Chapter: 13

MEK4560
The Finite Element Method in Solid Mechanics II

(April 25, 2008)

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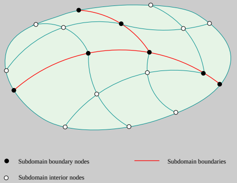

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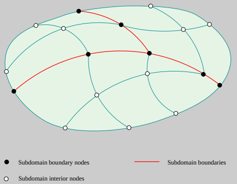

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13. Plasticity

The topic of the present chapter is *elastic-plastic* materials, and some comments on Finite element analysis taking elastic-plastic effects into account. The focus is on the nonlinear effect in the material law, the effects of large deformations and large strains are neglected.

As an introduction we introduce some topics related to *nonlinear materials* and *nonlinear analysis*.

In the textbook [Cook et al., 2002][2] a discussion of plasticity is in Chapter 17.3-17.6. The book [Lemaitre and Chaboche, 1990][4] is good source for material models. A good exposition of Finite Element methods for nonlinear structures are [Belytschko et al., 2000][1]. Nonlinear solid mechanics is also discussed in [Holzapfel, 2000][3], ([Simo and Hughes, 1998][6] and [Malvern, 1969][5]).

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[2] R. D. Cook, D. S. Malkus, M. E. Plesha, and R. J. Witt. *Concepts and Applications of Finite Element Analysis*. Number ISBN: 0-471-35605-0. John Wiley & Sons, Inc., 4th edition, October 2002.

[4] J. Lemaitre and J.-L. Chaboche. *Mechanics of solid materials*. Cambridge university press, 1990.

[1] Ted Belytschko, Wing Kam Liu, and Brian Moran. *Nonlinear Finite Elements for Continua and Structures*. John Wiley & Sons, Ltd., 2000.

[3] Herhard A. Holzapfel. *Nonlinear Solid Mechanics. A Continuum Approach for Engineering*. John Wiley & Sons, 1st edition, March 2000.

[6] J.C. Simo and T.J.R. Hughes. *Computational Inelasticity*. Springer-Verlag, New York, 1998.

[5] L. E. Malvern. *Introduction to the Mechanics of Continuous Medium*. Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

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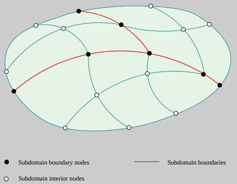

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13.1. Introduction

A material is nonlinear when the relation between stress, $\boldsymbol{\sigma}$, and strains, $\boldsymbol{\varepsilon}$, is expressed using a *strain dependent* matrix, $\boldsymbol{C}(\boldsymbol{\varepsilon})$.

In a mathematical model of a material the *constitutive* relations model the stress as a function of the deformation history. This is a *material specific* part of the model, and different materials have different constitutive relations.

I one dimensional solid mechanics the constitutive equation is the *stress-strain* model for the material.

A stress-strain relation approximate the observed physical behavior to a material, subject to certain assumptions. A *phenomenological* approach is used. The observed macroscopic behavior is a result of microscopic interactions in the material. These interactions on the atomic or molecular level is not modeled, the effect on the macroscopic level is modeled by fitting macroscopic functions to experimental data.

In the sequel we concentrate on one dimensional models and briefly mention the extension to three dimensions.

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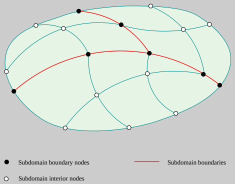

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The choice of the material model is crucial in an analysis, but the choice may not be obvious. It is up to the user to:

- choose a suitable material law, or
- develop/implement a suitable law (many commercial products allow the user to implement a user defied material law).

It is important to understand:

- the material model,
- the assumptions used in its derivation,
- is it suitable for the material in the construction,
- is it suitable for the loads and deformations,
- numerical aspects.

The Finite Element program use a set of *stresses* and *strains*. If the material data use a different par of stress and strain the material data must be converted, this is briefly discussed in subsection 13.2.

The numerical aspects of the material law are related to

- algorithms for stress updates,
- algorithms for the tangent stiffness matrix. (An exact Jacobi matrix is required to obtain second oder rate of convergence in Newton's method.)

13.2. Stress-strain curves

Stress-strain curves for one dimensional stress can be obtain from a *tensile* test. Constitutive relations are derived partially based on these curves.

