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AP Calc BC

Mr. Mellina/Ms. Lombardi

Chapter 10 – Parametric & Polar Equations

Topics:

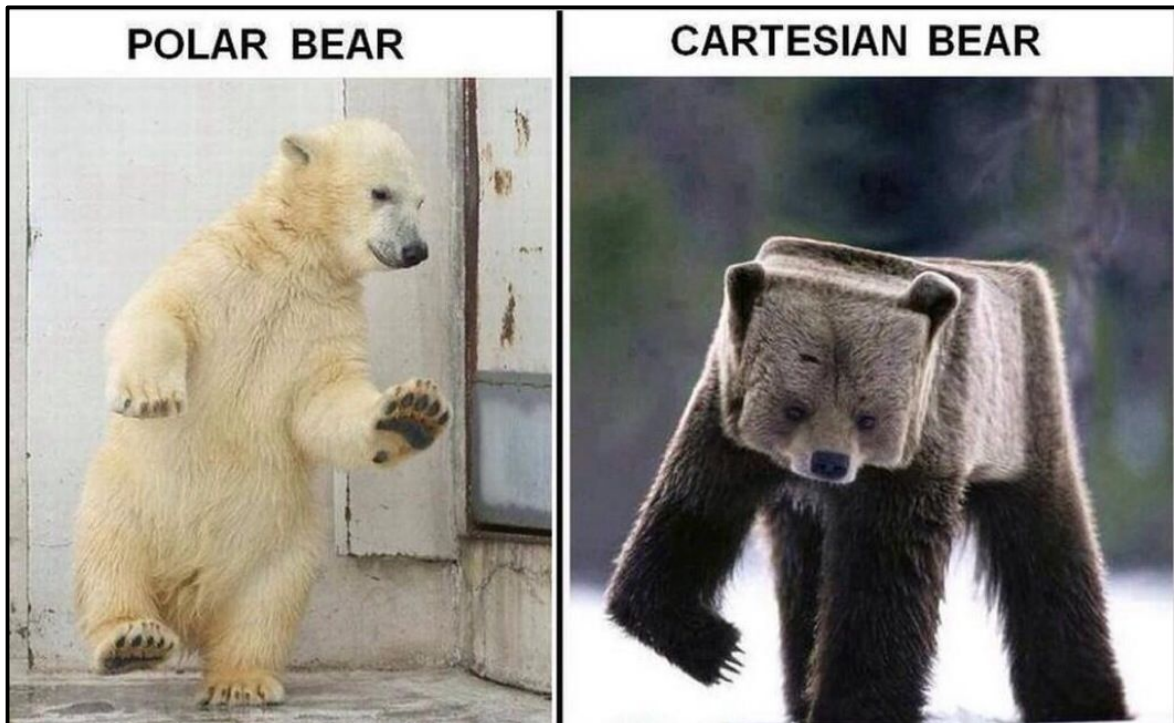
10.2 – Plane Curves and Parametric Equations

10.3 – Parametric Equations and Calculus

12.2/12.3 – Differentiation & Integration/PVA of Vector Valued Functions

10.4 – Polar Coordinates and Polar Graphs

10.5 – Area and Arc Length in Polar Coordinates



10.2 – Plane Curves and Parametric Equations

Topics

- *Sketch the graph of a curve given by a set of parametric equations.*
- *Eliminate the parameter in a set of parametric equations.*
- *Find a set of parametric equations to represent a curve.*

Warm Up!

An object moves along a line in such a way that its x - and y -coordinates at time t are $x = 1 - t$ and $y = 1 + 2t$. When and where does the object cross the circle $x^2 + (y - 1)^2 = 25$?

Definition of a Plane Curve

If f and g are continuous functions of t on an interval I , then the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

are **parametric equations** and t is the **parameter**. The set of points (x, y) obtained as t varies over the interval I is the **graph** of the parametric equations. Taken together, the parametric equations and the graph are a **plane curve**, denoted by C .

When sketching a curve represented by a set of parametric equations, you can plot points in the xy -plane. Each set of coordinates (x, y) is determined from a value chosen for the parameter t . By plotting the resulting points in order of increasing values of t , the curve is traced out in a specific direction. This is called the **orientation** of the curve.

Example 1: Sketching a Curve

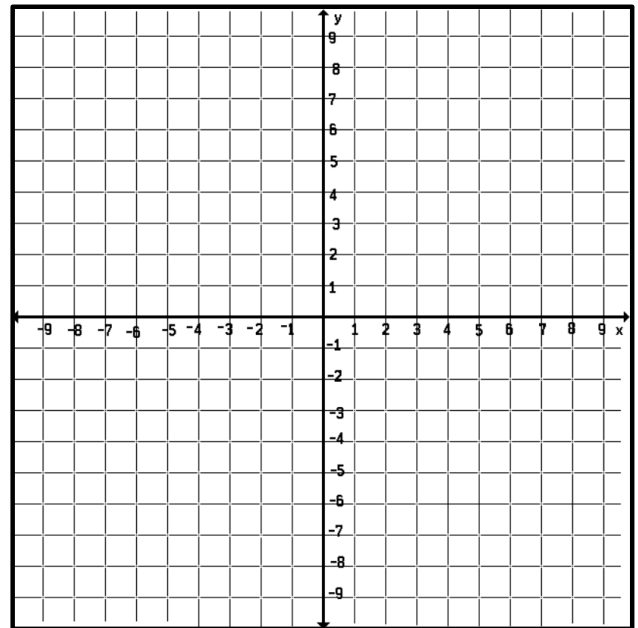
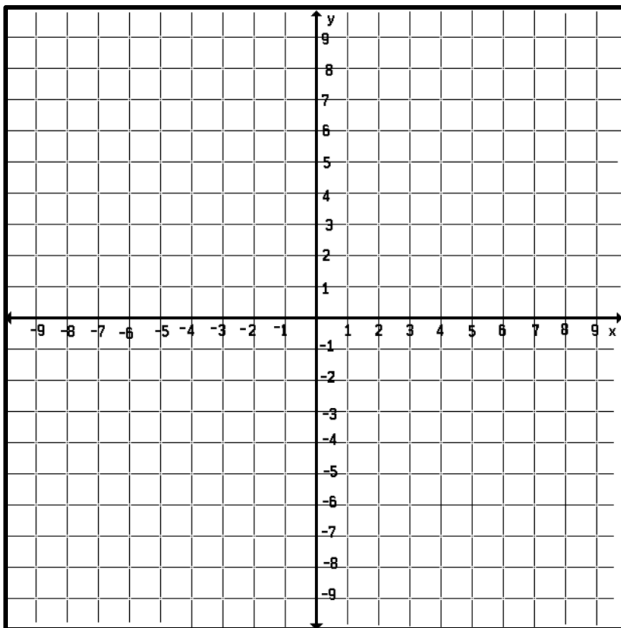
Sketch the curve described by the parametric equations

a. $x = f(t) = t^2 - 4$
 $y = g(t) = \frac{t}{2}$

b. $x = f(t) = 4t^2 - 4$
 $y = g(t) = t$

t	-2	-1	0	1	2	3
x						
y						

t	-1	-0.5	0	0.5	1	1.5
x						
y						

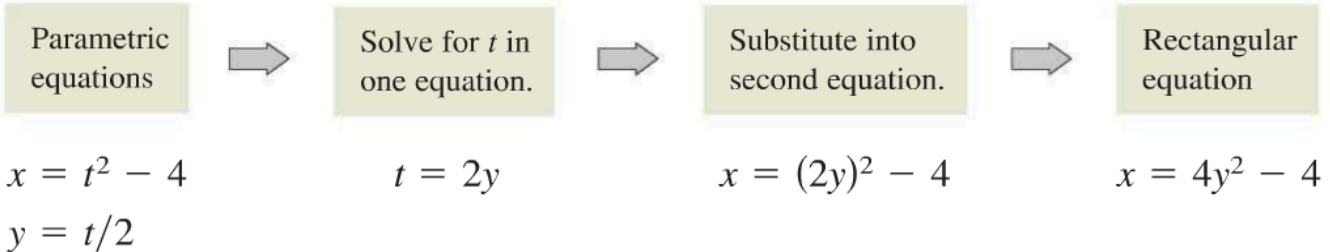


According to the Vertical Line Test, both graphs above do not define y as a function of x . This points out one benefit of parametric equations – they can be used to represent graphs that are more general than graphs of functions.

