better job, we need to choose a sharper resolution and repeat the measurement. In this way we usually converge to a value which we identify with the "true" measure of the object. These are objects for which the limit

RATIONAL OBJECTS

$$M_d = \lim_{l \to 0} M_d(l) \,, \tag{1.2}$$

exists and is finite. We call this class of objects rational, since the outcome of a physical measurement is always a rational number. Only in the realm of mathematical idealizations we can take the limit (1.2) completely, and thus obtaining a measure which is a real number.

For example, this is what we have to do if we want to calculate[1] the cir-

EXAMPLE: LENGTH OF THE CIRCUMFERENCE

Great	Britain	Circle	Island
l (km)	$L(l)$ km	l (km)	$L(l)$ (km)
500	2600	500	3000
100	3800	130	3133
54	5770	65	3139
17	8640	16	3141

Table 1.1: Measurements of the coasts of Great Britain and Circle Island.

cumference of the circle: we consider an inscribed n-agon of edge length l and we estimate as nl the measure of the circumference length. Then we repeat the procedure with a polygon with more sides and shorter edge lengths to get the successive estimate. In this way we construct a sequence that, as $n \to \infty$ and $l \to 0$, converges to the limit 2π,[2] which we consider the "exact" measure of the length of an ideal circle.

IS THE MEASURE ALWAYS DEFINED?

The question is now: does the limit (1.2) always exist? The answer is no! Incredibly only recently[3] we have realized that for most physical objects the limit (1.2) does not exist.

HOW LONG IS THE COAST OF GREAT BRITAIN?

The classical thought experiment relevant to this problem concerns the measurement of the coast of an island. We consider the coast of Great Britain and of the imaginary Circle Island. We want to determine the length of their coasts. We start by setting up a (giant) ruler with $l = 500$km and we determine the first estimates; then we chose a smaller ruler and we obtain a second estimate. Just few other measurements will be sufficient to understand what's going on; we report them in Table 1.1. The first thing we notice by looking at the data is that the length measurements of the coast of Circle Island seem to converge to a finite value (obviously $10^3\pi$), while the length measurements of the coast of Great Britain seem to increase unboundedly. If we make a log–log plot of the data of Table 1.1 we find the result shown in Figure 1.1. The fit shows that $\log L(l) \sim -0.36 \log l$ for Great Britain and

[1] When we measure geometrical objects we are actually prescribing a way to calculate their measure.

[2] In this way we have actually defined the real number π and we have given a way to approximately calculate it.

[3] Mostly after the fundamental works of B.B. Mandelbrot [1]

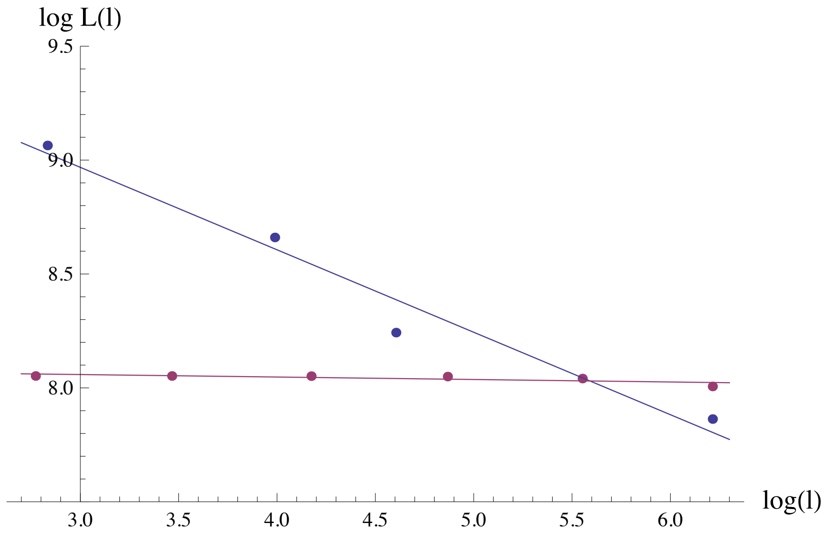

Figure 1.1: Log–log plot of the length of the coast of Great Britain (blue) and the length of Circle Island (red) as a function of the ruler length.

$\log L(l) \sim -0.01 \log l$ for Circle Island. We conclude that the length of Circle Island is well defined while that of Great Britain is not, i.e. the limit (1.2) does not exist in the second case.

Comment on Portugal–Spain border!

