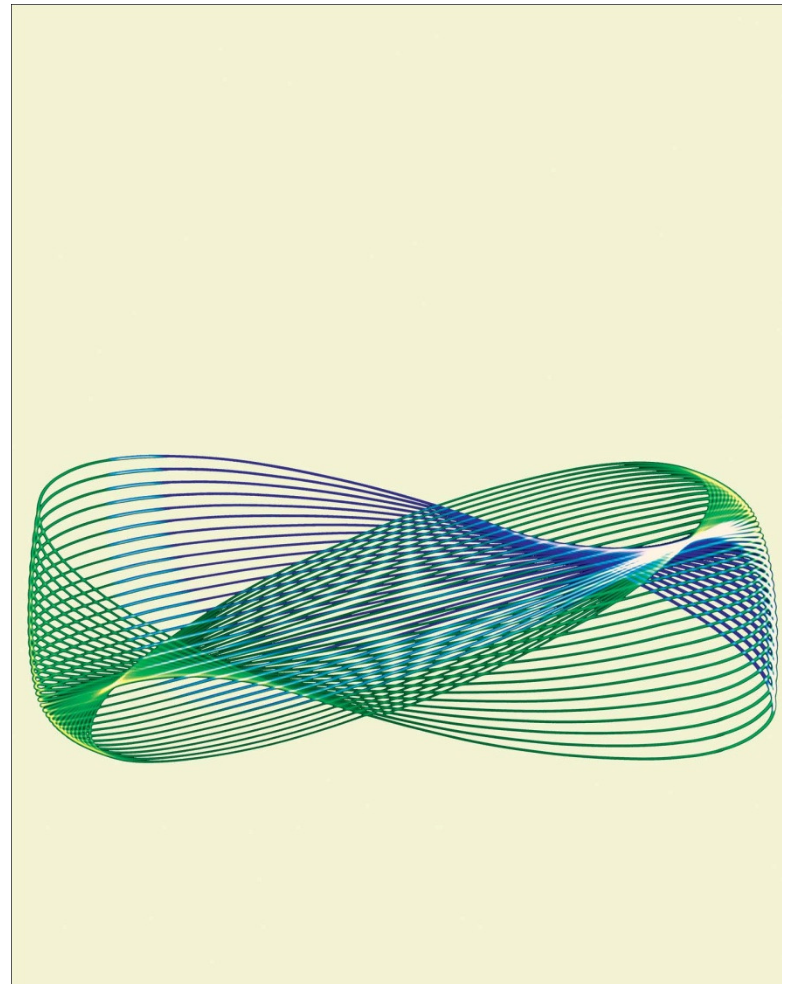
Chapter 14. Oscillations

This striking computer-generated image demonstrates an important type of motion: oscillatory motion. Examples of oscillatory motion include a car bouncing up and down, a ringing bell, and the current in an antenna



Chapter Goal: To understand systems that oscillate with simple harmonic motion.

Chapter 14. Oscillations

Topics:

- Simple Harmonic Motion
- Simple Harmonic Motion and Circular Motion
- Energy in Simple Harmonic Motion
- The Dynamics of Simple Harmonic Motion
- Vertical Oscillations
- The Pendulum
- Damped Oscillations
- Driven Oscillations and Resonance

Chapter 14. Reading Quizzes

What is the name of the quantity represented by the symbol ω

A. Angular momentum
B. Angular frequency
C. Phase constant
D. Uniform circular motion
E. Centripetal acceleration

What term is used to describe an oscillator that “runs down” and eventually stops?

A. Tired oscillator
B. Out of shape oscillator
C. Damped oscillator
D. Resonant oscillator
E. Driven oscillator

The starting conditions of an oscillator are characterized by

A. the initial acceleration.
B. the phase constant.
C. the phase angle.
D. the frequency.

Chapter 14. Basic Content and Examples

Simple Harmonic Motion

A system can oscillate in many ways, but we will be especially interested in the smooth sinusoidal oscillation called Simple Harmonic Motion (SHM). The characteristic equation for SHM is a cosine function.

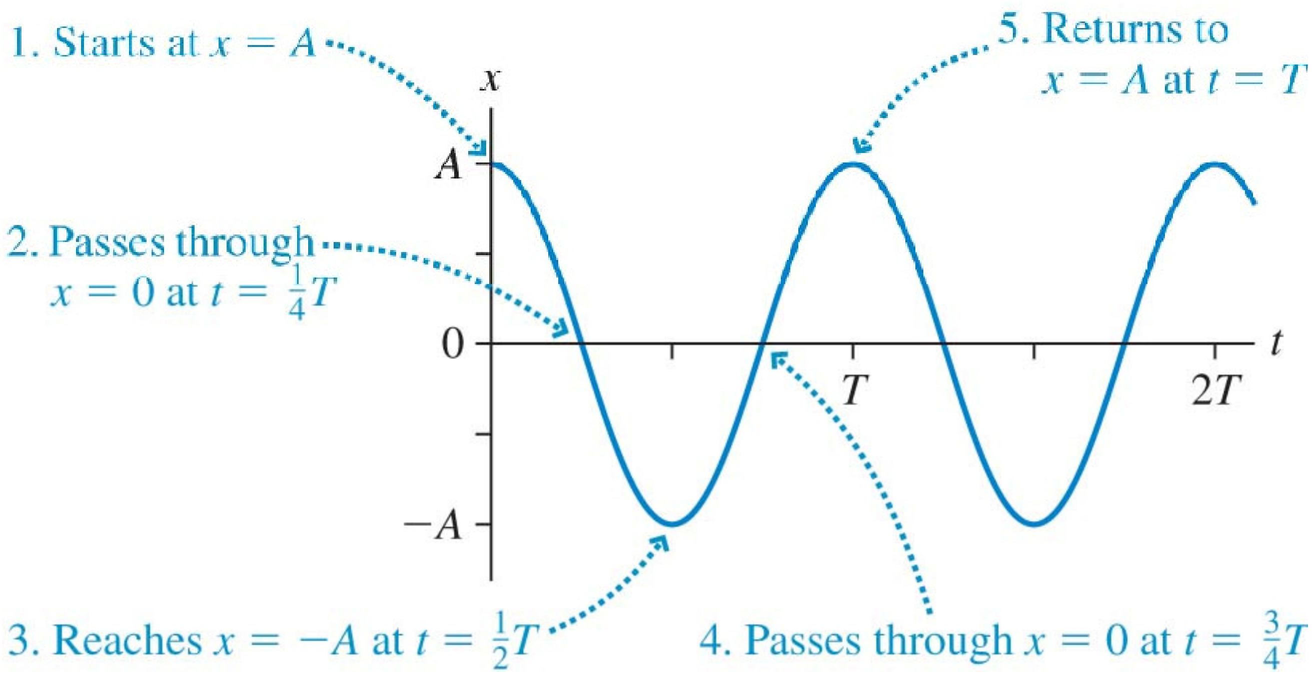
$$x(t) = A\cos\left(\frac{2\pi t}{T}\right)$$

The argument of the cosine function is in *radians*. The time to complete one full cycle, or one oscillation, is called the period, T. The frequency, f, is the number of cycles per second. Frequency and period are related by

$$f = \frac{1}{T} \qquad \text{or} \qquad T = \frac{1}{f}$$

Simple Harmonic Motion

FIGURE 14.4 The position-versus-time graph for simple harmonic motion.



Simple Harmonic Motion

The oscillation frequency f is measured in cycles per second, or Hertz.

We may also define an angular frequency ω in radians per second, to describe the oscillation.

$$\omega \text{ (in rad/s)} = \frac{2\pi}{T} = 2\pi f \text{ (in Hz)}$$

The position of an object oscillating with SHM can then be written as

$$x(t) = A\cos\omega t$$

The maximum speed of this object is

$$v_{\text{max}} = \frac{2\pi A}{T} = 2\pi f A = \omega A$$

EXAMPLE 14.2 A system in simple harmonic motion

QUESTION:

EXAMPLE 14.2 A system in simple harmonic motion

An air-track glider is attached to a spring, pulled 20.0 cm to the right, and released at $t = 0$ s. It makes 15 oscillations in 10.0 s.

a. What is the period of oscillation?
b. What is the object's maximum speed?
c. What are the position and velocity at $t = 0.800$ s?

EXAMPLE 14.2 A system in simple harmonic motion

MODEL An object oscillating on a spring is in SHM.