*It often happens that two different sets of parametric equations have the same graph (as seen above). However, comparing the values of t in both graphs above, you can see the second graph is traced out more **rapidly** (considering t as time) than the first graph. So, in applications, different parametric representations can be used to represent various **speeds** at which objects travel along a given path.*

Eliminating the Parameter

Finding a rectangular equation that represents the graph of a set of parametric equations is called **eliminating the parameter**. For instance, you can eliminate the parameter from the set of parametric equations in Example 1 as follows.



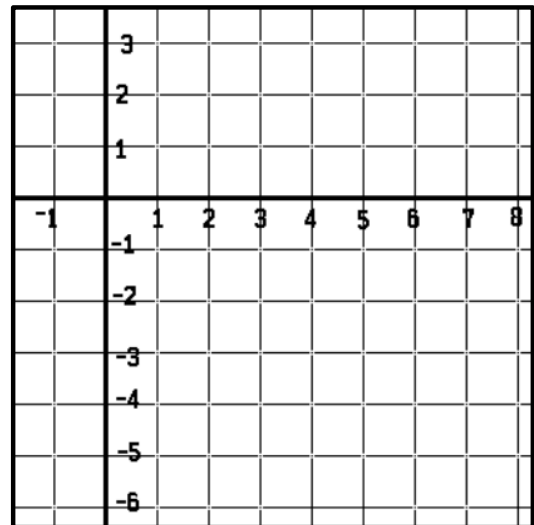
Once you have eliminated the parameter, you can recognize that the equation $x = 4y^2 - 4$ represents a parabola with a horizontal axis and vertex at $(-4, 0)$, as shown in Figure 10.20.

The range of x and y implied by the parametric equations may be altered by the change to rectangular form. In such instances, the domain of the rectangular equation must be adjusted so that its graph matches the graph of the parametric equations. Such a situation is demonstrated in the next example.

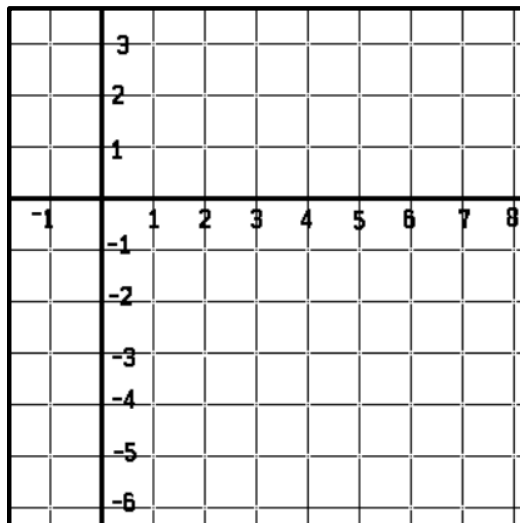
Example 2: Using Parametric Equations

Sketch the curve represented by the parametric equations (indicate the orientation of the curve), and write the corresponding rectangular equation by eliminating the parameter.

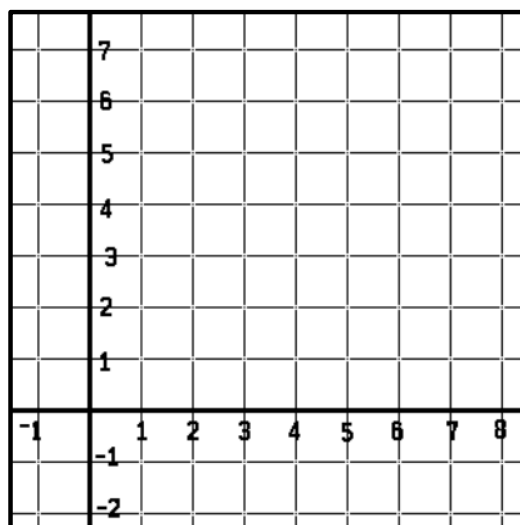
a. $x = \frac{1}{\sqrt{t+1}}$ and $y = \frac{t}{t+1}$, $t > -1$



b. $x = \sqrt{t}$ and $y = t - 5$



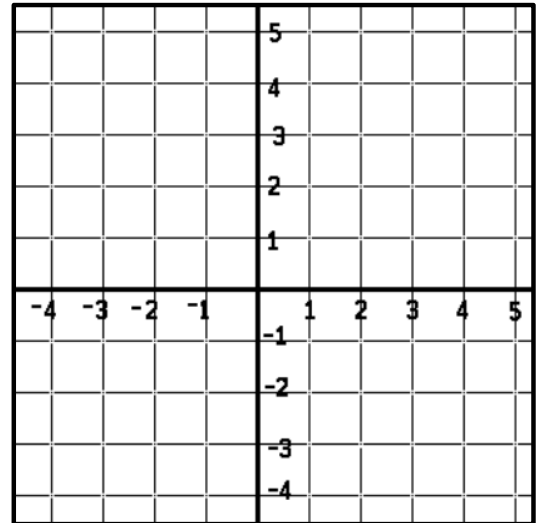
c. $x = |t - 1|$ and $y = t + 2$



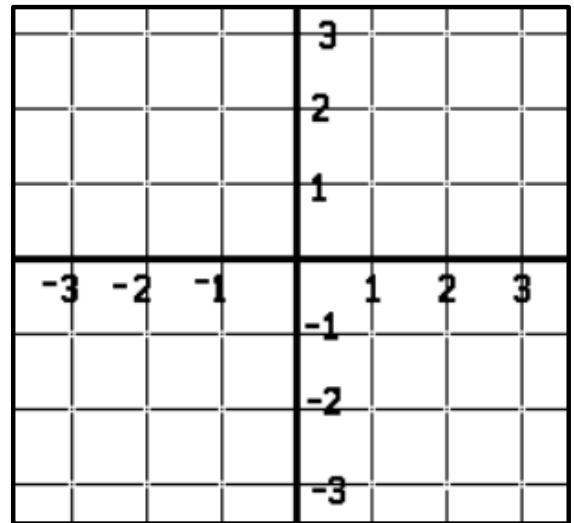
Example 3: Using Trigonometry to Eliminate a Parameter

It is not necessary for the parameter in a set of parametric equations to represent time. This example uses an *angle* as the parameter. Sketch the curve represented by the following parametric equations by eliminating the parameter and finding the corresponding rectangular equation.

a. $x = 3 \cos \theta$ and $y = 4 \sin \theta$, $0 \leq \theta < 2\pi$



b. $x = \cos \theta$ and $y = 2 \sin 2\theta$



10.3 – Parametric Equations and Calculus

Topics

- Find the slope of a tangent line to a curve given by a set of parametric equations.
- Find the arc length of a curve given by a set of parametric equations.
- Find the area of a surface of revolution (parametric form).

Warm Up!

True or False?

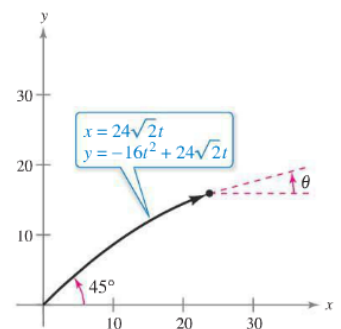
- The graph of the parametric equations $x = t^2$ and $y = t^2$ is the line $y = x$.
- If y is a function of t and x is a function of t , then y is a function of x .
- The curve represented by the parametric equations $x = t$ and $y = \cos t$ can be written as an equation of the form $y = f(x)$.

