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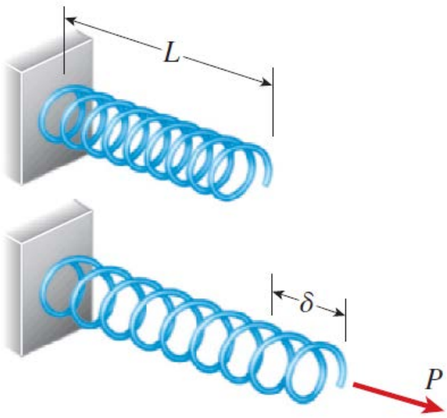
Chapter 2 Axially Loaded Members

Structural components subjected only to **tension** or **compression**: solid bars with straight longitudinal axes, cables and coil springs – can be seen in truss members, connecting rods, spokes in bicycle wheels, columns in buildings, and struts in aircraft engine mounts

2.2 Changes in Lengths of Axially Loaded Members

⊙ Springs



1. If the material of the spring is linear elastic,

   $$P = k\delta$$

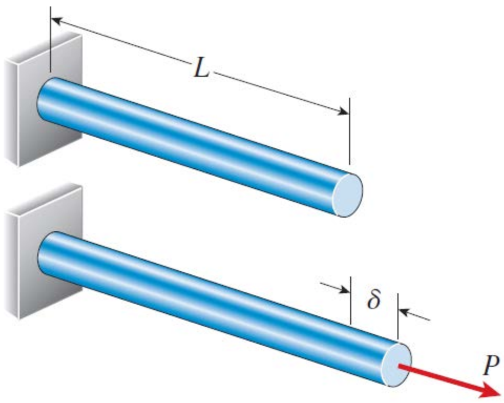
   where k is the **stiffness** of the spring, or spring constant, defined as the f________ required to produce a unit e____________.

2. The inverse relationship is

   $$\delta = \frac{1}{k}P = fP$$

   where f is the **flexibility** of the spring, or compliance, defined as the e__________ produced by a unit f_______.

⊙ Prismatic Bars



1. If the load acts through the centroid of the end cross section (proof at pp. 31-32), at a point away from the ends (Saint-Venant's principle), the normal stress is uniformly distributed and thus

   $$\sigma = \frac{P}{A}$$

2. If the bar is h__________, the axial strain is

   $$\varepsilon = \frac{\delta}{L}$$

3. Using Hooke's law $\sigma = E\varepsilon$, the elongation is derived as

   $\delta = \overline{\quad\quad}$ where EA:axial r______

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4. The relationship above works for both tension and compression with a proper sign convention (plus sign for P and δ in tension and negative in compression)

5. Stiffness and flexibility of a prismatic bar:

$$k = \frac{EA}{L}$$

$$f = \frac{L}{EA}$$

⊙ Cables

1. Important structural elements used when lifting/pulling/sustaining heavy objects (elevators, suspension bridges, cable-stayed bridges)

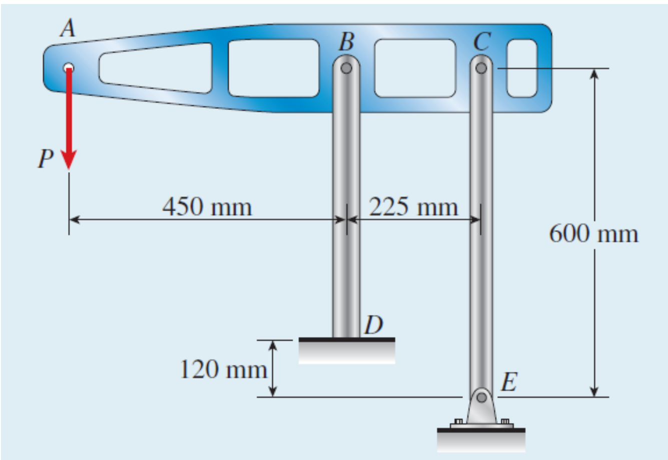
2. Cannot resist c_____________, thus not used against b_____________

3. Elongation of a cable is greater than that of a solid bar with the same material because the wires in the cable "tighten up" (instead of deforming in axial direction) → the "effective" modulus of elasticity is less than that of the material

⊙ **Example 2-2**: Cross-sectional areas of the bars BD and CE are 1,020 mm^2 and 520 mm^2, respectively. The bars are made of steel having a modulus of elasticity $E = 205\ \mathrm{GPa}$. Assume the beam ABC is rigid.

a) Maximum allowable load P if the displacement of point A is limited to 1.0 mm

b) If $P = 25\ \mathrm{kN}$, what is the required cross-sectional area of bar CE so that the displacement at point A is equal to 1.0 mm?



