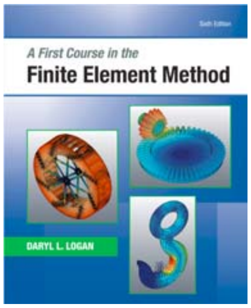
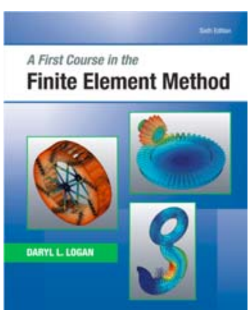
Chapter 11 – Three-Dimensional Stress Analysis



Learning Objectives

- To introduce concepts of three-dimensional stress and strain.
- To develop the tetrahedral solid-element stiffness matrix.
- To describe how body and surface tractions are treated.
- To illustrate a numerical example of the tetrahedral element stiffness matrix.

Chapter 11 – Three-Dimensional Stress Analysis



Learning Objectives

- To describe the isoparametric formulation of the stiffness matrix for three dimensional hexahedral (brick) elements, including the linear (eight-noded) brick, and the quadratic (20 noded) brick.
- To present some commercial computer program examples of three-dimensional solid models and results for real-world applications.
- To present a comparison of the four-noded tetrahedral, the ten-noded tetrahedral, the eight-noded brick, and the twenty-noded brick.

Three-Dimensional Stress Analysis

Introduction

In this chapter, we consider the three-dimensional, or solid, element.

This element is useful for the stress analysis of general three-dimensional bodies that require more precise analysis than is possible though two-dimensional and/or axisymmetric analysis.

Examples of three dimensional problems are arch dams, thick-walled pressure vessels, and solid forging parts as used, for instance, in the heavy equipment and automotive industries.

Three-Dimensional Stress Analysis

Introduction

The tetrahedron is the basic three-dimensional element, and it is used in the development of the shape functions, stiffness matrix, and force matrices in terms of a global coordinate system.

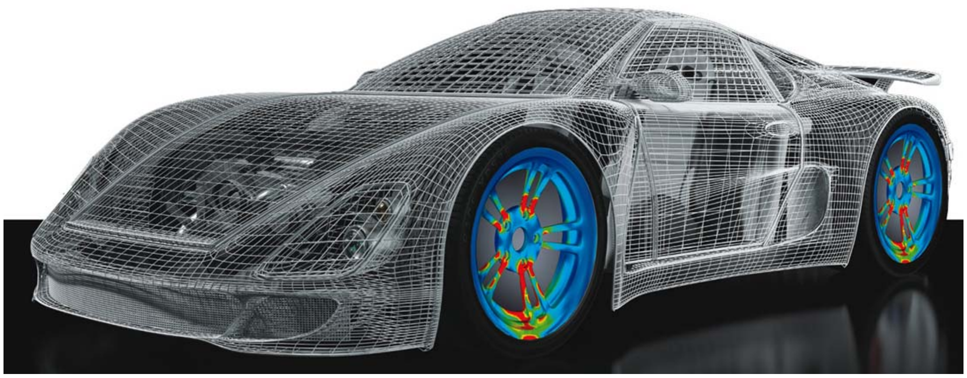
We follow this development with the isoparametric formulation of the stiffness matrix for the hexahedron, or brick element.

Finally, we will provide some typical three-dimensional applications.

Three-Dimensional Stress Analysis

Introduction

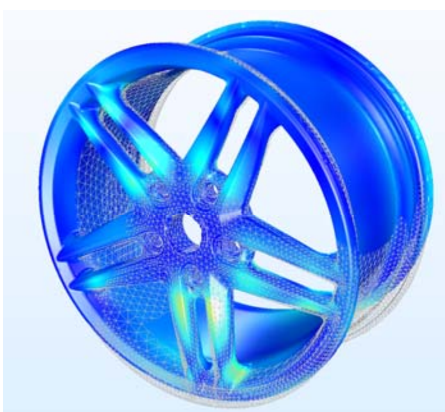
Finite elements can be use for analyzing linear and nonlinear stress characteristics of structures and mechanical components.

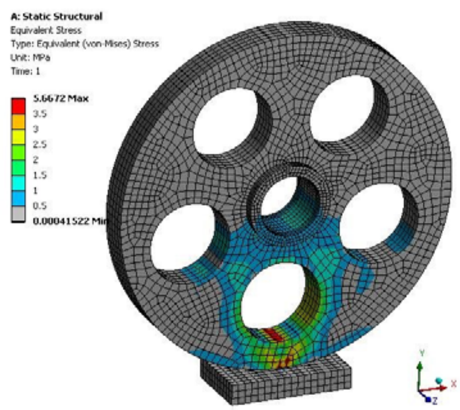


Three-Dimensional Stress Analysis

Introduction

Finite element discretization, stresses, and deformations of a wheel rim in a structural analysis.

