Chapter 2: Derivatives

In this chapter we will cover:

2.0.1 The tangent line and the velocity problems. The derivative at a point and rates of change.

2.0.2 The derivative as a function. Differentiability.

2.1 Derivatives of constant, power, polynomials and exponential functions.

2.2 The Product Rule and the Quotient Rule.

2.3 Derivatives of trigonometric functions.

2.4 Chain Rule. The derivative of general exponential functions.

2.5 Implicit differentiation.

2.6 Derivatives of logarithmic functions. Logarithmic differentiation.

2.9 Related rates

2.10 Chapter Review

2.0.1. The tangent line and the velocity problems. The derivative at a point and rates of change.

A. The tangent problem:

Consider the graph of a function $f(x)$, such as the graph shown below:

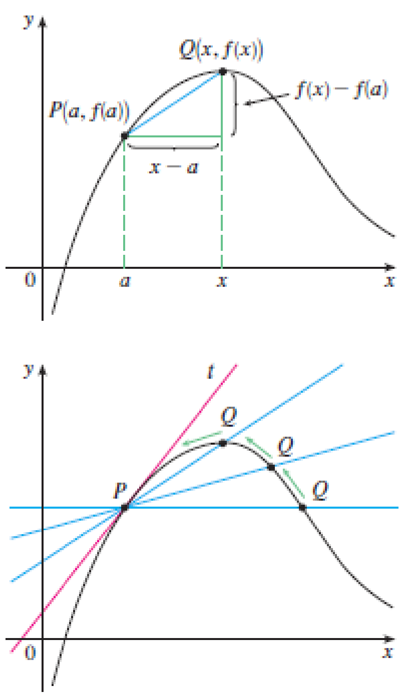

Figure 1: The graph of a generic function $f(x)$ and calculation of the tangent line

How can we define and find the tangent line to this graph at a point on the curve $P(a, f(a))$?

The definition of the tangent line:

We remember that to find tangent line at the point P , we consider a generic point $Q(x, f(x))$ on the graph, and calculate first the slope of the secant line PQ: $m_{PQ} = \dfrac{f(x)-f(a)}{x-a}$, as shown in the figure.

Then the tangent line at $P(a, f(a))$ is the line which passes through P and which has the slope:

(1) $m_T \equiv \lim_{Q\to P} m_{PQ} = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$, provided that this limit exists.

Example 1:

Calculate the equation of the tangent line to the graph of $f(x)=x^2$ at $P(1,1)$, $P(0,0)$ and $P(2,4)$.

Note:

We sometimes refer to the slope of the tangent line to the graph of $f(x)$ at a point $P(a, f(a))$ as the slope of the function at $P(a, f(a))$. This is because if we zoom in enough near the point P, then the function $f(x)$ (if smooth at the point) will appear to be a line.

It is useful to see that formula (1) can also be written as:

(2) $m_T = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, by making in (1) the substitution $h = x-a$.

Example 2: Let $f(x)=\dfrac{1}{x}$. Calculate the equation of the tangent line at the point $P(1,1)$ on the graph of this function.

B. The velocity problem:

We have seen in 1.1 that for an object which moves along a straight line with the position given by $s=f(t)$, we have that the average velocity of the object over a time interval $[t_1, t_2]$ is given by:

(3) $v^{[t_1,t_2]}{}_{Avg} \equiv \dfrac{f(t_2)-f(t_1)}{t_2-t_1}$ (change in position divided by the change in time over this time interval) and

define the instantaneous velocity (or instantaneous speed) at $t=a$ as

(4) $v(a) \equiv \lim_{h\to 0} v_{Avg}^{[a,a+h]} = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ where t is an arbitrary time $t \neq a$.

Example 3: Do problem 13 from the exercise set 2.7.

Equations (1), (2) and (4) , which are very similar, all give the instantaneous rate of change of the function $f(x)$ at $P(a, f(a))$. This quantity is called the derivative of the function $f(x)$ at a and is denoted by $f'(a)$.

Definition 1:

The derivative of a function $f(x)$ at a point in its domain $a \in D$ is given by:

(5) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$, if this limit exists.

