

**Name:** \_\_\_\_\_

**Period:** \_\_\_\_\_

**Date:** \_\_\_\_\_

**AP Calc AB**

**Mr. Mellina**

## **Chapter 2: Limits & Continuity**

*Sections:*

- ❖ *2.1 Rates of Change of Limits*
- ❖ *2.2 Limits Involving Infinity*
- ❖ *2.3 Continuity*
- ❖ *2.4 Rates of Change and Tangent Lines*

*HW Sets*

*Set A (Section 2.1) Page 66, #'s 1-15 odd, 16, 25-33 odd.*

*Set B (Section 2.1) Pages 67 & 68, #'s 43, 44, 45-50, 57, 58.*

*Set C (Section 2.2) Page 76, #'s 1-29 odd, 35-38, 41, 53.*

*Set D (Section 2.3) Page 84, #'s 1-10 all, 11-15 odd, 19-35 odd.*



**KNOW  
LIMITS**

## Introduction to Chapter 2

In previous math and science course, you have used the formula  $d = rt$  (or distance = rate multiplied by time) to determine the speed, distance, or time of an object. The “rate” found was the object’s average speed.

A moving body’s **average speed** during an interval of time is found by dividing the total distance covered by the elapsed time. Speed is always positive because the distance covered is positive as is the elapsed time.

To find the **average velocity**, we also divide the total distance covered by the elapsed time. Average velocity, however, can be a negative value when the distanced covered is negative (ex: an object falling from a cliff).

If an object is dropped from an initial height of  $h_0$ , we can use the position function  $s(t) = -16t^2 + h_0$  to model the height,  $s$  (in feet), of an object that has fallen for  $t$  seconds.

### Example 1

Wile E. Coyote, once again trying to catch the Road Runner, waits for the nastily speedy bird atop a 900 foot cliff. With his Acme Rocket Pac strapped to his back, Wile E. is poised to leap from the cliff, fire up his rocket pack, and finally partake of a juicy road runner roast. Finally, the Road Runner zips by and Wile E. leaps from the cliff. Alas, as always, the rocket malfunctions and fails to fire, sending poor Wile E. plummeting to the road below disappearing into a cloud of dust.

- a. What is the position function for the Wile E. Coyote?
- b. Find Wile E.’s average velocity for the first 3 seconds.
- c. Find Wile E.’s average velocity between  $t = 2$  and  $t = 3$  seconds.
- d. Find Wile E.’s velocity at the instant  $t = 3$  seconds.

The problem with part (d) is that we are trying to find the instantaneous velocity. Since we have not yet studied the concept of a limit, we could not find the answer to part d. For now, let's solve this problem by studying what happens to the velocity as we get "close" to 3 seconds.

e. Find the average velocity between  $t = 2.5$  and  $t = 3$  seconds.

f. Find the average velocity between  $t = 2.9$  and  $t = 3$  seconds.

g. Find the average velocity between  $t = 2.99$  and  $t = 3$  seconds.

h. Find the average velocity between  $t = 2.999$  and  $t = 3$  seconds.

So, even though we cannot find the average velocity at exactly  $t = 3$  seconds, we can discover what Wile E.'s velocity is approaching at  $t = 3$  seconds. What do you think the answer for part (d) should be?

## 2.1 Rates of Change and Limits

### Topics

- ❖ *Average and Instantaneous Speed*
  - ❖ *Definition of Limit*
  - ❖ *Properties of Limits*
- ❖ *One-Sided and Two-Sided Limits*
  - ❖ *Sandwich Theorem*

### Warm Up!

Find  $f(2)$  without a calculator

a.  $f(x) = \frac{4x^2 - 5}{x^3 + 4}$

b.  $f(x) = \begin{cases} 3x - 1, & x < 2 \\ \frac{1}{x^2 - 1}, & x \geq 2 \end{cases}$

### Example 1: Average and Instantaneous Speed

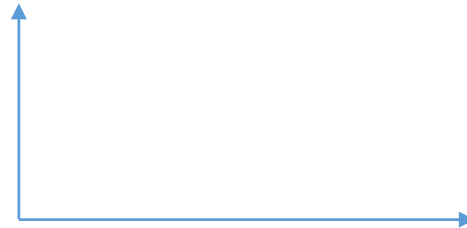
Suppose you drive 200 miles, and it takes you 4 hours

- a. What is your average speed?
- b. If you look at your speedometer during this trip, it might read 65 mph. What is this speed called?