Figure 1.1 tells us that the length of the coast of Great Britain scales, as the ruler length is varied, as a power law: FRACTALS

$$L(l) = L(l_0) \left(\frac{l}{l_0} \right)^{1-d_f} , \qquad (1.3)$$

where we introduced a reference scale l_0 and we defined the *fractal dimension* d_f. Objects whose length behaves as in (1.3) are called *fractals* since they have a fractional dimension; in fact we have $d_f = 1.36$ for the coast of Great Britain while $d_f = 1.01$ for Circle Island.

The generalization of equation (1.3) is: FRACTAL DIMENSION

$$M_d(l) = M_d(l_0) \left(\frac{l}{l_0} \right)^{d-d_f} . \qquad (1.4)$$

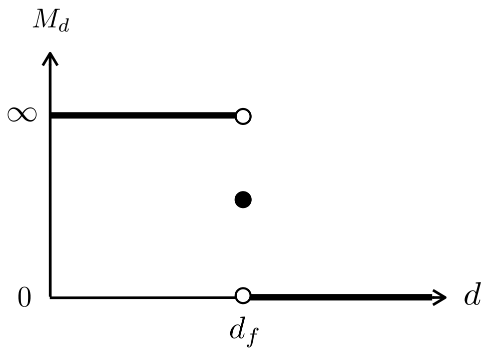

Figure 1.2: Hausdorff property.

We can solve (1.4) for d_f to find:

$$d_f = d + \lim_{l \to 0} \frac{\log M_d(l)}{\log l_0/l} \,. \tag{1.5}$$

Thus in general THE CONCEPT OF MEASURE IS SCALE DEPENDENT, BUT NOT JUST FOR THE REASON THAT MOST QUANTITIES ARE INCOMMENSURABLE! Still AN OBJECT CAN HAVE A ILL DEFINED NOTION OF MEASURE BUT A WELL DEFINED NOTION OF FRACTAL DIMENSION! In fact the limit (1.2) exist only when the topological dimension and the fractal dimension coincide $d = d_f$. For equation (1.4) to be true the following scaling law must be valid:

$$N(l) = N(l_0) \left(\frac{l_0}{l}\right)^{d_f} \,, \tag{1.6}$$

from which we find the following alternative expression for the fractal dimension:

$$d_f = \lim_{l \to 0} \frac{\log\, N(l)}{\log\, l_0/l} \,. \tag{1.7}$$

Relation (1.6) is the key to define the notion of dimension for objects where their measure depends on the resolution we are using to measure them.[4]

Note that an object of fractal dimension d_f has a finite d_f–dimensional mea- HAUSDORFF PROPERTY

[4]One can introduce an early version of the concept of *anomalous dimension* η of an object. This is simply the difference between the fractal and the topological dimension. In the case of the Koch curve the anomalous dimension is positive $\eta = 0.26186...$, while in the case of the Cantor set is negative $\eta = -0.36907...$

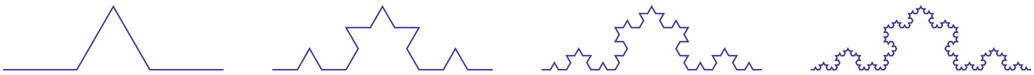

Figure 1.3: The Koch curve.

sure, i.e. the limit (1.2) exists. In particular, it is important to note that the measure of an object M_d is zero if $d > d_f$ while is infinite if $d < d_f$. This last property has been used by Hausdorff to precisely define the notion of fractal dimension [2].

We consider now some classical examples of fractals and we calculate their fractal dimensions. Consider first a square of area $A(l_0) = l_0^2$. If we change unit length to $l = l_0/2$ we find $A(l) = 4l^2$ and so $N(l) = 4$. We find that the dimension of the square is $d = \frac{\log 4}{\log 2} = 2$. EXAMPLE: SQUARE

The simplest example of fractal object is the famous Koch curve. It's construction is shown in Figure 1.3. If at the scale l_0 the length of the curve is $L(l_0) = 4l_0$ at the scale $l = l_0/3^n$ we have $N(l) = 4^n$ and we find that the fractal dimension is $d = \frac{\log 4}{\log 3} = 1.26186...$ The Koch curve appears to be more than a line but less than a two dimensional object. Can we find out what's the fractal measure $M_{1.26186...}$ of the Koch curve? EXAMPLE: KOCH CURVE