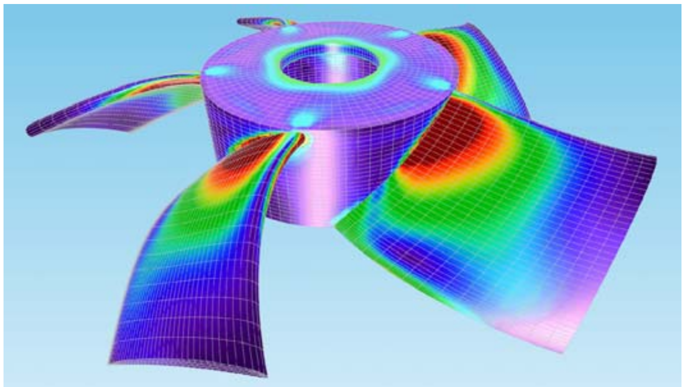




Three-Dimensional Stress Analysis

Introduction

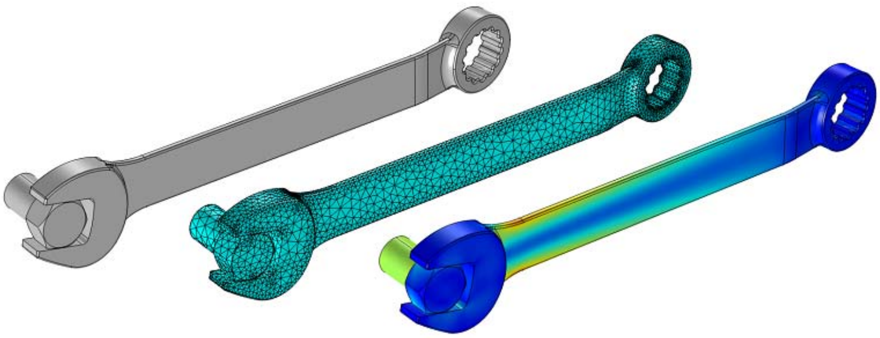
Cooling fan blade vibration, which was predicted by structural mechanics mode shape analysis.



Three-Dimensional Stress Analysis

Introduction

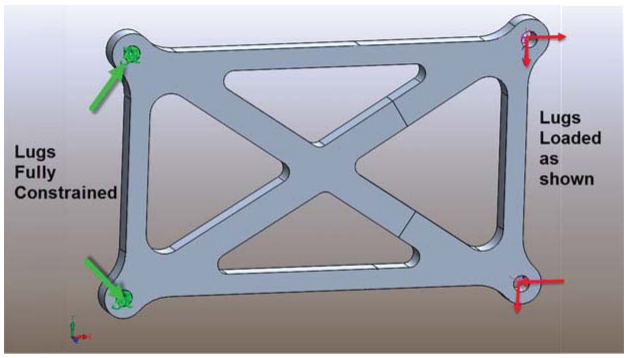
The model of a wrench and bolt that has been imported, meshed, and solved for an applied load.



Three-Dimensional Stress Analysis

Introduction

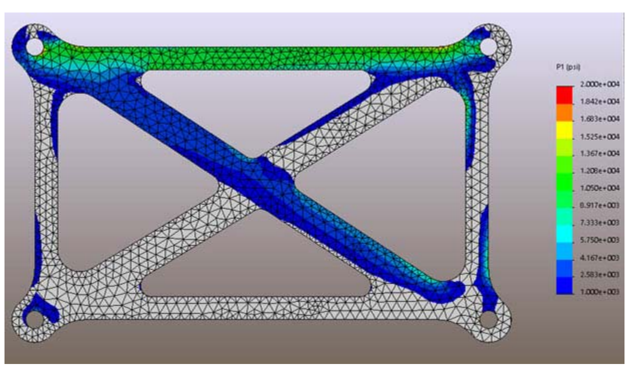
Below is a cross brace, supported at the left-hand edge by two lugs and loaded at the right-hand edge through two additional lugs. The structure is sitting in the global XY plane.



Three-Dimensional Stress Analysis

Introduction

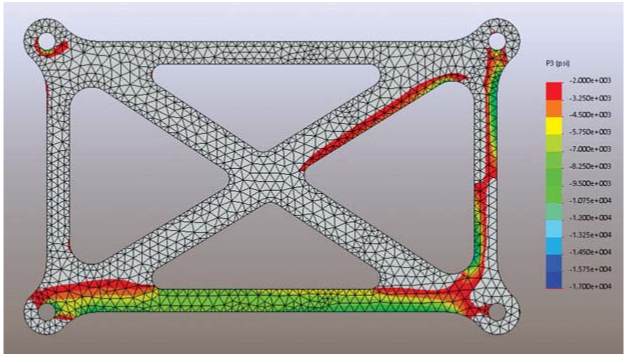
The figure shows stress contour plot of the maximum principal stress P1. This is a useful way of seeing the biggest tensile stress flow direction.



Three-Dimensional Stress Analysis

Introduction

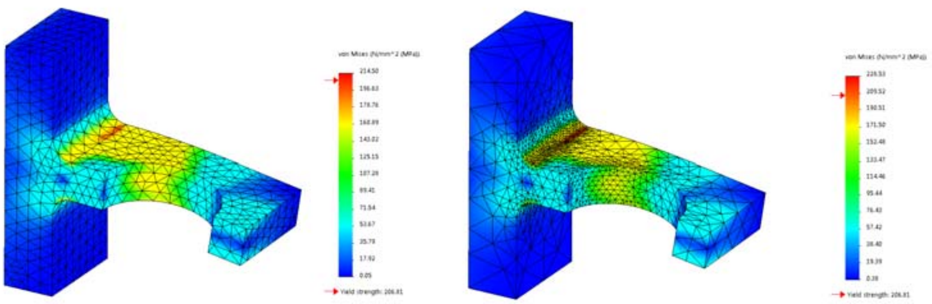
The figure shows the P3 stress distribution through the structure. The plot shows only compressive stresses.



Three-Dimensional Stress Analysis

Introduction

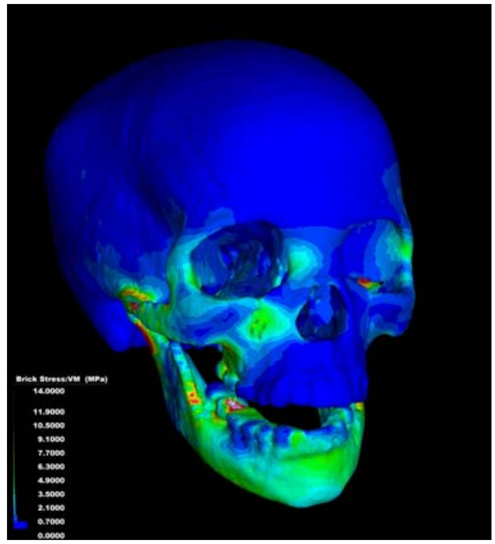
A mesh with smaller elements will allow the solver to more accurately calculate stress distribution across an area. Smaller elements mean more nodes for calculation. More calculation helps to average the stresses in a more accurate manner through the material.