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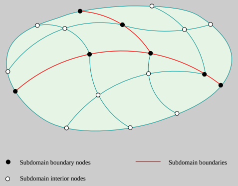

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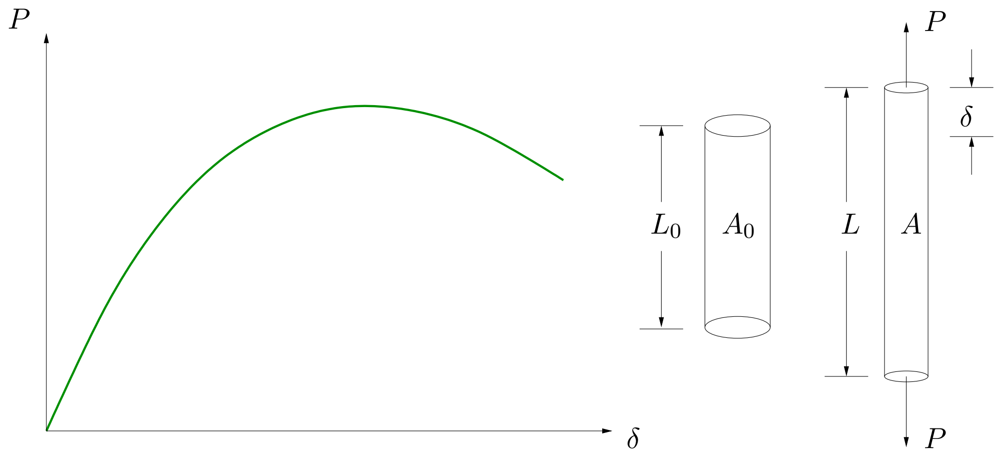

In the tensile test the force, P, and the elongation, δ is measured, and the force is plotted as a function of the elongation. In order to extract meaningful information from the plot, the effect of the *geometry*, i.e. (A, L), of the specimen must be removed. How? We have to make some choices. Use the initial length L_0 and area A_0, or the current L and A? I.e. *which stress and strain measure to use*?

If the change in area and length is small, the tensor for small strains used in linearized elasticity is used. Otherwise, models incorporating large strains must be used.

In any case, it is important to know the definitions of the stress and strains used in the model.

One alternative is to define the strech as

$$\lambda_x = \frac{L}{L_0} = \frac{L_0 + \delta}{L_0}$$

where $L = L_0 + \delta$. The nominal, or engineering, stress is given by:

$$\tau = \frac{P}{A_0}$$

where A_0 is the initial cross section area. The engineering strains are:

$$\varepsilon = \lambda_x - 1 = \frac{\delta}{L_0}$$

Using this we can define the relation between nominal stress and engineering strains.

Example: Assume that the load-displacement relation is modeled using:

$$P(\delta) = 1 - (\delta - 1)^2$$

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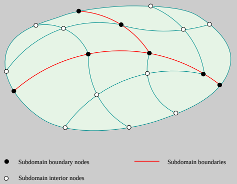

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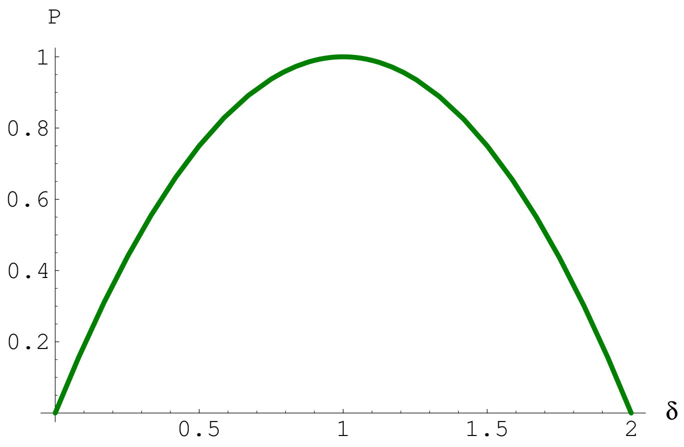


