GLENCOE MATHEMATICS

Algebra 2

Chapter 11
Resource Masters

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Consumable Workbooks

Many of the worksheets contained in the Chapter Resource Masters booklets are available as consumable workbooks.

Study Guide and Intervention Workbook	0-07-828029-X
Skills Practice Workbook	0-07-828023-0
Practice Workbook	0-07-828024-9

ANSWERS FOR WORKBOOKS The answers for Chapter 11 of these workbooks can be found in the back of this Chapter Resource Masters booklet.

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Algebra 2
Chapter 11 Resource Masters

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Contents

Teacher's Guide to Using the Chapter 11 Resource Masters

The ***Fast File*** Chapter Resource system allows you to conveniently file the resources you use most often. The *Chapter 11 Resource Masters* includes the core materials needed for Chapter 11. These materials include worksheets, extensions, and assessment options. The answers for these pages appear at the back of this booklet.

All of the materials found in this booklet are included for viewing and printing in the *Algebra 2 TeacherWorks* CD-ROM.

Vocabulary Builder Pages vii–viii include a student study tool that presents up to twenty of the key vocabulary terms from the chapter. Students are to record definitions and/or examples for each term. You may suggest that students highlight or star the terms with which they are not familiar.

WHEN TO USE Give these pages to students before beginning Lesson 11-1. Encourage them to add these pages to their Algebra 2 Study Notebook. Remind them to add definitions and examples as they complete each lesson.

Study Guide and Intervention

Each lesson in *Algebra 2* addresses two objectives. There is one Study Guide and Intervention master for each objective.

WHEN TO USE Use these masters as reteaching activities for students who need additional reinforcement. These pages can also be used in conjunction with the Student Edition as an instructional tool for students who have been absent.

Skills Practice There is one master for each lesson. These provide computational practice at a basic level.

WHEN TO USE These masters can be used with students who have weaker mathematics backgrounds or need additional reinforcement.

Practice There is one master for each lesson. These problems more closely follow the structure of the Practice and Apply section of the Student Edition exercises. These exercises are of average difficulty.

WHEN TO USE These provide additional practice options or may be used as homework for second day teaching of the lesson.

Reading to Learn Mathematics

One master is included for each lesson. The first section of each master asks questions about the opening paragraph of the lesson in the Student Edition. Additional questions ask students to interpret the context of and relationships among terms in the lesson. Finally, students are asked to summarize what they have learned using various representation techniques.

WHEN TO USE This master can be used as a study tool when presenting the lesson or as an informal reading assessment after presenting the lesson. It is also a helpful tool for ELL (English Language Learner) students.

Enrichment There is one extension master for each lesson. These activities may extend the concepts in the lesson, offer an historical or multicultural look at the concepts, or widen students' perspectives on the mathematics they are learning. These are not written exclusively for honors students, but are accessible for use with all levels of students.

WHEN TO USE These may be used as extra credit, short-term projects, or as activities for days when class periods are shortened.

Assessment Options

The assessment masters in the *Chapter 11 Resource Masters* offer a wide range of assessment tools for intermediate and final assessment. The following lists describe each assessment master and its intended use.

Chapter Assessment

CHAPTER TESTS

- *Form 1* contains multiple-choice questions and is intended for use with basic level students.
- *Forms 2A and 2B* contain multiple-choice questions aimed at the average level student. These tests are similar in format to offer comparable testing situations.
- *Forms 2C and 2D* are composed of free-response questions aimed at the average level student. These tests are similar in format to offer comparable testing situations. Grids with axes are provided for questions assessing graphing skills.
- *Form 3* is an advanced level test with free-response questions. Grids without axes are provided for questions assessing graphing skills.

  All of the above tests include a free-response Bonus question.
- The **Open-Ended Assessment** includes performance assessment tasks that are suitable for all students. A scoring rubric is included for evaluation guidelines. Sample answers are provided for assessment.
- A **Vocabulary Test**, suitable for all students, includes a list of the vocabulary words in the chapter and ten questions assessing students' knowledge of those terms. This can also be used in conjunction with one of the chapter tests or as a review worksheet.

Intermediate Assessment

- Four free-response **quizzes** are included to offer assessment at appropriate intervals in the chapter.
- A **Mid-Chapter Test** provides an option to assess the first half of the chapter. It is composed of both multiple-choice and free-response questions.

Continuing Assessment

- The **Cumulative Review** provides students an opportunity to reinforce and retain skills as they proceed through their study of Algebra 2. It can also be used as a test. This master includes free-response questions.
- The **Standardized Test Practice** offers continuing review of algebra concepts in various formats, which may appear on the standardized tests that they may encounter. This practice includes multiple-choice, grid-in, and quantitative-comparison questions. Bubble-in and grid-in answer sections are provided on the master.

Answers

- Page A1 is an answer sheet for the Standardized Test Practice questions that appear in the Student Edition on pages 628–629. This improves students' familiarity with the answer formats they may encounter in test taking.
- The answers for the lesson-by-lesson masters are provided as reduced pages with answers appearing in red.
- Full-size answer keys are provided for the assessment masters in this booklet.

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11

Reading to Learn Mathematics

Vocabulary Builder

This is an alphabetical list of the key vocabulary terms you will learn in Chapter 11. As you study the chapter, complete each term's definition or description. Remember to add the page number where you found the term. Add these pages to your Algebra Study Notebook to review vocabulary at the end of the chapter.

Vocabulary Term	Found on Page	Definition/Description/Example
arithmetic mean AR·ihth·MEH·tihk		
arithmetic sequence		
arithmetic series		
Binomial Theorem		
common difference		
common ratio		
factorial		
Fibonacci sequence fih·buh·NAH·chee		
geometric mean		
geometric sequence		

(continued on the next page)

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11 Reading to Learn Mathematics

Vocabulary Builder *(continued)*

Vocabulary Term	Found on Page	Definition/Description/Example
geometric series		
index of summation		
inductive hypothesis		
infinite geometric series		
iteration IH·tuh·RAY·shuhn		
mathematical induction		
partial sum		
Pascal's triangle pas·KALZ		
recursive formula rih·KUHR·sihv		
sigma notation SIHG·muh		

11-1 Study Guide and Intervention

Arithmetic Sequences

Arithmetic Sequences An **arithmetic sequence** is a sequence of numbers in which each **term** after the first term is found by adding the **common difference** to the preceding term.

***n*th Term of an Arithmetic Sequence**	$a_n = a_1 + (n - 1)d$, where a_1 is the first term, d is the common difference, and n is any positive integer

Example 1 **Find the next four terms of the arithmetic sequence 7, 11, 15, ...**

Find the common difference by subtracting two consecutive terms.

$11 - 7 = 4$ and $15 - 11 = 4$, so $d = 4$.

Now add 4 to the third term of the sequence, and then continue adding 4 until the four terms are found. The next four terms of the sequence are 19, 23, 27, and 31.

Example 2 **Find the thirteenth term of the arithmetic sequence with $a_1 = 21$ and $d = -6$.**

Use the formula for the nth term of an arithmetic sequence with $a_1 = 21$, $n = 13$, and $d = -6$.

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_{13} = 21 + (13 - 1)(-6)$ $n = 13$, $a_1 = 21$, $d = -6$

$a_{13} = -51$ Simplify.

The thirteenth term is −51.

Example 3 **Write an equation for the *n*th term of the arithmetic sequence −14, −5, 4, 13, ...**

In this sequence $a_1 = -14$ and $d = 9$. Use the formula for a_n to write an equation.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$= -14 + (n - 1)9$ $a_1 = -14$, $d = 9$

$= -14 + 9n - 9$ Distributive Property

$= 9n - 23$ Simplify.

Evens

Exercises

Find the next four terms of each arithmetic sequence.

1. 106, 111, 116, ... **2.** −28, −31, −34, ... **3.** 207, 194, 181, ...

Find the first five terms of each arithmetic sequence described.

4. $a_1 = 101$, $d = 9$ **5.** $a_1 = -60$, $d = 4$ **6.** $a_1 = 210$, $d = -40$

Find the indicated term of each arithmetic sequence.

7. $a_1 = 4$, $d = 6$, $n = 14$ **8.** $a_1 = -4$, $d = -2$, $n = 12$

9. $a_1 = 80$, $d = -8$, $n = 21$ **10.** a_{10} for 0, −3, −6, −9, ...

Write an equation for the *n*th term of each arithmetic sequence.

11. 18, 25, 32, 39, ... **12.** −110, −85, −60, −35, ... **13.** 6.2, 8.1, 10.0, 11.9, ...

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11-1 Study Guide and Intervention *(continued)*

Arithmetic Sequences

Arithmetic Means The **arithmetic means** of an arithmetic sequence are the terms between any two nonsuccessive terms of the sequence.

To find the k arithmetic means between two terms of a sequence, use the following steps.

Step 1 Let the two terms given be a_1 and a_n, where $n = k + 2$.
Step 2 Substitute in the formula $a_n = a_1 + (n - 1)d$.
Step 3 Solve for d, and use that value to find the k arithmetic means:
$a_1 + d, a_1 + 2d, \ldots, a_1 + kd$.

Example **Find the five arithmetic means between 37 and 121.**

You can use the nth term formula to find the common difference. In the sequence, 37, ?, ?, ?, ?, ?, 121, …, a_1 is 37 and a_7 is 121.

$a_n = a_1 + (n - 1)d$ Formula for the nth term
$121 = 37 + (7 - 1)d$ $a_1 = 37, a_7 = 121, n = 7$
$121 = 37 + 6d$ Simplify.
$84 = 6d$ Subtract 37 from each side.
$d = 14$ Divide each side by 6.

Now use the value of d to find the five arithmetic means.

$$37 \xrightarrow[+14]{} 51 \xrightarrow[+14]{} 65 \xrightarrow[+14]{} 79 \xrightarrow[+14]{} 93 \xrightarrow[+14]{} 107 \xrightarrow[+14]{} 121$$

The arithmetic means are 51, 65, 79, 93, and 107.

2-10even

Exercises

Find the arithmetic means in each sequence.

1. 5, ?, ?, ?, −3
2. 18, ?, ?, ?, −2
3. 16, ?, ?, 37
4. 108, ?, ?, ?, ?, 48
5. −14, ?, ?, ?, −30
6. 29, ?, ?, ?, 89
7. 61, ?, ?, ?, ?, 116
8. 45, ?, ?, ?, ?, ?, 81
9. −18, ?, ?, ?, 14
10. −40, ?, ?, ?, ?, ?, −82
11. 100, ?, ?, 235
12. 80, ?, ?, ?, ?, −30
13. 450, ?, ?, ?, 570
14. 27, ?, ?, ?, ?, ?, 57
15. 125, ?, ?, ?, 185
16. 230, ?, ?, ?, ?, ?, 128
17. −20, ?, ?, ?, ?, 370
18. 48, ?, ?, ?, 100

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11-1 Skills Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 7, 11, 15, …

2. −10, −5, 0, …

3. 101, 202, 303, …

4. 15, 7, −1, …

5. −67, −60, −53, …

6. −12, −15, −18, …

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 6, d = 9$

8. $a_1 = 27, d = 4$

9. $a_1 = -12, d = 5$

10. $a_1 = 93, d = -15$

11. $a_1 = -64, d = 11$

12. $a_1 = -47, d = -20$

Find the indicated term of each arithmetic sequence.

13. $a_1 = 2, d = 6, n = 12$

14. $a_1 = 18, d = 2, n = 8$

15. $a_1 = 23, d = 5, n = 23$

16. $a_1 = 15, d = -1, n = 25$

17. a_{31} for 34, 38, 42, …

18. a_{42} for 27, 30, 33, …

Complete the statement for each arithmetic sequence.

19. 55 is the ___?___th term of 4, 7, 10, … .

20. 163 is the ___?___th term of −5, 2, 9, … .

Write an equation for the *n*th term of each arithmetic sequence.

21. 4, 7, 10, 13, …

22. −1, 1, 3, 5, …

23. −1, 3, 7, 11, …

24. 7, 2, −3, −8, …

Find the arithmetic means in each sequence.

25. 6, ___?___, ___?___, ___?___, 38

26. 63, ___?___, ___?___, ___?___, 147

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11-1 Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 5, 8, 11, …

2. $-4, -6, -8, \ldots$

3. 100, 93, 86, …

4. $-24, -19, -14, \ldots$

5. $\frac{7}{2}, 6, \frac{17}{2}, 11, \ldots$

6. 4.8, 4.1, 3.4, …

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 7, d = 7$

8. $a_1 = -8, d = 2$

9. $a_1 = -12, d = -4$

10. $a_1 = \frac{1}{2}, d = \frac{1}{2}$

11. $a_1 = -\frac{5}{6}, d = -\frac{1}{3}$

12. $a_1 = 10.2, d = -5.8$

Find the indicated term of each arithmetic sequence.

13. $a_1 = 5, d = 3, n = 10$

14. $a_1 = 9, d = 3, n = 29$

15. a_{18} for $-6, -7, -8, \ldots.$

16. a_{37} for 124, 119, 114, … .

17. $a_1 = \frac{9}{5}, d = -\frac{3}{5}, n = 10$

18. $a_1 = 14.25, d = 0.15, n = 31$

Complete the statement for each arithmetic sequence.

19. 166 is the __?__th term of 30, 34, 38, …

20. 2 is the __?__th term of $\frac{3}{5}, \frac{4}{5}, 1, \ldots$

Write an equation for the *n*th term of each arithmetic sequence.

21. $-5, -3, -1, 1, \ldots$

22. $-8, -11, -14, -17, \ldots$

23. $1, -1, -3, -5, \ldots$

24. $-5, 3, 11, 19, \ldots$

Find the arithmetic means in each sequence.

25. -5, __?__, __?__, __?__, 11

26. 82, __?__, __?__, __?__, 18

27. EDUCATION Trevor Koba has opened an English Language School in Isehara, Japan. He began with 26 students. If he enrolls 3 new students each week, in how many weeks will he have 101 students?

28. SALARIES Yolanda interviewed for a job that promised her a starting salary of $32,000 with a $1250 raise at the end of each year. What will her salary be during her sixth year if she accepts the job?

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11-1 Reading to Learn Mathematics

Arithmetic Sequences

Pre-Activity **How are arithmetic sequences related to roofing?**

Read the introduction to Lesson 11-1 at the top of page 578 in your textbook.

Describe how you would find the number of shingles needed for the fifteenth row. (Do not actually calculate this number.) Explain why your method will give the correct answer.

Reading the Lesson

1. Consider the formula $a_n = a_1 + (n - 1)d$.

a. What is this formula used to find?

b. What do each of the following represent?

a_n: ______________________________

a_1: ______________________________

n: ______________________________

d: ______________________________

2. Consider the equation $a_n = -3n + 5$.

a. What does this equation represent?

b. Is the graph of this equation a straight line? Explain your answer.

c. The functions represented by the equations $a_n = -3n + 5$ and $f(x) = -3x + 5$ are alike in that they have the same formula. How are they different?

Helping You Remember

3. A good way to remember something is to explain it to someone else. Suppose that your classmate Shala has trouble remembering the formula $a_n = a_1 + (n - 1)d$ correctly. She thinks that the formula should be $a_n = a_1 + nd$. How would you explain to her that she should use $(n - 1)d$ rather than nd in the formula?

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11-1 Enrichment

Fibonacci Sequence

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in n months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.
2. Each pair of rabbits produces one pair each month.
3. No rabbits die.

Let F_n represent the number of pairs of rabbits at the end of n months. If you begin with one pair of newborn rabbits, $F_0 = F_1 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1$, or 2. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1$, or 3.

The chart below shows the number of rabbits each month for several months.

Month	Adult Pairs	Newborn Pairs	Total
F_0	0	1	1
F_1	1	0	1
F_2	1	1	2
F_3	2	1	3
F_4	3	2	5
F_5	5	3	8

Solve.

1. Starting with a single pair of newborn rabbits, how many pairs of rabbits would there be at the end of 12 months?

2. Write the first 10 terms of the sequence for which $F_0 = 3$, $F_1 = 4$, and $F_n = F_{n-2} + F_{n-1}$.

3. Write the first 10 terms of the sequence for which $F_0 = 1$, $F_1 = 5$, $F_n = F_{n-2} + F_{n-1}$.

11-2 Study Guide and Intervention

Arithmetic Series

Arithmetic Series An **arithmetic series** is the sum of consecutive terms of an arithmetic sequence.

Sum of an Arithmetic Series	The sum S_n of the first n terms of an arithmetic series is given by the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$

Example 1 **Find S_n for the arithmetic series with $a_1 = 14$, $a_n = 101$, and $n = 30$.**

Use the sum formula for an arithmetic series.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) && \text{Sum formula} \\ S_{30} &= \frac{30}{2}(14 + 101) && n = 30,\ a_1 = 14,\ a_n = 101 \\ &= 15(115) && \text{Simplify.} \\ &= 1725 && \text{Multiply.} \end{aligned}$$

The sum of the series is 1725.

1-3 all, 10-15 all

Example 2 **Find the sum of all positive odd integers less than 180.**

The series is 1 + 3 + 5 + … + 179.

Find n using the formula for the nth term of an arithmetic sequence.

$$\begin{aligned} a_n &= a_1 + (n - 1)d && \text{Formula for } n\text{th term} \\ 179 &= 1 + (n - 1)2 && a_n = 179,\ a_1 = 1,\ d = 2 \\ 179 &= 2n - 1 && \text{Simplify.} \\ 180 &= 2n && \text{Add 1 to each side.} \\ n &= 90 && \text{Divide each side by 2.} \end{aligned}$$

Then use the sum formula for an arithmetic series.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) && \text{Sum formula} \\ S_{90} &= \frac{90}{2}(1 + 179) && n = 90,\ a_1 = 1,\ a_n = 179 \\ &= 45(180) && \text{Simplify.} \\ &= 8100 && \text{Multiply.} \end{aligned}$$

The sum of all positive odd integers less than 180 is 8100.

Exercises

Find S_n for each arithmetic series described.

1. $a_1 = 12$, $a_n = 100$, $n = 12$

2. $a_1 = 50$, $a_n = -50$, $n = 15$

3. $a_1 = 60$, $a_n = -136$, $n = 50$

4. $a_1 = 20$, $d = 4$, $a_n = 112$

5. $a_1 = 180$, $d = -8$, $a_n = 68$

6. $a_1 = -8$, $d = -7$, $a_n = -71$

7. $a_1 = 42$, $n = 8$, $d = 6$

8. $a_1 = 4$, $n = 20$, $d = 2\frac{1}{2}$

9. $a_1 = 32$, $n = 27$, $d = 3$

Find the sum of each arithmetic series.

10. $8 + 6 + 4 + \ldots + -10$

11. $16 + 22 + 28 + \ldots + 112$

12. $-45 + (-41) + (-37) + \ldots + 35$

Find the first three terms of each arithmetic series described.

13. $a_1 = 12$, $a_n = 174$, $S_n = 1767$

14. $a_1 = 80$, $a_n = -115$, $S_n = -245$

15. $a_1 = 6.2$, $a_n = 12.6$, $S_n = 84.6$

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11-2 Study Guide and Intervention *(continued)*

Arithmetic Series

Sigma Notation A shorthand notation for representing a series makes use of the Greek letter Σ. The **sigma notation** for the series 6 + 12 + 18 + 24 + 30 is $\sum_{n=1}^{5} 6n$.

Example **Evaluate $\sum_{k=1}^{18} (3k + 4)$.**

The sum is an arithmetic series with common difference 3. Substituting $k = 1$ and $k = 18$ into the expression $3k + 4$ gives $a_1 = 3(1) + 4 = 7$ and $a_{18} = 3(18) + 4 = 58$. There are 18 terms in the series, so $n = 18$. Use the formula for the sum of an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{18} = \frac{18}{2}(7 + 58)$ $n = 18$, $a_1 = 7$, $a_n = 58$

$= 9(65)$ Simplify.

$= 585$ Multiply.

So $\sum_{k=1}^{18} (3k + 4) = 585$.

1-6 all

Exercises

Find the sum of each arithmetic series.

1. $\sum_{n=1}^{20} (2n + 1)$

2. $\sum_{n=5}^{25} (x - 1)$

3. $\sum_{k=1}^{18} (2k - 7)$

4. $\sum_{r=10}^{75} (2r - 200)$

5. $\sum_{x=1}^{15} (6x + 3)$

6. $\sum_{t=1}^{50} (500 - 6t)$

7. $\sum_{k=1}^{80} (100 - k)$

8. $\sum_{n=20}^{85} (n - 100)$

9. $\sum_{s=1}^{200} 3s$

10. $\sum_{m=14}^{28} (2m - 50)$

11. $\sum_{p=1}^{36} (5p - 20)$

12. $\sum_{j=12}^{32} (25 - 2j)$

13. $\sum_{n=18}^{42} (4n - 9)$

14. $\sum_{n=20}^{50} (3n + 4)$

15. $\sum_{j=5}^{44} (7j - 3)$

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11-2 Skills Practice

Arithmetic Series

Find S_n for each arithmetic series described.

1. $a_1 = 1, a_n = 19, n = 10$

2. $a_1 = -5, a_n = 13, n = 7$

3. $a_1 = 12, a_n = -23, n = 8$

4. $a_1 = 7, n = 11, a_n = 67$

5. $a_1 = 5, n = 10, a_n = 32$

6. $a_1 = -4, n = 10, a_n = -22$

7. $a_1 = -8, d = -5, n = 12$

8. $a_1 = 1, d = 3, n = 15$

9. $a_1 = 100, d = -7, a_n = 37$

10. $a_1 = -9, d = 4, a_n = 27$

11. $d = 2, n = 26, a_n = 42$

12. $d = -12, n = 11, a_n = -52$

Find the sum of each arithmetic series.

13. $1 + 4 + 7 + 10 + \ldots + 43$

14. $5 + 8 + 11 + 14 + \ldots + 32$

15. $3 + 5 + 7 + 9 + \ldots + 19$

16. $-2 + (-5) + (-8) + \ldots + (-20)$

17. $\sum_{n=1}^{5}(2n - 3)$

18. $\sum_{n=1}^{18}(10 + 3n)$

19. $\sum_{n=2}^{10}(4n + 1)$

20. $\sum_{n=5}^{12}(4 - 3n)$

Find the first three terms of each arithmetic series described.

21. $a_1 = 4, a_n = 31, S_n = 175$

22. $a_1 = -3, a_n = 41, S_n = 228$

23. $n = 10, a_n = 41, S_n = 230$

24. $n = 19, a_n = 85, S_n = 760$

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11-2 Practice

Arithmetic Series

Find S_n for each arithmetic series described.

1. $a_1 = 16, a_n = 98, n = 13$

2. $a_1 = 3, a_n = 36, n = 12$

3. $a_1 = -5, a_n = -26, n = 8$

4. $a_1 = 5, n = 10, a_n = -13$

5. $a_1 = 6, n = 15, a_n = -22$

6. $a_1 = -20, n = 25, a_n = 148$

7. $a_1 = 13, d = -6, n = 21$

8. $a_1 = 5, d = 4, n = 11$

9. $a_1 = 5, d = 2, a_n = 33$

10. $a_1 = -121, d = 3, a_n = 5$

11. $d = 0.4, n = 10, a_n = 3.8$

12. $d = -\frac{2}{3}, n = 16, a_n = 44$

Find the sum of each arithmetic series.

13. $5 + 7 + 9 + 11 + \ldots + 27$

14. $-4 + 1 + 6 + 11 + \ldots + 91$

15. $13 + 20 + 27 + \ldots + 272$

16. $89 + 86 + 83 + 80 + \ldots + 20$

17. $\sum_{n=1}^{4} (1 - 2n)$

18. $\sum_{j=1}^{6} (5 + 3n)$

19. $\sum_{n=1}^{5} (9 - 4n)$

20. $\sum_{k=4}^{10} (2k + 1)$

21. $\sum_{n=3}^{8} (5n - 10)$

22. $\sum_{n=1}^{101} (4 - 4n)$

Find the first three terms of each arithmetic series described.

23. $a_1 = 14, a_n = -85, S_n = -1207$

24. $a_1 = 1, a_n = 19, S_n = 100$

25. $n = 16, a_n = 15, S_n = -120$

26. $n = 15, a_n = 5\frac{4}{5}, S_n = 45$

27. STACKING A health club rolls its towels and stacks them in layers on a shelf. Each layer of towels has one less towel than the layer below it. If there are 20 towels on the bottom layer and one towel on the top layer, how many towels are stacked on the shelf?

28. BUSINESS A merchant places $1 in a jackpot on August 1, then draws the name of a regular customer. If the customer is present, he or she wins the $1 in the jackpot. If the customer is not present, the merchant adds $2 to the jackpot on August 2 and draws another name. Each day the merchant adds an amount equal to the day of the month. If the first person to win the jackpot wins $496, on what day of the month was her or his name drawn?

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11-2 Reading to Learn Mathematics

Arithmetic Series

Pre-Activity **How do arithmetic series apply to amphitheaters?**

Read the introduction to Lesson 11-2 at the top of page 583 in your textbook.

Suppose that an amphitheater can seat 50 people in the first row and that each row thereafter can seat 9 more people than the previous row. Using the vocabulary of arithmetic sequences, describe how you would find the number of people who could be seated in the first 10 rows. (Do not actually calculate the sum.)

Reading the Lesson

1. What is the relationship between an arithmetic sequence and the corresponding arithmetic series?

2. Consider the formula $S_n = \frac{n}{2}(a_1 + a_n)$. Explain the meaning of this formula in words.

3. a. What is the purpose of sigma notation?

b. Consider the expression $\sum_{i=2}^{12}(4i - 2)$.

This form of writing a sum is called ___________________________.

The variable i is called the ___________________________.

The first value of i is ___________.

The last value of i is ___________.

How would you read this expression?

Helping You Remember

4. A good way to remember something is to relate it to something you already know. How can your knowledge of how to find the average of two numbers help you remember the formula $S_n = \frac{n}{2}(a_1 + a_n)$?

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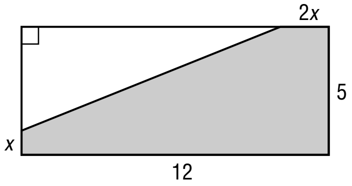
11-2 Enrichment

Geometric Puzzlers

For the problems on this page, you will need to use the Pythagorean Theorem and the formulas for the area of a triangle and a trapezoid.