Optional Note: In general, for a function $y = f(x)$, the difference quotient:

(5) $\frac{\Delta y}{\Delta x} \equiv \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ represents the **average (relative) rate of change** of this function, when x changes by Δx (from x_1 to x_2).

The (relative) instantaneous rate of change of the quantity $y = f(x)$ at $x = a$ is $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$, which is also the slope of the tangent line to the graph of $f(x)$ at a .

Example 4: Do example 6 on page 148.

End Optional

Example 5:

a) Find the derivative of the function $f(x) = x^2 - 4x + 9$ at a point a .

b) Find an equation of the tangent line to the parabola $f(x) = x^2 - 4x + 9$ at $P(3,6)$.

Example 5:

Calculate f'(0), f'(1) , f'(-1) and f'(2) for $f(x) = x^2$ and interpret these as the instantaneous rate of change of $f(x)$ at these points.

Note that when the derivative is large in magnitude at a, then the y values of the function near a change rapidly, and when the derivative is close to 0, then the curve is relatively flat near a, and the function changes very slowly near a.

Homework: Problems 3,4, 7, 8, 9,10,11,15,16,17,22,24,27,29,31,32,33,36,37,53,54 from Exercise set 2.7.

2.0.2 The derivative as a function. Differentiability.

Note that in 2.0.1 we calculated the derivative of a function $f(x)$ at a specific point a in its domain.

It is often best to calculate (if possible) $f'(x)$ at any point x in its domain. Therefore, define:

(1) $$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ when this limit exists.

Note:

1. We can try to use also (1') $f'(x) = \lim_{t\to x} \dfrac{f(t)-f(x)}{t-x}$ as a generalization of (5_2) from 2.0.1 although the formula (1) above is more natural, and therefore almost always used.
2. Given any number x, the expression shown in (1) defines a new function $f'(x)$, called the derivative of f at x. This can be interpreted as the slope of the tangent line to the graph of f at x. The domain of f' (smaller than the domain of f) is the set $\{x \in D_f \mid f'(x) \text{ exists}\}$.

Example 1:

If $f(x) = x^3 - x$, calculate $f'(x)$. Then draw the graphs of $f(x)$ and $f'(x)$ on the same axes and confirm that $f'(x)$ gives the slope of the tangent line to the graph of $f(x)$ at each point x.

Example 2:

For $f(x) = \sqrt{x}$, calculate $f'(x)$. Find the domains of $f'(x)$ and of $f(x)$. Graph both functions on their respective domains.

Definition 1:

a) A function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists.
b) A function $f(x)$ is differentiable on (a,b) (or $(-\infty,a)$ or (a,∞) or $(-\infty,\infty)$) if $f'(x)$ exists at each point of the corresponding interval.

Example 3:

Where is $f(x) = |x|$ differentiable and where it is not differentiable?

Question: In general, how can a function $f(x)$ fail to be differentiable?

First, let us establish the following important result:

Theorem 1: Consider a function $f: D \to R$ and $a \in D$.

If $f(x)$ is differentiable at a, then $f(x)$ is continuous at a.

Proof: Consider $f(x) = f(a) + \frac{f(x) - f(a)}{x - a} \cdot (x - a)$ and take the limit of this equality , remembering that

$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$ exists.

End of proof.

Therefore, one way in which a function is not differentiable at a is when the function is discontinuous at a .

To understand best how a function may fail to be differentiable at a, define:

(2) $f'_l(a) = \lim_{\substack{x \to a \\ x < a}} \frac{f(x) - f(a)}{x - a}$ and (3) $f'_r(a) = \lim_{\substack{x \to a \\ x > a}} \frac{f(x) - f(a)}{x - a}$.

Since differentiability at a means that $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists, then we can establish:

Theorem 2: Consider a function $f: D \to R$ and $a \in D$. $f(x)$ is differentiable at a if and only if:

$f'_l(a)$ and $f'_r(a)$ exists and if $f'_l(a) = f'_r(a)$ **.**

Therefore, a function $f(x)$ can fail to be differentiable if one of the following takes place:

- $f(x)$ is discontinuous at a (think also of the "extreme" case in which $f(x)$ has the same slope on either side of the point but it is discontinuous at a, such as : $f(x) = \begin{cases} 2x + 1, \text{ for } x < 0 \\ 2x + 2, \text{ for } x \geq 0 \end{cases}$, which is discontinuous at 0. Show that this function is not differentiable at 0, by calculating $f'_l(0)$ and $f'_r(0)$ **).**
- $f'_l(a) = \pm\infty$ (or it does not exist) or $f'_r(a) = \pm\infty$ (or it does not exist) . Note that an infinite (semi) tangent line means that the (semi) tangent line is vertical.
- $f'_l(a)$ and $f'_r(a)$ both exist but $f'_l(a) \neq f'_r(a)$**.** (this case corresponds to a corner in the graph of $f(x)$).