Slope and Tangent Lines

Now that you can represent a graph in the plane by a set of parametric equations, it is natural to ask how to use calculus to study plane curves. Consider the projectile represented by the parametric equations

$$x = 24\sqrt{2}t \quad \text{and} \quad y = -16t^2 + 24\sqrt{2}t$$

as shown in Figure 10.29. From the discussion at the beginning of Section 10.2, you know that these equations enable you to locate the position of the projectile at a given time. You also know that the object is initially projected at an angle of 45° , or a slope of $m = \tan 45^\circ = 1$. But how can you find the slope at some other time t ? The next theorem answers this question by giving a formula for the slope of the tangent line as a function of t .



At time t , the angle of elevation of the projectile is θ .

Figure 10.29

THEOREM 10.7 Parametric Form of the Derivative

If a smooth curve C is given by the equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

then the slope of C at (x, y) is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right] = \frac{\frac{d}{dt} \left[\frac{dy}{dx} \right]}{dx/dt}$$

Second derivative

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \left[\frac{d^2y}{dx^2} \right] = \frac{\frac{d}{dt} \left[\frac{d^2y}{dx^2} \right]}{dx/dt}.$$

Third derivative

Example 1: Finding a Derivative

Find $\frac{dy}{dx}$.

a. $x = \sin t, y = \cos t$

b. $x = 2e^\theta, y = e^{-\theta/2}$

Example 2: Finding Slope and Concavity

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, and find the slope and concavity (if possible) at the given value of the parameter.

a. $x = \sqrt{t}, y = \frac{1}{4}(t^2 - 4)$ at the point $(2,3)$

b. $x = t^2 + 5t + 4, y = 4t, \text{ at } t = 0$

c. $x = 4 \cos \theta, y = 4 \sin \theta, \text{ at } \theta = \frac{\pi}{4}$

Example 3: Finding Equations of Tangent Lines

Find the equation of the tangent lines at the point to the curves at each given point.

a. $x = t^2 - 4, y = t^2 - 2t, (0,0), (-3, -1), (-3,3)$

b. $x = 2 \cot \theta$, $y = 2 \sin^2 \theta$, $\left(-\frac{2}{\sqrt{3}}, \frac{3}{2}\right)$, $(0, 2)$, $\left(2\sqrt{3}, \frac{1}{2}\right)$

THEOREM 10.8 Arc Length in Parametric Form

If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the arc length of C over the interval is given by

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt.$$

Example 5: Arc Length

Find the arc length of the curve on the given interval. Check answer in the Calculator.

a. $x = 3t + 5$, $y = 7 - 2t$, on $-1 \leq t \leq 3$

b. $x = 6t^2$, $y = 2t^3$, on $1 \leq t \leq 4$

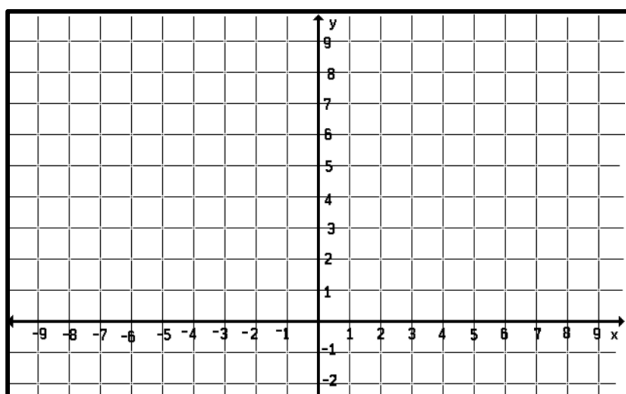
c. $x = \arcsin t$, $y = \ln \sqrt{1 - t^2}$, on $0 \leq t \leq \frac{1}{2}$

Extra Practice (Calculator Active)

Consider the parametric function below.

$$\begin{cases} x = t^2 - 4 \\ y = 3 \sin t \end{cases} \text{ on } 0 \leq t \leq \pi$$

a. Sketch the curve



b. Find the highest point on the curve.

c. Find the length of the curve on the interval.

12.2/12.3 – Differentiation and Integration of Vector Valued Functions/Velocity & Acceleration

Topics

- *Differentiate a Vector-Valued Function*
- *Integrate a Vector-Valued Function*
- *Describe the velocity and acceleration associated with a a vector-valued function.*
 - *Use a vector-valued function to analyze projective motion.*

Warm Up!

Calculator Active

A particle moves along the x -axis so that its velocity at time t is given by

$$v(t) = -(t + 1) \sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

- (a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?
- (b) Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.
- (c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.
- (d) During the time interval $0 \leq t \leq 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

Vector Functions

Vectors are quantities that have both **magnitude (size)** and **direction**. They can be used to indicate motion in a two-dimensional plane. We use the symbol $\langle a, b \rangle$ to represent a vector that stretches from the origin to the coordinate (a, b) . Because vectors indicate motion, we can use calculus to measure the motion of the path traveled by the tip of the vector. The following relationships should be learned and memorized.

1. The particle's **position vector** is $\vec{r}(t) = \langle x(t), y(t) \rangle$.
2. The **magnitude** of the position vector is the length of the vector $L = \sqrt{(x(t))^2 + (y(t))^2}$.
3. The **velocity vector** is $\vec{v}(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$.
4. The **speed** is the magnitude of the velocity vector $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$.
5. The **acceleration vector** is $\vec{a}(t) = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \right\rangle$.
6. The **displacement** from t_1 to t_2 is given by integrating the velocity vector $\vec{v}(t) = \langle v_1(t), v_2(t) \rangle$ or $\left\langle \int_{t_1}^{t_2} v_1(t) dt, \int_{t_1}^{t_2} v_2(t) dt \right\rangle$.
7. The **total distance traveled** by the position vector is given by $\int_{t_1}^{t_2} \sqrt{(v_1(t))^2 + (v_2(t))^2} dt$.

Example 1: Multiple Choice Examples

1998 BC21 (non-calculator)

21. The length of the path described by the parametric equations $x = \frac{1}{3}t^3$ and $y = \frac{1}{2}t^2$, where $0 \leq t \leq 1$, is given by

(A) $\int_0^1 \sqrt{t^2 + 1} dt$

(B) $\int_0^1 \sqrt{t^2 + t} dt$

(C) $\int_0^1 \sqrt{t^4 + t^2} dt$

(D) $\frac{1}{2} \int_0^1 \sqrt{4 + t^4} dt$

(E) $\frac{1}{6} \int_0^1 t^2 \sqrt{4t^2 + 9} dt$

2003 BC84 (calculator active)

84. A particle moves in the xy -plane so that its position at any time t is given by $x(t) = t^2$ and $y(t) = \sin(4t)$. What is the speed of the particle when $t = 3$?
- (A) 2.909 (B) 3.062 (C) 6.884 (D) 9.016 (E) 47.393

2008 BC1 (non-calculator)

1. At time $t \geq 0$, a particle moving in the xy -plane has velocity vector given by $v(t) = \langle t^2, 5t \rangle$. What is the acceleration vector of the particle at time $t = 3$?
- (A) $\langle 9, \frac{45}{2} \rangle$ (B) $\langle 6, 5 \rangle$ (C) $\langle 2, 0 \rangle$ (D) $\sqrt{306}$ (E) $\sqrt{61}$

1998 BC10 (non-calculator)

10. A particle moves on a plane curve so that at any time $t > 0$ its x -coordinate is $t^3 - t$ and its y -coordinate is $(2t - 1)^3$. The acceleration vector of the particle at $t = 1$ is
- (A) (0,1) (B) (2,3) (C) (2,6) (D) (6,12) (E) (6,24)

Example 2: FRQ Questions

2006 Form B BC2 (Calculator active)

An object moving along a curve in the xy -plane is at position $(x(t), y(t))$ at time t , where

$$\frac{dx}{dt} = \tan(e^{-t}) \quad \text{and} \quad \frac{dy}{dt} = \sec(e^{-t}) \quad \text{for } t \geq 0.$$

At time $t = 1$, the object is at a position $(2, -3)$.