EXAMPLE 14.2 A system in simple harmonic motion

SOLVE a. The oscillation frequency is

$$f = \frac{15\ \text{oscillations}}{10.0\ \text{s}} = 1.50\ \text{oscillations/s} = 1.50\ \text{Hz}$$

Thus the period is $T = 1/f = 0.667$ s.

b. The oscillation amplitude is $A = 0.200$ m. Thus

$$v_{\text{max}} = \frac{2\pi A}{T} = \frac{2\pi(0.200\ \text{m})}{0.667\ \text{s}} = 1.88\ \text{m/s}$$

EXAMPLE 14.2 A system in simple harmonic motion

c. The object starts at $x = +A$ at $t = 0$ s. This is exactly the oscillation described by Equations 14.2 and 14.6. The position at $t = 0.800$ s is

$$x = A\cos\left(\frac{2\pi t}{T}\right) = (0.200\ \text{m})\cos\left(\frac{2\pi(0.800\ \text{s})}{0.667\ \text{s}}\right)$$

$$= (0.200\ \text{m})\cos(7.54\ \text{rad}) = 0.0625\ \text{m} = 6.25\ \text{cm}$$

EXAMPLE 14.2 A system in simple harmonic motion

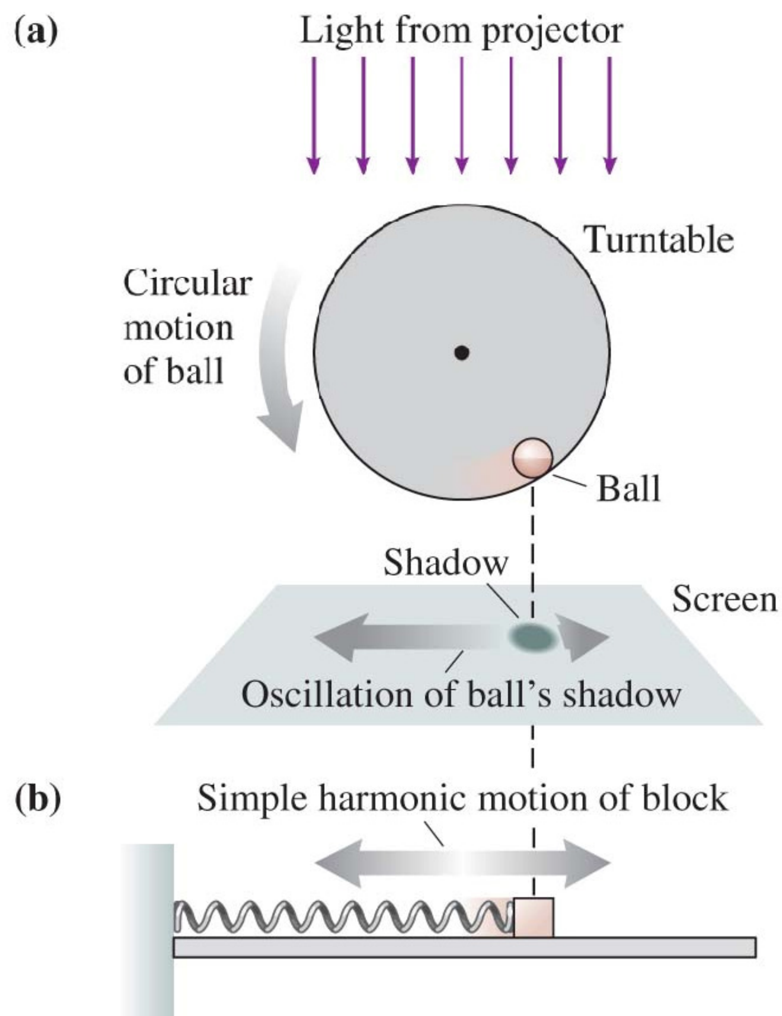
The velocity at this instant of time is

$$v_x = -v_{\max}\sin\left(\frac{2\pi t}{T}\right) = -(1.88\ \mathrm{m/s})\sin\left(\frac{2\pi(0.800\ \mathrm{s})}{0.667\ \mathrm{s}}\right)$$

$$= -(1.88\ \mathrm{m/s})\sin(7.54\ \mathrm{rad}) = -1.79\ \mathrm{m/s} = -179\ \mathrm{cm/s}$$

At $t = 0.800$ s, which is slightly more than one period, the object is 6.25 cm to the right of equilibrium and moving to the *left* at 179 cm/s. Notice the use of radians in the calculations.

FIGURE 14.6 A projection of the circular motion of a rotating ball matches the simple harmonic motion of an object on a spring.



Simple Harmonic Motion

If the initial position of an object in SHM is not A, then we may still use the cosine function, with a phase constant measured in radians.

In this case, the two primary kinematic equations of SHM are:

$$x(t) = A\cos(\omega t + \phi_0)$$

$$v_x(t) = -\omega A\sin(\omega t + \phi_0) = -v_{\max}\sin(\omega t + \phi_0)$$

EXAMPLE 14.4 Using the initial conditions

QUESTION:

EXAMPLE 14.4 Using the initial conditions

An object on a spring oscillates with a period of 0.80 s and an amplitude of 10 cm. At $t = 0$ s, it is 5.0 cm to the left of equilibrium and moving to the left. What are its position and direction of motion at $t = 2.0$ s?

EXAMPLE 14.4 Using the initial conditions

MODEL An object oscillating on a spring is in simple harmonic motion.

EXAMPLE 14.4 Using the initial conditions

SOLVE We can find the phase constant ϕ_0 from the initial condition $x_0 = -5.0\text{ cm} = A\cos\phi_0$. This condition gives

$$\phi_0 = \cos^{-1}\left(\frac{x_0}{A}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \pm\frac{2}{3}\pi\text{ rad} = \pm 120°$$

Because the oscillator is moving to the *left* at $t = 0$, it is in the upper half of the circular-motion diagram and must have a phase constant between 0 and π rad. Thus ϕ_0 is $\frac{2}{3}\pi$ rad. The angular frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.80\text{ s}} = 7.85\text{ rad/s}$$

EXAMPLE 14.4 Using the initial conditions

Thus the object's position at time $t = 2.0$ s is

$$\begin{aligned} x(t) &= A\cos(\omega t + \phi_0) \\ &= (10\text{ cm})\cos\left((7.85\text{ rad/s})(2.0\text{ s}) + \frac{2}{3}\pi\right) \\ &= (10\text{ cm})\cos(17.8\text{ rad}) = 5.0\text{ cm} \end{aligned}$$

The object is now 5.0 cm to the right of equilibrium. But which way is it moving? There are two ways to find out. The direct way is to calculate the velocity at $t = 2.0$ s:

$$v_x = -\omega A\sin(\omega t + \phi_0) = +68\text{ cm/s}$$

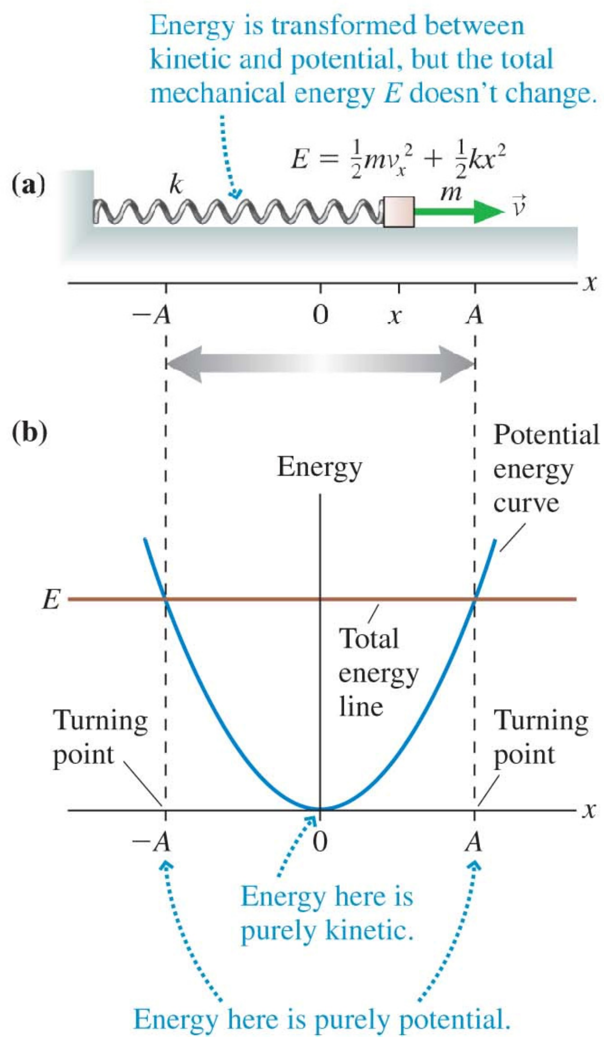
EXAMPLE 14.4 Using the initial conditions

The velocity is positive, so the motion is to the right. Alternatively, we could note that the phase at $t = 2.0$ s is $\phi = 17.8$ rad. Dividing by π, you can see that

$$\phi = 17.8 \text{ rad} = 5.67\pi \text{ rad} = (4\pi + 1.67\pi) \text{ rad}$$

The 4π rad represents two complete revolutions. The "extra" phase of 1.67π rad falls between π and 2π rad, so the particle in the circular-motion diagram is in the lower half of the circle and moving to the right.

FIGURE 14.10 The energy is transformed between kinetic energy and potential energy as the object oscillates, but the mechanical energy $E = K + U$ doesn't change.

