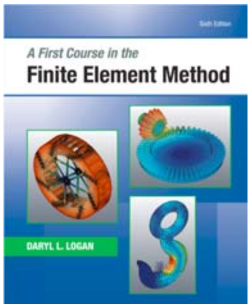
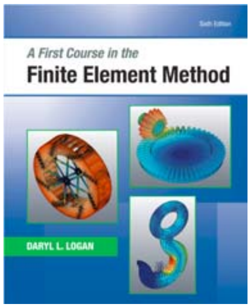
Chapter 10 – Isoparametric Elements



Learning Objectives

- To formulate the isoparametric formulation of the bar element stiffness matrix
- To present the isoparametric formulation of the plane four-noded quadrilateral (Q4) element stiffness matrix
- To describe two methods for numerical integration— Newton-Cotes and Gaussian Quadrature —used for evaluation of definite integrals
- To solve an explicit example showing the evaluation of the stiffness matrix for the plane quadrilateral element by the four-point Gaussian quadrature rule

Chapter 10 – Isoparametric Elements



Learning Objectives

- To illustrate by example how to evaluate the stresses at a given point in a plane quadrilateral element using Gaussian quadrature
- To evaluate the stiffness matrix of the three-noded bar using Gaussian quadrature and compare the result to that found by explicit evaluation of the stiffness matrix for the bar
- To describe some higher-order shape functions for the three-noded linear strain bar, the improved bilinear quadratic (Q6), the eight- and nine-noded quadratic quadrilateral (Q8 and Q9) elements, and the twelve-noded cubic quadrilateral (Q12) element
- To compare the performance of the CST, Q4, Q6, Q8, and Q9 elements to beam elements

Isoparametric Elements

Introduction

In this chapter, we introduce the isoparametric formulation of the element stiffness matrices.

After considering the linear-strain triangular element (LST) in Chapter 8, we can see that the development of element matrices and equations expressed in terms of a global coordinate system becomes an enormously difficult task (if even possible) except for the simplest of elements such as the constant-strain triangle of Chapter 6.

Hence, the isoparametric formulation was developed.

Isoparametric Elements

Introduction

The isoparametric method may appear somewhat tedious (and confusing initially), but it will lead to a simple computer program formulation, and it is generally applicable for two- and three-dimensional stress analysis and for nonstructural problems.

The isoparametric formulation allows elements to he created that are nonrectangular and have curved sides.

Numerous commercial computer programs (as described in Chapter 1) have adapted this formulation for their various libraries of elements.

Isoparametric Elements

Introduction

First, we will illustrate the isoparametric formulation to develop the simple bar element stiffness matrix.

Use of the bar element makes it relatively easy to understand the method because simple expressions result.

Then, we will consider the development of the isoparametric formulation of the simple quadrilateral element stiffness matrix.

Isoparametric Elements

Introduction

Next, we will introduce numerical integration methods for evaluating the quadrilateral element stiffness matrix.

Then, we will illustrate the adaptability of the isoparametric formulation to common numerical integration methods.

Finally, we will consider some higher-order elements and their associated shape functions.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

The term ***isoparametric*** is derived from the use of the same shape functions (or interpolation functions) [*N*] to define the element's geometric shape as are used to define the displacements within the element.

Thus, when the interpolation function is $u = a_1 + a_2 s$ for the displacement, we use $x = a_1 + a_2 s$ for the description of the nodal coordinate of a point on the bar element and, hence, the physical shape of the element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Isoparametric element equations are formulated using a natural (or intrinsic) coordinate system ***s*** that is defined by element geometry and not by the element orientation in the global-coordinate system.

In other words, axial coordinate ***s*** is attached to the bar and remains directed along the axial length of the bar, regardless of how the bar is oriented in space.

There is a relationship (called a ***transformation mapping)*** between the natural coordinate systems and the global coordinate system ***x*** for each element of a specific structure.

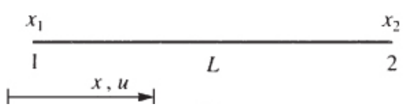
Isoparametric Elements

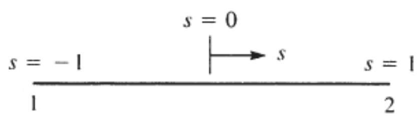
Isoparametric Formulation of the Bar Element

First, the natural coordinate ***s*** is attached to the element, with the origin located at the center of the element.

The ***s*** axis need not be parallel to the *x* axis-this is only for convenience.

Consider the bar element to have two degrees of freedom-axial displacements u_1 and u_2 at each node associated with the global *x* axis.





Isoparametric Elements

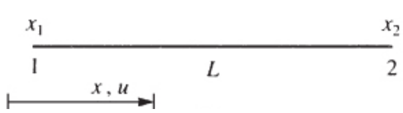
Isoparametric Formulation of the Bar Element

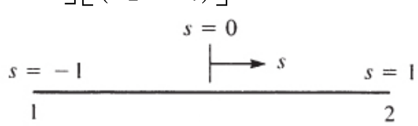
For the special case when the *s* and *x* axes are parallel to each other, the *s* and *x* coordinates can be related by:

$$x = x_c + \frac{L}{2}s$$

Using the global coordinates x_1 and x_2 with $x_c = (x_1 + x_2)/2$, we can express the natural coordinate *s* in terms of the global coordinates as:

$$s = \left[x - \frac{(x_1 + x_2)}{2}\right]\left[\frac{2}{(x_2 - x_1)}\right]$$





Isoparametric Elements

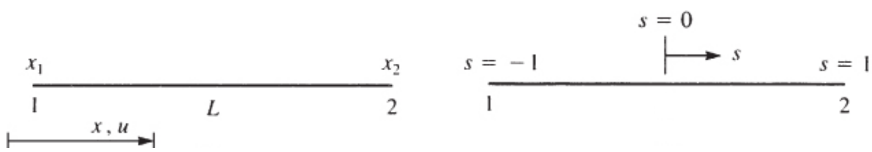
Isoparametric Formulation of the Bar Element

The shape functions used to define a position within the bar are found in a manner similar to that used in Chapter 3 to define displacement within a bar (Section 3.1).

We begin by relating the natural coordinate to the global coordinate by:

$$x = a_1 + a_2 s$$

Note that $-1 \leq s \leq 1$.



Isoparametric Elements

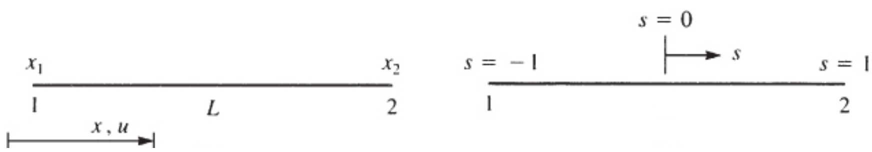
Isoparametric Formulation of the Bar Element

Solving for the *a's* in terms of x_1 and x_2, we obtain:

$$x = \left(\frac{1}{2}\right)\left[(1-s)x_1 + (1+s)x_2\right]$$

In matrix form:

$$\{x\} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



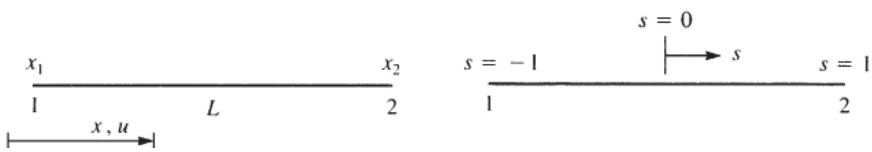
Isoparametric Elements

Isoparametric Formulation of the Bar Element

The linear shape functions map the *s* coordinate of any point in the element to the *x* coordinate.

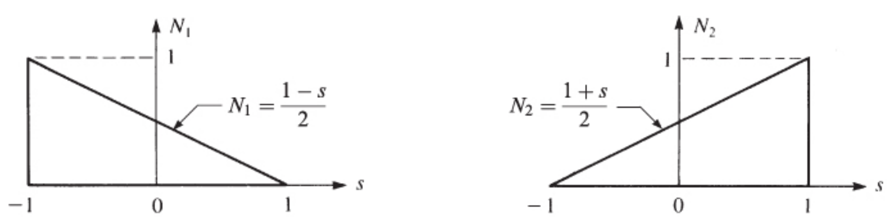
For instance, when $s = -1$, then $x = x_1$ and when $s = 1$, then $x = x_2$

$$\{x\} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

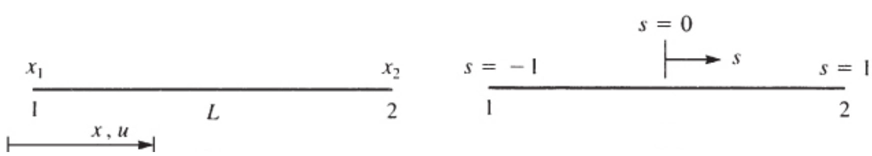


Isoparametric Elements

Isoparametric Formulation of the Bar Element

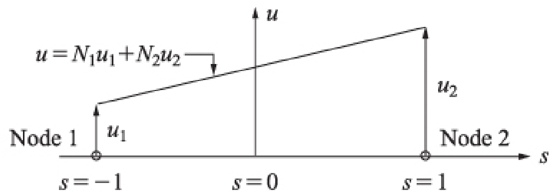


$$\{x\} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

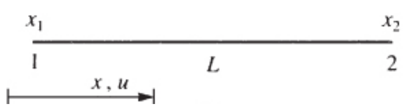


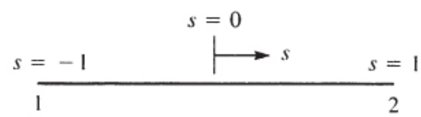
Isoparametric Elements

Isoparametric Formulation of the Bar Element



$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$





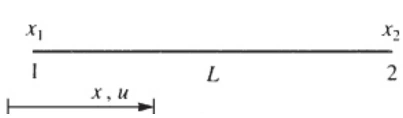
Isoparametric Elements

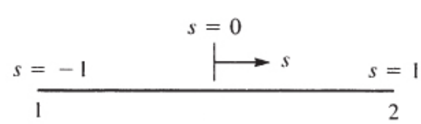
Isoparametric Formulation of the Bar Element

When a particular coordinate *s* is substituted into [N] yields the displacement of a point on the bar in terms of the nodal degrees of freedom u_1 and u_2.

Since *u* and *x* are defined by the same shape functions at the same nodes, the element is called *isoparametric.*

$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$





Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

We now want to formulate element matrix [*B*] to evaluate [*k*].

We use the isoparametric formulation to illustrate its manipulations.

For a simple bar element, no real advantage may appear evident.

However, for higher-order elements, the advantage will become clear because relatively simple computer program formulations will result.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

To construct the element stiffness matrix, determine the strain, which is defined in terms of the derivative of the displacement with respect to *x.*

The displacement *u,* however, is now a function of *s* so we must apply the chain rule of differentiation to the function *u* as follows:

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} \qquad \varepsilon_x = \frac{du}{dx} \quad \Rightarrow \quad \varepsilon_x = \frac{du}{dx} = \frac{du}{ds}\bigg/\frac{dx}{ds}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

The derivative of *u* with respect to *s* is: $\frac{du}{ds} = \frac{u_2 - u_1}{2}$

The derivative of *x* with respect to *s* is: $\frac{dx}{ds} = \frac{x_2 - x_1}{2} = \frac{L}{2}$

Therefore the strain is: $\{\varepsilon_x\} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Since $\{\varepsilon\} = [B]\{d\}$, the strain-displacement matrix [*B*] is:

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 3 - Strain-Displacement and Stress-Strain Relationships

Recall that use of linear shape functions results in a constant [*B*] matrix, and hence, in a constant strain within the element.

For higher-order elements, such as the quadratic bar with three nodes, [*B*] becomes a function of natural coordinates *s.*

The stress matrix is again given by Hooke's law as:

$$\{\sigma\} = E\{\varepsilon\} = E[B]\{d\}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix is: $[k] = \int_0^L [B]^T E[B] A\, dx$

However, in general, we must transform the coordinate *x* to *s* because [*B*] is, in general, a function of *s.*

$$\int_0^L f(x)dx = \int_{-1}^{1} f(s)\left|[J]\right| ds$$

where [*J*] is called the ***Jacobian*** matrix.

In the one-dimensional case, we have |[*J*]| = *J.*

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

For the simple bar element: $\left|[J]\right| = \frac{dx}{ds} = \frac{L}{2}$

The Jacobian determinant relates an element length (*dx*) in the global-coordinate system to an element length (*ds*) in the natural-coordinate system.

In general, |[J]| is a function of *s* and depends on the numerical values of the nodal coordinates.

This can be seen by looking at for the equations for a quadrilateral element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix in natural coordinates is:

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\,ds$$

For the one-dimensional case, we have used the modulus of elasticity *E* = [*D*].

Performing the simple integration, we obtain:

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

For higher-order one-dimensional elements, the integration in closed form becomes difficult if not impossible.

Even the simple rectangular element stiffness matrix is difficult to evaluate in closed form.

However, the use of numerical integration, as described in Section 10.3, illustrates the distinct advantage of the isoparametric formulation of the equations.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Determine the body-force matrix using the natural coordinate system *s.* The body-force matrix is:

$$\{f_{\mathbf{b}}\} = \int_V [N]^{\mathrm{T}} \{X_{\mathbf{b}}\}\, dV \qquad \{f_{\mathbf{b}}\} = \int_{x_1}^{x_2} [N]^{\mathrm{T}} \{X_{\mathbf{b}}\}\, A\, dx$$

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_{\mathbf{b}}\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_{\mathbf{b}}\} \frac{L}{2} ds = \frac{ALX_{\mathbf{b}}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The physical interpretation of the results for $\{f_b\}$ is that since *AL* represents the volume of the element and X_b the body force per unit volume, then ALX_b is the total body force acting on the element.

The factor ½ indicates that this body force is equally distributed to the two nodes of the element.

$$\{f_{\mathbf{b}}\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_{\mathbf{b}}\} \frac{L}{2} ds = \frac{ALX_{\mathbf{b}}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Determine the surface-force matrix using the natural coordinate system *s.* The surface-force matrix is:

$$\{f_s\} = \int_S [N_s]^T \{T_x\}\, dS$$

Assuming the cross section is constant and the traction is uniform over the perimeter and along the length of the element, we obtain:

$$\{f_s\} = \int_0^L [N_s]^T \{T_x\}\, dx$$

where we now assume {*Tx*} is in units of force per unit length.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Since $\{T_x\}$ is in force-per-unit-length $\{T_x\}L$ is now the total force.

The ½ indicates that the uniform surface traction is equally distributed to the two nodes of the element.

Isoparametric Elements

Isoparametric Formulation of the Bar Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting for N_1 and N_2 and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \frac{1-s}{2} \\ \frac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Note that if $\{T_x\}$ were a function of x (or s), then the amounts of force allocated to each node would generally not be equal and would be found through integration.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Recall that the term *isoparametric* is derived from the use of the same interpolation functions to define the element shape as are used to define the displacements within the element.

The approximation for displacement is:

$$u = a_1 + a_2 s + a_3 t + a_4 st$$

The description of a coordinate point in the plane element is:

$$x = a_1 + a_2 s + a_3 t + a_4 st$$

The natural-coordinate systems s-t defined by element geometry and not by the element orientation in the global-coordinate system x-y.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Much as in the bar element example, there is a transformation mapping between the two coordinate systems for each element of a specific structure, and this relationship must be used in the element formulation.

We will now formulate the isoparametric formulation of the simple linear plane quadrilateral element stiffness matrix.

This formulation is general enough to be applied to more complicated (higher-order) elements such as a quadratic plane element with three nodes along an edge, which can have straight or quadratic curved sides.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Higher-order elements have additional nodes and use different shape functions as compared to the linear element, but the steps in the development of the stiffness matrices are the same.

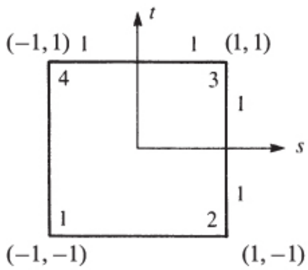
We will briefly discuss these elements after examining the linear plane element formulation.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

The natural ***s-t*** coordinates are attached to the element, with the origin at the center of the element.