#### ***Average Speed***

*Average speed of a moving body during an interval of time is found by dividing the \_\_\_\_\_ covered by the elapsed \_\_\_\_\_. If  $y = f(t)$  is a distance or position function of a moving body at time  $t$ , then the **average rate of change (or average speed)** is the ratio:*

$$\text{---} = \frac{\text{---}}{\text{---}}$$



*where the elapsed time is the interval of time  $t$  to  $t + h$ , or simply  $h$ .*

**Example 2: Finding an Average Speed**

A rock breaks loose from the top of a tall cliff and falls  $y = 16t^2$  feet in the first  $t$  seconds. Find the following.

- a. Its average speed during the first 2 seconds of fall?
  
  
  
  
  
  
  
  
  
  
- b. Its average speed during the first 4 seconds of fall
  
  
  
  
  
  
  
  
  
  
- c. What is the *instantaneous* speed at 2 seconds?

***Instantaneous Speed***  
*A moving body's instantaneous speed is the speed of the moving object at a given \_\_\_\_\_ of \_\_\_\_\_.*

*Above we can see that the velocity approaches 64ft/sec as  $h$  becomes very small. We say that the velocity has a \_\_\_\_\_ value of 64 as  $h$  \_\_\_\_\_ zero.*

- d. Confirm part (c) from above algebraically

## Limits

Most limits of interest in the real world can be viewed as numerical limits of values of functions. Limits give us a language for describing how the outputs of a function behave as inputs approach some particular value. In Example 2, the average speed was not defined at  $h=0$  but approached the limit 64 as  $h$  approached 0.

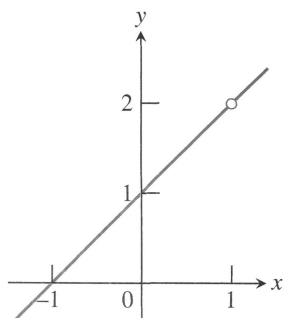
$$\lim_{x \rightarrow 1} f(x) =$$

“The limit of  $f(x)$  as  $x$  approaches  $c$  equals  $L$ ”

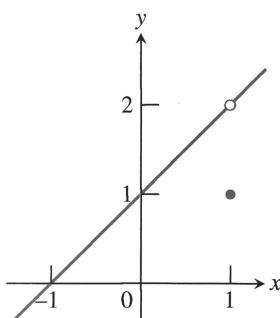
The limit of a function is:

- An \_\_\_\_\_, \_\_\_\_\_-value.
- A \_\_\_\_\_, \_\_\_\_\_ value.
- A \_\_\_\_\_-value that may or may not be \_\_\_\_\_ and we don't care!
- An \_\_\_\_\_ the function gets really \_\_\_\_\_ as the \_\_\_\_\_ gets really close to a number.

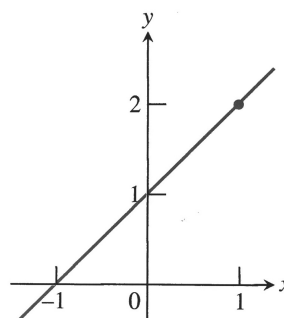
The existence of a limit as  $x \rightarrow c$  never depends on how the function may or may not be defined at  $c$ .



(a)  $f(x) = \frac{x^2 - 1}{x - 1}$



(b)  $g(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$



(c)  $h(x) = x + 1$

$$\lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} g(x) = \lim_{x \rightarrow 1} h(x) = 2$$

## Theorem 1: Properties of Limits

If  $L$ ,  $M$ ,  $c$ , and  $k$  are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \text{ and } \lim_{x \rightarrow c} g(x) = M, \text{ then}$$

1. **Sum Rule:** The limit of the sum of two functions is the sum of their limits

$$\lim_{x \rightarrow c} (f(x) + g(x)) =$$

2. **Difference Rule:** The limit of the difference of two functions is the difference of their limits.