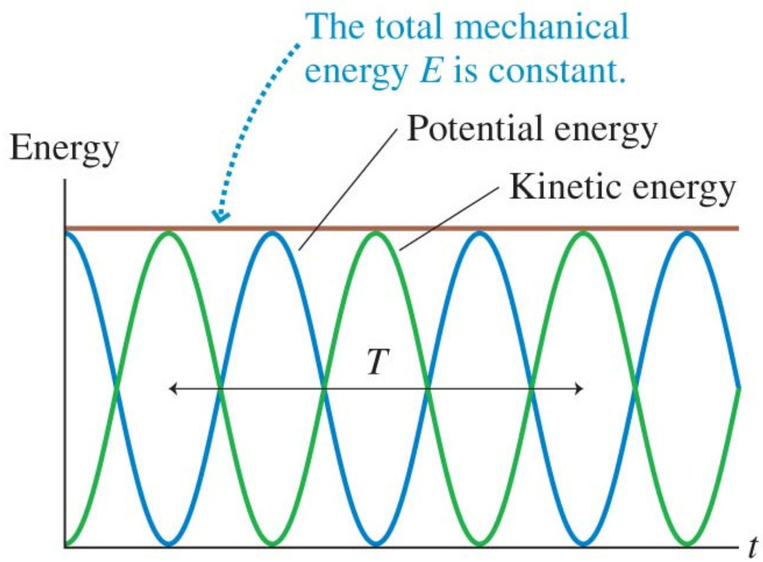


Energy in Simple Harmonic Motion

Energy is conserved in SHM.

$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}m(v_{\text{max}})^2 \quad \text{(conservation of energy)}$$

FIGURE 14.11 Kinetic energy, potential energy, and the total mechanical energy for simple harmonic motion.



EXAMPLE 14.5 Using conservation of energy

QUESTION:

EXAMPLE 14.5 Using conservation of energy

A 500 g block on a spring is pulled a distance of 20 cm and released. The subsequent oscillations are measured to have a period of 0.80 s. At what position or positions is the block's speed 1.0 m/s?

EXAMPLE 14.5 Using conservation of energy

MODEL The motion is SHM. Energy is conserved.

EXAMPLE 14.5 Using conservation of energy

SOLVE The block starts from the point of maximum displacement, where $E = U = \frac{1}{2}kA^2$. At a later time, when the position is x and the velocity is v_x, energy conservation requires

$$\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

Solving for x, we find

$$x = \sqrt{A^2 - \frac{mv_x^2}{k}} = \sqrt{A^2 - \left(\frac{v}{\omega}\right)^2}$$

where we used $k/m = \omega^2$ from Equation 14.24. The angular frequency is easily found from the period: $\omega = 2\pi/T = 7.85$ rad/s. Thus

$$x = \sqrt{(0.20\text{ m})^2 - \left(\frac{1.0\text{ m/s}}{7.85\text{ rad/s}}\right)^2} = \pm 0.15\text{ m} = \pm 15\text{ cm}$$

There are two positions because the block has this speed on either side of equilibrium.

Dynamics of Simple Harmonic Motion

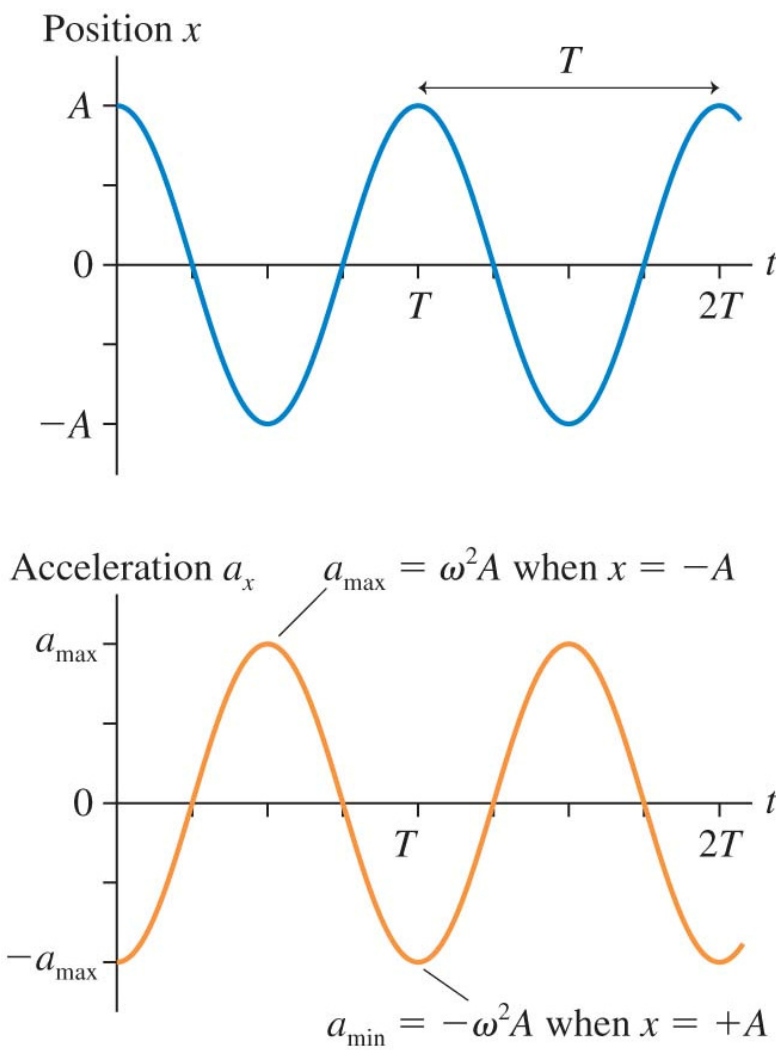
The acceleration of an object in SHM is maximum when the displacement is most negative, minimum when the displacement is at a maximum, and zero when $x = 0$. The derivative of the velocity is.

$$a_x = \frac{dv_x}{dt} = \frac{d}{dt}(-\omega A \sin \omega t) = -\omega^2 A \cos \omega t$$

Because $x = A \cos \omega t$, this can be written as

$$a_x = -\omega^2 x$$

FIGURE 14.13 Position and acceleration graphs for an oscillating spring. We've chosen $\phi_0 = 0$.



Dynamics of Simple Harmonic Motion

When we combine **Hooke's Law** for a mass on a spring with **Newton's second law**, we obtain the *equation of motion* for a mass on a spring.

$$a_x = -\frac{k}{m}x$$

The solution of this equation of motion is

$$x(t) = A\cos(\omega t + \phi_0)$$

where the angular frequency

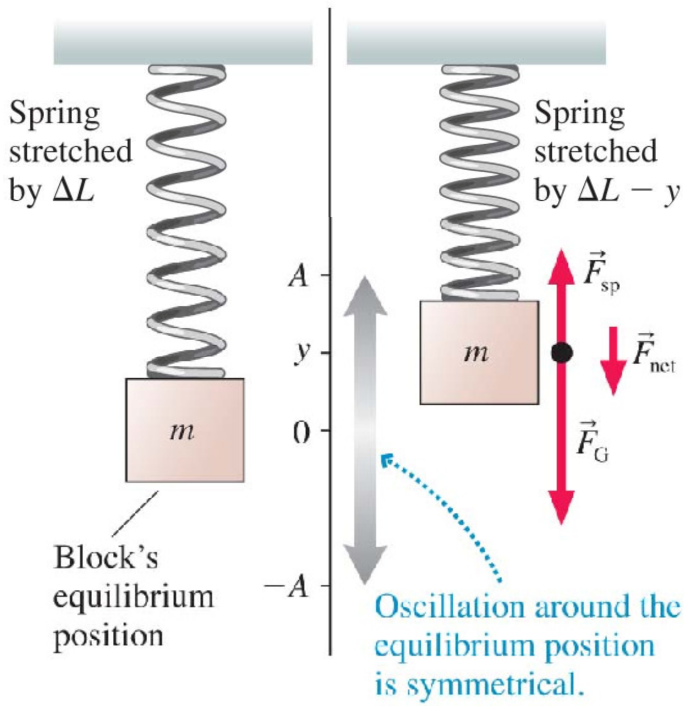
$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

is determined by the mass and the spring constant.

Vertical Oscillations

Motion for a mass hanging from a spring is the same as for horizontal SHM, but the equilibrium position is affected.

FIGURE 14.17 The block oscillates around the equilibrium position.



$$\Delta L = \frac{mg}{k}$$

EXAMPLE 14.7 Bungee oscillations

QUESTION:

EXAMPLE 14.7 Bungee oscillations

An 83 kg student hangs from a bungee cord with spring constant 270 N/m. The student is pulled down to a point where the cord is 5.0 m longer than its unstretched length, then released. Where is the student, and what is his velocity 2.0 s later?