Three-Dimensional Stress Analysis

Introduction

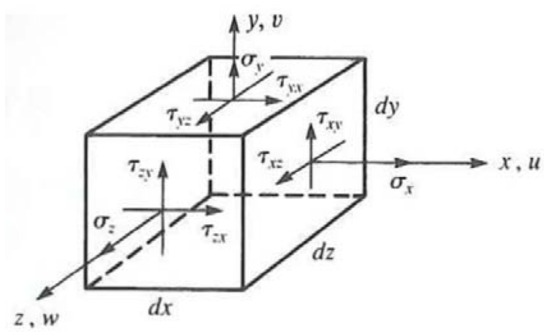
Finite element model of human skull showing stress distribution in bite at 2nd molar.



Three-Dimensional Stress Analysis

Introduction

We begin by considering the three-dimensional infinitesimal element in Cartesian coordinates with dimensions *dx*, *dy*, and *dz*.

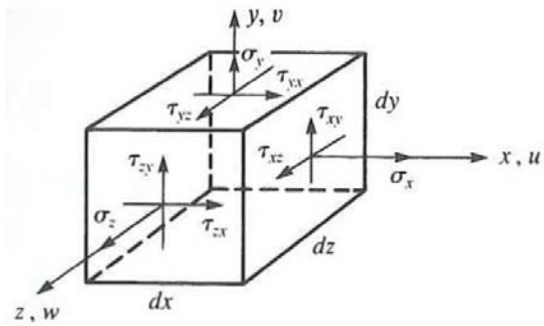


Normal stresses are perpendicular to the faces of the elements given by σ_x, σ_y, and σ_z.

Three-Dimensional Stress Analysis

Introduction

We begin by considering the three-dimensional infinitesimal element in Cartesian coordinates with dimensions *dx*, *dy*, and *dz*.



Shear stresses act parallel to the faces and are represented by τ_{xy}, τ_{yz}, and τ_{zx}.

Three-Dimensional Stress Analysis

Introduction

From moment equilibrium of the element, we show in Appendix C that:

$$\tau_{xy} = \tau_{yx} \qquad \tau_{yz} = \tau_{zy} \qquad \tau_{zx} = \tau_{xz}$$

The element strain-displacement relationships are:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

Three-Dimensional Stress Analysis

Introduction

Where *u*, *v*, and *w* are the displacements associated with the *x*, *y*, and *z* directions. The shears strains γ are now given by:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad = \gamma_{yx}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad = \gamma_{zy}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad = \gamma_{zx}$$

Three-Dimensional Stress Analysis

Introduction

The stresses and strains can be represented by column matrices as:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_x \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_x \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

For an isotropic material the stress/strain as: $\{\sigma\} = [D]\{\varepsilon\}$

Three-Dimensional Stress Analysis

Introduction

The constitutive matrix [*D*] is given by:

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

We now develop the tetrahedral stress element stiffness matrix by again using the steps outlined in Chapter 1.

The development is seen to be an extension of the plane element previously described in Chapter 6.

Three-Dimensional Stress Analysis

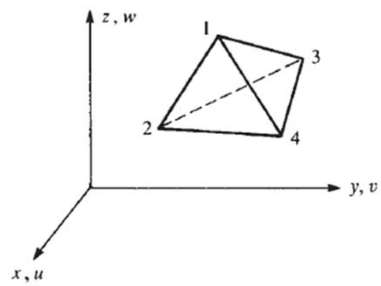
The Tetrahedral Element

Step 1 - Discretize and Select Element Types

Consider the tetrahedral element with corner nodes 1-4.

The nodes of the element must be numbered such that when viewed from the last node, the first three nodes are numbered in a counterclockwise manner.

This ordering of nodes avoids calculation of negative volumes.



Three-Dimensional Stress Analysis

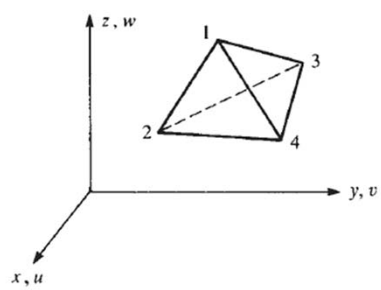
The Tetrahedral Element

Step 1 - Discretize and Select Element Types

Consider the tetrahedral element with corner nodes 1-4.

If the last node is say, node 4, the first three nodes are numbered in a counterclockwise manner, such as 1, 2, 3, 4 or 2, 3, 1, 4.

Using an isoparametric formulation to evaluate the [k] matrix for the tetrahedral element enables us to use the element node numbering in any order (we will discuss this later).



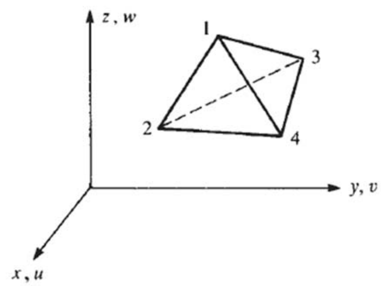
Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 1 - Discretize and Select Element Types

The unknown nodal displacements are given by:

$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$



Hence, there are 3 degrees of freedom per node, or 12 total degrees of freedom per element.

