Chapter 11

Angular Momentum; General Rotation

Units of Chapter 11

- Angular Momentum—Objects Rotating About a Fixed Axis
- Vector Cross Product; Torque as a Vector
- Angular Momentum of a Particle
- Angular Momentum and Torque for a System of Particles; General Motion
- Angular Momentum and Torque for a Rigid Object

Units of Chapter 11

- Conservation of Angular Momentum
- The Spinning Top and Gyroscope
- Rotating Frames of Reference; Inertial Forces
- The Coriolis Effect

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

The rotational analog of linear momentum is angular momentum, L:

$$L = I\omega.$$

Then the rotational analog of Newton's second law is:

$$\Sigma\tau = \frac{dL}{dt}.$$

This form of Newton's second law is valid even if I is not constant.

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

In the absence of an external torque, angular momentum is conserved:

$$\frac{dL}{dt} = 0 \text{ and } L = I\omega = \text{constant.}$$

More formally,

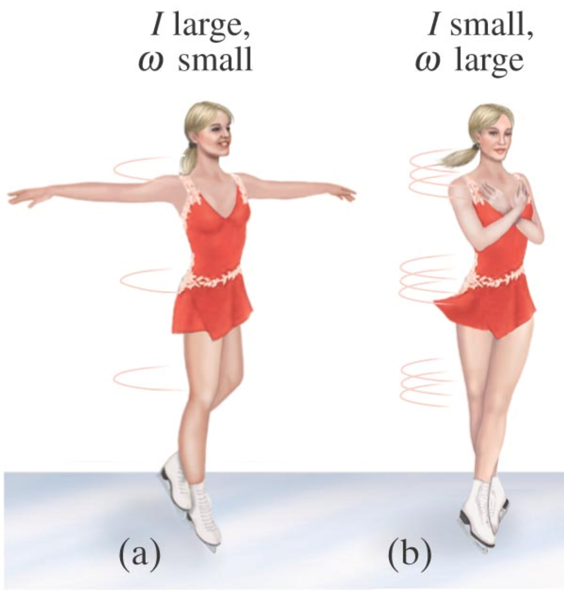
the total angular momentum of a rotating object remains constant if the net external torque acting on it is zero.

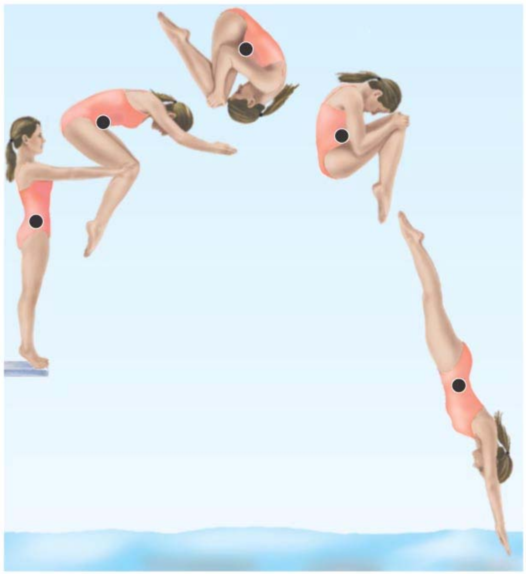
11-1 Angular Momentum—Objects Rotating About a Fixed Axis

This means:

$$I\omega = I_0\omega_0 = \text{constant.}$$

Therefore, if an object's moment of inertia changes, its angular speed changes as well.

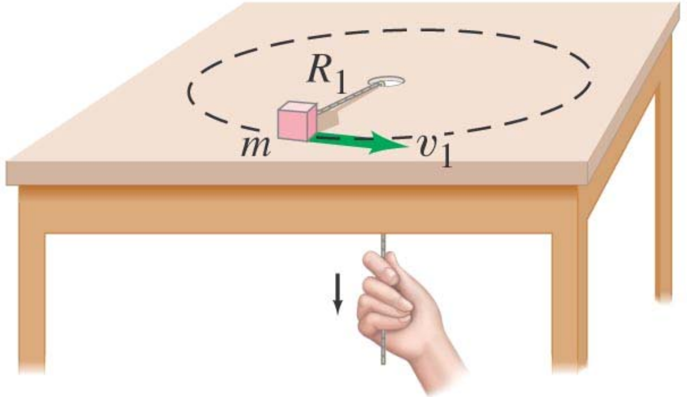




11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Example 11-1: Object rotating on a string of changing length.

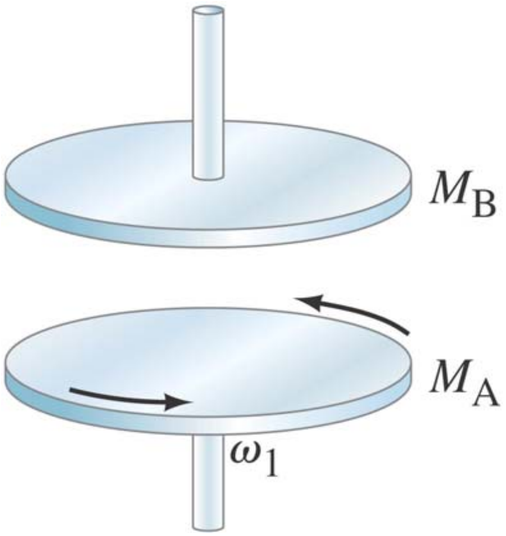
A small mass m attached to the end of a string revolves in a circle on a frictionless tabletop. The other end of the string passes through a hole in the table. Initially, the mass revolves with a speed v_1 = 2.4 m/s in a circle of radius R_1 = 0.80 m. The string is then pulled slowly through the hole so that the radius is reduced to R_2 = 0.48 m. What is the speed, v_2, of the mass now?



11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Example 11-2: Clutch.

A simple clutch consists of two cylindrical plates that can be pressed together to connect two sections of an axle, as needed, in a piece of machinery. The two plates have masses M_A = 6.0 kg and M_B = 9.0 kg, with equal radii R_0 = 0.60 m. They are initially separated. Plate M_A is accelerated from rest to an angular velocity ω_1 = 7.2 rad/s in time Δt = 2.0 s. Calculate (a) the angular momentum of M_A, and (b) the torque required to have accelerated M_A from rest to ω_1. (c) Next, plate M_B, initially at rest but free to rotate without friction, is placed in firm contact with freely rotating plate M_A, and the two plates both rotate at a constant angular velocity ω_2, which is considerably less than ω_1. Why does this happen, and what is ω_2?