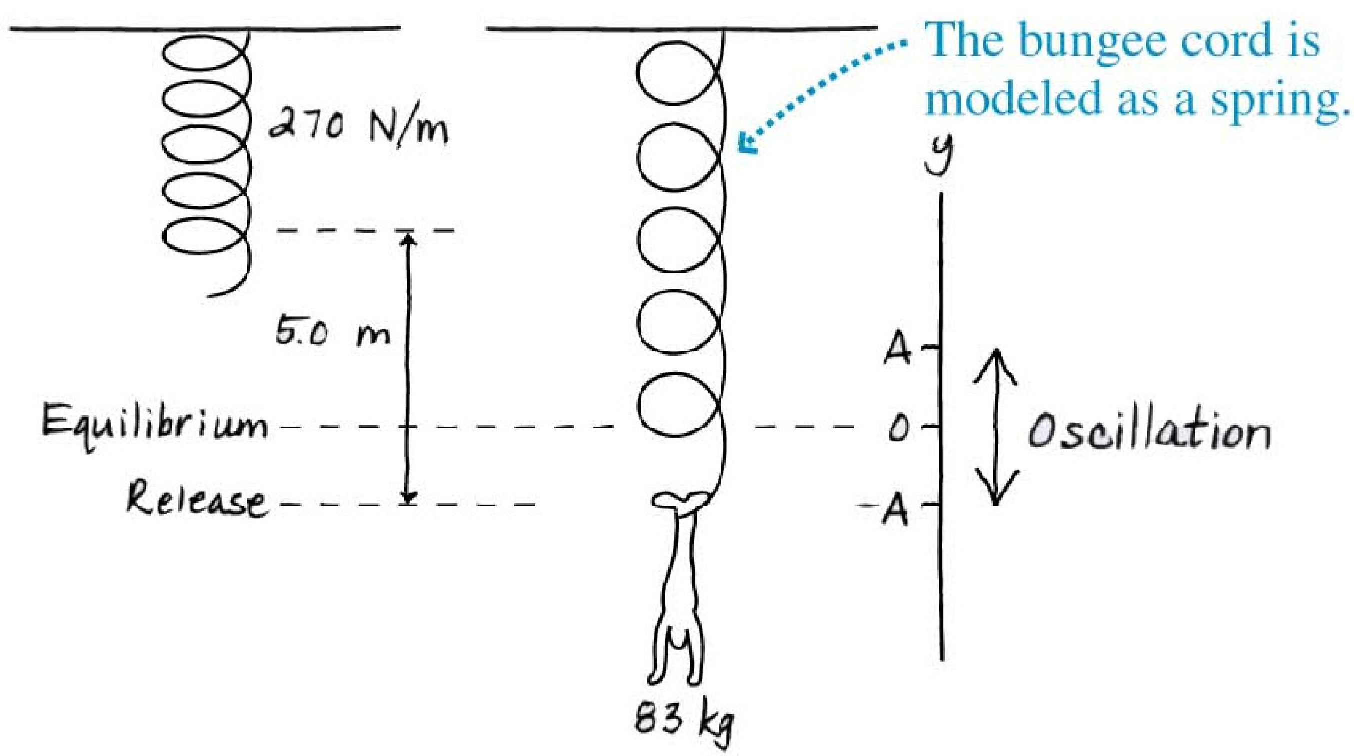
EXAMPLE 14.7 Bungee oscillations

MODEL A bungee cord can be modeled as a spring. Vertical oscillations on the bungee cord are SHM.

EXAMPLE 14.7 Bungee oscillations

VISUALIZE **FIGURE 14.18** shows the situation.

FIGURE 14.18 A student on a bungee cord oscillates about the equilibrium position.



EXAMPLE 14.7 Bungee oscillations

SOLVE Although the cord is stretched by 5.0 m when the student is released, this is *not* the amplitude of the oscillation. Oscillations occur around the equilibrium position, so we have to begin by finding the equilibrium point where the student hangs motionless. The cord stretch at equilibrium is given by Equation 14.41:

$$\Delta L = \frac{mg}{k} = 3.0 \text{ m}$$

Stretching the cord 5.0 m pulls the student 2.0 m below the equilibrium point, so $A = 2.0$ m. That is, the student oscillates with amplitude $A = 2.0$ m about a point 3.0 m beneath the bungee cord's original end point.

EXAMPLE 14.7 Bungee oscillations

The student's position as a function of time, as measured from the equilibrium position, is

$$y(t) = (2.0\text{ m})\cos(\omega t + \phi_0)$$

where $\omega = \sqrt{k/m} = 1.80$ rad/s The initial condition

$$y_0 = A\cos\phi_0 = -A$$

requires the phase constant to be $\phi_0 = \pi$ rad. At $t = 2.0$ s the student's position and velocity are

$$y = (2.0\text{ m})\cos\big((1.80\text{ rad/s})(2.0\text{ s}) + \pi\text{ rad}\big) = 1.8\text{ m}$$

$$v_y = -\omega A\sin(\omega t + \phi_0) = -1.6\text{ m/s}$$

EXAMPLE 14.7 Bungee oscillations

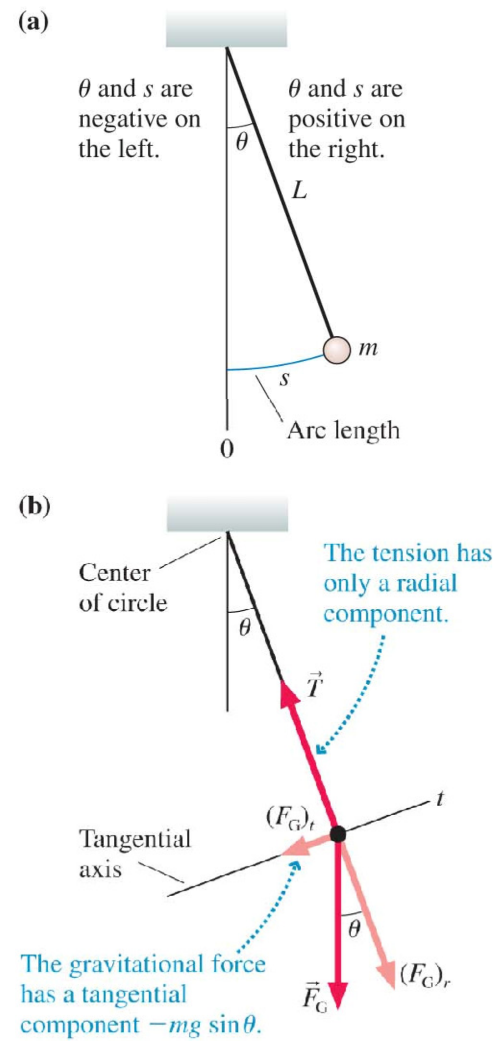
The student is 1.8 m *above* the equilibrium position, or 1.2 m *below* the original end of the cord. Because his velocity is negative, he's passed through the highest point and is heading back down.

The Pendulum

Consider a mass m attached to a string of length L. If it is displaced from its lowest position by an angle θ, Newton's second law for the tangential component, parallel to the motion, is

$$(F_{\text{net}})_t = \sum F_t = (F_G)_t = -mg\sin\theta = ma_t$$

FIGURE 14.19 The motion of a pendulum.



The Pendulum

Suppose we restrict the pendulum's oscillations to small angles (< 10°). Then we may use the **small angle approximation** sin $\theta \approx \theta$, where θ is measured in radians. Since $\theta = s/L$, the net force on the mass is

$$(F_{\text{net}})_t = -\frac{mg}{L}s$$

and the angular frequency of the motion is found to be

$$\omega = 2\pi f = \sqrt{\frac{g}{L}}$$

EXAMPLE 14.9 The maximum angle of a pendulum

QUESTION:

EXAMPLE 14.9 The maximum angle of a pendulum

A 300 g mass on a 30-cm-long string oscillates as a pendulum. It has a speed of 0.25 m/s as it passes through the lowest point. What maximum angle does the pendulum reach?

EXAMPLE 14.9 The maximum angle of a pendulum

MODEL Assume that the angle remains small, in which case the motion is simple harmonic motion.

EXAMPLE 14.9 The maximum angle of a pendulum

SOLVE The angular frequency of the pendulum is

$$\omega = \sqrt{\frac{g}{L}} = \sqrt{\frac{9.8\ \text{m/s}^2}{0.30\ \text{m}}} = 5.72\ \text{rad/s}$$

The speed at the lowest point is $v_{max} = \omega A$, so the amplitude is

$$A = s_{max} = \frac{v_{max}}{\omega} = \frac{0.25\ \text{m/s}}{5.72\ \text{rad/s}} = 0.0437\ \text{m}$$

The maximum angle, at the maximum arc length s_{max}, is

$$\theta_{max} = \frac{s_{max}}{L} = \frac{0.04347\ \text{m}}{0.30\ \text{m}} = 0.145\ \text{rad} = 8.3°$$

EXAMPLE 14.9 The maximum angle of a pendulum

ASSESS Because the maximum angle is less than 10°, our analysis based on the small-angle approximation is valid.