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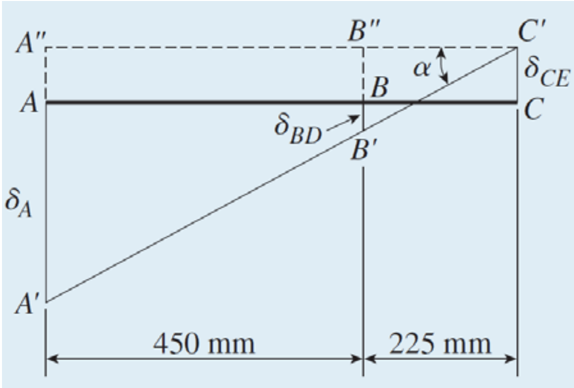

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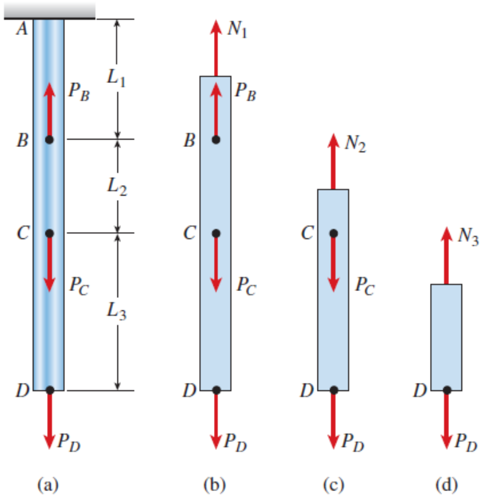
2.3 Changes in Lengths under Non-uniform Conditions

⦿ Bars with Intermediate Axial Loads

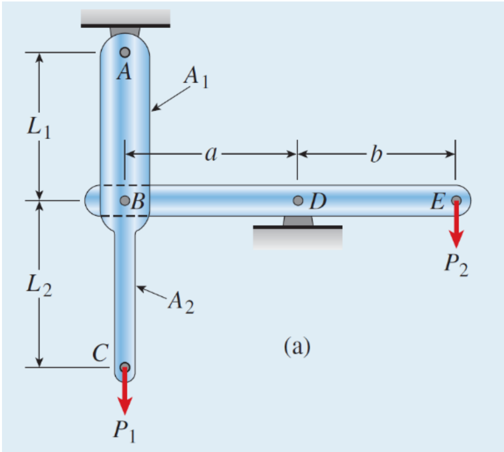
1. Procedure for a prismatic bar
   A. Identify the segments based on the intermediate axial loads
   B. Determine the internal axial forces from equilibrium
   C. Determine the change in each segment using $\delta_i = N_i L_i / EA$
   D. Add the changes to find the total elongation, i.e. $\delta = \sum_{i=1}^{n} \delta_i$



2. If the bar consists of multiple prismatic bars with different materials and/or cross-sectional areas, $\delta = \sum_{i=1}^{n} N_i L_i / E_i A_i$

⦿ **Example 2-3:** $L_1 = 20.0$ in., $L_2 = 34.8$ in., $A_1 = 0.25$ in.^{2}, $A_2 = 0.15$ in.^{2}, $E = 29.0 \times 10^6$ psi, $a = 28$ in., and $b = 25$ in.

Calculate the vertical displacement δ_C at C when the load $P_1 = 2{,}100$ lb and $P_2 = 5{,}600$ lb (Self-weight neglected)