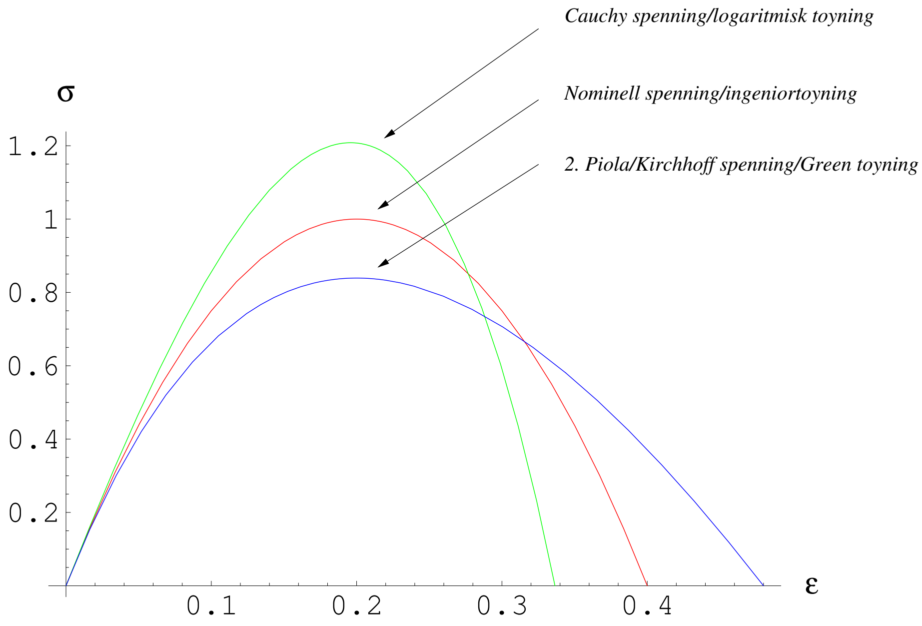
If the deformation is volume preserving, i.e.

$$AL = A_0 L_0,$$

the relations between stress and strain measures are summarized in the table below:

Strain	*Stress*
Engineering strain, $\varepsilon = \lambda_x - 1$	Nominal stress, $\sigma = \frac{P}{A_0}$
Logarithmic strain, $\varepsilon = \ln(\lambda_x)$	Cauchy (true) stress, $\sigma = \frac{\lambda_x P}{A_0}$
Green strains, $\varepsilon = \frac{1}{2}(\lambda_x^2 - 1)$	2. Piola-Kirchhoff stress, $\sigma = \frac{P}{\lambda_x A_0}$

The material law for different stress and strain measures are indicated below:



Note that the material coefficients are different for the different stress-strain relations.

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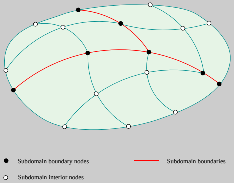

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13.3. One dimensional elasticity

A fundamental property of an elastic material is that the stress level only depend on the current value. A consequence of this is that loading and unloading follow the same curve and that the construction returns to its initial configuration after deformation.

Elastic materials has a one-to-one relation between stress and strains.

13.4. One dimensional plasticity

Materials that exhibit permanent deformation after a complete unloading is called *plastics materials*. Many materials show a linearly elastic behavior up to a level called the *yield limit*:

- *metals* (steel),
- *concrete*,
- *earth*

If the material is loaded above the yield limit plastic behavior result, *plastic strains*.

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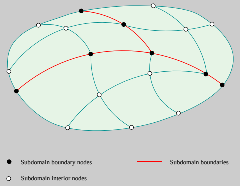

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Elastic-plastic materials can be divided into two sub groups:

- materials independent of the velocity, or strain rates, and
- materials dependent of the velocity.

A stress-strain curve for a typical elastic-plastic material is shown in the figure below.

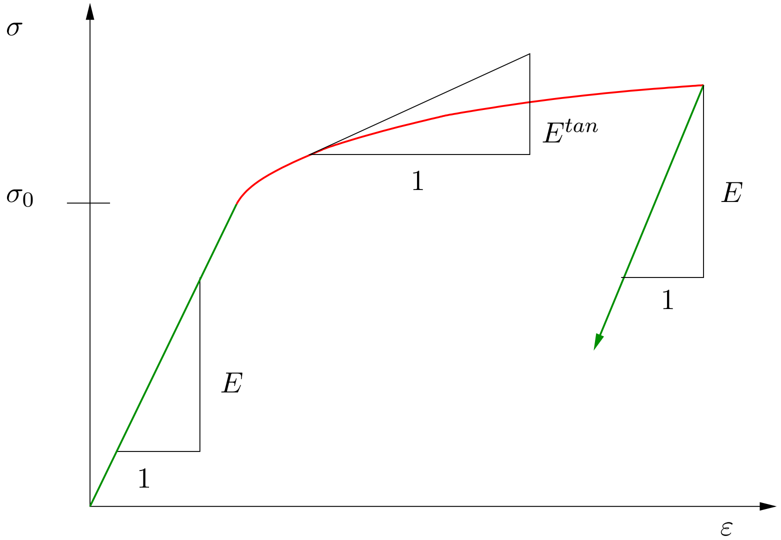
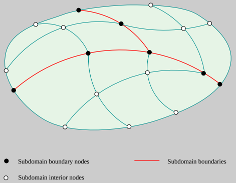


Figure 13.1: Stress-strain curve for a typical elastic-plastic material.