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

For a compatible displacement field, the element displacement functions *u*, *v*, and *w* must be linear along each edge. We select the linear displacement function as:

$$u(x,y,z) = a_1 + a_2 x + a_3 y + a_4 z$$
$$v(x,y,z) = a_5 + a_6 x + a_7 y + a_8 z$$
$$w(x,y,z) = a_9 + a_{10} x + a_{11} y + a_{12} z$$

In the same manner as in Chapter 6, we can express the *a*'s in terms of the known nodal coordinates (x_1, y_1, z_1, ..., z_4) and the unknown nodal displacements (u_1, v_1, w_1, ..., w_4) of the element.

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

Skipping the straightforward but tedious details, we obtain:

$$\begin{aligned} u(x,y,z) = \frac{1}{6V}\Big[&(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z)u_1 \\ &+(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z)u_2 \\ &+(\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z)u_3 \\ &+(\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z)u_4\Big] \end{aligned}$$

where $6V$ is obtained by evaluating the determinant:

$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_3 & z_4 \end{vmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

The coefficients α_i, β_i, γ_i, and δ_i (i = 1, 2, 3, 4) are given by;

$$\alpha_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

The coefficients α_i, β_i, γ_i, and δ_i (i = 1, 2, 3, 4) are given by;

$$\alpha_2 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_2 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_2 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_2 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

The coefficients α_i, β_i, γ_i, and δ_i (i = 1, 2, 3, 4) are given by;

$$\alpha_3 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} \qquad \beta_3 = -\begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_4 & z_4 \end{vmatrix}$$

$$\gamma_3 = \begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_4 & z_4 \end{vmatrix} \qquad \delta_3 = -\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_4 & y_4 \end{vmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

The coefficients α_i, β_i, γ_i, and δ_i (i = 1, 2, 3, 4) are given by;

$$\alpha_4 = -\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \qquad \beta_4 = \begin{vmatrix} 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \end{vmatrix}$$

$$\gamma_4 = -\begin{vmatrix} 1 & x_1 & z_1 \\ 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \end{vmatrix} \qquad \delta_4 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

Expressions for v and w are obtained by simply substituting v_i s for all u_i s and then w_i s for all u_i s.

The displacement expression for u can be written in expanded form in terms of the shape functions and unknown nodal displacements as:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 & 0 \\ 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 & 0 \\ 0 & 0 & N_1 & 0 & 0 & N_2 & 0 & 0 & N_3 & 0 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_4 \\ v_4 \\ w_4 \end{Bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 2 - Select Displacement Functions

The shape functions are given by:

$$N_1 = \frac{1}{6V}\left(\alpha_1 + \beta_1 x + \gamma_1 y + \delta_1 z\right)$$

$$N_2 = \frac{1}{6V}\left(\alpha_2 + \beta_2 x + \gamma_2 y + \delta_2 z\right)$$

$$N_3 = \frac{1}{6V}\left(\alpha_3 + \beta_3 x + \gamma_3 y + \delta_3 z\right)$$

$$N_4 = \frac{1}{6V}\left(\alpha_4 + \beta_4 x + \gamma_4 y + \delta_4 z\right)$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

The element strains for the three-dimensional stress state are given by:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_x \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{Bmatrix}$$

$$\{\varepsilon\} = [B]\{d\}$$

$$\{\varepsilon\} = \left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4] \right] \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{Bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

The submatrix $[B_1]$ is defined by:

$$[B_1] = \begin{bmatrix} N_{1,x} & 0 & 0 \\ 0 & N_{1,y} & 0 \\ 0 & 0 & N_{1,z} \\ N_{1,y} & N_{1,x} & 0 \\ 0 & N_{1,z} & N_{1,y} \\ N_{1,z} & 0 & N_{1,x} \end{bmatrix}$$

where the comma indicates differentiation with respect to that variable.

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 3 - Define the Strain-Displacement and Stress-Strain Relationships

Submatrices $[B_2]$, $[B_3]$, and $[B_4]$ are defined by simply indexing the subscript from 1 to 2, 3, and then 4, respectively.

Substituting in the shape functions, $[B_1]$ is expressed as:

$$[B_1] = \frac{1}{6V}\begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \gamma_1 & 0 \\ 0 & 0 & \delta_1 \\ \gamma_1 & \beta_1 & 0 \\ 0 & \delta_1 & \gamma_1 \\ \delta_1 & 0 & \beta_1 \end{bmatrix} \quad \cdots \rightarrow \quad [B_4] = \frac{1}{6V}\begin{bmatrix} \beta_4 & 0 & 0 \\ 0 & \gamma_4 & 0 \\ 0 & 0 & \delta_4 \\ \gamma_4 & \beta_4 & 0 \\ 0 & \delta_4 & \gamma_4 \\ \delta_4 & 0 & \beta_4 \end{bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 4 - Derive the Element Stiffness Matrix and Equations

The element stresses are related to the element strains by:

$$\{\sigma\} = [D]\{\varepsilon\} \qquad \{\sigma\} = [D][B]\{d\}$$

The stiffness matrix can be defined as:

$$[k] = \iiint_V [B]^T [D][B] dV$$

Because both matrices [*B*] and [*D*] are constant for the simple tetrahedral element, the element stiffness matrix can be simplified to:

$$[k] = [B]^T [D][B]V$$

where *V* is the volume of the tetrahedral element.

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Treatment of Body and Surface Forces

The element body force matrix is given by: $\{f_b\} = \int_V [N]^T \{X\} dV$

where

$$\{X\} = \begin{Bmatrix} X_b \\ Y_b \\ Z_b \end{Bmatrix}$$

For constant body forces, the nodal components of the total resultant body forces can be shown to be distributed to the nodes in four equal parts, that is:

$$\{f_b\} = \frac{V}{4}\left[X_b \ Y_b \ Z_b \,\vdots\, X_b \ Y_b \ Z_b \,\vdots\, X_b \ Y_b \ Z_b \,\vdots\, X_b \ Y_b \ Z_b\right]^T$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Treatment of Body and Surface Forces

The surface force matrix is given by: $\{f_s\} = \iint\limits_S [N_S]^T \{T\} dS$

where $[N_S]$ is the shape function matrix evaluated on the surface where the surface traction occurs.