$$\lim_{x \rightarrow c} (f(x) - g(x)) =$$

### *Properties of Limits continued...*

3. *Product Rule: The limit of a product of two function is the product of their limits*

$$\lim_{x \rightarrow c} (f(x) \cdot g(x)) =$$

4. *Constant Multiple Rule: The limit of a constant times a function is the constant times the limit of the functions.*

$$\lim_{x \rightarrow c} (kf(x)) =$$

5. *Quotient Rule: The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.*

$$\lim_{x \rightarrow c} \left( \frac{f(x)}{g(x)} \right) =$$

6. *Power Rule: If  $r$  and  $s$  are integers,  $s \neq 0$ , then*

$$\lim_{x \rightarrow c} (f(x))^{r/s} =$$

### **Example 3: Using Properties of Limits**

Use the given information to evaluate the limits:  $\lim_{x \rightarrow c} f(x) = 2$  and  $\lim_{x \rightarrow c} g(x) = 3$

a.  $\lim_{x \rightarrow c} [5g(x)]$

b.  $\lim_{x \rightarrow c} [f(x) + g(x)]$

c.  $\lim_{x \rightarrow c} [f(x)g(x)]$

d.  $\lim_{x \rightarrow c} \left[ \frac{f(x)}{g(x)} \right]$

### Methods of Evaluating Limits

1. \_\_\_\_\_ Analysis.
  - a. Does the appear to approach a y-value as x approaches the limit?
2. \_\_\_\_\_ Analysis.
  - a. Evaluating the function at x-values very close to the limit on both sides.
3. \_\_\_\_\_
  - a. Direct Substitution
  - b. If Direct substitution yields an indeterminate form  $\left(\frac{0}{0}\right)$ , then use some algebraic technique to simplify

### Example 4: Evaluating Limits

Evaluate the following limit algebraically as x approaches the real number provided.

a.  $\lim_{x \rightarrow 3} [x^2(2 - x)]$

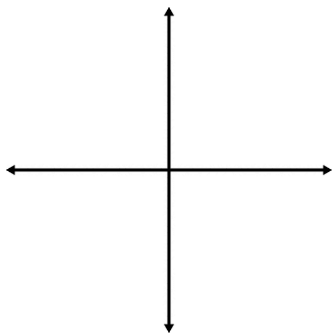
b.  $\lim_{x \rightarrow 1} \frac{\sqrt{5-x}-2}{x-1}$

c.  $\lim_{x \rightarrow 0} \left[ \frac{1 - \cos x}{x} \right]$

### Example 5: Special Limits

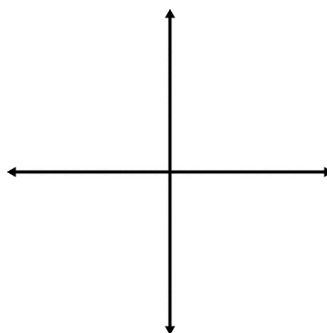
Investigate the following limit graphically and with numerical analysis.

a.  $\lim_{x \rightarrow 0} \left[ \frac{\sin x}{x} \right]$



x	y

b.  $\lim_{x \rightarrow 0} \left[ \frac{\sin 5x}{5x} \right]$



x	y

c.  $\lim_{\odot \rightarrow 0} \frac{\sin \odot}{\odot}$

d.  $\lim_{x \rightarrow 0} \left[ \frac{\sin 5x}{x} \right]$



### Example 6: Using the Product Rule

Determine the following limit

a.  $\lim_{x \rightarrow 0} \frac{\tan x}{x}$

b.  $\lim_{x \rightarrow 0} \frac{\sin^2 x}{x}$

c.  $\lim_{x \rightarrow 2} \frac{x^3 - 1}{x - 2}$

#### One-Sided and Two-Sided Limits

Sometimes the values of a function  $f$  tend to different limits as  $x$  approaches a number  $c$  from opposite sides. When this happens, we call the limit of  $f$  as  $x$  approaches  $c$  from the right the \_\_\_\_\_ hand limit of  $f$  at  $c$  and the limit as  $x$  approaches  $c$  from the left the \_\_\_\_\_ hand limit of  $f$  at  $c$ . Here is the notation we use:

Right-hand: "the limit of  $f$  as  $x$  approaches  $c$  from the right"

Left-hand: "the limit of  $f$  as  $x$  approaches  $c$  from the left"

#### Theorem 3

A function  $f(x)$  has a limit as  $x$  approaches  $c$  if and only if the right-hand and left-hand limits at  $c$  \_\_\_\_\_ and are \_\_\_\_\_.