1. A rectangle measures 5 by 12 units. The upper left corner is cut off as shown in the diagram.

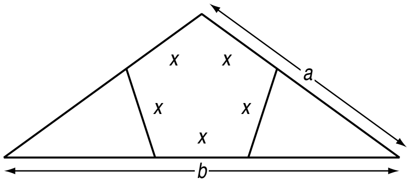


a. Find the area $A(x)$ of the shaded pentagon.

b. Find x and $2x$ so that $A(x)$ is a maximum. What happens to the cut-off triangle?

2. A triangle with sides of lengths a, a, and b is isosceles. Two triangles are cut off so that the remaining pentagon has five equal sides of length x. The value of x can be found using this equation.

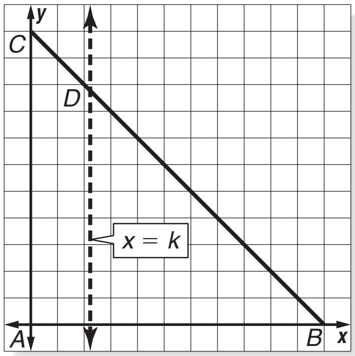
$(2b - a)x^2 + (4a^2 - b^2)(2x - a) = 0$



a. Find x when $a = 10$ and $b = 12$.

b. Can a be equal to $2b$?

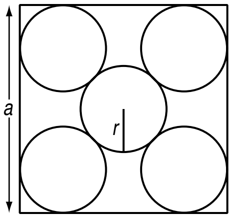
3. The coordinates of the vertices of a triangle are $A(0, 0)$, $B(11, 0)$, and $C(0, 11)$. A line $x = k$ cuts the triangle into two regions having equal area.



a. What are the coordinates of point D?

b. Write and solve an equation for finding the value of k.

4. Inside a square are five circles with the same radius.



a. Connect the center of the top left circle to the center of the bottom right circle. Express this length in terms of r.

b. Draw the square with vertices at the centers of the four outside circles. Express the diagonal of this square in terms of r and a.

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11-3 Study Guide and Intervention

Geometric Sequences

Geometric Sequences A **geometric sequence** is a sequence in which each term after the first is the product of the previous term and a constant called the **constant ratio**.

***n*th Term of a Geometric Sequence**	$a_n = a_1 \cdot r^{n-1}$, where a_1 is the first term, r is the common ratio, and n is any positive integer

Example 1 **Find the next two terms of the geometric sequence 1200, 480, 192, ….**

Since $\frac{480}{1200} = 0.4$ and $\frac{192}{480} = 0.4$, the sequence has a common ratio of 0.4. The next two terms in the sequence are $192(0.4) = 76.8$ and $76.8(0.4) = 30.72$.

Example 2 **Write an equation for the *n*th term of the geometric sequence 3.6, 10.8, 32.4, ….**

In this sequence $a_1 = 3.6$ and $r = 3$. Use the nth term formula to write an equation.

$$\begin{aligned} a_n &= a_1 \cdot r^{n-1} && \text{Formula for } n\text{th term} \\ &= 3.6 \cdot 3^{n-1} && a_1 = 3.6,\ r = 3 \end{aligned}$$

An equation for the nth term is $a_n = 3.6 \cdot 3^{n-1}$.

Exercises

Find the next two terms of each geometric sequence.

1. 6, 12, 24, …

2. 180, 60, 20, …

3. 2000, −1000, 500, …

4. 0.8, −2.4, 7.2, …

5. 80, 60, 45, …

6. 3, 16.5, 90.75, …

Find the first five terms of each geometric sequence described.

7. $a_1 = \frac{1}{9}, r = 3$

8. $a_1 = 240, r = -\frac{3}{4}$

9. $a_1 = 10, r = \frac{5}{2}$

Find the indicated term of each geometric sequence.

10. $a_1 = -10, r = 4, n = 2$

11. $a_1 = -6, r = -\frac{1}{2}, n = 8$

12. $a_3 = 9, r = -3, n = 7$

13. $a_4 = 16, r = 2, n = 10$

14. $a_4 = -54, r = -3, n = 6$

15. $a_1 = 8, r = \frac{2}{3}, n = 5$

Write an equation for the *n*th term of each geometric sequence.

16. 500, 350, 245, …

17. 8, 32, 128, …

18. 11, −24.2, 53.24, …

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11-3 Study Guide and Intervention *(continued)*

Geometric Sequences

Geometric Means The **geometric means** of a geometric sequence are the terms between any two nonsuccessive terms of the sequence.

To find the k geometric means between two terms of a sequence, use the following steps.

Step 1 Let the two terms given be a_1 and a_n, where $n = k + 2$.
Step 2 Substitute in the formula $a_n = a_1 \cdot r^{n-1}$ $(= a_1 \cdot r^{k+1})$.
Step 3 Solve for r, and use that value to find the k geometric means:
$a_1 \cdot r, a_1 \cdot r^2, \ldots, a_1 \cdot r^k$

Example **Find the three geometric means between 8 and 40.5.**

Use the nth term formula to find the value of r. In the sequence 8, __?__, __?__, __?__, 40.5, a_1 is 8 and a_5 is 40.5.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term
$40.5 = 8 \cdot r^{5-1}$ $n = 5, a_1 = 8, a_5 = 40.5$
$5.0625 = r^4$ Divide each side by 8.
$r = \pm 1.5$ Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find the geometric means.

$r = 1.5$	$r = -1.5$
$a_2 = 8(1.5)$ or 12	$a_2 = 8(-1.5)$ or -12
$a_3 = 12(1.5)$ or 18	$a_3 = -12(-1.5)$ or 18
$a_4 = 18(1.5)$ or 27	$a_4 = 18(-1.5)$ or -27

The geometric means are 12, 18, and 27, or −12, 18, and −27.

Exercises

Find the geometric means in each sequence.

1. 5, __?__, __?__, __?__, 405

2. 5, __?__, __?__, 20.48

3. $\frac{3}{5}$, __?__, __?__, __?__, 375

4. −24, __?__, __?__, $\frac{1}{9}$

5. 12, __?__, __?__, __?__, __?__, __?__, $\frac{3}{16}$

6. 200, __?__, __?__, __?__, 414.72

7. $\frac{35}{49}$, __?__, __?__, __?__, __?__, −12,005

8. 4, __?__, __?__, __?__, $156\frac{1}{4}$

9. $-\frac{1}{81}$, __?__, __?__, __?__, __?__, __?__, −9

10. 100, __?__, __?__, __?__, 384.16

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11-3 Skills Practice

Geometric Sequences

Find the next two terms of each geometric sequence.

1. $-1, -2, -4, \ldots$

2. $6, 3, \frac{3}{2}, \ldots$

3. $-5, -15, -45, \ldots$

4. $729, -243, 81, \ldots$

5. $1536, 384, 96, \ldots$

6. $64, 160, 400, \ldots$

Find the first five terms of each geometric sequence described.

7. $a_1 = 6, r = 2$

8. $a_1 = -27, r = 3$

9. $a_1 = -15, r = -1$

10. $a_1 = 3, r = 4$

11. $a_1 = 1, r = \frac{1}{2}$

12. $a_1 = 216, r = -\frac{1}{3}$

Find the indicated term of each geometric sequence.

13. $a_1 = 5, r = 2, n = 6$

14. $a_1 = 18, r = 3, n = 6$

15. $a_1 = -3, r = -2, n = 5$

16. $a_1 = -20, r = -2, n = 9$

17. a_8 for $-12, -6, -3, \ldots$

18. a_7 for $80, \frac{80}{3}, \frac{80}{9}, \ldots$

Write an equation for the *n*th term of each geometric sequence.

19. $3, 9, 27, \ldots$

20. $-1, -3, -9, \ldots$

21. $2, -6, 18, \ldots$

22. $5, 10, 20, \ldots$

Find the geometric means in each sequence.

23. $4, \underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ }, 64$

24. $1, \underline{\ ?\ }, \underline{\ ?\ }, \underline{\ ?\ }, 81$

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11-3 Practice

Geometric Sequences

Find the next two terms of each geometric sequence.

1. $-15, -30, -60, \ldots$

2. $80, 40, 20, \ldots$

3. $90, 30, 10, \ldots$

4. $-1458, 486, -162, \ldots$

5. $\frac{1}{4}, \frac{3}{8}, \frac{9}{16}, \ldots$

6. $216, 144, 96, \ldots$

Find the first five terms of each geometric sequence described.

7. $a_1 = -1, r = -3$

8. $a_1 = 7, r = -4$

9. $a_1 = -\frac{1}{3}, r = 2$

10. $a_1 = 12, r = \frac{2}{3}$

Find the indicated term of each geometric sequence.

11. $a_1 = 5, r = 3, n = 6$

12. $a_1 = 20, r = -3, n = 6$

13. $a_1 = -4, r = -2, n = 10$

14. a_8 for $-\frac{1}{250}, -\frac{1}{50}, -\frac{1}{10}, \ldots$

15. a_{12} for $96, 48, 24, \ldots$

16. $a_1 = 8, r = \frac{1}{2}, n = 9$

17. $a_1 = -3125, r = -\frac{1}{5}, n = 9$

18. $a_1 = 3, r = \frac{1}{10}, n = 8$

Write an equation for the *n*th term of each geometric sequence.

19. $1, 4, 16, \ldots$

20. $-1, -5, -25, \ldots$

21. $1, \frac{1}{2}, \frac{1}{4}, \ldots$

22. $-3, -6, -12, \ldots$

23. $7, -14, 28, \ldots$

24. $-5, -30, -180, \ldots$

Find the geometric means in each sequence.

25. 3, ?, ?, ?, 768

26. 5, ?, ?, ?, 1280

27. 144, ?, ?, ?, 9

28. 37,500, ?, ?, ?, ?, −12

29. BIOLOGY A culture initially contains 200 bacteria. If the number of bacteria doubles every 2 hours, how many bacteria will be in the culture at the end of 12 hours?

30. LIGHT If each foot of water in a lake screens out 60% of the light above, what percent of the light passes through 5 feet of water?

31. INVESTING Raul invests $1000 in a savings account that earns 5% interest compounded annually. How much money will he have in the account at the end of 5 years?

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11-3 Reading to Learn Mathematics

Geometric Sequences

Pre-Activity **How do geometric sequences apply to a bouncing ball?**

Read the introduction to Lesson 11-3 at the top of page 588 in your textbook.

Suppose that you drop a ball from a height of 4 feet, and that each time it falls, it bounces back to 74% of the height from which it fell. Describe how would you find the height of the third bounce. (Do not actually calculate the height of the bounce.)

Reading the Lesson

1. Explain the difference between an arithmetic sequence and a geometric sequence.

2. Consider the formula $a_n = a_1 \cdot r^{n-1}$.

a. What is this formula used to find?

b. What do each of the following represent?

a_n: ______________________________

a_1: ______________________________

r: ______________________________

n: ______________________________

3. a. In the sequence 5, 8, 11, 14, 17, 20, the numbers 8, 11, 14, and 17 are

______________ between 5 and 20.

b. In the sequence 12, 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, the numbers 4, $\frac{4}{3}$, and $\frac{4}{9}$ are

______________ between 12 and $\frac{4}{27}$.

Helping You Remember

4. Suppose that your classmate Ricardo has trouble remembering the formula $a_n = a_1 \cdot r^{n-1}$ correctly. He thinks that the formula should be $a_n = a_1 \cdot r^n$. How would you explain to him that he should use r^{n-1} rather than r^n in the formula?

Lesson 11-3

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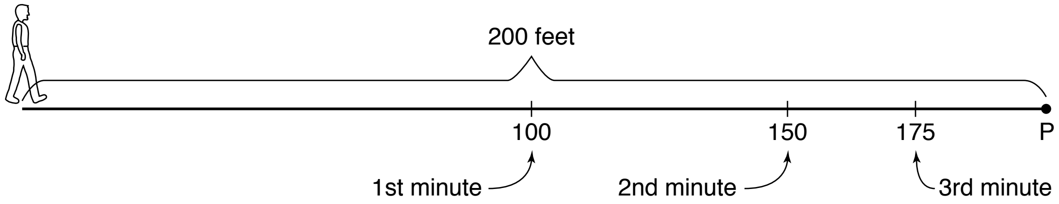
11-3 Enrichment

Half the Distance

Suppose you are 200 feet from a fixed point, *P*. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.



An interesting sequence results because according to the problem, you never actually reach the point *P*, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

Example **How many minutes are needed to get within 0.1 foot of a point 200 feet away?**

Count the number of times you divide by 2.

Enter: 200 ÷ 2 ENTER ÷ 2 ENTER ÷ 2 ENTER, and so on

Result: 0.0976562

You divided by 2 eleven times. The time needed is 11 minutes.

Use the method illustrated above to solve each problem.

1. If it is about 2500 miles from Los Angeles to New York, how many minutes would it take to get within 0.1 mile of New York? How far from New York are you at that time?

2. If it is 25,000 miles around Earth, how many minutes would it take to get within 0.5 mile of the full distance around Earth? How far short would you be?

3. If it is about 250,000 miles from Earth to the Moon, how many minutes would it take to get within 0.5 mile of the Moon? How far from the surface of the Moon would you be?

4. If it is about 30,000,000 feet from Honolulu to Miami, how many minutes would it take to get to within 1 foot of Miami? How far from Miami would you be at that time?

5. If it is about 93,000,000 miles to the sun, how many minutes would it take to get within 500 miles of the sun? How far from the sun would you be at that time?

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11-4 Study Guide and Intervention

Geometric Series

Geometric Series A **geometric series** is the indicated sum of consecutive terms of a geometric sequence.

Sum of a Geometric Series	The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, where $r \neq 1$.

Example 1 **Find the sum of the first four terms of the geometric sequence for which $a_1 = 120$ and $r = \frac{1}{3}$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_4 = \frac{120\left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - \frac{1}{3}}$ $n = 4$, $a_1 = 120$, $r = \frac{1}{3}$

≈ 177.78 Use a calculator.

The sum of the series is 177.78.

Example 2 **Find the sum of the geometric series $\sum_{j=1}^{7} 4 \cdot 3^{j-2}$.**

Since the sum is a geometric series, you can use the sum formula.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_7 = \frac{\frac{4}{3}(1 - 3^7)}{1 - 3}$ $n = 7$, $a_1 = \frac{4}{3}$, $r = 3$

≈ 1457.33 Use a calculator.

The sum of the series is 1457.33.

Exercises

Find S_n for each geometric series described.

1. $a_1 = 2$, $a_n = 486$, $r = 3$

2. $a_1 = 1200$, $a_n = 75$, $r = \frac{1}{2}$

3. $a_1 = \frac{1}{25}$, $a_n = 125$, $r = 5$

4. $a_1 = 3$, $r = \frac{1}{3}$, $n = 4$

5. $a_1 = 2$, $r = 6$, $n = 4$

6. $a_1 = 2$, $r = 4$, $n = 6$

7. $a_1 = 100$, $r = -\frac{1}{2}$, $n = 5$

8. $a_3 = 20$, $a_6 = 160$, $n = 8$

9. $a_4 = 16$, $a_7 = 1024$, $n = 10$

Find the sum of each geometric series.

10. 6 + 18 + 54 + … to 6 terms

11. $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ to 10 terms

12. $\sum_{j=4}^{8} 2^j$

13. $\sum_{k=1}^{7} 3 \cdot 2^{k-1}$

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11-4 Study Guide and Intervention *(continued)*

Geometric Series

Specific Terms You can use one of the formulas for the sum of a geometric series to help find a particular term of the series.

Example 1 **Find a_1 in a geometric series for which $S_6 = 441$ and $r = 2$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$441 = \frac{a_1(1 - 2^6)}{1 - 2}$ $S_6 = 441, r = 2, n = 6$

$441 = \frac{-63a_1}{-1}$ Subtract.

$a_1 = \frac{441}{63}$ Divide.

$a_1 = 7$ Simplify.

The first term of the series is 7.

Example 2 **Find a_1 in a geometric series for which $S_n = 244$, $a_n = 324$, and $r = -3$.**

Since you do not know the value of n, use the alternate sum formula.

$S_n = \frac{a_1 - a_n r}{1 - r}$ Alternate sum formula

$244 = \frac{a_1 - (324)(-3)}{1 - (-3)}$ $S_n = 244, a_n = 324, r = -3$

$244 = \frac{a_1 + 972}{4}$ Simplify.

$976 = a_1 + 972$ Multiply each side by 4.

$a_1 = 4$ Subtract 972 from each side.

The first term of the series is 4.

Example 3 **Find a_4 in a geometric series for which $S_n = 796.875$, $r = \frac{1}{2}$, and $n = 8$.**

First use the sum formula to find a_1.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$796.875 = \frac{a_1\left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$ $S_8 = 796.875, r = \frac{1}{2}, n = 8$

$796.875 = \frac{0.99609375a_1}{0.5}$ Use a calculator.

$a_1 = 400$

Since $a_4 = a_1 \cdot r^3$, $a_4 = 400\left(\frac{1}{2}\right)^3 = 50$. The fourth term of the series is 50.

Exercises

Find the indicated term for each geometric series described.

1. $S_n = 726, a_n = 486, r = 3; a_1$

2. $S_n = 850, a_n = 1280, r = -2; a_1$

3. $S_n = 1023.75, a_n = 512, r = 2; a_1$

4. $S_n = 118.125, a_n = -5.625, r = -\frac{1}{2}; a_1$

5. $S_n = 183, r = -3, n = 5; a_1$

6. $S_n = 1705, r = 4, n = 5; a_1$

7. $S_n = 52{,}084, r = -5, n = 7; a_1$

8. $S_n = 43{,}690, r = \frac{1}{4}, n = 8; a_1$

9. $S_n = 381, r = 2, n = 7; a_4$

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11-4 Skills Practice

Geometric Series

Find S_n for each geometric series described.

1. $a_1 = 2, a_5 = 162, r = 3$

2. $a_1 = 4, a_6 = 12{,}500, r = 5$

3. $a_1 = 1, a_8 = -1, r = -1$

4. $a_1 = 4, a_n = 256, r = -2$

5. $a_1 = 1, a_n = 729, r = -3$

6. $a_1 = 2, r = -4, n = 5$

7. $a_1 = -8, r = 2, n = 4$

8. $a_1 = 3, r = -2, n = 12$

9. $a_1 = 8, r = 3, n = 5$

10. $a_1 = 6, a_n = \frac{3}{8}, r = \frac{1}{2}$

11. $a_1 = 8, r = \frac{1}{2}, n = 7$

12. $a_1 = 2, r = -\frac{1}{2}, n = 6$

Find the sum of each geometric series.

13. $4 + 8 + 16 + \ldots$ to 5 terms

14. $-1 - 3 - 9 - \ldots$ to 6 terms

15. $3 + 6 + 12 + \ldots$ to 5 terms

16. $-15 + 30 - 60 + \ldots$ to 7 terms

17. $\sum_{n=1}^{4} 3^{n-1}$

18. $\sum_{n=1}^{5} (-2)^{n-1}$

19. $\sum_{n=1}^{4} \left(\frac{1}{3}\right)^{n-1}$

20. $\sum_{n=1}^{9} 2(-3)^{n-1}$

Find the indicated term for each geometric series described.

21. $S_n = 1275, a_n = 640, r = 2; a_1$

22. $S_n = -40, a_n = -54, r = -3; a_1$

23. $S_n = 99, n = 5, r = -\frac{1}{2}; a_1$

24. $S_n = 39{,}360, n = 8, r = 3; a_1$

NAME ____ DATE ____________ PERIOD _____

11-4 Practice

Geometric Series

Find S_n for each geometric series described.

1. $a_1 = 2, a_6 = 64, r = 2$

2. $a_1 = 160, a_6 = 5, r = \frac{1}{2}$

3. $a_1 = -3, a_n = -192, r = -2$

4. $a_1 = -81, a_n = -16, r = -\frac{2}{3}$

5. $a_1 = -3, a_n = 3072, r = -4$

6. $a_1 = 54, a_6 = \frac{2}{9}, r = \frac{1}{3}$

7. $a_1 = 5, r = 3, n = 9$

8. $a_1 = -6, r = -1, n = 21$

9. $a_1 = -6, r = -3, n = 7$

10. $a_1 = -9, r = \frac{2}{3}, n = 4$

11. $a_1 = \frac{1}{3}, r = 3, n = 10$

12. $a_1 = 16, r = -1.5, n = 6$

Find the sum of each geometric series.

13. $162 + 54 + 18 + \ldots$ to 6 terms

14. $2 + 4 + 8 + \ldots$ to 8 terms

15. $64 - 96 + 144 - \ldots$ to 7 terms

16. $\frac{1}{9} - \frac{1}{3} + 1 - \ldots$ to 6 terms

17. $\sum_{n=1}^{8} (-3)^{n-1}$

18. $\sum_{n=1}^{9} 5(-2)^{n-1}$

19. $\sum_{n=1}^{5} -1(4)^{n-1}$

20. $\sum_{n=1}^{6} \left(\frac{1}{2}\right)^{n-1}$

21. $\sum_{n=1}^{10} 2560\left(\frac{1}{2}\right)^{n-1}$

22. $\sum_{n=1}^{4} 9\left(\frac{2}{3}\right)^{n-1}$

Find the indicated term for each geometric series described.

23. $S_n = 1023, a_n = 768, r = 4; a_1$

24. $S_n = 10{,}160, a_n = 5120, r = 2; a_1$

25. $S_n = -1365, n = 12, r = -2; a_1$

26. $S_n = 665, n = 6, r = 1.5; a_1$

27. CONSTRUCTION A pile driver drives a post 27 inches into the ground on its first hit. Each additional hit drives the post $\frac{2}{3}$ the distance of the prior hit. Find the total distance the post has been driven after 5 hits.

28. COMMUNICATIONS Hugh Moore e-mails a joke to 5 friends on Sunday morning. Each of these friends e-mails the joke to 5 of her or his friends on Monday morning, and so on. Assuming no duplication, how many people will have heard the joke by the end of Saturday, not including Hugh?

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11-4 Reading to Learn Mathematics

Geometric Series

Pre-Activity **How is e-mailing a joke like a geometric series?**

Read the introduction to Lesson 11-4 at the top of page 594 in your textbook.

- Suppose that you e-mail the joke on Monday to five friends, rather than three, and that each of those friends e-mails it to five friends on Tuesday, and so on. Write a sum that shows that total number of people, including yourself, who will have read the joke by Thursday. (Write out the sum using plus signs rather than sigma notation. Do not actually find the sum.)
- Use exponents to rewrite the sum you found above. (Use an exponent in each term, and use the same base for all terms.)

Reading the Lesson

1. Consider the formula $S_n = \dfrac{a_1(1 - r^n)}{1 - r}$.

a. What is this formula used to find?

b. What do each of the following represent?

S_n: ______________________________

a_1: ______________________________

r: ______________________________

c. Suppose that you want to use the formula to evaluate $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

n = __________ a_1 = __________ r = __________ r^n = __________

d. Suppose that you want to use the formula to evaluate the sum $\sum_{n=1}^{6} 8(-2)^{n-1}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

n = __________ a_1 = __________ r = __________ r^n = __________

Helping You Remember

2. This lesson includes three formulas for the sum of the first n terms of a geometric series. All of these formulas have the same denominator and have the restriction $r \neq 1$. How can this restriction help you to remember the denominator in the formulas?

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11-4 Enrichment

Annuities

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that $30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount A. After the third payment, the amount left is

$1.09[1.09A - 30{,}000(1 + 1.09)] - 30{,}000 = 1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$.

The results are summarized in the table below.

Payment Number	Number of Dollars Left After Payment
1	$A - 30{,}000$
2	$1.09A - 30{,}000(1 + 1.09)$
3	$1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$

1. Use the pattern shown in the table to find the number of dollars left after the fourth payment.

2. Find the amount left after the tenth payment.

The amount left after the 14th payment is $1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13})$. However, there should be no money left after the 14th and final payment.

$1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}) = 0$

Notice that $1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}$ is a geometric series where $a_1 = 1$, $a_n = 1.09^{13}$, $n = 14$ and $r = 1.09$.

Using the formula for S_n,

$$1 + 1.09 + 1.09^2 + \ldots + 1.09^{13} = \frac{a_1 - a_1r^n}{1 - r} = \frac{1 - 1.09^{14}}{1 - 1.09} = \frac{1 - 1.09^{14}}{-0.09}.$$

3. Show that when you solve for A you get $A = \frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$.

Therefore, to provide $30,000 for 14 years where the annual interest rate is 9%, you need $\frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$ dollars.

4. Use a calculator to find the value of A in problem 3.

In general, if you wish to provide P dollars for each of n years at an annual rate of r%, you need A dollars where

$$\left(1 + \frac{r}{100}\right)^{n-1}A - P\left[1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2 + \ldots + \left(1 + \frac{r}{100}\right)^{n-1}\right] = 0.$$

You can solve this equation for A, given P, n, and r.

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11-5 Study Guide and Intervention

Infinite Geometric Series

Infinite Geometric Series A geometric series that does not end is called an **infinite geometric series**. Some infinite geometric series have sums, but others do not because the **partial sums** increase without approaching a limiting value.

Sum of an Infinite Geometric Series	$S = \frac{a_1}{1 - r}$ for $-1 < r < 1$. If $\|r\| \geq 1$, the infinite geometric series does not have a sum.

Example **Find the sum of each infinite geometric series, if it exists.**

a. $\mathbf{75 + 15 + 3 + \ldots}$

First, find the value of r to determine if the sum exists. $a_1 = 75$ and $a_2 = 15$, so $r = \frac{15}{75}$ or $\frac{1}{5}$. Since $\left|\frac{1}{5}\right| < 1$, the sum exists. Now use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1 - r} \qquad \text{Sum formula}$$

$$= \frac{75}{1 - \frac{1}{5}} \qquad a_1 = 75,\ r = \frac{1}{5}$$

$$= \frac{75}{\frac{4}{5}} \text{ or } 93.75 \qquad \text{Simplify.}$$

The sum of the series is 93.75.

b. $\displaystyle\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1}$

In this infinite geometric series, $a_1 = 48$ and $r = -\frac{1}{3}$.

$$S = \frac{a_1}{1 - r} \qquad \text{Sum formula}$$

$$= \frac{48}{1 - \left(-\frac{1}{3}\right)} \qquad a_1 = 48,\ r = -\frac{1}{3}$$

$$= \frac{48}{\frac{4}{3}} \text{ or } 36 \qquad \text{Simplify.}$$

Thus $\displaystyle\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1} = 36$.