See also Figure 7 on page 159 in the textbook for graphical illustration of these cases.

Homework: Problems 1,3,4,7,8,9,10.11,16,17,19,21,22,26,27,28,37,38,44,45,56 from Exercise Set 2.8 .

2.1 Derivatives of constant, power, polynomials and exponential functions.

From now on and for a few sections we intend to establish formulas for derivatives of elementary functions (algebraic and trigonometric), for transcendental functions and for combinations of these.

This will allow us to quickly calculate the derivatives of these functions, and these calculations will be used in many applications (as it gives the relative instantaneous rate of change of these expressions).

Finally, we will collect all these formulas in a table of derivatives, to organize and help us with memorizing these.

In this section , we start with deriving formulas for the derivatives of constant, power, polynomial and exponential function.

Using formula (1) from 2.0.2, it is easy to see that:

(1) $\boxed{\dfrac{d(c)}{dx} = 0}$, for any constant $c \in R$.

Similarly: $\boxed{\dfrac{d(x)}{dx} = 1,\ \dfrac{d(x^2)}{dx} = 2x \text{ and } \dfrac{d(x^3)}{dx} = 3x^2}$

This seems to suggest the more general formula:

(2) $\boxed{\dfrac{d(x^n)}{dx} = nx^{n-1}}$, for $n \in N$, which is easy to prove if we use the algebraic formula :

$$a^n - b^n = (a-b)\cdot\left(a^{n-1} + a^{n-2}\cdot b + a^{n-3}\cdot b^2 + \ldots + a\cdot b^{n-2} + b^{n-1}\right)$$

(or the binomial formula : $(a+h)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}h^k$ where $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$).

Formula (2) is called the Power Rule for derivatives for $n \in N$. We can show that the power rule (2) holds for $n \in R$ as well .

First, check that (2) holds for $n = -1, -2$, which suggests that it holds for $n \in Z$ (a proof will be given in the next section) . Also check that (2) holds for $n = 1/2$ and $n = 1/3$.

The following general rules can also easily be verified:

(3) $\boxed{\dfrac{d(c \cdot f(x))}{dx} = c \cdot f'(x)}$, for any constant $c \in R$, and :

(4) $\boxed{\dfrac{d(f(x)+g(x))}{dx} = f'(x) + g'(x)}$ and (5) $\boxed{\dfrac{d(f(x)-g(x))}{dx} = f'(x) - g'(x)}$

by recalling that $(f+g)(x) \equiv f(x) + g(x)$ and that $(f-g)(x) \equiv f(x) - g(x)$.

Formulas (3) to (5) allow us to extend formulas (1) and (2) to other expressions (see examples 5 and 6 in the textbook).

Let us turn now to the derivative of an exponential function $f(x)=a^x$, $f:R\to(0,\infty)$ since $a>0$.

First, using the algebraic property: $a^{x+h}=a^x\cdot a^h$, we find that :

(5) $f'(x)=(a^x)'=a^x\cdot f'(0)=f(x)\cdot f'(0)$, therefore, we need to determine $f'(0)$ the general derivative $f'(x)=(a^x)'$.

There are many possible equivalent definitions of the number e, which is the basis of the natural logarithmic function (such as: (6) $e\equiv\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=\lim_{h\to 0}(1+h)^{\frac{1}{h}}$).

Here, we define e as the irrational number $e\in(2,3)$ such that

(7) $\lim_{h\to 0}\frac{e^h-1}{h}=1$ (note that "informally (7) is equivalent with (6)).

(7) means that e^x is the particular exponential function a^x for which $f'(0)=1$ (the slope of the tangent line at origin for the function $f(x)=e^x$ is 1) (see Figure 1 below).