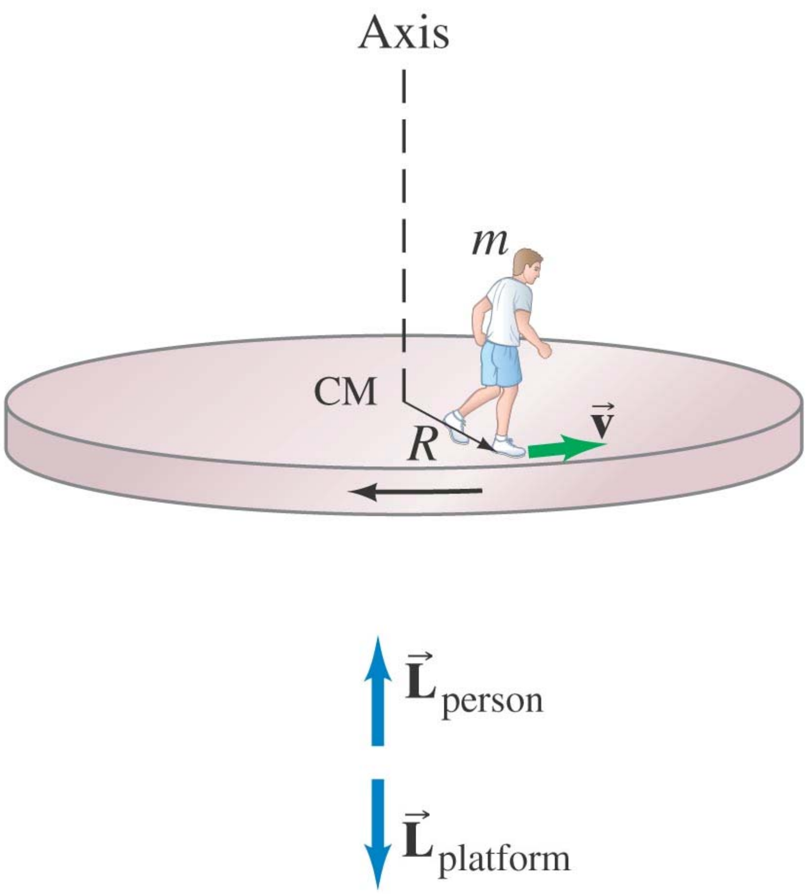
11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Example 11-3: Neutron star.

Astronomers detect stars that are rotating extremely rapidly, known as neutron stars. A neutron star is believed to form from the inner core of a larger star that collapsed, under its own gravitation, to a star of very small radius and very high density. Before collapse, suppose the core of such a star is the size of our Sun ($r \approx 7 \times 10^5$ km) with mass 2.0 times as great as the Sun, and is rotating at a frequency of 1.0 revolution every 100 days. If it were to undergo gravitational collapse to a neutron star of radius 10 km, what would its rotation frequency be? Assume the star is a uniform sphere at all times, and loses no mass.

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Angular momentum is a vector; for a symmetrical object rotating about a symmetry axis it is in the same direction as the angular velocity vector.



11-1 Angular Momentum—Objects Rotating About a Fixed Axis

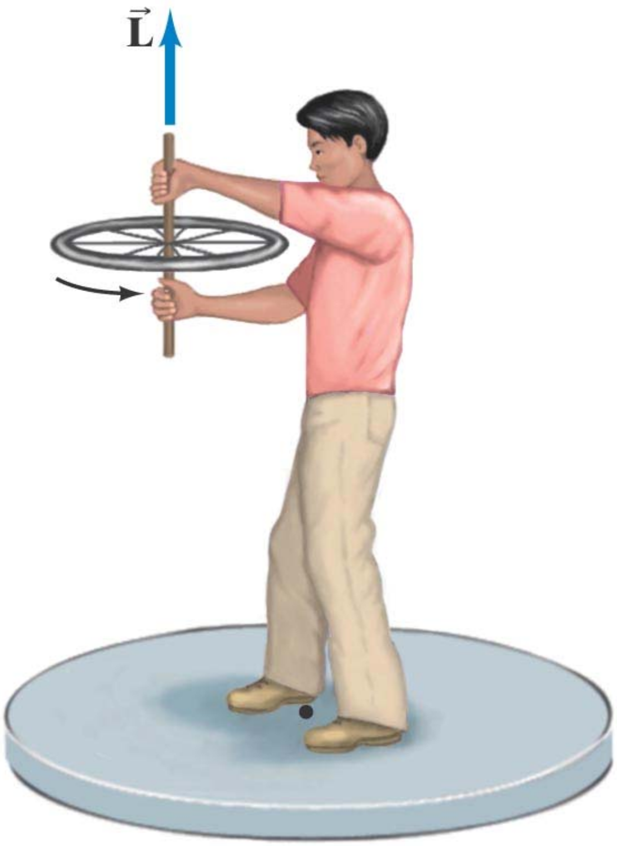
Example 11-4: Running on a circular platform.

Suppose a 60-kg person stands at the edge of a 6.0-m-diameter circular platform, which is mounted on frictionless bearings and has a moment of inertia of 1800 kg·m^2. The platform is at rest initially, but when the person begins running at a speed of 4.2 m/s (with respect to the Earth) around its edge, the platform begins to rotate in the opposite direction. Calculate the angular velocity of the platform.

11-1 Angular Momentum—Objects Rotating About a Fixed Axis

Conceptual Example 11-5: Spinning bicycle wheel.

Your physics teacher is holding a spinning bicycle wheel while he stands on a stationary frictionless turntable. What will happen if the teacher suddenly flips the bicycle wheel over so that it is spinning in the opposite direction?