For example, consider the case of a uniform pressure p acting on a face with nodes 1-3 of the element.

$$\{f_s\} = \iint\limits_S [N]^T_{\substack{\text{evaluated on}\\ \text{surface 1,2,3}}} \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix} dS$$

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 4 - Derive the Element Stiffness Matrix and Equations

Treatment of Body and Surface Forces

Simplifying and integrating we can show that:

$$\{f_s\} = \frac{S_{123}}{3} \begin{Bmatrix} p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ p_x \\ p_y \\ p_z \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Where S_{123} is the area of the surface associated with nodes 1-3.

The use of volume coordinates facilitates the integration surface forces.

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 5 - Assemble the Element Equations and Introduce Boundary Conditions

The final assembled or global equation written in matrix form is:

$$\{F\} = [K]\{d\}$$

where **{*F*}** is the equivalent global nodal loads obtained by lumping distributed edge loads and element body forces at the nodes and **[*K*]** is the global structure stiffness matrix.

Three-Dimensional Stress Analysis

The Tetrahedral Element

Step 6 - Solve for the Nodal Displacements

Once the element equations are assembled and modified to account for the boundary conditions, a set of simultaneous algebraic equations that can be written in expanded matrix form as:

$$\{F\} = [K]\{d\}$$

Step 7 - Solve for the Element Forces (Stresses)

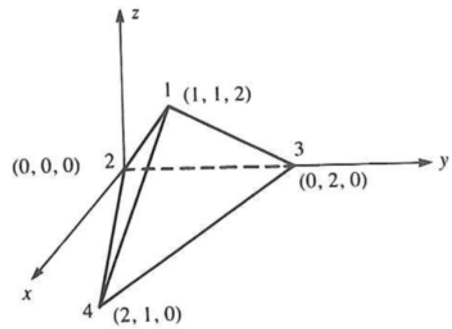
For the structural stress-analysis problem, important secondary quantities of strain and stress (or moment and shear force) can be obtained in terms of the displacements determined in Step 6.

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

Evaluate the matrices necessary to determine the stiffness matrix for the tetrahedral element shown below.

Let E = 30 x 10^6 *psi* and ν = 0.30. The coordinates are in units of inches.



Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

To evaluate the element stiffness matrix, we first determine the element volume V and all α's, β's, γ's, and δ's we have:

$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_3 & z_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{vmatrix} = 8\,in^3$$

Also, we obtain:

$$\alpha_1 = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & 0 \end{vmatrix} = 0 \qquad \beta_1 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

To evaluate the element stiffness matrix, we first determine the element volume V and all α's, β's, γ's, and δ's we have:

$$6V = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_3 & z_4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{vmatrix} = 8\,in^3$$

Also, we obtain:

$$\gamma_1 = -\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = 0 \qquad \delta_1 = -\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 4$$

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

The remaining values for the α's, β's, γ's, and δ's are:

$$\alpha_2 = 8 \quad \beta_2 = -2 \quad \gamma_2 = -4 \quad \delta_2 = -1$$

$$\alpha_3 = 0 \quad \beta_3 = -2 \quad \gamma_3 = 4 \quad \delta_3 = -1$$

$$\alpha_4 = 0 \quad \beta_4 = 4 \quad \gamma_4 = 0 \quad \delta_4 = -2$$

Note that α's typically have units of cubic inches or cubic meters, where β's, γ's, and δ's have units of square inches or square meters.

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

Next, the shape functions are:

$$N_1 = \frac{4z}{8} \qquad N_2 = \frac{8 - 2x - 4y - z}{8}$$

$$N_3 = \frac{-2x + 4y - z}{8} \qquad N_4 = \frac{4x - 2z}{8}$$

Note that $N_1 + N_2 + N_3 + N_4 = 1$ is again satisfied.

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

The 6 x 3 submatrices of the matrix [*B*] are now evaluated as:

$$[B_1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad [B_2] = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{8} \\ -\frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{8} & -\frac{1}{2} \\ -\frac{1}{8} & 0 & -\frac{1}{4} \end{bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

The 6 x 3 submatrices of the matrix [*B*] are now evaluated as:

$$[B_3] = \begin{bmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{8} \\ \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{8} & \frac{1}{2} \\ -\frac{1}{8} & 0 & -\frac{1}{4} \end{bmatrix} \qquad [B_4] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

Next, the [*D*] matrix is evaluated as:

$$[D] = \frac{30 \times 10^6}{(1+0.3)(1-0.6)} \begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0.3 & 0 & 0 & 0 \\ 0.3 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix}$$

Three-Dimensional Stress Analysis

The Tetrahedral Element – Example 1

Finally, substituting the results for *V*, [*B*], and [*D*], we obtain the element stiffness matrix.