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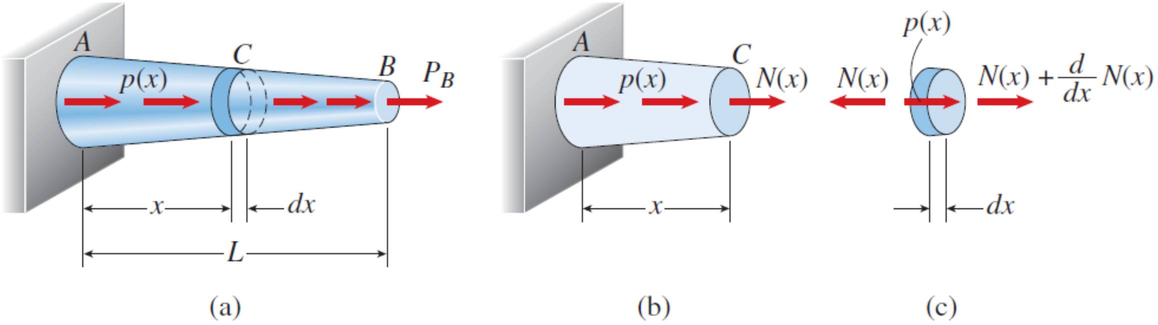
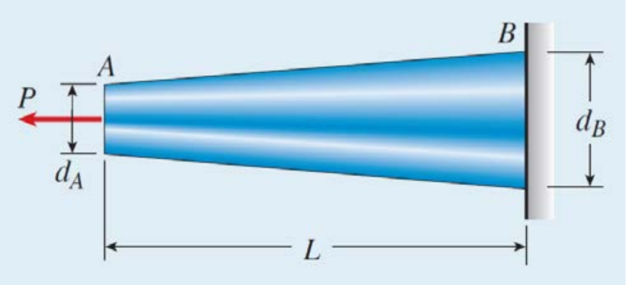
⦿ Bars with Continuously Varying Loads or Dimensions



1. Suppose the axial load $P(x)$, Young's modulus $E(x)$ or cross-sectional area $A(x)$ vary over the length
2. Consider the cylinder with the infinitesimal length dx → Figure (c)
3. The elongation of the cylinder is $d\delta =$ ———
4. The total elongation is $\delta = \int_0^L d\delta = \int_0^L \qquad dx$
5. Limitations: works when the material is linear elastic and the stress is uniformly distributed – the latter many not hold if the angle of the tapered bar is large

⦿ **Example 2-4:** Determine the elongation of a tapered bar AB of solid circular cross section



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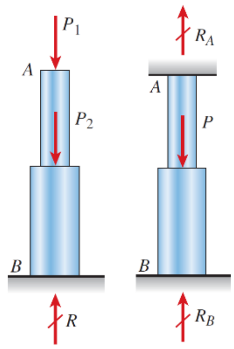
2.4 Statically Indeterminate Structures

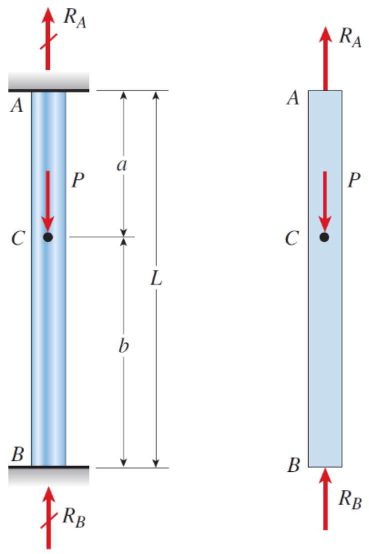


⦿ Statistically Indeterminate Structures

1. Statically determinate: r________ and i_________ f_________ can be determined solely from free-body diagrams and equations of e____________
   Statically indeterminate: otherwise

2. Example: axially loaded bar with fixed boundary conditions at both ends (→)

   $R_A - P + R_B = 0$ (1)



3. How to solve?
   Introduce **equation of c**______________, i.e.

   $\delta_{AB} =$ (2)

4. Equations (1) and (2) are defined in terms of different terms (force and displacement) → Use f_______-d________ relationship $\delta_{AC} = \frac{R_A a}{EA}$ and $\delta_{CB} = -\frac{R_B b}{EA}$

5. Now, Eq. (2) becomes $\delta_{AB} = \delta_{AC} + \delta_{CB} = \frac{R_A a}{EA} - \frac{R_B b}{EA} = 0$ (3)

6. Solving Eq. (1) and (3) together, one can get $R_A = \frac{Pb}{L}$ and $R_B = \frac{Pa}{L}$

7. With these reactions, one can obtain the internal forces and the elongations. For example, the vertical displacement of point *C* is $\delta_C = \delta_{AC} = \frac{R_A a}{EA} = \frac{Pab}{LEA}$

8. Stresses can be found too. For example, the normal stress in the segment *AC* is $\sigma_{AC} = \frac{R_A}{A} = \frac{Pb}{AL}$

⦿ Terminologies

1. **(a) Equilibrium equations**: static or kinetic equations

   (b) Compatibility equations: geometric equations, kinematic equations, equations of consistent deformations

   (c) Force-displacement relations: constitutive relations

2. **Stiffness method** (displacement method) vs **flexibility method** (force method)