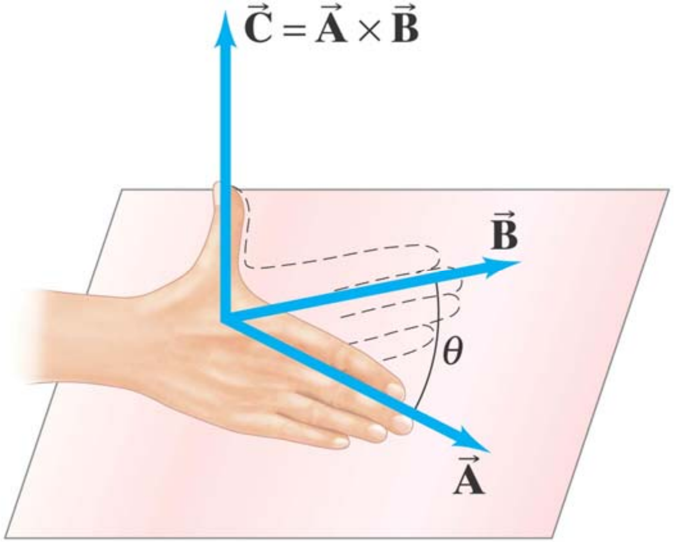


11-2 Vector Cross Product; Torque as a Vector

The vector cross product is defined as:

$$C = \left|\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right| = AB\sin\theta.$$

The direction of the cross product is defined by a right-hand rule:



11-2 Vector Cross Product; Torque as a Vector

The cross product can also be written in determinant form:

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{\mathbf{i}} + (A_z B_x - A_x B_z)\hat{\mathbf{j}} + (A_x B_y - A_y B_x)\hat{\mathbf{k}}.$$

11-2 Vector Cross Product; Torque as a Vector

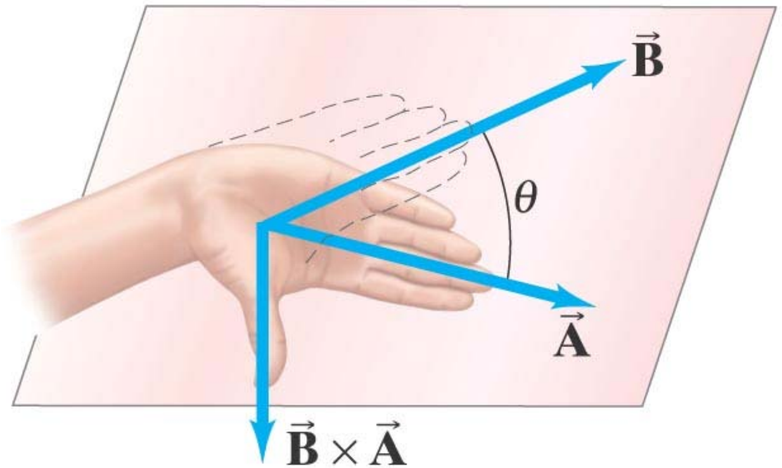
Some properties of the cross product:

$$\vec{\mathbf{A}} \times \vec{\mathbf{A}} = 0$$

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}}$$

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) + (\vec{\mathbf{A}} \times \vec{\mathbf{C}})$$

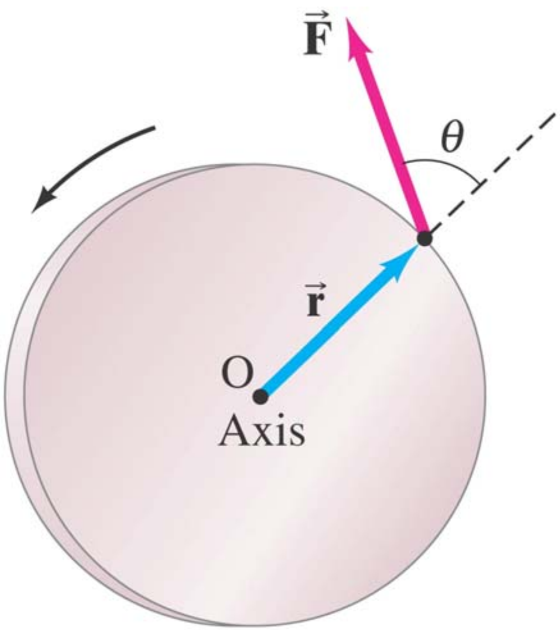
$$\frac{d}{dt}(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = \frac{d\vec{\mathbf{A}}}{dt} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \frac{d\vec{\mathbf{B}}}{dt}.$$



11-2 Vector Cross Product; Torque as a Vector

Torque can be defined as the vector product of the force and the vector from the point of action of the force to the axis of rotation:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}.$$

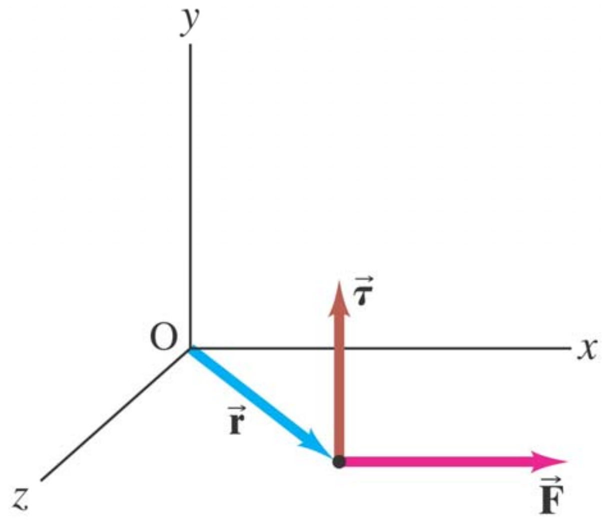


11-2 Vector Cross Product; Torque as a Vector

For a particle, the torque can be defined around a point O:

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}.$$

Here, $\vec{\mathrm{r}}$ is the position vector from the particle relative to O.