$$[k] = [B]^T [D][B]V$$

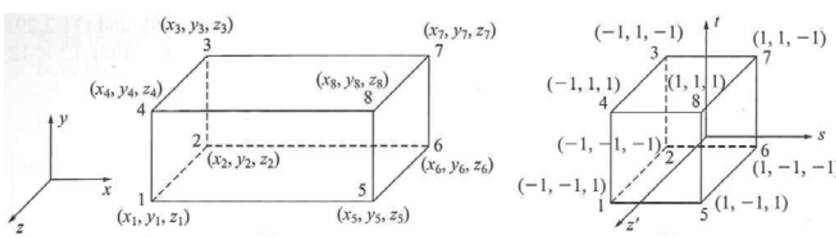
$$[k] = 10^6 \begin{bmatrix} 0.3846 & 0 & 0 & -0.0962 & 0 & -0.1923 & -0.0962 & 0 & -0.1923 & -0.1923 & 0 & 0.3846 \\ 0 & 0.3846 & 0 & 0 & -0.0962 & -0.3846 & 0 & -0.0962 & 0.3846 & 0 & -0.1923 & 0 \\ 0 & 0 & 1.3462 & -0.2885 & -0.5769 & -0.3365 & -0.2885 & 0.5769 & -0.3365 & 0.5769 & 0 & -0.6731 \\ -0.0962 & 0 & -0.2885 & 0.7452 & 0.4808 & 0.1202 & -0.0240 & -0.0962 & 0.1202 & -0.6250 & -0.3846 & 0.0481 \\ 0 & -0.0962 & -0.5769 & 0.4808 & 1.4663 & 0.2404 & 0.0962 & -1.2260 & 0.0481 & -0.5769 & -0.1442 & 0.2885 \\ -0.1923 & -0.3846 & -0.3365 & 0.1202 & 0.2404 & 0.5649 & 0.1202 & -0.0481 & -0.2043 & -0.0481 & 0.1923 & -0.0240 \\ -0.0962 & 0 & -0.2885 & -0.0240 & 0.0962 & 0.1202 & 0.7452 & -0.4808 & 0.1202 & -0.6250 & 0.3846 & 0.0481 \\ 0 & -0.0962 & 0.5769 & -0.0962 & -1.2260 & -0.0481 & -0.4808 & 1.4663 & -0.2404 & 0.5769 & -0.1442 & -0.2885 \\ -0.1923 & 0.3846 & -0.3365 & 0.1202 & 0.0481 & -0.2043 & 0.1202 & -0.2404 & 0.5649 & -0.0481 & -0.1923 & -0.0240 \\ -0.1923 & 0 & 0.5769 & -0.6250 & -0.5769 & -0.0481 & -0.6250 & 0.5769 & -0.0481 & 1.4423 & 0 & -0.4808 \\ 0 & -0.1923 & 0 & -0.3846 & -0.1442 & 0.1923 & 0.3846 & -0.1442 & -0.1923 & 0 & 0.4808 & 0 \\ 0.3846 & 0 & -0.6731 & 0.0481 & 0.2885 & -0.0240 & 0.0481 & -0.2885 & -0.0240 & -0.4808 & 0 & 0.7212 \end{bmatrix}$$

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The basic (linear) hexahedral element has eight corner nodes with isoparametric natural coordinates given by *s*, *t*, and *z'*.

The formulation of the stiffness matrix follows step analogous to the isoparametric formulation of the stiffness matrix for the plane element in Chapter 10.



Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The function used to describe the element geometry for *x* in terms of the generalized degrees of freedom a_i s is:

$$x = a_1 + a_2 s + a_3 t + a_4 z' + a_5 st + a_6 tz' + a_7 z's + a_8 stz'$$

$$y = a_9 + a_{10} s + a_{11} t + a_{12} z' + a_{13} st + a_{14} tz' + a_{15} z's + a_{16} stz'$$

$$z = a_{17} + a_{18} s + a_{19} t + a_{20} z' + a_{21} st + a_{22} tz' + a_{23} z's + a_{24} stz'$$

First, expand the coordinate definition from Chapter 10 to include the *z* coordinate as follows:

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^{8} \left(\begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \right)$$

Three-Dimensional Stress Analysis

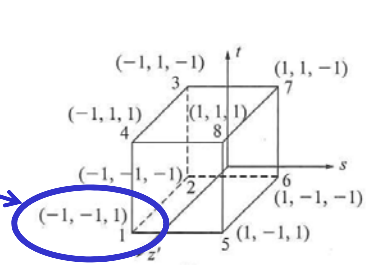
Isoparametric Formulation and Hexahedral Element

The shape functions are:

$$N_i = \frac{1}{8}(1+ss_i)(1+tt_i)(1+z'z'_i)$$

where s_i, t_i, and $z'_i = \pm 1$ and i = 1, 2, … , 8. For N_1:

$$N_1 = \frac{1}{8}(1+ss_i)(1+tt_i)(1+z'z'_i)$$



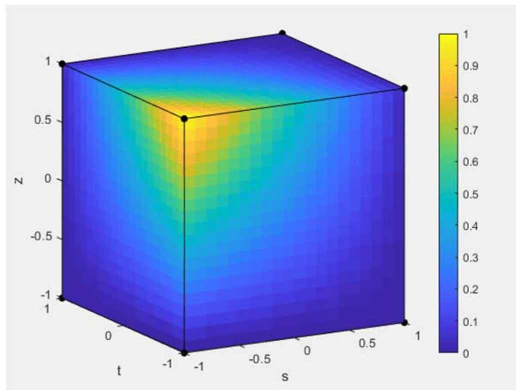
where s_1 = -1, t_1 = -1 and z'_1 = 1, we obtain:

$$N_1 = \frac{1}{8}(1-s)(1-t)(1+z')$$

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The shape functions are:



$$N_1 = \frac{1}{8}(1-s)(1-t)(1+z')$$

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

Explicit forms of the other shape functions follow similarly.

The shape functions map the natural coordinates (*s, t, z'*) of any point in the element to any point in the global coordinates (*x, y, z*).

For instance, when we let i = 8 and substitute s_8 = 1, t_8 = 1, and z_8= 1, we obtain:

$$N_8 = \frac{1}{8}(1+s)(1+t)(1+z')$$

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

Similar expressions are obtained for the other shape functions.

Then evaluating all shape functions at node 8, we obtain N_8 = 1, and all other shape functions equal zero at node 8.

We see that N_1 = 0 when s = 1 or when t = 1. Therefore, we obtain:

$$x = x_8 \qquad y = y_8 \qquad z = z_8$$

We see that indeed our geometric shape functions map any point in the natural-coordinate system to one in the global-coordinate system.