### Example 7: Exploring Right- and Left-Hand Limits

Let  $f(x)$  be the graph provided. Find the limit.

a.  $\lim_{x \rightarrow 2^-} f(x)$

b.  $\lim_{x \rightarrow 2^+} f(x)$

c.  $f(2)$

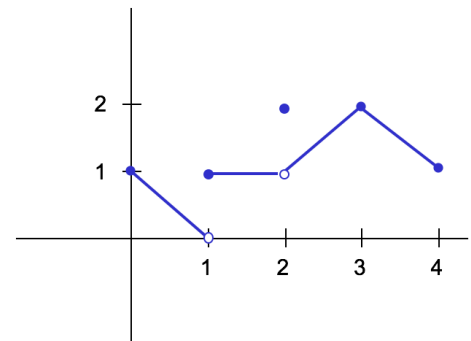
d.  $\lim_{x \rightarrow 2} f(x)$

e.  $\lim_{x \rightarrow 1^-} f(x)$

f.  $\lim_{x \rightarrow 1^+} f(x)$

g.  $\lim_{x \rightarrow 1} f(x)$

h.  $\lim_{x \rightarrow 3} f(x)$



## **2.2 Limits Involving Infinity**

### *Topics*

- ❖ *Finite Limits as  $x \rightarrow \pm\infty$*
- ❖ *Sandwich Theorem Revisited*
- ❖ *Infinite Limits as  $x \rightarrow a$*
- ❖ *End Behavior Models*
- ❖ *“Seeing” Limits as  $x \rightarrow \pm\infty$*

### **Warm Up!**

Consider the function  $f(x) = \frac{1}{x}$

- a. What happens to  $f(x)$  as the denominator gets larger?
  
  
  
  
  
  
  
  
  
  
- b. Are there any asymptotes for  $f(x)$ ? If so, find them.

#### ***Finite Limits as $x \rightarrow \pm\infty$***

*We use  $\infty$  to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. When we say, “the limit of  $f$  as  $x$  approaches infinity” we mean the limit of  $f$  as  $x$  moves increasingly far to the \_\_\_\_\_ on a number line. When we say, “the limit of  $f$  as  $x$  approaches negative infinity” we mean the limit of  $f$  as  $x$  moves increasingly far to the \_\_\_\_\_.*

#### **Example 1: Finite Limits as $x \rightarrow \pm\infty$**

Find the following limits

a.  $\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)$

b.  $\lim_{x \rightarrow -\infty} \left(\frac{1}{x}\right)$

#### ***Horizontal Asymptote***

*The line  $y = b$  is a horizontal asymptote of the graph of a function  $y = f(x)$  if either*

$$\lim_{x \rightarrow \infty} f(x) =$$

$$\lim_{x \rightarrow -\infty} f(x) =$$

c.  $\lim_{x \rightarrow \infty} \left[ \frac{2x^4}{x^2+1} \right]$

**End Behavior Model**

For a rational function  $\frac{ax^n+\dots}{bx^d+\dots}$ , where \_\_\_\_ is the degree of the numerator and \_\_\_\_ is the degree of the denominator. The end behavior model can be written as:

	End Behavior Model	End Behavior	Asymptote
Degrees are equal $n = d$			
Larger degree in denominator $n < d$			
Larger degree in numerator $n > d$			

**Example 2: Looking for Horizontal Asymptotes**

For each example below, write the end behavior model, evaluate the limits to + & - infinity, and determine any horizontal or slant asymptotes.

a.  $f(x) = \frac{2x+5}{3x^2-6x+1}$

b.  $f(x) = \frac{2x^2-3x+5}{x^2+1}$

c.  $f(x) = \frac{x^2+x+9}{3x-3}$

**Properties of Limits as  $x \rightarrow \pm\infty$**

Limits at infinity have the same properties as those of infinite limits. Review the previous section for properties.

A general rule for functions that are divided is

- If the denominator \_\_\_\_\_ faster than the numerator, the limit as  $x$  approaches infinity will be \_\_\_\_\_.
- If the numerator \_\_\_\_\_ faster than the denominator, then as  $x$  approaches infinity, the limit will increase or decrease without bound.
- If the numerator and denominator \_\_\_\_\_ at the same rate, look for the a limit of a non-zero constant.