Tactics: Identifying and analyzing simple harmonic motion

TACTICS BOX 14.1 **Identifying and analyzing simple harmonic motion**

1. If the net force acting on a particle is a linear restoring force, the motion will be simple harmonic motion around the equilibrium position.
2. The position as a function of time is $x(t) = A\cos(\omega t + \phi_0)$. The velocity as a function of time is $v_x(t) = -\omega A\sin(\omega t + \phi_0)$. The maximum speed is $v_{max} = \omega A$. The equations are given here in terms of x, but they can be written in terms of y, θ, or some other parameter if the situation calls for it.

Tactics: Identifying and analyzing simple harmonic motion

3. The amplitude A and the phase constant ϕ_0 are determined by the initial conditions through $x_0 = A\cos\phi_0$ and $v_{0x} = -\omega A\sin\phi_0$.
4. The angular frequency ω (and hence the period $T = 2\pi/\omega$) depends on the physics of the particular situation. But ω does *not* depend on A or ϕ_0.
5. Mechanical energy is conserved. Thus $\frac{1}{2}mv_x^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \frac{1}{2}m(v_{max})^2$. Energy conservation provides a relationship between position and velocity that is independent of time.

Exercises 7–12, 15–19

Damped Oscillations

• An oscillation that runs down and stops is called a damped oscillation.
• One possible reason for dissipation of energy is the drag force due to air resistance.
• This is difficult to calculate exactly but a good model for a slowly moving object is

$$\vec{D} = -b\vec{v} \qquad \text{(model of the drag force)}$$

Damped Oscillations

When a mass on a spring experiences the force of the spring as given by Hooke's Law, as well as a drag force of magnitude $|D|=bv$, the solution is

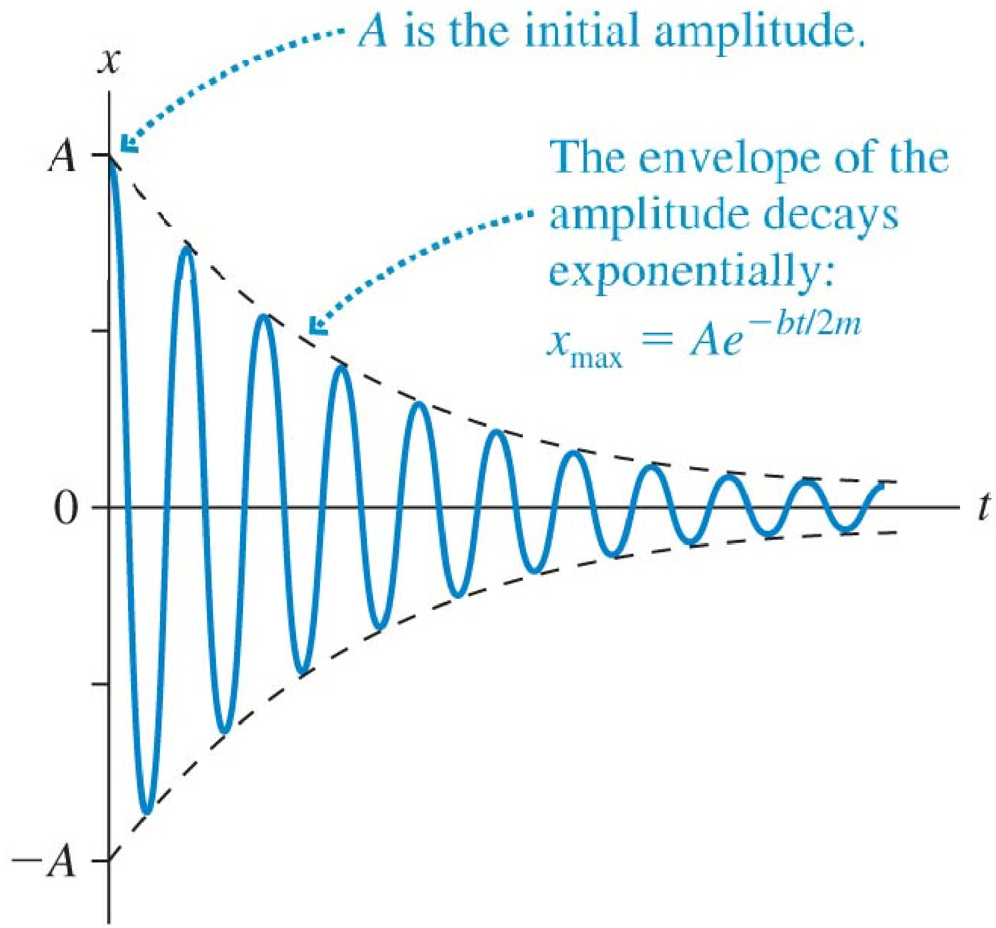
$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0) \qquad \text{(damped oscillator)}$$

where the angular frequency is given by

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

Damped Oscillations

FIGURE 14.23 Position-versus-time graph for a damped oscillator.

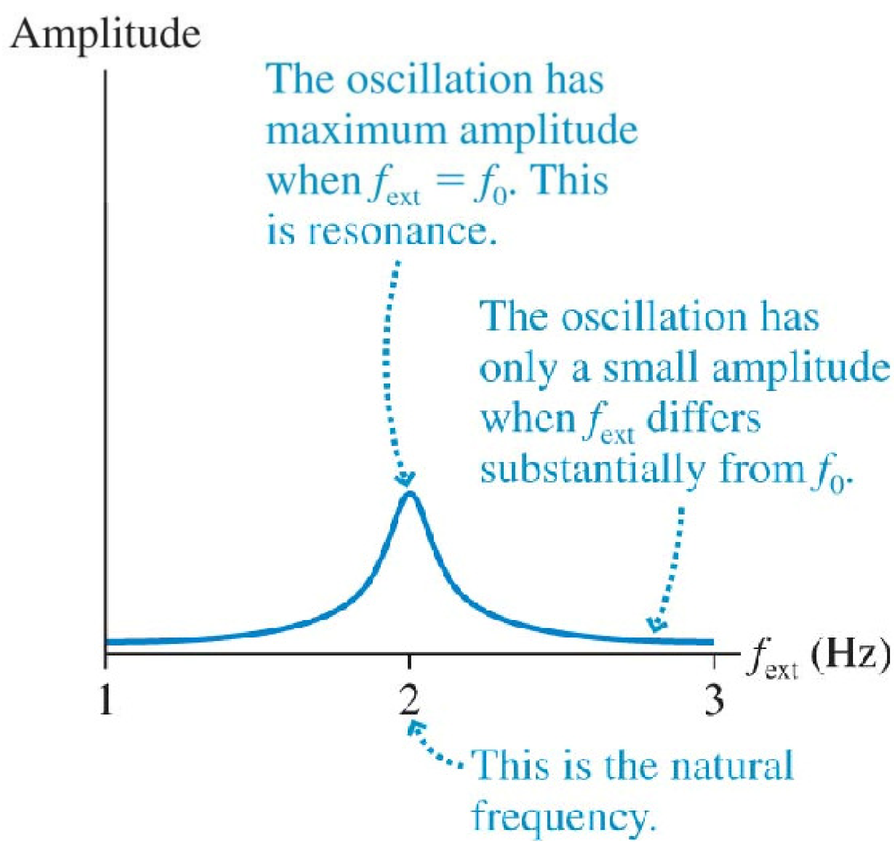


Driven Oscillations and Resonance

- Consider an oscillating system that, when left to itself, oscillates at a frequency f_0. We call this the **natural frequency** of the oscillator.
- Suppose that this system is subjected to a *periodic* external force of frequency f_{ext}. This frequency is called the **driving frequency**.
- The amplitude of oscillations is generally not very high if f_{ext} differs much from f_0.
- As f_{ext} gets closer and closer to f_0, the amplitude of the oscillation rises dramatically.

Driven Oscillations and Resonance

FIGURE 14.26 The response curve shows the amplitude of a driven oscillator at frequencies near its natural frequency of 2.0 Hz.



Driven Oscillations and Resonance

A singer or musical instrument can shatter a crystal goblet by matching the goblet's natural oscillation frequency.

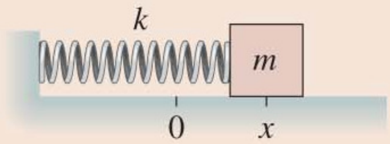
Chapter 14. Summary Slides

General Principles

Dynamics

SHM occurs when a linear restoring force acts to return a system to an equilibrium position.

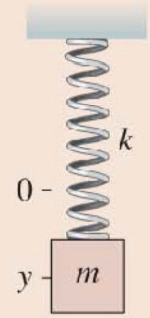
Horizontal spring



$$(F_{\text{net}})_x = -kx$$

Vertical spring

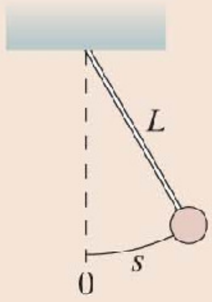
The origin is at the equilibrium position $\Delta L = mg/k$.



$$(F_{\text{net}})_y = -ky$$

$$\omega = \sqrt{\frac{k}{m}} \qquad T = 2\pi\sqrt{\frac{m}{k}}$$

Pendulum



$$(F_{\text{net}})_t = -\left(\frac{mg}{L}\right)s$$

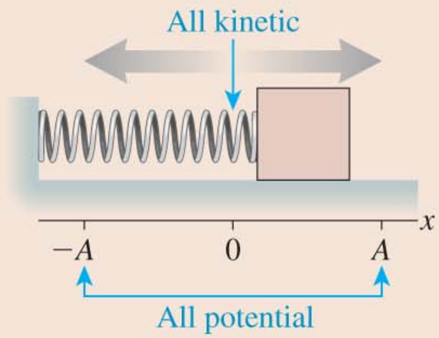
$$\omega = \sqrt{\frac{g}{L}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

General Principles

Energy

If there is **no friction** or dissipation, kinetic and potential energy are alternately transformed into each other, but the total mechanical energy $E = K + U$ is conserved.