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The displacement functions in terms of the generalized degrees of freedom are of the same form as used to describe the element geometry given by:

$$u = a_1 + a_2 s + a_3 t + a_4 z' + a_5 st + a_6 tz' + a_7 z's + a_8 stz'$$

$$v = a_9 + a_{10} s + a_{11} t + a_{12} z' + a_{13} st + a_{14} tz' + a_{15} z's + a_{16} stz'$$

$$w = a_{17} + a_{18} s + a_{19} t + a_{20} z' + a_{21} st + a_{22} tz' + a_{23} z's + a_{24} stz'$$

There are now a total of 24 degrees of freedom in the linear hexahedral element.

Therefore, we use the same shape functions as used to describe the geometry.

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The displacement functions is given by:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_{i=1}^{8} \left(\begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \right)$$

with the same shape functions, the size of the shape function matrix now 3 X 24.

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The Jacobian matrix in three dimensions is:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{bmatrix}$$

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The strain-displacement relationships, in terms of global coordinates are:

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial f}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{vmatrix}}{\left|[J]\right|} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial z'} & \frac{\partial f}{\partial z'} & \frac{\partial z}{\partial z'} \end{vmatrix}}{\left|[J]\right|} \qquad \frac{\partial f}{\partial z} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial f}{\partial t} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial f}{\partial z'} \end{vmatrix}}{\left|[J]\right|}$$

Substituting *u, v,* and then *w* for *f* and using the definitions of the strains, we can express the strains in terms of natural coordinates (s, *t, z'*).

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

The matrix [*B*] is now a function of s, *t,* and *z'* and is of order 6 x 24.

The 24 x 24 stiffness matrix is now given by:

$$[k] = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B] \left|[J]\right| ds\, dt\, dz'$$

It is best to evaluate [*k*] by numerical integration; that is, we evaluate (integrate) the eight-node hexahedral element stiffness matrix using a 2 x 2 x 2 rule (or two-point rule). Actually, eight points defined in Table 11-1are used to evaluate [*k*] as

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

Gauss points and weights for the linear hexahedral are:

Points, i	s_i	t_i	z'_i	w_i
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	1
2	$1/\sqrt{3}$	$-1/\sqrt{3}$	$1/\sqrt{3}$	1
3	$1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1
4	$-1/\sqrt{3}$	$1/\sqrt{3}$	$1/\sqrt{3}$	1
5	$-1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1
6	$1/\sqrt{3}$	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1
7	$1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	1
8	$-1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$	1

Three-Dimensional Stress Analysis

Isoparametric Formulation and Hexahedral Element

Using the eight Gauss points to evaluate [k] give:

$$[k]=\sum_{i=1}^{8}\left[B\left(s_i,t_i,z'_i\right)\right]^T[D]\left[B\left(s_i,t_i,z'_i\right)\right]\left|\left[J\left(s_i,t_i,z'_i\right)\right]\right|W_i$$

As is true with the bilinear quadrilateral element, the eight-noded linear hexahedral element cannot model beam-bending action well because the element sides remain straight during the element deformation.

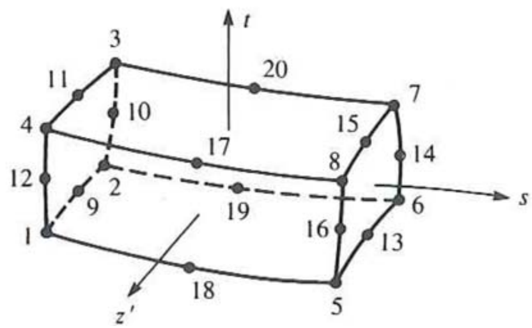
During the bending process, the elements will be stretched and can shear lock.

The quadratic hexahedral element described subsequently remedies the shear locking problem.

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The quadratic hexahedral element has a total of 20 nodes with the inclusion of a total of 12 mid-side nodes



Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The function describing the element geometry for *x* in terms of the 20 *a*'s is:

$$x = a_1 + a_2 s + a_3 t + a_4 z' + a_5 st + a_6 tz' + a_7 z's + a_8 s^2 + a_9 t^2$$
$$+ a_{10} z'^2 + a_{11} s^2 t + a_{12} st^2 + a_{13} t^2 z' + a_{14} tz'^2 + a_{15} z'^2 s$$
$$+ a_{16} z' s^2 + a_{17} stz' + a_{18} s^2 tz' + a_{19} st^2 z' + a_{20} stz'^2$$

Similar expressions describe the *y* and *z* coordinates.

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The function describing the element geometry for *x* in terms of the 20 *a*'s is:

$$x = a_1 + a_2 s + a_3 t + a_4 z' + a_5 st + a_6 tz' + a_7 z's + a_8 s^2 + a_9 t^2$$
$$+ a_{10} z'^2 + a_{11} s^2 t + a_{12} st^2 + a_{13} t^2 z' + a_{14} tz'^2 + a_{15} z'^2 s$$
$$+ a_{16} z' s^2 + a_{17} stz' + a_{18} s^2 tz' + a_{19} st^2 z' + a_{20} stz'^2$$

The *x*-displacement function *u* is described by the same polynomial used for the *x* element geometry.

Similar expressions are used for displacement functions *v* and *w.*

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The function describing the element geometry for *x* in terms of the 20 *a*'s is:

$$x = a_1 + a_2 s + a_3 t + a_4 z' + a_5 st + a_6 tz' + a_7 z's + a_8 s^2 + a_9 t^2$$
$$+ a_{10} z'^2 + a_{11} s^2 t + a_{12} st^2 + a_{13} t^2 z' + a_{14} tz'^2 + a_{15} z'^2 s$$
$$+ a_{16} z' s^2 + a_{17} stz' + a_{18} s^2 tz' + a_{19} st^2 z' + a_{20} stz'^2$$

In order to satisfy interelement compatibility, the three cubic terms s^3 , t^3, and z'^3 are not included.