**Example 3: Using the Quotient Rule of Limits Approaching Infinity**

Find the limit.

a.  $\lim_{x \rightarrow \infty} \left( \frac{5x + \sin x}{x} \right)$

b.  $\lim_{x \rightarrow \infty} \left( \frac{\sin x}{2x^2 + x} \right)$

**Infinite Limits as  $x \rightarrow a$** 

If the values of a function  $f(x)$  outgrow all positive bounds as  $x$  approaches a finite number  $a$ , we say that  $\lim_{x \rightarrow a} f(x) = \infty$ . Similarly, if the values of  $f$  become large and negative, exceeding all negative bounds as  $x$  approaches  $a$ , we say that  $\lim_{x \rightarrow a} f(x) = -\infty$ .

**Vertical Asymptotes**

The line  $x = a$  is a vertical asymptote of the graph of a function  $y = f(x)$  if either

$$\lim_{x \rightarrow a^+} f(x) = \infty$$

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

**Example 4: Finding Vertical Asymptotes**

Find the vertical asymptotes of the following functions. Describe the behavior to the left and right of each vertical asymptote.

a.  $f(x) = \frac{1}{x}$

b.  $f(x) = \frac{1}{x^2}$

c.  $f(x) = \frac{1}{x^2 - 4}$

d.  $f(x) = \frac{6x - 3x^2}{x^2 - 4}$

## 2.3 Continuity

### Topics

- ❖ Continuity at a Point
- ❖ Continuous Functions
- ❖ Algebraic Combinations
  - ❖ Composites
- ❖ Intermediate Value Theorem for Continuous Functions

### Warm Up!

Complete the following without a calculator

1. Find  $\lim_{x \rightarrow -1} \left( \frac{3x^2 - 2x + 1}{x^3 + 4} \right)$

2. Let  $f(x) = \text{int}x$ . Find each

a.  $\lim_{x \rightarrow -1^-} f(x)$

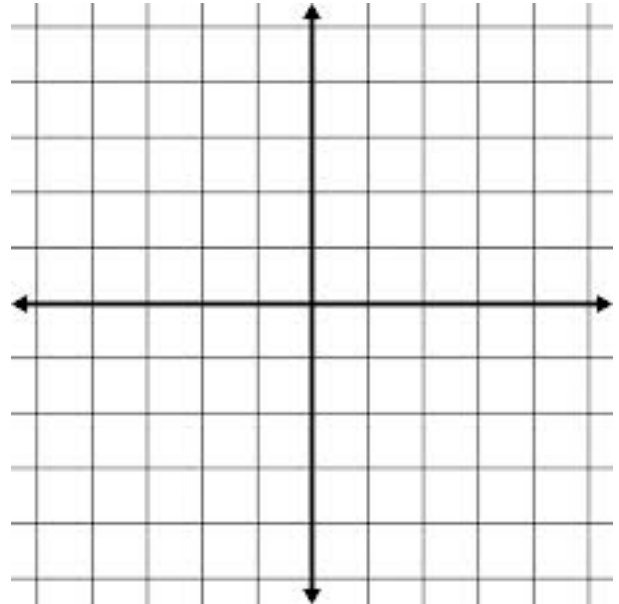
b.  $\lim_{x \rightarrow -1^+} f(x)$

c.  $\lim_{x \rightarrow -1} f(x)$

d.  $f(-1)$

### Example 1: Investigating Continuity

Sketch the graph of  $f(x) = \left\{ \begin{array}{ll} x, & (-\infty, 0] \\ x^2 - 4, & (0, \infty) \end{array} \right\}$



Does  $f(x)$  appear to be continuous?

### Continuity

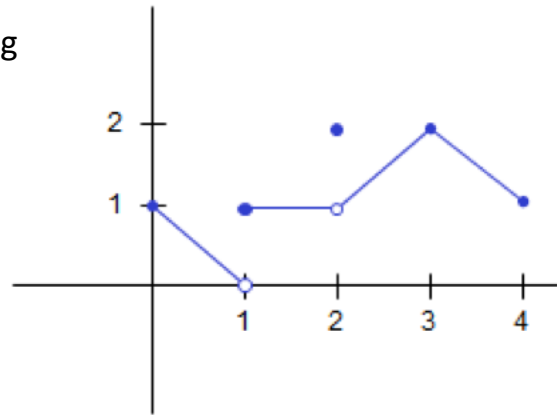
Most of the techniques of calculus require that functions be \_\_\_\_\_. A function is continuous if you can draw it in one motion without picking up your pencil.

