Chapter 11.- Mixed-Strategy Nash Equilibrium

- As we have seen, some games do not have a Nash equilibrium in **pure strategies**.
- However, existence of Nash equilibrium would follow if we extend this notion to **mixed strategies**.
- All we need is for each player's mixed strategy to be a best response to the mixed strategies of all other players.

- **Example: Matching pennies game.-** We saw before that this game does not have a Nash equilibrium in pure strategies.

1 \ 2	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Matching Pennies

- Intuitively: Given the "pure conflict" nature of the matching pennies game, letting my opponent know for sure which strategy I will choose is never optimal, since this will give my opponent the ability to hurt me for sure.
- This is why **randomizing is optimal.**

- Consider the following profile of mixed strategies:

$$\sigma_1 = \left(\frac{1}{2}, \frac{1}{2}\right) \text{ and } \sigma_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

- Note that

$$u_1(H, \sigma_2) = 1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$$

$$u_1(T, \sigma_2) = -1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 0$$

- And therefore,

$$u_1(\sigma_1, \sigma_2) = \frac{1}{2} \cdot u_1(H, \sigma_2) + \frac{1}{2} \cdot u_1(T, \sigma_2) = 0$$

- Since payoffs are symmetrical, we also have

$$u_2(\sigma_1, \sigma_2) = 0$$

- Note that:
- Each player is ***indifferent*** between his two strategies (H or T) if the other player randomizes according to $\sigma_j = \left(\frac{1}{2}, \frac{1}{2}\right)$ (both *H* and *T* yield a payoff of zero). *Both strategies are best responses to $\sigma_j = \left(\frac{1}{2}, \frac{1}{2}\right)$.*
- Playing the mixed strategy $\sigma_i = \left(\frac{1}{2}, \frac{1}{2}\right)$ also yields a payoff of zero and therefore is also a best response to $\sigma_j = \left(\frac{1}{2}, \frac{1}{2}\right)$.

- Therefore, if the other player chooses H or T with probability ½ each, then each player is perfectly content with also randomizing between H and T with probability ½.

- This constitutes a Nash equilibrium in mixed strategies.

- **Definition:** Consider a (mixed) strategy profile
$$\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$$
where σ_i is a mixed strategy for player i. The profile σ is a **mixed-strategy Nash equilibrium** if and only if **playing $\boldsymbol{\sigma_i}$ is a best response to $\boldsymbol{\sigma_{-i}}$**. That is:
$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \text{ for each } s'_i \in S_i$$

- **Fact #1 about mixed-strategy Nash Equilibrium:** A mixed strategy is σ_i is a best response to σ_{-i} only if σ_i assigns positive probability exclusively to strategies $s_i \in S_i$ that are best-responses to σ_{-i}.

- **Facts about mixed-strategy Nash equilibria:**

1. In any mixed-strategy Nash equilibrium $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, **players assign positive probability only to rationalizable strategies**. That is, $\sigma_i(s_i) > 0$ only if s_i is rationalizable.
2. In any mixed-strategy Nash equilibrium $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, the mixed strategy σ_i assigns positive probability exclusively to strategies $s_i \in S_i$ that are best-responses to σ_{-i}. That is:

   If $\sigma_i(s_i) > 0$, then it must be that:

   $$u_i(s_i, \sigma_{-i}) \geq u_i(s'_i, \sigma_{-i}) \text{ for every } s'_i \in S_i.$$

3. In any mixed-strategy Nash equilibrium $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, each player i is ***indifferent*** between all the strategies s_i that he can play with positive probability according to σ_i. That is, for each $i = 1, \dots, n$:

$$u_i(s_i, \sigma_{-i}) = u_i(s'_i, \sigma_{-i})$$

for all s_i, s'_i such that $\sigma_i(s_i) > 0$ and $\sigma_i(s'_i) > 0$

- Using these facts, we can characterize a step-by-step procedure to find mixed-strategy Nash equilibria in two player games (things get a bit more complicated in games with three or more players).