The ***s*** and ***t*** axes need not be orthogonal, and neither has to be parallel to the *x* or *y* axis.

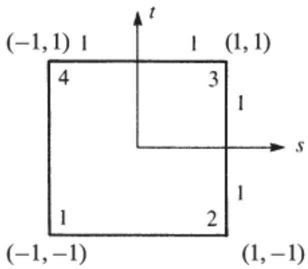
The orientation of ***s-t*** coordinates is such that the four corner nodes and the edges of the quadrilateral are bounded by +1 or -1

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

The natural ***s-t*** coordinates are attached to the element, with the origin at the center of the element.



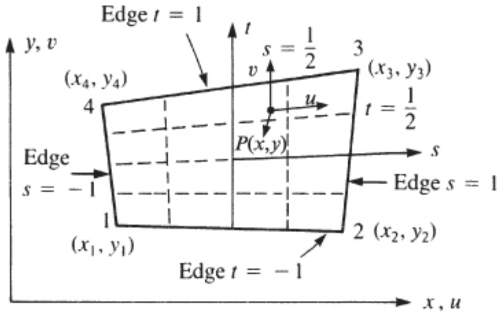
This orientation will later allow us to take advantage more fully of common numerical integration schemes.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Consider the quadrilateral to have eight degrees of freedom u_1, v_1, ... , u_4, and v_4 associated with the global *x* and *y* directions. The element then has straight sides but is otherwise of arbitrary shape.



Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

For the special case when the distorted element becomes a rectangular element with sides parallel to the global ***x-y*** coordinates, the ***s-t*** coordinates can be related to the global element coordinates *x* and *y* by

$$x = x_c + bs \qquad y = y_c + ht$$

where x_c and y_c are the global coordinates of the element centroid.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Assuming global coordinates *x* and *y* are related to the natural coordinates *s* and *t* as follows:

$$x = a_1 + a_2 s + a_3 t + a_4 st \qquad y = a_5 + a_6 s + a_7 t + a_8 st$$

Solving for the *a's* in terms of x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, we obtain

$$x = \frac{1}{4}\left[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4\right]$$

$$y = \frac{1}{4}\left[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4\right]$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

In matrix form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

where:

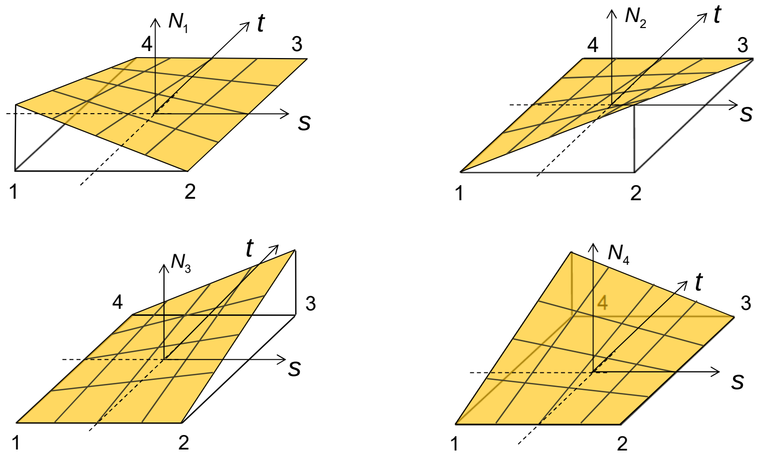
$$N_1 = \frac{(1-s)(1-t)}{4} \qquad N_2 = \frac{(1+s)(1-t)}{4}$$

$$N_3 = \frac{(1+s)(1+t)}{4} \qquad N_4 = \frac{(1-s)(1+t)}{4}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

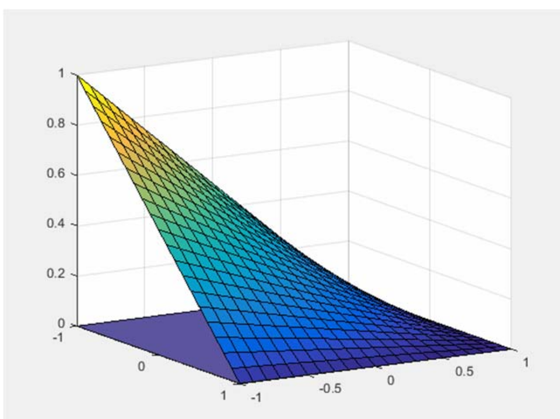
Step 1 Select Element Type



Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

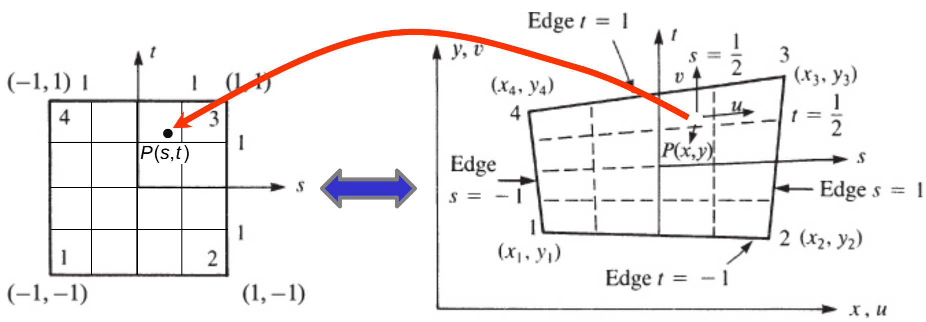


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

These shape functions are seen to map the ***s*** and ***t*** coordinates of any point in the square element to those ***x*** and ***y*** coordinates in the quadrilateral element.

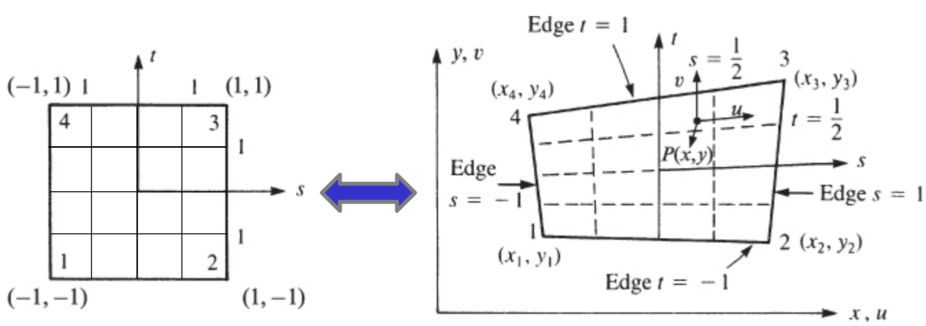


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Consider square element node 1 coordinates, where s = -1 and t = -1 then $x = x_1$ and $y = y_1$.

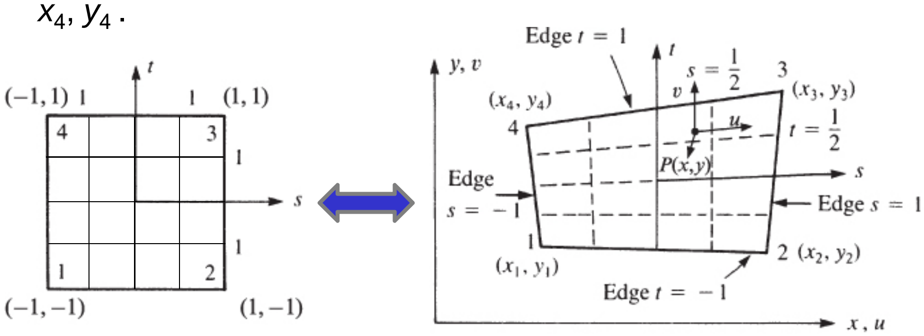


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Other local nodal coordinates at nodes 2, 3, and 4 on the square element in *s*-*t* isoparametric coordinates are mapped into a quadrilateral element in global coordinates x_2, y_2 through x_4, y_4 .

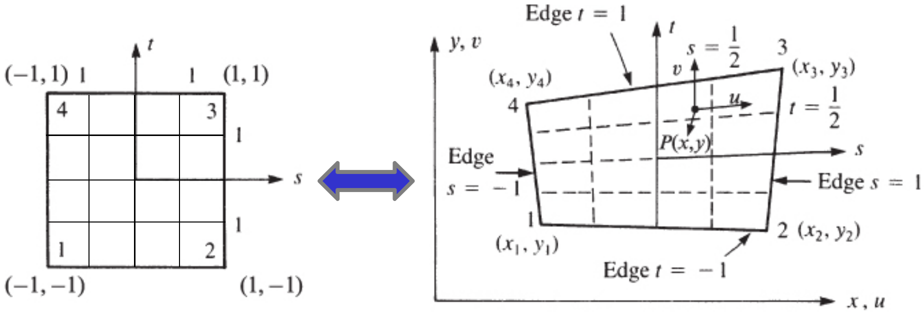


Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

Also observe the property that $N_1 + N_2 + N_3 + N_4 = 1$ for all values of s and t.



Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 1 Select Element Type

We have always developed the element interpolation functions either by assuming some relationship between the natural and global coordinates in terms of the generalized coordinates ***a***'s or, similarly, by assuming a displacement function in terms of the ***a***'*s.*

However, physical intuition can often guide us in directly expressing shape functions based on the following two criteria set forth in Section 3.2 and used on numerous occasions:

$$\sum_{i=1}^{n} N_i = 1 \qquad i = 1, 2, ,\ldots, n$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 2 Select of Displacement Functions

The displacement functions within an element are now similarly defined by the same shape functions as are used to define the element geometric shape:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We now want to formulate element matrix [*B*] to evaluate [*k*].

However, because it becomes tedious and difficult (if not impossible) to write the shape functions in terms of the ***x*** and ***y*** coordinates, as seen in Chapter 8, we will carry out the formulation in terms of the isoparametric coordinates ***s*** and ***t***.

This may appear tedious, but it is easier to use the ***s***- and ***t***-coordinate expressions.

This approach also leads to a simple computer program formulation.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

To construct an element stiffness matrix, we must determine the strains, which are defined in terms of the derivatives of the displacements with respect to the ***x*** and ***y*** coordinates.

The displacements, however, are now functions of the ***s*** and ***t*** coordinates.

The derivatives $\partial u/\partial x$ and $\partial v/\partial y$ are now expressed in terms of ***s*** and ***t***.

Therefore, we need to apply the chain rule of differentiation.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The chain rule yields:

$$\frac{\partial f}{\partial s} = \boxed{\frac{\partial f}{\partial x}}\frac{\partial x}{\partial s} + \boxed{\frac{\partial f}{\partial y}}\frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \boxed{\frac{\partial f}{\partial x}}\frac{\partial x}{\partial t} + \boxed{\frac{\partial f}{\partial y}}\frac{\partial y}{\partial t}$$

The strains can then be found; for example, ε_x = ∂***u***/∂***x***

We want to get solve the two equations for *∂f/∂x and ∂f/∂y.*

$$\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \boxed{\frac{\partial f}{\partial x}} \\ \boxed{\frac{\partial f}{\partial y}} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Consider Cramer's rule for small systems:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Consider Cramer's rule for small systems:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \qquad \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{Bmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Consider Cramer's rule for small systems:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \qquad \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial f}{\partial s} \\ \frac{\partial f}{\partial t} \end{Bmatrix}$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Consider Cramer's rule for small systems:

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Consider Cramer's rule for small systems:

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} = \frac{\frac{\partial f}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial f}{\partial t}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} = \frac{\frac{\partial f}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial f}{\partial t}}{|J|}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Consider Cramer's rule for small systems:

$$\frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} = \frac{\frac{\partial x}{\partial s}\frac{\partial f}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial f}{\partial s}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} = \frac{\frac{\partial x}{\partial s}\frac{\partial f}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial f}{\partial s}}{|J|}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The determinant in the denominator is the determinant of the *Jacobian* matrix [*J*].

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial y}{\partial s}\frac{\partial x}{\partial t}$$

We now want to express the element strains as: $\{\varepsilon\} = [B]\{d\}$

Where [*B*] must now be expressed as a function of ***s*** and ***t***.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The usual relationship between strains and displacements given in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial(\;)}{\partial x} & 0 \\ 0 & \dfrac{\partial(\;)}{\partial y} \\ \dfrac{\partial(\;)}{\partial y} & \dfrac{\partial(\;)}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Where the rectangular matrix on the right side is an *operator matrix;* that is, $\partial(\;)/\partial \boldsymbol{x}$ and $\partial(\;)/\partial \boldsymbol{y}$ represent the partial derivatives of any variable we put inside the parentheses.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Evaluating the determinant in the numerators, we have

$$\frac{\partial(\;)}{\partial x} = \frac{1}{\left|[J]\right|}\left[\frac{\partial y}{\partial t}\frac{\partial(\;)}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\;)}{\partial t}\right]$$

$$\frac{\partial(\;)}{\partial y} = \frac{1}{\left|[J]\right|}\left[\frac{\partial x}{\partial s}\frac{\partial(\;)}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\;)}{\partial s}\right]$$

Where $|[\boldsymbol{J}]|$ is the determinant of $[\boldsymbol{J}]$.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We can obtain the strains expressed in terms of the natural coordinates (***s-t***) as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{\left|[J]\right|} \begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We can express the previous equation in terms of the shape functions and global coordinates in compact matrix form as:

$$\{\varepsilon\} = [D'][N]\{d\}$$

$$[D'] = \frac{1}{\left|[J]\right|} \begin{bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We can express the previous equation in terms of the shape functions and global coordinates in compact matrix form as:

$$\{\varepsilon\} = [D'][N]\{d\}$$

$$[D'] = \frac{1}{\|[J]\|}\left\{\begin{matrix} a\dfrac{\partial(\;)}{\partial s} - b\dfrac{\partial(\;)}{\partial t} & 0 \\ 0 & c\dfrac{\partial(\;)}{\partial t} - d\dfrac{\partial(\;)}{\partial s} \\ c\dfrac{\partial(\;)}{\partial t} - d\dfrac{\partial(\;)}{\partial s} & a\dfrac{\partial(\;)}{\partial s} - b\dfrac{\partial(\;)}{\partial t} \end{matrix}\right\}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We can express the previous equation in terms of the shape functions and global coordinates in compact matrix form as:

$$a = \frac{\partial y}{\partial t} \qquad b = \frac{\partial y}{\partial s} \qquad c = \frac{\partial x}{\partial s} \qquad d = \frac{\partial x}{\partial t}$$

$$[D'] = \frac{1}{\|[J]\|}\left\{\begin{matrix} a\dfrac{\partial(\;)}{\partial s} - b\dfrac{\partial(\;)}{\partial t} & 0 \\ 0 & c\dfrac{\partial(\;)}{\partial t} - d\dfrac{\partial(\;)}{\partial s} \\ c\dfrac{\partial(\;)}{\partial t} - d\dfrac{\partial(\;)}{\partial s} & a\dfrac{\partial(\;)}{\partial s} - b\dfrac{\partial(\;)}{\partial t} \end{matrix}\right\}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The shape function matrix [N] is the 2 x 8 {d} is the column matrix.

$$\underset{3\times 8}{[B]} = \underset{3\times 2}{[D']} \underset{2\times 8}{[N]}$$

The matrix multiplications yield

$$\left[B(s,t)\right] = \frac{1}{\left|[J]\right|}\left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\right]$$

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Here *i* is a dummy variable equal to 1, 2, 3, and 4, and

$$a = \frac{\partial y}{\partial t} = \frac{1}{4}\left[y_1(s-1) + y_2(-s-1) + y_3(1+s) + y_4(1-s)\right]$$

$$b = \frac{\partial y}{\partial s} = \frac{1}{4}\left[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)\right]$$

$$c = \frac{\partial x}{\partial s} = \frac{1}{4}\left[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)\right]$$

$$d = \frac{\partial x}{\partial t} = \frac{1}{4}\left[x_1(s-1) + x_2(-s-1) + x_3(1+s) + x_4(1-s)\right]$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Using the shape functions, we have

$$N_{1,s} = \frac{1}{4}(t-1) \qquad N_{1,t} = \frac{1}{4}(s-1)$$

where the comma followed by the variable *s* or *t* indicates differentiation with respect to that variable; that is, $N_{1,s} = \partial N_1/\partial \mathbf{s}$ and so on.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

The determinant |[J]| is a polynomial in *s* and *t* and is tedious to evaluate even for the simplest case of the linear plane quadrilateral element.