Exercises

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = -7,\ r = \frac{5}{8}$

2. $1 + \frac{5}{4} + \frac{25}{16} + \ldots$

3. $a_1 = 4,\ r = \frac{1}{2}$

4. $\frac{2}{9} + \frac{5}{27} + \frac{25}{162} + \ldots$

5. $15 + 10 + 6\frac{2}{3} + \ldots$

6. $18 - 9 + 4\frac{1}{2} - 2\frac{1}{4} + \ldots$

7. $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots$

8. $1000 + 800 + 640 + \ldots$

9. $6 - 12 + 24 - 48 + \ldots$

10. $\displaystyle\sum_{n=1}^{\infty} 50\left(\frac{4}{5}\right)^{n-1}$

11. $\displaystyle\sum_{k=1}^{\infty} 22\left(-\frac{1}{2}\right)^{k-1}$

12. $\displaystyle\sum_{s=1}^{\infty} 24\left(\frac{7}{12}\right)^{s-1}$

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11-5 Study Guide and Intervention *(continued)*

Infinite Geometric Series

Repeating Decimals A repeating decimal represents a fraction. To find the fraction, write the decimal as an infinite geometric series and use the formula for the sum.

Example **Write each repeating decimal as a fraction.**

a. $0.\overline{42}$

Write the repeating decimal as a sum.

$$0.\overline{42} = 0.42424242\ldots$$
$$= \frac{42}{100} + \frac{42}{10{,}000} + \frac{42}{1{,}000{,}000} + \ldots$$

In this series $a_1 = \frac{42}{100}$ and $r = \frac{1}{100}$.

$$S = \frac{a_1}{1 - r} \qquad \text{Sum formula}$$
$$= \frac{\frac{42}{100}}{1 - \frac{1}{100}} \qquad a_1 = \frac{42}{100},\ r = \frac{1}{100}$$
$$= \frac{\frac{42}{100}}{\frac{99}{100}} \qquad \text{Subtract.}$$
$$= \frac{42}{99} \text{ or } \frac{14}{33} \qquad \text{Simplify.}$$

Thus $0.\overline{42} = \frac{14}{33}$.

b. $0.5\overline{24}$

Let $S = 0.5\overline{24}$.

$$S = 0.5242424\ldots \qquad \text{Write as a repeating decimal.}$$
$$1000S = 524.242424\ldots \qquad \text{Multiply each side by 1000.}$$
$$10S = 5.242424\ldots \qquad \text{Mulitply each side by 10.}$$
$$990S = 519 \qquad \text{Subtract the third equation from the second equation.}$$
$$S = \frac{519}{990} \text{ or } \frac{173}{330} \qquad \text{Simplify.}$$

Thus, $0.5\overline{24} = \frac{173}{330}$

Exercises

Write each repeating decimal as a fraction.

1. $0.\overline{2}$

2. $0.\overline{8}$

3. $0.\overline{30}$

4. $0.\overline{87}$

5. $0.\overline{10}$

6. $0.\overline{54}$

7. $0.\overline{75}$

8. $0.\overline{18}$

9. $0.\overline{62}$

10. $0.\overline{72}$

11. $0.0\overline{72}$

12. $0.0\overline{45}$

13. $0.0\overline{6}$

14. $0.0\overline{138}$

15. $0.\overline{0138}$

16. $0.0\overline{81}$

17. $0.2\overline{45}$

18. $0.4\overline{36}$

19. $0.5\overline{4}$

20. $0.8\overline{63}$

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11-5 Skills Practice

Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 1, r = \frac{1}{2}$

2. $a_1 = 5, r = -\frac{2}{5}$

3. $a_1 = 8, r = 2$

4. $a_1 = 6, r = \frac{1}{2}$

5. $4 + 2 + 1 + \frac{1}{2} + \ldots$

6. $540 - 180 + 60 - 20 + \ldots$

7. $5 + 10 + 20 + \ldots$

8. $-336 + 84 - 21 + \ldots$

9. $125 + 25 + 5 + \ldots$

10. $9 - 1 + \frac{1}{9} - \ldots$

11. $\frac{3}{4} + \frac{9}{4} + \frac{27}{4} + \ldots$

12. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

13. $5 + 2 + 0.8 + \ldots$

14. $9 + 6 + 4 + \ldots$

15. $\sum_{n=1}^{\infty} 10\left(\frac{1}{2}\right)^{n-1}$

16. $\sum_{n=1}^{\infty} 6\left(-\frac{1}{3}\right)^{n-1}$

17. $\sum_{n=1}^{\infty} 15\left(\frac{2}{5}\right)^{n-1}$

18. $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)\left(\frac{1}{3}\right)^{n-1}$

Write each repeating decimal as a fraction.

19. $0.\overline{4}$

20. $0.\overline{8}$

21. $0.\overline{27}$

22. $0.\overline{67}$

23. $0.\overline{54}$

24. $0.\overline{375}$

25. $0.\overline{641}$

26. $0.\overline{171}$

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11-5 Practice

Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 35, r = \frac{2}{7}$

2. $a_1 = 26, r = \frac{1}{2}$

3. $a_1 = 98, r = -\frac{3}{4}$

4. $a_1 = 42, r = \frac{6}{5}$

5. $a_1 = 112, r = -\frac{3}{5}$

6. $a_1 = 500, r = \frac{1}{5}$

7. $a_1 = 135, r = -\frac{1}{2}$

8. $18 - 6 + 2 - \dots$

9. $2 + 6 + 18 + \dots$

10. $6 + 4 + \frac{8}{3} + \dots$

11. $\frac{4}{25} + \frac{2}{5} + 1 + \dots$

12. $10 + 1 + 0.1 + \dots$

13. $100 + 20 + 4 + \dots$

14. $-270 + 135 - 67.5 + \dots$

15. $0.5 + 0.25 + 0.125 + \dots$

16. $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \dots$

17. $0.8 + 0.08 + 0.008 + \dots$

18. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \dots$

19. $3 + \frac{9}{7} + \frac{27}{49} + \dots$

20. $0.3 - 0.003 + 0.00003 - \dots$

21. $0.06 + 0.006 + 0.0006 + \dots$

22. $\frac{2}{3} - 2 + 6 - \dots$

23. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^{n-1}$

24. $\sum_{n=1}^{\infty} \frac{2}{3}\left(-\frac{3}{4}\right)^{n-1}$

25. $\sum_{n=1}^{\infty} 18\left(\frac{2}{3}\right)^{n-1}$

26. $\sum_{n=1}^{\infty} 5(-0.1)^{n-1}$

Write each repeating decimal as a fraction.

27. $0.\overline{6}$

28. $0.\overline{09}$

29. $0.\overline{43}$

30. $0.\overline{27}$

31. $0.\overline{243}$

32. $0.\overline{84}$

33. $0.\overline{990}$

34. $0.\overline{150}$

35. PENDULUMS On its first swing, a pendulum travels 8 feet. On each successive swing, the pendulum travels $\frac{4}{5}$ the distance of its previous swing. What is the total distance traveled by the pendulum when it stops swinging?

36. ELASTICITY A ball dropped from a height of 10 feet bounces back $\frac{9}{10}$ of that distance. With each successive bounce, the ball continues to reach $\frac{9}{10}$ of its previous height. What is the total vertical distance (both up and down) traveled by the ball when it stops bouncing? (*Hint*: Add the total distance the ball falls to the total distance it rises.)

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11-5 Reading to Learn Mathematics

Infinite Geometric Series

Pre-Activity **How does an infinite geometric series apply to a bouncing ball?**

Read the introduction to Lesson 11-5 at the top of page 599 in your textbook.

Note the following powers of 0.6: $0.6^1 = 0.6$; $0.6^2 = 0.36$; $0.6^3 = 0.216$; $0.6^4 = 0.1296$; $0.6^5 = 0.07776$; $0.6^6 = 0.046656$; $0.6^7 = 0.0279936$. If a ball is dropped from a height of 10 feet and bounces back to 60% of its previous height on each bounce, after how many bounces will it bounce back to a height of less than 1 foot?

Reading the Lesson

1. Consider the formula $S = \frac{a_1}{1 - r}$.

a. What is the formula used to find?

b. What do each of the following represent?

S: ______________________________

a_1: ______________________________

r: ______________________________

c. For what values of r does an infinite geometric sequence have a sum?

d. Rewrite your answer for part d as an absolute value inequality.

2. For each of the following geometric series, give the values of a_1 and r. Then state whether the sum of the series exists. (Do not actually find the sum.)

a. $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$

$a_1 =$ __________ $r =$ __________

Does the sum exist? __________

b. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \ldots$

$a_1 =$ __________ $r =$ __________

Does the sum exist? __________

c. $\sum_{i=1}^{\infty} 3^i$

$a_1 =$ __________ $r =$ __________

Does the sum exist? __________

Helping You Remember

3. One good way to remember something is to relate it to something you already know. How can you use the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$ that you learned in Lesson 11-4 for finding the sum of a geometric series to help you remember the formula for finding the sum of an infinite geometric series?

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11-5 Enrichment

Convergence and Divergence

Convergence and divergence are terms that relate to the existence of a sum of an infinite series. If a sum exists, the series is convergent. If not, the series is divergent. Consider the series $12 + 3 + \frac{3}{4} + \frac{3}{16} + \ldots$. This is a geometric series with $r = \frac{1}{4}$. The sum is given by the formula $S = \frac{a_1}{1 - r}$. Thus, the sum is $12 \div \frac{3}{4}$ or 16. This series is convergent since a sum exists. Notice that the first two terms have a sum of 15. As more terms are added, the sum comes closer (or converges) to 16.

Recall that a geometric series has a sum if and only if $-1 < r < 1$. Thus, a geometric series is convergent if r is between -1 and 1, and divergent if r has another value. An infinite arithmetic series cannot have a sum unless all of the terms are equal to zero.

Example **Determine whether each series is convergent or divergent.**

a. $2 + 5 + 8 + 11 + \ldots$ divergent

b. $-2 + 4 + (-8) + 16 + \ldots$ divergent

c. $16 + 8 + 4 + 2 + \ldots$ convergent

Determine whether each series is convergent or divergent. If the series is convergent, find the sum.

1. $5 + 10 + 15 + 20 + \ldots$

2. $16 + 8 + 4 + 2 + \ldots$

3. $1 + 0.1 + 0.01 + 0.001 + \ldots$

4. $4 + 2 + 0 - 2 - \ldots$

5. $2 - 4 + 8 - 16 + \ldots$

6. $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \ldots$

7. $4 + 2.4 + 1.44 + 0.864 + \ldots$

8. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 + \ldots$

9. $-\frac{5}{3} + \frac{10}{9} - \frac{20}{27} + \frac{40}{81} - \ldots$

10. $48 + 12 + 3 + \frac{3}{4} + \ldots$

Bonus: Is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ convergent or divergent?

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11-6 Study Guide and Intervention

Recursion and Special Sequences

Special Sequences In a **recursive formula**, each succeeding term is formulated from one or more previous terms. A recursive formula for a sequence has two parts:

1. the value(s) of the first term(s), and

2. an equation that shows how to find each term from the term(s) before it.

Example **Find the first five terms of the sequence in which $a_1 = 6$, $a_2 = 10$, and $a_n = 2a_{n-2}$ for $n \geq 3$.**

$a_1 = 6$
$a_2 = 10$
$a_3 = 2a_1 = 2(6) = 12$
$a_4 = 2a_2 = 2(10) = 20$
$a_5 = 2a_3 = 2(12) = 24$

The first five terms of the sequence are 6, 10, 12, 20, 24.

Exercises

Find the first five terms of each sequence.

1. $a_1 = 1, a_2 = 1, a_n = 2(a_{n-1} + a_{n-2}), n \geq 3$

2. $a_1 = 1, a_n = \frac{1}{1 + a_{n-1}}, n \geq 2$

3. $a_1 = 3, a_n = a_{n-1} + 2(n - 2), \text{n} \geq 2$

4. $a_1 = 5, a_n = a_{n-1} + 2, n \geq 2$

5. $a_1 = 1, a_n = (n - 1)a_{n-1}, n \geq 2$

6. $a_1 = 7, a_n = 4a_{n-1} - 1, n \geq 2$

7. $a_1 = 3, a_2 = 4, a_n = 2a_{n-2} + 3a_{n-1}, n \geq 3$

8. $a_1 = 0.5, a_n = a_{n-1} + 2n, n \geq 2$

9. $a_1 = 8, a_2 = 10, a_n = \frac{a_{n-2}}{a_{n-1}}, n \geq 3$

10. $a_1 = 100, a_n = \frac{a_{n-1}}{n}, n \geq 2$

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11-6 Study Guide and Intervention *(continued)*

Recursion and Special Sequences

Iteration Combining composition of functions with the concept of recursion leads to the process of **iteration**. Iteration is the process of composing a function with itself repeatedly.

Example **Find the first three iterates of $f(x) = 4x - 5$ for an initial value of $x_0 = 2$.**

To find the first iterate, find the value of the function for $x_0 = 2$

$x_1 = f(x_0)$ Iterate the function.
$= f(2)$ $x_0 = 2$
$= 4(2) - 5$ or 3 Simplify.

To find the second iteration, find the value of the function for $x_1 = 3$.

$x_2 = f(x_1)$ Iterate the function.
$= f(3)$ $x_1 = 3$
$= 4(3) - 5$ or 7 Simplify.

To find the third iteration, find the value of the function for $x_2 = 7$.

$x_3 = f(x_2)$ Iterate the function.
$= f(7)$ $x_2 = 7$
$= 4(7) - 5$ or 23 Simplify.

The first three iterates are 3, 7, and 23.

Exercises

Find the first three iterates of each function for the given initial value.

1. $f(x) = x - 1; x_0 = 4$

2. $f(x) = x^2 - 3x; x_0 = 1$

3. $f(x) = x^2 + 2x + 1; x_0 = -2$

4. $f(x) = 4x - 6; x_0 = -5$

5. $f(x) = 6x - 2; x_0 = 3$

6. $f(x) = 100 - 4x; x_0 = -5$

7. $f(x) = 3x - 1; x_0 = 47$

8. $f(x) = x^3 - 5x^2; x_0 = 1$

9. $f(x) = 10x - 25; x_0 = 2$

10. $f(x) = 4x^2 - 9; x_0 = -1$

11. $f(x) = 2x^2 + 5; x_0 = -4$

12. $f(x) = \frac{x - 1}{x + 2}; x_0 = 1$

13. $f(x) = \frac{1}{2}(x + 11); x_0 = 3$

14. $f(x) = \frac{3}{x}; x_0 = 9$

15. $f(x) = x - 4x^2; x_0 = 1$

16. $f(x) = x + \frac{1}{x}; x_0 = 2$

17. $f(x) = x^3 - 5x^2 + 8x - 10; x_0 = 1$

18. $f(x) = x^3 - x^2; x_0 = -2$

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11-6 Skills Practice

Recursion and Special Sequences

Find the first five terms of each sequence.

1. $a_1 = 4, a_{n+1} = a_n + 7$

2. $a_1 = -2, a_{n+1} = a_n + 3$

3. $a_1 = 5, a_{n+1} = 2a_n$

4. $a_1 = -4, a_{n+1} = 6 - a_n$

5. $a_1 = 1, a_{n+1} = a_n + n$

6. $a_1 = -1, a_{n+1} = n - a_n$

7. $a_1 = -6, a_{n+1} = a_n + n + 1$

8. $a_1 = 8, a_{n+1} = a_n - n - 2$

9. $a_1 = -3, a_{n+1} = 2a_n + 7$

10. $a_1 = 4, a_{n+1} = -2a_n - 5$

11. $a_1 = 0, a_2 = 1, a_{n+1} = a_n + a_{n-1}$

12. $a_1 = -1, a_2 = -1, a_{n+1} = a_n - a_{n-1}$

13. $a_1 = 3, a_2 = -5, a_{n+1} = -4a_n + a_{n-1}$

14. $a_1 = -3, a_2 = 2, a_{n+1} = a_{n-1} - a_n$

Find the first three iterates of each function for the given initial value.

15. $f(x) = 2x - 1, x_0 = 3$

16. $f(x) = 5x - 3, x_0 = 2$

17. $f(x) = 3x + 4, x_0 = -1$

18. $f(x) = 4x + 7, x_0 = -5$

19. $f(x) = -x - 3, x_0 = 10$

20. $f(x) = -3x + 6, x_0 = 6$

21. $f(x) = -3x + 4, x_0 = 2$

22. $f(x) = 6x - 5, x_0 = 1$

23. $f(x) = 7x + 1, x_0 = -4$

24. $f(x) = x^2 - 3x, x_0 = 5$

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11-6 Practice

Recursion and Special Sequences

Find the first five terms of each sequence.

1. $a_1 = 3, a_{n+1} = a_n + 5$

2. $a_1 = -7, a_{n+1} = a_n + 8$

3. $a_1 = -3, a_{n+1} = 3a_n + 2$

4. $a_1 = -8, a_{n+1} = 10 - a_n$

5. $a_1 = 4, a_{n+1} = n - a_n$

6. $a_1 = -3, a_{n+1} = 3a_n$

7. $a_1 = 4, a_{n+1} = -3a_n + 4$

8. $a_1 = 2, a_{n+1} = -4a_n - 5$

9. $a_1 = 3, a_2 = 1, a_{n+1} = a_n - a_{n-1}$

10. $a_1 = -1, a_2 = 5, a_{n+1} = 4a_{n-1} - a_n$

11. $a_1 = 2, a_2 = -3, a_{n+1} = 5a_n - 8a_{n-1}$

12. $a_1 = -2, a_2 = 1, a_{n+1} = -2a_n + 6a_{n-1}$

Find the first three iterates of each function for the given initial value.

13. $f(x) = 3x + 4, x_0 = -1$

14. $f(x) = 10x + 2, x_0 = -1$

15. $f(x) = 8 + 3x, x_0 = 1$

16. $f(x) = 8 - x, x_0 = -3$

17. $f(x) = 4x + 5, x_0 = -1$

18. $f(x) = 5(x + 3), x_0 = -2$

19. $f(x) = -8x + 9, x_0 = 1$

20. $f(x) = -4x^2, x_0 = -1$

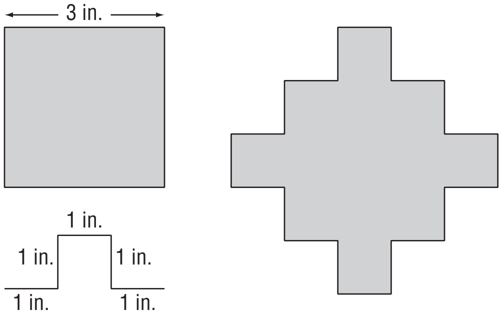
21. $f(x) = x^2 - 1, x_0 = 3$

22. $f(x) = 2x^2; x_0 = 5$

23. INFLATION Iterating the function $c(x) = 1.05x$ gives the future cost of an item at a constant 5% inflation rate. Find the cost of a $2000 ring in five years at 5% inflation.

FRACTALS For Exercises 24–27, use the following information.

Replacing each side of the square shown with the combination of segments below it gives the figure to its right.



24. What is the perimeter of the original square?

25. What is the perimeter of the new shape?

26. If you repeat the process by replacing each side of the new shape by a proportional combination of 5 segments, what will the perimeter of the third shape be?

27. What function $f(x)$ can you iterate to find the perimeter of each successive shape if you continue this process?

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11-6 Reading to Learn Mathematics

Recursion and Special Sequences

Pre-Activity **How is the Fibonacci sequence illustrated in nature?**

Read the introduction to Lesson 11-6 at the top of page 606 in your textbook.

What are the next three numbers in the sequence that gives the number of shoots corresponding to each month?

Reading the Lesson

1. Consider the sequence in which $a_1 = 4$ and $a_n = 2a_{n-1} + 5$.

a. Explain why this is a *recursive* formula.

b. Explain in your own words how to find the first four terms of this sequence. (Do not actually find any terms after the first.)

c. What happens to the terms of this sequence as n increases?

2. Consider the function $f(x) = 3x - 1$ with an initial value of $x_0 = 2$.

a. What does it mean to *iterate* this function?

b. Fill in the blanks to find the first three iterates. The blanks that follow the letter x are for subscripts.

$x_1 = f(x_{\_}) = f(\underline{\quad}) = 3(\underline{\quad}) - 1 = \underline{\quad} - 1 = \underline{\quad}$

$x_2 = f(x_{\_}) = f(\underline{\quad}) = 3(\underline{\quad}) - 1 = \underline{\quad}$

$x_3 = f(x_{\_}) = f(\underline{\quad}) = 3(\underline{\quad}) - 1 = \underline{\quad}$

c. As this process continues, what happens to the values of the iterates?

Helping You Remember

3. Use a dictionary to find the meanings of the words *recurrent* and *iterate*. How can the meanings of these words help you to remember the meaning of the mathematical terms *recursive* and *iteration*? How are these ideas related?

11-6 Enrichment

Continued Fractions

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$$

Example 1 **Evaluate the continued fraction above. Start at the bottom and work your way up.**

Step 1: $4 + \frac{1}{5} = \frac{20}{5} + \frac{1}{5} = \frac{21}{5}$

Step 2: $\frac{1}{\frac{21}{5}} = \frac{5}{21}$

Step 3: $3 + \frac{5}{21} = \frac{63}{21} + \frac{5}{21} = \frac{68}{21}$

Step 4: $\frac{1}{\frac{68}{21}} = \frac{21}{68}$

Step 5: $2 + \frac{21}{68} = 2\frac{21}{68}$

Example 2 **Change $\frac{25}{11}$ into a continued fraction.**

Follow the steps.

Step 1: $\frac{25}{11} = \frac{22}{11} + \frac{3}{11} = 2 + \frac{3}{11}$

Step 2: $\frac{3}{11} = \frac{1}{\frac{11}{3}}$

Step 3: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3}$

Step 4: $\frac{2}{3} = \frac{1}{\frac{3}{2}}$

Step 5: $\frac{3}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2}$

Stop, because the numerator is 1.

Thus, $\frac{25}{11}$ can be written as $2 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}$

Evaluate each continued fraction.

1. $1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{3}}}}$

2. $0 + \cfrac{1}{6 + \cfrac{1}{4 + \frac{1}{2}}}$

3. $2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \frac{1}{10}}}}$

4. $5 + \cfrac{1}{7 + \cfrac{1}{9 + \frac{1}{11}}}$

Change each fraction into a continued fraction.

5. $\frac{75}{31}$

6. $\frac{29}{8}$

7. $\frac{13}{19}$

11-7 Study Guide and Intervention

The Binomial Theorem

Pascal's Triangle **Pascal's triangle** is the pattern of coefficients of powers of binomials displayed in triangular form. Each row begins and ends with 1 and each coefficient is the sum of the two coefficients above it in the previous row.

Pascal's Triangle	$(a + b)^0$						1					
	$(a + b)^1$					1		1				
	$(a + b)^2$				1		2		1			
	$(a + b)^3$			1		3		3		1		
	$(a + b)^4$		1		4		6		4		1	
	$(a + b)^5$	1		5		10		10		5		1

Example **Use Pascal's triangle to find the number of possible sequences consisting of 3 *a*s and 2 *b*s.**

The coefficient 10 of the a^3b^2-term in the expansion of $(a + b)^5$ gives the number of sequences that result in three *a*s and two *b*s.

Exercises

Expand each power using Pascal's triangle.

1. $(a + 5)^4$

2. $(x - 2y)^6$

3. $(j - 3k)^5$

4. $(2s + t)^7$

5. $(2p + 3q)^6$

6. $\left(a - \frac{b}{2}\right)^4$

7. Ray tosses a coin 15 times. How many different sequences of tosses could result in 4 heads and 11 tails?

8. There are 9 true/false questions on a quiz. If twice as many of the statements are true as false, how many different sequences of true/false answers are possible?

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11-7 Study Guide and Intervention *(continued)*

The Binomial Theorem

The Binomial Theorem

Binomial Theorem	If n is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n$

Another useful form of the Binomial Theorem uses **factorial** notation and sigma notation.

Factorial	If n is a positive integer, then $n! = n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$.
Binomial Theorem, Factorial Form	$(a + b)^n = \frac{n!}{n!0!}a^nb^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0b^n$ $= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}b^k$

Example 1 **Evaluate $\frac{11!}{8!}$.**

$$\frac{11!}{8!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 11 \cdot 10 \cdot 9 = 990$$

Example 2 **Expand $(a - 3b)^4$.**

$$(a - 3b)^4 = \sum_{k=0}^{4} \frac{4!}{(4-k)!k!}a^{4-k}(-3b)^k$$
$$= \frac{4!}{4!0!}a^4 + \frac{4!}{3!1!}a^3(-3b)^1 + \frac{4!}{2!2!}a^2(-3b)^2 + \frac{4!}{1!3!}a(-3b)^3 + \frac{4!}{0!4!}(-3b)^4$$
$$= a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$$

Exercises

Evaluate each expression.

1. $5!$

2. $\frac{9!}{7!2!}$

3. $\frac{10!}{6!4!}$

Expand each power.

4. $(a - 3)^6$

5. $(r + 2s)^7$

6. $(4x + y)^4$

7. $\left(2 - \frac{m}{2}\right)^5$

Find the indicated term of each expansion.

8. third term of $(3x - y)^5$

9. fifth term of $(a + 1)^7$

10. fourth term of $(j + 2k)^8$

11. sixth term of $(10 - 3t)^7$

12. second term of $\left(m + \frac{2}{3}\right)^9$

13. seventh term of $(5x - 2)^{11}$

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11-7 Skills Practice

The Binomial Theorem

Evaluate each expression.

1. $8!$

2. $10!$

3. $12!$

4. $\frac{15!}{13!}$

5. $\frac{6!}{3!}$

6. $\frac{10!}{2!8!}$

7. $\frac{9!}{3!6!}$

8. $\frac{20!}{15!5!}$

Expand each power.

9. $(x - y)^3$

10. $(a + b)^5$

12. $(m + 1)^4$

11. $(g - h)^4$

13. $(r + 4)^3$

14. $(a - 5)^4$

15. $(y - 7)^3$

16. $(d + 2)^5$

17. $(x - 1)^4$

18. $(2a + b)^4$

19. $(c - 4d)^3$

20. $(2a + 3)^3$

Find the indicated term of each expansion.