- **Procedure for finding mixed-strategy equilibria in discrete, two-player games:**

1. **Step 1:** Find the set of rationalizable strategies in the game using iterated dominance.
2. **Step 2:** Restricting attention to rationalizable strategies, write equations for each player to characterize mixing distributions that make each player indifferent between the relevant pure strategies.
3. **Step 3:** Solve these equations to determine equilibrium mixing distributions.

- **Example: A lobbying game.-** Suppose two firms simultaneously and independently decide whether to lobby (L) or not lobby (N) the government in hopes of trying to generate favorable legislation. Suppose payoffs are:

X \ Y	L	N
L	–5, –5	25, 0
N	0, 15	10, 10

- This game has two pure-strategy Nash equilibria:

X \ Y	L	N
L	−5, −5	25, 0
N	0, 15	10, 10

- **Question:** Does it also have a mixed-strategy Nash equilibrium?
- Since this game has only two players and two strategies, this question is easy to answer.
- **Step 1:** Note that both strategies are rationalizable for each player.

- **Step 2:** With only two players and two strategies, a profile of mixed strategies σ_1, σ_2 is a Nash equilibrium if and only if:

I. Player 1 is indifferent between L and N when player 2 uses σ_2.

II. Player 2 is indifferent between L and N when player 1 uses σ_1.

- That is, if and only if σ_1, σ_2 are such that:

$$u_1(L, \sigma_2) = u_1(N, \sigma_2)$$

and

$$u_2(\sigma_1, L) = u_2(\sigma_1, N)$$

- Since each player has only two strategies (L and N), any mixed strategy is fully described by

$$\sigma_i = (\sigma_i(L), 1 - \sigma_i(L))$$

- Where:

$$\sigma_i(L) = \Pr(Player\ i\ chooses\ L)$$
$$1 - \sigma_i(L) = \Pr(Player\ i\ chooses\ N)$$

- Therefore,

$$u_1(L, \sigma_2) = -5 \cdot \sigma_2(L) + 25 \cdot \left(1 - \sigma_2(L)\right) = 25 - 30 \cdot \sigma_2(L)$$
$$u_1(N, \sigma_2) = 0 \cdot \sigma_2(L) + 10 \cdot \left(1 - \sigma_2(L)\right) = 10 - 10 \cdot \sigma_2(L)$$
$$u_2(\sigma_1, L) = -5 \cdot \sigma_1(L) + 15 \cdot \left(1 - \sigma_1(L)\right) = 15 - 20 \cdot \sigma_1(L)$$
$$u_2(\sigma_1, N) = 0 \cdot \sigma_1(L) + 10 \cdot \left(1 - \sigma_1(L)\right) = 10 - 10 \cdot \sigma_1(L)$$

- In any mixed-strategy Nash equilibrium, we must have $u_1(L, \sigma_2) = u_1(N, \sigma_2)$. That is:

$$25 - 30 \cdot \sigma_2(L) = 10 - 10 \cdot \sigma_2(L)$$

- This will be satisfied if:

$$\sigma_2(L) = \frac{3}{4}$$

- And we also must have $u_2(\sigma_1, L) = u_2(\sigma_1, N)$. That is:

$$15 - 20 \cdot \sigma_1(L) = 10 - 10 \cdot \sigma_1(L)$$

- This will be satisfied if:

$$\sigma_1(L) = \frac{1}{2}$$

- Therefore, this game has a mixed-strategy equilibrium (σ_1, σ_2), where:

$$\sigma_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and

$$\sigma_2 = \left(\frac{3}{4}, \frac{1}{4}\right)$$

- This example also illustrates that some games may have Nash equilibria in pure strategies AND also in mixed strategies.

- **Example: A tennis-service game.-** Consider two tennis players.

- Player 1 (the server) must decide whether to serve to the opponent’s forehand (F), center (C) or backhand (B).

- Simultaneously, Player 2 (the receiver) must decide whether to favor the forehand, center of backhand side.

- Suppose payoffs are given by:

1 \ 2	F	C	B
F	0, 5	2, 3	2, 3
C	2, 3	0, 5	3, 2
B	5, 0	3, 2	2, 3

- We begin by noting that this game does not have any pure-strategy Nash equilibrium.