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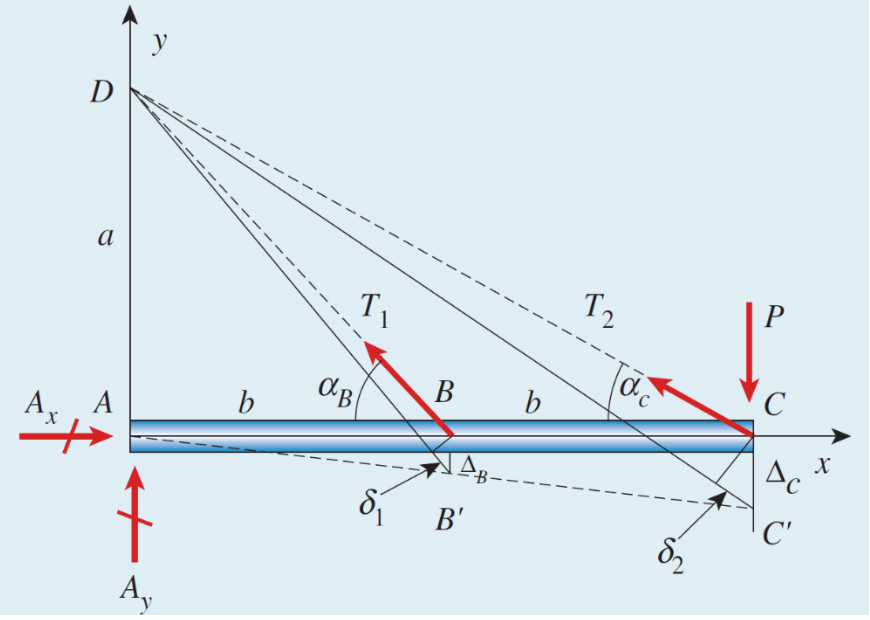
⊙ **Example 2-5**: A vertical load *P* applied on a horizontal *rigid* bar *ABC* is supported by wires *BD* and *CD* with diameters d_1 and d_2, modulus of elasticity E_1 and E_2, and lengths L_1 and L_2 respectively. Obtain formulas for the allowable load P when the allowable stresses in the wires *BD* and *CD* are σ_1 and σ_2, respectively.



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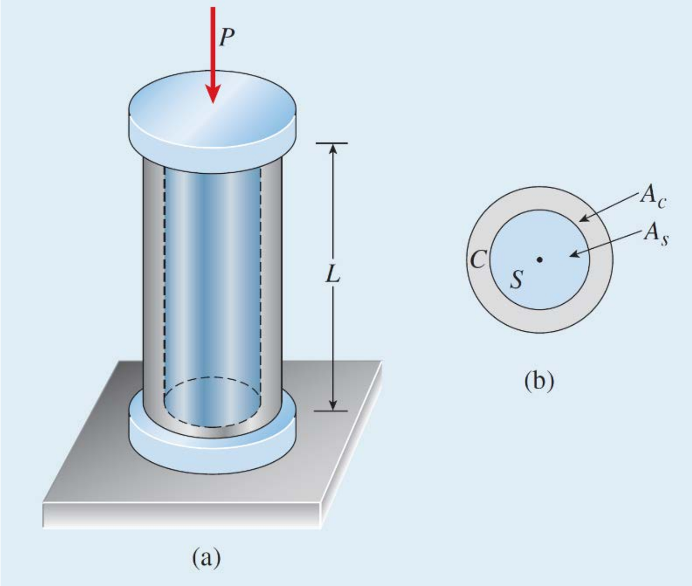
⊙ **Example 2-6**: A solid circular steel cylinder S (with Young's modulus E_s and cross-sectional area A_s) is encased in a hollow circular copper tube C (with E_c and A_c). When the cylinder and tube are compressed between the *rigid* plates of a testing machine y compressive forces P, determine the compressive forces P_s and P_c, the corresponding stresses, and the shortening of the assembly, δ.



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2.5 Thermal Effects~~, Misfits, and Prestrains~~

Other sources of deformations than external loads: thermal effects, misfits (imperfections), prestrains, settlements, intertial loads, etc.