- a) Write an equation for the line tangent to the curve at position $(2, -3)$.

- b) Find the acceleration vector and the speed of the object at time $t = 1$.

- c) Find the total distance traveled by the object over the time interval $1 \leq t \leq 2$.

- d) Is there a time $t \geq 0$ at which the object is on the y -axis? Explain why or why not.

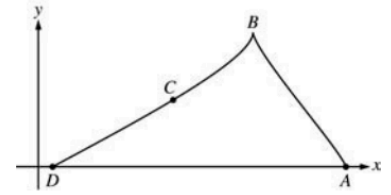
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2003 SCORING GUIDELINES

(Calculator Active)

Question 2

A particle starts at point A on the positive x -axis at time $t = 0$ and travels along the curve from A to B to C to D , as shown above. The coordinates of the particle's position $(x(t), y(t))$ are differentiable functions of t , where

$$x'(t) = \frac{dx}{dt} = -9\cos\left(\frac{\pi t}{6}\right)\sin\left(\frac{\pi\sqrt{t+1}}{2}\right) \text{ and } y'(t) = \frac{dy}{dt} \text{ is not explicitly given.}$$



At time $t = 9$, the particle reaches its final position at point D on the positive x -axis.

- At point C , is $\frac{dy}{dt}$ positive? At point C , is $\frac{dx}{dt}$ positive? Give a reason for each answer.
- The slope of the curve is undefined at point B . At what time t is the particle at point B ?
- The line tangent to the curve at the point $(x(8), y(8))$ has equation $y = \frac{5}{9}x - 2$. Find the velocity vector and the speed of the particle at this point.
- How far apart are points A and D , the initial and final positions, respectively, of the particle?

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2010 SCORING GUIDELINES (Form B)

Question 2

(Calculator Active)

The velocity vector of a particle moving in the plane has components given by

$$\frac{dx}{dt} = 14 \cos(t^2) \sin(e^t) \quad \text{and} \quad \frac{dy}{dt} = 1 + 2 \sin(t^2), \quad \text{for } 0 \leq t \leq 1.5.$$

At time $t = 0$, the position of the particle is $(-2, 3)$.

- (a) For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.
- (b) Write an equation for the line tangent to the path of the particle at $t = 1$.
- (c) Find the speed of the particle at $t = 1$.
- (d) Find the acceleration vector of the particle at $t = 1$.

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2011 SCORING GUIDELINES

(Calculator Active)

Question 1

At time t , a particle moving in the xy -plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin(t^2)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

- (a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.
- (b) Find the slope of the line tangent to the path of the particle at time $t = 3$.
- (c) Find the position of the particle at time $t = 3$.
- (d) Find the total distance traveled by the particle over the time interval $0 \leq t \leq 3$.

10.4 – Polar Coordinates & Polar Graphs

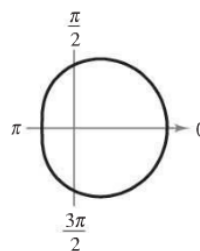
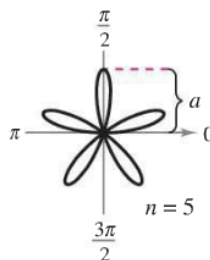
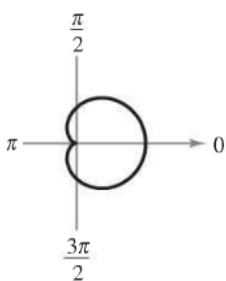
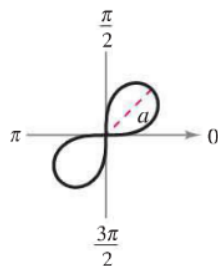
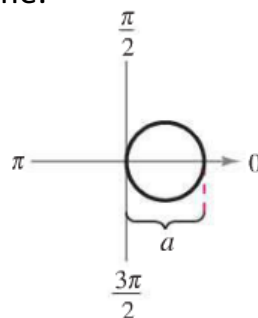
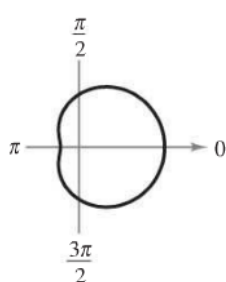
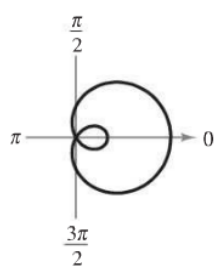
Topics

- *Understand the polar coordinate system*
- *Rewrite rectangular coordinates and equations in polar form and vice-versa.*
 - *Sketch the graph of an equation given in polar form.*
 - *Find the slope of a tangent line to a polar graph.*
 - *Identify several types of special polar graphs.*

Warm Up!

- a. Brainstorm with a neighbor everything you can remember about polar coordinates.

- b. Match the Polar Graph with its name.



Word Bank

- Limaçon with inner loop
- Cardioid
- Dimpled limaçon
- Convex limaçon
- Rose Curve
- Circle
- Lemniscate

Polar Coordinates

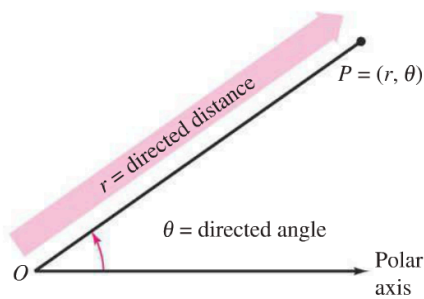
So far, you have been representing graphs as collections of points (x, y) on the rectangular coordinate system. The corresponding equations for these graphs have been in either rectangular or parametric form. In this section, you will study a coordinate system called the **polar coordinate system**.

To form the polar coordinate system in the plane, fix a point O , called the **pole** (or **origin**), and construct from O an initial ray called the **polar axis**, as shown in Figure 10.35. Then each point P in the plane can be assigned **polar coordinates** (r, θ) , as follows.

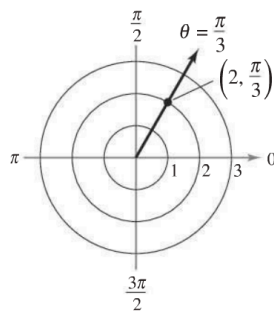
$r =$ directed distance from O to P

$\theta =$ directed angle, counterclockwise from polar axis to segment \overline{OP}

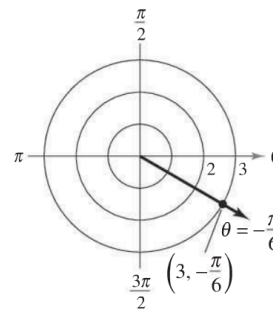
Figure 10.36 shows three points on the polar coordinate system. Notice that in this system, it is convenient to locate points with respect to a grid of concentric circles intersected by **radial lines** through the pole.



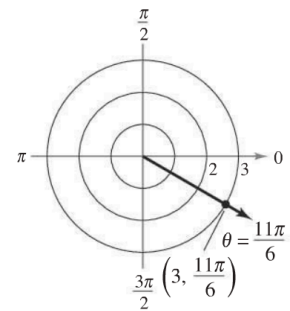
Polar coordinates
Figure 10.35



(a)
Figure 10.36



(b)



(c)

THEOREM 10.10 Coordinate Conversion

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) of the point as follows.