- To see why, note that best-responses are given by:

1 \ 2	F	C	B
F	0, 5	2, 3	2, 3
C	2, 3	0, 5	3, 2
B	5, 0	3, 2	2, 3

- So there is no pair of mutual best-responses in pure strategies.

- **Question:** Find the mixed-strategy Nash equilibria in this game.
- **Step 1:** Using iterated dominance, find the set of rationalizable strategies *R.*
  - To find the reduced game R^1:
  - Note first that all three strategies $\{F, C, B\}$ are best-responses for player 2, so they will all survive.
  - For player 1, $\{C, B\}$ are best-responses. And we can show easily that F is dominated by a mixed strategy between $\{C, B\}$. From here, we have:

$$R^1 = \{C, B\} \times \{F, C, B\}$$

- (cont…)
  - To find R^2, we note that in the reduced game R^1, the only dominated strategy is F, for player 2. Player 1 does not have any dominated strategy in R^1. Therefore,

$$R^2 = \{C, B\} \times \{C, B\}$$

  - It is easy to verify that there are no dominated strategies in R^2. Therefore the game cannot be reduced any further and we have

$$R = \{C, B\} \times \{C, B\}$$

- The set of rationalizable strategies is:

1 \ 2	C	B
C	0, 5	3, 2
B	3, 2	2, 3

- To find mixed-strategy Nash equilibria, we need to look for mixing distributions:

$$\sigma_1 = \left(0, \sigma_1(C), 1 - \sigma_1(C)\right)$$
$$\sigma_2 = \left(0, \sigma_2(C), 1 - \sigma_2(C)\right)$$

(where each player randomizes only between “C” and “B” and play “F” with zero probability) such that both players are indifferent between C and B.

- That is, we must have:

$$u_1(C, \sigma_2) = u_1(B, \sigma_2)$$

and

$$u_2(\sigma_1, C) = u_2(\sigma_1, B)$$

- Expected payoffs are given by:

$$u_1(C, \sigma_2) = 0 \cdot \sigma_2(C) + 3 \cdot \big(1 - \sigma_2(C)\big) = 3 - 3 \cdot \sigma_2(C)$$
$$u_1(B, \sigma_2) = 3 \cdot \sigma_2(C) + 2 \cdot \big(1 - \sigma_2(C)\big) = 2 + 1 \cdot \sigma_2(C)$$
$$u_2(\sigma_1, C) = 5 \cdot \sigma_1(C) + 2 \cdot \big(1 - \sigma_1(C)\big) = 2 + 3 \cdot \sigma_1(C)$$
$$u_2(\sigma_1, B) = 2 \cdot \sigma_1(C) + 3 \cdot \big(1 - \sigma_1(C)\big) = 3 - 1 \cdot \sigma_1(C)$$

- Therefore, $\sigma_1(C)$ and $\sigma_2(C)$ need to satisfy:

$$3 - 3 \cdot \sigma_2(C) = 2 + 1 \cdot \sigma_2(C)$$

and

$$2 + 3 \cdot \sigma_1(C) = 3 - 1 \cdot \sigma_1(C)$$

- This yields:

$$\sigma_2(C) = \frac{1}{4} \quad \text{and} \quad \sigma_1(C) = \frac{1}{4}$$

- Therefore, the mixed-strategy Nash equilibrium in this game is given by the mixing distributions:

$$\sigma_1 = \left(0, \frac{1}{4}, \frac{3}{4}\right)$$

and

$$\sigma_2 = \left(0, \frac{1}{4}, \frac{3}{4}\right)$$

- **Example:** Find the set of Nash equilibria (pure and mixed) in this game:

1 \ 2	X	Y	Z
U	2, 0	1, 1	4, 2
M	3, 4	1, 2	2, 3
D	1, 3	0, 2	3, 0

(b)

- We begin with the pure-strategy equilibria:

1 \ 2	X	Y	Z
U	2, 0	1, 1	4, 2
M	3, 4	1, 2	2, 3
D	1, 3	0, 2	3, 0

(b)