However, we can evaluate |[J]| as

$$\left\|[J]\right\| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4] \qquad \{Y_c\}^T = [y_1 \quad y_2 \quad y_3 \quad y_4]$$

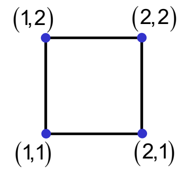
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a square global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2 \quad 2 \quad 1] \qquad \{Y_c\}^T = [1 \quad 1 \quad 2 \quad 2]$$



$$|[J]| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} = 2 = 2A$$

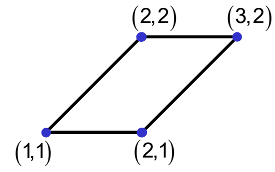
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a skewed global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2 \quad 3 \quad 2] \qquad \{Y_c\}^T = [1 \quad 1 \quad 2 \quad 2]$$



$$|[J]| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} = 2 = 2A$$

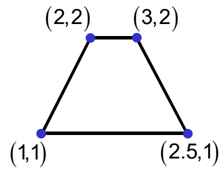
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a trapezoidal global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2.5 \quad 2 \quad 1.5] \qquad \{Y_c\}^T = [1 \quad 1 \quad 2 \quad 2]$$



$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\} = 2-t$$

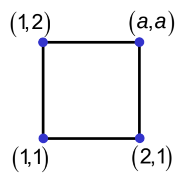
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2 \quad a \quad 1] \qquad \{Y_c\}^T = [1 \quad 1 \quad a \quad 2]$$



$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$= 2a - 2s - 2t + as + at - 2$$

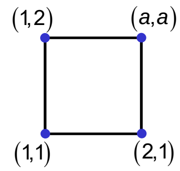
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a global element with the following coordinates:

$$\{X_c\}^T = [1 \quad 2 \quad a \quad 1] \qquad \{Y_c\}^T = [1 \quad 1 \quad a \quad 2]$$



Let's evaluate the |[J]| at (s, t) = (1, 1) :

$$\|[J]\| = 2a - 2s - 2t + as + at - 2 \ = 4a - 6$$

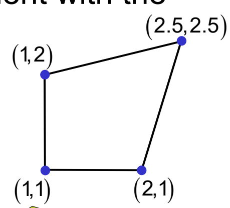
For the |[J]| to be positive, a > 3/2. If a < 3/2, then the |[J]| is negative.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

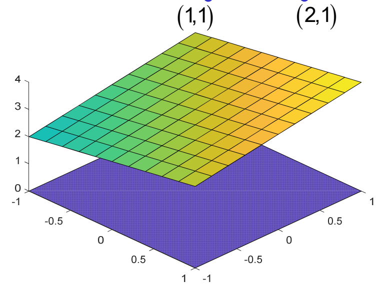
Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a element with the following coordinates:



Let's assume a > 3/2, say a = 2.5, then the |[J]| is:

$$\|[J]\| = \frac{1}{2}(6 + s + t)$$



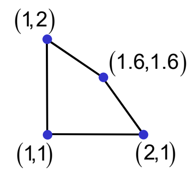
|[J]| is positive over the entire element.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

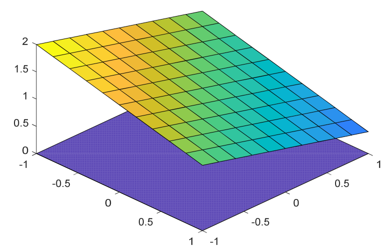
Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a element with the following coordinates:



Let's assume *a* > 3/2, say *a* = 1.6, then the |[J]| is:

$$\left|[J]\right| = \frac{1}{5}(6 - 2t - 2s)$$



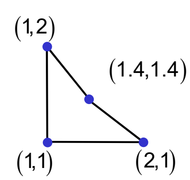
|[J]| is positive over the entire element.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

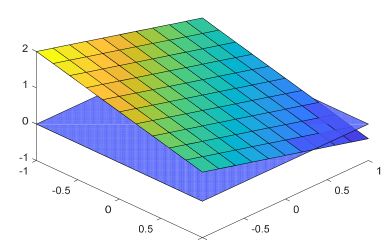
Step 3 Strain-Displacement and Stress-Strain Relationships

Let's compute the determinant |[J]| for a element with the following coordinates:



Let's assume *a* < 3/2, say *a* = 1.4, then the |[J]| is:

$$\left|[J]\right| = \frac{1}{5}(4 - 3t - 3s)$$



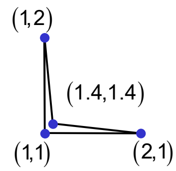
|[J]| is negative at (*s*, *t*) = (1, 1).

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

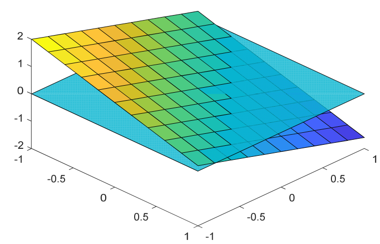
Let's compute the determinant |[J]| for a element with the following coordinates:



Let's assume *a* < 3/2, say *a* = 1.1, then the |[J]| is:

$$\left|[J]\right| = \frac{1}{10}\left(2 - 9t - 0s\right)$$

|[J]| is negative at (*s*, *t*) = (1, 1).



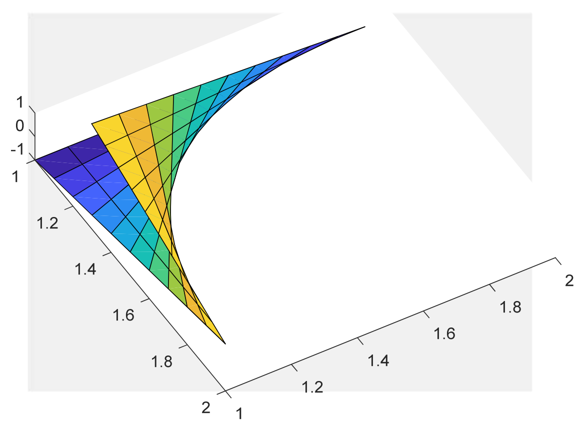
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

When the |[J]| is negative the mapping between local coordinates to global coordinates is not 1-to-1.

Here is a plot of the mapping as the vales of *a* range from 2 to 1.1:



Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 3 Strain-Displacement and Stress-Strain Relationships

We observe that |[*J*]| is a function of ***s*** and ***t*** and the known global coordinates x_1, x_2, ... , y_4.

Hence, [***B***] is a function of ***s*** and ***t*** in both the numerator and the denominator and of the known global coordinates x_1 through y_4.

The stress-strain relationship is a function of ***s*** and ***t***.

$$\{\sigma\} = [D][B]\{d\}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

We now want to express the stiffness matrix in terms of ***s-t*** coordinates.

For an element with a constant thickness *h,* we have

$$[k] = \int\int_A [B]^T [D][B]\, h\, dx\, dy$$

However, [*B*] is now a function of ***s*** and ***t***, we must integrate with respect to ***s*** and ***t***.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

Once again, to transform the variables and the region from ***x*** and ***y*** to ***s*** and ***t***, we must have a standard procedure that involves the determinant of [*J*].

$$\iint_A f(x,y)\,dx\,dy \quad \Rightarrow \quad \int_{-1}^{1}\int_{-1}^{1} f(s,t)\left|[J]\right| ds\,dt$$

where the inclusion of |[J]| in the integrand on the right side of equation results from a theorem of integral calculus.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

We also observe that the Jacobian (the determinant of the Jacobian matrix) relates an element area (*dx dy*) in the global coordinate system to an elemental area (*ds dt*) in the natural coordinate system.

For rectangles and parallelograms, *J* is the constant value *J* = *A*/4, where *A* represents the physical surface area of the element.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\,dt$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

The |[*J*]| and [*B*] are complicated expressions within the integral.

Integration to determine the element stiffness matrix is usually done numerically.

The stiffness matrix is of the order 8 x 8.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

Body Forces - The element body-force matrix will now be determined from

$$\underset{(8\times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8\times 2)}{[N]^T} \underset{(2\times 1)}{\{X_b\}}\, h \left|[J]\right| ds\, dt$$

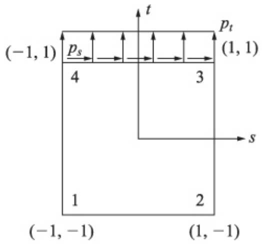
Like the stiffness matrix, the body-force matrix has to be evaluated by numerical integration.

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

Surface Forces - The surface-force matrix, say, along edge t = 1 with overall length *L,* is



$$\{f_s\} = \frac{L}{2}\int_{-1}^{1} [N_s]^T \{T\}\, h\, ds$$
$$(4\times 1) \qquad (4\times 2)\ (2\times 1)$$

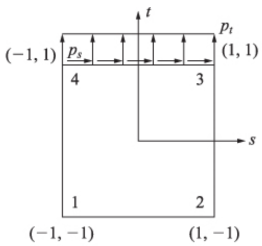
$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \frac{hL}{2}\int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T_{along\ t=1} \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} ds$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Step 4 Derive the Element Stiffness Matrix and Equations

Surface Forces - For the case of uniform (constant) p_s, and p_t, along edge t = 1, the total surface-force matrix is



$$\{f_s\} = \frac{L}{2}\int_{-1}^{1} [N_s]^T \{T\}\, h\, ds$$
$$(4\times 1) \qquad (4\times 2)\ (2\times 1)$$

$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \frac{hL}{2}\begin{Bmatrix} p_s \\ p_t \\ p_s \\ p_t \end{Bmatrix}$$

Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Steps 5 - 7

Steps 5 through 7, which involve assembling the global stiffness matrix and equations, determining the unknown nodal displacements, and calculating the stress, are identical to those in presented in previous chapters.

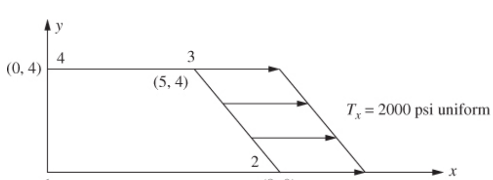
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Example 1

For the four-noded linear plane quadrilateral element shown below with a uniform surface traction along side 2-3, evaluate the force matrix by using the energy equivalent nodal forces.

Let the thickness of the element be h = 0.1 in.



$$\{f_s\} = \frac{hL}{2}\int_{-1}^{1}[N_s]^T\{T\}\,ds$$

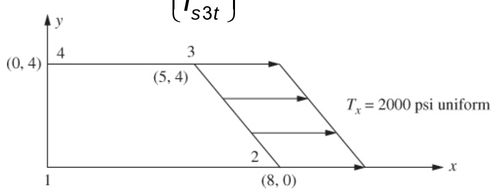
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Example 1

With length of side 2-3 given by: $L = \sqrt{(5-8)^2 + (4-0)^2} = 5$

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = \frac{hL}{2} \int_{-1}^{1} \begin{bmatrix} N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_3 \end{bmatrix}^T_{along\ s=1} \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} dt$$



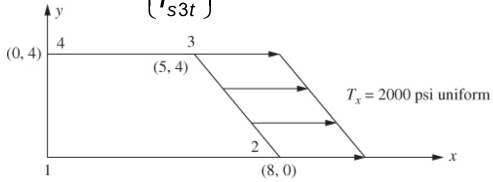
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Example 1

Substituting for ***L***, the surface traction matrix, and the thickness *h* = 0.1 we obtain

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = \frac{(0.1\text{in.})5\text{in.}}{2} \int_{-1}^{1} \begin{bmatrix} N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_3 \end{bmatrix}^T_{along\ s=1} \begin{Bmatrix} 2{,}000 \\ 0 \end{Bmatrix} dt$$



Isoparametric Elements

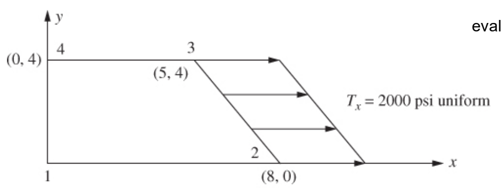
Isoparametric Formulation of the Quadrilateral Element

Example 1

Simplifying gives:

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = 0.25\,\text{in.}^2 \int_{-1}^{1} \begin{bmatrix} 2{,}000 N_2 \\ 0 \\ 2{,}000 N_3 \\ 0 \end{bmatrix} dt = 500\,\text{lb.} \int_{-1}^{1} \begin{bmatrix} N_2 \\ 0 \\ N_3 \\ 0 \end{bmatrix} dt$$

evaluated along $s = 1$ evaluated along $s = 1$



Isoparametric Elements

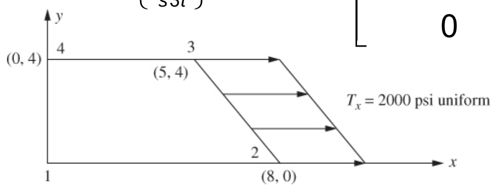
Isoparametric Formulation of the Quadrilateral Element

Example 1

Substituting the shape functions, we have

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = 500\,\text{lb.} \int_{-1}^{1} \begin{bmatrix} \dfrac{s-t-st+1}{4} \\ 0 \\ \dfrac{s+t+st+1}{4} \\ 0 \end{bmatrix} dt = 250\,\text{lb.} \int_{-1}^{1} \begin{bmatrix} 1-t \\ 0 \\ t+1 \\ 0 \end{bmatrix} dt$$

evaluated along $s = 1$



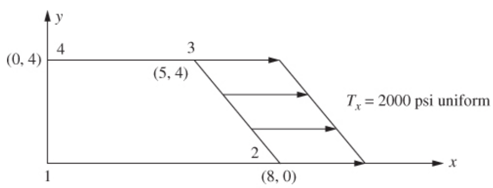
Isoparametric Elements

Isoparametric Formulation of the Quadrilateral Element

Example 1

Performing the integration gives:

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = 250\,\text{lb.}\int_{-1}^{1} \begin{bmatrix} 1-t \\ 0 \\ t+1 \\ 0 \end{bmatrix} dt \quad = 500\,\text{lb.} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad = \begin{bmatrix} 500 \\ 0 \\ 500 \\ 0 \end{bmatrix} \text{lb.}$$



Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

In this section, we will describe two methods for numerical evaluation of definite integrals, because it has proven most useful for finite element work.

The Newton-Cotes methods for one and two intervals of integration are the well known trapezoid and Simpson's one-third rule, respectively.

We will then describe Gauss' method for numerical evaluation of definite integrals.

After describing both methods, we will then understand why the Gaussian quadrature method is used in finite element work.

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

The Newton-Cotes method is a common technique for evaluation of definite integrals.

To evaluate the integral $\mathrm{I} = \int_{-1}^{1} y\,dx$

we assume the sampling points of *y(x)* are spaced at equal intervals.

Since the limits of integration are from -1 to 1 using the isoparametric formulation, the Newton-Cotes formula is given by

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;= h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

The constants C_i are the Newton-Cotes constants for numerical integration with *i* intervals.

The number of intervals will be one less than the number of sampling points, *n.*

The term *h* is the interval between the limits of integration (for limits of integration between -1 and 1 this makes *h* = 2).

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;= h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

The Newton-Cotes constants have been published and are summarized in the table below for *i* = 1 to 6.

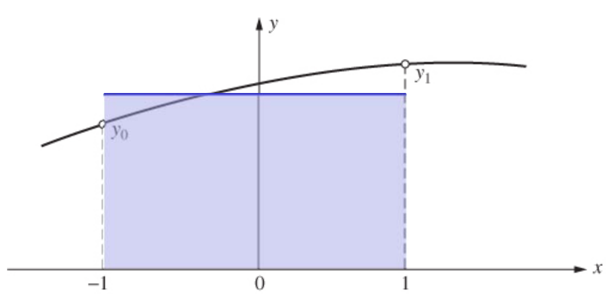
Intervals, *i*	No. of Points, *n*	C_0	C_1	C_2	C_3	C_4	C_5	C_6
1	2	1/2	1/2			(trapezoid rule)		
2	3	1/6	4/6	1/6		(Simpson's 1/3 rule)		
3	4	1/8	3/8	3/8	1/8	(Simpson's 3/8 rule)		
4	5	7/90	32/90	12/90	32/90	7/90		
5	6	19/288	75/288	50/288	50/288	75/288	19/288	
6	7	41/840	216/840	27/840	272/840	27/840	216/840	41/840

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i = h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

The case *i* = 1 corresponds to the well known trapezoid rule illustrated below.