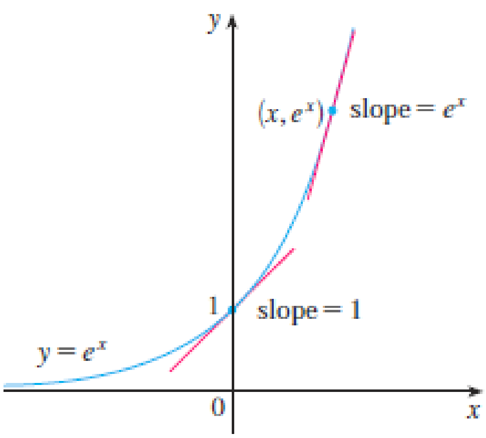

Figure 1: Definition of the number e .

Using (5), this produces the useful formula:

(8) $\boxed{\frac{d(e^x)}{dx}=e^x}$.

Example 1: Do example 8 and 9 in the textbook.

Homework: Problems 1,3,4,6,9,10,11,13,15,16,18,20,23,27,33,36,38,47,51,52,60,67,68,70,71,75,77 from Exercise Set 3.1 .

2.2 The Product Rule and the Quotient Rule:

These two general formulas for derivatives (the product rule and the quotient rule) will allow us to calculate derivatives of combinations of functions learnt in 2.1.

The general formulas are:

(1) $$\boxed{\frac{d\left(f(x)\cdot g(x)\right)}{dx} = f'(x)\cdot g(x) + f(x)\cdot g'(x)}$$ (the product rule) and

(2) $$\boxed{\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}}$$ (the quotient rule).

Memorize these formulas correctly (pay special attention to the $-$ sign in formula (2))

Proof of **(1):**

$$\frac{d\left(f(x)\cdot g(x)\right)}{dx} \equiv \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h} =$$

$$\lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h} =$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot g(x+h) + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\cdot f(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$$

Proof of **(2):**

$$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} \equiv \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)} =$$

$$\lim_{h\to 0}\frac{f(x+h)\cdot g(x) - f(x)\cdot g(x) + f(x)\cdot g(x) - f(x)\cdot g(x+h)}{h\cdot g(x)\cdot g(x+h)} =$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot\frac{g(x)}{g(x)\cdot g(x+h)} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\cdot\frac{f(x)}{g(x)\cdot g(x+h)} = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{g^2(x)}$$

Note that for (2) we need to keep track of which expression (function) is the numerator and which is the denominator, while for (1) the order of the terms of the product is not important.

Example 1: Do Examples 1, 3, 4 and 5 in section 3.2 in the textbook.

Memorize the formulas shown at the end of the section 3.2 .

Homework: Problems 1,3,6,15,16,22,23,25,27,29,31,33,35,43,46,49,54 from Exercise Set 3.2 in the textbook.

2.3 Derivatives of trigonometric functions.

Goal: Derive formulas for $\frac{d}{dx}(\sin(x)) = (\sin(x))'$ $\quad \frac{d}{dx}(\cos(x)) = (\cos(x))'$

$\frac{d}{dx}(\tan(x)) = (\tan(x))'$ $\quad \frac{d}{dx}(\cot(x)) = (\cot(x))'$

Before applying formula (1) in 2.0.2. to derive these, note the graphs of $f(x) = \sin(x)$ and of its derivative (built by measuring the slope of the tangent lines to the graph of $f(x)$ at different points) below:

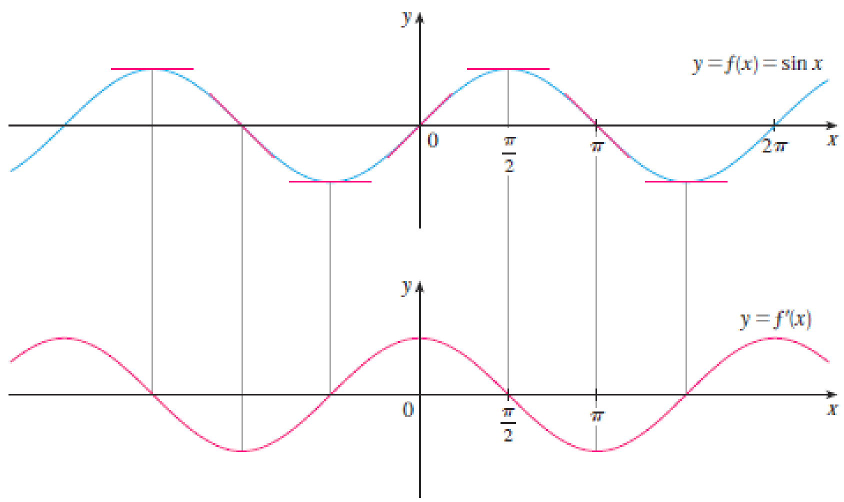

Figure 1: The graphs of $f(x) = \sin(x)$ and of its derivative $f'(x)$

Therefore a good guess for $(\sin(x))'$ is $(\sin(x))' = \cos(x)$. Let us verify this algebraically:

Using formula (1) in 2.0.2 :

(1)
$$\frac{d(\sin(x))}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} =$$
$$\lim_{h\to 0}\left[\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right].$$

Therefore, in order to find $(\sin(x))'$, we need to prove a formula for $\lim_{h\to 0}\frac{\sin(h)}{h}$ (which we guessed as being 1 in 1.2) and for $\lim_{h\to 0}\frac{\cos(h)-1}{h}$. For proving that $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$, consider the sector of a unit circle shown in Figure 2 below:

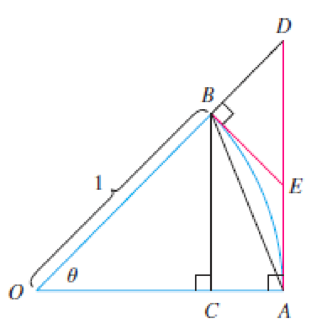

Figure 2: A sector of an unit circle: note that $BC = \sin(\theta)$, $OC = \cos(\theta)$, $AO = 1$ and $AD = \tan(\theta)$

Also, we have that:

$$Area(\Delta\,\mathrm{OBC}) \le Area(\text{sector OAB}) \le Area(\Delta OAD) \to$$
$$\frac{\sin(\theta)\cos(\theta)}{2} \le \frac{\theta}{2} \le \frac{\tan(\theta)}{2} \to \cos(\theta) \le \frac{\theta}{\sin(\theta)} \le \frac{1}{\cos(\theta)} \to \lim_{\theta\to 0}\frac{\theta}{\sin(\theta)} = 1$$

(by using the squeeze theorem).

Therefore:

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=\lim_{\theta\to 0}\frac{1}{\frac{\theta}{\sin(\theta)}}=1 \text{ (from the limit laws)} .$$

Therefore, we have shown that: **(2)** $\lim_{h\to 0}\frac{\sin(h)}{h}=\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1$. (2) is an important result which we will use to find $(\sin(x))'$, and an important limit of its own. Note that (2) implies more generally that

(3) $\lim_{u\to 0}\frac{\sin(u)}{u}=1$, where u is any expression which approaches 0.

Example 1: See Example 5 in section 3.3 in the textbook.

In order to find $\lim_{h\to 0}\frac{\cos(h)-1}{h}$, note that :

$\lim_{h\to 0}\frac{\cos(h)-1}{h}=\lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)}=-\lim_{h\to 0}\frac{\sin(h)}{h}\frac{\sin(h)}{\cos(h)+1}=0$. Therefore:

(4) $\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$ which implies the more general result:

(5) $\lim_{u\to 0}\frac{\cos(u)-1}{u}=0$.

Returning now to finding a formula for $(\sin(x))'$, substitute (2) and (4) in (1) to find that

(6) $\frac{d(\sin(x))}{dx}=\cos(x)$, as guessed (compare with the graph in Figure 1).

Similarly:

$$\frac{d(\cos(x))}{dx}=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}=\lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(x)\sin(h)-\cos(x)}{h}=$$
$$\lim_{h\to 0}\left[\cos(x)\frac{\cos(h)-1}{h}-\sin(x)\frac{\sin(h)}{h}\right]=-\sin(x)$$

(after using again (2) and (4)).

Therefore:

(7) $\frac{d(\cos(x))}{dx}=-\sin(x)$.

(6) and (7) are important formulas for the derivatives of sin(x) and cos(x) which we will use often.