- **Mixed-strategy equilibria**: first, using iterated dominance we look for the set of rationalizable strategies *R*
  - Player 1: “M” is a best response to “X” and “Y”, while “U” is a best response to “Z”. The strategy “D” is dominated by “U”.
  - Player 2: “Z” is a best response to “U”, “X” is a best response to “M” and “D”.
  - We need to check if “Y” is a dominated strategy. Same procedure we followed in Chapter 6 shows that it is NOT a dominated strategy.
  - Therefore:

$$R^1 = \{U, M\} \times \{X, Y, Z\}$$

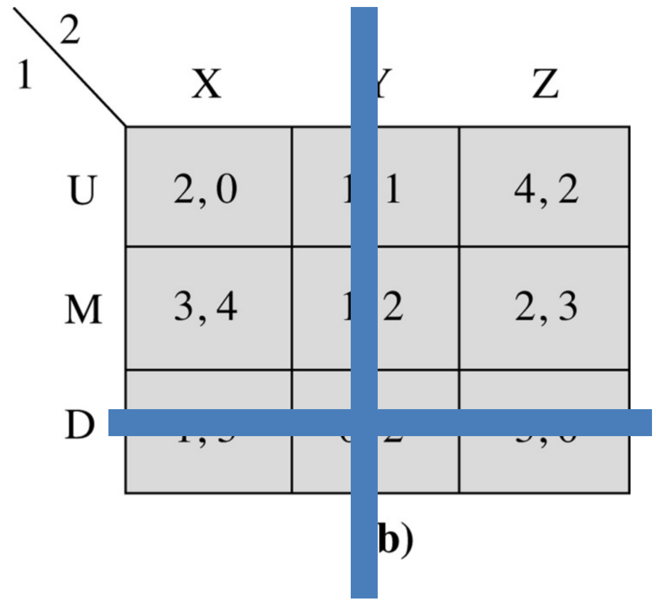
- Matrix form of the reduced game R^1 is:

1 \ 2	X	Y	Z
U	2, 0	1, 1	4, 2
M	3, 4	1, 2	2, 3
D	[illegible]	[illegible]	[illegible]

(b)

  - Player 1 has no dominated strategies in the reduced game given by R^1.
  - For Player 2, “Y” is dominated by “Z” in the reduced game given by R^1.
  - Therefore, $R^2 = \{U, M\} \times \{X, Z\}$.

- $R^2 = \{U, M\} \times \{X, Z\}$. Reduced game:



b)

  - Player 1 has no dominated strategies in the reduced game given by R^2.
  - Player 2 has no dominated strategies in the reduced game given by R^2.
  - Therefore, no further reduction can be done and we have $R^2 = R$. Therefore,

$$\boldsymbol{R = \{U, M\} \times \{X, Z\} = \{(U, X), (U, Z), (M, X), (M, Z)\}}$$

- Focusing on the rationalizable strategies *R,* we now need to find well-defined mixing probabilities

$$\sigma_1 = (\sigma_1(U), 1 - \sigma_1(U), 0)$$
$$\sigma_2 = (\sigma_2(X), 0, 1 - \sigma_2(X))$$

such that both players are indifferent between their actions (X and Z for player 2, and U and M for player 1). That is:

$$u_1(U, \sigma_2) = u_1(M, \sigma_2)$$

and

$$u_2(\sigma_1, X) = u_2(\sigma_1, Z)$$

- The reduced game R looks like this:

1 \ 2	X	Z
U	2, 0	4, 2
M	3, 4	2, 3

- From here we have:

$$u_1(U,\sigma_2) = 2\cdot\sigma_2(X) + 4\cdot\big(1-\sigma_2(X)\big) = 4 - 2\cdot\sigma_2(X)$$
$$u_1(M,\sigma_2) = 3\cdot\sigma_2(X) + 2\cdot\big(1-\sigma_2(X)\big) = 2 + 1\cdot\sigma_2(X)$$

- And:

$$u_2(\sigma_1,X) = 0\cdot\sigma_1(U) + 4\cdot\big(1-\sigma_1(U)\big) = 4 - 4\cdot\sigma_1(U)$$
$$u_2(\sigma_1,Z) = 2\cdot\sigma_1(U) + 3\cdot\big(1-\sigma_1(U)\big) = 3 - 1\cdot\sigma_1(U)$$