21. fourth term of $(m + n)^{10}$

22. seventh term of $(x - y)^8$

23. third term of $(b + 6)^5$

24. sixth term of $(s - 2)^9$

25. fifth term of $(2a + 3)^6$

26. second term of $(3x - y)^7$

11-7 Practice

The Binomial Theorem

Evaluate each expression.

1. $7!$

2. $11!$

3. $\frac{9!}{5!}$

4. $\frac{20!}{18!}$

5. $\frac{8!}{6!2!}$

6. $\frac{8!}{5!3!}$

7. $\frac{12!}{6!6!}$

8. $\frac{41!}{3!38!}$

Expand each power.

9. $(n + v)^5$

10. $(x - y)^4$

11. $(x + y)^6$

12. $(r + 3)^5$

13. $(m - 5)^5$

14. $(x + 4)^4$

15. $(3x + y)^4$

16. $(2m - y)^4$

17. $(w - 3z)^3$

18. $(2d + 3)^6$

19. $(x + 2y)^5$

20. $(2x - y)^5$

21. $(a - 3b)^4$

22. $(3 - 2z)^4$

23. $(3m - 4n)^3$

24. $(5x - 2y)^4$

Find the indicated term of each expansion.

25. seventh term of $(a + b)^{10}$

26. sixth term of $(m - n)^{10}$

27. ninth term of $(r - s)^{14}$

28. tenth term of $(2x + y)^{12}$

29. fourth term of $(x - 3y)^6$

30. fifth term of $(2x - 1)^9$

31. GEOMETRY How many line segments can be drawn between ten points, no three of which are collinear, if you use exactly two of the ten points to draw each segment?

32. PROBABILITY If you toss a coin 4 times, how many different sequences of tosses will give exactly 3 heads and 1 tail or exactly 1 head and 3 tails?

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11-7 Reading to Learn Mathematics

The Binomial Theorem

Pre-Activity **How does a power of a binomial describe the numbers of boys and girls in a family?**

Read the introduction to Lesson 11-7 at the top of page 612 in your textbook.

- If a family has four children, list the sequences of births of girls and boys that result in three girls and one boy.
- Describe a way to figure out how many such sequences there are without listing them.

Lesson 11-7

Reading the Lesson

1. Consider the expansion of $(w + z)^5$.

a. How many terms does this expansion have?

b. In the second term of the expansion, what is the exponent of w?

What is the exponent of z?

What is the coefficient of the second term?

c. In the fourth term of the expansion, what is the exponent of w?

What is the exponent of z?

What is the coefficient of the fourth term?

d. What is the last term of this expansion?

2. a. State the definition of a *factorial* in your own words. (Do not use mathematical symbols in your definition.)

b. Write out the product that you would use to calculate 10!. (Do not actually calculate the product.)

c. Write an expression involving factorials that could be used to find the coefficient of the third term of the expansion of $(m - n)^6$. (Do not actually calculate the coefficient.)

Helping You Remember

3. Without using Pascal's triangle or factorials, what is an easy way to remember the first two and last two coefficients for the terms of the binomial expansion of $(a + b)^n$?

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11-7 Enrichment

Patterns in Pascal's Triangle

You have learned that the coefficients in the expansion of $(x + y)^n$ yield a number pyramid called **Pascal's triangle.**

Row 1 ⟶ 1
Row 2 ⟶ 1 1
Row 3 ⟶ 1 2 1
Row 4 ⟶ 1 3 4 1
Row 5 ⟶ 1 4 6 4 1
Row 6 ⟶ 1 5 10 10 5 1
Row 7 ⟶ 1 6 15 20 15 6 1

As many rows can be added to the bottom of the pyramid as you please.

This activity explores some of the interesting properties of this famous number pyramid.

1. Pick a row of Pascal's triangle.

- **a.** What is the sum of all the numbers in all the rows *above* the row you picked?
- **b.** What is the sum of all the numbers in the row you picked?
- **c.** How are your answers for parts **a** and **b** related?
- **d.** Repeat parts **a** through **c** for at least three more rows of Pascal's triangle. What generalization seems to be true?
- **e.** See if you can prove your generalization.

2. Pick any row of Pascal's triangle that comes after the first.

- **a.** Starting at the left end of the row, add the first number, the third number, the fifth number, and so on. State the sum.
- **b.** In the same row, add the second number, the fourth number, and so on. State the sum.
- **c.** How do the sums in parts **a** and **b** compare?
- **d.** Repeat parts **a** through **c** for at least three other rows of Pascal's triangle. What generalization seems to be true?

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11-8 Study Guide and Intervention

Proof and Mathematical Induction

Mathematical Induction Mathematical induction is a method of proof used to prove statements about positive integers.

Mathematical Induction Proof	**Step 1** Show that the statement is true for some integer n. **Step 2** Assume that the statement is true for some positive integer k where $k \geq n$. This assumption is called the **inductive hypothesis**. **Step 3** Show that the statement is true for the next integer $k + 1$.

Example **Prove that $5 + 11 + 17 + \ldots + (6n - 1) = 3n^2 + 2n$.**

Step 1 When $n = 1$, the left side of the given equation is $6(1) - 1 = 5$. The right side is $3(1)^2 + 2(1) = 5$. Thus the equation is true for $n = 1$.

Step 2 Assume that $5 + 11 + 17 + \ldots + (6k - 1) = 3k^2 + 2k$ for some positive integer k.

Step 3 Show that the equation is true for $n = k + 1$. First, add $[6(k + 1) - 1]$ to each side.

$$\begin{aligned} 5 + 11 + 17 + \ldots + (6k - 1) + [6(k + 1) - 1] &= 3k^2 + 2k + [6(k + 1) - 1] \\ &= 3k^2 + 2k + 6k + 5 && \text{Add.} \\ &= 3k^2 + 6k + 3 + 2k + 2 && \text{Rewrite.} \\ &= 3(k^2 + 2k + 1) + 2(k + 1) && \text{Factor.} \\ &= 3(k + 1)^2 + 2(k + 1) && \text{Factor.} \end{aligned}$$

The last expression above is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus the equation is true for $n = k + 1$.

This proves that $5 + 11 + 17 + \ldots + (6n - 1) = 3n^2 + 2n$ for all positive integers n.

Exercises

Prove that each statement is true for all positive integers.

1. $3 + 7 + 11 + \ldots + (4n - 1) = 2n^2 + n$.

2. $500 + 100 + 20 + \ldots + 4 \cdot 5^{4 - n} = 625\left(1 - \frac{1}{5^n}\right)$.

11-8 Study Guide and Intervention *(continued)*

Proof by Mathematical Induction

Counterexamples To show that a formula or other generalization is *not* true, find a **counterexample**. Often this is done by substituting values for a variable.

Example 1 **Find a counterexample for the formula $2n^2 + 2n + 3 = 2^{n+2} - 1$.**
Check the first few positive integers.

n	Left Side of Formula	Right Side of Formula	
1	$2(1)^2 + 2(1) + 3 = 2 + 2 + 3$ or 7	$2^{1+2} - 1 = 2^3 - 1$ or 7	*true*
2	$2(2)^2 + 2(2) + 3 = 8 + 4 + 3$ or 15	$2^{2+2} - 1 = 2^4 - 1$ or 15	*true*
3	$2(3)^2 + 2(3) + 3 = 18 + 6 + 3$ or 27	$2^{3+2} - 1 = 2^5 - 1$ or 31	*false*

The value $n = 3$ provides a counterexample for the formula.

Example 2 **Find a counterexample for the statement $x^2 + 4$ is either prime or divisible by 4.**

n	$x^2 + 4$	*True?*	*n*	$x^2 + 4$	*True?*
1	1 + 4 or 5	*Prime*	6	36 + 4 or 40	*Div. by* 4
2	4 + 4 or 8	*Div. by* 4	7	49 + 4 or 53	*Prime*
3	9 + 4 or 13	*Prime*	8	64 + 4 or 68	*Div. by* 4
4	16 + 4 or 20	*Div. by* 4	9	81 + 4 or 85	*Neither*
5	25 + 4 or 29	*Prime*			

The value $n = 9$ provides a counterexample.

Exercises

Find a counterexample for each statement.

1. $1 + 5 + 9 + \ldots + (4n - 3) = 4n - 3$

2. $100 + 110 + 120 + \ldots + (10n + 90) = 5n^2 + 95$

3. $900 + 300 + 100 + \ldots + 100(3^{3-n}) = 900 \cdot \frac{2n}{n+1}$

4. $x^2 + x + 1$ is prime.

5. $2n + 1$ is a prime number.

6. $7n - 5$ is a prime number.

7. $\frac{1}{2} + 1 + \frac{3}{2} + \ldots + \frac{n}{2} = n - \frac{1}{2}$

8. $5n^2 + 1$ is divisible by 3.

9. $n^2 - 3n + 1$ is prime for $n > 2$.

10. $4n^2 - 1$ is divisible by either 3 or 5.

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11-8 Skills Practice

Proof and Mathematical Induction

Prove that each statement is true for all positive integers.

1. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

2. $2 + 4 + 6 + \ldots + 2n = n^2 + n$

3. $6^n - 1$ is divisible by 5.

Find a counterexample for each statement.

4. $3^n + 3n$ is divisible by 6.

5. $1 + 4 + 8 + \ldots + 2^n = \frac{n(n + 1)(2n + 1)}{6}$

11-8 Practice

Proof and Mathematical Induction

Prove that each statement is true for all positive integers.

1. $1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$

2. $1 + 4 + 9 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

3. $18^n - 1$ is a multiple of 17.

Find a counterexample for each statement.

4. $1 + 4 + 7 + \ldots + (3n - 2) = n^3 - n^2 + 1$

5. $5^n - 2n - 3$ is divisible by 3.

6. $1 + 3 + 5 + \ldots + (2n - 1) = \frac{n^2 + 3n - 2}{2}$

7. $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^4 - n^3 + 1$

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11-8 Reading to Learn Mathematics

Proof and Mathematical Induction

Pre-Activity **How does the concept of a ladder help you prove statements about numbers?**

Read the introduction to Lesson 11-8 at the top of page 618 in your textbook.

What are two ways in which a ladder could be constructed so that you could not reach every step of the ladder?

Reading the Lesson

1. Fill in the blanks to describe the three steps in a proof by mathematical induction.

Step 1 Show that the statement is ____________ for the number ________.

Step 2 Assume that the statement is ____________ for some positive ____________ k.

This assumption is called the ________________________.

Step 3 Show that the statement is ____________ for the next integer __________.

2. Suppose that you wanted to prove that the following statement is true for all positive integers.

$$3 + 6 + 9 + \ldots + 3n = \frac{3n(n + 1)}{2}$$

a. Which of the following statements shows that the statement is true for $n = 1$?

i. $3 = \frac{3 \cdot 2 + 1}{2}$ **ii.** $3 = \frac{3 \cdot 1 \cdot 2}{2}$ **iii.** $3 = \frac{3 + 1 + 2}{2}$

b. Which of the following is the statement for $n = k + 1$?

i. $3 + 6 + 9 + \ldots + 3^k = \frac{3k(k + 1)}{2}$

ii. $3 + 6 + 9 + \ldots + 3^{k + 1} = \frac{3k(k + 1)}{2}$

iii. $3 + 6 + 9 + \ldots + 3^{k + 1} = 3(k + 1)(k + 2)$

iv. $3 + 6 + 9 + \ldots + 3(k + 1) = \frac{3(k + 1)(k + 2)}{2}$

Helping You Remember

3. Many students confuse the roles of n and k in a proof by mathematical induction. What is a good way to remember the difference in the ways these variables are used in such a proof?

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11-8 Enrichment

Proof by Induction

Mathematical induction is a useful tool when you want to prove that a statement is true for all natural numbers.

The three steps in using induction are:

1. Prove that the statement is true for $n = 1$.
2. Prove that if the statement is true for the natural number n, it must also be true for $n + 1$.
3. Conclude that the statement is true for all natural numbers.

Follow the steps to complete each proof.

Theorem A: The sum of the first n odd natural numbers is equal to n^2.

1. Show that the theorem is true for $n = 1$.

2. Suppose $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Show that $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$.

3. Summarize the results of problems 1 and 2.

Theorem B: Show that $a^n - b^n$ is exactly divisible by $a - b$ for n equal to 1, 2, 3, and all natural numbers.

4. Show that the theorem is true for $n = 1$.

5. The expression $a^{n+1} - b^{n+1}$ can be rewritten as $a(a^n - b^n) + b^n(a - b)$. Verify that this is true.

6. Suppose $a - b$ is a factor of $a^n - b^n$. Use the result in problem 5 to show that $a - b$ must then also be a factor of $a^{n+1} - b^{n+1}$.

7. Summarize the results of problems 4 through 6.

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11-1 Study Guide and Intervention

Arithmetic Sequences

Arithmetic Sequences An **arithmetic sequence** is a sequence of numbers in which each **term** after the first term is found by adding the **common difference** to the preceding term.

nth Term of an Arithmetic Sequence	$a_n = a_1 + (n - 1)d$, where a_1 is the first term, d is the common difference, and n is any positive integer

Example 1 **Find the next four terms of the arithmetic sequence 7, 11, 15, ….**

Find the common difference by subtracting two consecutive terms.

11 − 7 = 4 and 15 − 11 = 4, so $d = 4$.

Now add 4 to the third term of the sequence, and then continue adding 4 until the four terms are found. The next four terms of the sequence are 19, 23, 27, and 31.

Example 2 **Find the thirteenth term of the arithmetic sequence with $a_1 = 21$ and $d = -6$.**

Use the formula for the nth term of an arithmetic sequence with $a_1 = 21$, $n = 13$, and $d = -6$.

$a_n = a_1 + (n - 1)d$ Formula for nth term

$a_{13} = 21 + (13 - 1)(-6)$ $n = 13$, $a_1 = 21$, $d = -6$

$a_{13} = -51$ Simplify.

The thirteenth term is −51.

Example 3 **Write an equation for the nth term of the arithmetic sequence −14, −5, 4, 13, ….**

In this sequence $a_1 = -14$ and $d = 9$. Use the formula for a_n to write an equation.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$= -14 + (n - 1)9$ $a_1 = -14$, $d = 9$

$= -14 + 9n - 9$ Distributive Property

$= 9n - 23$ Simplify.

Exercises

Find the next four terms of each arithmetic sequence.

1. 106, 111, 116, … **121, 126, 131, 136**

2. −28, −31, −34, … **−37, −40, −43, −46**

3. 207, 194, 181, … **168, 155, 142, 129**

Find the first five terms of each arithmetic sequence described.

4. $a_1 = 101$, $d = 9$ **101, 110, 119, 128, 137**

5. $a_1 = -60$, $d = 4$ **−60, −56, −52, −48, −44**

6. $a_1 = 210$, $d = -40$ **210, 170, 130, 90, 50**

Find the indicated term of each arithmetic sequence.

7. $a_1 = 4$, $d = 6$, $n = 14$ **82**

8. $a_1 = -4$, $d = -2$, $n = 12$ **−26**

9. $a_1 = 80$, $d = -8$, $n = 21$ **−80**

10. a_{10} for 0, −3, −6, −9, … **−27**

Write an equation for the nth term of each arithmetic sequence.

11. 18, 25, 32, 39, … **$7n + 11$**

12. −110, −85, −60, −35, … **$25n - 135$**

13. 6.2, 8.1, 10.0, 11.9, … **$1.9n + 4.3$**

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11-1 Study Guide and Intervention (continued)

Arithmetic Sequences

Arithmetic Means The **arithmetic means** of an arithmetic sequence are the terms between any two nonsuccessive terms of the sequence.

To find the k arithmetic means between two terms of a sequence, use the following steps.

Step 1	Let the two terms given be a_1 and a_n, where $n = k + 2$.
Step 2	Substitute in the formula $a_n = a_1 + (n - 1)d$.
Step 3	Solve for d, and use that value to find the k arithmetic means: $a_1 + d, a_1 + 2d, \ldots, a_1 + kd$.

Example **Find the five arithmetic means between 37 and 121.**

You can use the nth term formula to find the common difference. In the sequence, 37, ?, ?, ?, ?, ?, 121, …, a_1 is 37 and a_7 is 121.

$a_n = a_1 + (n - 1)d$ Formula for the nth term

$121 = 37 + (7 - 1)d$ $a_1 = 37$, $a_7 = 121$, $n = 7$

$121 = 37 + 6d$ Simplify.

$84 = 6d$ Subtract 37 from each side.

$d = 14$ Divide each side by 6.

Now use the value of d to find the five arithmetic means.

$$37 \underbrace{\ \ }_{+14} 51 \underbrace{\ \ }_{+14} 65 \underbrace{\ \ }_{+14} 79 \underbrace{\ \ }_{+14} 93 \underbrace{\ \ }_{+14} 107 \underbrace{\ \ }_{+14} 121$$

The arithmetic means are 51, 65, 79, 93, and 107.

Exercises

Find the arithmetic means in each sequence.

1. 5, ?, ?, ?, −3 **3, 1, −1**

2. 18, ?, ?, ?, −2 **13, 8, 3**

3. 16, ?, ?, 37 **23, 30**

4. 108, ?, ?, ?, ?, 48 **96, 84, 72, 60**

5. −14, ?, ?, ?, −30 **−18, −22, −26**

6. 29, ?, ?, ?, 89 **44, 59, 74**

7. 61, ?, ?, ?, ?, 116 **72, 83, 94, 105**

8. 45, ?, ?, ?, ?, ?, 81 **51, 57, 63, 69, 75**

9. −18, ?, ?, ?, 14 **−10, −2, 6**

10. −40, ?, ?, ?, ?, ?, −82 **−47, −54, −61, −68, −75**

11. 100, ?, ?, 235 **145, 190**

12. 80, ?, ?, ?, ?, −30 **58, 36, 14, −8**

13. 450, ?, ?, ?, 570 **480, 510, 540**

14. 27, ?, ?, ?, ?, ?, 57 **32, 37, 42, 47, 52**

15. 125, ?, ?, ?, 185 **140, 155, 170**

16. 230, ?, ?, ?, ?, ?, 128 **213, 196, 179, 162, 145**

17. −20, ?, ?, ?, ?, 370 **58, 136, 214, 292**

18. 48, ?, ?, ?, 100 **61, 74, 87**

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11-1 Skills Practice

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 7, 11, 15, … **19, 23, 27, 31**

2. −10, −5, 0, … **5, 10, 15, 20**

3. 101, 202, 303, … **404, 505, 606, 707**

4. 15, 7, −1, … **−9, −17, −25, −33**

5. −67, −60, −53, … **−46, −39, −32, −25**

6. −12, −15, −18, … **−21, −24, −27, −30**

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 6, d = 9$ **6, 15, 24, 33, 42**

8. $a_1 = 27, d = 4$ **27, 31, 35, 39, 43**

9. $a_1 = -12, d = 5$ **−12, −7, −2, 3, 8**

10. $a_1 = 93, d = -15$ **93, 78, 63, 48, 33**

11. $a_1 = -64, d = 11$ **−64, −53, −42, −31, −20**

12. $a_1 = -47, d = -20$ **−47, −67, −87, −107, −127**

Find the indicated term of each arithmetic sequence.

13. $a_1 = 2, d = 6, n = 12$ **68**

14. $a_1 = 18, d = 2, n = 8$ **32**

15. $a_1 = 23, d = 5, n = 23$ **133**

16. $a_1 = 15, d = -1, n = 25$ **−9**

17. a_{31} for 34, 38, 42, … **154**

18. a_{42} for 27, 30, 33, … **150**

Complete the statement for each arithmetic sequence.

19. 55 is the __?__th term of 4, 7, 10, … . **18**

20. 163 is the __?__th term of −5, 2, 9, … . **25**

Write an equation for the *n*th term of each arithmetic sequence.

21. 4, 7, 10, 13, … $\mathbf{a_n = 3n + 1}$

22. −1, 1, 3, 5, … $\mathbf{a_n = 2n - 3}$

23. −1, 3, 7, 11, … $\mathbf{a_n = 4n - 5}$

24. 7, 2, −3, −8, … $\mathbf{a_n = -5n + 12}$

Find the arithmetic means in each sequence.

25. 6, __?__, __?__, __?__, 38 **14, 22, 30**

26. 63, __?__, __?__, __?__, 147 **84, 105, 126**

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11-1 Practice (Average)

Arithmetic Sequences

Find the next four terms of each arithmetic sequence.

1. 5, 8, 11, … **14, 17, 20, 23**

2. −4, −6, −8, … **−10, −12, −14, −16**

3. 100, 93, 86, … **79, 72, 65, 58**

4. −24, −19, −14, … **−9, −4, 1, 6**

5. $\frac{7}{2}, 6, \frac{17}{2}, 11, \ldots$ $\mathbf{\frac{27}{2}, 16, \frac{37}{2}, 21}$

6. 4.8, 4.1, 3.4, … **2.7, 2, 1.3, 0.6**

Find the first five terms of each arithmetic sequence described.

7. $a_1 = 7, d = 7$ **7, 14, 21, 28, 35**

8. $a_1 = -8, d = 2$ **−8, −6, −4, −2, 0**

9. $a_1 = -12, d = -4$ **−12, −16, −20, −24, −28**

10. $a_1 = \frac{1}{2}, d = \frac{1}{2}$ $\mathbf{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}}$

11. $a_1 = -\frac{5}{6}, d = -\frac{1}{3}$ $\mathbf{-\frac{5}{6}, -\frac{7}{6}, -\frac{3}{2}, -\frac{11}{6}, -\frac{13}{6}}$

12. $a_1 = 10.2, d = -5.8$ **10.2, 4.4, −1.4, −7.2, −13**

Find the indicated term of each arithmetic sequence.

13. $a_1 = 5, d = 3, n = 10$ **32**

14. $a_1 = 9, d = 3, n = 29$ **93**

15. a_{18} for −6, −7, −8, … . **−23**

16. a_{37} for 124, 119, 114, … . **−56**

17. $a_1 = \frac{9}{5}, d = -\frac{3}{5}, n = 10$ $\mathbf{-\frac{18}{5}}$

18. $a_1 = 14.25, d = 0.15, n = 31$ **18.75**

Complete the statement for each arithmetic sequence.

19. 166 is the __?__th term of 30, 34, 38, … **35**

20. 2 is the __?__th term of $\frac{3}{5}, \frac{4}{5}, 1, \ldots$ **8**

Write an equation for the *n*th term of each arithmetic sequence.

21. −5, −3, −1, 1, … $\mathbf{a_n = 2n - 7}$

22. −8, −11, −14, −17, … $\mathbf{a_n = -3n - 5}$

23. 1, −1, −3, −5, … $\mathbf{a_n = -2n + 3}$

24. −5, 3, 11, 19, … $\mathbf{a_n = 8n - 13}$

Find the arithmetic means in each sequence.

25. −5, __?__, __?__, __?__, 11 **−1, 3, 7**

26. 82, __?__, __?__, __?__, 18 **66, 50, 34**

27. EDUCATION Trevor Koba has opened an English Language School in Isehara, Japan. He began with 26 students. If he enrolls 3 new students each week, in how many weeks will he have 101 students? **26 wk**

28. SALARIES Yolanda interviewed for a job that promised her a starting salary of $32,000 with a $1250 raise at the end of each year. What will her salary be during her sixth year if she accepts the job? **$38,250**

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11-1 Reading to Learn Mathematics

Arithmetic Sequences

Pre-Activity **How are arithmetic sequences related to roofing?**

Read the introduction to Lesson 11-1 at the top of page 578 in your textbook.

Describe how you would find the number of shingles needed for the fifteenth row. (Do not actually calculate this number.) Explain why your method will give the correct answer. **Sample answer: Add 3 times 14 to 2. This works because the first row has 2 shingles and 3 more are added 14 times to go from the first row to the fifteenth row.**

Reading the Lesson

1. Consider the formula $a_n = a_1 + (n - 1)d$.

a. What is this formula used to find?
a particular term of an arithmetic sequence

b. What do each of the following represent?

a_n: **the *n*th term**

a_1: **the first term**

n: **a positive integer that indicates which term you are finding**

d: **the common difference**

2. Consider the equation $a_n = -3n + 5$.

a. What does this equation represent? **Sample answer: It gives the *n*th term of an arithmetic sequence with first term 2 and common difference −3.**

b. Is the graph of this equation a straight line? Explain your answer. **Sample answer: No; the graph is a set of points that fall on a line, but the points do not fill the line.**

c. The functions represented by the equations $a_n = -3n + 5$ and $f(x) = -3x + 5$ are alike in that they have the same formula. How are they different? **Sample answer: They have different domains. The domain of the first function is the set of positive integers. The domain of the second function is the set of all real numbers.**

Helping You Remember

3. A good way to remember something is to explain it to someone else. Suppose that your classmate Shala has trouble remembering the formula $a_n = a_1 + (n - 1)d$ correctly. She thinks that the formula should be $a_n = a_1 + nd$. How would you explain to her that she should use $(n - 1)d$ rather than nd in the formula? **Sample answer: Each term after the first in an arithmetic sequence is found by adding *d* to the previous term. You would add *d* once to get to the second term, twice to get to the third term, and so on. So *d* is added *n* − 1 times, not *n* times, to get the *n*th term.**

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11-1 Enrichment

Fibonacci Sequence

Leonardo Fibonacci first discovered the sequence of numbers named for him while studying rabbits. He wanted to know how many pairs of rabbits would be produced in n months, starting with a single pair of newborn rabbits. He made the following assumptions.

1. Newborn rabbits become adults in one month.
2. Each pair of rabbits produces one pair each month.
3. No rabbits die.

Let F_n represent the number of pairs of rabbits at the end of n months. If you begin with one pair of newborn rabbits, $F_0 = F_1 = 1$. This pair of rabbits would produce one pair at the end of the second month, so $F_2 = 1 + 1$, or 2. At the end of the third month, the first pair of rabbits would produce another pair. Thus, $F_3 = 2 + 1$, or 3.

The chart below shows the number of rabbits each month for several months.

Month	Adult Pairs	Newborn Pairs	Total
F_0	0	1	1
F_1	1	0	1
F_2	1	1	2
F_3	2	1	3
F_4	3	2	5
F_5	5	3	8

Solve.