By using now the quotient rule (formula (2) in 2.2) , we can easily derive that (exercise):

(8) $\frac{d(\tan(x))}{dx} = \frac{1}{\cos^2(x)}$ and that: **(9)** $\frac{d(\cot(x))}{dx} = -\frac{1}{\sin^2(x)}$.

Add formulas (6), (7), (8) and (9) to the table of derivatives and memorize this table.

Example 2: Example 2 in section 3.3 in the textbook and exercises 1,2,10, 39 and 46 from exercise set 3.3.

Homework: exercises 7,9,12,13,15,25,27,40,42,44,49 and 52 from exercise set 3.3.

2.4. The Chain Rule:

Goal: Derive and use a rule for a composed function of the form: $\frac{d(f(u(x)))}{dx}$.

Motivation: Until now, we know how to calculate the derivatives of most of the simple algebraic, trigonometric and exponential functions. However, what if we need to calculate the derivative of composed functions such as: $\sqrt{x^2+1}$ or $\sin(x^2)$. The chain rule derived in this section will provide a general formula to calculate such derivatives (and others).

Note that:

(1) $$\frac{d(f(u(x)))}{dx} = \lim_{h\to 0} \frac{f((u(x+h)) - f(u(x))}{h} = \lim_{h\to 0} \frac{f((u(x+h)) - f(u(x))}{u(x+h)-u(x)} \cdot \frac{u(x+h)-u(x)}{h} = \frac{df(u(x))}{d(u(x))} \cdot \frac{d(u(x))}{dx}$$

Formula (1) is the chain rule, written shortly as:

(2) $\frac{d(f(u(x)))}{dx} = \frac{df(u(x))}{d(u(x))} \cdot \frac{d(u(x))}{dx}$ and which reads : The derivative of a function applied to another function is the derivative of the outer function, evaluated at the inner function times the derivative of the inner function.

Example 1:

1. Do Example 1 in section 3.4: Calculate $f'(x)$ for $f(x) = \sqrt{x^2+1}$

2. Do Example 2, that is calculate $f'(x)$ for $f(x) = \sin(x^2)$ and $g'(x)$ for $g(x) = \sin^2(x)$.

Using (1), we can easily derive:

$$\frac{d(a^x)}{dx} = \frac{d(e^{x\ln(a)})}{dx} = e^{x\ln(a)} \cdot (x\ln(a))' = e^{x\ln(a)} \cdot \ln(a) = a^x \cdot \ln(a) .$$

Therefore: **(3)** $\frac{d(a^x)}{dx} = a^x \cdot \ln(a)$. This is the generalization of formula for $\frac{d(e^x)}{dx}$, derived in 2.1, and in agreement with this.

Using the chain rule (2), we easily generalize all differentiation formula derived so far in the table of differentiation, by tagging an extra u'(x) for each, for example:

$\frac{d(x^n)}{dx} = n \cdot x^{n-1}$ for n=1,2,3 .... becomes:

(4) $\frac{d((u(x))^n)}{dx} = n \cdot (u(x))^{n-1} \cdot u'(x)$, for n=1,2,3,... and so on for the others.

Exercise:

Generalize as such all formulas in the table of derivatives to obtain the derivatives of composed functions.

Example 2: Do Examples 3,4 and 8 in section 3.4 in the textbook.

Homework: exercises 1,2,3,4,7,9,12,21,28,39,40,47,51,63, 65 and 73 from exercise set 3.4 in the textbook.

2.5 Implicit Differentiation:

In sections 2.1 to 2.4 we learnt how to calculate the derivative of a function $y = y(x)$, which is given explicitly in terms of another variable x, such as:

(1) $y = \sqrt{x^3 + 1}$ or $y = x\sin(x)$, or, in general: $y = f(x)$.

Many functions, however, are defined by an implicit relation between x and y of the form: (2) $E(x, y) = 0$. In this case we can still calculate $y'(x)$ by taking the derivative of (2) and often using chain rule . This method is called implicit differentiation.

Example 1:

Example 1 on page 210. Consider the implicit equation: (3) $x^2 + y^2 = 25$ (where we consider that $y = y(x)$).

a) Calculate $y'(x)$

b) Find the equation of the tangent line to the circle (3) $x^2 + y^2 = 25$ at the point (3,4) .

c) Look also at solution 2 in the textbook in which $y'(x)$ is found by first solving for $y(x)$ explicitly .