11-2 Vector Cross Product; Torque as a Vector

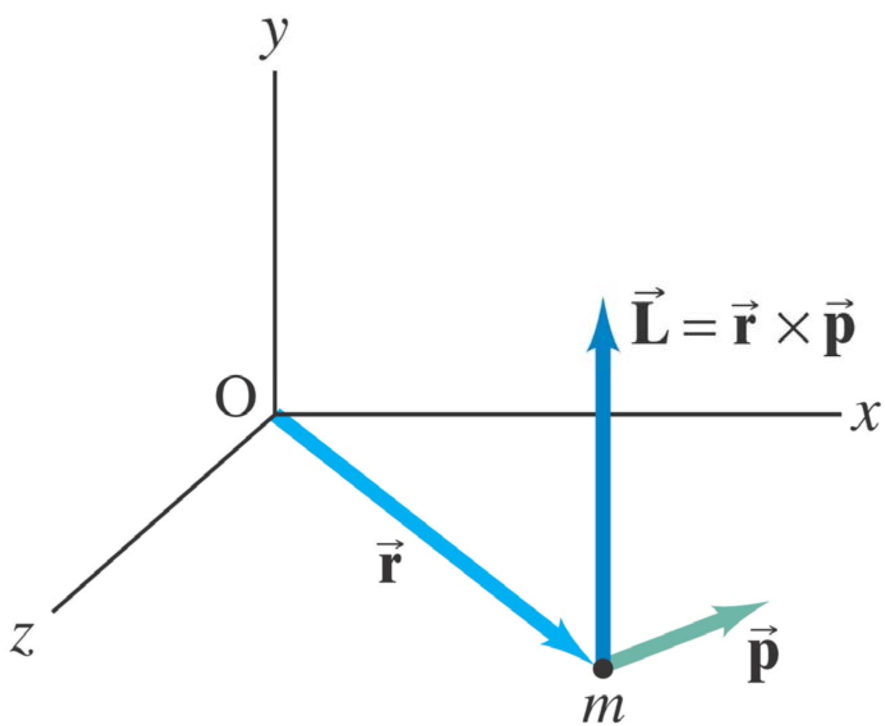
Example 11-6: Torque vector.

Suppose the vector $\vec{\mathbf{r}}$ is in the xz plane, and is given by $\vec{\mathbf{r}}$ = (1.2 m)$\hat{\mathbf{i}}$ + 1.2 m)$\hat{\mathbf{k}}$. Calculate the torque vector $\vec{\tau}$ if $\vec{\mathbf{F}}$ = (150 N)$\hat{\mathbf{i}}$.

11-3 Angular Momentum of a Particle

The angular momentum of a particle about a specified axis is given by:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}.$$



11-3 Angular Momentum of a Particle

If we take the derivative of $\vec{L}$, we find:

$$\frac{d\vec{\mathbf{L}}}{dt} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}.$$

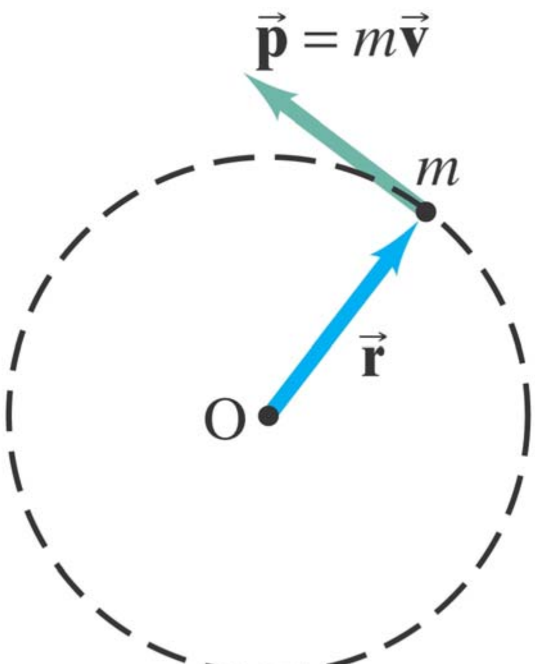
Since $\vec{\mathbf{r}} \times \Sigma\vec{\mathbf{F}} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt} = \frac{d\vec{\mathbf{L}}}{dt},$

we have: $\Sigma\vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt}.$

11-3 Angular Momentum of a Particle

Conceptual Example 11-7: A particle's angular momentum.

What is the angular momentum of a particle of mass m moving with speed v in a circle of radius r in a counterclockwise direction?



11-4 Angular Momentum and Torque for a System of Particles; General Motion

The angular momentum of a system of particles can change only if there is an external torque—torques due to internal forces cancel.

$$\frac{d\vec{\mathbf{L}}}{dt} = \sum \vec{\boldsymbol{\tau}}_{\text{ext}}.$$

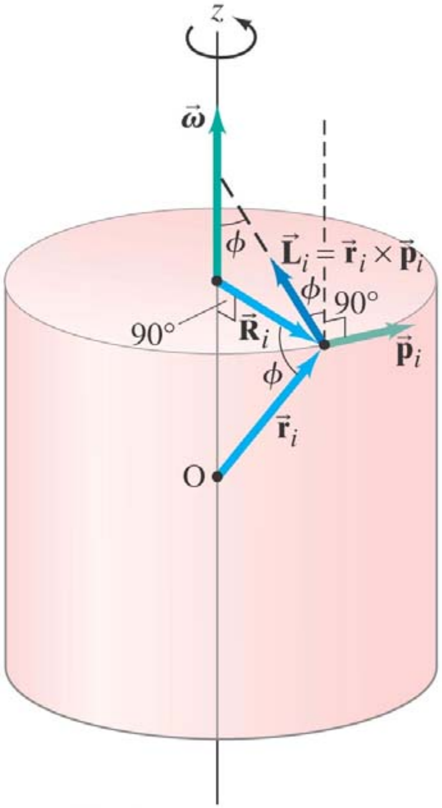
This equation is valid in any inertial reference frame. It is also valid for the center of mass, even if it is accelerating:

$$\frac{d\vec{\mathbf{L}}_{\text{CM}}}{dt} = \sum \vec{\boldsymbol{\tau}}_{\text{CM}}.$$

11-5 Angular Momentum and Torque for a Rigid Object

For a rigid object, we can show that its angular momentum when rotating around a particular axis is given by:

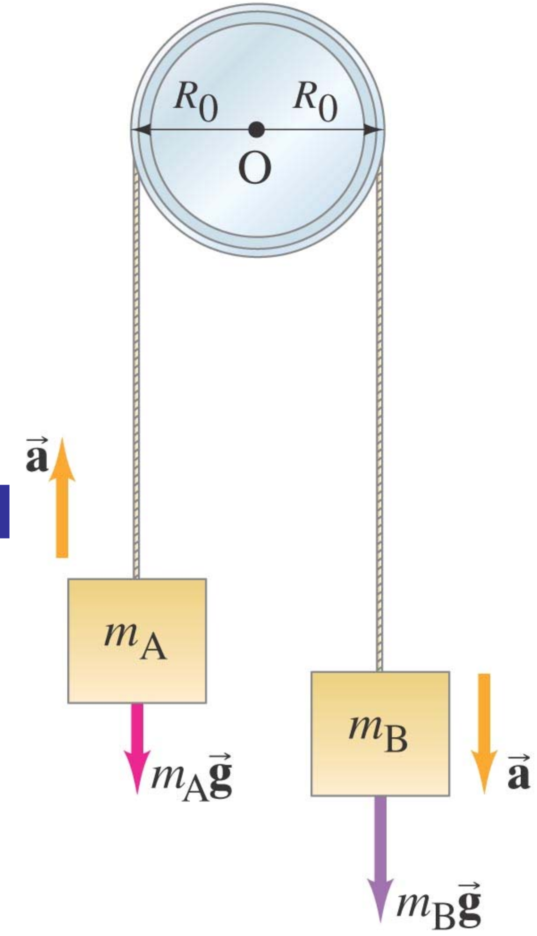
$$L_{\omega} = I\omega.$$



11-5 Angular Momentum and Torque for a Rigid Object

Example 11-8: Atwood's machine.

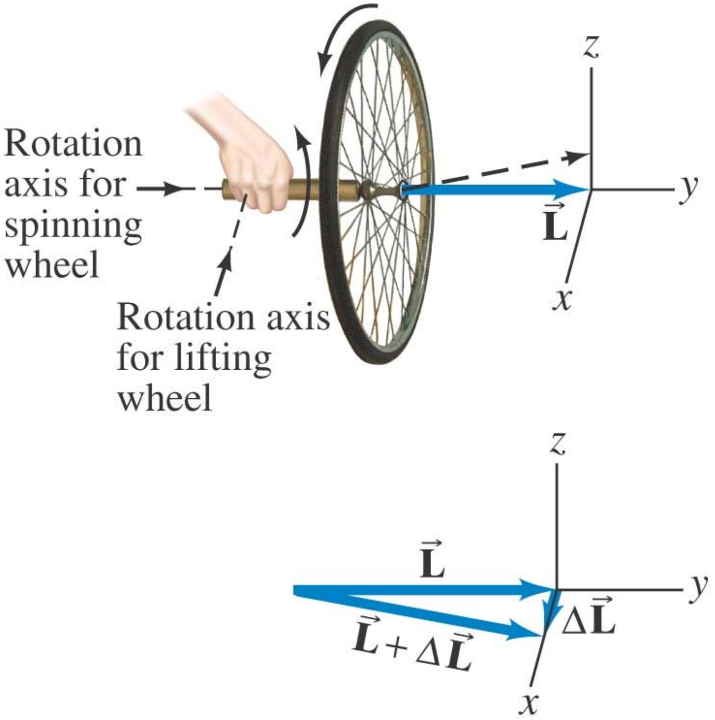
An Atwood machine consists of two masses, m_A and m_B, which are connected by an inelastic cord of negligible mass that passes over a pulley. If the pulley has radius R_0 and moment of inertia I about its axle, determine the acceleration of the masses m_A and m_B, and compare to the situation where the moment of inertia of the pulley is ignored.



11-5 Angular Momentum and Torque for a Rigid Object

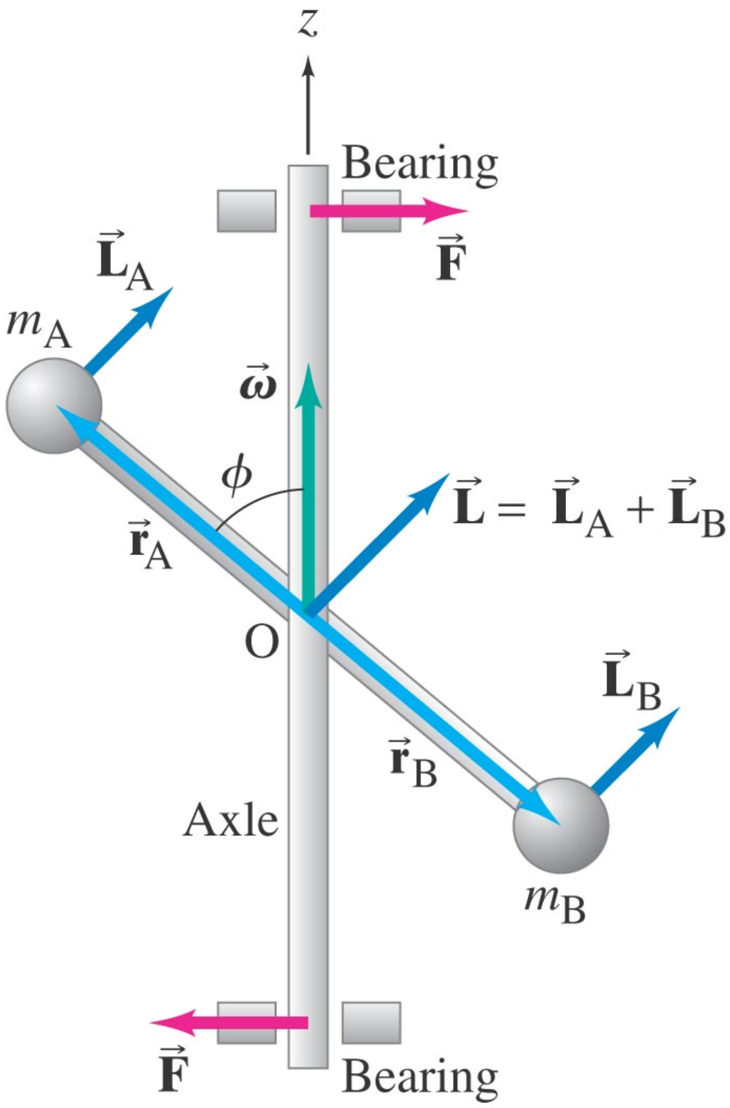
Conceptual Example 11-9: Bicycle wheel.