The main steps in developing a model of plasticity are:

1. A decomposition of each increment of strains into an elastic, reversible part, $d\varepsilon^e$, and an irreversible plastic part $d\varepsilon^p$:
$$d\varepsilon = d\varepsilon^e + d\varepsilon^p$$
2. A yield function, $f(\boldsymbol{\sigma}, \boldsymbol{q})$, modeling the plastic deformation. $\boldsymbol{q}$ is a set of internal variables.
3. A flow rule governing the plastic flow, i.e. determines the plastic strain increments, $d\varepsilon^p$.
4. Evolution equations for internal variables, including a stain-hardening model governing the evolution of the yield function.

Elastic-plastic materials are *path-Dependant* and *dissipative*. A major part of the work used to deform a plastic material is irreversible, i.e. transformed to other forms of energy, in particular heat. The stress depend on the deformation history and can not be written as a function of the strain. It is specified as a relation between rates of stresses and strains.

Figure 13.1 show a typical stress-strain curve for elastic-plastic materials, e.g. a metal under one dimensional stress. Initially the material is (linearly) elastic until the initial yield stress, denoted σ_0, is attained. Then, the elastic deformation is followed by an elastic-plastic deformation where permanent, irreversible plastic deformations are induced by further loading. Reducing the stress is called unloading, and here it is assumed that the response is governed by the elastic law.

Above we introduced the decomposition

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p$$

of the strain increments. Division of both sides with respect to a time increment dt, the rate relation becomes:

$$\dot{\varepsilon} = \dot{\varepsilon^e} + \dot{\varepsilon^p}$$

The stress increment (rate) is related by the elastic modulus to the increment (rate) of the elastic strain:

$$d\sigma = E d\varepsilon^e, \qquad \text{or} \qquad \dot{\sigma} = E\dot{\varepsilon}^e$$

In the elastic-plastic, nonlinear, regime the relations are

$$d\sigma = E d\varepsilon^e = E^{tan} d\varepsilon, \qquad \text{or} \qquad \dot{\sigma} = E\dot{\varepsilon}^e = E^{tan}\dot{\varepsilon}$$

where E^{tan} is the elastic-plastic tangent module, the slope of the stress-strain curve, see Figure 13.1.

The relations are *homogeneous* in strains and strain rates, i.e. if time is scaled by an arbitrary factor, the constitutive relations remains unchanged. Thus the material response are rate independent. In the sequel the rate form is used.

The plastic strain rate is given by a flow rule, often specified using a *flow potential* denoted Ψ:

$$\dot{\varepsilon^p} = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma}$$

where $\dot{\lambda}$ is the *plastic rate parameter*. An example of a potential is:

$$\Psi = |\sigma| = \bar{\sigma} = \sigma\,\mathrm{sign}(\sigma), \qquad \frac{\partial \Psi}{\partial \sigma} = \mathrm{sign}(\sigma)$$

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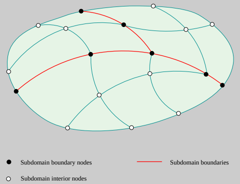

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$\bar{\sigma}$ is the *effective stress*.

Isotropic flow law: The yield condition is

$$f = \bar{\sigma} - \sigma_Y(\bar{\varepsilon}) \tag{13.1}$$

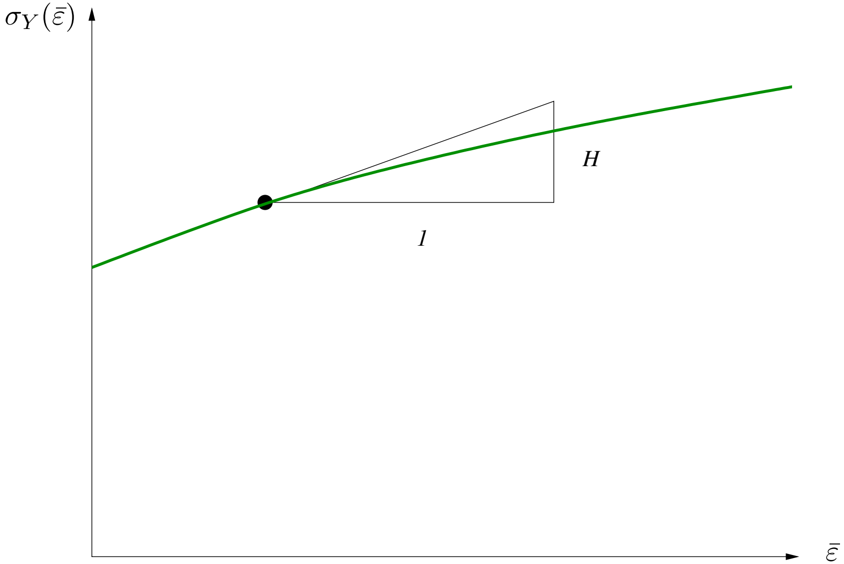
where σ_Y is the *yield strength* in one dimensional tension and $\bar{\varepsilon}$ is the *effective plastic strain*. Note that the yield strength depend on the effective plastic strain, this is called *hardening*.