Polar-to-Rectangular

$$x = r \cos \theta$$

$$y = r \sin \theta$$

Rectangular-to-Polar

$$\tan \theta = \frac{y}{x}$$

$$r^2 = x^2 + y^2$$

Example 1: Graphing and Converting Polar Coordinates

The polar coordinates of a point are given. Plot the point and find the corresponding rectangular coordinates for the point.

a. $(8, \frac{\pi}{2})$

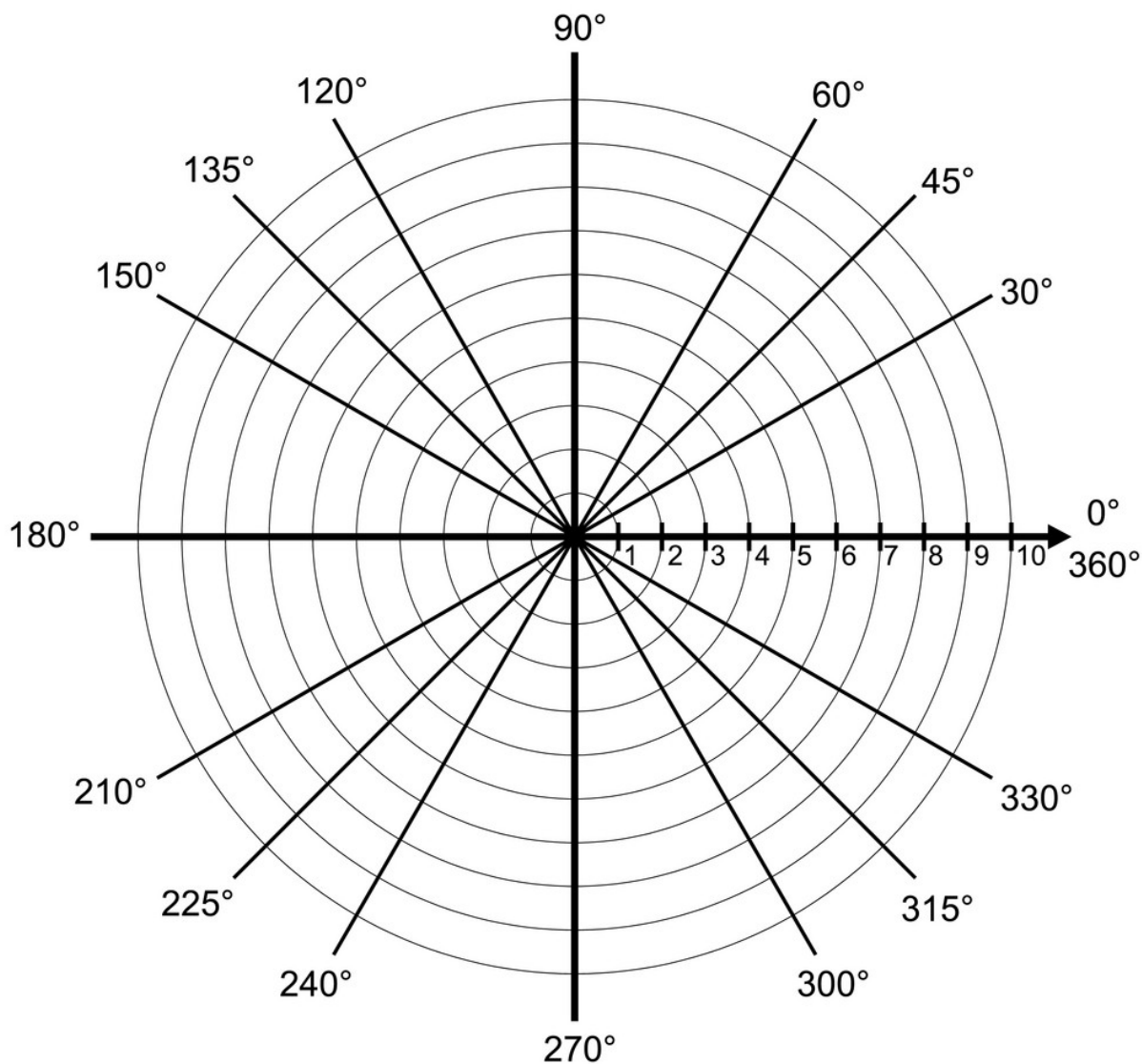
b. $(-2, \frac{5\pi}{3})$

c. $(-4, -\frac{3\pi}{4})$

d. $(0, \frac{7\pi}{6})$

e. $(7, \frac{5\pi}{4})$

f. $(-2, \frac{11\pi}{6})$



Example 2: Rectangular-to-Polar Conversion

The rectangular coordinates of a point are given. Find two sets of polar coordinates for the point for $0 \leq \theta < 2\pi$.

a. $(1, 0)$

b. $(0, -9)$

c. $(-3, 4)$

d. $(6, -2)$

e. $(-5, -5\sqrt{3})$

f. $(3, -\sqrt{3})$

Example 3: Rectangular-to-Polar Conversion

Convert the rectangular equation to polar form and use a calculator to sketch the polar graph.

a. $x^2 + y^2 = 9$

b. $x^2 - y^2 = 9$

c. $y = 8$

d. $x = 12$

e. $3x - y + 2 = 0$

f. $xy = 4$

Example 4: Polar-to-Rectangular Conversion

Convert the polar equation to rectangular form.

a. $r = 4$

b. $r = 3 \sin \theta$

c. $r = \theta$

d. $\theta = \frac{5\pi}{6}$

e. $r = 3 \sec \theta$

f. $r = \sec \theta \tan \theta$

Slope and Tangent Lines

To find the slope of a tangent line to a polar graph, consider a differentiable function given by $r = f(\theta)$. To find the slope in polar form, use the parametric equations

$$x = r \cos \theta = f(\theta) \cos \theta \quad \text{and} \quad y = r \sin \theta = f(\theta) \sin \theta.$$

Using the parametric form of dy/dx given in Theorem 10.7, you have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$$

which establishes the next theorem.

THEOREM 10.11 Slope in Polar Form

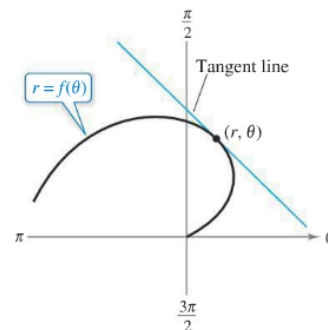
If f is a differentiable function of θ , then the *slope* of the tangent line to the graph of $r = f(\theta)$ at the point (r, θ) is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}$$

provided that $dx/d\theta \neq 0$ at (r, θ) . (See Figure 10.44.)

1. Solutions of $\frac{dy}{d\theta} = 0$ yield horizontal tangents, provided that $\frac{dx}{d\theta} \neq 0$.
2. Solutions of $\frac{dx}{d\theta} = 0$ yield vertical tangents, provided that $\frac{dy}{d\theta} \neq 0$.

If $dy/d\theta$ and $dx/d\theta$ are *simultaneously* 0, then no conclusion can be drawn about tangent lines.



Tangent line to polar curve
Figure 10.44

Example 5: Horizontal and Vertical Tangency

Find the points of horizontal and vertical tangency to the polar curve.

a. $r = 1 - \sin \theta$

10.5 – Area and Arc Length in Polar Coordinates

Topics

- *Find the area of a region bounded by a polar graph.*
- *Find the points of intersection of two polar graphs.*
 - *Find the arc length of a polar graph.*

Warm Up!

Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

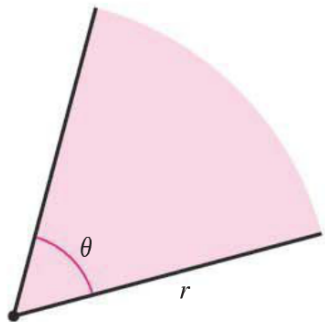
- If (r_1, θ_1) and (r_2, θ_2) represent the same point on the polar coordinate system, then $|r_1| = |r_2|$.
- If (r, θ_1) and (r, θ_2) represent the same point on the polar coordinate system, then $\theta_1 = \theta_2 + 2n\pi$ for some integer n .
- If $x > 0$, then the point (x, y) on the rectangular coordinate system can be represented by (r, θ) on the polar coordinate system, where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$.
- The polar equations $r = \sin 2\theta$, $r = -\sin 2\theta$, and $r = \sin(-2\theta)$ all have the same graph.