Suppose you are holding a bicycle wheel by a handle connected to its axle. The wheel is spinning rapidly so its angular momentum points horizontally as shown. Now you suddenly try to tilt the axle upward (so the CM moves vertically). You expect the wheel to go up (and it would if it weren't rotating), but it unexpectedly swerves to the right! Explain.



11-5 Angular Momentum and Torque for a Rigid Object

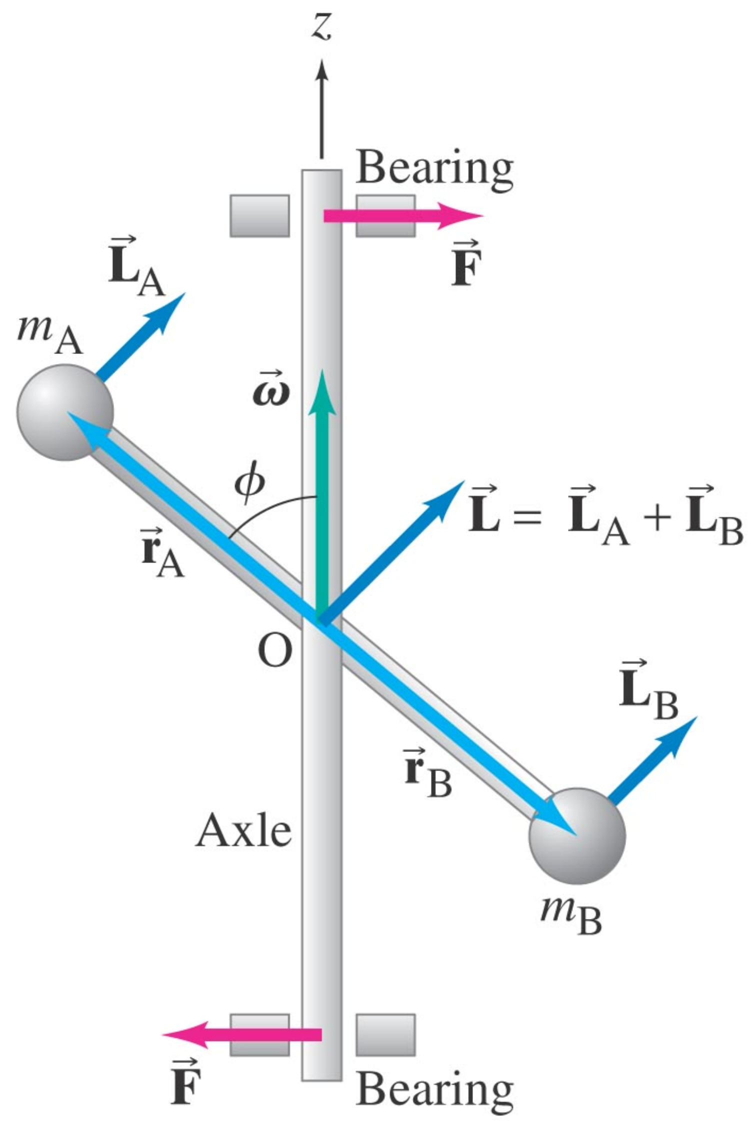
A system that is rotationally imbalanced will not have its angular momentum and angular velocity vectors in the same direction. A torque is required to keep an unbalanced system rotating.



11-5 Angular Momentum and Torque for a Rigid Object

Example 11-10: Torque on unbalanced system.

Determine the magnitude of the net torque τ_{net} needed to keep the illustrated system turning.



11-6 Conservation of Angular Momentum

If the net torque on a system is constant,

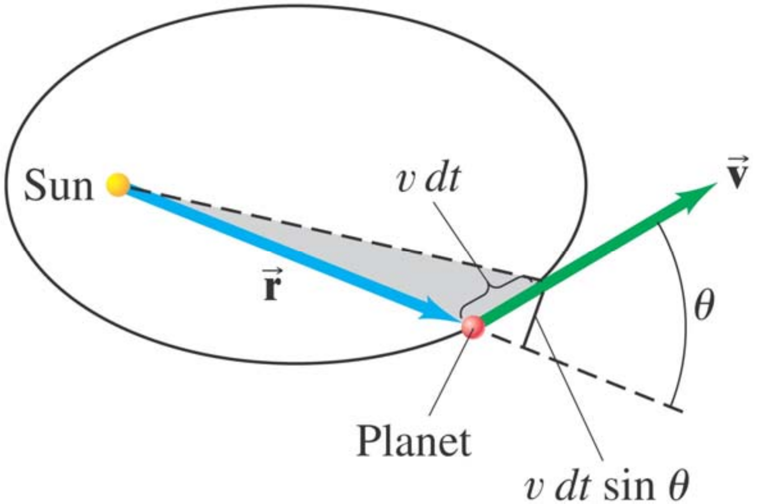
$$\frac{d\vec{\mathbf{L}}}{dt} = 0 \quad \text{and} \quad \vec{\mathbf{L}} = \text{constant.} \qquad \left[\Sigma\vec{\boldsymbol{\tau}} = 0\right]$$

The total angular momentum of a system remains constant if the net external torque acting on the system is zero.

11-6 Conservation of Angular Momentum

Example 11-11: Kepler’s second law derived.

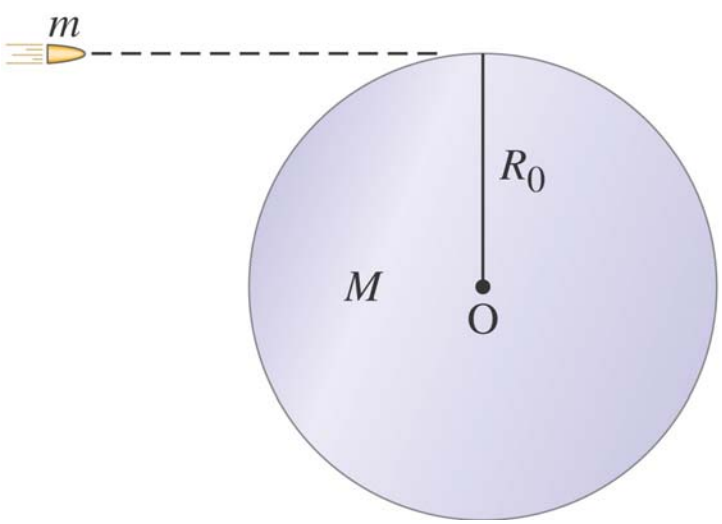
Kepler’s second law states that each planet moves so that a line from the Sun to the planet sweeps out equal areas in equal times. Use conservation of angular momentum to show this.