The history of the plastic deformation is characterized by the effective plastic strain, given by

$$\bar{\varepsilon} = \int \dot{\bar{\varepsilon}}\, dt, \qquad \dot{\bar{\varepsilon}} = \sqrt{\dot{\varepsilon}^p \dot{\varepsilon}^p}$$

$\bar{\varepsilon}$ is an example of an internal variable used to characterize the inelastic deformation.

The yield behavior Equation 13.1 is called isotropic hardening: the yield strength in tension and compression is equal and given by σ_Y. A typical hardening curve is shown below:



The slope of the curve is the plastic module, $H = \frac{d\sigma_Y}{d\bar{\varepsilon}}$.

For this model we have the relation

$$\dot{\varepsilon}^p = \dot{\bar{\varepsilon}}\,\mathrm{sign}(\sigma) = \dot{\bar{\varepsilon}}\frac{\partial f}{\partial \sigma}$$

Since $\dot{\bar{\varepsilon}} = \dot{\lambda}$

$$\frac{\partial f}{\partial \sigma} = \frac{\partial \Psi}{\partial \sigma}$$

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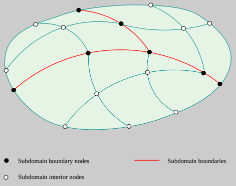

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This is called an *associative* plastic model, the plastic flow is in the direction normal to the yield surface. We do not go into details here, but this is important in multiaxial plasticity.

Plastic deformations occur only when the yield condition $f = 0$ is met. During plastic loading, the stress must remain on the yield surface $\dot{f} = 0$. Enforcement of this leads to the *consistency* condition:

$$\dot{f} = \dot{\bar{\sigma}} - \dot{\sigma}_Y(\bar{\varepsilon}) = 0$$

which gives

$$\dot{\bar{\sigma}} = \frac{d\sigma_Y(\bar{\varepsilon})}{d\bar{\varepsilon}}\dot{\bar{\varepsilon}} = H\dot{\bar{\varepsilon}}, \qquad \text{where} \qquad H = \frac{d\sigma_Y}{d\bar{\varepsilon}}$$

is the *plastic modulus*. The relations between stress and strain rates can be found:

$$\dot{\sigma} = E\dot{\varepsilon}^e = E^{tan}\dot{\varepsilon} = H\dot{\varepsilon}^p$$

$$\frac{1}{E^{tan}} = \frac{1}{E} + \frac{1}{H} \qquad \text{or} \qquad E^{tan} = E\left(1 - \frac{E}{E+H}\right)$$

where we have used that

$$\dot{\varepsilon}^e = \frac{\dot{\sigma}}{E}, \quad \dot{\varepsilon} = \frac{\dot{\sigma}}{E^{tan}}, \quad \dot{\varepsilon}^p = \frac{\dot{\sigma}}{H} \qquad \text{and} \qquad \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p.$$

The loading-unloading conditions can also be written

$$\dot{\lambda} \geq 0, \qquad f \leq 0, \qquad \dot{\lambda} f = 0$$

The first states that the plastic rate parameter is non-negative, the second that the stress must lie on or below the yield surface. The last condition states that the stress is on the yield surface during plastic loading, $\dot{\lambda} > 0$ and that the rate parameter is zero when the loading is elastic.

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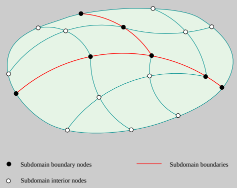

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Kinematic and isotropic flow model: In cyclic loading the isotropic flow law is a poor model for many materials. The figure below show a phenomenon observed in cyclic plasticity known as the *Bauschinger effect*.