- Both players will be indifferent between their relevant strategies if and only if:

$4 - 2 \cdot \sigma_2(X) = 2 + 1 \cdot \sigma_2(X)$ (for player 1)

$4 - 4 \cdot \sigma_1(U) = 3 - 1 \cdot \sigma_1(U)$ (for player 2)

- The first condition will hold if and only if

$$\sigma_2(X) = \frac{2}{3}$$

- And the second condition will hold if and only if

$$\sigma_1(U) = \frac{1}{3}$$

- Therefore, this game has one mixed-strategy Nash equilibrium where players randomize according to the distributions:

$$\sigma_1 = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$$

$$\sigma_2 = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$$

- **Mixed-strategy Nash Equilibrium in Continuous Games:** As in discrete games, the key feature is that players must randomize in a way that makes other players indifferent between their relevant strategies.
- **Example: Bertrand competition with capacity constraints.**
- Consider a duopoly industry of a homogenous good with two firms who compete in prices.
- Suppose the market consists of **10 consumers**, each of which will purchase **one unit of the good**. Suppose that **each consumer is willing to pay at most $1 for the good**.

- For simplicity, suppose the **production cost is zero for both firms**.
- If this setup fully describes the model, then it is a very simple case of **Bertrand competition**. As we learned previously, **the equilibrium prices would be those that yield a profit of zero**.
- Since production cost is zero, this mean that the Nash equilibrium prices would be:

$$p_1 = 0 \text{ and } p_2 = 0$$

as we learned previously, this would be the UNIQUE Nash equilibrium in the game.

- Suppose now that both firms have a **capacity constraint**. Specifically, suppose each firm can produce **at most eight units of the good**.
- This will change the features of the model drastically: Now **the firm with the cheapest price cannot capture the entire market because of the capacity constraint**.
- Conversely, **the firm with the highest price can still capture two consumers**.
- As a result, the Nash equilibrium properties of this model will change. As we will see, it will no longer have an equilibrium in pure strategies. Instead, it will have a unique equilibrium in mixed strategies.

- **With capacity constraints, the game no longer has an equilibrium in pure strategies:** We begin by noting that by setting the highest possible price ($p_i = 1$), firm i ensures itself a profit of at least \$2 (since at the very least it will sell two units due to the capacity constraint of the opponent).
- Suppose $p_1 = p_2 > 0$. Can this be a Nash equilibrium? No, because it would be better for either firm to undercut the other firm's price by an infinitesimal amount. This will always yield a higher payoff than choosing the same price as the opponent.

- Suppose $p_1 = p_2 = 0$. Can this be a Nash equilibrium? **It used to be the Nash equilibrium without capacity constraints, but not any more**. Why? Because if my opponent sets a price of zero, my best response now is to set a price of $1. This will ensure me a profit of $2 instead of $0, which is what I would obtain if I set my price to zero.
- Therefore, combining the two cases above, there cannot be a Nash equilibrium in pure strategies where $p_1 = p_2$

- Can there be a pure-strategy Nash equilibrium in which $p_i < p_j \leq 1$? First note that if one firm chooses a price higher than the other firm, then the only rational price to choose is the highest possible price (since you would have two captive costumers).
- That is, **if $p_i < p_j$ in equilibrium, then it must be the case that $p_j = 1$**. But if **$p_j = 1$, it is not optimal for firm i to charge strictly less than 1. Firm i would like to keep raising p_i by infinitesimal amounts to become closer and closer to $1. So the best response by i would not be well-defined.**

- Therefore since there is no pure-strategy Nash equilibrium where $p_1 = p_2$ and there is no pure-strategy Nash equilibrium where $p_i < p_j$, **we conclude that this game does not possess a pure-strategy Nash equilibrium.**
- How about a **mixed-strategy Nash equilibrium**?
- Notice that the strategy space is continuous, which makes the problem a bit “trickier”. Still, we can describe the mixed-strategy Nash equilibrium using the same principle as in discrete games: **In equilibrium, both players must be indifferent between all their relevant strategies**.