1. Starting with a single pair of newborn rabbits, how many pairs of rabbits would there be at the end of 12 months?
233

2. Write the first 10 terms of the sequence for which $F_0 = 3$, $F_1 = 4$, and $F_n = F_{n-2} + F_{n-1}$.
3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322

3. Write the first 10 terms of the sequence for which $F_0 = 1$, $F_1 = 5$, $F_n = F_{n-2} + F_{n-1}$.
1, 5, 6, 11, 17, 28, 45, 73, 118, 191, 309

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11-2 Study Guide and Intervention

Arithmetic Series

Arithmetic Series An **arithmetic series** is the sum of consecutive terms of an arithmetic sequence.

Sum of an Arithmetic Series	The sum S_n of the first n terms of an arithmetic series is given by the formula $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ or $S_n = \frac{n}{2}(a_1 + a_n)$

Example 1 **Find S_n for the arithmetic series with $a_1 = 14$, $a_n = 101$, and $n = 30$.**

Use the sum formula for an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{30} = \frac{30}{2}(14 + 101)$ $n = 30$, $a_1 = 14$, $a_n = 101$

$= 15(115)$ Simplify.

$= 1725$ Multiply.

The sum of the series is 1725.

Example 2 **Find the sum of all positive odd integers less than 180.**

The series is 1 + 3 + 5 + … + 179.

Find n using the formula for the nth term of an arithmetic sequence.

$a_n = a_1 + (n - 1)d$ Formula for nth term

$179 = 1 + (n - 1)2$ $a_n = 179$, $a_1 = 1$, $d = 2$

$179 = 2n - 1$ Simplify.

$180 = 2n$ Add 1 to each side.

$n = 90$ Divide each side by 2.

Then use the sum formula for an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{90} = \frac{90}{2}(1 + 179)$ $n = 90$, $a_1 = 1$, $a_n = 179$

$= 45(180)$ Simplify.

$= 8100$ Multiply.

The sum of all positive odd integers less than 180 is 8100.

Exercises

Find S_n for each arithmetic series described.

1. $a_1 = 12$, $a_n = 100$, $n = 12$ **672**
2. $a_1 = 50$, $a_n = -50$, $n = 15$ **0**
3. $a_1 = 60$, $a_n = -136$, $n = 50$ **−1900**
4. $a_1 = 20$, $d = 4$, $a_n = 112$ **1584**
5. $a_1 = 180$, $d = -8$, $a_n = 68$ **1860**
6. $a_1 = -8$, $d = -7$, $a_n = -71$ **−395**
7. $a_1 = 42$, $n = 8$, $d = 6$ **504**
8. $a_1 = 4$, $n = 20$, $d = 2\frac{1}{2}$ **555**
9. $a_1 = 32$, $n = 27$, $d = 3$ **1917**

Find the sum of each arithmetic series.

10. 8 + 6 + 4 + … + −10 **−10**
11. 16 + 22 + 28 + … + 112 **1088**
12. −45 + (−41) + (−37) + … + 35 **−105**

Find the first three terms of each arithmetic series described.

13. $a_1 = 12$, $a_n = 174$, $S_n = 1767$ **12, 21, 30**
14. $a_1 = 80$, $a_n = -115$, $S_n = -245$ **80, 65, 50**
15. $a_1 = 6.2$, $a_n = 12.6$, $S_n = 84.6$ **6.2, 7.0, 7.8**

Lesson 11-2

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11-2 Study Guide and Intervention *(continued)*

Arithmetic Series

Sigma Notation A shorthand notation for representing a series makes use of the Greek letter Σ. The **sigma notation** for the series 6 + 12 + 18 + 24 + 30 is $\sum_{n=1}^{5} 6n$.

Example **Evaluate $\sum_{k=1}^{18} (3k + 4)$.**

The sum is an arithmetic series with common difference 3. Substituting $k = 1$ and $k = 18$ into the expression $3k + 4$ gives $a_1 = 3(1) + 4 = 7$ and $a_{18} = 3(18) + 4 = 58$. There are 18 terms in the series, so $n = 18$. Use the formula for the sum of an arithmetic series.

$S_n = \frac{n}{2}(a_1 + a_n)$ Sum formula

$S_{18} = \frac{18}{2}(7 + 58)$ $n = 18$, $a_1 = 7$, $a_n = 58$

$= 9(65)$ Simplify.

$= 585$ Multiply.

So $\sum_{k=1}^{18} (3k + 4) = 585$.

Exercises

Find the sum of each arithmetic series.

1. $\sum_{n=1}^{20} (2n + 1)$ **440**
2. $\sum_{n=5}^{25} (x - 1)$ **294**
3. $\sum_{k=1}^{18} (2k - 7)$ **216**
4. $\sum_{r=10}^{75} (2r - 200)$ **−7590**
5. $\sum_{x=1}^{15} (6x + 3)$ **765**
6. $\sum_{t=1}^{50} (500 - 6t)$ **17,350**
7. $\sum_{k=1}^{80} (100 - k)$ **4760**
8. $\sum_{n=20}^{85} (n - 100)$ **−3135**
9. $\sum_{s=1}^{200} 3s$ **60,300**
10. $\sum_{m=14}^{28} (2m - 50)$ **−120**
11. $\sum_{p=1}^{36} (5p - 20)$ **2610**
12. $\sum_{j=12}^{32} (25 - 2j)$ **−399**
13. $\sum_{n=18}^{42} (4n - 9)$ **2775**
14. $\sum_{n=20}^{50} (3n + 4)$ **3379**
15. $\sum_{j=5}^{44} (7j - 3)$ **6740**

Answers

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11-2 Skills Practice

Arithmetic Series

Find S_n for each arithmetic series described.

1. $a_1 = 1, a_n = 19, n = 10$ **100**

2. $a_1 = -5, a_n = 13, n = 7$ **28**

3. $a_1 = 12, a_n = -23, n = 8$ **−44**

4. $a_1 = 7, n = 11, a_n = 67$ **407**

5. $a_1 = 5, n = 10, a_n = 32$ **185**

6. $a_1 = -4, n = 10, a_n = -22$ **−130**

7. $a_1 = -8, d = -5, n = 12$ **−426**

8. $a_1 = 1, d = 3, n = 15$ **330**

9. $a_1 = 100, d = -7, a_n = 37$ **685**

10. $a_1 = -9, d = 4, a_n = 27$ **90**

11. $d = 2, n = 26, a_n = 42$ **442**

12. $d = -12, n = 11, a_n = -52$ **88**

Find the sum of each arithmetic series.

13. $1 + 4 + 7 + 10 + \ldots + 43$ **330**

14. $5 + 8 + 11 + 14 + \ldots + 32$ **185**

15. $3 + 5 + 7 + 9 + \ldots + 19$ **99**

16. $-2 + (-5) + (-8) + \ldots + (-20)$ **−77**

17. $\sum_{n=1}^{5} (2n - 3)$ **15**

18. $\sum_{n=1}^{18} (10 + 3n)$ **693**

19. $\sum_{n=2}^{10} (4n + 1)$ **225**

20. $\sum_{n=5}^{12} (4 - 3n)$ **−172**

Find the first three terms of each arithmetic series described.

21. $a_1 = 4, a_n = 31, S_n = 175$ **4, 7, 10**

22. $a_1 = -3, a_n = 41, S_n = 228$ **−3, 1, 5**

23. $n = 10, a_n = 41, S_n = 230$ **5, 9, 13**

24. $n = 19, a_n = 85, S_n = 760$ **−5, 0, 5**

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11-2 Practice (Average)

Arithmetic Series

Find S_n for each arithmetic series described.

1. $a_1 = 16, a_n = 98, n = 13$ **741**

2. $a_1 = 3, a_n = 36, n = 12$ **234**

3. $a_1 = -5, a_n = -26, n = 8$ **−124**

4. $a_1 = 5, n = 10, a_n = -13$ **−40**

5. $a_1 = 6, n = 15, a_n = -22$ **−120**

6. $a_1 = -20, n = 25, a_n = 148$ **1600**

7. $a_1 = 13, d = -6, n = 21$ **−987**

8. $a_1 = 5, d = 4, n = 11$ **275**

9. $a_1 = 5, d = 2, a_n = 33$ **285**

10. $a_1 = -121, d = 3, a_n = 5$ **−2494**

11. $d = 0.4, n = 10, a_n = 3.8$ **20**

12. $d = -\frac{2}{3}, n = 16, a_n = 44$ **784**

Find the sum of each arithmetic series.

13. $5 + 7 + 9 + 11 + \ldots + 27$ **192**

14. $-4 + 1 + 6 + 11 + \ldots + 91$ **870**

15. $13 + 20 + 27 + \ldots + 272$ **5415**

16. $89 + 86 + 83 + 80 + \ldots + 20$ **1308**

17. $\sum_{n=1}^{4} (1 - 2n)$ **−16**

18. $\sum_{j=1}^{6} (5 + 3n)$ **93**

19. $\sum_{n=1}^{5} (9 - 4n)$ **−15**

20. $\sum_{k=4}^{10} (2k + 1)$ **105**

21. $\sum_{n=3}^{8} (5n - 10)$ **105**

22. $\sum_{n=1}^{101} (4 - 4n)$ **−20,200**

Find the first three terms of each arithmetic series described.

23. $a_1 = 14, a_n = -85, S_n = -1207$

14, 11, 8

24. $a_1 = 1, a_n = 19, S_n = 100$

1, 3, 5

25. $n = 16, a_n = 15, S_n = -120$

−30, −27, −24

26. $n = 15, a_n = 5\frac{4}{5}, S_n = 45$

$\mathbf{\frac{1}{5}, \frac{3}{5}, 1}$

27. STACKING A health club rolls its towels and stacks them in layers on a shelf. Each layer of towels has one less towel than the layer below it. If there are 20 towels on the bottom layer and one towel on the top layer, how many towels are stacked on the shelf? **210 towels**

28. BUSINESS A merchant places $1 in a jackpot on August 1, then draws the name of a regular customer. If the customer is present, he or she wins the $1 in the jackpot. If the customer is not present, the merchant adds $2 to the jackpot on August 2 and draws another name. Each day the merchant adds an amount equal to the day of the month. If the first person to win the jackpot wins $496, on what day of the month was her or his name drawn? **August 31**

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11-2 Reading to Learn Mathematics

Arithmetic Series

Pre-Activity **How do arithmetic series apply to amphitheaters?**

Read the introduction to Lesson 11-2 at the top of page 583 in your textbook.

Suppose that an amphitheater can seat 50 people in the first row and that each row thereafter can seat 9 more people than the previous row. Using the vocabulary of arithmetic sequences, describe how you would find the number of people who could be seated in the first 10 rows. (Do not actually calculate the sum.) **Sample answer: Find the first 10 terms of an arithmetic sequence with first term 50 and common difference 9. Then add these 10 terms.**

Reading the Lesson

1. What is the relationship between an arithmetic sequence and the corresponding arithmetic series? **Sample answer: An arithmetic sequence is a list of terms with a common difference between successive terms. The corresponding arithmetic series is the sum of the terms of the sequence.**

2. Consider the formula $S_n = \frac{n}{2}(a_1 + a_n)$. Explain the meaning of this formula in words.
Sample answer: To find the sum of the first *n* terms of an arithmetic sequence, find half the number of terms you are adding. Multiply this number by the sum of the first term and the *n*th term.

3. a. What is the purpose of sigma notation?
Sample answer: to write a series in a concise form

b. Consider the expression $\sum_{i=2}^{12}(4i - 2)$.

This form of writing a sum is called **sigma notation**.

The variable i is called the **index of summation**.

The first value of i is **2**.

The last value of i is **12**.

How would you read this expression? **The sum of 4*i* –2 as *i* goes from 2 to 12.**

Helping You Remember

4. A good way to remember something is to relate it to something you already know. How can your knowledge of how to find the average of two numbers help you remember the formula $S_n = \frac{n}{2}(a_1 + a_n)$? **Sample answer: Rewrite the formula as** $S_n = n \cdot \frac{a_1 + a_n}{2}$**. The average of the first and last terms is given by the expression** $\frac{a_1 + a_n}{2}$**. The sum of the first *n* terms is the average of the first and last terms multiplied by the number of terms.**

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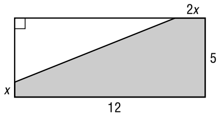
11-2 Enrichment

Geometric Puzzlers

For the problems on this page, you will need to use the Pythagorean Theorem and the formulas for the area of a triangle and a trapezoid.

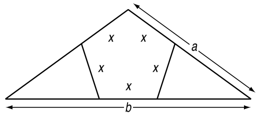
1. A rectangle measures 5 by 12 units. The upper left corner is cut off as shown in the diagram.



a. Find the area $A(x)$ of the shaded pentagon.
$A(x) = 60 - (5 - x)(6 - x)$

b. Find x and $2x$ so that $A(x)$ is a maximum. What happens to the cut-off triangle?
$x = 5$ and $2x = 10$; the triangle will not exist.

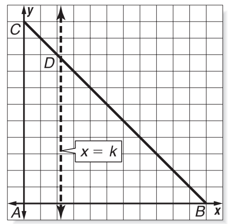
2. A triangle with sides of lengths a, a, and b is isosceles. Two triangles are cut off so that the remaining pentagon has five equal sides of length x. The value of x can be found using this equation.
$(2b - a)x^2 + (4a^2 - b^2)(2x - a) = 0$



a. Find x when $a = 10$ and $b = 12$.
$x \approx 4.46$

b. Can a be equal to $2b$?
Yes, but it would not be possible to have a pentagon of the type described.

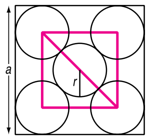
3. The coordinates of the vertices of a triangle are $A(0, 0)$, $B(11, 0)$, and $C(0, 11)$. A line $x = k$ cuts the triangle into two regions having equal area.



a. What are the coordinates of point D?
$(k, 11 - k)$

b. Write and solve an equation for finding the value of k.
$\frac{1}{2}k(11 + 11 - k) = 22$;
$k = 11 - \sqrt{77}$

4. Inside a square are five circles with the same radius.



a. Connect the center of the top left circle to the center of the bottom right circle. Express this length in terms of r.
$4r$

b. Draw the square with vertices at the centers of the four outside circles. Express the diagonal of this square in terms of r and a.
$(a - 2r)\sqrt{2}$

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11-3 Study Guide and Intervention

Geometric Sequences

Geometric Sequences A **geometric sequence** is a sequence in which each term after the first is the product of the previous term and a constant called the **constant ratio**.

nth Term of a Geometric Sequence	$a_n = a_1 \cdot r^{n-1}$, where a_1 is the first term, r is the common ratio, and n is any positive integer

Example 1 **Find the next two terms of the geometric sequence 1200, 480, 192, … .**

Since $\frac{480}{1200} = 0.4$ and $\frac{192}{480} = 0.4$, the sequence has a common ratio of 0.4. The next two terms in the sequence are $192(0.4) = 76.8$ and $76.8(0.4) = 30.72$.

Example 2 **Write an equation for the nth term of the geometric sequence 3.6, 10.8, 32.4, … .**

In this sequence $a_1 = 3.6$ and $r = 3$. Use the nth term formula to write an equation.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$= 3.6 \cdot 3^{n-1}$ $a_1 = 3.6$, $r = 3$

An equation for the nth term is $a_n = 3.6 \cdot 3^{n-1}$.

Exercises

Find the next two terms of each geometric sequence.

1. 6, 12, 24, …
48, 96

2. 180, 60, 20, …
$\frac{20}{3}, \frac{20}{9}$

3. 2000, −1000, 500, …
−250, 125

4. 0.8, −2.4, 7.2, …
−21.6, 64.8

5. 80, 60, 45, …
33.75, 25.3125

6. 3, 16.5, 90.75, …
499.125, 2745.1875

Find the first five terms of each geometric sequence described.

7. $a_1 = \frac{1}{9}, r = 3$
$\frac{1}{9}, \frac{1}{3}, 1, 3, 9$

8. $a_1 = 240, r = -\frac{3}{4}$
$240, -180, 135, -101\frac{1}{4}, 75\frac{15}{16}$

9. $a_1 = 10, r = \frac{5}{2}$
$10, 25, 62\frac{1}{2}, 156\frac{1}{4}, 390\frac{5}{8}$

Find the indicated term of each geometric sequence.

10. $a_1 = -10, r = 4, n = 2$
−40

11. $a_1 = -6, r = -\frac{1}{2}, n = 8$
$\frac{3}{64}$

12. $a_3 = 9, r = -3, n = 7$
729

13. $a_4 = 16, r = 2, n = 10$
1024

14. $a_4 = -54, r = -3, n = 6$
−486

15. $a_1 = 8, r = \frac{2}{3}, n = 5$
$\frac{128}{81}$

Write an equation for the nth term of each geometric sequence.

16. 500, 350, 245, …
$500 \cdot 0.7^{n-1}$

17. 8, 32, 128, …
$8 \cdot 4^{n-1}$

18. 11, −24.2, 53.24, …
$11 \cdot (-2.2)^{n-1}$

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11-3 Study Guide and Intervention *(continued)*

Geometric Sequences

Geometric Means The **geometric means** of a geometric sequence are the terms between any two nonsuccessive terms of the sequence.

To find the k geometric means between two terms of a sequence, use the following steps.

Step 1	Let the two terms given be a_1 and a_n, where $n = k + 2$.
Step 2	Substitute in the formula $a_n = a_1 \cdot r^{n-1}$ $(= a_1 \cdot r^{k+1})$.
Step 3	Solve for r, and use that value to find the k geometric means: $a_1 \cdot r, a_1 \cdot r^2, \ldots, a_1 \cdot r^k$

Example **Find the three geometric means between 8 and 40.5.**

Use the nth term formula to find the value of r. In the sequence 8, ?, ?, ?, 40.5, a_1 is 8 and a_5 is 40.5.

$a_n = a_1 \cdot r^{n-1}$ Formula for nth term

$40.5 = 8 \cdot r^{5-1}$ $n = 5$, $a_1 = 8$, $a_5 = 40.5$

$5.0625 = r^4$ Divide each side by 8.

$r = \pm 1.5$ Take the fourth root of each side.

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of r to find the geometric means.

$r = 1.5$	$r = -1.5$
$a_2 = 8(1.5)$ or 12	$a_2 = 8(-1.5)$ or −12
$a_3 = 12(1.5)$ or 18	$a_3 = -12(-1.5)$ or 18
$a_4 = 18(1.5)$ or 27	$a_4 = 18(-1.5)$ or −27

The geometric means are 12, 18, and 27, or −12, 18, and −27.

Exercises

Find the geometric means in each sequence.

1. 5, ?, ?, ?, 405
±15, 45, ±135

2. 5, ?, ?, 20.48
8, 12.8

3. $\frac{3}{5}$, ?, ?, ?, 375
±3, 15, ±75

4. −24, ?, ?, $\frac{1}{9}$
$4, -\frac{2}{3}$

5. 12, ?, ?, ?, ?, ?, $\frac{3}{16}$
$\pm 6, 3, \pm\frac{3}{2}, \frac{3}{4}, \pm\frac{3}{8}$

6. 200, ?, ?, ?, 414.72
±240, 288, ±345.6

7. $\frac{35}{49}$, ?, ?, ?, ?, −12,005
$-\frac{35}{7}, 35, -245, 1715$

8. 4, ?, ?, ?, $156\frac{1}{4}$
$\pm 10, 25, \pm 62\frac{1}{2}$

9. $-\frac{1}{81}$, ?, ?, ?, ?, ?, −9
$\pm\frac{1}{27}, -\frac{1}{9}, \pm\frac{1}{3}, -1, \pm 3$

10. 100, ?, ?, ?, 384.16
±140, 196, ±274.4

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11-3 Skills Practice

Geometric Sequences

Find the next two terms of each geometric sequence.

1. $-1, -2, -4, \ldots$ **$-8, -16$**

2. $6, 3, \frac{3}{2}, \ldots$ **$\frac{3}{4}, \frac{3}{8}$**

3. $-5, -15, -45, \ldots$ **$-135, -405$**

4. $729, -243, 81, \ldots$ **$-27, 9$**

5. $1536, 384, 96, \ldots$ **$24, 6$**

6. $64, 160, 400, \ldots$ **$1000, 2500$**

Find the first five terms of each geometric sequence described.

7. $a_1 = 6, r = 2$
$6, 12, 24, 48, 96$

8. $a_1 = -27, r = 3$
$-27, -81, -243, -729, -2187$

9. $a_1 = -15, r = -1$
$-15, 15, -15, 15, -15$

10. $a_1 = 3, r = 4$
$3, 12, 48, 192, 768$

11. $a_1 = 1, r = \frac{1}{2}$
$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

12. $a_1 = 216, r = -\frac{1}{3}$
$216, -72, 24, -8, \frac{8}{3}$

Find the indicated term of each geometric sequence.

13. $a_1 = 5, r = 2, n = 6$ **160**

14. $a_1 = 18, r = 3, n = 6$ **4374**

15. $a_1 = -3, r = -2, n = 5$ **-48**

16. $a_1 = -20, r = -2, n = 9$ **-5120**

17. a_8 for $-12, -6, -3, \ldots$ **$-\frac{3}{32}$**

18. a_7 for $80, \frac{80}{3}, \frac{80}{9}, \ldots$ **$\frac{80}{729}$**

Write an equation for the *n*th term of each geometric sequence.

19. $3, 9, 27, \ldots$ **$a_n = 3^n$**

20. $-1, -3, -9, \ldots$ **$a_n = -1(3)^{n-1}$**

21. $2, -6, 18, \ldots$ **$a_n = 2(-3)^{n-1}$**

22. $5, 10, 20, \ldots$ **$a_n = 5(2)^{n-1}$**

Find the geometric means in each sequence.

23. 4, __?__, __?__, __?__, 64 **$\pm 8, 16, \pm 32$**

24. 1, __?__, __?__, __?__, 81 **$\pm 3, 9, \pm 27$**

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11-3 Practice (Average)

Geometric Sequences

Find the next two terms of each geometric sequence.

1. $-15, -30, -60, \ldots$ **$-120, -240$**

2. $80, 40, 20, \ldots$ **$10, 5$**

3. $90, 30, 10, \ldots$ **$\frac{10}{3}, \frac{10}{9}$**

4. $-1458, 486, -162, \ldots$ **$54, -18$**

5. $\frac{1}{4}, \frac{3}{8}, \frac{9}{16}, \ldots$ **$\frac{27}{32}, \frac{81}{64}$**

6. $216, 144, 96, \ldots$ **$64, \frac{128}{3}$**

Find the first five terms of each geometric sequence described.

7. $a_1 = -1, r = -3$
$-1, 3, -9, 27, -81$

8. $a_1 = 7, r = -4$
$7, -28, 112, -448, 1792$

9. $a_1 = -\frac{1}{3}, r = 2$
$-\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3}, -\frac{8}{3}, -\frac{16}{3}$

10. $a_1 = 12, r = \frac{2}{3}$
$12, 8, \frac{16}{3}, \frac{32}{9}, \frac{64}{27}$

Find the indicated term of each geometric sequence.

11. $a_1 = 5, r = 3, n = 6$ **1215**

12. $a_1 = 20, r = -3, n = 6$ **-4860**

13. $a_1 = -4, r = -2, n = 10$ **2048**

14. a_8 for $-\frac{1}{250}, -\frac{1}{50}, -\frac{1}{10}, \ldots$ **$-\frac{625}{2}$**

15. a_{12} for $96, 48, 24, \ldots$ **$\frac{3}{64}$**

16. $a_1 = 8, r = \frac{1}{2}, n = 9$ **$\frac{1}{32}$**

17. $a_1 = -3125, r = -\frac{1}{5}, n = 9$ **$-\frac{1}{125}$**

18. $a_1 = 3, r = \frac{1}{10}, n = 8$ **$\frac{3}{10{,}000{,}000}$**

Write an equation for the *n*th term of each geometric sequence.

19. $1, 4, 16, \ldots$ **$a_n = (4)^{n-1}$**

20. $-1, -5, -25, \ldots$ **$a_n = -1(5)^{n-1}$**

21. $1, \frac{1}{2}, \frac{1}{4}, \ldots$ **$a_n = \left(\frac{1}{2}\right)^{n-1}$**

22. $-3, -6, -12, \ldots$ **$a_n = -3(2)^{n-1}$**

23. $7, -14, 28, \ldots$ **$a_n = 7(-2)^{n-1}$**

24. $-5, -30, -180, \ldots$ **$a_n = -5(6)^{n-1}$**

Find the geometric means in each sequence.

25. 3, __?__, __?__, __?__, 768 **$12, 48, 192$**

26. 5, __?__, __?__, __?__, 1280 **$\pm 20, 80, \pm 320$**

27. 144, __?__, __?__, __?__, 9
$\pm 72, 36, \pm 18$

28. 37,500, __?__, __?__, __?__, __?__, -12
$-7500, 1500, -300, 60$

29. BIOLOGY A culture initially contains 200 bacteria. If the number of bacteria doubles every 2 hours, how many bacteria will be in the culture at the end of 12 hours? **12,800**

30. LIGHT If each foot of water in a lake screens out 60% of the light above, what percent of the light passes through 5 feet of water? **1.024%**

31. INVESTING Raul invests $1000 in a savings account that earns 5% interest compounded annually. How much money will he have in the account at the end of 5 years? **$1276.28**

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11-3 Reading to Learn Mathematics

Geometric Sequences

Pre-Activity **How do geometric sequences apply to a bouncing ball?**

Read the introduction to Lesson 11-3 at the top of page 588 in your textbook.

Suppose that you drop a ball from a height of 4 feet, and that each time it falls, it bounces back to 74% of the height from which it fell. Describe how would you find the height of the third bounce. (Do not actually calculate the height of the bounce.)

Sample answer: Multiply 4 by 0.74 three times.

Reading the Lesson

1. Explain the difference between an arithmetic sequence and a geometric sequence.

Sample answer: In an arithmetic sequence, each term after the first is found by adding the common difference to the previous term. In a geometric sequence, each term after the first is found by multiplying the previous term by the common ratio.

2. Consider the formula $a_n = a_1 \cdot r^{n-1}$.

a. What is this formula used to find? **a particular term of a geometric sequence**

b. What do each of the following represent?

a_n: **the *n*th term**

a_1: **the first term**

r: **the common ratio**

n: **a positive integer that indicates which term you are finding**

3. a. In the sequence 5, 8, 11, 14, 17, 20, the numbers 8, 11, 14, and 17 are **arithmetic means** between 5 and 20.

b. In the sequence 12, 4, $\frac{4}{3}$, $\frac{4}{9}$, $\frac{4}{27}$, the numbers 4, $\frac{4}{3}$, and $\frac{4}{9}$ are **geometric means** between 12 and $\frac{4}{27}$.