⦿ Thermal Effects (expands when heated; contracted when cooled)

1. For most structural materials, thermal strain ε_T is proportional to the temperature change ΔT

   $\varepsilon_T = \alpha(\Delta T)$

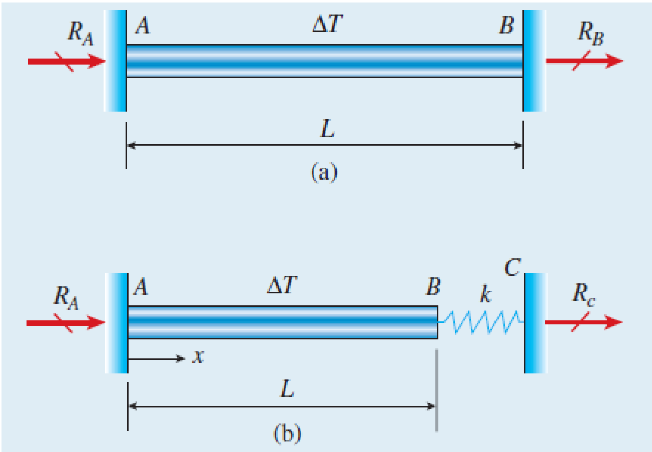
2. α: **coefficient of thermal expansion** $(1/K = 1/°\mathrm{C})$ or $(1/°\mathrm{F})$

3. Stress that would cause the thermal strain caused by ΔT: $\sigma = E\alpha(\Delta T)$

   e.g. the temperature change $\Delta T = 100°\mathrm{F}$ causes the same strain produced by $29{,}000\ \mathrm{psi}$ (typical allowable stress of stainless steel)

4. Temperature-displacement relation: $\delta_T = \varepsilon_T L = \alpha(\Delta T)L$

⦿ **Example 2-7:** A prismatic bar *AB* of length *L*, made of linearly elastic material with modulus *E* and thermal expansion coefficient α is held between immovable supports.



(a) When the temperature is raised by ΔT, derive a formula for the thermal stress σ_T

(b) Modify the formula in (a) if the rigid support *B* is replaced by an elastic support having a spring constant *k*. Only the bar is subjected to the temperature change ΔT.

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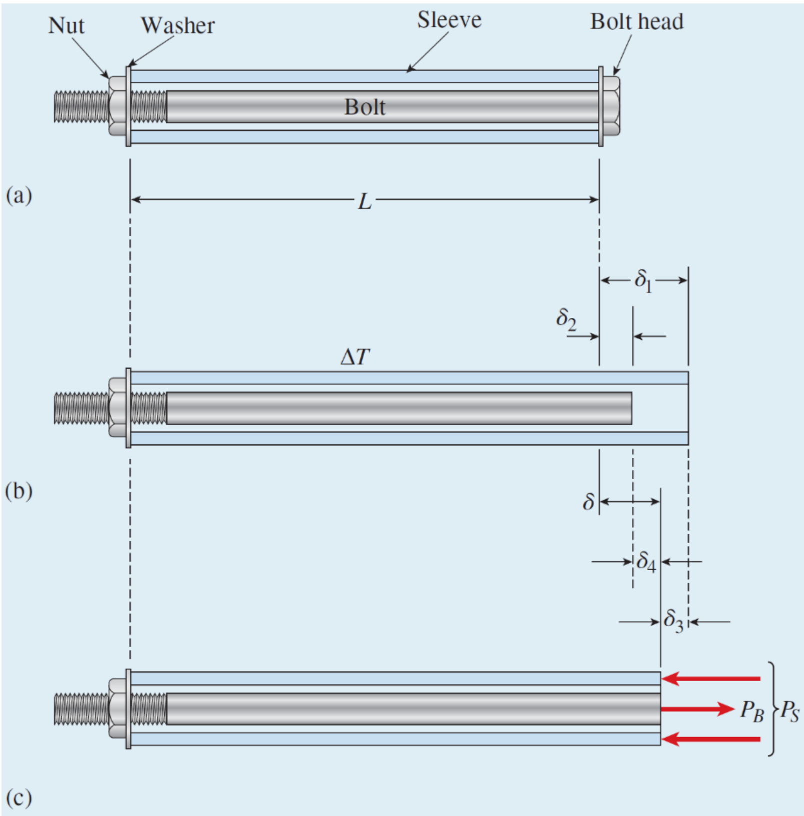
⊙ **Example 2-8:** A sleeve in the form of a circular tube of length L is placed around a bolt and fitted between washers at each end. Assume $\alpha_S > \alpha_B$. (a) If the temperature is raised by ΔT, what are the stresses in the sleeve and bolt, respectively? (b) What is the increase δ in the length L of the sleeve and bolt?