$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i = 2\left(\frac{y_0 + y_1}{2}\right) = \left[y_0 + y_1\right]$$

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

The case *i* = 2 corresponds to the well-known Simpson one-third rule.

It has been shown that the formulas for *i* = 3 and *i* = 5 have the same accuracy as the formulas for *i* = 2 and *i* = 4, respectively.

Therefore, it is recommended that the even formulas with *i* = 2 and *i* = 4 be used in practice.

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;= h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

To obtain greater accuracy one can then use a smaller interval (include more evaluations of the function to be integrated).

This can be accomplished by using a higher-order Newton-Cotes formula, thus increasing the number of intervals *i*.

It has been shown that we need to use *n* equally spaced sampling points to integrate exactly a polynomial of order at most *n* - 1.

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;= h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

On the other hand, using Gaussian quadrature we will show that we use unequally spaced sampling points *n* and integrate exactly a polynomial of order at most *2n* - 1.

For instance, using the Newton-Cotes formula with *n* = 2 sampling points, the highest order polynomial we can integrate exactly is a linear one.

However, using Gaussian quadrature, we can integrate a cubic polynomial exactly.

Isoparametric Elements

Newton-Cotes and Gaussian Quadrature

Gaussian quadrature is then more accurate with fewer sampling points than Newton-Cotes quadrature

This is because Gaussian quadrature is based on optimizing the position of the sampling points (not making them equally spaced as in the Newton-Cotes method) and also optimizing the weights W_i (see the table below).

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y(x_i)$$

Order N	Points u_i	Weights w_i
1	0.000000000	2.00000000
2	±0.577350269	1.00000000
3	0.000000000	0.88888889
	±0.774596669	0.55555556
4	±0.339981044	0.65214515
	±0.861136312	0.34785485

Isoparametric Elements

Newton-Cotes Example

Using the Newton-Cotes method with i = 2 intervals (n = 3 sampling points), evaluate the integrals:

$$\mathrm{I} = \int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right) \right] dx \qquad \mathrm{I} = \int_{-1}^{1} \left[3^x - x \right] dx$$

Using three sampling points means we evaluate the function inside the integrand at x = -1, x = 0, and x = 1, and multiply each evaluated function by the respective Newton-Cotes numbers.

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i = 2\left[\tfrac{1}{6}y_0 + \tfrac{4}{6}y_1 + \tfrac{1}{6}y_2\right]$$

Isoparametric Elements

Newton-Cotes Example

Using the Newton-Cotes method with i = 2 intervals (n = 3 sampling points), evaluate the integrals:

$$\mathrm{I} = \int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right) \right] dx \qquad \mathrm{I} = \int_{-1}^{1} \left[3^x - x \right] dx$$

$$\mathrm{I} = \int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right) \right] dx \approx 2\left[\tfrac{1}{6}(1.8775826) + \tfrac{4}{6}(1) + \tfrac{1}{6}(1.8775826)\right]$$

$$= 2.5850550$$

0.027% error

$$\mathrm{I} = \int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right) \right] dx = 2.5843688$$

Isoparametric Elements

Newton-Cotes Example

Using the Newton-Cotes method with i = 2 intervals (n = 3 sampling points), evaluate the integrals:

$$\mathrm{I} = \int_{-1}^{1} \left[x^2 + \cos\left(\frac{x}{2}\right) \right] dx \qquad \mathrm{I} = \int_{-1}^{1} \left[3^x - x \right] dx$$

$$\mathrm{I} = \int_{-1}^{1} \left[3^x - x \right] dx \approx 2\left[\tfrac{1}{6}(1.3333333) + \tfrac{4}{6}(1) + \tfrac{1}{6}(2) \right]$$

$$= 2.4444444 \qquad \boxed{0.706\% \text{ error}}$$

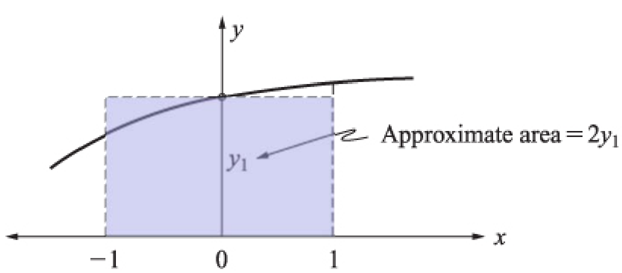
$$\mathrm{I} = \int_{-1}^{1} \left[3^x - x \right] dx = 2.427305$$

Isoparametric Elements

Gaussian Quadrature

To evaluate the integral: $\mathrm{I} = \int_{-1}^{1} y\, dx$

where $y = y(x)$, we might choose (sample or evaluate) y at the midpoint $y(0) = y_1$ and multiply by the length of the interval, as shown below to arrive at $\mathrm{I} = 2y_1$, a result that is exact if the curve happens to be a straight line.



Isoparametric Elements

Gaussian Quadrature

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y\left(x_i\right)$$

That is, to approximate the integral, we evaluate the function at several sampling points *n,* multiply each value y_i by the appropriate weight W_i , and add the terms.

Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained.

Sampling points are located symmetrically with respect to the center of the interval.

Isoparametric Elements

Gaussian Quadrature

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y\left(x_i\right)$$

In general, Gaussian quadrature using *n* points (Gauss points) is exact if the integrand is a polynomial of degree $2n - 1$ or less.

In using *n* points, we effectively replace the given function $y = f(x)$ by a polynomial of degree $2n- 1$.

The accuracy of the numerical integration depends on how well the polynomial fits the given curve.

Isoparametric Elements

Gaussian Quadrature

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y(x_i)$$

If the function $f(x)$ is not a polynomial, Gaussian quadrature is inexact, but it becomes more accurate as more Gauss points are used.

Also, it is important to understand that the ratio of two polynomials is, in general, not a polynomial; therefore, Gaussian quadrature will not yield exact integration of the ratio.

Isoparametric Elements

Gaussian Quadrature - Two-Point Formula

To illustrate the derivation of a two-point (n = 2) consider:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = W_1 y(x_1) + W_2 y(x_2)$$

There are four unknown parameters to determine: W_1, W_2, x_1, and x_2.

Therefore, we assume a cubic function for y as follows:

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$

Isoparametric Elements

Gaussian Quadrature - Two-Point Formula

In general, with four parameters in the two-point formula, we would expect the Gauss formula to exactly predict the area under the curve.

$$A = \int_{-1}^{1} \left(C_0 + C_1 x + C_2 x^2 + C_3 x^3 \right) dx = 2C_0 + \frac{2}{3} C_2$$

However, we will assume, based on Gauss's method, that $W_1 = W_2$ and that $x_1 = x_2$ as we use two symmetrically located Gauss points at $x = \pm a$ with equal weights.

The area predicted by Gauss's formula is

$$A_G = W\, y(-a) + W\, y(a) = 2W \left(C_0 + C_2 a^2 \right)$$

Isoparametric Elements

Gaussian Quadrature - Two-Point Formula

If the error, $e = A - A_G$, is to vanish for any C_0 and C_2, we must have, in the error expression:

$$e = A - A_G = \left(2C_0 + \tfrac{2}{3} C_2 \right) - \left(C_0 + C_2 a^2 \right) 2W$$

$$\frac{\partial e}{\partial C_0} = 0 = 2 - 2W \qquad \Rightarrow \qquad W = 1$$

$$\frac{\partial e}{\partial C_2} = 0 = \frac{2}{3} - 2a^2 W \qquad \Rightarrow \qquad a = \sqrt{1/3} = 0.5773....$$

Now $W = 1$ and $a = 0.5773$ ... are the W_i's and a_i's (x_i's) for the two-point Gaussian quadrature as given in the table.

Isoparametric Elements

Gaussian Quadrature Example

Use three-point Gaussian Quadrature evaluate the integrals:

$$I = \int_{-1}^{1}\left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx \qquad I = \int_{-1}^{1}\left[3^x - x\right]dx$$

$$I \approx \sum_{i=1}^{3} W_i\left[x_i^{\,2} + \cos\left(\frac{x_i}{2}\right)\right]$$

$$\approx \frac{5}{9}(1.5259328) + \frac{8}{9}(1.0) + \frac{5}{9}(1.5259328) = 2.5843698$$

0.00004% error

Order N	Points u_i	Weights w_i
1	0.000000000	2.00000000
2	±0.577350269	1.00000000
3	0.000000000	0.88888889
	±0.774596669	0.55555556
4	±0.339981044	0.65214515
	±0.861136312	0.34785485

Isoparametric Elements

Gaussian Quadrature Example

Use three-point Gaussian Quadrature evaluate the integrals:

$$I = \int_{-1}^{1}\left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx \qquad I = \int_{-1}^{1}\left[3^x - x\right]dx$$

$$I \approx \sum_{i=1}^{3} W_i\left[3^{x_i} - x_i\right]$$

$$\approx \frac{5}{9}(1.2015923) + \frac{8}{9}(1.0) + \frac{5}{9}(1.5673475) = 2.4271888$$

0.00477% error

Order N	Points u_i	Weights w_i
1	0.000000000	2.00000000
2	±0.577350269	1.00000000
3	0.000000000	0.88888889
	±0.774596669	0.55555556
4	±0.339981044	0.65214515
	±0.861136312	0.34785485

Isoparametric Elements

Gaussian Quadrature Example

In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$I = \int_{-1}^{1} \int_{-1}^{1} f(s,t)\,ds\,dt \approx \int_{-1}^{1} \left[\sum_{i=1}^{n} W_i f(s_i,t) \right] dt$$

$$\approx \sum_{j=1}^{n} W_j \left[\sum_{i=1}^{n} W_i f(s_i,t_j) \right]$$

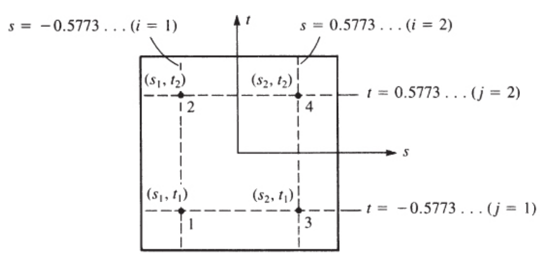
$$\approx \sum_{i=1}^{n} \sum_{j=1}^{n} W_i W_j f(s_i,t_j)$$

Isoparametric Elements

Gaussian Quadrature Example

For example, a four-point Gauss rule (often described as a 2 x 2 rule) is shown below with i = 1, 2 and j = 1, 2 yields

$$I \approx \sum_{i=1}^{2} \sum_{j=1}^{2} W_i W_j f(s_i,t_j) \approx W_1 W_1 f(s_1,t_1) + W_1 W_2 f(s_1,t_2) + W_2 W_1 f(s_2,t_1) + W_2 W_2 f(s_2,t_2)$$



The four sampling points are at s_i *and* t_i = ±0.5773... and W_i = 1.0

Isoparametric Elements

Gaussian Quadrature Example

In three dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other two as

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(s,t,z)\,ds\,dt\,dz \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} W_i W_j W_k\, f\left(s_i, t_j, z_k\right)$$

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

For the two-dimensional element, we have shown in previous chapters that

$$[k] = \iint_A [B]^T [D][B]\,h\,dx\,dy$$

where, in general, the integrand is a function of *x* and *y* and nodal coordinate values.

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

We have shown that [*k*] for a quadrilateral element can be evaluated in terms of a local set of coordinates ***s***-***t***, with limits from -1 to 1within the element.

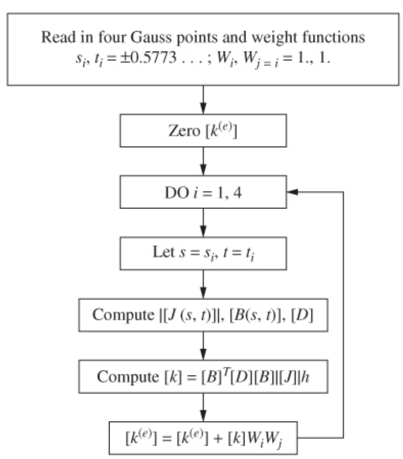
$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

Each coefficient of the integrand $[B]^T$ [*D*] [*B*] |[*J*]| evaluated by numerical integration in the same manner as *f*(s, *t*) was integrated.

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

A flowchart to evaluate [*k*] for an element using four-point Gaussian quadrature is shown here.



Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

The explicit form for four-point Gaussian quadrature (now using the single summation notation with i = 1, 2, 3, 4), we have

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B] h \left|[J]\right| ds\, dt$$

$$= \left[B(s_1,t_1)\right]^T [D]\left[B(s_1,t_1)\right]\left|\left[J(s_1,t_1)\right]\right| W_1 W_1$$

$$+ \left[B(s_2,t_2)\right]^T [D]\left[B(s_2,t_2)\right]\left|\left[J(s_2,t_2)\right]\right| W_2 W_2$$

$$+ \left[B(s_3,t_3)\right]^T [D]\left[B(s_3,t_3)\right]\left|\left[J(s_3,t_3)\right]\right| W_3 W_3$$

$$+ \left[B(s_4,t_4)\right]^T [D]\left[B(s_4,t_4)\right]\left|\left[J(s_4,t_4)\right]\right| W_4 W_4$$

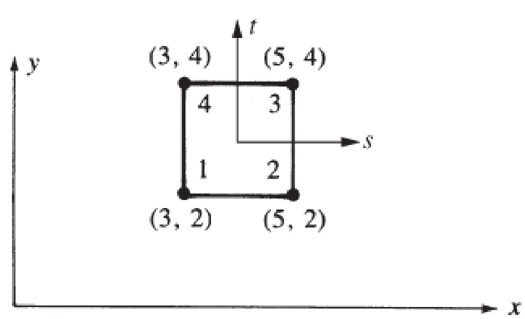
where s_1=t_1= -0.5773, s_2=-0.5773, t_2=0.5773, s_3=0.5773, t_3=-0.5773, and s_4=t_4=0.5773 and W_1=W_2=W_3=W_4=1.0

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

Evaluate the stiffness matrix for the quadrilateral element shown below using the four-point Gaussian quadrature rule.

Let E = 30 x 10^6 psi and ν = 0.25. The global coordinates are shown in inches. Assume h = 1 in.



Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

Using the four-point rule, the four points are:

$$(s_1, t_1) = (-0.5773, -0.5773)$$
$$(s_2, t_2) = (-0.5773, \; 0.5773)$$
$$(s_3, t_3) = (\; 0.5773, -0.5773)$$
$$(s_4, t_4) = (\; 0.5773, \; 0.5773)$$

With $W_1 = W_2 = W_3 = W_4 = 1.0$

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

$$\begin{aligned}[k] = &\left[B(s_1,t_1)\right]^T [D]\left[B(s_1,t_1)\right]\left|\left[J(s_1,t_1)\right]\right| \\ &+\left[B(s_2,t_2)\right]^T [D]\left[B(s_2,t_2)\right]\left|\left[J(s_2,t_2)\right]\right| \\ &+\left[B(s_3,t_3)\right]^T [D]\left[B(s_3,t_3)\right]\left|\left[J(s_3,t_3)\right]\right| \\ &+\left[B(s_4,t_4)\right]^T [D]\left[B(s_4,t_4)\right]\left|\left[J(s_4,t_4)\right]\right|\end{aligned}$$

First evaluate $|[J]|$ at each Gauss, for example:

$$\left|\left[J(-0.5773, -0.5773)\right]\right|$$

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

Recall:

$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4] \qquad \{Y_c\}^T = [y_1 \quad y_2 \quad y_3 \quad y_4]$$

For this example:

$$\{X_c\}^T = [3 \quad 5 \quad 5 \quad 3] \qquad \{Y_c\}^T = [2 \quad 2 \quad 4 \quad 4]$$

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

Recall:

$$\left|\left[J(-0.5773, -0.5773)\right]\right|$$

$$= \frac{1}{8}[3 \quad 5 \quad 5 \quad 3] \begin{bmatrix} 0 & 1-(-0.5773) & (-0.5773)-(-0.5773) & (-0.5773)-1 \\ (-0.5773)-1 & 0 & (-0.5773)+1 & -(-0.5773)-(-0.5773) \\ (-0.5773)-(-0.5773) & -(-0.5773)-1 & 0 & (-0.5773)+1 \\ 1-(-0.5773) & (-0.5773)+(-0.5773) & -(-0.5773)-1 & 0 \end{bmatrix} \begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$= 1.000$$

Similarly:

$$\left|\left[J(-0.5773, 0.5773)\right]\right| = 1.000$$

$$\left|\left[J(0.5773, -0.5773)\right]\right| = 1.000$$

$$\left|\left[J(0.5773, 0.5773)\right]\right| = 1.000$$

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

To evaluate [*B*] consider:

$$\left[B(-0.5773,-0.5773)\right] = \frac{1}{\left|\left[J(-0.5773,-0.5773)\right]\right|}\left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\right]$$

where

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

For this example:

$$a = \frac{1}{4}\left[y_1(s-1) + y_2(-s-1) + y_3(1+s) + y_4(1-s)\right]$$

$$= \frac{1}{4}\Big[2\big((-0.5773)-1\big) + 2\big(-(-0.5773)-1\big)$$
$$+ 4\big(1+(-0.5773)\big) + 4\big(1-(-0.5773)\big)\Big]$$

$$= 1.000$$

Similar computations are used to obtain *b, c,* and *d.*

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

The shape functions are computed as:

$$N_{1,s} = \frac{1}{4}(t-1) = \frac{1}{4}((-0.5773)-1) = -0.3943$$

$$N_{1,t} = \frac{1}{4}(s-1) = \frac{1}{4}((-0.5773)-1) = -0.3943$$

Similarly, $[B_2]$, $[B_3]$, and $[B_4]$ must be evaluated like $[B_1]$ at (-0.5773, -0.5773).

We then repeat the calculations to evaluate [*B*] at the other Gauss points.

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

Using a computer program written specifically to evaluate [*B*], at each Gauss point and then [*k*], we obtain the final form of [*B*(-0.5773, -0.5773)], as

$$\left[B(-0.5773, -0.5773)\right]$$

$$= \left[\begin{array}{cc|cc|cc|cc} -0.3943 & 0 & 0.3943 & 0 & 0.1057 & 0 & -0.1057 & 0 \\ 0 & -0.3943 & 0 & -0.1057 & 0 & 0.1057 & 0 & 0.3943 \\ -0.3943 & -0.3943 & -0.1057 & 0.3943 & 0.1057 & 0.1057 & 0.3943 & -0.1057 \end{array}\right]$$

With similar expressions for [*B*(-0.5773, 0.5773)], and so on.

Isoparametric Elements

Evaluation of the Stiffness Matrix by Gaussian Quadrature

The matrix [D] is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \times 10^6 \, psi$$

Finally, [*k*] is:

$$[k] = 10^4 \left[\begin{array}{cccc|cccc} 1466 & 500 & -866 & -100 & -733 & -500 & 133 & 100 \\ 500 & 1466 & 100 & 133 & -500 & -733 & -100 & -866 \\ -866 & 100 & 1466 & -500 & 133 & -100 & -733 & 500 \\ -100 & 133 & -500 & 1466 & 100 & -866 & 500 & -733 \\ \hline -733 & -500 & 133 & 100 & 1466 & 500 & -866 & -100 \\ -500 & -733 & -100 & -866 & 500 & 1466 & 100 & 133 \\ 133 & -100 & -733 & 500 & -866 & 100 & 1466 & -500 \\ 100 & -866 & 500 & -733 & -100 & 133 & -500 & 1466 \end{array}\right]$$

Isoparametric Elements

Evaluation of Element Stresses

The stresses are not constant within the quadrilateral element.

$$\{\sigma\} = [D][B]\{d\}$$

In practice, the stresses are evaluated at the same Gauss points used to evaluate the stiffness matrix [*k*].

The common method used in computer programs is to evaluate the stresses in all elements at a shared node and then use an average of these element nodal stresses to represent the stress at the node.

Isoparametric Elements

Evaluation of Element Stresses

The stresses are not constant within the quadrilateral element.

$$\{\sigma\} = [D][B]\{d\}$$

Stress plots obtained in these programs are based on this average nodal method.

The following example illustrates the use of Gaussian quadrature to evaluate the stress matrix at the *s* = 0, *t* = 0 locations of the element.

Isoparametric Elements

Evaluation of Element Stresses

For the rectangular element shown below, assume plane stress conditions with E = 30 X 10^6 psi , ν = 0.3, and displacements u_1 = 0, v_1 = 0, u_2 = 0.001 in., v_2 = 0.0015 in., u_3 = 0.003 in., v_3 = 0.0016 in., u_4 = 0, and v_4 = 0.

Evaluate the stresses, σ_x, σ_x, and τ_{xy} at *s* = 0, *t* = 0.

First, evaluate [*B*] at s = 0, *t* = 0.

$$|B| = \frac{1}{\|[J]\|}\left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\right]$$

$$= \frac{1}{\|[J(0,0)]\|}\left[[B_1(0,0)] \quad [B_2(0,0)] \quad [B_3(0,0)] \quad [B_4(0,0)]\right]$$

Isoparametric Elements

Evaluation of Element Stresses

For the rectangular element shown below, assume plane stress conditions with E = 30 X 10^6 psi , ν = 0.3, and displacements u_1 = 0, v_1 = 0, u_2 = 0.001 in., v_2 = 0.0015 in., u_3 = 0.003 in., v_3 = 0.0016 in., u_4 = 0, and v_4 = 0.

$$\left|\left[J(0,0)\right]\right| = \frac{1}{8}\begin{bmatrix}3 & 5 & 5 & 3\end{bmatrix}\begin{bmatrix}0 & 1 & 0 & -1\\ -1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1\\ 1 & 0 & -1 & 0\end{bmatrix}\begin{Bmatrix}2\\2\\4\\4\end{Bmatrix}$$

$$\left|\left[J(0,0)\right]\right| = 1$$

Isoparametric Elements

Evaluation of Element Stresses

Recall, $[B_i]$ is:

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

with: $a = 1 \quad b = 0 \quad c = 1 \quad d = 0$

Differentiating the shape functions with respect to s and t and then evaluating at s = 0, t = 0, we obtain:

$$N_{1,s} = -\frac{1}{4} \quad N_{2,s} = \frac{1}{4} \quad N_{3,s} = \frac{1}{4} \quad N_{4,s} = -\frac{1}{4}$$

$$N_{1,t} = -\frac{1}{4} \quad N_{2,t} = -\frac{1}{4} \quad N_{3,t} = \frac{1}{4} \quad N_{4,t} = \frac{1}{4}$$

Isoparametric Elements

Evaluation of Element Stresses

Therefore [B] is:

$$[B_1] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \quad [B_2] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad [B_3] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad [B_4] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

The element stress matrix $\{\sigma\}$ is then obtained by substituting [B] and the plane stress [D] matrix into the definition as:

$$\{\sigma\} = [D][B]\{d\}$$

Isoparametric Elements

Evaluation of Element Stresses

$$\{\sigma\} = [D][B]\{d\}$$

$$= \frac{30\times10^6}{1-0.09}\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

$$\times\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 & 0.25 & 0.25 & 0.25 & -0.25 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.001 \\ 0.0015 \\ 0.003 \\ 0.0016 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 3.321 \\ 1.071 \\ 1.417 \end{Bmatrix} 10^4\ psi$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

In general, higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element.

This results in higher-order strain variations and convergence occurs at a faster rate using fewer elements.

The trade-off is that there is a substantial increase in required computational power.

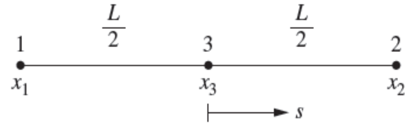
Another advantage of higher-order elements is that curved boundaries of irregularly-shaped bodies can be approximated more closely than simple straight-sided linear elements.

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

We have been working with the linear strain bar element throughout the text.

The linear strain bar (also called a quadratic isoparametric bar element) shown below has three coordinates of nodes in the global coordinates.



Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

For the three-noded linear strain bar isoparametric element we will determine the shape functions, N_1, N_2, and N_3, and the strain-displacement matrix [B].

Assume the general axial displacement function to be a quadratic:

$$x = a_1 + a_2 s + a_3 s^2$$

Evaluating the *a's* in terms of the nodal coordinates, we obtain

$$x(-1) = x_1 = a_1 - a_2 + a_3$$

$$x(0) = x_3 = a_1$$

$$x(1) = x_2 = a_1 + a_2 + a_3$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

Substituting the values for a_1, a_2, and a_3 into the general equation for x, we obtain

$$x = a_1 + a_2 s + a_3 s^2 = x_3 + \left(\frac{x_2 - x_1}{2}\right)s + \left(\frac{x_1 + x_2 - 2x_3}{2}\right)s^2$$

Combining like terms gives:

$$x = \left(\frac{s(s-1)}{2}\right)x_1 + \left(\frac{s(s+1)}{2}\right)x_2 + \left(1 - s^2\right)x_3$$

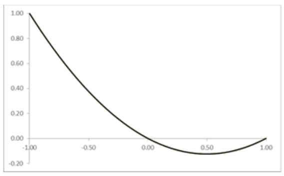
$$\{x\} = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \dfrac{s(s-1)}{2} & \dfrac{s(s+1)}{2} & 1 - s^2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

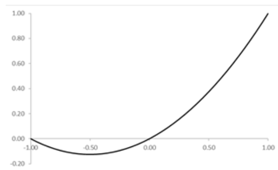
Isoparametric Elements

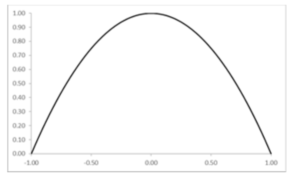
Higher-Order Shape Functions – Linear Strain Bar

Therefore the shape functions

$$N_1 = \frac{s(s-1)}{2} \qquad N_2 = \frac{s(s+1)}{2} \qquad N_3 = 1 - s^2$$







Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

Now determine the strain-displacement matrix [*B*] as:

$$\varepsilon = \frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} = [B]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Using an isoparametric formulation the displacement function is:

$$u = u_3 + \frac{u_2}{2}s - \frac{u_1}{2}s + \frac{u_1}{2}s^2 + \frac{u_2}{2}s^2 - \frac{2u_3}{2}s^2$$

$$\frac{du}{ds} = \frac{u_2}{2} - \frac{u_1}{2} + u_1 s + u_2 s - 2u_3 s = \left(s - \frac{1}{2}\right)u_1 + \left(s + \frac{1}{2}\right)u_2 - (2s)u_3$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

We have previously showed that: $\frac{dx}{ds} = \frac{L}{2} = \left\|[J]\right\|$

This relationship holds for the higher-order one-dimensional elements as well as for the two-noded constant strain bar element as long as node 3 is at the geometry center of the bar.

Using this relationship gives:

$$\frac{du}{dx} = \frac{du}{ds}\frac{ds}{dx} = \frac{2}{L}\left[\left(s - \frac{1}{2}\right)u_1 + \left(s + \frac{1}{2}\right)u_2 - (2s)u_3\right]$$

$$= \left(\frac{2s-1}{L}\right)u_1 + \left(\frac{2s+1}{L}\right)u_2 - \left(\frac{4s}{L}\right)u_3$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

In matrix form:

$$\frac{du}{dx} = \begin{bmatrix} \frac{2s-1}{L} & \frac{2s+1}{L} & -\frac{4s}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

The axial strain becomes:

$$\varepsilon_x = \frac{du}{dx} = \begin{bmatrix} \frac{2s-1}{L} & \frac{2s+1}{L} & -\frac{4s}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [B] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Where the gradient matrix [*B*] is:

$$[B] = \begin{bmatrix} \frac{2s-1}{L} & \frac{2s+1}{L} & -\frac{4s}{L} \end{bmatrix}$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

Substituting the expression for [*B*] in the stiffness matrix, we obtain

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\,ds$$

$$= \frac{AEL}{2}\int_{-1}^{1}\begin{bmatrix} \frac{(2s-1)^2}{L^2} & \frac{(2s-1)(2s+1)}{L^2} & \frac{(-4s)(2s-1)}{L^2} \\ \frac{(2s-1)(2s+1)}{L^2} & \frac{(2s-1)^2}{L^2} & \frac{(-4s)(2s+1)}{L^2} \\ \frac{(-4s)(2s-1)}{L^2} & \frac{(-4s)(2s+1)}{L^2} & \frac{(2s-1)^2}{L^2} \end{bmatrix} ds$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

Substituting the expression for [*B*] in the stiffness matrix, we obtain

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\,ds$$

$$= \frac{AE}{2L}\int_{-1}^{1}\begin{bmatrix} 4s^2-4s+1 & 4s^2-1 & -8s^2+4s \\ 4s^2-1 & 4s^2+4s+1 & -8s^2-4s \\ -8s^2+4s & -8s^2-4s & 16s^2 \end{bmatrix} ds$$

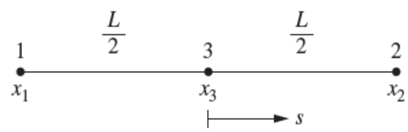
$$= \frac{AE}{2L}\left.\begin{bmatrix} \frac{4}{3}s^3-2s^2+s & \frac{4}{3}s^3-s & -\frac{8}{3}s^3+2s^2 \\ \frac{4}{3}s^3-s & \frac{4}{3}s^3+2s^2+s & -\frac{8}{3}s^3-2s^2 \\ -\frac{8}{3}s^3+2s^2 & -\frac{8}{3}s^3-2s^2 & \frac{16}{3}s^3 \end{bmatrix}\right|_{-1}^{1} = \frac{AE}{2L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ 0.667 & 4.67 & -5.33 \\ -5.33 & -5.33 & 10.67 \end{bmatrix}$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

Now let's illustrate how to evaluate the stiffness matrix for the three-noded bar element using two-point Gaussian quadrature.

We will compare the results to that obtained by the explicit integration performed.



Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

Recall the stiffness matrix for this element is:

$$[k] = \frac{L}{2}\int_{-1}^{1}[B]^T E[B]A\,ds$$

$$= \frac{AE}{2L}\int_{-1}^{1}\begin{bmatrix} 4s^2-4s+1 & 4s^2-1 & -8s^2+4s \\ 4s^2-1 & 4s^2+4s+1 & -8s^2-4s \\ -8s^2+4s & -8s^2-4s & 16s^2 \end{bmatrix} ds$$

Using two-point Gaussian quadrature, we evaluate the stiffness matrix at two points:

$$s_1 = -\frac{1}{\sqrt{3}} = -0.57735 \qquad s_2 = \frac{1}{\sqrt{3}} = 0.57735$$

$$w_1 = 1 \qquad w_2 = 1$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

We then evaluate each term in the integrand at each Gauss point and multiply each term by its weight (here weights are1).

We then add those Gauss point evaluations together to obtain the final term for each element of the stiffness matrix.

For two-point evaluation, there will be two terms added together to obtain each element of the stiffness matrix. We proceed to evaluate the stiffness matrix term by term.