Helping You Remember

4. Suppose that your classmate Ricardo has trouble remembering the formula $a_n = a_1 \cdot r^{n-1}$ correctly. He thinks that the formula should be $a_n = a_1 \cdot r^n$. How would you explain to him that he should use r^{n-1} rather than r^n in the formula?

Sample answer: Each term after the first in a geometric sequence is found by multiplying the previous term by *r*. There are *n* – 1 terms before the *n*th term, so you would need to multiply by *r* a total of *n* – 1 times, not *n* times, to get the *n*th term.

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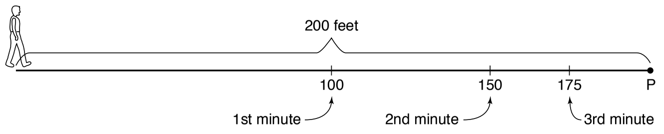
11-3 Enrichment

Half the Distance

Suppose you are 200 feet from a fixed point, *P*. Suppose that you are able to move to the halfway point in one minute, to the next halfway point one minute after that, and so on.



An interesting sequence results because according to the problem, you never actually reach the point *P*, although you do get arbitrarily close to it.

You can compute how long it will take to get within some specified small distance of the point. On a calculator, you enter the distance to be covered and then count the number of successive divisions by 2 necessary to get within the desired distance.

Example **How many minutes are needed to get within 0.1 foot of a point 200 feet away?**

Count the number of times you divide by 2.

Enter: 200 ÷ 2 ENTER ÷ 2 ENTER ÷ 2 ENTER, and so on

Result: 0.0976562

You divided by 2 eleven times. The time needed is 11 minutes.

Use the method illustrated above to solve each problem.

1. If it is about 2500 miles from Los Angeles to New York, how many minutes would it take to get within 0.1 mile of New York? How far from New York are you at that time? **15 minutes, 0.0762934 mile**

2. If it is 25,000 miles around Earth, how many minutes would it take to get within 0.5 mile of the full distance around Earth? How far short would you be? **16 minutes; 0.3814697 mile**

3. If it is about 250,000 miles from Earth to the Moon, how many minutes would it take to get within 0.5 mile of the Moon? How far from the surface of the Moon would you be? **19 minutes, 0.4768372 mile**

4. If it is about 30,000,000 feet from Honolulu to Miami, how many minutes would it take to get to within 1 foot of Miami? How far from Miami would you be at that time? **25 minutes, 0.8940697 foot**

5. If it is about 93,000,000 miles to the sun, how many minutes would it take to get within 500 miles of the sun? How far from the sun would you be at that time? **18 minutes, 354.766846 miles**

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11-4 Study Guide and Intervention

Geometric Series

Geometric Series A **geometric series** is the indicated sum of consecutive terms of a geometric sequence.

Sum of a Geometric Series	The sum S_n of the first n terms of a geometric series is given by $S_n = \frac{a_1(1 - r^n)}{1 - r}$ or $S_n = \frac{a_1 - a_1r^n}{1 - r}$, where $r \neq 1$.

Example 1 **Find the sum of the first four terms of the geometric sequence for which $a_1 = 120$ and $r = \frac{1}{3}$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_4 = \frac{120\left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - \frac{1}{3}}$ $n = 4$, $a_1 = 120$, $r = \frac{1}{3}$

≈ 177.78 Use a calculator.

The sum of the series is 177.78.

Example 2 **Find the sum of the geometric series $\sum_{j=1}^{7} 4 \cdot 3^{j-2}$.**

Since the sum is a geometric series, you can use the sum formula.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$S_7 = \frac{\frac{4}{3}(1 - 3^7)}{1 - 3}$ $n = 7$, $a_1 = \frac{4}{3}$, $r = 3$

≈ 1457.33 Use a calculator.

The sum of the series is 1457.33.

Exercises

Find S_n for each geometric series described.

1. $a_1 = 2$, $a_n = 486$, $r = 3$ **728**

2. $a_1 = 1200$, $a_n = 75$, $r = \frac{1}{2}$ **2325**

3. $a_1 = \frac{1}{25}$, $a_n = 125$, $r = 5$ **156.24**

4. $a_1 = 3$, $r = \frac{1}{3}$, $n = 4$ **4.44**

5. $a_1 = 2$, $r = 6$, $n = 4$ **518**

6. $a_1 = 2$, $r = 4$, $n = 6$ **2730**

7. $a_1 = 100$, $r = -\frac{1}{2}$, $n = 5$ **68.75**

8. $a_3 = 20$, $a_6 = 160$, $n = 8$ **1275**

9. $a_4 = 16$, $a_7 = 1024$, $n = 10$ **87,381.25**

Find the sum of each geometric series.

10. 6 + 18 + 54 + … to 6 terms **2184**

11. $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ to 10 terms **255.75**

12. $\sum_{j=4}^{8} 2^j$ **496**

13. $\sum_{k=1}^{7} 3 \cdot 2^{k-1}$ **381**

Lesson 11-4

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11-4 Study Guide and Intervention *(continued)*

Geometric Series

Specific Terms You can use one of the formulas for the sum of a geometric series to help find a particular term of the series.

Example 1 **Find a_1 in a geometric series for which $S_6 = 441$ and $r = 2$.**

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$441 = \frac{a_1(1 - 2^6)}{1 - 2}$ $S_6 = 441$, $r = 2$, $n = 6$

$441 = \frac{-63a_1}{-1}$ Subtract.

$a_1 = \frac{441}{63}$ Divide.

$a_1 = 7$ Simplify.

The first term of the series is 7.

Example 2 **Find a_1 in a geometric series for which $S_n = 244$, $a_n = 324$, and $r = -3$.**

Since you do not know the value of n, use the alternate sum formula.

$S_n = \frac{a_1 - a_nr}{1 - r}$ Alternate sum formula

$244 = \frac{a_1 - (324)(-3)}{1 - (-3)}$ $S_n = 244$, $a_n = 324$, $r = -3$

$244 = \frac{a_1 + 972}{4}$ Simplify.

$976 = a_1 + 972$ Multiply each side by 4.

$a_1 = 4$ Subtract 972 from each side.

The first term of the series is 4.

Example 3 **Find a_4 in a geometric series for which $S_n = 796.875$, $r = \frac{1}{2}$, and $n = 8$.**

First use the sum formula to find a_1.

$S_n = \frac{a_1(1 - r^n)}{1 - r}$ Sum formula

$796.875 = \frac{a_1\left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$ $S_8 = 796.875$, $r = \frac{1}{2}$, $n = 8$

$796.875 = \frac{0.99609375a_1}{0.5}$ Use a calculator.

$a_1 = 400$

Since $a_4 = a_1 \cdot r^3$, $a_4 = 400\left(\frac{1}{2}\right)^3 = 50$. The fourth term of the series is 50.

Exercises

Find the indicated term for each geometric series described.

1. $S_n = 726$, $a_n = 486$, $r = 3$; a_1 **6**

2. $S_n = 850$, $a_n = 1280$, $r = -2$; a_1 **−10**

3. $S_n = 1023.75$, $a_n = 512$, $r = 2$; a_1 $\frac{1}{4}$

4. $S_n = 118.125$, $a_n = -5.625$, $r = -\frac{1}{2}$; a_1 **180**

5. $S_n = 183$, $r = -3$, $n = 5$; a_1 **3**

6. $S_n = 1705$, $r = 4$, $n = 5$; a_1 **5**

7. $S_n = 52{,}084$, $r = -5$, $n = 7$; a_1 **4**

8. $S_n = 43{,}690$, $r = \frac{1}{4}$, $n = 8$; a_1 **32, 768**

9. $S_n = 381$, $r = 2$, $n = 7$; a_4 **24**

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11-4 Skills Practice

Geometric Series

Find S_n for each geometric series described.

1. $a_1 = 2, a_5 = 162, r = 3$ **242**

2. $a_1 = 4, a_6 = 12{,}500, r = 5$ **15,624**

3. $a_1 = 1, a_8 = -1, r = -1$ **0**

4. $a_1 = 4, a_n = 256, r = -2$ **172**

5. $a_1 = 1, a_n = 729, r = -3$ **547**

6. $a_1 = 2, r = -4, n = 5$ **410**

7. $a_1 = -8, r = 2, n = 4$ **−120**

8. $a_1 = 3, r = -2, n = 12$ **−4095**

9. $a_1 = 8, r = 3, n = 5$ **968**

10. $a_1 = 6, a_n = \frac{3}{8}, r = \frac{1}{2}$ **$\frac{93}{8}$**

11. $a_1 = 8, r = \frac{1}{2}, n = 7$ **$\frac{127}{8}$**

12. $a_1 = 2, r = -\frac{1}{2}, n = 6$ **$\frac{21}{16}$**

Find the sum of each geometric series.

13. $4 + 8 + 16 + \ldots$ to 5 terms **124**

14. $-1 - 3 - 9 - \ldots$ to 6 terms **−364**

15. $3 + 6 + 12 + \ldots$ to 5 terms **93**

16. $-15 + 30 - 60 + \ldots$ to 7 terms **−645**

17. $\sum_{n=1}^{4} 3^{n-1}$ **40**

18. $\sum_{n=1}^{5} (-2)^{n-1}$ **11**

19. $\sum_{n=1}^{4} \left(\frac{1}{3}\right)^{n-1}$ **$\frac{40}{27}$**

20. $\sum_{n=1}^{9} 2(-3)^{n-1}$ **9842**

Find the indicated term for each geometric series described.

21. $S_n = 1275, a_n = 640, r = 2; a_1$ **5**

22. $S_n = -40, a_n = -54, r = -3; a_1$ **2**

23. $S_n = 99, n = 5, r = -\frac{1}{2}; a_1$ **144**

24. $S_n = 39{,}360, n = 8, r = 3; a_1$ **12**

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11-4 Practice (Average)

Geometric Series

Find S_n for each geometric series described.

1. $a_1 = 2, a_6 = 64, r = 2$ **126**

2. $a_1 = 160, a_6 = 5, r = \frac{1}{2}$ **315**

3. $a_1 = -3, a_n = -192, r = -2$ **−129**

4. $a_1 = -81, a_n = -16, r = -\frac{2}{3}$ **−55**

5. $a_1 = -3, a_n = 3072, r = -4$ **2457**

6. $a_1 = 54, a_6 = \frac{2}{9}, r = \frac{1}{3}$ **$\frac{728}{9}$**

7. $a_1 = 5, r = 3, n = 9$ **49,205**

8. $a_1 = -6, r = -1, n = 21$ **−6**

9. $a_1 = -6, r = -3, n = 7$ **−3282**

10. $a_1 = -9, r = \frac{2}{3}, n = 4$ **$-\frac{65}{3}$**

11. $a_1 = \frac{1}{3}, r = 3, n = 10$ **$\frac{29{,}524}{3}$**

12. $a_1 = 16, r = -1.5, n = 6$ **−66.5**

Find the sum of each geometric series.

13. $162 + 54 + 18 + \ldots$ to 6 terms **$\frac{728}{3}$**

14. $2 + 4 + 8 + \ldots$ to 8 terms **510**

15. $64 - 96 + 144 - \ldots$ to 7 terms **463**

16. $\frac{1}{9} - \frac{1}{3} + 1 - \ldots$ to 6 terms **$-\frac{182}{9}$**

17. $\sum_{n=1}^{8} (-3)^{n-1}$ **−1640**

18. $\sum_{n=1}^{9} 5(-2)^{n-1}$ **855**

19. $\sum_{n=1}^{5} -1(4)^{n-1}$ **−341**

20. $\sum_{n=1}^{6} \left(\frac{1}{2}\right)^{n-1}$ **$\frac{63}{32}$**

21. $\sum_{n=1}^{10} 2560\left(\frac{1}{2}\right)^{n-1}$ **5115**

22. $\sum_{n=1}^{4} 9\left(\frac{2}{3}\right)^{n-1}$ **$\frac{65}{3}$**

Find the indicated term for each geometric series described.

23. $S_n = 1023, a_n = 768, r = 4; a_1$ **3**

24. $S_n = 10{,}160, a_n = 5120, r = 2; a_1$ **80**

25. $S_n = -1365, n = 12, r = -2; a_1$ **1**

26. $S_n = 665, n = 6, r = 1.5; a_1$ **32**

27. CONSTRUCTION A pile driver drives a post 27 inches into the ground on its first hit. Each additional hit drives the post $\frac{2}{3}$ the distance of the prior hit. Find the total distance the post has been driven after 5 hits. **$70\frac{1}{3}$ in.**

28. COMMUNICATIONS Hugh Moore e-mails a joke to 5 friends on Sunday morning. Each of these friends e-mails the joke to 5 of her or his friends on Monday morning, and so on. Assuming no duplication, how many people will have heard the joke by the end of Saturday, not including Hugh? **97,655 people**

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11-4 Reading to Learn Mathematics

Geometric Series

Pre-Activity **How is e-mailing a joke like a geometric series?**

Read the introduction to Lesson 11-4 at the top of page 594 in your textbook.

- Suppose that you e-mail the joke on Monday to five friends, rather than three, and that each of those friends e-mails it to five friends on Tuesday, and so on. Write a sum that shows that total number of people, including yourself, who will have read the joke by Thursday. (Write out the sum using plus signs rather than sigma notation. Do not actually find the sum.)
  $1 + 5 + 25 + 125$
- Use exponents to rewrite the sum you found above. (Use an exponent in each term, and use the same base for all terms.)
  $5^0 + 5^1 + 5^2 + 5^3$

Reading the Lesson

1. Consider the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$.

a. What is this formula used to find? **the sum of the first *n* terms of a geometric series**

b. What do each of the following represent?

S_n: **the sum of the first *n* terms**

a_1: **the first term**

r: **the common ratio**

c. Suppose that you want to use the formula to evaluate $3 - 1 + \frac{1}{3} - \frac{1}{9} + \frac{1}{27}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

$n =$ **5** $a_1 =$ **3** $r =$ **$-\frac{1}{3}$** $r^n =$ **$\left(-\frac{1}{3}\right)^5$ or $-\frac{1}{243}$**

d. Suppose that you want to use the formula to evaluate the sum $\sum_{n=1}^{6} 8(-2)^{n-1}$. Indicate the values you would substitute into the formula in order to find S_n. (Do not actually calculate the sum.)

$n =$ **6** $a_1 =$ **8** $r =$ **-2** $r^n =$ **$(-2)^6$ or 64**

Helping You Remember

2. This lesson includes three formulas for the sum of the first n terms of a geometric series. All of these formulas have the same denominator and have the restriction $r \neq 1$. How can this restriction help you to remember the denominator in the formulas?
Sample answer: If $r = 1$, then $r - 1 = 0$. Because division by 0 is undefined, a formula with $r - 1$ in the denominator will not apply when $r = 1$.

Lesson 11-4

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11-4 Enrichment

Annuities

An annuity is a fixed amount of money payable at given intervals. For example, suppose you wanted to set up a trust fund so that \$30,000 could be withdrawn each year for 14 years before the money ran out. Assume the money can be invested at 9%.

You must find the amount of money that needs to be invested. Call this amount A. After the third payment, the amount left is

$1.09[1.09A - 30{,}000(1 + 1.09)] - 30{,}000 = 1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$.

The results are summarized in the table below.

Payment Number	Number of Dollars Left After Payment
1	$A - 30{,}000$
2	$1.09A - 30{,}000(1 + 1.09)$
3	$1.09^2A - 30{,}000(1 + 1.09 + 1.09^2)$

1. Use the pattern shown in the table to find the number of dollars left after the fourth payment. **$1.09^3A - 30{,}000(1 + 1.09 + 1.09^2 + 1.09^3)$**

2. Find the amount left after the tenth payment.
$1.09^9A - 30{,}000(1 + 1.09 + 1.09^2 + 1.09^3 + \ldots + 1.09^9)$

The amount left after the 14th payment is $1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13})$. However, there should be no money left after the 14th and final payment.

$1.09^{13}A - 30{,}000(1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}) = 0$

Notice that $1 + 1.09 + 1.09^2 + \ldots + 1.09^{13}$ is a geometric series where $a_1 = 1$, $a_n = 1.09^{13}$, $n = 14$ and $r = 1.09$.

Using the formula for S_n,

$1 + 1.09 + 1.09^2 + \ldots + 1.09^{13} = \frac{a_1 - a_1r^n}{1 - r} = \frac{1 - 1.09^{14}}{1 - 1.09} = \frac{1 - 1.09^{14}}{-0.09}$.

3. Show that when you solve for A you get $A = \frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$.
$1.09^{13}A - 30{,}000\left(\frac{1 - 1.09^{14}}{-0.09}\right) = 0$ results in stated expression for A.

Therefore, to provide \$30,000 for 14 years where the annual interest rate is 9%, you need $\frac{30{,}000}{0.09}\left(\frac{1.09^{14} - 1}{1.09^{13}}\right)$ dollars.

4. Use a calculator to find the value of A in problem 3. **\$254,607**

In general, if you wish to provide P dollars for each of n years at an annual rate of r%, you need A dollars where

$\left(1 + \frac{r}{100}\right)^{n-1}A - P\left[1 + \left(1 + \frac{r}{100}\right) + \left(1 + \frac{r}{100}\right)^2 + \ldots + \left(1 + \frac{r}{100}\right)^{n-1}\right] = 0.$

You can solve this equation for A, given P, n, and r.

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11-5 Study Guide and Intervention

Infinite Geometric Series

Infinite Geometric Series A geometric series that does not end is called an **infinite geometric series**. Some infinite geometric series have sums, but others do not because the **partial sums** increase without approaching a limiting value.

Sum of an Infinite Geometric Series	$S = \frac{a_1}{1 - r}$ for $-1 < r < 1$. If $\lvert r \rvert \geq 1$, the infinite geometric series does not have a sum.

Example **Find the sum of each infinite geometric series, if it exists.**

a. $75 + 15 + 3 + \ldots$

First, find the value of r to determine if the sum exists. $a_1 = 75$ and $a_2 = 15$, so $r = \frac{15}{75}$ or $\frac{1}{5}$. Since $\left|\frac{1}{5}\right| < 1$, the sum exists. Now use the formula for the sum of an infinite geometric series.

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$
$$= \frac{75}{1 - \frac{1}{5}} \quad a_1 = 75,\ r = \frac{1}{5}$$
$$= \frac{75}{\frac{4}{5}} \text{ or } 93.75 \quad \text{Simplify.}$$

The sum of the series is 93.75.

b. $\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1}$

In this infinite geometric series, $a_1 = 48$ and $r = -\frac{1}{3}$.

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$
$$= \frac{48}{1 - \left(-\frac{1}{3}\right)} \quad a_1 = 48,\ r = -\frac{1}{3}$$
$$= \frac{48}{\frac{4}{3}} \text{ or } 36 \quad \text{Simplify.}$$

Thus $\sum_{n=1}^{\infty} 48\left(-\frac{1}{3}\right)^{n-1} = 36$.

Exercises

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = -7$, $r = \frac{5}{8}$ **$-18\frac{2}{3}$**

2. $1 + \frac{5}{4} + \frac{25}{16} + \ldots$ **does not exist**

3. $a_1 = 4$, $r = \frac{1}{2}$ **8**

4. $\frac{2}{9} + \frac{5}{27} + \frac{25}{162} + \ldots$ **$1\frac{1}{3}$**

5. $15 + 10 + 6\frac{2}{3} + \ldots$ **45**

6. $18 - 9 + 4\frac{1}{2} - 2\frac{1}{4} + \ldots$ **12**

7. $\frac{1}{10} + \frac{1}{20} + \frac{1}{40} + \ldots$ **$\frac{1}{5}$**

8. $1000 + 800 + 640 + \ldots$ **5000**

9. $6 - 12 + 24 - 48 + \ldots$ **does not exist**

10. $\sum_{n=1}^{\infty} 50\left(\frac{4}{5}\right)^{n-1}$ **250**

11. $\sum_{k=1}^{\infty} 22\left(-\frac{1}{2}\right)^{k-1}$ **$14\frac{2}{3}$**

12. $\sum_{s=1}^{\infty} 24\left(\frac{7}{12}\right)^{s-1}$ **$57\frac{3}{5}$**

Lesson 11-5

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11-5 Study Guide and Intervention *(continued)*

Infinite Geometric Series

Repeating Decimals A repeating decimal represents a fraction. To find the fraction, write the decimal as an infinite geometric series and use the formula for the sum.

Example **Write each repeating decimal as a fraction.**

a. $0.\overline{42}$

Write the repeating decimal as a sum.

$$0.\overline{42} = 0.42424242\ldots$$
$$= \frac{42}{100} + \frac{42}{10{,}000} + \frac{42}{1{,}000{,}000} + \ldots$$

In this series $a_1 = \frac{42}{100}$ and $r = \frac{1}{100}$.

$$S = \frac{a_1}{1 - r} \quad \text{Sum formula}$$
$$= \frac{\frac{42}{100}}{1 - \frac{1}{100}} \quad a_1 = \frac{42}{100},\ r = \frac{1}{100}$$
$$= \frac{\frac{42}{100}}{\frac{99}{100}} \quad \text{Subtract.}$$
$$= \frac{42}{99} \text{ or } \frac{14}{33} \quad \text{Simplify.}$$

Thus $0.\overline{42} = \frac{14}{33}$.

b. $0.5\overline{24}$

Let $S = 0.5\overline{24}$.

$$S = 0.5242424\ldots \quad \text{Write as a repeating decimal.}$$
$$1000S = 524.242424\ldots \quad \text{Multiply each side by 1000.}$$
$$10S = 5.242424\ldots \quad \text{Mulitply each side by 10.}$$
$$990S = 519 \quad \text{Subtract the third equation from the second equation.}$$
$$S = \frac{519}{990} \text{ or } \frac{173}{330} \quad \text{Simplify.}$$

Thus, $0.5\overline{24} = \frac{173}{330}$

Exercises

Write each repeating decimal as a fraction.

1. $0.\overline{2}$ **$\frac{2}{9}$**

2. $0.\overline{8}$ **$\frac{8}{9}$**

3. $0.\overline{30}$ **$\frac{10}{33}$**

4. $0.\overline{87}$ **$\frac{29}{33}$**

5. $0.\overline{10}$ **$\frac{10}{99}$**

6. $0.\overline{54}$ **$\frac{6}{11}$**

7. $0.\overline{75}$ **$\frac{25}{33}$**

8. $0.\overline{18}$ **$\frac{2}{11}$**

9. $0.\overline{62}$ **$\frac{62}{99}$**

10. $0.\overline{72}$ **$\frac{8}{11}$**

11. $0.0\overline{72}$ **$\frac{4}{55}$**

12. $0.0\overline{45}$ **$\frac{1}{22}$**

13. $0.0\overline{6}$ **$\frac{1}{15}$**

14. $0.0\overline{138}$ **$\frac{23}{1665}$**

15. $0.\overline{0138}$ **$\frac{46}{3333}$**

16. $0.0\overline{81}$ **$\frac{9}{110}$**

17. $0.2\overline{45}$ **$\frac{27}{110}$**

18. $0.4\overline{36}$ **$\frac{24}{55}$**

19. $0.5\overline{4}$ **$\frac{49}{90}$**

20. $0.8\overline{63}$ **$\frac{19}{22}$**

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11-5 Skills Practice

Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 1, r = \frac{1}{2}$ **2**

2. $a_1 = 5, r = -\frac{2}{5}$ **$\frac{25}{7}$**

3. $a_1 = 8, r = 2$ **does not exist**

4. $a_1 = 6, r = \frac{1}{2}$ **12**

5. $4 + 2 + 1 + \frac{1}{2} + \ldots$ **8**

6. $540 - 180 + 60 - 20 + \ldots$ **405**

7. $5 + 10 + 20 + \ldots$ **does not exist**

8. $-336 + 84 - 21 + \ldots$ **−268.8**

9. $125 + 25 + 5 + \ldots$ **156.25**

10. $9 - 1 + \frac{1}{9} - \ldots$ **$\frac{81}{10}$**

11. $\frac{3}{4} + \frac{9}{4} + \frac{27}{4} + \ldots$ **does not exist**

12. $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ **$\frac{1}{2}$**

13. $5 + 2 + 0.8 + \ldots$ **$\frac{25}{3}$**

14. $9 + 6 + 4 + \ldots$ **27**

15. $\sum_{n=1}^{\infty} 10\left(\frac{1}{2}\right)^{n-1}$ **20**

16. $\sum_{n=1}^{\infty} 6\left(-\frac{1}{3}\right)^{n-1}$ **$\frac{9}{2}$**

17. $\sum_{n=1}^{\infty} 15\left(\frac{2}{5}\right)^{n-1}$ **25**

18. $\sum_{n=1}^{\infty} \left(-\frac{4}{3}\right)\left(\frac{1}{3}\right)^{n-1}$ **−2**

Write each repeating decimal as a fraction.