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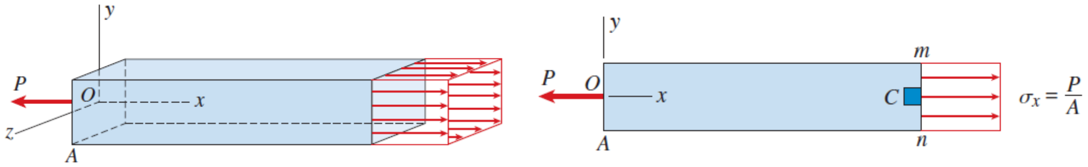
2.6 Stresses on Inclined Sections

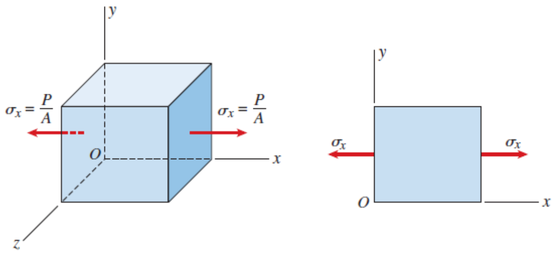
⊙ Stresses on Section Perpendicular to Axis



1. Normal stress $\sigma = P/A$ (prismatic, homogeneous and axial force at the centroid, away from the stress concentration at ends)

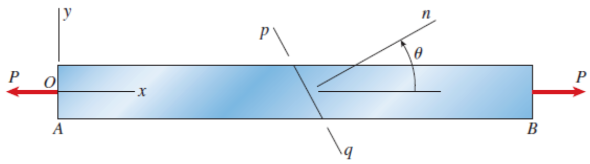
2. Normal stress only, i.e no s___________

3. "Stress element"(→): an infinitesimal block with its right-hand face lying in cross section ("C" in the figure above)



⊙ Stresses on Inclined Sections

1. Orientation of the inclined section: determined by θ ~ angle between x-axis and the normal vector n

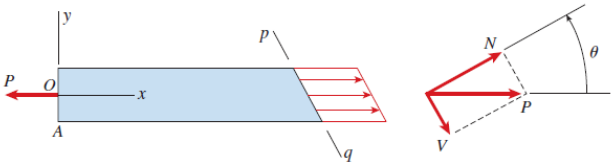


(a)

2. Two force components for the inclined section pq:

   $N = P \cdot$

   $V = P \cdot$

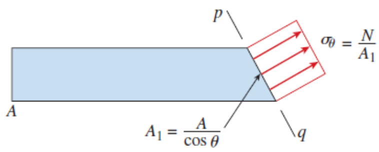


(b)

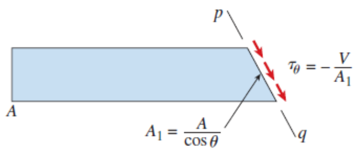
3. Then the normal and shear stresses on the inclined section are:

   $\sigma =$ ——, $\tau =$ ——

   where $A_1 =$

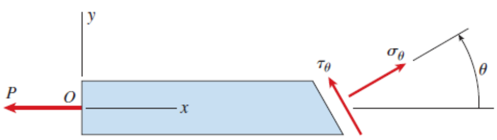


(c)



(d)

4. Sign conventions: normal stress positive in tension, and shear stress positive when it produces counterclockwise rotation



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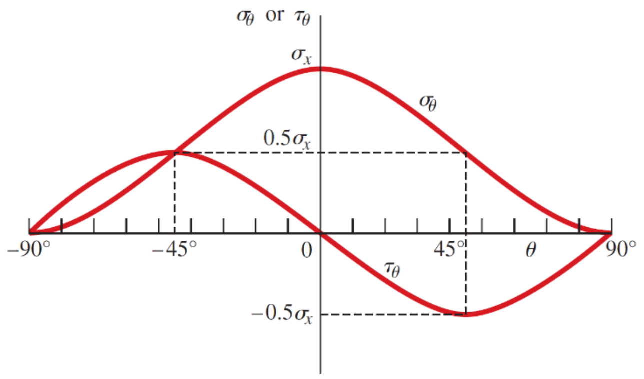
5. Thus, the normal and shear stresses on inclined section with the orientation θ are

$$\sigma_\theta = \frac{P}{A}\cos^2\theta = \sigma_x\cos^2\theta = \frac{\sigma_x}{2}(1+\cos 2\theta)$$

$$\tau_\theta = -\frac{P}{A}\sin\theta\cos\theta = -\sigma_x\sin\theta\cos\theta = -\frac{\sigma_x}{2}(\sin 2\theta)$$