Another example is the Cantor set or dust. We start with a line of length l_0 and we cut away the central third of it. In this way we generate a fractal of dimension $d = \frac{\log 2}{\log 3} = 0.63093...$ We have constructed a fractal of dimension less than the dimension of the line we started with. Can we construct the Cantor set starting from a set of points and adding more and more points? EXAMPLE: CANTOR DUST

The Koch island is a geometrical figure that has a finite area but an infinite perimeter and can be used to model for Great Britain. The perimeter is made by three Koch curves each of which has infinite length, the area instead satisfies the recursive relation EXAMPLE: KOCH ISLAND

$$A_{n+1} = A_n + 3 \cdot 4^{n-1} \frac{\sqrt{3}}{4} \left(\frac{a}{3^n} \right) ,$$

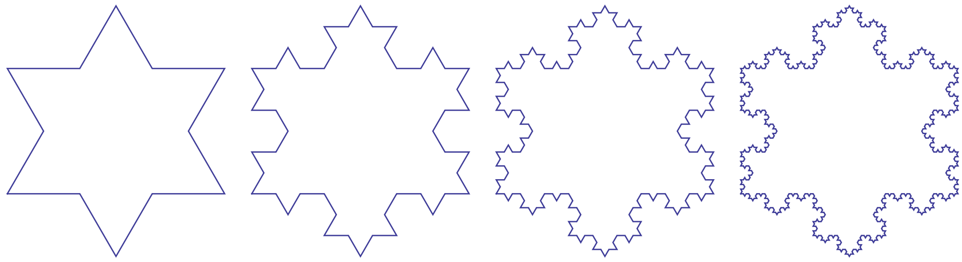

Figure 1.4: The Koch island.

where a is the length of the edge of the step zero triangle. This can be solved to give

$$A_\infty = A_1 + \frac{\sqrt{3}}{12}a^2 \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^{n-1} = \frac{\sqrt{3}}{4}a^2 + \frac{\sqrt{3}}{4}\frac{3}{5}a^2 = \frac{2\sqrt{3}}{5}a^2 \,,$$

where we summed the geometric series.[5]

NATURAL FRACTALS

Fractal are everywhere! For example see an interesting study of metabolic rates in animals [4].

d_f IS DEFINED FOR SELF–SIMILARITY OBJECTS

The fractal dimension d_f is defined only for *self–similar* objects. An object appears d–dimensional if, for a certain range of scales, it is self–similar.

SCALE DEPENDENCE

But now another question: can we assign a fractal dimension to every physical or mathematical object? No! We must point out that equation (1.7) is well defined only for self–similar or self–affine objects: in all the other cases also the concept of dimension is not well defined, or in other words is scale dependent. We may find physical objects that have different fractal dimensions at different scales or resolutions. We will see that this is indeed

[5]It is worth noticing that geometric series arise when the problem at hand displays self–similarity. To understand this we have to notice that the series $S = \sum_{n=0}^{\infty} x^n$ is self–similar: $S = xS + 1$ (with scale factor x and unit translation).

the case: statistical and quantum fields have different dimensions at the UV and IR scales respectively. THIS IS A VERY IMPORTANT PHENOMENA THAT APPEARS ALL OVER NATURE.

There exist other definitions of dimension, all related to each other, apart the "compass" or "self-similar" dimension defined in (1.7). In particular the "box-counting" and "Hausdorff" dimensions can be defined for systems that are not precisely self–similar [2] and they can be used to define scale–dependent fractal dimensions. MANY DEFINITIONS OF DIMENSION

1.2 Similarity and self–similarity

HOW TO DEFINE A FRACTAL

We have seen that a physical object has a well defined fractal dimension only if it is self–similar and that it has a non zero finite d–dimensional volume only if it's fractal dimension is d. We see now how to use similarity transformations to define fractals [3].

Two objects are *similar* if they have the same shape regardless of their size. A *similarity* $\mathcal{S}$ is a transformation which can be written as a composition of a *translation*, a *rotation* and a *scale transformation*. A geometrical object, i.e. a subset A of $\mathbb{R}^d$, is transformed by a similarity to the subset $\mathcal{S}(A)$ of $\mathbb{R}^d$. More generally, one can consider the similarity $\mathcal{S}$ to be the union of n similarities: SIMILARITY

$$\mathcal{S}(A) = \mathcal{S}_1(A) \cup \mathcal{S}_2(A) \cup ... \cup \mathcal{S}_n(A)\,, \tag{1.8}$$

so to extend the concept of similarity to a broader class of geometrical objects.