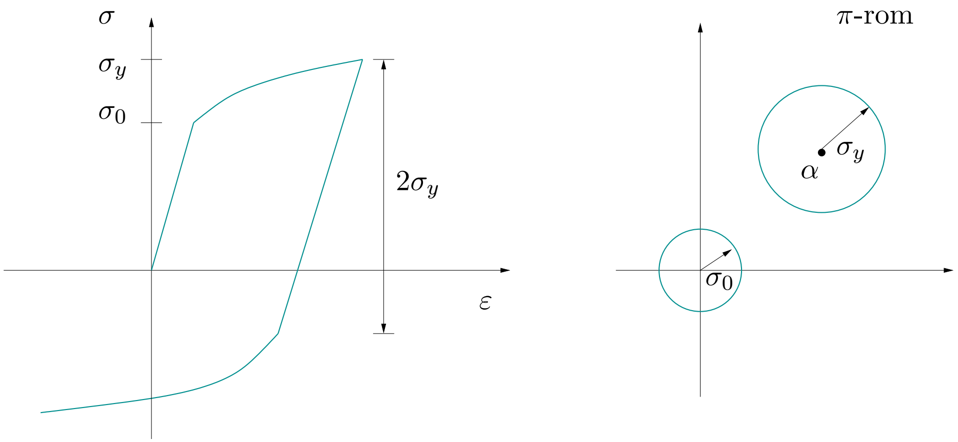

Figure 13.2: Combined isotropic-kinematic hardening. The Bauschinger effect is shown to the left, the translation and expansion of the flow surface is to the right (multiaxial stress).

Note that the yield strength in compression is reduced compared to tension and that the center of the yield surface is moved in the direction of the plastic flow. The figure Figure 13.2 show a multiaxial stress state; the expansion of the circular yield surface is related to isotropic hardening, while the translation is related to *kinematic* hardening.

In order to model kinematic hardening a new variable α is introduced, it is called the *backstress*, it is introduced both in the yield condition and the plastic flow relation. The plastic flow law for this model is

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial \Psi}{\partial \sigma}, \qquad \Psi = |\sigma - \alpha|$$

and the yield criteria is

$$f = |\sigma - \alpha| - \sigma_Y(\bar{\varepsilon})$$

Note that

$$\frac{\partial \Psi}{\partial \sigma} = \frac{\partial f}{\partial \sigma} = \mathrm{sign}(\sigma - \alpha) \qquad \text{and} \qquad \dot{\bar{\varepsilon}} = \dot{\lambda}$$

In addition an equations modeling the evolution of α is required. A simple model is

$$\dot{\alpha} = \kappa \dot{\varepsilon}^p$$

Differentiating the yield criteria:

$$\dot{f} = (\dot{\sigma} - \dot{\alpha})\,\mathrm{sign}(\sigma - \alpha) - H\dot{\bar{\varepsilon}} = 0 \qquad \text{thus} \qquad \dot{\bar{\varepsilon}} = \frac{1}{H}(\dot{\sigma} - \dot{\alpha})\,\mathrm{sign}(\sigma - \alpha)$$

Furthermore

$$\dot{\sigma} = E\varepsilon^e = E(\dot{\varepsilon} - \dot{\varepsilon}^p) = E(\dot{\varepsilon} - \dot{\bar{\varepsilon}}\,\mathrm{sign}(\sigma - \alpha))$$

Subtracting the *backstress* we obtain

$$\dot{\sigma} - \dot{\alpha} = \frac{E\dot{\varepsilon}}{1 + \frac{E}{H} + \frac{\kappa}{H}}$$

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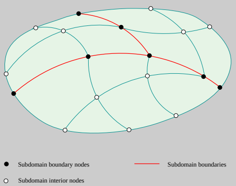

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giving the following formulas for the plastic strain

$$\dot{\bar{\varepsilon}} = \frac{E\dot{\varepsilon}\,\mathrm{sign}(\sigma-\alpha)}{H+E+\kappa}$$

and the tangent operator

$$\dot{\sigma} = E^{tan}\dot{\varepsilon} = E\left(1 - \frac{E}{H+E+\kappa}\right)\dot{\varepsilon}$$

Summary, one dimensional plasticity:

- Strain rate:
$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p$$
- Stress rate:
$$\dot{\sigma} = E\varepsilon^e = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$$
- Plastic flow rule:
$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial\Psi}{\partial\sigma}, \quad \dot{\bar{\varepsilon}} = \dot{\lambda}, \quad \sigma' = \sigma - \alpha, \quad \Psi = |\sigma'|$$
- Evolution equation for backstress:
$$\dot{\alpha} = \kappa\dot{\varepsilon}^p$$