⦿ Maximum Normal and Shear Stresses

1. Plots of σ_θ and τ_θ (→)



2. Maximum normal stress:

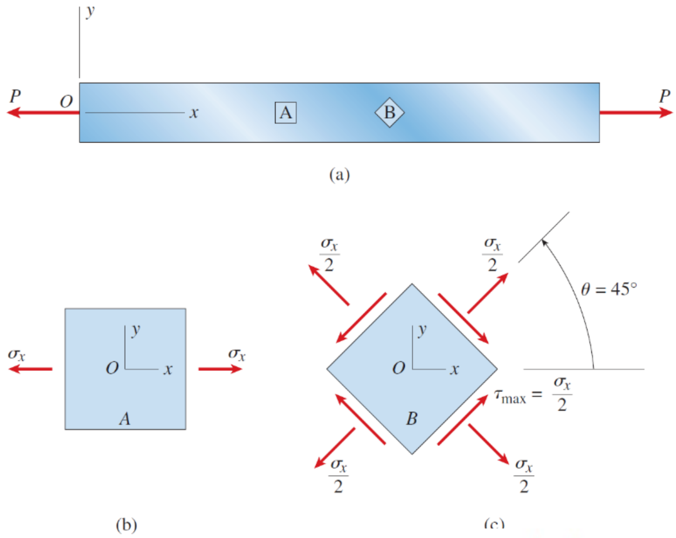
   $\sigma_{max} =$ at $\theta =$

   Corresponding shear stress =

3. Maximum shear stress:

   $\tau_{max} =$ —— at $\theta = \pm$

   Corresponding normal stress =



4. The most important orientations for uniaxial stress condition is $\theta =$ and $\theta = \pm$

5. Even though $\sigma_{max} = 2\tau_{max}$, a bar can fail in shear if the material is more vulnerable to the shear failure. Examples: shear failure of wood block in compression (→), and "slip bands" of flat bar of low carbon steel (→→)

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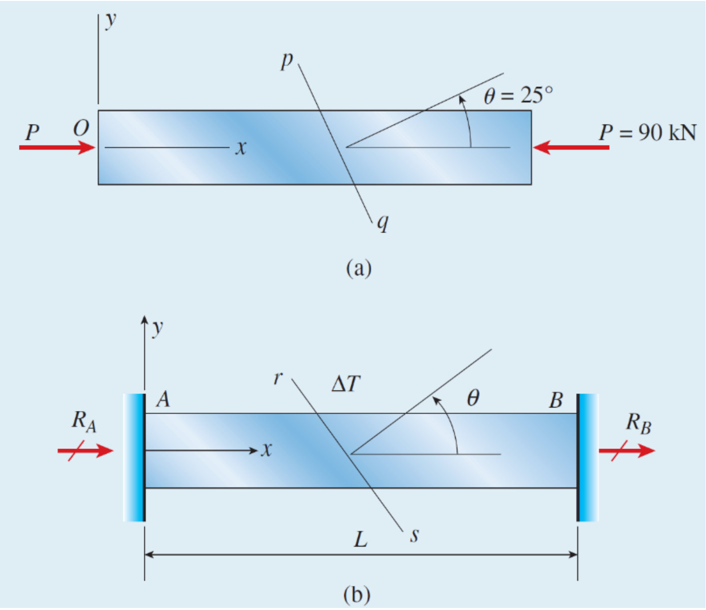
⦿ **Example 2-10**: A prismatic bar with $L = 0.5\text{ m}$ and $A = 1{,}200\text{ mm}^2$ is compressed by an axial load $P = 90\text{ kN}$. (a) Complete state of stress acting on section *pq* (angle $\theta = 25°$); (b) Bar fixed between supports *A* and *B* under temperature change $\Delta T = 33\text{°C}$. The compressive shear stress on section *rs* is 65 MPa. Find the shear stress τ_θ on plane *rs*, and angle θ ($E = 110\text{ GPa}$, $\alpha = 20 \times 10^{-6}/\text{°C}$)