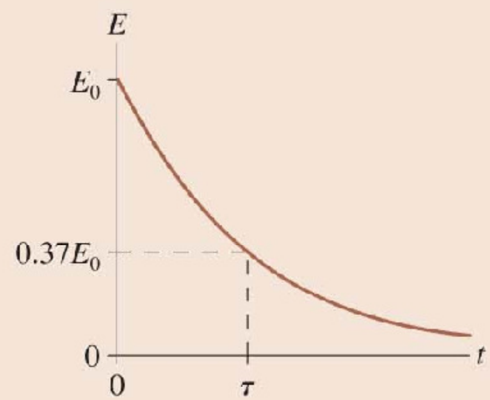
$$E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$$
$$= \frac{1}{2}m(v_{max})^2$$
$$= \frac{1}{2}kA^2$$



In a **damped system,** the energy decays exponentially

$$E = E_0 e^{-t/\tau}$$

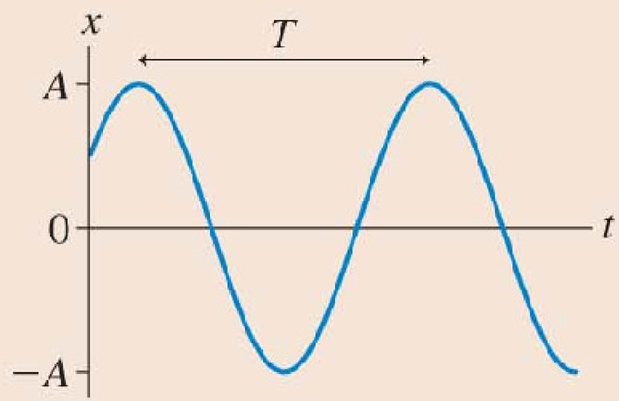
where τ is the time constant.



Important Concepts

Simple harmonic motion (SHM) is a sinusoidal oscillation with period T and amplitude A.

Frequency $f = \dfrac{1}{T}$



Angular frequency

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Position $x(t) = A\cos(\omega t + \phi_0)$

$$= A\cos\left(\frac{2\pi t}{T} + \phi_0\right)$$

Velocity $v_x(t) = -v_{\max}\sin(\omega t + \phi_0)$ with maximum speed $v_{\max} = \omega A$

Acceleration $a_x = -\omega^2 x$

Important Concepts

SHM is the projection onto the x-axis of **uniform circular motion.**

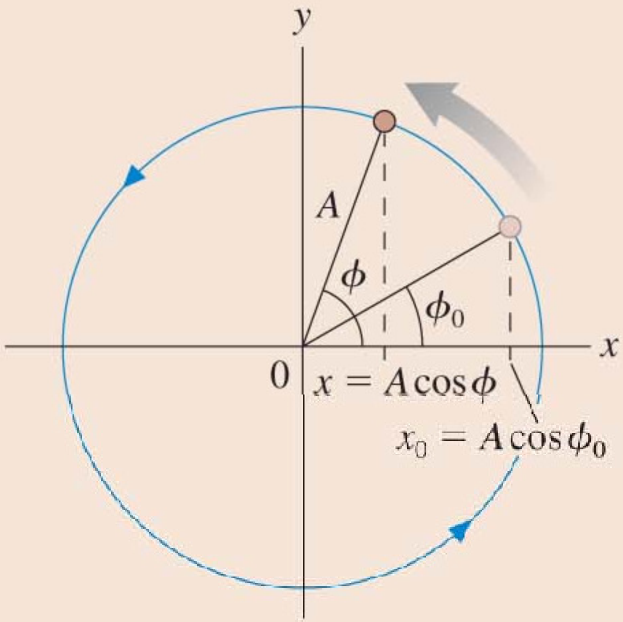
$\phi = \omega t + \phi_0$ is the phase

The position at time t is

$$x(t) = A\cos\phi$$
$$= A\cos(\omega t + \phi_0)$$

The phase constant ϕ_0 determines the initial conditions:

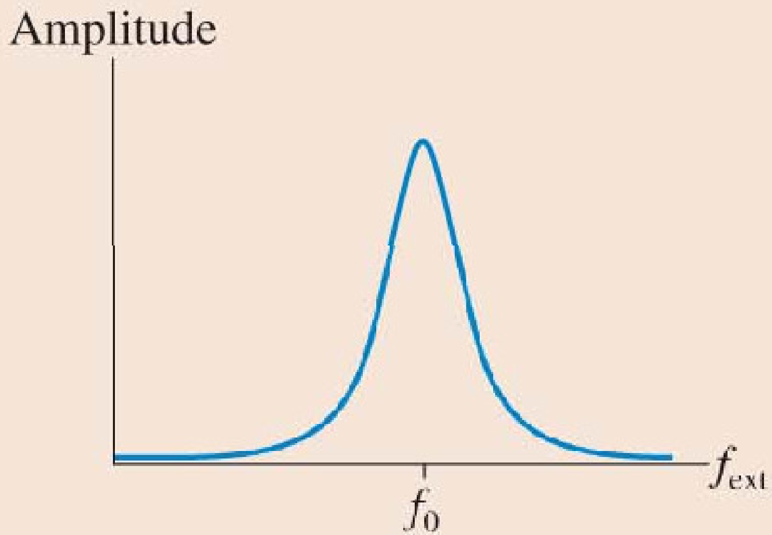
$$x_0 = A\cos\phi_0 \qquad v_{0x} = -\omega A\sin\phi_0$$



Applications

Resonance

When a system is driven by a periodic external force, it responds with a large-amplitude oscillation if $f_{ext} \approx f_0$, where f_0 is the system's natural oscillation frequency, or **resonant frequency.**



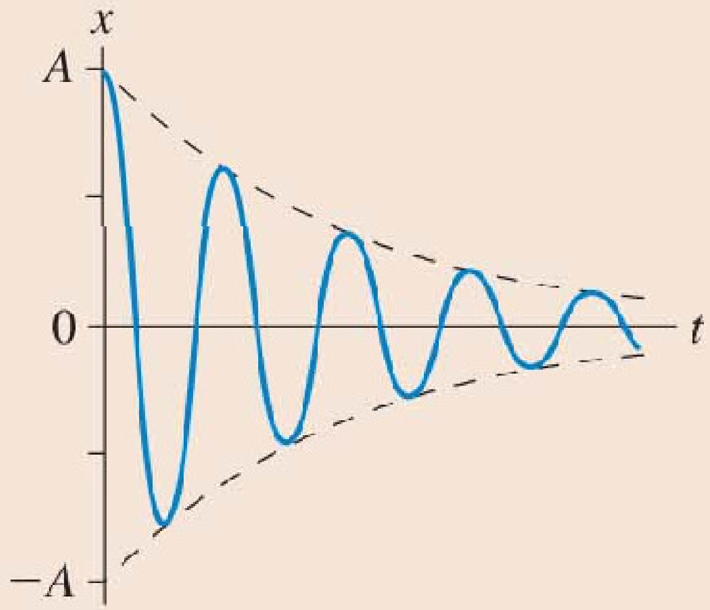
Applications

Damping

If there is a drag force $\vec{D} = -b\vec{v}$, where b is the damping constant, then (for lightly damped systems)

$$x(t) = Ae^{-bt/2m}\cos(\omega t + \phi_0)$$

The time constant for energy loss is $\tau = m/b$.