Instead the three quartic terms s^2tz', st^2z', and stz'^2 are used.

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

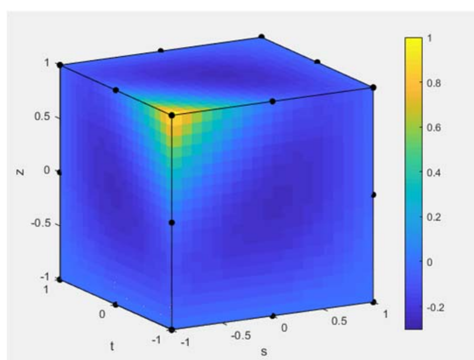
The development of the stiffness matrix follows the same steps we outlined before for the linear hexahedral element, where the shape functions now take on new forms.

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

Again, letting s_i, t_i, and $z'_i = \pm 1$, we have for the corner nodes (i = 1, 2, ..., 8):

$$N_i = \frac{(1+ss_i)(1+tt_i)(1+z'z'_i)}{8}(ss_i + tt_i + z'z'_i - 2)$$

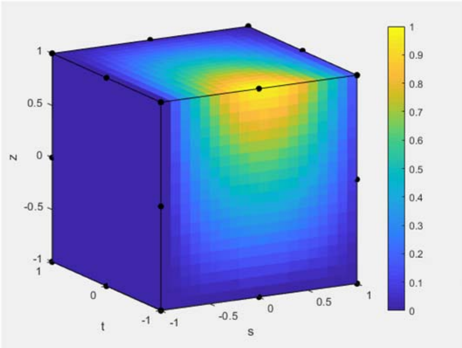


Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

For the mid-side nodes at s_i = 0, t_i = ±1, and z'_i = ±1, (i = 17, 18, 19, and 20), we get:

$$N_i = \frac{\left(1-s^2\right)\left(1+tt_i\right)\left(1+z'z'_i\right)}{4}$$

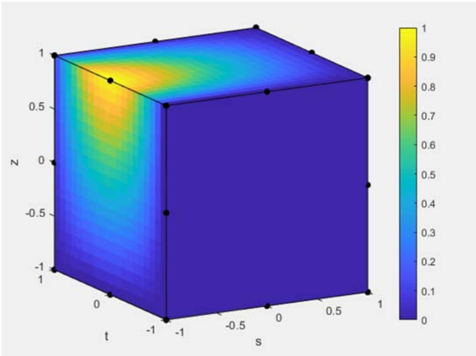


Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

For the mid-side nodes at s_i = ±1, t_i = 0, and z'_i = ±1, (i = 10, 12, 14, and 16), we get:

$$N_i = \frac{\left(1+ss_i\right)\left(1-t^2\right)\left(1+z'z'_i\right)}{4}$$

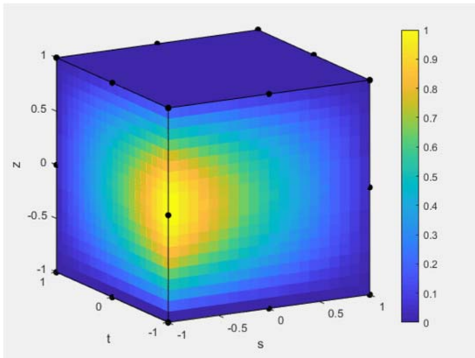


Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

For the mid-side nodes at $s_i = \pm 1$, $t_i = \pm 1$, and $z'_i = 0$, (i = 9, 11, 13, and 15), we get:

$$N_i = \frac{(1 - ss_i)(1 + tt_i)(1 + z'^2)}{4}$$



Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The [B] matrix is now a 60 x 60 matrix.

The stiffness matrix of the quadratic hexahedral element is of order 60 x 60.

This is consistent with the fact that the element has 20 nodes and 3 degrees of freedom (u_i, v_i, and w_i) per node

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The stiffness matrix for this 20-node quadratic solid element can be evaluated using a Gaussian quadrature with ther 3 x 3 x 3 rule (27 points).

However, a special 14-point rule may be a better choice.

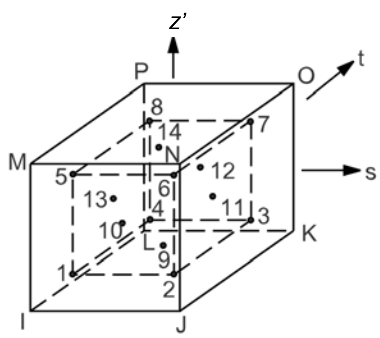
As with the eight-noded plane element of Chapter 10, the 20-node solid element is also called a serendipity element.

Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The 20-node solid uses a different type of integration point scheme.

This scheme places points close to each of the 8 corner nodes and close to the centers of the 6 faces for a total of 14 points.



Three-Dimensional Stress Analysis

Quadratic Hexahedral Element

The 20-node solid uses a different type of integration point scheme.

This scheme places points close to each of the 8 corner nodes and close to the centers of the 6 faces for a total of 14 points.

Type	Integration Point Location	Weighting Factor
8 Corner Points	s = ±0.75878 69106 39328	0.33518 00554 01662
	t = ±0.75878 69106 39329	
	z' = ±0.75878 69106 39329	
6 Center Points	s = ±0.79582 24257 54222, t=z'=0.0	0.88642 65927 97784
	t = ±0.79582 24257 54222, s=z'=0.0	
	z' = ±0.79582 24257 54222, s=t=0.0	

Isoparametric Elements

Problems

25. Work problems ***11.1***, ***11.3***, ***11.6a***, and ***11.9*** in your textbook.

26. Use the solid elements in SAP2000 to solve problem ***11.14*** in your textbook.

End of Chapter 11