In the plane one can easily characterize a similarity as transformation which send the point (x, y) to the point $\mathcal{S}(x, y)$ as follows: SIMILARITY IN $d = 2$

$$\mathcal{S}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix},$$

where the parameters a, b refer to the translation, the angle θ to the rotation and the scale factor λ to the scale transformation.

Fractals like the Koch curve and the Cantor set are instead *self–similar*. SELF–SIMILARITY: FRACTALS AS FIXED POINTS

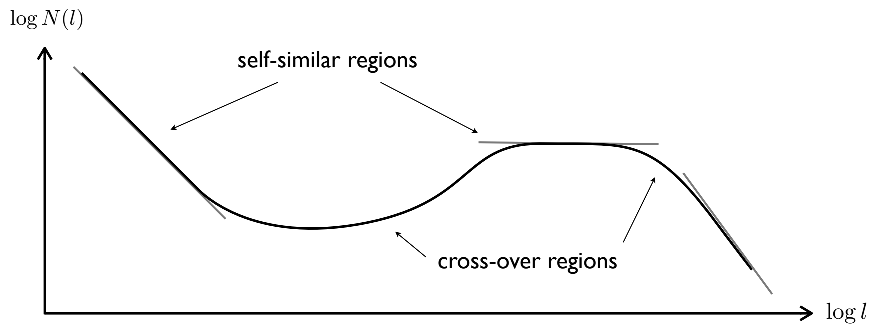

Figure 1.5: Flow of the fractal dimension, which is defined in self–similar regimes.

A geometrical object, A_* of $\mathbb{R}^d$, is said to be self-similar if there exist a similarity $\mathcal{S}$ such that:

$$\mathcal{S}(A_*) = A_*. \tag{1.9}$$

This equation, as we will see, represents the first example of a fixed point equation where one characterizes a set, or more generally a configuration, as the fixed point of a particular transformation. We say that A_* is a fixed point of $\mathcal{S}$.

One can allow for a different scaling factor in the diverse dimensions, it that case one speaks of an *affine linear transformation* $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_n$ and about *self–affine* geometrical objects obeying $\mathcal{A}(A) = A$. Affine linear transformations are a composition of a linear mapping with a translation. AFFINE TRANSFORMATIONS

We look now at specific examples, in particular at how one can define the set of points of, say, the Koch curve as the fixed-point of a particular similarity EXAMPLE: SIMILARITY TRANSFORMATION OF THE KOCH CURVE