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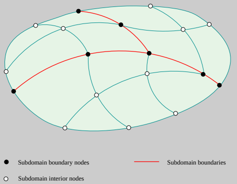

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Summary, one dimensional plasticity:

- Yield condition:

$$f = |\sigma - \alpha| - \sigma_y(\bar{\varepsilon}) = 0$$

- Loading-unloading conditions:

$$\dot{\lambda} \geq 0, \quad f \leq 0, \quad \dot{\lambda} f = 0$$

- Consistency condition:

$$\dot{f} = 0, \quad \Longrightarrow \quad \dot{\bar{\varepsilon}} = \dot{\lambda} = \frac{E\dot{\varepsilon}\,\mathrm{sign}\,\sigma'}{E + H + \kappa}$$

- Tangent modulus:

$$\dot{\sigma} = E^{tan}\dot{\varepsilon}, \qquad E^{tan} = E - \beta\frac{E^2}{E + (H + \kappa)}$$

($\beta = 1$ for plastic loading, $\beta = 0$ elastic loading or unloading.)

13.5. Multiaxial plasticity, von Mises yield surface

In multiaxial plasticity the one dimensional model is extended. One model is a formulation called *von Mises* yield surface, or J_2 plasticity (second invariant of the deviatoric stress tensor)

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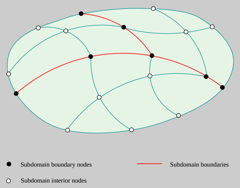

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The main assumption of the model is that the plastic flow of the material is unaffected by pressure. This was shown experimentally by Bridgman in 1949. The yield condition and and the plastic flow direction is based on the *deviatoriske* part of the stress tensor

$$\boldsymbol{\sigma}^{dev} = \boldsymbol{\sigma} - \frac{1}{3}\,\mathrm{trace}(\boldsymbol{\sigma})\boldsymbol{I}$$

The yield condition is taken to be

$$f(\boldsymbol{\Sigma}, \boldsymbol{q}) = \bar{\sigma} - \sigma_y(\bar{\varepsilon}) = 0, \quad \boldsymbol{\Sigma} = \boldsymbol{\sigma} - \boldsymbol{\alpha}, \quad \boldsymbol{\Sigma}^{dev} = \boldsymbol{\sigma}^{dev} - \boldsymbol{\alpha}, \quad \bar{\sigma} = \sqrt{\frac{3}{2}\boldsymbol{\Sigma}^{dev} : \boldsymbol{\Sigma}^{dev}}$$

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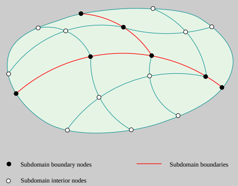

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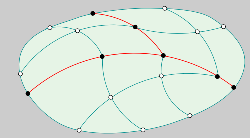
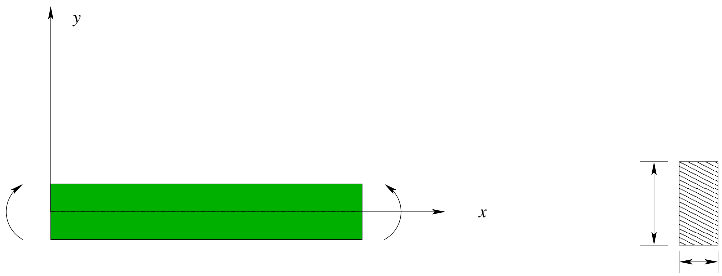
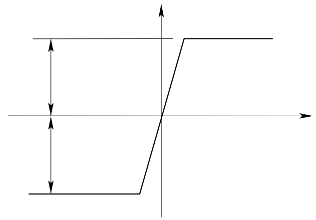


13.6. Plastisk ledd i en rektangulær bjelke

Figuren under viser en bjelke utsatt for rent moment. Materialkurven viser et materiale som er *elasto-plastisk*.





Material egenskaper	Geometri egenskaper	Last
$E = 30 \times 10^6$	$b = 1$	$M = 1.0M_{yp}$ to $1.5M_{yp}$
$\nu = 0.4$	$h = 2$	$M_{yp} = 2400$
$\sigma_{yp} = 36000$	$I_z = \frac{bh^3}{12} = 0.6667$	

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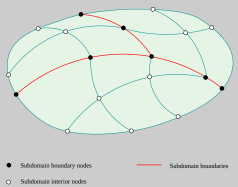

Problem: Vis at bjelken er elastisk opp til $M = M_{yp} = \sigma_{yp}\frac{h^2}{6}$ og blir helt plastisk når $M = M_{ult} = 1.5M_{yp}$.