Note that in many cases for an equation of the form (2), $y(x)$ cannot be found (or it cannot be easily found) explicitly. Therefore the method outlined in points a) and b) above is the only method we can use to find $y'(x)$.

Example 2: Example 2 on page 211.

Consider the implicit equation : (4) $x^3 + y^3 = 6xy$.

a) Find $y'(x)$ by implicit differentiation ;

b) Find the equation of the tangent line to this curve (the folium of Descartes) at (3,3).

c) at what points in the first quadrant (x_0, y_0) (with $x_0 > 0$ and $y_0 > 0$) on this curve is the tangent line horizontal ?

Example 3:

Consider (5) $y^3 + 7y = x^3$. Calculate $y'(0)$ and $y'(2)$.

Important Note:

The simplicity of the process above hides some difficulties, though. That is, in an implicit formula of the type (2), $y(x)$ may not be defined (or it may not be a function of x at x), or $f'(x)$ may not exist at x. However, if (2) determines $y = f(x)$ **as a function of** x (that is, if $y = f(x)$ can be found **uniquely** from (2)), and if this $f(x)$ is differentiable at x, then the derivative y' found as in the process above by implicit differentiation" gives us the correct value of $f'(x)$.

Example 4:

Consider (6) $x^2 + y^2 = 1$. …

By implicit differentiation, we find $y'(x) = -\frac{x}{y}$. This is correct.

What is, however, $y'\left(\frac{1}{2}\right)$?

Note that (6) does not necessarily gives y as a function y=f(x), unless extra conditions are specified (such as $y \geq 0$).

Besides, even if we assume that (6) gives a function y=f(x) at some x_0, this function may not be differentiable at x_0 !...

In the problems in our textbook, though, we assume that an implicit equation (2) determines a differentiable function at any x_0 where we want to calculate derivatives, such that implicit differentiation gives the correct derivative.

Example 5: Example 4 on page 213:

Find y'' if $x^4 + y^4 = 16$.

Example 6: Do problem 48 from exercise set 3.5 to show that (7) $\left(y^r\right)' = \left(y^{\frac{m}{n}}\right)' = \frac{m}{n} y^{\frac{m}{n}-1} = r\, y^{r-1}$ using implicit differentiation.

From now on, we consider (7) correct and we use it.

(7) can also be shown for any **real power** r as well. .

Homework:

Problems 1,2,3,5,6,10,16,21,28,33,35,36,43 and 65 from Exercise Set 3.5 .

2.6 Derivatives of logarithmic functions. Logarithmic Differentiation .

In this section we will complete the table of derivatives with formulas for the derivatives of the logarithmic functions: $\log_a(x)$ and $\ln(x)$.

We will also learn a technique called logarithmic differentiation which allows us to calculate relatively quickly the derivatives of complicated expressions, typically involving products, fractions and / or powers and exponents.

I. The derivatives of $\log_a(x)$ and $\ln(x)$:

First we show that: **(1)** $\frac{d(\log_a(x))}{dx} = \frac{1}{x\ln(a)}$. (note that in (1) we need : $a > 0$ and $x > 0$. Why?)

To show (1), recall that **(2)** $y = \log_a(x) \leftrightarrow a^y = x$

Use implicit differentiation in (2) and recall that (see 2.4): $\frac{d\left(a^{u(x)}\right)}{dx} = a^{u(x)} \cdot \ln(a) \cdot u'(x)$. Therefore:

$a^y \cdot \ln(a) \cdot y'(x) = 1 \rightarrow y'(x) = \frac{1}{a^y \ln(a)} = \frac{1}{x \ln(a)}$ (from (2)), which shows (1).

Consider now $a = e$ in (1) to find that: **(3)** $\frac{d(\ln(x))}{dx} = \frac{1}{x}$.

Formulas (1) and (3) give the derivatives of logarithmic functions and they can be generalized for composite functions as:

(4) $\frac{d(\log_a(u(x)))}{dx} = \frac{u'(x)}{u(x)\ln(a)}$ and **(5)** $\frac{d(\ln(u(x)))}{dx} = \frac{u'(x)}{u(x)}$.