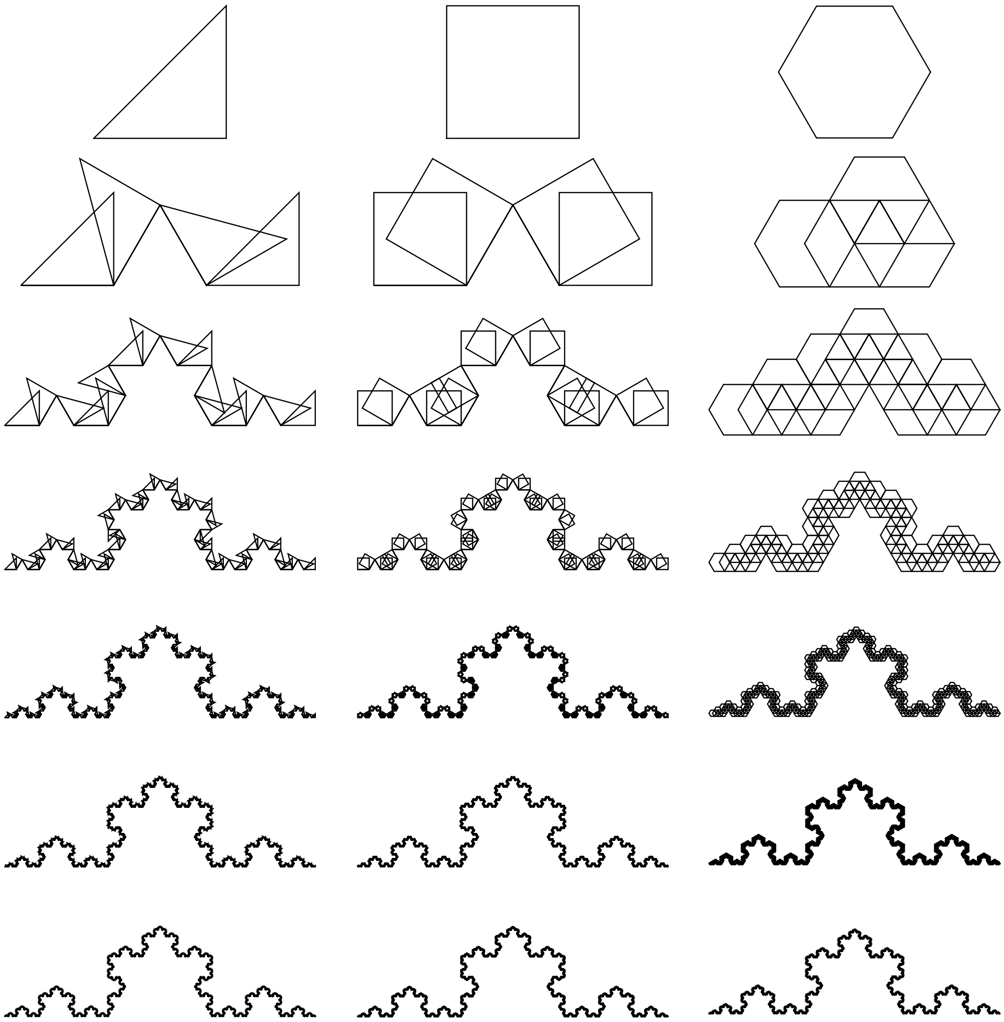

Figure 1.6: Approaching the Koch curve fixed-point.

transformation. Then the following similarities

$$\begin{aligned}
\mathcal{S}_1 \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\
\mathcal{S}_2 \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix} \\
\mathcal{S}_3 \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{\sqrt{3}}{6} \end{pmatrix} \\
\mathcal{S}_4 \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}
\end{aligned}$$

characterize the Koch curve as the solution of the fixed point equation $\mathcal{S}(K) = \mathcal{S}_1(K) \cup \mathcal{S}_2(K) \cup \mathcal{S}_3(K) \cup \mathcal{S}_4(K) = K$. As we will show, the fixed point of the similarity transformation is attractive in the whole domain of allowed two dimensional sets. Thus any subset A of $\mathbb{R}^2$ is attracted to the Koch curve K! This is clearly shown in Figure 1.6.

1.3 Scale invariance and power laws

DISCRETE SCALE INVARIANCE

We can consider the Koch curve as the graph of a function as shown in Figure 1.7. The Koch function is invariant under discrete scale transformations $x \to \lambda x$ for any $\lambda = \frac{1}{3^n}$ with $n = 0, 1, 2, ...$ Looking at the picture we see that, for example, $y\left(\frac{1}{6}\right) = \frac{1}{3} y\left(\frac{1}{2}\right)$ or $y\left(\frac{1}{3}\frac{1}{2}\right) = \frac{1}{3} y\left(\frac{1}{2}\right)$ so the scale transformation for the Koch curve is:

$$y(x) = \lambda^{-1} y(\lambda x) \,. \tag{1.10}$$

which defines an HOMOGENEOUS FUNCTION or SELF–SIMILAR FUNCTION.

GENERAL SCALE INVARIANCE

In the general case we consider a SELF–AFFINE FUNCTION which scales anisotropically. The x coordinate rescales as $x \to \lambda x$ while the y coordinate rescales as $y \to \lambda^{\alpha} y$, where α is the anisotropic exponent. We generalize (1.10) to:

$$y(x) = \lambda^{-\alpha} y(\lambda x) \,. \tag{1.11}$$

In case the function is continuously scale invariant we can solve (1.11) by choosing $\lambda = \frac{1}{x}$ to obtain:

$$y(x) = y(1)\, x^{\alpha} \,,$$

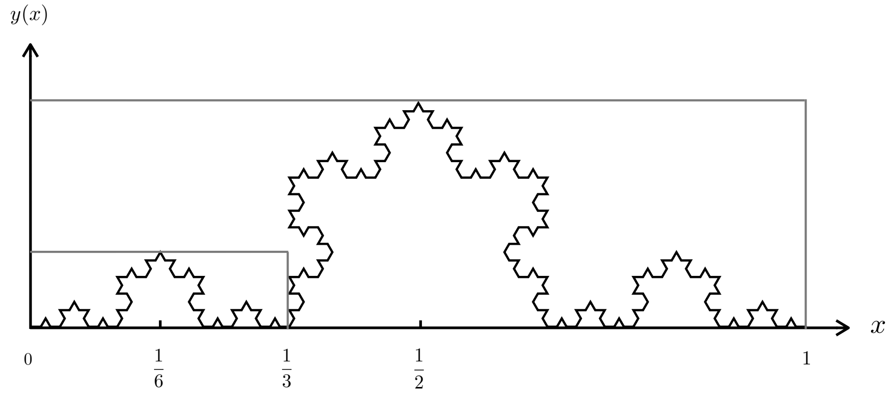

Figure 1.7: Discretely scale invariant Koch curve.

which is a POWER LAW. Conversely, a power law is a homogeneous function. CONTINUOUSLY SCALE INVARIANT FUNCTIONS ARE POWER LAWS.