Løsning:

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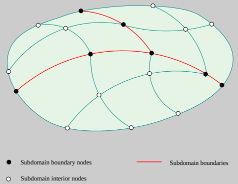
C*** STR. OF MATLS., TIMOSHENKO, PART 2, 3RD ED., PG. 349, ART. 64
C*** USING BILINEAR KINEMATIC HARDENING PLASTICITY BEHAVIOR TO DESCRIBE
C*** THE MATERIAL NONLINEARITY
ANTYPE,STATIC
ET,1,BEAM23
R,1,2,(2/3),2              ! AREA = 2, IZZ = 2/3, H = 2
MP,EX,1,30E6
MP,NUXY,1,0.3
TB,BKIN,1,1                ! BILINEAR KINEMATIC HARDENING
TBTEMP,70
TBDATA,1,36000,0           ! YIELD POINT AND ZERO TANGENT MODULUS
N,1                        ! DEFINE NODES
N,2,10
E,1,2                      ! DEFINE ELEMENT
D,1,ALL                    ! BOUNDARY CONDITIONS AND LOADS
SAVE                       ! SAVE DATABASE
FINISH
/SOLU
SOLCONTROL,0
NEQIT,5                    ! MAXIMUM 5 EQUILIBRIUM ITERATIONS PER STEP
NCNV,0                     ! DO NOT TERMINATE THE ANALYSIS IF THE SOLUTION FAILS
                           !  TO CONVERGE
OUTRES,EPPL,1              ! STORE PLASTIC STRAINS FOR EVERY SUBSTEP
CNVTOL,U                   ! CONVERGENCE CRITERION BASED UPON DISPLACEMENTS AND
CNVTOL,ROT                 !  ROTATIONS
```

```
*DO,I,1,4
F,2,MZ,(20000+(I*4000))   ! APPLY MOMENT LOAD
SOLVE
*ENDDO
FINISH
/POST26
NSOL,2,2,U,Y,UY2          ! NODE 2 DISPLACEMENT
ESOL,3,1,,LEPPL,1,EPPLAXL ! AXIAL PLASTIC STRAIN
PRVAR,2,3
FINISH
/CLEAR, NOSTART ! CLEAR PREVIOUS DATABASE BEFORE STARTING PART2
/PREP7
C*** USING BILINEAR ISOTROPIC HARDENING PLASTICITY BEHAVIOR TO DESCRIBE
C*** THE MATERIAL NONLINEARITY
RESUME
TBDELE,BKIN,1             ! DELETE NONLINEAR MATERIAL TABLE BKIN
TB,BISO,1,1               ! BILINEAR ISOTROPIC HARDENING
TBTEMP,70
TBDATA,1,36000,0          ! YIELD POINT AND ZERO TANGENT MODULUS
FINISH
/SOLU
SOLCONTROL,0
NEQIT,5                   ! MAXIMUM 5 EQUILIBRIUM ITERATIONS PER STEP
NCNV,0                    ! DO NOT TERMINATE THE ANALYSIS IF THE SOLUTION FAILS
                          !  TO CONVERGE
OUTRES,EPPL,1             ! STORE PLASTIC STRAINS FOR EVERY SUBSTEP
CNVTOL,U                  ! CONVERGENCE CRITERION BASED UPON DISPLACEMENTS AND
CNVTOL,ROT                !  ROTATIONS
*DO,I,1,4
F,2,MZ,(20000+(I*4000))   ! APPLY MOMENT LOAD
SOLVE
*ENDDO
FINISH
/POST26
NSOL,2,2,U,Y,UY2          ! NODE 2 DISPLACEMENT
ESOL,3,1,,LEPPL,1,EPPLAXL ! AXIAL PLASTIC STRAIN
PRVAR,2,3
```

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```
/OUT,VM24,VRT
/OUT
FINISH
*LIST,VM24,VRT
```

Svar/kommentarer: Vi ser at bjelken kollapser ved $M = 1.5\,M_{yp}$. Den vil også gjøre det for verdier som er litt lavere enn $M = 1.5\,M_{yp}$. Det er fordi at spenningene evalueres i diskret punkter (integrasjonspunkter) over bjelkens tverrsnitt.

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