11-6 Conservation of Angular Momentum

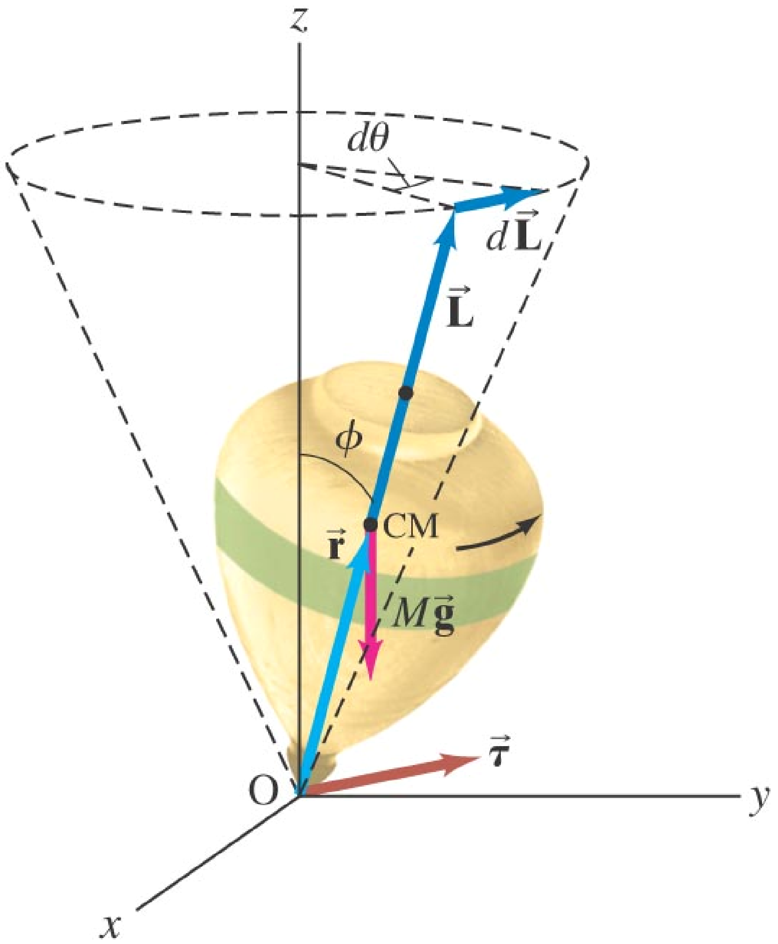
Example 11-12: Bullet strikes cylinder edge.

A bullet of mass m moving with velocity v strikes and becomes embedded at the edge of a cylinder of mass M and radius R_0. The cylinder, initially at rest, begins to rotate about its symmetry axis, which remains fixed in position. Assuming no frictional torque, what is the angular velocity of the cylinder after this collision? Is kinetic energy conserved?



11-7 The Spinning Top and Gyroscope

A spinning top will precess around its point of contact with a surface, due to the torque created by gravity when its axis of rotation is not vertical.

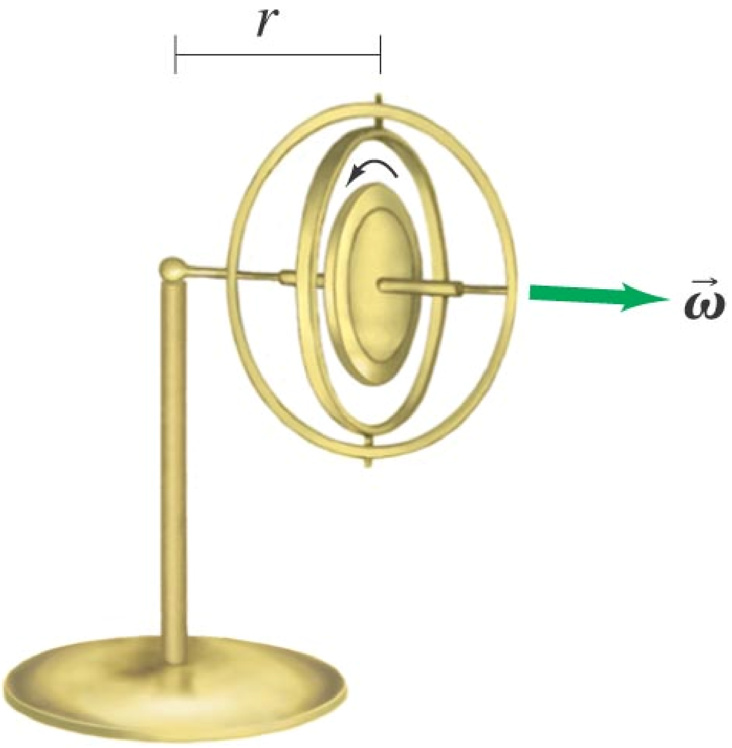


11-7 The Spinning Top and Gyroscope

The angular velocity of the precession is given by:

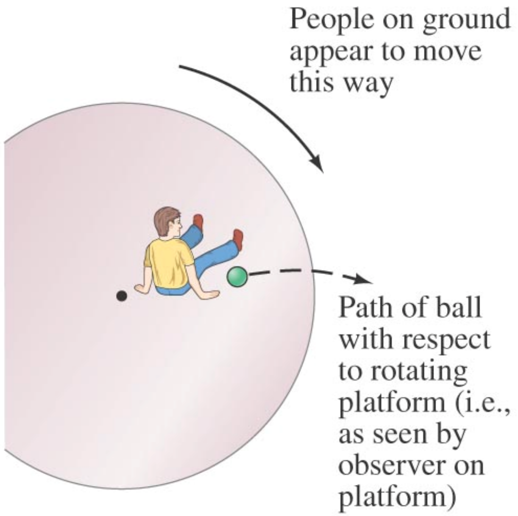
$$\Omega = \frac{Mgr}{I\omega}.$$

This is also the angular velocity of precession of a toy gyroscope, as shown.