The simplicity of formulas (3) and (5) makes the natural logarithm (and the natural exponential) very useful in Calculus (and in other sciences).

Add formulas (1), (3), (4) and (5) to the table of derivatives and memorize all formulas in the table of derivatives.

Example 1 (page 218): Find $y'(x)$ if $y(x) = \ln\left(x^3 + 1\right)$.

Example 2: (page 219): Find $y'(x)$ if $y(x) = \ln(\sin(x))$.

Example 3: (page 219): Find $y'(x)$ if $y(x) = \sqrt{\ln(x)}$.

Example 4: (page 219): Find $y'(x)$ if $y(x) = \log_{10}(2 + \sin(x))$.

II. Logarithmic differentiation:

To calculate relatively quickly the derivatives of more complicated functions involving products, quotients, powers and/or exponents, we can apply the natural logarithm to both sides of the equation first. This technique is called logarithmic differentiation. Note that there are some cases of the form $y = f(x)^{g(x)},\ y' = ?$, when y'(x) cannot be calculated without logarithmic differentiation.

Example 7: (page 219): Calculate $y'(x)$ if $y(x) = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

Example 8 (problem 43 from Exercise set 2.6): Calculate $y'(x)$ if $y(x) = x^x$.

Homework: problems 2,5,6,10,13,15,20,24,26,31,33,39,40,44,46 and 49 from exercise set 2.6 .

2.9 Related rates :

In many practical problems, quantities change (usually with time). If we know the rate of change of one quantity: $\frac{dx}{dt}$ and a relation (1) $y = y(x)$ or (2) $E(x, y) = 0$, then we can find $\frac{dy}{dt}$ by taking the derivative (with respect to t) of (1) or (2). The rates (derivatives) $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are related, and therefore these rates of change are called **related rates**.

Example 1: (page 243):

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 $cm^3/\sec$. Considering that this process continues indefinitely, how fast is the radius of the balloon increasing when the diameter of the balloon is 50 cm ?

When solving related rates problems, it is useful to follow the following guidelines:

Step 1: Translate all given information in terms of mathematical quantities: Let t be time.

Draw, if possible, a diagram which is valid at ALL times. On this figure label or find the quantities of interest $(x(t), y(t), \ldots)$ and other given quantities (constants, etc) .

Write a formula for the given $\frac{dx}{dt}$. Express the required rate of change $\left(\frac{dy}{dt} \text{ in general}\right)$ in terms of derivatives.

Step 2: Find the relation between y and x: (3) $E(x, y) = 0$ valid at ALL times.

Step 3: Differentiate (3) and solve for $\frac{dy}{dt}$.

Example 2 (page 245):

A ladder 10 ft. long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft./sec , how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft. from the wall?

Example 3 (page 245) :

EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Example 4 (page 247):

V EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Homework: problems 2,5,7,10,11,12,15,16,21,23,26,37,44,45,46 from problem set 3.9.

2.10 Chapter Review

In this chapter, we :

- Learned the definition of the derivative of a function at a point: $f'(a)$, and its interpretation and use in terms of the slope of the tangent line m_T, instantaneous velocity $v(a)$ and instantaneous rate of change of a function: $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (Section 2.0.1).
- Learned the definition of $f'(x)$, its interpretation as a function, and the notion of differentiable functions. Cases of non-differentiability and the graphical meaning of differentiable and non-differentiable functions (Section 2.0.2).
- Derived and learned (memorized) formulas for the derivatives of x^a, e^x, a^x, polynomials, $f \pm g, \mathrm{c} \cdot f, f \cdot g, \frac{f}{g}, f(g(x))$, trigonometric and logarithmic functions and organized these in a table of derivatives (Sections 2.1, 2.2, 2.3 and 2.4 and 2.6);
- Learned how to calculate derivatives of functions defined implicitly using implicit differentiation (Section 2.5);
- Learned how to solve related rates problems and how to use logarithmic differentiation (Section 2.9 and 2.6).

Learn the main results (definitions, theorems and formulas from these sections) (best write a review guide to include these) , then solve the review problems from pages 263 to 267 in the textbook, including the Concept Check, True/ False and Exercises types of problems.