19. $0.\overline{4}$ **$\frac{4}{9}$**

20. $0.\overline{8}$ **$\frac{8}{9}$**

21. $0.\overline{27}$ **$\frac{3}{11}$**

22. $0.\overline{67}$ **$\frac{67}{99}$**

23. $0.\overline{54}$ **$\frac{6}{11}$**

24. $0.\overline{375}$ **$\frac{125}{333}$**

25. $0.\overline{641}$ **$\frac{641}{999}$**

26. $0.\overline{171}$ **$\frac{57}{333}$**

Lesson 11-5

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11-5 Practice (Average)

Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. $a_1 = 35, r = \frac{2}{7}$ **49**

2. $a_1 = 26, r = \frac{1}{2}$ **52**

3. $a_1 = 98, r = -\frac{3}{4}$ **56**

4. $a_1 = 42, r = \frac{6}{5}$ **does not exist**

5. $a_1 = 112, r = -\frac{3}{5}$ **70**

6. $a_1 = 500, r = \frac{1}{5}$ **625**

7. $a_1 = 135, r = -\frac{1}{2}$ **90**

8. $18 - 6 + 2 - \ldots$ **$\frac{27}{2}$**

9. $2 + 6 + 18 + \ldots$ **does not exist**

10. $6 + 4 + \frac{8}{3} + \ldots$ **18**

11. $\frac{4}{25} + \frac{2}{5} + 1 + \ldots$ **does not exist**

12. $10 + 1 + 0.1 + \ldots$ **$\frac{100}{9}$**

13. $100 + 20 + 4 + \ldots$ **125**

14. $-270 + 135 - 67.5 + \ldots$ **−180**

15. $0.5 + 0.25 + 0.125 + \ldots$ **1**

16. $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \ldots$ **$\frac{7}{9}$**

17. $0.8 + 0.08 + 0.008 + \ldots$ **$\frac{8}{9}$**

18. $\frac{1}{12} - \frac{1}{6} + \frac{1}{3} - \ldots$ **does not exist**

19. $3 + \frac{9}{7} + \frac{27}{49} + \ldots$ **$\frac{21}{4}$**

20. $0.3 - 0.003 + 0.00003 - \ldots$ **$\frac{30}{101}$**

21. $0.06 + 0.006 + 0.0006 + \ldots$ **$\frac{1}{15}$**

22. $\frac{2}{3} - 2 + 6 - \ldots$ **does not exist**

23. $\sum_{n=1}^{\infty} 3\left(\frac{1}{4}\right)^{n-1}$ **4**

24. $\sum_{n=1}^{\infty} \frac{2}{3}\left(-\frac{3}{4}\right)^{n-1}$ **$\frac{8}{21}$**

25. $\sum_{n=1}^{\infty} 18\left(\frac{2}{3}\right)^{n-1}$ **54**

26. $\sum_{n=1}^{\infty} 5(-0.1)^{n-1}$ **$\frac{50}{11}$**

Write each repeating decimal as a fraction.

27. $0.\overline{6}$ **$\frac{2}{3}$**

28. $0.\overline{09}$ **$\frac{1}{11}$**

29. $0.\overline{43}$ **$\frac{43}{99}$**

30. $0.\overline{27}$ **$\frac{3}{11}$**

31. $0.\overline{243}$ **$\frac{9}{37}$**

32. $0.\overline{84}$ **$\frac{28}{33}$**

33. $0.\overline{990}$ **$\frac{110}{111}$**

34. $0.\overline{150}$ **$\frac{50}{333}$**

35. PENDULUMS On its first swing, a pendulum travels 8 feet. On each successive swing, the pendulum travels $\frac{4}{5}$ the distance of its previous swing. What is the total distance traveled by the pendulum when it stops swinging? **40 ft**

36. ELASTICITY A ball dropped from a height of 10 feet bounces back $\frac{9}{10}$ of that distance. With each successive bounce, the ball continues to reach $\frac{9}{10}$ of its previous height. What is the total vertical distance (both up and down) traveled by the ball when it stops bouncing? (*Hint*: Add the total distance the ball falls to the total distance it rises.) **190 ft**

Answers

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11-5 Reading to Learn Mathematics

Infinite Geometric Series

Pre-Activity **How does an infinite geometric series apply to a bouncing ball?**

Read the introduction to Lesson 11-5 at the top of page 599 in your textbook.

Note the following powers of 0.6: $0.6^1 = 0.6$; $0.6^2 = 0.36$; $0.6^3 = 0.216$; $0.6^4 = 0.1296$; $0.6^5 = 0.07776$; $0.6^6 = 0.046656$; $0.6^7 = 0.0279936$. If a ball is dropped from a height of 10 feet and bounces back to 60% of its previous height on each bounce, after how many bounces will it bounce back to a height of less than 1 foot? **5 bounces**

Reading the Lesson

1. Consider the formula $S = \frac{a_1}{1 - r}$.

a. What is the formula used to find? **the sum of an infinite geometric series**

b. What do each of the following represent?

S: **the sum**

a_1: **the first term**

r: **the common ratio**

c. For what values of r does an infinite geometric sequence have a sum? $-1 < r < 1$

d. Rewrite your answer for part d as an absolute value inequality. $|r| < 1$

2. For each of the following geometric series, give the values of a_1 and r. Then state whether the sum of the series exists. (Do not actually find the sum.)

a. $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \ldots$ $a_1 =$ $\frac{2}{3}$ $r =$ $\frac{1}{3}$

Does the sum exist? **yes**

b. $2 - 1 + \frac{1}{2} - \frac{1}{4} + \ldots$ $a_1 =$ 2 $r =$ $-\frac{1}{2}$

Does the sum exist? **yes**

c. $\sum_{i=1}^{\infty} 3^i$ $a_1 =$ 3 $r =$ 3

Does the sum exist? **no**

Helping You Remember

3. One good way to remember something is to relate it to something you already know. How can you use the formula $S_n = \frac{a_1(1 - r^n)}{1 - r}$ that you learned in Lesson 11-4 for finding the sum of a geometric series to help you remember the formula for finding the sum of an infinite geometric series? **Sample answer: If $-1 < r < 1$, then as n gets large, r^n approaches 0, so $1 - r^n$ approaches 1. Therefore, S_n approaches $\frac{a_1 \cdot 1}{1 - r}$, or $\frac{a_1}{1 - r}$.**

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11-5 Enrichment

Convergence and Divergence

Convergence and divergence are terms that relate to the existence of a sum of an infinite series. If a sum exists, the series is convergent. If not, the series is divergent. Consider the series $12 + 3 + \frac{3}{4} + \frac{3}{16} + \ldots$. This is a geometric series with $r = \frac{1}{4}$. The sum is given by the formula $S = \frac{a_1}{1 - r}$. Thus, the sum is $12 \div \frac{3}{4}$ or 16. This series is convergent since a sum exists. Notice that the first two terms have a sum of 15. As more terms are added, the sum comes closer (or converges) to 16.

Recall that a geometric series has a sum if and only if $-1 < r < 1$. Thus, a geometric series is convergent if r is between -1 and 1, and divergent if r has another value. An infinite arithmetic series cannot have a sum unless all of the terms are equal to zero.

Example **Determine whether each series is convergent or divergent.**

a. $\mathbf{2 + 5 + 8 + 11 + \ldots}$ divergent

b. $\mathbf{-2 + 4 + (-8) + 16 + \ldots}$ divergent

c. $\mathbf{16 + 8 + 4 + 2 + \ldots}$ convergent

Determine whether each series is convergent or divergent. If the series is convergent, find the sum.

1. $5 + 10 + 15 + 20 + \ldots$
divergent

2. $16 + 8 + 4 + 2 + \ldots$
convergent; 32

3. $1 + 0.1 + 0.01 + 0.001 + \ldots$
convergent; 1.11

4. $4 + 2 + 0 - 2 - \ldots$
divergent

5. $2 - 4 + 8 - 16 + \ldots$
divergent

6. $1 - \frac{1}{5} + \frac{1}{25} - \frac{1}{125} + \ldots$
convergent; $\frac{5}{6}$

7. $4 + 2.4 + 1.44 + 0.864 + \ldots$
convergent; 10

8. $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1 + \ldots$
divergent

9. $-\frac{5}{3} + \frac{10}{9} - \frac{20}{27} + \frac{40}{81} - \ldots$
convergent; −1

10. $48 + 12 + 3 + \frac{3}{4} + \ldots$
convergent; 64

Bonus: Is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$ convergent or divergent? **divergent**

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11-6 Study Guide and Intervention

Recursion and Special Sequences

Special Sequences In a **recursive formula**, each succeeding term is formulated from one or more previous terms. A recursive formula for a sequence has two parts:

1. the value(s) of the first term(s), and
2. an equation that shows how to find each term from the term(s) before it.

Example **Find the first five terms of the sequence in which $a_1 = 6$, $a_2 = 10$, and $a_n = 2a_{n-2}$ for $n \geq 3$.**

$a_1 = 6$
$a_2 = 10$
$a_3 = 2a_1 = 2(6) = 12$
$a_4 = 2a_2 = 2(10) = 20$
$a_5 = 2a_3 = 2(12) = 24$

The first five terms of the sequence are 6, 10, 12, 20, 24.

Exercises

Find the first five terms of each sequence.

1. $a_1 = 1, a_2 = 1, a_n = 2(a_{n-1} + a_{n-2}), n \geq 3$ **1, 1, 4, 10, 28**

2. $a_1 = 1, a_n = \frac{1}{1 + a_{n-1}}, n \geq 2$ **$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}$**

3. $a_1 = 3, a_n = a_{n-1} + 2(n - 2)$, n ≥ 2 **3, 3, 5, 9, 15**

4. $a_1 = 5, a_n = a_{n-1} + 2, n \geq 2$ **5, 7, 9, 11, 13**

5. $a_1 = 1, a_n = (n - 1)a_{n-1}, n \geq 2$ **1, 1, 2, 6, 24**

6. $a_1 = 7, a_n = 4a_{n-1} - 1, n \geq 2$ **7, 27, 107, 427, 1707**

7. $a_1 = 3, a_2 = 4, a_n = 2a_{n-2} + 3a_{n-1}, n \geq 3$ **3, 4, 18, 62, 222**

8. $a_1 = 0.5, a_n = a_{n-1} + 2n, n \geq 2$ **0.5, 4.5, 10.5, 18.5, 28.5**

9. $a_1 = 8, a_2 = 10, a_n = \frac{a_{n-2}}{a_{n-1}}, n \geq 3$ **8, 10, 0.8, 12.5, 0.064**

10. $a_1 = 100, a_n = \frac{a_{n-1}}{n}, n \geq 2$ **$100, 50, \frac{50}{3}, \frac{50}{12}, \frac{50}{60}$**

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11-6 Study Guide and Intervention *(continued)*

Recursion and Special Sequences

Iteration Combining composition of functions with the concept of recursion leads to the process of **iteration**. Iteration is the process of composing a function with itself repeatedly.

Example **Find the first three iterates of $f(x) = 4x - 5$ for an initial value of $x_0 = 2$.**

To find the first iterate, find the value of the function for $x_0 = 2$

$x_1 = f(x_0)$ Iterate the function.
$= f(2)$ $x_0 = 2$
$= 4(2) - 5$ or 3 Simplify.

To find the second iteration, find the value of the function for $x_1 = 3$.

$x_2 = f(x_1)$ Iterate the function.
$= f(3)$ $x_1 = 3$
$= 4(3) - 5$ or 7 Simplify.

To find the third iteration, find the value of the function for $x_2 = 7$.

$x_3 = f(x_2)$ Iterate the function.
$= f(7)$ $x_2 = 7$
$= 4(7) - 5$ or 23 Simplify.

The first three iterates are 3, 7, and 23.

Exercises

Find the first three iterates of each function for the given initial value.

1. $f(x) = x - 1; x_0 = 4$ **3, 2, 1**

2. $f(x) = x^2 - 3x; x_0 = 1$ **−2, 10, 70**

3. $f(x) = x^2 + 2x + 1; x_0 = -2$ **1, 4, 25**

4. $f(x) = 4x - 6; x_0 = -5$ **−26, −110, −446**

5. $f(x) = 6x - 2; x_0 = 3$ **16, 94, 562**

6. $f(x) = 100 - 4x; x_0 = -5$ **120, −380, 1620**

7. $f(x) = 3x - 1; x_0 = 47$ **140, 419, 1256**

8. $f(x) = x^3 - 5x^2; x_0 = 1$ **−4, −144, −3,089,664**

9. $f(x) = 10x - 25; x_0 = 2$ **−5, −75, −775**

10. $f(x) = 4x^2 - 9; x_0 = -1$ **−5, 91, 33,115**

11. $f(x) = 2x^2 + 5; x_0 = -4$ **37, 2743, 15,048,103**

12. $f(x) = \frac{x - 1}{x + 2}; x_0 = 1$ **$0, -\frac{1}{2}, -1$**

13. $f(x) = \frac{1}{2}(x + 11); x_0 = 3$ **7, 9, 10**

14. $f(x) = \frac{3}{x}; x_0 = 9$ **$\frac{1}{3}, 9, \frac{1}{3}$**

15. $f(x) = x - 4x^2; x_0 = 1$ **−3, −39, −6123**

16. $f(x) = x + \frac{1}{x}; x_0 = 2$ **2.5, 2.9, about 3.245**

17. $f(x) = x^3 - 5x^2 + 8x - 10; x_0 = 1$ **−6, −454, −94,610,886**

18. $f(x) = x^3 - x^2; x_0 = -2$ **−12, −1872, −6,563,711,232**

NAME ______________________________ DATE ____________ PERIOD _____

11-6 Skills Practice

Recursion and Special Sequences

Find the first five terms of each sequence.

1. $a_1 = 4, a_{n+1} = a_n + 7$ **4, 11, 18, 25, 32**

2. $a_1 = -2, a_{n+1} = a_n + 3$ **−2, 1, 4, 7, 10**

3. $a_1 = 5, a_{n+1} = 2a_n$ **5, 10, 20, 40, 80**

4. $a_1 = -4, a_{n+1} = 6 - a_n$ **−4, 10, −4, 10, −4**

5. $a_1 = 1, a_{n+1} = a_n + n$ **1, 2, 4, 7, 11**

6. $a_1 = -1, a_{n+1} = n - a_n$ **−1, 2, 0, 3, 1**

7. $a_1 = -6, a_{n+1} = a_n + n + 1$ **−6, −4, −1, 3, 8**

8. $a_1 = 8, a_{n+1} = a_n - n - 2$ **8, 5, 1, −4, −10**

9. $a_1 = -3, a_{n+1} = 2a_n + 7$ **−3, 1, 9, 25, 57**

10. $a_1 = 4, a_{n+1} = -2a_n - 5$ **4, −13, 21, −47, 89**

11. $a_1 = 0, a_2 = 1, a_{n+1} = a_n + a_{n-1}$ **0, 1, 1, 2, 3**

12. $a_1 = -1, a_2 = -1, a_{n+1} = a_n - a_{n-1}$ **−1, −1, 0, 1, 1**

13. $a_1 = 3, a_2 = -5, a_{n+1} = -4a_n + a_{n-1}$ **3, −5, 23, −97, 411**

14. $a_1 = -3, a_2 = 2, a_{n+1} = a_{n-1} - a_n$ **−3, 2, −5, 7, −12**

Find the first three iterates of each function for the given initial value.

15. $f(x) = 2x - 1, x_0 = 3$ **5, 9, 17**

16. $f(x) = 5x - 3, x_0 = 2$ **7, 32, 157**

17. $f(x) = 3x + 4, x_0 = -1$ **1, 7, 25**

18. $f(x) = 4x + 7, x_0 = -5$ **−13, −45, −173**

19. $f(x) = -x - 3, x_0 = 10$ **−13, 10, −13**

20. $f(x) = -3x + 6, x_0 = 6$ **−12, 42, −120**

21. $f(x) = -3x + 4, x_0 = 2$ **−2, 10, −26**

22. $f(x) = 6x - 5, x_0 = 1$ **1, 1, 1**

23. $f(x) = 7x + 1, x_0 = -4$ **−27, −188, −1315**

24. $f(x) = x^2 - 3x, x_0 = 5$ **10, 70, 4690**

NAME ______________________________ DATE ____________ PERIOD _____

11-6 Practice (Average)

Recursion and Special Sequences

Find the first five terms of each sequence.

1. $a_1 = 3, a_{n+1} = a_n + 5$ **3, 8, 13, 18, 23**

2. $a_1 = -7, a_{n+1} = a_n + 8$ **−7, 1, 9, 17, 25**

3. $a_1 = -3, a_{n+1} = 3a_n + 2$ **−3, −7, −19, −55, −163**

4. $a_1 = -8, a_{n+1} = 10 - a_n$ **−8, 18, −8, 18, −8**

5. $a_1 = 4, a_{n+1} = n - a_n$ **4, −3, 5, −2, 6**

6. $a_1 = -3, a_{n+1} = 3a_n$ **−3, −9, −27,−81, −243**

7. $a_1 = 4, a_{n+1} = -3a_n + 4$ **4, −8, 28, −80, 244**

8. $a_1 = 2, a_{n+1} = -4a_n - 5$ **2, −13, 47, −193, 767**

9. $a_1 = 3, a_2 = 1, a_{n+1} = a_n - a_{n-1}$ **3, 1, −2, −3, −1**

10. $a_1 = -1, a_2 = 5, a_{n+1} = 4a_{n-1} - a_n$ **−1, 5, −9, 29, −65**

11. $a_1 = 2, a_2 = -3, a_{n+1} = 5a_n - 8a_{n-1}$ **2, −3, −31, −131, −407**

12. $a_1 = -2, a_2 = 1, a_{n+1} = -2a_n + 6a_{n-1}$ **−2, 1, −14, 34, −152**

Find the first three iterates of each function for the given initial value.

13. $f(x) = 3x + 4, x_0 = -1$ **1, 7, 25**

14. $f(x) = 10x + 2, x_0 = -1$ **−8, −78, −778**

15. $f(x) = 8 + 3x, x_0 = 1$ **11, 41, 131**

16. $f(x) = 8 - x, x_0 = -3$ **11, −3, 11**

17. $f(x) = 4x + 5, x_0 = -1$ **1, 9, 41**

18. $f(x) = 5(x + 3), x_0 = -2$ **5, 40, 215**

19. $f(x) = -8x + 9, x_0 = 1$ **1, 1, 1**

20. $f(x) = -4x^2, x_0 = -1$ **−4; −64; −16,384**

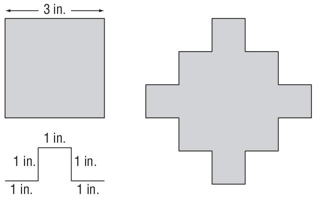
21. $f(x) = x^2 - 1, x_0 = 3$ **8, 63, 3968**

22. $f(x) = 2x^2; x_0 = 5$ **50; 5000; 50,000,000**

23. INFLATION Iterating the function $c(x) = 1.05x$ gives the future cost of an item at a constant 5% inflation rate. Find the cost of a $2000 ring in five years at 5% inflation. **$2552.56**

FRACTALS For Exercises 24–27, use the following information.

Replacing each side of the square shown with the combination of segments below it gives the figure to its right.



24. What is the perimeter of the original square? **12 in.**

25. What is the perimeter of the new shape? **20 in.**

26. If you repeat the process by replacing each side of the new shape by a proportional combination of 5 segments, what will the perimeter of the third shape be? **$33\frac{1}{3}$ in.**

27. What function $f(x)$ can you iterate to find the perimeter of each successive shape if you continue this process? **$f(x) = \frac{5}{3}x$**

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11-6 Reading to Learn Mathematics

Recursion and Special Sequences

Pre-Activity How is the Fibonacci sequence illustrated in nature?

Read the introduction to Lesson 11-6 at the top of page 606 in your textbook.

What are the next three numbers in the sequence that gives the number of shoots corresponding to each month? **8, 13, 21**

Reading the Lesson

1. Consider the sequence in which $a_1 = 4$ and $a_n = 2a_{n-1} + 5$.

a. Explain why this is a *recursive* formula. **Sample answer: Each term is found from the value of the previous term.**

b. Explain in your own words how to find the first four terms of this sequence. (Do not actually find any terms after the first.) **Sample answer: The first term is 4. To find the second term, double the first term and add 5. To find the third term, double the second term and add 5. To find the fourth term, double the third term and add 5.**

c. What happens to the terms of this sequence as n increases? **Sample answer: They keep getting larger and larger.**

2. Consider the function $f(x) = 3x - 1$ with an initial value of $x_0 = 2$.

a. What does it mean to *iterate* this function? **to compose the function with itself repeatedly**

b. Fill in the blanks to find the first three iterates. The blanks that follow the letter x are for subscripts.

$x_1 = f(x_{\underline{0}}) = f(\underline{\ 2\ }) = 3(\underline{\ 2\ }) - 1 = \underline{\ 6\ } - 1 = \underline{\ 5\ }$

$x_2 = f(x_{\underline{1}}) = f(\underline{\ 5\ }) = 3(\underline{\ 5\ }) - 1 = \underline{\ 14\ }$

$x_3 = f(x_{\underline{2}}) = f(\underline{\ 14\ }) = 3(\underline{\ 14\ }) - 1 = \underline{\ 41\ }$

c. As this process continues, what happens to the values of the iterates? **Sample answer: They keep getting larger and larger.**

Helping You Remember

3. Use a dictionary to find the meanings of the words *recurrent* and *iterate*. How can the meanings of these words help you to remember the meaning of the mathematical terms *recursive* and *iteration*? How are these ideas related? **Sample answer: *Recurrent* means happening repeatedly, while *iterate* means to repeat a process or operation. A *recursive* formula is used repeatedly to find the value of one term of a sequence based on the previous term. *Iteration* means to compose a function with it self repeatedly. Both ideas have to do with repetition—doing the same thing over and over again.**

Lesson 11-6

NAME ______________________ DATE ____________ PERIOD _____

11-6 Enrichment

Continued Fractions

The fraction below is an example of a continued fraction. Note that each fraction in the continued fraction has a numerator of 1.

$$2 + \cfrac{1}{3 + \cfrac{1}{4 + \frac{1}{5}}}$$

Example 1 **Evaluate the continued fraction above. Start at the bottom and work your way up.**

Step 1: $4 + \frac{1}{5} = \frac{20}{5} + \frac{1}{5} = \frac{21}{5}$

Step 2: $\frac{1}{\frac{21}{5}} = \frac{5}{21}$

Step 3: $3 + \frac{5}{21} = \frac{63}{21} + \frac{5}{21} = \frac{68}{21}$

Step 4: $\frac{1}{\frac{68}{21}} = \frac{21}{68}$

Step 5: $2 + \frac{21}{68} = 2\frac{21}{68}$

Example 2 **Change $\frac{25}{11}$ into a continued fraction.**

Follow the steps.

Step 1: $\frac{25}{11} = \frac{22}{11} + \frac{3}{11} = 2 + \frac{3}{11}$

Step 2: $\frac{3}{11} = \frac{1}{\frac{11}{3}}$

Step 3: $\frac{11}{3} = \frac{9}{3} + \frac{2}{3} = 3 + \frac{2}{3}$

Step 4: $\frac{2}{3} = \frac{1}{\frac{3}{2}}$

Step 5: $\frac{3}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2}$

Stop, because the numerator is 1.

Thus, $\frac{25}{11}$ can be written as $2 + \cfrac{1}{3 + \cfrac{1}{1 + \frac{1}{2}}}$

Evaluate each continued fraction.

1. $1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \frac{1}{3}}}}$ **$1\frac{17}{24}$**

2. $0 + \cfrac{1}{6 + \cfrac{1}{4 + \frac{1}{2}}}$ **$\frac{9}{56}$**

3. $2 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{8 + \frac{1}{10}}}}$ **$2\frac{496}{2065}$**

4. $5 + \cfrac{1}{7 + \cfrac{1}{9 + \frac{1}{11}}}$ **$5\frac{100}{711}$**

Change each fraction into a continued fraction.

5. $\frac{75}{31}$

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}}$$

6. $\frac{29}{8}$

$$3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}$$

7. $\frac{13}{19}$

$$0 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{6}}}$$

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11-7 Study Guide and Intervention

The Binomial Theorem

Pascal's Triangle **Pascal's triangle** is the pattern of coefficients of powers of binomials displayed in triangular form. Each row begins and ends with 1 and each coefficient is the sum of the two coefficients above it in the previous row.

Pascal's Triangle	$(a + b)^0$	1
	$(a + b)^1$	1 1
	$(a + b)^2$	1 2 1
	$(a + b)^3$	1 3 3 1
	$(a + b)^4$	1 4 6 4 1
	$(a + b)^5$	1 5 10 10 5 1

Example **Use Pascal's triangle to find the number of possible sequences consisting of 3 as and 2 bs.**

The coefficient 10 of the a^3b^2-term in the expansion of $(a + b)^5$ gives the number of sequences that result in three as and two bs.

Exercises

Expand each power using Pascal's triangle.

1. $(a + 5)^4$ $a^4 + 20a^3 + 150a^2 + 500a + 625$

2. $(x - 2y)^6$ $x^6 - 12x^5y + 60x^4y^2 - 160x^3y^3 + 240x^2y^4 - 192xy^5 + 64y^6$

3. $(j - 3k)^5$ $j^5 - 15j^4k + 90j^3k^2 - 270j^2k^3 + 405jk^4 - 243k^5$

4. $(2s + t)^7$ $128s^7 + 448s^6t + 672s^5t^2 + 560s^4t^3 + 280s^3t^4 + 84s^2t^5 + 14st^6 + t^7$

5. $(2p + 3q)^6$ $64p^6 + 576p^5q + 2160p^4q^2 + 4320p^3q^3 + 4860p^2q^4 + 2916pq^5 + 729q^6$

6. $\left(a - \frac{b}{2}\right)^4$ $a^4 - 2a^3b + \frac{3}{2}a^2b^2 - \frac{1}{2}ab^3 + \frac{1}{16}b^4$

7. Ray tosses a coin 15 times. How many different sequences of tosses could result in 4 heads and 11 tails? 1365

8. There are 9 true/false questions on a quiz. If twice as many of the statements are true as false, how many different sequences of true/false answers are possible? 84

Lesson 11-7

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11-7 Study Guide and Intervention *(continued)*

The Binomial Theorem

The Binomial Theorem

Binomial Theorem	If n is a nonnegative integer, then $(a + b)^n = 1a^nb^0 + \frac{n}{1}a^{n-1}b^1 + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3 + \ldots + 1a^0b^n$

Another useful form of the Binomial Theorem uses **factorial** notation and sigma notation.

Factorial	If n is a positive integer, then $n! = n(n - 1)(n - 2) \cdot \ldots \cdot 2 \cdot 1$.
Binomial Theorem, Factorial Form	$(a + b)^n = \frac{n!}{n!0!}a^nb^0 + \frac{n!}{(n-1)!1!}a^{n-1}b^1 + \frac{n!}{(n-2)!2!}a^{n-2}b^2 + \ldots + \frac{n!}{0!n!}a^0b^n$ $= \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}a^{n-k}b^k$

Example 1 **Evaluate $\frac{11!}{8!}$.**

$$\frac{11!}{8!} = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$
$$= 11 \cdot 10 \cdot 9 = 990$$

Example 2 **Expand $(a - 3b)^4$.**

$$(a - 3b)^4 = \sum_{k=0}^{4} \frac{4!}{(4-k)!k!}a^{4-k}(-3b)^k$$
$$= \frac{4!}{4!0!}a^4 + \frac{4!}{3!1!}a^3(-3b)^1 + \frac{4!}{2!2!}a^2(-3b)^2 + \frac{4!}{1!3!}a(-3b)^3 + \frac{4!}{0!4!}(-3b)^4$$
$$= a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$$

Exercises

Evaluate each expression.