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

We will evaluate the stiffness matrix term by term as follows:

$$[k_{11}] = \sum_{i=1}^{2} w_i (2s_i - 1)^2 = [2(-0.57735) - 1]^2 + [2(0.57735) - 1]^2$$

$$= 4.6667$$

$$[k_{12}] = \sum_{i=1}^{2} w_i (2s_i - 1)(2s_i + 1)$$

$$= [2(-0.57735) - 1][2(-0.57735) + 1]^2 + [2(0.57735) - 1][2(0.57735) + 1]^2$$

$$= 0.6667$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

We will evaluate the stiffness matrix term by term as follows:

$$\left[k_{13}\right] = \sum_{i=1}^{2} w_i \left(-4s_i\right)\left(2s_i - 1\right) = \left[-4\left(-0.57735\right)\right]\left[2\left(-0.57735\right) - 1\right] + \left[-4\left(0.57735\right)\right]\left[2\left(0.57735\right) - 1\right]$$

$$= -5.3333$$

$$\left[k_{22}\right] = \sum_{i=1}^{2} w_i \left(2s_i + 1\right)^2 = \left[2\left(-0.57735\right) + 1\right]^2 + \left[2\left(0.57735\right) + 1\right]^2$$

$$= 4.6667$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

We will evaluate the stiffness matrix term by term as follows:

$$\left[k_{23}\right] = \sum_{i=1}^{2} w_i \left(-4s_i\right)\left(2s_i + 1\right) = \left[-4\left(-0.57735\right)\right]\left[2\left(-0.57735\right) + 1\right] + \left[-4\left(0.57735\right)\right]\left[2\left(0.57735\right) + 1\right]$$

$$= -5.3333$$

$$\left[k_{33}\right] = \sum_{i=1}^{2} w_i \left(16s_i\right)^2 = 16\left(-0.57735\right)^2 + 16\left(0.57735\right)^2$$

$$= 10.6667$$

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

By symmetry, the $[k_{21}]$ equals the $[k_{21}]$, etc. Therefore, from the evaluations of the terms, the final stiffness matrix is

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4.67 & 0.667 & -5.33 \\ 0.667 & 4.67 & -5.33 \\ -5.33 & -5.33 & 10.67 \end{bmatrix}$$

The results obtained from Gaussian quadrature are identical to those obtained analytically by direct explicit integration of each term in the stiffness matrix.

Isoparametric Elements

Higher-Order Shape Functions – Linear Strain Bar

To further illustrate elements with improved physical behavior, we start with the Q6 element, and then to further illustrate the concept of higher-order elements, we will consider the quadratic (Q8 and Q9) elements and cubic (Q12) element shape functions.

We then compare results for a cantilever beam model meshed with the numerous element types described in this and previous chapters, such as the CST, Q4, Q6, Q8, and Q9 elements.

Isoparametric Elements

Higher-Order Shape Functions – Bilinear Quadratic (Q6)

An improved element to remove the shear locking inherent in the Q4 element is to add two internal degrees of freedom per displacement function (g_1 –g_4) to the Q4 element displacement functions.

This element is then called a Q6 element.

$$u(s,t) = \boxed{\sum_{i=1}^{4} N_i u_i} + g_1\left(1-s^2\right) + g_2\left(1-t^2\right)$$

$$v(s,t) = \boxed{\sum_{i=1}^{4} N_i v_i} + g_3\left(1-s^2\right) + g_4\left(1-t^2\right)$$

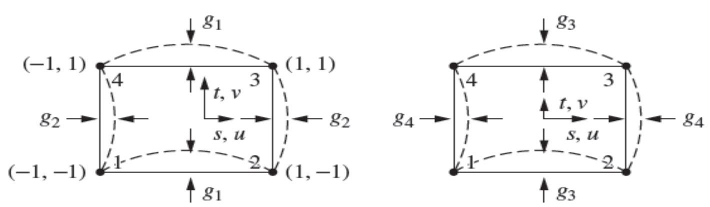
These are the shape functions derived for the isoparametric Q4 element

Isoparametric Elements

Higher-Order Shape Functions – Bilinear Quadratic (Q6)

The displacement field is enhanced by modes that describe the state of constant curvature (also called bubble modes) that are represented by g_1 through g_4.

These corrections allow the elements to curve between the nodes and can then model bending with either *s* or *t axis* as the neutral axis.

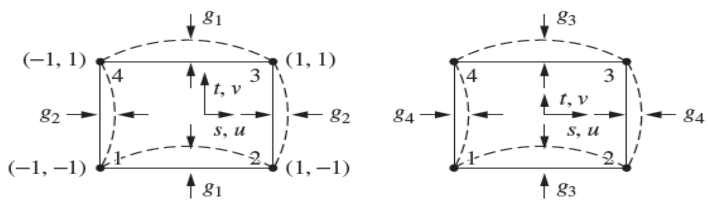


Isoparametric Elements

Higher-Order Shape Functions – Bilinear Quadratic (Q6)

The magnitude of these modes is determined by minimizing the internal strain energy in the element.

The additional degrees of freedom are condensed out before the element stiffness matrix is developed.

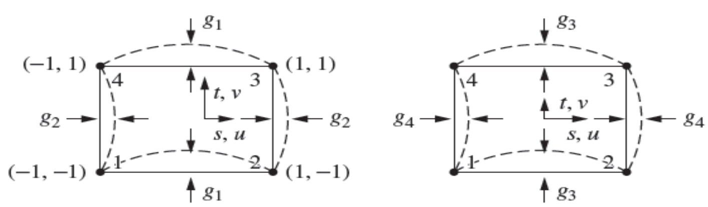


Isoparametric Elements

Higher-Order Shape Functions – Bilinear Quadratic (Q6)

Hence, only the degrees of freedom associated with the four corner nodes appear.

The element can model pure bending exactly if it is a rectangular shape.

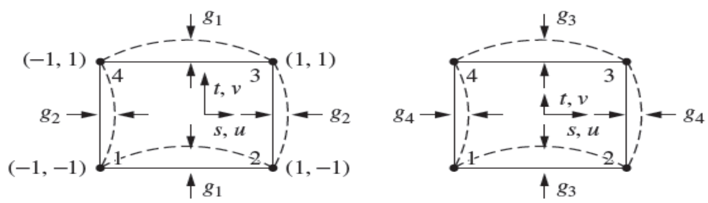


Isoparametric Elements

Higher-Order Shape Functions – Bilinear Quadratic (Q6)

Because the g_1 –g_4 degrees of freedom are internal and not nodal degrees of freedom, they are not connected to other elements.

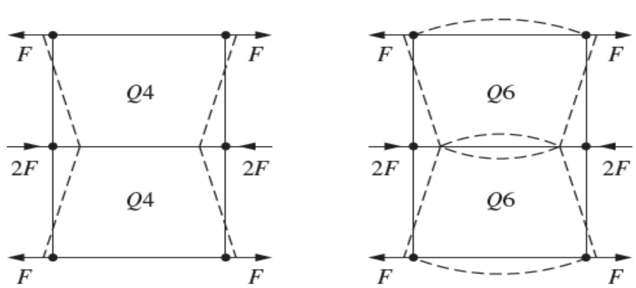
There is a possibility that the edges of two adjacent elements may have different curvatures and thus the displacement field along this common edge may be incompatible.



Isoparametric Elements

Higher-Order Shape Functions – Bilinear Quadratic (Q6)

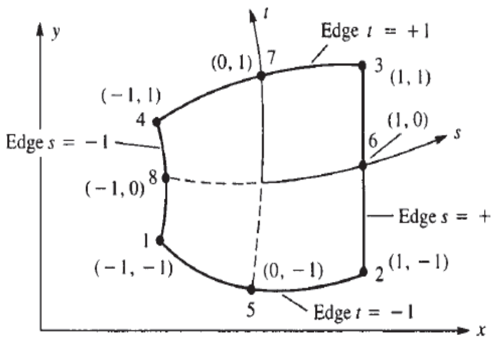
This incompatibility will occur under certain loading conditions, such as shown:



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

A quadratic isoparametric element with four corner nodes and four additional mid-side nodes. This eight-noded element is often called a Q8 element.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

The shape functions of the quadratic element are based on the incomplete cubic polynomial such that coordinates *x* and *y* are:

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$

$$y = a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} st^2$$

These functions have been chosen so that the number of generalized degrees of freedom (2 per node times 8 nodes equals 16) are identical to the total number of *a's.*

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

The shape functions of the quadratic element are based on the incomplete cubic polynomial such that coordinates *x* and *y* are:

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$

$$y = a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} st^2$$

The literature also refers to this eight-noded element as a "serendipity" element as it is based on an incomplete cubic, but it yields good results in such cases as beam bending.

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

The shape functions of the quadratic element are based on the incomplete cubic polynomial such that coordinates *x* and *y* are:

$$x = a_1 + a_2 s + a_3 t + a_4 st + a_5 s^2 + a_6 t^2 + a_7 s^2 t + a_8 st^2$$

$$y = a_9 + a_{10} s + a_{11} t + a_{12} st + a_{13} s^2 + a_{14} t^2 + a_{15} s^2 t + a_{16} st^2$$

We are also reminded that because we are considering an isoparametric formulation, displacements *u* and *v* are of identical form as *x* and *y*, respectively.

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

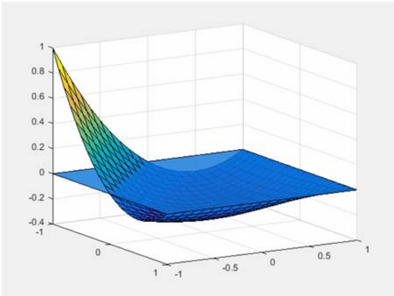
To describe the shape functions, two forms are required: one for corner nodes and one for mid-side nodes. The four corner nodes are:

$$N_1 = \frac{1}{4}(1-s)(1-t)(-s-t-1)$$

$$N_2 = \frac{1}{4}(1+s)(1-t)(s-t-1)$$

$$N_3 = \frac{1}{4}(1+s)(1+t)(s+t-1)$$

$$N_4 = \frac{1}{4}(1-s)(1+t)(-s+t-1)$$



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

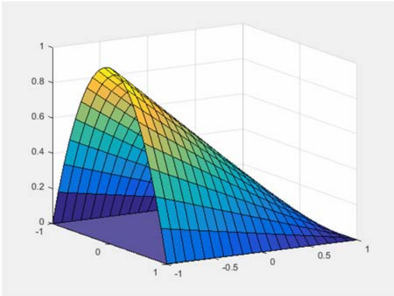
The four mid-side nodes are:

$$N_5 = \frac{1}{2}(1+s)(1-t)(1-s)$$

$$N_6 = \frac{1}{2}(1+s)(1+t)(1-t)$$

$$N_7 = \frac{1}{2}(1+s)(1+t)(1-s)$$

$$N_8 = \frac{1}{2}(1-s)(1+t)(1-t)$$



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

The displacement functions are given by:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & N_6 & 0 & N_7 & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}$$

$$\varepsilon_x = \frac{du}{dx} = [D'][N]\{d\} \qquad [B] = [D'][N]$$

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

Recall the [*D*'] operator is:

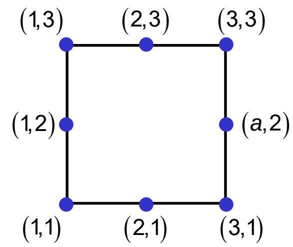
$$[D'] = \frac{1}{|[J]|} \begin{Bmatrix} \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} & 0 \\ 0 & \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} \\ \frac{\partial x}{\partial s}\frac{\partial(\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial(\ )}{\partial s} & \frac{\partial y}{\partial t}\frac{\partial(\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial(\ )}{\partial t} \end{Bmatrix}$$

$$\varepsilon_x = \frac{du}{dx} = [D'][N]\{d\} \qquad [B] = [D'][N]$$

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

Let's compute the determinant |[J]| for a global element with the following coordinates:



$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ a\ \ 2\ \ 1]$$

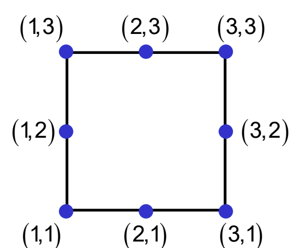
$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2]$$

The |[J]| is: $\left\|[J]\right\| = \frac{1}{2}\left(a - at^2 + 3t^2 - 1\right)$

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

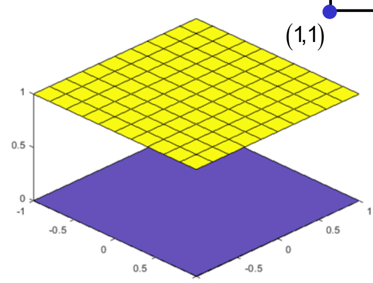
Let's compute the determinant |[J]| for a element with $(x_6, y_6) = (3, 2)$:



$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 1]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2]$$

The |[J]| is:



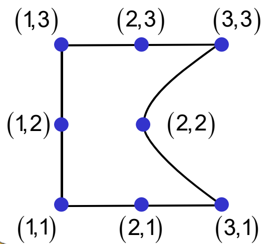
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

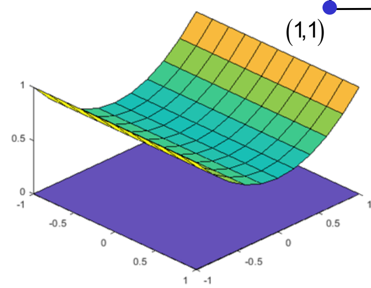
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (2, 2):

$$\{X_c\}^T = [1 \ \ 3 \ \ 3 \ \ 1 \ \ 2 \ \ 2 \ \ 2 \ \ 1]$$

$$\{Y_c\}^T = [1 \ \ 1 \ \ 3 \ \ 3 \ \ 1 \ \ 2 \ \ 3 \ \ 2]$$



The |[J]| is:



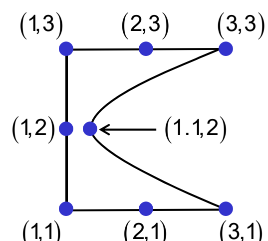
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

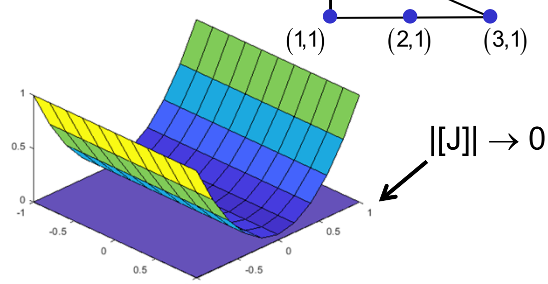
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (1.1, 2):

$$\{X_c\}^T = [1 \ \ 3 \ \ 3 \ \ 1 \ \ 2 \ \ 1.1 \ \ 2 \ \ 1]$$

$$\{Y_c\}^T = [1 \ \ 1 \ \ 3 \ \ 3 \ \ 1 \ \ 2 \ \ 3 \ \ 2]$$



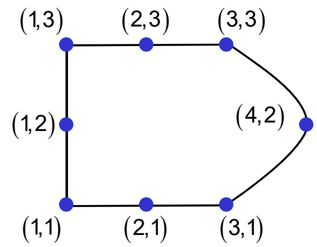
The |[J]| is:



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

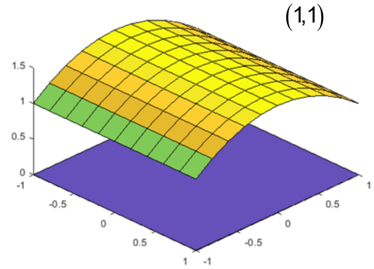
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (4, 2):



$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 4\ \ 2\ \ 1]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2]$$

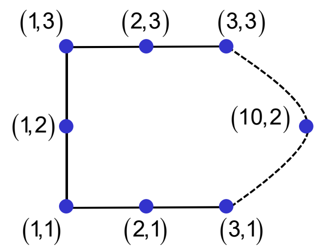
The |[J]| is:



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

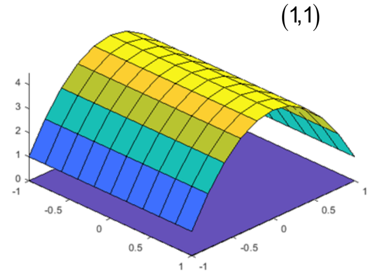
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (10, 2):



$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 10\ \ 2\ \ 1]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2]$$

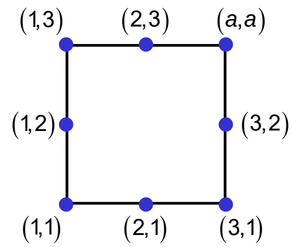
The |[J]| is:



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

Let's compute the determinant |[J]| for a element with the following coordinates:



$$\{X_c\}^T = [1 \ \ 3 \ \ a \ \ 1 \ \ 2 \ \ 3 \ \ 2 \ \ 1]$$

$$\{Y_c\}^T = [\ 1 \ \ 1 \ \ a \ \ 3 \ \ 1 \ \ 2 \ \ 3 \ \ 2]$$

Let's evaluate the |[J]| at (s, t) = (1, 1) :

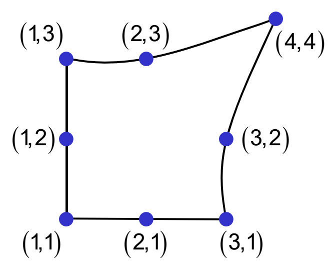
$$\left|[J]\right| = 3a - 8$$

For the |[J]| to be positive, a > 8/3. If a < 8/3, then the |[J]| is negative.

Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

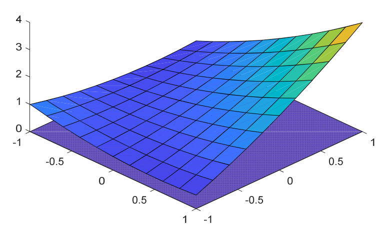
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (4, 4):



$$\{X_c\}^T = [1 \ \ 3 \ \ 4 \ \ 1 \ \ 2 \ \ 3 \ \ 2 \ \ 1]$$

$$\{Y_c\}^T = [\ 1 \ \ 1 \ \ 4 \ \ 3 \ \ 1 \ \ 2 \ \ 3 \ \ 2]$$

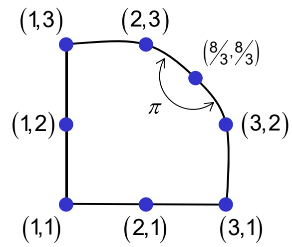
At (s, t) = (1, 1) |[J]| > 0.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

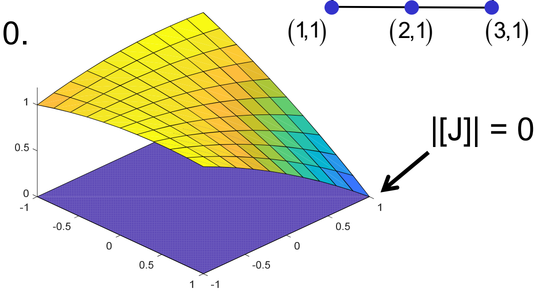
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (8/3, 8/3):



$$\{X_c\}^T = \begin{bmatrix} 1 & 3 & \tfrac{8}{3} & 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

$$\{Y_c\}^T = \begin{bmatrix} 1 & 1 & \tfrac{8}{3} & 3 & 1 & 2 & 3 & 2 \end{bmatrix}$$

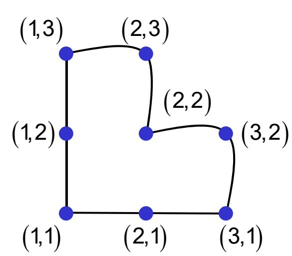
At (s, t) = (1, 1) |[J]| = 0.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

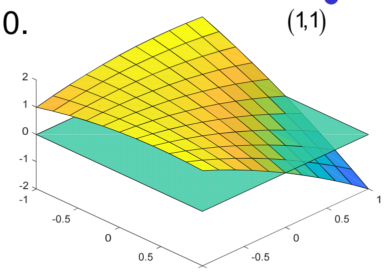
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (2, 2):



$$\{X_c\}^T = \begin{bmatrix} 1 & 3 & 2 & 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$

$$\{Y_c\}^T = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 2 & 3 & 2 \end{bmatrix}$$

At (s, t) = (1, 1) |[J]| < 0.



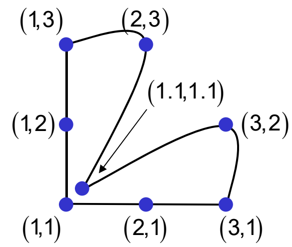
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

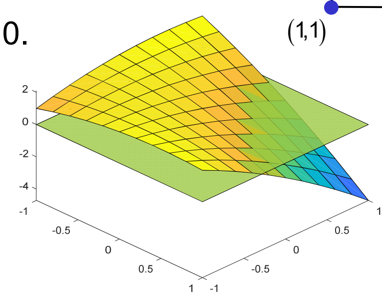
Let's compute the determinant |[J]| for a element with (x_3, y_3) = (1.1, 1.1):

$$\{X_c\}^T = [1\ \ 3\ \ 1.1\ \ 1\ \ 2\ \ 3\ \ 2\ \ 1]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 1.1\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2]$$



At (s, t) = (1, 1) |[J]| < 0.

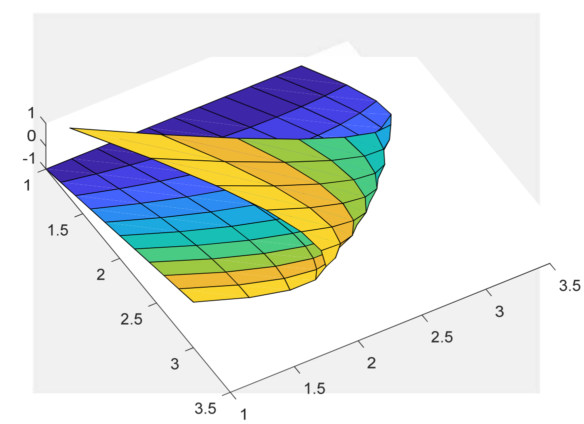


Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

When the |[J]| is negative the mapping between local coordinates to global coordinates is not 1-to-1.

Here is a plot of the mapping as the vales of *a* range from 3 to 1.1:

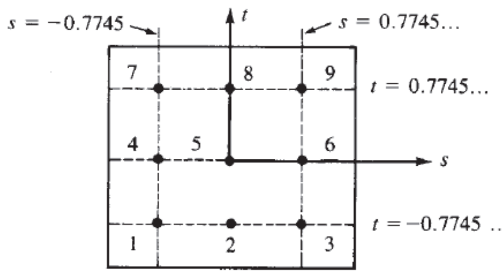


Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q8)

To evaluate the matrix [*B*] and the matrix [*k*] for the eight-noded quadratic isoparametric element, we now use the nine-point Gauss rule (often described as a 3 X 3 rule).

Results using 2 x 2 and 3 x 3 rules have shown significant differences, and the 3 x 3 rule is recommended.

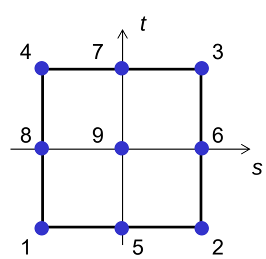


Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

By adding a ninth node at *s* = 0, *t* = 0, we can create an element called a Q9.

This is an internal node that is not connected to any other nodes. We then add the $a_{17}s^2t^2$ and $a_{18}s^2t^2$ terms to *x* and *y* equations, respectively, and to *u* and *v.*

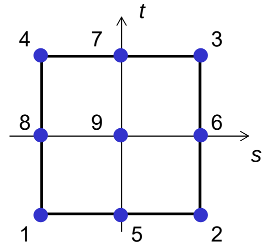


Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

The element is then called a Lagrange element as the shape functions can be derived using Lagrange interpolation formulas.

The shape function for the Q9 element have three general forms: a set for the corner nodes, a set for the nodes along the edges, and the center node.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

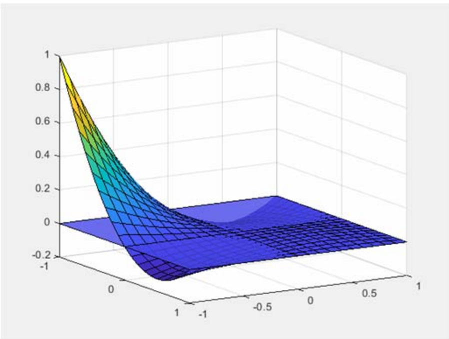
The shape functions for the four corner nodes are:

$$N_1 = \frac{1}{4}(s^2 - s)(t^2 - t)$$

$$N_2 = \frac{1}{4}(s^2 + s)(t^2 - t)$$

$$N_3 = \frac{1}{4}(s^2 + s)(t^2 + t)$$

$$N_4 = \frac{1}{4}(s^2 - s)(t^2 + t)$$



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

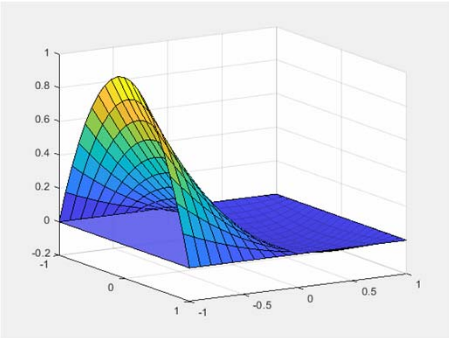
The four mid-side nodes are:

$$N_5 = \frac{1}{2}\left(1-s^2\right)\left(t^2-t\right)$$

$$N_6 = \frac{1}{2}\left(s^2+s\right)\left(1-t^2\right)$$

$$N_7 = \frac{1}{2}\left(1-s^2\right)\left(t^2+t\right)$$

$$N_8 = \frac{1}{2}\left(s^2-s\right)\left(1-t^2\right)$$

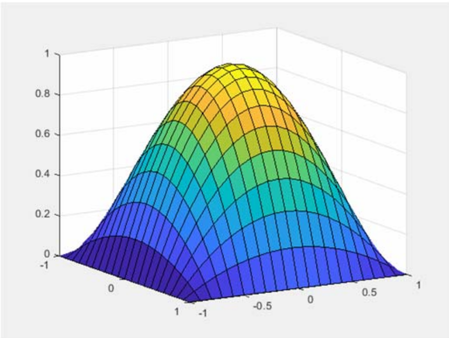


Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

The center node is:

$$N_9 = \left(1-s^2\right)\left(1-t^2\right)$$



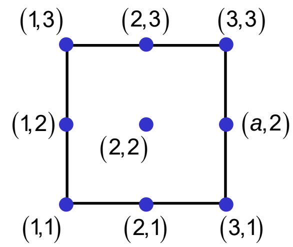
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

Let's compute the determinant |[J]| for a element with (x_6, y_6) = (a, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ a\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is: $\|[J]\| = \frac{1}{2}\left(a - 3s + 2as - at^2 + 6st^2 + 3t^2 - 2ast^2 - 1\right)$

At (s, t) = (1, 0): $\|[J]\| = \frac{1}{2}(3a - 7)$

For the |[J]| to be positive, a > 7/3.

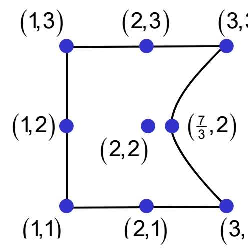
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

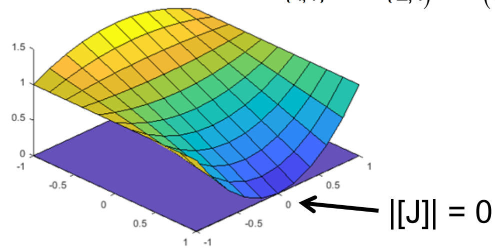
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (7/3, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ \tfrac{7}{3}\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is:



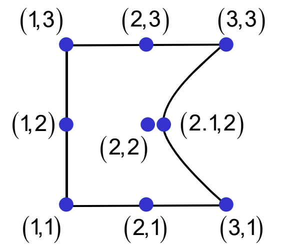
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

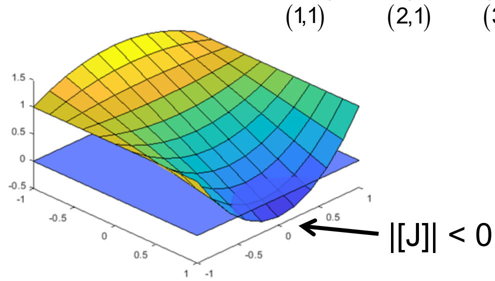
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (2.1, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 2.1\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is:



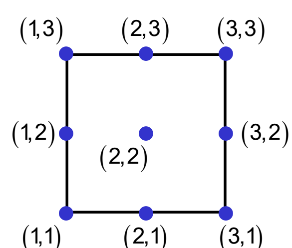
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

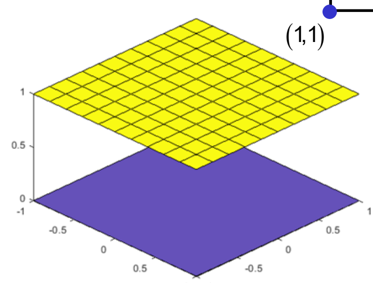
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (3, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is:



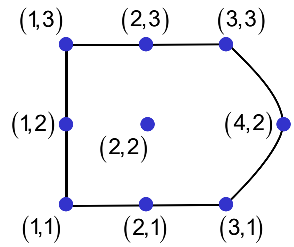
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

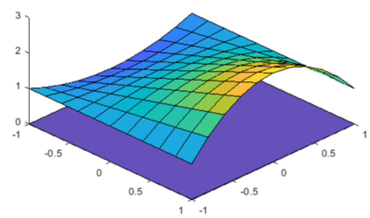
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (4, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 4\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is:



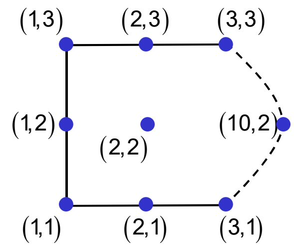
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

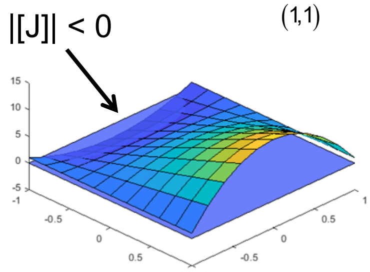
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (10, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 10\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is:



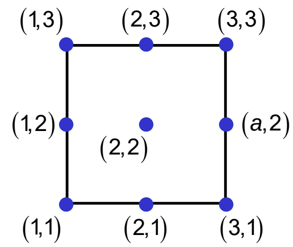
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

Let's compute the determinant |[J]| for a element with (x_6, y_6) = (a, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ a\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



The |[J]| is: $\left\|[J]\right\| = \frac{1}{2}\left(a - 3s + 2as - at^2 + 6st^2 + 3t^2 - 2ast^2 - 1\right)$

At (s, t) = (-1, 0): $\left\|[J]\right\| = \frac{1}{2}(5 - a)$

For the |[J]| to be positive, $a < 5$.