Example 1: Tangent Lines of Polar Curves

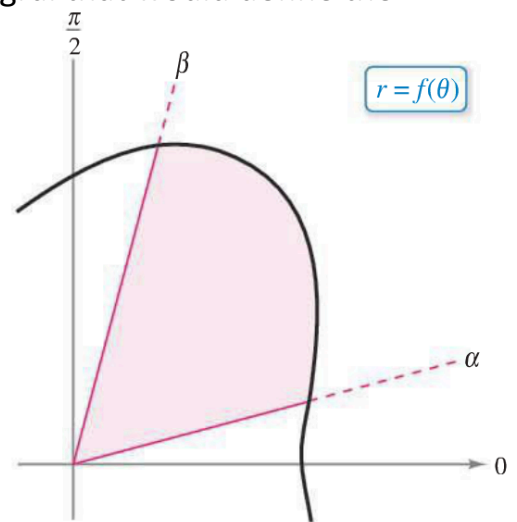
Find the slope of the curve at the point where $r = 3 \sin 2\theta$ at the point where $\theta = \frac{\pi}{6}$ and use it to write an equation for the line tangent to the graph. Verify graphically.

Example 2: Investigating Area between two Polar Curves

Given the area of a sector equation below, write a definite integral that would define the shaded region to the right.



The area of a sector of a circle is $A = \frac{1}{2}\theta r^2$.



THEOREM 10.13 Area in Polar Coordinates

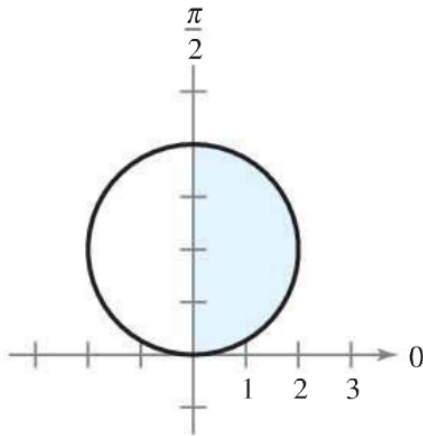
If f is continuous and nonnegative on the interval $[\alpha, \beta]$, $0 < \beta - \alpha \leq 2\pi$, then the area of the region bounded by the graph of $r = f(\theta)$ between the radial lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2} \int_{\alpha}^{\beta} [f(\theta)]^2 d\theta$$
$$= \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta. \quad 0 < \beta - \alpha \leq 2\pi$$

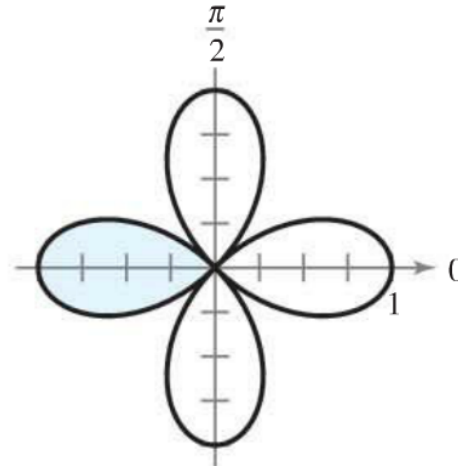
Example 3: Finding the Area of a Polar Region

Write an integral that represents the area of the shaded region of the figure.

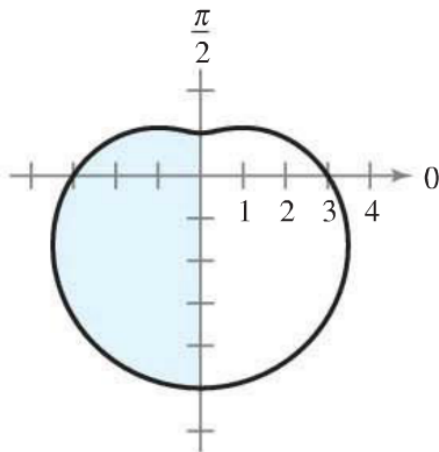
a. $r = 4 \sin \theta$



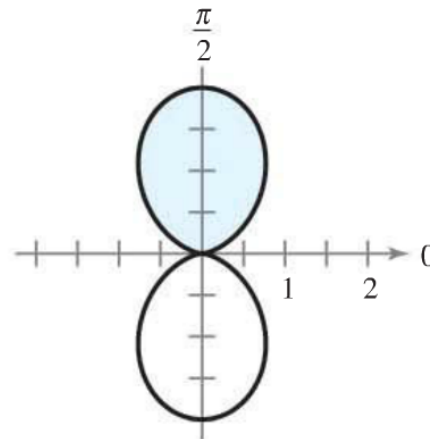
b. $r = \cos 2\theta$



c. $r = 3 - 2 \sin \theta$



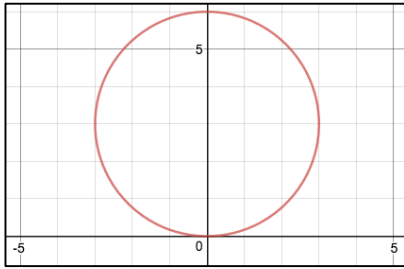
d. $r = 1 - \cos 2\theta$



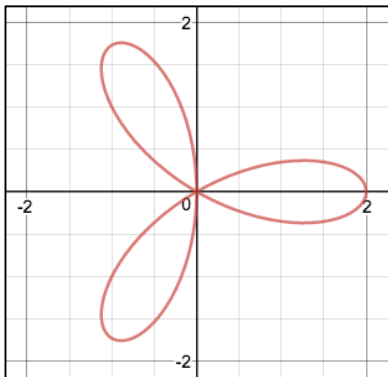
Example 4: Finding the Area of a Polar Region

Find the area of the region. Use a calculator to see the graph, but integrate without.

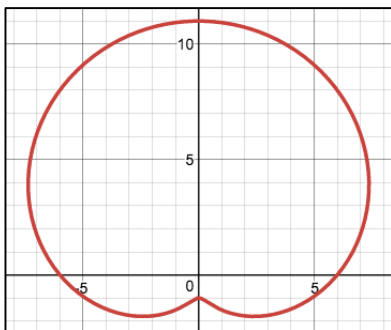
- a. Interior of $r = 6 \sin \theta$



- b. One pedal of $r = 2 \cos 3\theta$



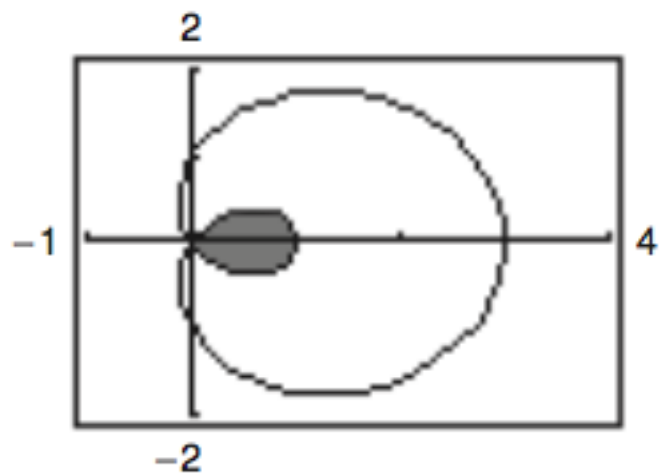
- c. Interior of $r = 6 + 5 \sin \theta$ (below the polar axis)



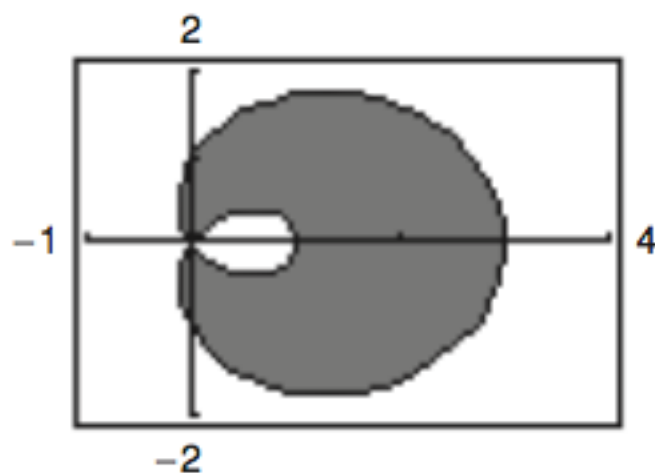
Example 4: Finding the Area of a Polar Region

Find the area of the given region using your calculator.