1. $5!$ 120
2. $\frac{9!}{7!2!}$ 36
3. $\frac{10!}{6!4!}$ 210

Expand each power.

4. $(a - 3)^6$ $a^6 - 18a^5 + 135a^4 - 540a^3 + 1215a^2 - 1458a + 729$

5. $(r + 2s)^7$ $r^7 + 14r^6s + 84r^5s^2 + 280r^4s^3 + 560r^3s^4 + 672r^2s^5 + 448rs^6 + 128s^7$

6. $(4x + y)^4$ $256x^4 + 256x^3y + 96x^2y^2 + 16xy^3 + y^4$

7. $\left(2 - \frac{m}{2}\right)^5$ $32 - 40m + 20m^2 - 5m^3 + \frac{5}{8}m^4 - \frac{1}{32}m^5$

Find the indicated term of each expansion.

8. third term of $(3x - y)^5$ $270x^3y^2$
9. fifth term of $(a + 1)^7$ $35a^3$
10. fourth term of $(j + 2k)^8$ $448j^5k^3$
11. sixth term of $(10 - 3t)^7$ $-510{,}300t^5$
12. second term of $\left(m + \frac{2}{3}\right)^9$ $6m^8$
13. seventh term of $(5x - 2)^{11}$ $92{,}400{,}000x^5$

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11-7 Skills Practice

The Binomial Theorem

Evaluate each expression.

1. 8! **40,320**

2. 10! **3,628,800**

3. 12! **479,001,600**

4. $\frac{15!}{13!}$ **210**

5. $\frac{6!}{3!}$ **120**

6. $\frac{10!}{2!8!}$ **45**

7. $\frac{9!}{3!6!}$ **84**

8. $\frac{20!}{15!5!}$ **15,504**

Expand each power.

9. $(x - y)^3$ $x^3 - 3x^2y + 3xy^2 - y^3$

10. $(a + b)^5$ $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

12. $(m + 1)^4$ $m^4 + 4m^3 + 6m^2 + 4m + 1$

11. $(g - h)^4$ $g^4 - 4g^3h + 6g^2h^2 - 4gh^3 + h^4$

13. $(r + 4)^3$ $r^3 + 12r^2 + 48r + 64$

14. $(a - 5)^4$ $a^4 - 20a^3 + 150a^2 - 500a + 625$

15. $(y - 7)^3$ $y^3 - 21y^2 + 147y - 343$

16. $(d + 2)^5$ $d^5 + 10d^4 + 40d^3 + 80d^2 + 80d + 32$

17. $(x - 1)^4$ $x^4 - 4x^3 + 6x^2 - 4x + 1$

18. $(2a + b)^4$ $16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$

19. $(c - 4d)^3$ $c^3 - 12c^2d + 48cd^2 - 64d^3$

20. $(2a + 3)^3$ $8a^3 + 36a^2 + 54a + 27$

Find the indicated term of each expansion.

21. fourth term of $(m + n)^{10}$ $120m^7n^3$

22. seventh term of $(x - y)^8$ $28x^2y^6$

23. third term of $(b + 6)^5$ $360b^3$

24. sixth term of $(s - 2)^9$ $-4032s^4$

25. fifth term of $(2a + 3)^6$ $4860a^2$

26. second term of $(3x - y)^7$ $-5103x^6y$

Lesson 11-7

NAME ______________________ DATE __________ PERIOD _____

11-7 Practice (Average)

The Binomial Theorem

Evaluate each expression.

1. 7! **5040**

2. 11! **39,916,800**

3. $\frac{9!}{5!}$ **3024**

4. $\frac{20!}{18!}$ **380**

5. $\frac{8!}{6!2!}$ **28**

6. $\frac{8!}{5!3!}$ **56**

7. $\frac{12!}{6!6!}$ **924**

8. $\frac{41!}{3!38!}$ **10,660**

Expand each power.

9. $(n + v)^5$ $n^5 + 5n^4v + 10n^3v^2 + 10n^2v^3 + 5nv^4 + v^5$

10. $(x - y)^4$ $x^4 - 4x^3y + 6x^2y^2 -4xy^3 + y^4$

11. $(x + y)^6$ $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

12. $(r + 3)^5$ $r^5 + 15r^4 + 90r^3 + 270r^2 + 405r + 243$

13. $(m - 5)^5$ $m^5 - 25m^4 + 250m^3 - 1250m^2 + 3125m - 3125$

14. $(x + 4)^4$ $x^4 + 16x^3 + 96x^2 + 256x + 256$

15. $(3x + y)^4$ $81x^4 + 108x^3y + 54x^2y^2 + 12xy^3 + y^4$

16. $(2m - y)^4$ $16m^4 - 32m^3y + 24m^2y^2 - 8my^3 + y^4$

17. $(w - 3z)^3$ $w^3 - 9w^2z + 27wz^2 - 27z^3$

18. $(2d + 3)^6$ $64d^6 + 576d^5 + 2160d^4 + 4320d^3 + 4860d^2 + 2916d + 729$

19. $(x + 2y)^5$ $x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$

20. $(2x - y)^5$ $32x^5 - 80x^4y + 80x^3y^2 - 40x^2y^3 + 10xy^4 - y^5$

21. $(a - 3b)^4$ $a^4 - 12a^3b + 54a^2b^2 - 108ab^3 + 81b^4$

22. $(3 - 2z)^4$ $16z^4 - 96z^3 + 216z^2 - 216z + 81$

23. $(3m - 4n)^3$ $27m^3 - 108m^2n + 144mn^2 - 64n^3$

24. $(5x - 2y)^4$ $625x^4 - 1000x^3y + 600x^2y^2 - 160xy^3 + 16y^4$

Find the indicated term of each expansion.

25. seventh term of $(a + b)^{10}$ $210a^4b^6$

26. sixth term of $(m - n)^{10}$ $-252m^5n^5$

27. ninth term of $(r - s)^{14}$ $3003r^6s^8$

28. tenth term of $(2x + y)^{12}$ $1760x^3y^9$

29. fourth term of $(x - 3y)^6$ $-540x^3y^3$

30. fifth term of $(2x - 1)^9$ $4032x^5$

31. GEOMETRY How many line segments can be drawn between ten points, no three of which are collinear, if you use exactly two of the ten points to draw each segment? **45**

32. PROBABILITY If you toss a coin 4 times, how many different sequences of tosses will give exactly 3 heads and 1 tail or exactly 1 head and 3 tails? **8**

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11-7 Reading to Learn Mathematics

The Binomial Theorem

Pre-Activity **How does a power of a binomial describe the numbers of boys and girls in a family?**

Read the introduction to Lesson 11-7 at the top of page 612 in your textbook.

- If a family has four children, list the sequences of births of girls and boys that result in three girls and one boy. **BGGG GBGG GGBG GGGB**
- Describe a way to figure out how many such sequences there are without listing them. **Sample answer: The boy could be the first, second, third, or fourth child, so there are four sequences with three girls and one boy.**

Reading the Lesson

1. Consider the expansion of $(w + z)^5$.

a. How many terms does this expansion have? **6**

b. In the second term of the expansion, what is the exponent of w? **4**

What is the exponent of z? **1**

What is the coefficient of the second term? **5**

c. In the fourth term of the expansion, what is the exponent of w? **2**

What is the exponent of z? **3**

What is the coefficient of the fourth term? **10**

d. What is the last term of this expansion? $\boldsymbol{z^5}$

2. a. State the definition of a *factorial* in your own words. (Do not use mathematical symbols in your definition.) **Sample answer: The factorial of any positive integer is the product of that integer and all the smaller integers down to one. The factorial of zero is one.**

b. Write out the product that you would use to calculate 10!. (Do not actually calculate the product.) $\mathbf{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

c. Write an expression involving factorials that could be used to find the coefficient of the third term of the expansion of $(m - n)^6$. (Do not actually calculate the coefficient.) $\dfrac{6!}{4!2!}$

Helping You Remember

3. Without using Pascal's triangle or factorials, what is an easy way to remember the first two and last two coefficients for the terms of the binomial expansion of $(a + b)^n$? **Sample answer: The first and last coefficients are always 1. The second and next-to-last coefficients are always *n*, the power to which the binomial is being raised.**

Lesson 11-7

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11-7 Enrichment

Patterns in Pascal's Triangle

You have learned that the coefficients in the expansion of $(x + y)^n$ yield a number pyramid called **Pascal's triangle.**

Row 1 ⟶ 1
Row 2 ⟶ 1 1
Row 3 ⟶ 1 2 1
Row 4 ⟶ 1 3 4 1
Row 5 ⟶ 1 4 6 4 1
Row 6 ⟶ 1 5 10 10 5 1
Row 7 ⟶ 1 6 15 20 15 6 1

As many rows can be added to the bottom of the pyramid as you please.

This activity explores some of the interesting properties of this famous number pyramid.

1. Pick a row of Pascal's triangle.

a. What is the sum of all the numbers in all the rows *above* the row you picked? **See students' work.**

b. What is the sum of all the numbers in the row you picked? **See students' work.**

c. How are your answers for parts **a** and **b** related? **The answer for Part b is 1 more than the answer for Part a.**

d. Repeat parts **a** through **c** for at least three more rows of Pascal's triangle. What generalization seems to be true? **It appears that the sum of the numbers in any row is 1 more than the sum of the numbers in all of the rows above it.**

e. See if you can prove your generalization. **Sum of numbers in row** $\boldsymbol{n = 2^n - 1; 2^0 + 2^1 + 2^2 + \ldots + 2^{n-2}}$**, which, by the formula for the sum of a geometric series, is** $\boldsymbol{2^{n-1} - 1}$**.**

2. Pick any row of Pascal's triangle that comes after the first.

a. Starting at the left end of the row, add the first number, the third number, the fifth number, and so on. State the sum. **See students' work.**

b. In the same row, add the second number, the fourth number, and so on. State the sum. **See students' work.**

c. How do the sums in parts **a** and **b** compare? **The sums are equal.**

d. Repeat parts **a** through **c** for at least three other rows of Pascal's triangle. What generalization seems to be true? **In any row of Pascal's triangle after the first, the sum of the odd numbered terms is equal to the sum of the even numbered terms.**

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11-8 Study Guide and Intervention

Proof and Mathematical Induction

Mathematical Induction Mathematical induction is a method of proof used to prove statements about positive integers.

Mathematical Induction Proof	**Step 1** Show that the statement is true for some integer n. **Step 2** Assume that the statement is true for some positive integer k where $k \geq n$. This assumption is called the **inductive hypothesis**. **Step 3** Show that the statement is true for the next integer $k + 1$.

Example **Prove that $5 + 11 + 17 + \ldots + (6n - 1) = 3n^2 + 2n$.**

Step 1 When $n = 1$, the left side of the given equation is $6(1) - 1 = 5$. The right side is $3(1)^2 + 2(1) = 5$. Thus the equation is true for $n = 1$.

Step 2 Assume that $5 + 11 + 17 + \ldots + (6k - 1) = 3k^2 + 2k$ for some positive integer k.

Step 3 Show that the equation is true for $n = k + 1$. First, add $[6(k + 1) - 1]$ to each side.

$$\begin{aligned} 5 + 11 + 17 + \ldots + (6k - 1) + [6(k + 1) - 1] &= 3k^2 + 2k + [6(k + 1) - 1] \\ &= 3k^2 + 2k + 6k + 5 && \text{Add.} \\ &= 3k^2 + 6k + 3 + 2k + 2 && \text{Rewrite.} \\ &= 3(k^2 + 2k + 1) + 2(k + 1) && \text{Factor.} \\ &= 3(k + 1)^2 + 2(k + 1) && \text{Factor.} \end{aligned}$$

The last expression above is the right side of the equation to be proved, where n has been replaced by $k + 1$. Thus the equation is true for $n = k + 1$.

This proves that $5 + 11 + 17 + \ldots + (6n - 1) = 3n^2 + 2n$ for all positive integers n.

Exercises

Prove that each statement is true for all positive integers.

1. $3 + 7 + 11 + \ldots + (4n - 1) = 2n^2 + n$.

Step 1 The statement is true for $n = 1$ since $4(1) - 1 = 3$ and $2(1)^2 + 1 = 3$.
Step 2 Assume that $3 + 7 + 11 + \ldots + (4k - 1) = 2k^2 + k$ for some positive integer k.
Step 3 Adding the $(k + 1)$st term to each side from step 2, we get $3 + 7 + 11 + \ldots + (4k - 1) + [4(k + 1) - 1] = 2k^2 + k + [4(k + 1) - 1]$. Simplifying the right side of the equation gives $2(k + 1)^2 + (k + 1)$, which is the statement to be proved.

2. $500 + 100 + 20 + \ldots + 4 \cdot 5^{4-n} = 625\left(1 - \frac{1}{5^n}\right)$.

Step 1 The statement is true for $n = 1$, since $4 \cdot 5^{4-1} = 4 \cdot 5^3 = 500$ and $625\left(1 - \frac{1}{5^1}\right) = \frac{4}{5}(625) = 500$.

Step 2 Assume that $500 + 100 + 20 + \ldots + 4 \cdot 5^{4-k} = 625\left(1 - \frac{1}{5^k}\right)$ for some positive integer k.
Step 3 Adding the $(k + 1)$st term to each side from step 2 and simplifying gives $500 + 100 + 20 + \ldots + 4 \cdot 5^{4-k} + 4 \cdot 5^{3-k} = 625\left(1 - \frac{1}{5^k}\right) + 4 \cdot 5^{3-k} = 625\left(1 - \frac{1}{5^{k+1}}\right)$, which is the statement to be proved.

Lesson 11-8

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11-8 Study Guide and Intervention *(continued)*

Proof by Mathematical Induction

Counterexamples To show that a formula or other generalization is *not* true, find a **counterexample**. Often this is done by substituting values for a variable.

Example 1 **Find a counterexample for the formula $2n^2 + 2n + 3 = 2^{n+2} - 1$.**

Check the first few positive integers.

n	Left Side of Formula	Right Side of Formula	
1	$2(1)^2 + 2(1) + 3 = 2 + 2 + 3$ or 7	$2^{1+2} - 1 = 2^3 - 1$ or 7	*true*
2	$2(2)^2 + 2(2) + 3 = 8 + 4 + 3$ or 15	$2^{2+2} - 1 = 2^4 - 1$ or 15	*true*
3	$2(3)^2 + 2(3) + 3 = 18 + 6 + 3$ or 27	$2^{3+2} - 1 = 2^5 - 1$ or 31	*false*

The value $n = 3$ provides a counterexample for the formula.

Example 2 **Find a counterexample for the statement $x^2 + 4$ is either prime or divisible by 4.**

n	$x^2 + 4$	True?	n	$x^2 + 4$	True?
1	1 + 4 or 5	*Prime*	6	36 + 4 or 40	*Div. by 4*
2	4 + 4 or 8	*Div. by 4*	7	49 + 4 or 53	*Prime*
3	9 + 4 or 13	*Prime*	8	64 + 4 or 68	*Div. by 4*
4	16 + 4 or 20	*Div. by 4*	9	81 + 4 or 85	*Neither*
5	25 + 4 or 29	*Prime*			

The value $n = 9$ provides a counterexample.

Exercises

Find a counterexample for each statement. **Sample answers are given.**

1. $1 + 5 + 9 + \ldots + (4n - 3) = 4n - 3$ **$n = 2$**

2. $100 + 110 + 120 + \ldots + (10n + 90) = 5n^2 + 95$ **$n = 2$**

3. $900 + 300 + 100 + \ldots + 100(3^{3-n}) = 900 \cdot \frac{2n}{n + 1}$ **$n = 3$**

4. $x^2 + x + 1$ is prime. **$n = 4$**

5. $2n + 1$ is a prime number. **$n = 4$**

6. $7n - 5$ is a prime number. **$n = 2$**

7. $\frac{1}{2} + 1 + \frac{3}{2} + \ldots + \frac{n}{2} = n - \frac{1}{2}$ **$n = 3$**

8. $5n^2 + 1$ is divisible by 3. **$n = 3$**

9. $n^2 - 3n + 1$ is prime for $n > 2$. **$n = 9$**

10. $4n^2 - 1$ is divisible by either 3 or 5. **$n = 6$**

Answers

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11-8 Skills Practice

Proof and Mathematical Induction

Prove that each statement is true for all positive integers.

1. $1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Step 1: When $n = 1$, $2n - 1 = 2(1) - 1 = 1 = 1^2$. So, the equation is true for $n = 1$.

Step 2: Assume that $1 + 3 + 5 + \ldots + (2k - 1) = k^2$ for some positive integer k.

Step 3: Show that the given equation is true for $n = k + 1$.

$$\begin{aligned} 1 + 3 + 5 + \ldots + (2k - 1) + [2(k + 1) - 1] &= k^2 + [2(k + 1) - 1] \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

So, $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all positive integers n.

2. $2 + 4 + 6 + \ldots + 2n = n^2 + n$

Step 1: When $n = 1$, $2n = 2(1) = 2 = 1^2 + 1$. So, the equation is true for $n = 1$.

Step 2: Assume that $2 + 4 + 6 + \ldots + 2k = k^2 + k$ for some positive integer k.

Step 3: Show that the given equation is true for $n = k + 1$.

$$\begin{aligned} 2 + 4 + 6 + \ldots. + 2k + 2(k + 1) &= k^2 + k + 2(k + 1) \\ &= (k^2 + 2k + 1) + (k + 1) \\ &= (k + 1)^2 + (k + 1) \end{aligned}$$

So, $2 + 4 + 6 + \ldots + 2n = n^2 + n$ for all positive integers n.

3. $6^n - 1$ is divisible by 5.

Step 1: When $n = 1$, $6^n - 1 = 6^1 - 1 = 5$. So, the statement is true for $n = 1$.

Step 2: Assume that $6^k - 1$ is divisible by 5 for some positive integer k. Then there is a whole number r such that $6^k - 1 = 5r$.

Step 3: Show that the statement is true for $n = k + 1$.

$$\begin{aligned} 6^k - 1 &= 5r \\ 6^k &= 5r + 1 \\ 6(6^k) &= 6(5r + 1) \\ 6^{k+1} &= 30r + 6 \\ 6^{k+1} - 1 &= 30r + 5 \\ 6^{k+1} - 1 &= 5(6r + 1) \end{aligned}$$

Since r is a whole number, $6r + 1$ is a whole number, and $6^{k+1} - 1$ is divisible by 5. The statement is true for $n = k + 1$. So, $6^n - 1$ is divisible by 5 for all positive integers n.

Find a counterexample for each statement.

4. $3^n + 3n$ is divisible by 6.

Sample answer: $n = 2$

5. $1 + 4 + 8 + \ldots + 2^n = \frac{n(n + 1)(2n + 1)}{6}$

Sample answer: $n = 3$

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11-8 Practice (Average)

Proof and Mathematical Induction

Prove that each statement is true for all positive integers.

1. $1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$

Step 1: When $n = 1$, then $2^{n-1} = 2^{1-1} = 2^0 = 1 = 2^1 - 1$. So, the equation is true for $n = 1$.

Step 2: Assume that $1 + 2 + 4 + 8 + \ldots + 2^{k-1} = 2^k - 1$ for some positive integer k.

Step 3: Show that the given equation is true for $n = k + 1$.

$$\begin{aligned} 1 + 2 + 4 + 8 + \ldots + 2^{k-1} + 2^{(k+1)-1} &= (2^k - 1) + 2^{(k+1)-1} \\ &= 2^k - 1 + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1 \end{aligned}$$

So, $1 + 2 + 4 + 8 + \ldots + 2^{n-1} = 2^n - 1$ for all positive integers n.

2. $1 + 4 + 9 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$

Step 1: When $n = 1$, $n^2 = 1^2 = 1 = \frac{1(1 + 1)(2 \cdot 1 + 1)}{6}$; true for $n = 1$.

Step 2: Assume that $1 + 4 + 9 + \ldots + k^2 = \frac{k(k + 1)(2k + 1)}{6}$ for some positive integer k.

Step 3: Show that the given equation is true for $n = k + 1$.

$$\begin{aligned} 1 + 4 + 9 + \ldots + k^2 + (k + 1)^2 &= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \\ &= \frac{k(k + 1)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} = \frac{(k + 1)[k(2k + 1) + 6(k + 1)]}{6} \\ &= \frac{(k + 1)(2k^2 + 7k + 6)}{6} = \frac{(k + 1)[(k + 2)(2k + 3)]}{6} \\ &= \frac{(k + 1)[(k + 1) + 1][2(k + 1) + 1]}{6} \end{aligned}$$

So, $1 + 4 + 9 + \ldots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$ for all positive integers n.

3. $18^n - 1$ is a multiple of 17.

Step 1: When $n = 1$, $18^n - 1 = 18 - 1$ or 17; true for $n = 1$.

Step 2: Assume that $18^k - 1$ is divisible by 17 for some positive integer k. This means that there is a whole number r such that $18^k - 1 = 17r$.

Step 3: Show that the statement is true for $n = k + 1$.

$18^k - 1 = 17r$, so $18^k = 17r + 1$, and $18(18^k) = 18(17r + 1)$. This is equivalent to $18^{k+1} = 306r + 18$, so $18^{k+1} - 1 = 306r + 17$, and $18^{k+1} - 1 = 17(18r + 1)$.

Since r is a whole number, $18r + 1$ is a whole number, and $18^{k+1} - 1$ is divisible by 17. The statement is true for $n = k + 1$. So, $18^n - 1$ is divisible by 17 for all positive integers n.

Find a counterexample for each statement.

4. $1 + 4 + 7 + \ldots + (3n - 2) = n^3 - n^2 + 1$

Sample answer: $n = 3$

5. $5^n - 2n - 3$ is divisible by 3.

Sample answer: $n = 3$

6. $1 + 3 + 5 + \ldots + (2n - 1) = \frac{n^2 + 3n - 2}{2}$

Sample answer: $n = 3$

7. $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^4 - n^3 + 1$

Sample answer: $n = 3$

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11-8 Reading to Learn Mathematics

Proof and Mathematical Induction

Pre-Activity **How does the concept of a ladder help you prove statements about numbers?**

Read the introduction to Lesson 11-8 at the top of page 618 in your textbook.

What are two ways in which a ladder could be constructed so that you could not reach every step of the ladder?

Sample answer: 1. The first step could be too far off the ground for you to climb on it. 2. The steps could be too far apart for you to go up from one step to the next.

Reading the Lesson

1. Fill in the blanks to describe the three steps in a proof by mathematical induction.

Step 1 Show that the statement is **true** for the number **1**.

Step 2 Assume that the statement is **true** for some positive **integer** k.

This assumption is called the **inductive hypothesis**.

Step 3 Show that the statement is **true** for the next integer **$k + 1$**.

2. Suppose that you wanted to prove that the following statement is true for all positive integers.

$$3 + 6 + 9 + \ldots + 3n = \frac{3n(n + 1)}{2}$$

a. Which of the following statements shows that the statement is true for $n = 1$? **ii**

i. $3 = \frac{3 \cdot 2 + 1}{2}$ **ii.** $3 = \frac{3 \cdot 1 \cdot 2}{2}$ **iii.** $3 = \frac{3 + 1 + 2}{2}$

b. Which of the following is the statement for $n = k + 1$? **iv**

i. $3 + 6 + 9 + \ldots + 3^k = \frac{3k(k + 1)}{2}$

ii. $3 + 6 + 9 + \ldots + 3^{k+1} = \frac{3k(k + 1)}{2}$

iii. $3 + 6 + 9 + \ldots + 3^{k+1} = 3(k + 1)(k + 2)$

iv. $3 + 6 + 9 + \ldots + 3(k + 1) = \frac{3(k + 1)(k + 2)}{2}$

Helping You Remember

3. Many students confuse the roles of n and k in a proof by mathematical induction. What is a good way to remember the difference in the ways these variables are used in such a proof? **Sample answer: The letter *n* stands for "number" and is used as a variable to represent *any* natural number. The letter *k* is used to represent a particular value of *n*.**

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11-8 Enrichment

Proof by Induction

Mathematical induction is a useful tool when you want to prove that a statement is true for all natural numbers.

The three steps in using induction are:

1. Prove that the statement is true for $n = 1$.

2. Prove that if the statement is true for the natural number n, it must also be true for $n + 1$.

3. Conclude that the statement is true for all natural numbers.

Follow the steps to complete each proof.

Theorem A: The sum of the first n odd natural numbers is equal to n^2.

1. Show that the theorem is true for $n = 1$.
$1 = (1)^2$

2. Suppose $1 + 3 + 5 + \ldots + (2n - 1) = n^2$. Show that $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = (n + 1)^2$.
Add $2n + 1$ to each side of the equation whose truth was assumed: $1 + 3 + 5 + \ldots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$

3. Summarize the results of problems 1 and 2.
The theorem is true for $n = 1$. *If the sum of the first* n odd numbers equals n^2, then it is true that the sum of the first $n + 1$ odd numbers equals $(n + 1)^2$. Therefore, the theorem is true for all natural numbers.

Theorem B: Show that $a^n - b^n$ is exactly divisible by $a - b$ for n equal to 1, 2, 3, and all natural numbers.

4. Show that the theorem is true for $n = 1$.
$(a^1 - b^1) \div (a - b) = 1$

5. The expression $a^{n+1} - b^{n+1}$ can be rewritten as $a(a^n - b^n) + b^n(a - b)$. Verify that this is true.
$a(a^n - b^n) + b^n(a - b) = a^{n+1} - ab^n + ab^n - b^{n+1} = a^{n+1} - b^{n+1}$

6. Suppose $a - b$ is a factor of $a^n - b^n$. Use the result in problem 5 to show that $a - b$ must then also be a factor of $a^{n+1} - b^{n+1}$.
$a^{n+1} - b^{n+1} = a(a^n - b^n) + b^n(a - b)$; $a - b$ is a factor of both addends on the right side. So, $a - b$ is also a factor of the left side.

7. Summarize the results of problems 4 through 6.
The theorem is true for $n = 1$. If $a - b$ is a factor of $a^n - b^n$, it is also a factor of $a^{n+1} - b^{n+1}$. So, the theorem is true for all natural numbers *n*.