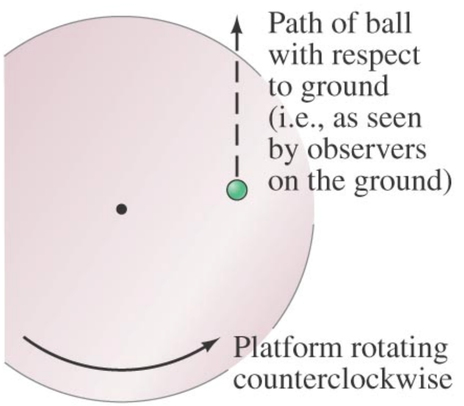


11-8 Rotating Frames of Reference; Inertial Forces

An inertial frame of reference is one in which Newton's laws hold; a rotating frame of reference is noninertial, and objects viewed from such a frame may move without a force acting on them.



Rotating reference frame



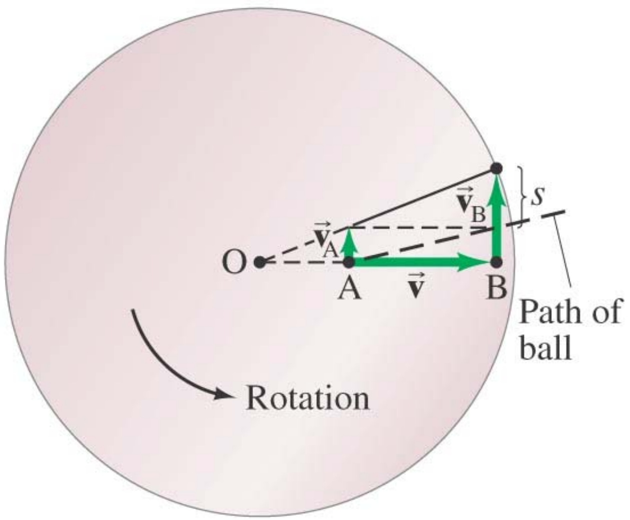
Inertial reference frame

11-8 Rotating Frames of Reference; Inertial Forces

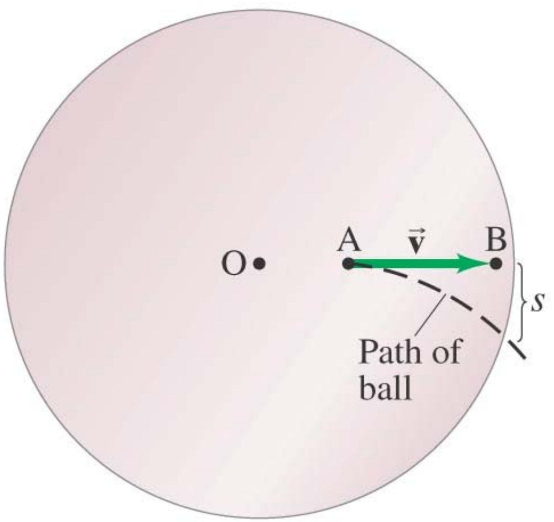
There is an apparent outward force on objects in rotating reference frames; this is a fictitious force, or a pseudoforce. The centrifugal “force” is of this type; there is no outward force when viewed from an inertial reference frame.

11-9 The Coriolis Effect

If an object is moving in a noninertial reference frame, there is another pesudoforce on it, as the tangential speed does not increase while the object moves farther from the axis of rotation. This results in a sideways drift.



Inertial reference frame

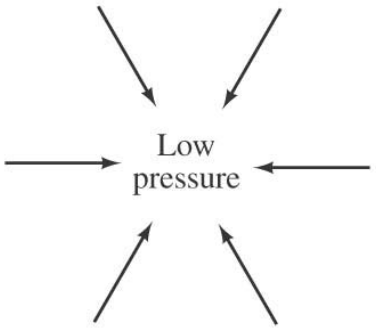


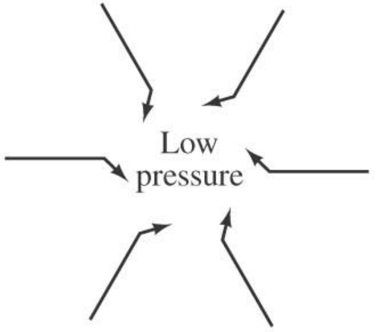
Rotating reference frame

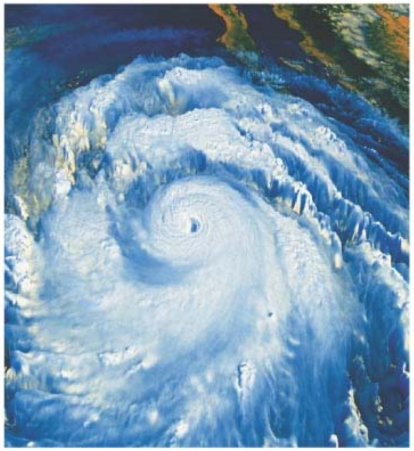
11-9 The Coriolis Effect

The Coriolis effect is responsible for the rotation of air around low-pressure areas—counterclockwise in the Northern Hemisphere and clockwise in the Southern. The Coriolis acceleration is:

$$a_{\mathrm{Cor}} = 2\omega v.$$







Summary of Chapter 11

- **Angular momentum of a rigid object:**

$$L = I\omega.$$

- **Newton's second law:**

$$\Sigma\tau = \frac{dL}{dt}.$$

- **Angular momentum is conserved.**

- **Torque:**

$$\vec{\boldsymbol{\tau}} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}.$$

Summary of Chapter 11

- Angular momentum of a particle:

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}.$$

- Net torque:

$$\Sigma\vec{\boldsymbol{\tau}} = \frac{d\vec{\mathbf{L}}}{dt}.$$

- If the net torque is zero, the vector angular momentum is conserved.