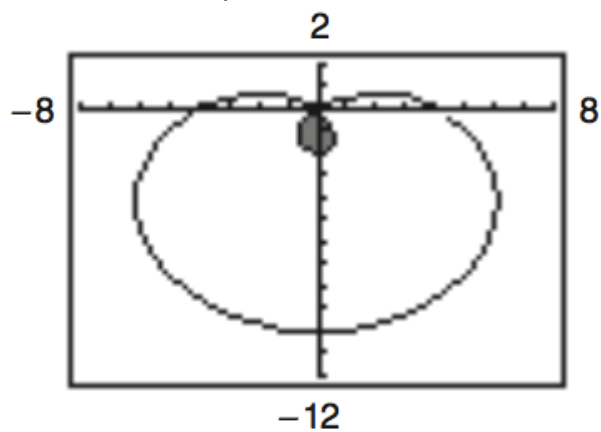
a. Inner loop of $r = 1 + 2 \cos \theta$



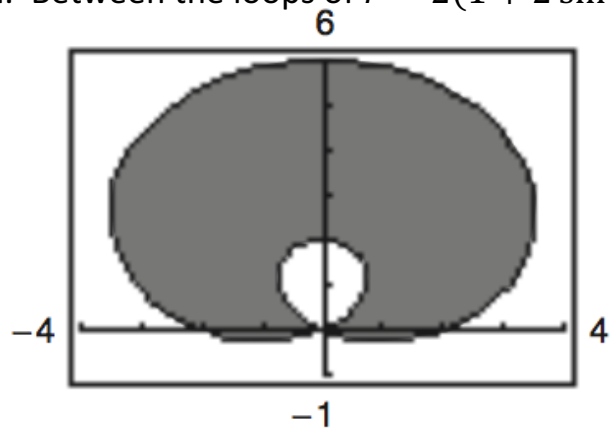
b. Between the loops of $r = 1 + 2 \cos \theta$



c. Inner loop of $r = 4 - 6 \sin \theta$



d. Between the loops of $r = 2(1 + 2 \sin \theta)$



Points of Intersection of Polar Graphs

Because a point may be represented in different ways in polar coordinates, care must be taken in determining the points of intersection of two polar graphs. For example, consider the points of intersection of the graphs of

$$r = 1 - 2 \cos \theta \quad \text{and} \quad r = 1$$

Example 5: Points of Intersection of Polar Graphs

Find the points of intersection of the graphs of the equations.

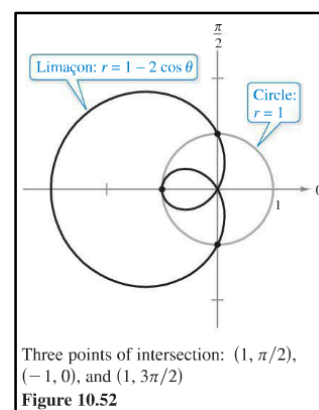
a. $r = 1 - 2 \cos \theta$
 $r = 1$

The corresponding points of intersection are $(1, \pi/2)$ and $(1, 3\pi/2)$. From Figure 10.52, however, you can see that there is a *third* point of intersection that did not show up when the two polar equations were solved simultaneously. (This is one reason why you should sketch a graph when finding the area of a polar region.) The reason the third point was not found is that it does not occur with the same coordinates in the two graphs. On the graph of $r = 1$, the point occurs with coordinates $(1, \pi)$, but on the graph of

$$r = 1 - 2 \cos \theta$$

the point occurs with coordinates $(-1, 0)$.

In addition to solving equations simultaneously and sketching a graph, note that because the pole can be represented by $(0, \theta)$, where θ is *any* angle, you should check separately for the pole when finding points of intersection.

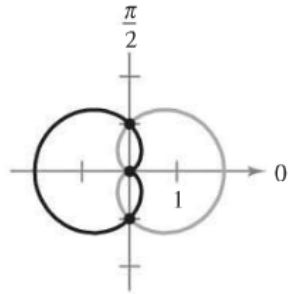


Three points of intersection: $(1, \pi/2)$, $(-1, 0)$, and $(1, 3\pi/2)$
Figure 10.52

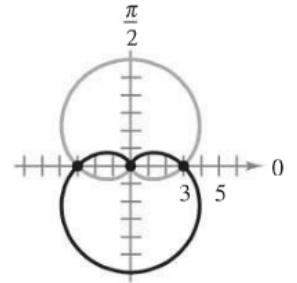
Example 6: Points of Intersection of Polar Graphs

Find the points of intersection of the graphs of the equations.

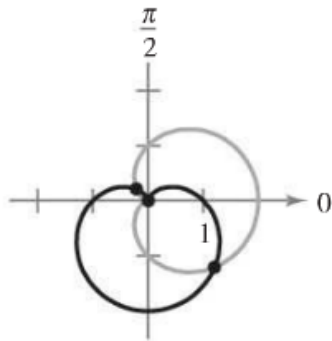
a. $r = 1 + \cos \theta$
 $r = 1 - \cos \theta$



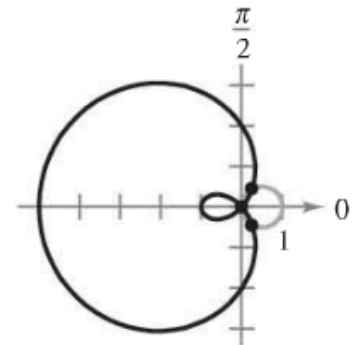
b. $r = 3(1 + \sin \theta)$
 $r = 3(1 - \sin \theta)$



c. $r = 1 + \cos \theta$
 $r = 1 - \sin \theta$



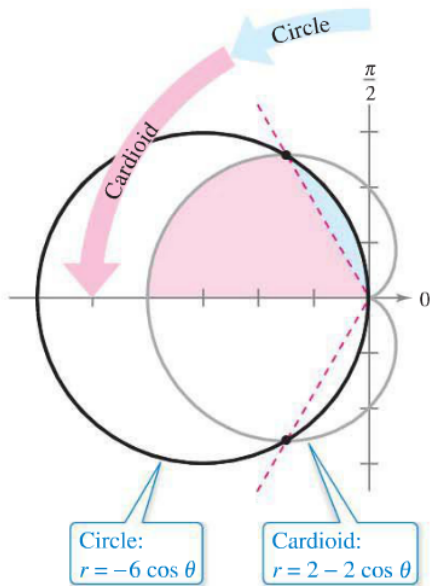
d. $r = 2 - 3 \cos \theta$
 $r = \cos \theta$



Example 7: Finding the Area of a Region Between Two Polar Curves

Find the area of the region common to the two regions bounded by the curves below.