Chapter 14. Questions

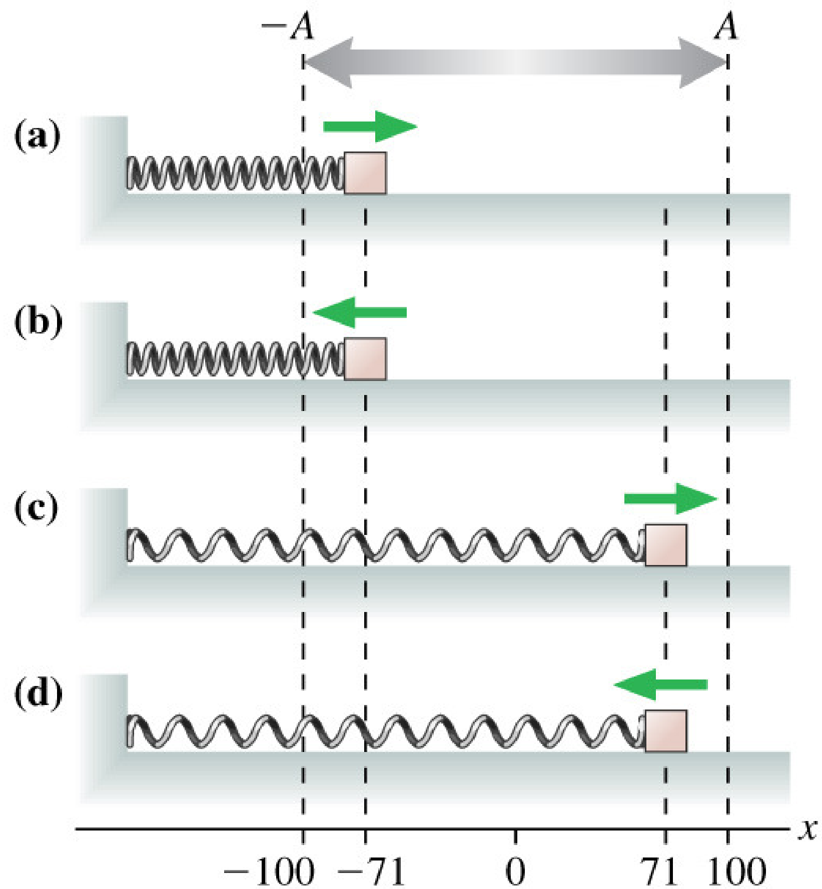
An object moves with simple harmonic motion. If the amplitude and the period are both doubled, the object's maximum speed is

A. quartered.
B. halved.
C. unchanged.
D. doubled.
E. quadrupled.

An object moves with simple harmonic motion. If the amplitude and the period are both doubled, the object's maximum speed is

A. quartered.
B. halved.
✔ **C. unchanged.**
D. doubled.
E. quadrupled.

The figure shows four oscillators at $t = 0$. Which one has the phase constant $\varphi_0 = \pi/4$ rad?



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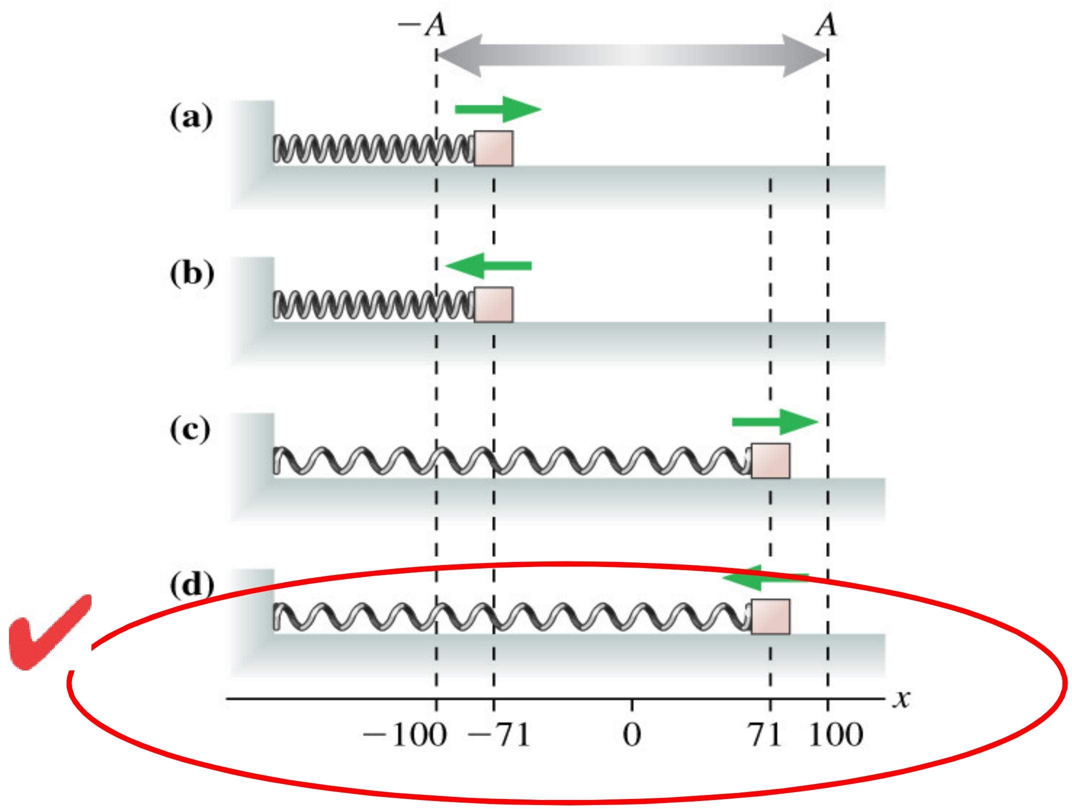
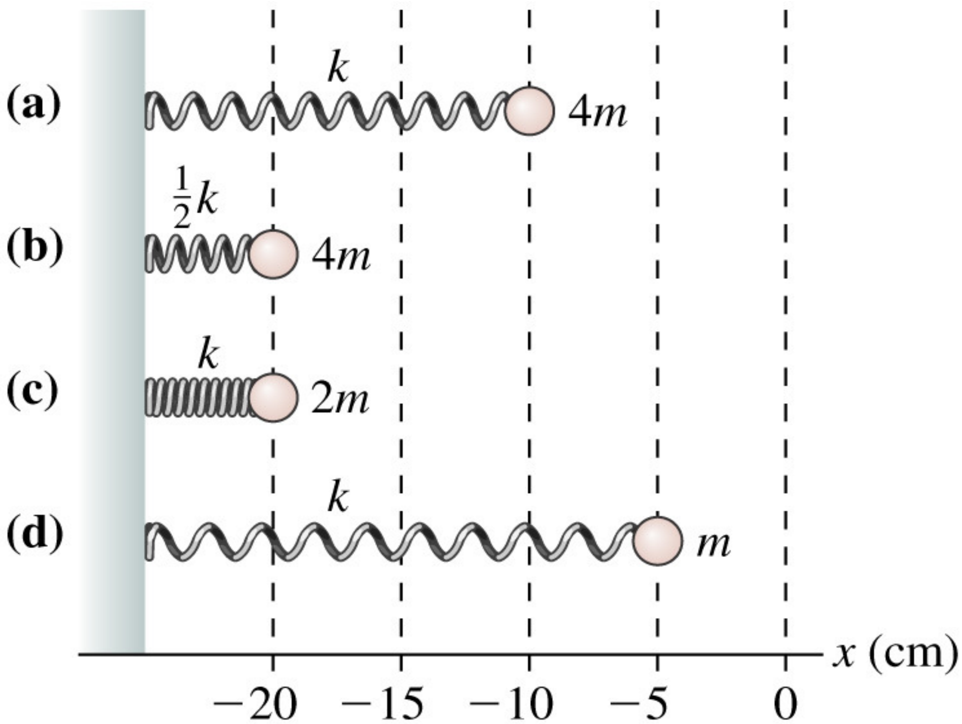


Four springs have been compressed from their equilibrium position at $x = 0$ cm. When released, they will start to oscillate. Rank in order, from highest to lowest, the maximum speeds of the oscillations.



A. c > b > a > d

B. c > b > a = d

C. a = d > b > c

D. d > a > b > c

E. b > c > a = d

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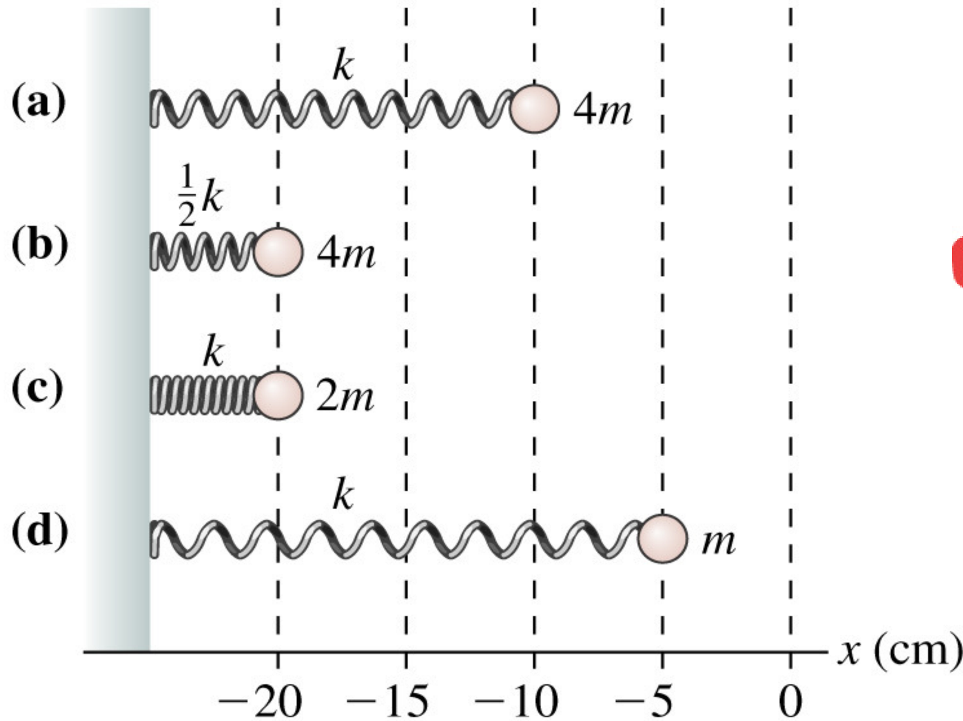