1.4 Random fractals

Up to now we have considered only deterministic fractals that are generated by iterating a deterministic rule. Self–similar physical systems are mainly described on a statistical basis, in this case physical objects are generally described by random or statistical fractals. The statistical properties of the system are scale invariant and we can associate a fractal dimension or an anomalous dimension to the random processes. RANDOM FRACTALS

The first example of random fractal is the RANDOM WALK. It is shown in the Figure 1.8. The fractal dimension is $d_f = 2$. RANDOM WALK

A random walk can be seen a RANDOM FUNCTION which has statistical self–similarity (affinity). Note that it is an example of continuous but nowhere differentiable function. One has: RANDOM FUNCTION

$$\langle y(x) \rangle = 2^{-1/2} \langle 2y(\lambda x) \rangle \ ,$$

thus the scaling exponent, here called Hurst exponent, is $\alpha = \frac{1}{2}$. The fractal dimension can be shown to be $d_f = 2 - \alpha = \frac{3}{2}$.

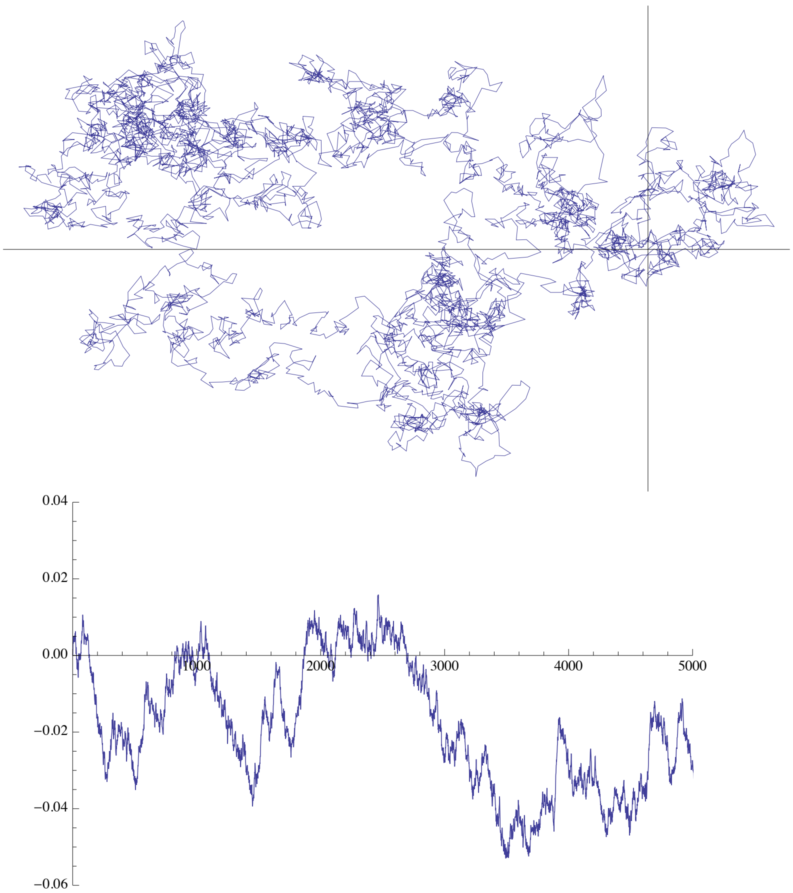

Figure 1.8: Random walk in the plane (top) and random walk as a random function (bottom).

Bibliography

[1] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman, San Francisco.

[2] H.O. Peitgen, H. Jurgens and D. Saupe, *Chaos and Fractals* (2004) Springer-Verlag, New York.

[3] J. Hutchinson, *Fractals and self-similarity*, Indiana Journal of Mathematics 30 (1987) 713-747.

[4] M. Sernetz, B. Gelléri, F. Hofman, *The organism as a bioreactor: interpretation of the reduction law of metabolism in terms of heterogeneous catalysis and fractal structure*, Journal of Theoretical Biology 117 (1985) 209-230.