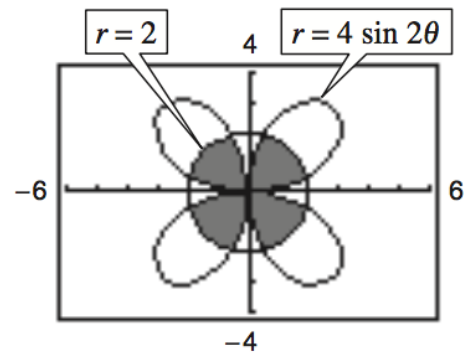
- a. $r = -6 \cos \theta$
 $r = 2 - 2 \cos \theta$



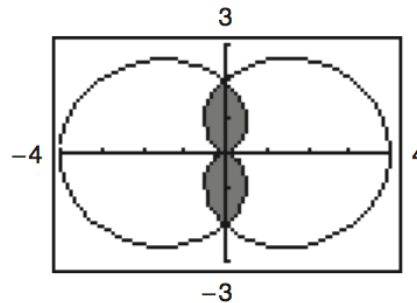
Example 8: Finding the Area of a Region Between Two Polar Curves

Find the area of the given region.

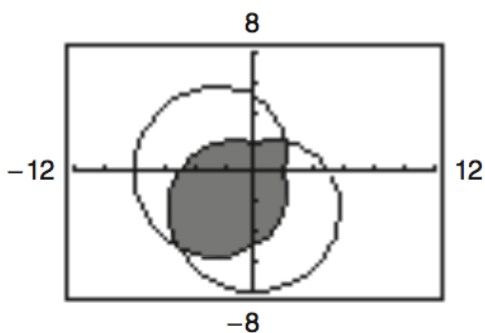
- a. Common interior of
 $r = 4 \cos 2\theta$
 $r = 2$



- b. Common interior of
 $r = 2(1 + \cos \theta)$
 $r = 2(1 - \cos \theta)$



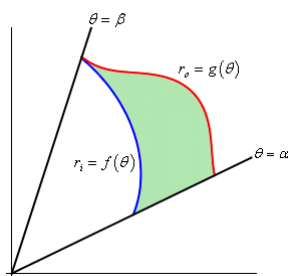
- c. Common interior of
 $r = 5 - 3 \sin \theta$
 $r = 5 - 3 \cos \theta$



Area Between 2 Polar Curves

To get the area between the polar curve $r = f(\theta)$ and the polar curve $r = g(\theta)$, we just subtract the area inside the inner curve from the area inside the outer curve.

If $f(\theta) \geq g(\theta)$,



$$\frac{1}{2} \int_a^b [f(\theta)^2 - g(\theta)^2] d\theta$$

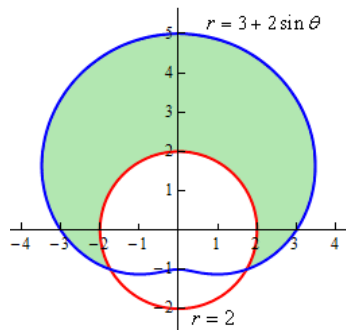
or

$$\frac{1}{2} \int_a^b [(r_{outer})^2 - (r_{inner})^2] d\theta$$

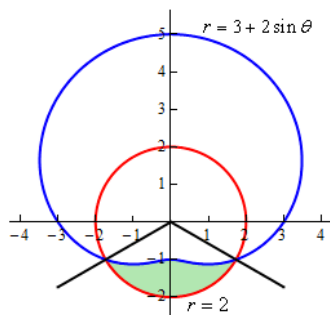
Example 9: Finding the Area of a Region Between Two Polar Curves

Find the area of the given region.

- a. Inside $r = 3 + 2 \sin \theta$ and outside $r = 2$



- b. Outside $r = 3 + 2 \sin \theta$ and inside $r = 2$



Polar AP MC Practice

Example 1. 1997 BC Exam #21 (non-calculator)

21. Which of the following is equal to the area of the region inside the polar curve $r = 2 \cos \theta$ and outside the polar curve $r = \cos \theta$?

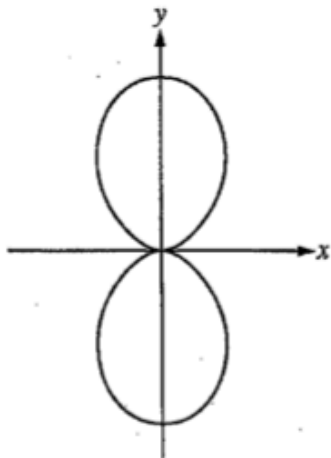
(A) $3 \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$ (B) $3 \int_0^{\pi} \cos^2 \theta \, d\theta$ (C) $\frac{3}{2} \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$ (D) $3 \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta$ (E) $3 \int_0^{\pi} \cos \theta \, d\theta$

Example 2. 1998 BC Exam #19 (non-calculator)

19. The area of the region inside the polar curve $r = 4 \sin \theta$ and outside the polar curve $r = 2$ is given by

(A) $\frac{1}{2} \int_0^{\pi} (4 \sin \theta - 2)^2 \, d\theta$ (B) $\frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 \sin \theta - 2)^2 \, d\theta$ (C) $\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (4 \sin \theta - 2)^2 \, d\theta$
(D) $\frac{1}{2} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} (16 \sin^2 \theta - 4) \, d\theta$ (E) $\frac{1}{2} \int_0^{\pi} (16 \sin^2 \theta - 4) \, d\theta$

Example 3. 2008 BC Exam #26 (non-calculator)



26. Which of the following expressions gives the total area enclosed by the polar curve $r = \sin^2 \theta$ shown in the figure above?

(A) $\frac{1}{2} \int_0^\pi \sin^2 \theta \, d\theta$

(B) $\int_0^\pi \sin^2 \theta \, d\theta$

(C) $\frac{1}{2} \int_0^\pi \sin^4 \theta \, d\theta$

(D) $\int_0^\pi \sin^4 \theta \, d\theta$

(E) $2 \int_0^\pi \sin^4 \theta \, d\theta$

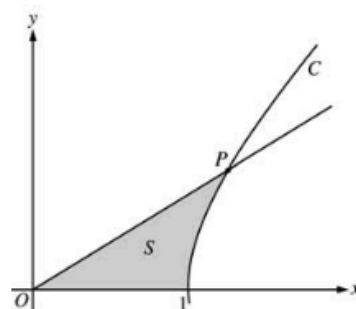
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2003 SCORING GUIDELINES**

(Calculator Active)

Question 3

The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1 + y^2}$. Let S be the shaded region bounded by the two graphs and the x -axis. The line and the curve intersect at point P .

- (a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P .
- (b) Set up and evaluate an integral expression with respect to y that gives the area of S .
- (c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2 \theta - \sin^2 \theta}$.
- (d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S .

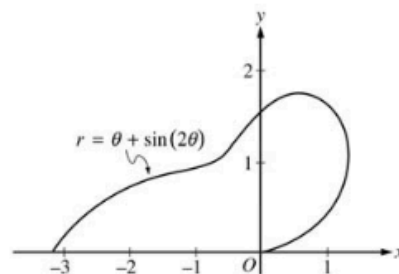


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2005 SCORING GUIDELINES

(Calculator Active)

Question 2

The curve above is drawn in the xy -plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \leq \theta \leq \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.



- (a) Find the area bounded by the curve and the x -axis.
- (b) Find the angle θ that corresponds to the point on the curve with x -coordinate -2 .
- (c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r ? What does this fact say about the curve?
- (d) Find the value of θ in the interval $0 \leq \theta \leq \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

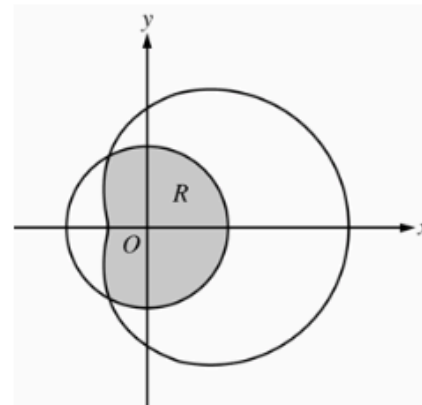
AP[®] CALCULUS BC
2007 SCORING GUIDELINES

(Calculator Active)

Question 3

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

- (a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R .
- (b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t , with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$.



Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

- (c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.