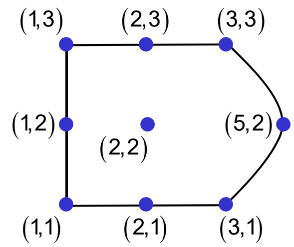
Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

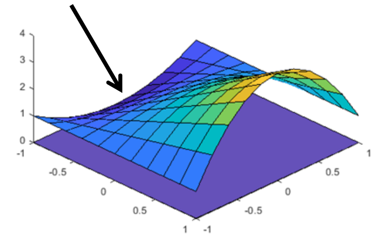
Let's compute the determinant |[J]| for a element with (x_6, y_6) = (5, 2):

$$\{X_c\}^T = [1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 5\ \ 2\ \ 1\ \ 2]$$

$$\{Y_c\}^T = [\ 1\ \ 1\ \ 3\ \ 3\ \ 1\ \ 2\ \ 3\ \ 2\ \ 2]$$



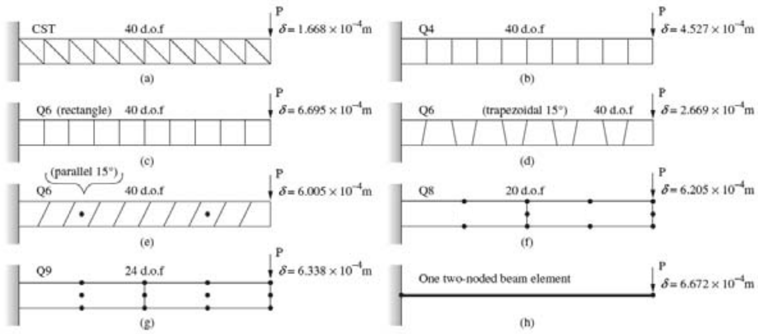
The |[J]| is:



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

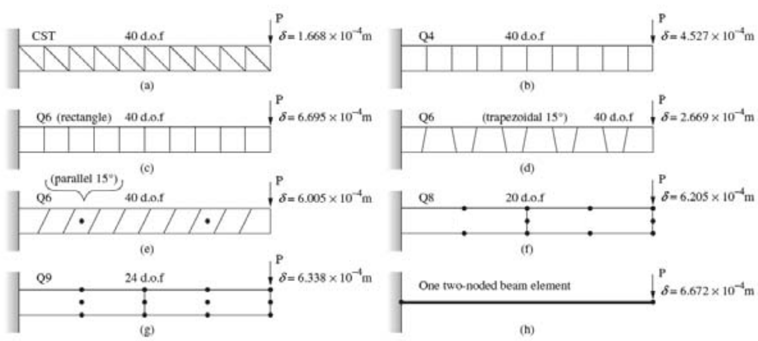
We now present a comparison of results for a cantilever beam meshed with the various plane elements as described in this and previous Chapters 6 and 8.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

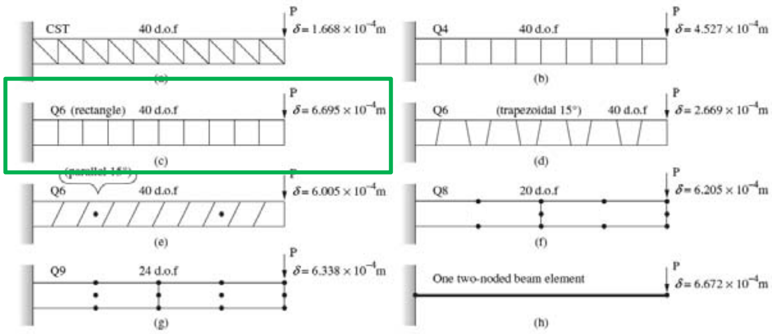
Below, the CST, Q4, Q6, Q8, and Q9 element mesh solutions are compared to the classical beam element.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

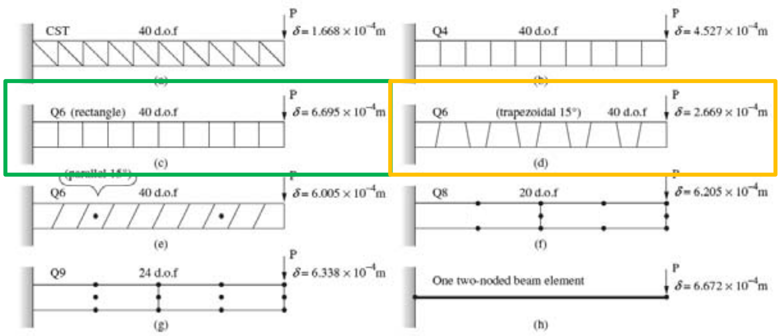
Note that the Q6 element (or Q4 incompatible) removes the shear locking that occurs with the Q4 element and yields excellent results for the displacement even with a single row of rectangular elements.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

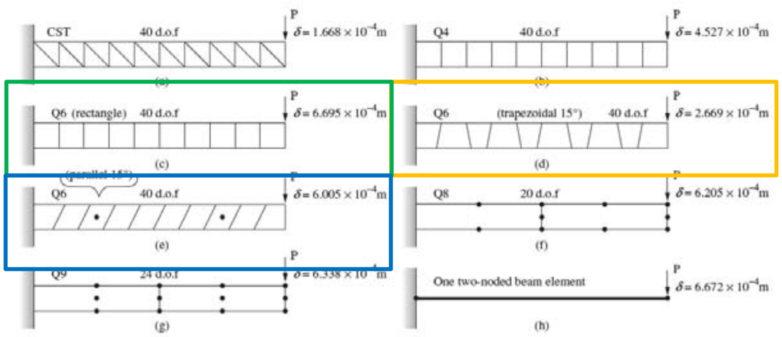
However, small angles of trapezoidal distortion (say 15° from the vertical) make the elements much too stiff.



Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

Also parallel distortion reduces accuracy of the elements but to a smaller amount than the trapezoidal distortion.

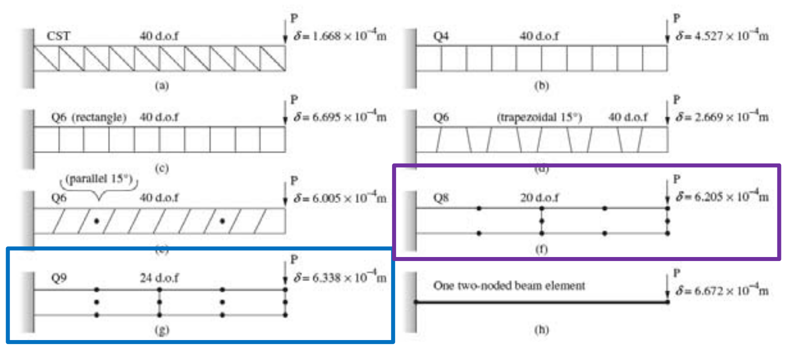


Isoparametric Elements

Higher-Order Shape Functions – Quadratic Rectangle (Q9)

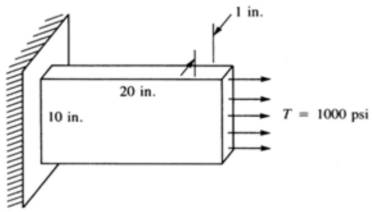
The Q8 and Q9 elements perform very well considering only one row and two elements or fewer total degrees of freedom (d.o.f) are used compared to the Q6 mesh.

The Q9 element with the additional internal node yields slightly better single row results than the Q8

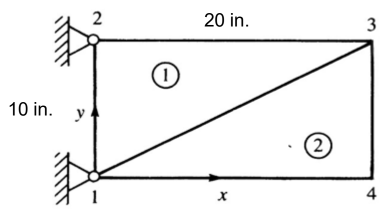


Plane Stress and Plane Strain Equations

Q8 Element Model

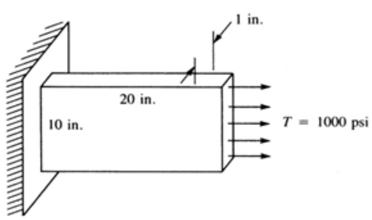


Rework this CST problem with rectangular Q8 elements.

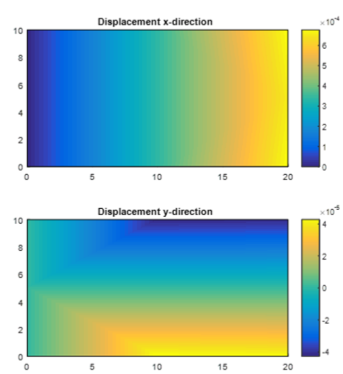


Plane Stress and Plane Strain Equations

Q8 Element Model

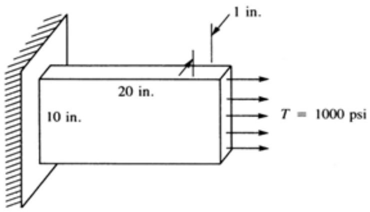


One Q8 element.

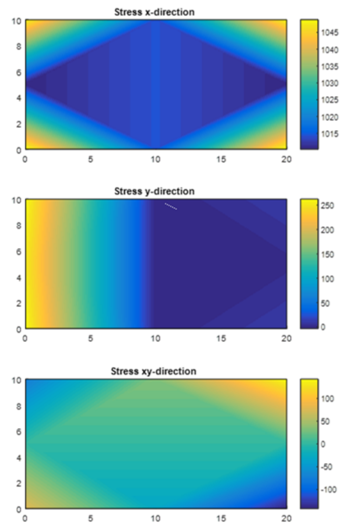


Plane Stress and Plane Strain Equations

Q8 Element Model

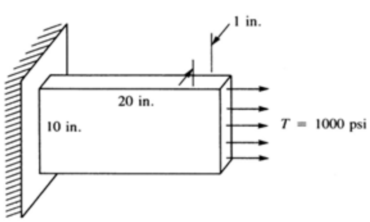


One Q8 element.

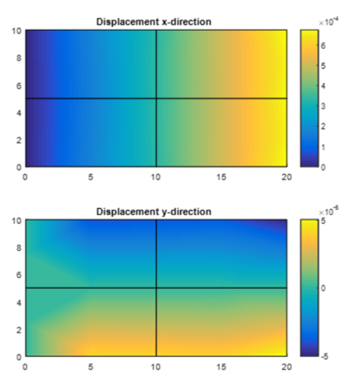


Plane Stress and Plane Strain Equations

Q8 Element Model

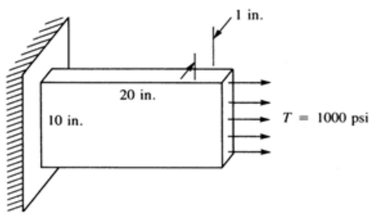


4 Q8 elements.

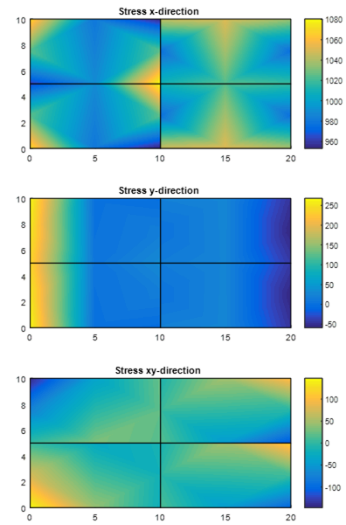


Plane Stress and Plane Strain Equations

Q8 Element Model

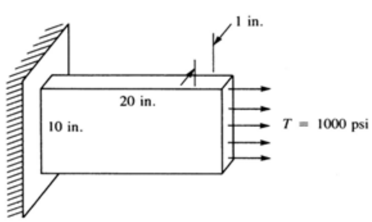


4 Q8 elements.

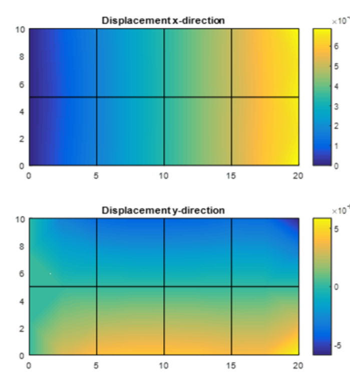


Plane Stress and Plane Strain Equations

Q8 Element Model

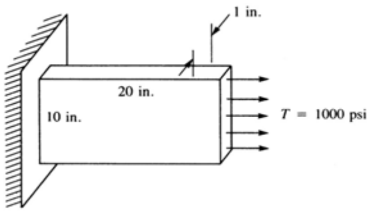


8 Q8 elements.

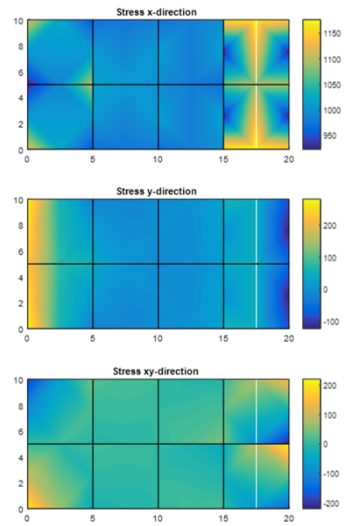


Plane Stress and Plane Strain Equations

Q8 Element Model



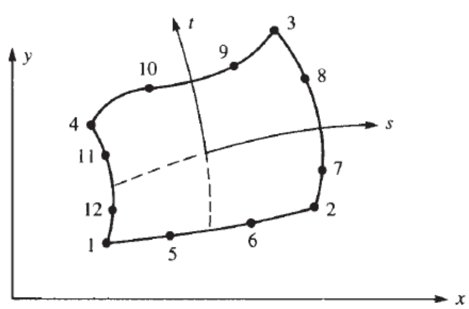
8 Q8 elements.



Isoparametric Elements

Higher-Order Shape Functions – Cubic Rectangle (Q12)

The cubic (Q12) element has four corner nodes and additional nodes taken to be at one-third and two-thirds of the length along each side.

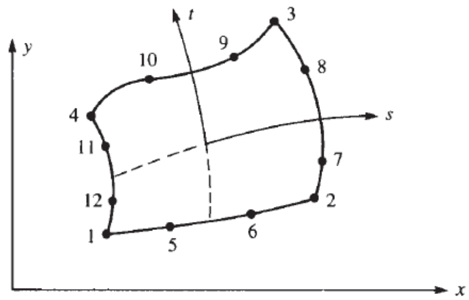


Isoparametric Elements

Higher-Order Shape Functions – Cubic Rectangle (Q12)

The shape functions of the cubic element are based on the incomplete quartic polynomial:

$$x = a_1 + a_2 s + a_3 t + a_4 s^2 + a_5 st + a_6 t^2 + a_7 s^2 t + a_8 st^2$$
$$+ a_9 s^3 + a_{10} t^3 + a_{11} s^3 t + a_{12} st^3$$

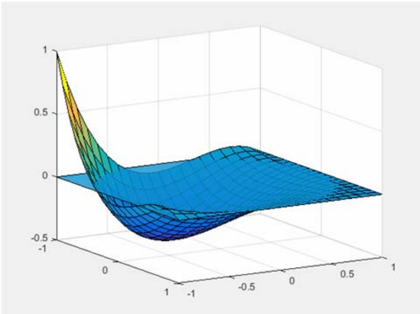


Isoparametric Elements

Higher-Order Shape Functions – Cubic Rectangle (Q12)

For the corner nodes (*i* = 1, 2, 3, 4),

$$N_i = \frac{1}{32}(1 + ss_i)(1 + tt_i)\left[9\left(s^2 + t^2\right) - 10\right] \qquad s_i = [-1, 1, 1, -1] \quad t_i = [-1, -1, 1, 1]$$

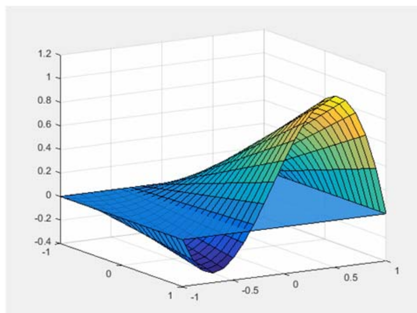


Isoparametric Elements

Higher-Order Shape Functions – Cubic Rectangle (Q12)

For nodes on sides *s* = ± 1 (*i* = 7, 8, 11, 12),

$$N_i = \frac{9}{32}(1+ss_i)(1+9tt_i)\left(1-t^2\right) \qquad s_i = \pm 1 \qquad t_i = \pm\tfrac{1}{3}$$

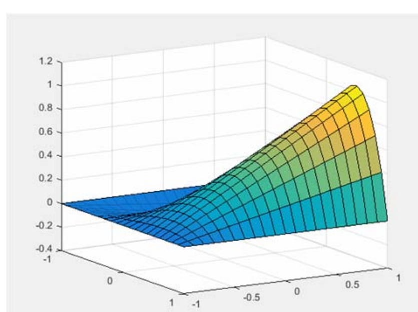


Isoparametric Elements

Higher-Order Shape Functions – Cubic Rectangle (Q12)

For nodes on sides *t* = ± 1 (*i* = 5, 6, 9, 10),

$$N_i = \frac{9}{32}(1+9ss_i)(1+tt_i)\left(1-s^2\right) \qquad s_i = \pm\tfrac{1}{3} \qquad t_i = \pm 1$$



Isoparametric Elements

Higher-Order Shape Functions – Cubic Rectangle (Q12)

Having the shape functions for the Q9 quadratic element or for the Q12 cubic element, we can obtain [*B*] and then set up [*k*] for numerical integration for plane element.

The cubic element requires a 3 X 3 rule (nine points) to evaluate the matrix exactly.

We then conclude that what is really desired is a library of shape functions that can be used in the general equations developed for stiffness matrices, distributed load, and body and can be applied not only to stress analysis but to nonstructural problems as well.

Isoparametric Elements

Problems

20. Work problems ***10.1***, ***10.6a***, ***10.8***, ***10.15dg***, and ***10.17b*** in your textbook.

21. Write a computer program to evaluation of the [*k*] stiffness matrix for the Q4 element by Gaussian quadrature. Check your stiffness matrix values with the Example 10.4 in the textbook. In addition, develop your code in such a way that it could be easily extended to the Q8, Q9, and Q12 elements.

End of Chapter 10