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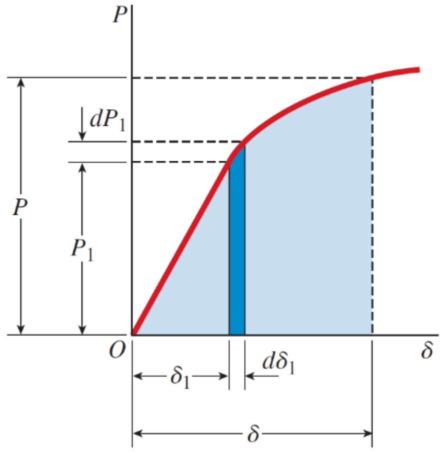
2.7 Strain Energy

⦿ Strain Energy

1. Suppose a tensile force P caused a prismatic bar to elongate by δ. The work done by the force P is

$$W = \int_0^\delta P_1 d\delta$$

where P_1 is the load at $\delta = \delta_1$ in the load-displacement diagram (→)

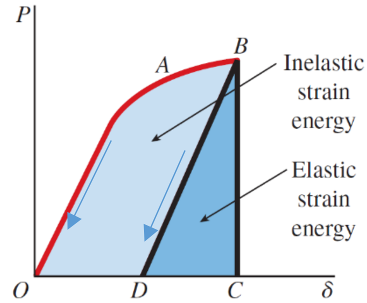


2. The strain caused by the work W increases the energy level (or the potential to do work) of the bar.

3. **Strain energy**: the energy absorbed by the bar during the loading process

4. From the principle of conservation of energy, the strain energy is

$$U = W = \int_0^\delta P_1 d\delta$$

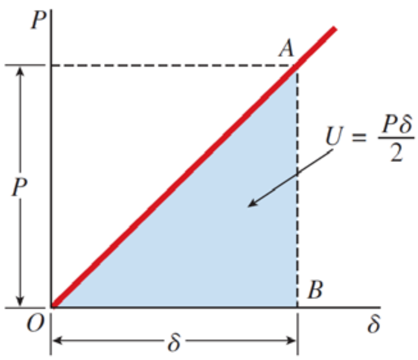


5. Unit of strain energy: $\mathrm{J = N \cdot m}$, ft-lb, ft-kips, in.-lb, in.-k.

6. **Elastic strain energy**: energy recovered in the form of work during unloading

   Inelastic strain energy: energy not recovered

⦿ Linearly Elastic Behavior



1. Strain energy: $U =$ ______

2. Strain energy in terms of load or elongation (using force-displacement relation):

   $U =$ or

⦿ Nonuniform Bars

1. Prismatic bar segments: $U = \sum_{i=1}^{n} U_i = \sum_{i=1}^{n} \frac{N_i^2 L_i}{2E_i A_i}$

2. Continuously varying bar: $U = \int_0^L \frac{[N(x)]^2 dx}{2EA(x)}$

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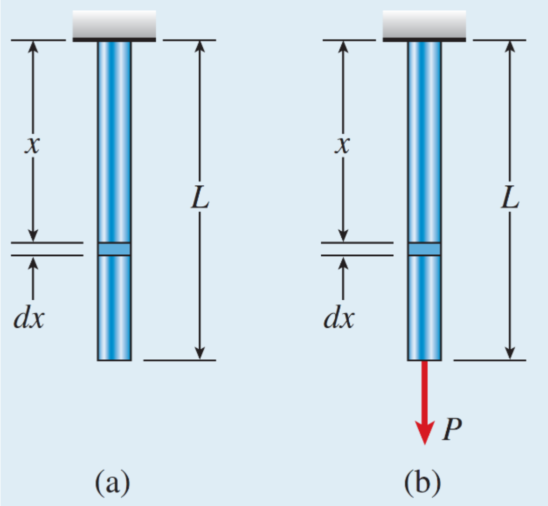
⦿ **Example 2-13:** Determine the strain energy of a prismatic bar under the loads: (a) the weight of the bar itself; and (b) the weight of the bar plus a load P at the lower end.



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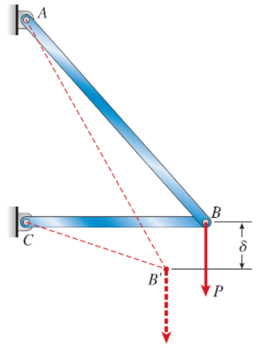
⦿ Displacements Caused by a Single Load



The displacement of a linearly elastic structure supporting only one load can be determined from its strain energy. From the system shown below, $U = P\delta/2$. Therefore, the displacement is $\delta = 2U/P$.

⦿ **Example 2-14:** Determine the vertical displacement δ_B of joint B of the truss under the vertical load P. Both members of the truss have the same axial rigidity EA.