A. c > b > a > d

✔ **B. c > b > a = d**

C. a = d > b > c

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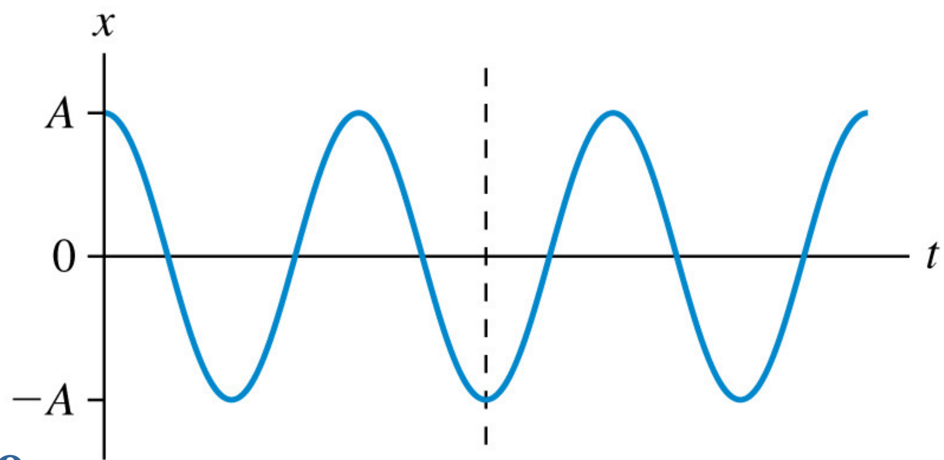
This is the position graph of a mass on a spring. What can you say about the velocity and the force at the instant indicated by the dotted line?



A. Velocity is zero; force is to the right.
B. Velocity is zero; force is to the left.
C. Velocity is negative; force is to the left.
D. Velocity is negative; force is to the right.
E. Velocity is positive; force is to the right.

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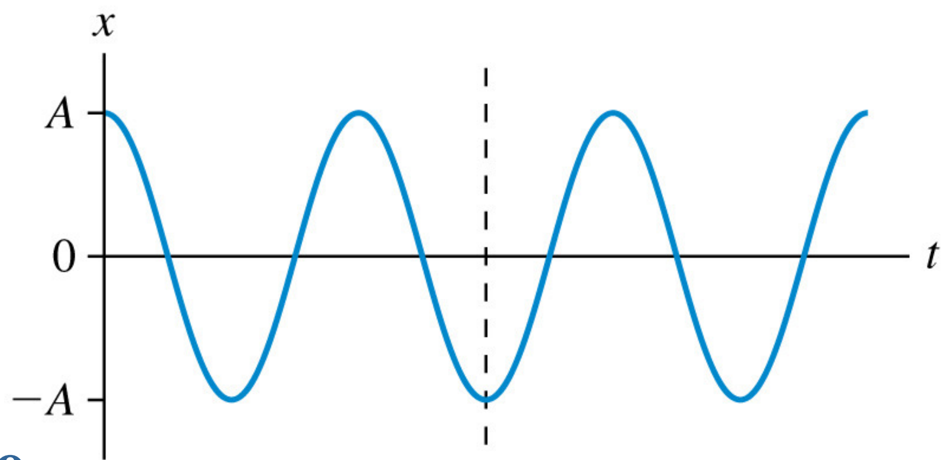

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One person swings on a swing and finds that the period is 3.0 s. Then a second person of equal mass joins him. With two people swinging, the period is

A. 6.0 s.

B. >3.0 s but not necessarily 6.0 s.

C. 3.0 s.

D. <3.0 s but not necessarily 1.5 s.

E. 1.5 s.

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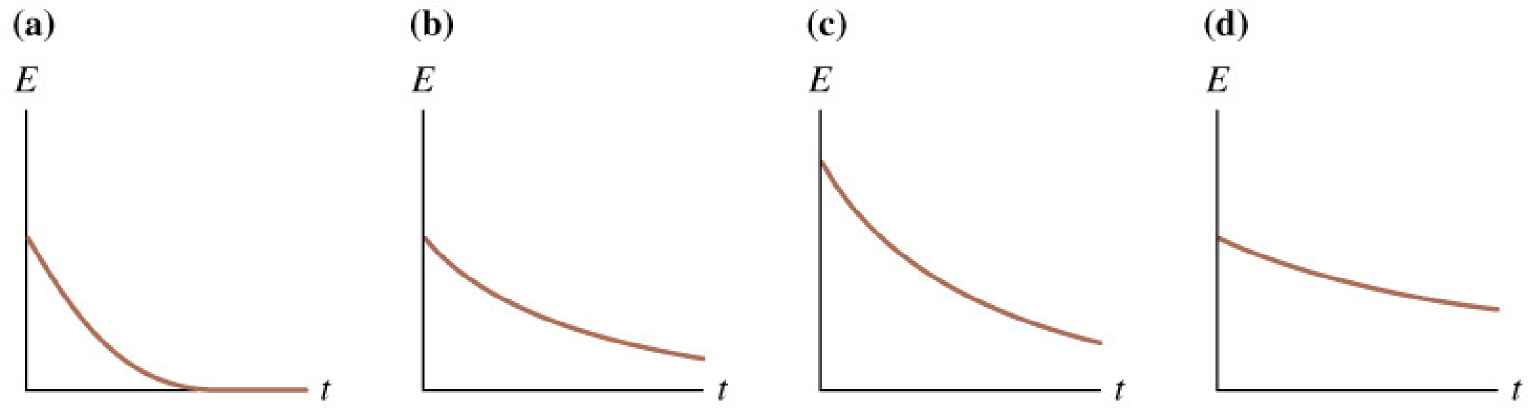
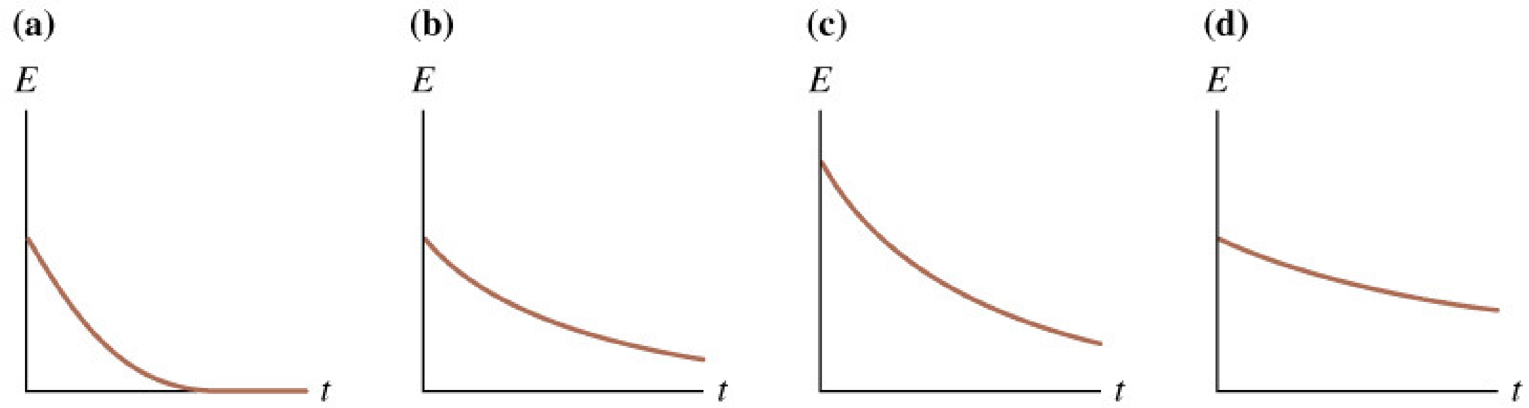


Rank in order, from largest to smallest, the time constants $\tau_a - \tau_d$ of the decays shown in the figure.

A. $\tau_c > \tau_b = \tau_d > \tau_a$
B. $\tau_a > \tau_b > \tau_c > \tau_d$
C. $\tau_a > \tau_b = \tau_d > \tau_c$
D. $\tau_d > \tau_b = \tau_c > \tau_a$
E. $\tau_d > \tau_b > \tau_c > \tau_a$



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B. $\tau_a > \tau_b > \tau_c > \tau_d$

C. $\tau_a > \tau_b = \tau_d > \tau_c$

✔ **D.** $\tau_d > \tau_b = \tau_c > \tau_a$

E. $\tau_d > \tau_b > \tau_c > \tau_a$