A function is continuous at a point if the \_\_\_\_\_ is the same as the \_\_\_\_\_ of the function

### Example 2: Investigating Continuity

Use the diagram at the right to answer the following

- Is the function continuous from  $[0,4]$ ?
- Does the function have any values of  $x$  that are discontinuities? If so, where?
- Is the function continuous at  $x = 0$  and  $x = 4$ ?



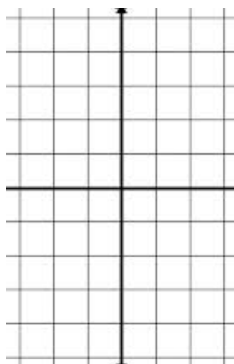
### Functions are Continuous at $c$ if

- $f(c)$  is \_\_\_\_\_.
- $\lim_{x \rightarrow c} f(x)$  \_\_\_\_\_.
- $\lim_{x \rightarrow c} f(x) =$  \_\_\_\_\_.

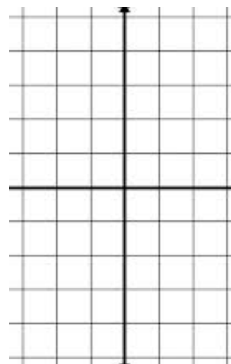
### Example 3: Understanding Discontinuity

Graph the following functions from the interval  $(-3,3)$ . Do you think you can trust the results you obtained graphically to determine if the function is continuous or not?

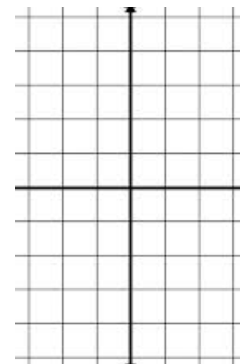
a.  $y = x^2 + 1$



b.  $y = \frac{1}{x-2}$



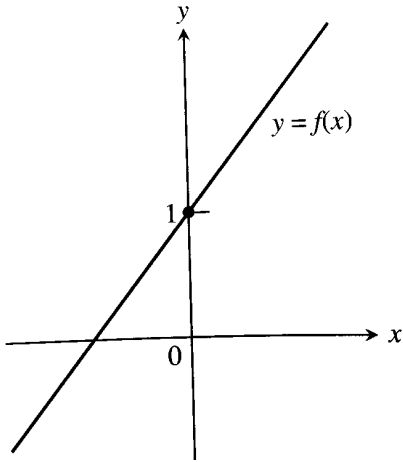
c.  $y = \frac{x^2-4}{x+2}$



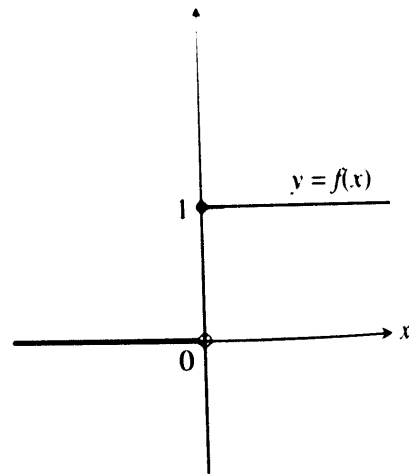
### Example 4: The Different Types of Discontinuity

Label the following graphs with the type of discontinuity: *Removable discontinuity*, *Non-Removable discontinuity*, *Jump discontinuity*, *infinite discontinuity*, or *Oscillating Discontinuity*.

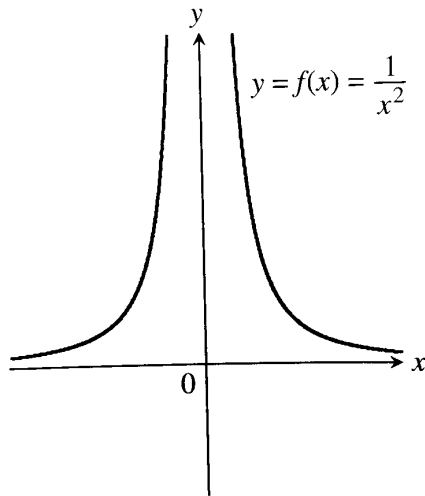
a.



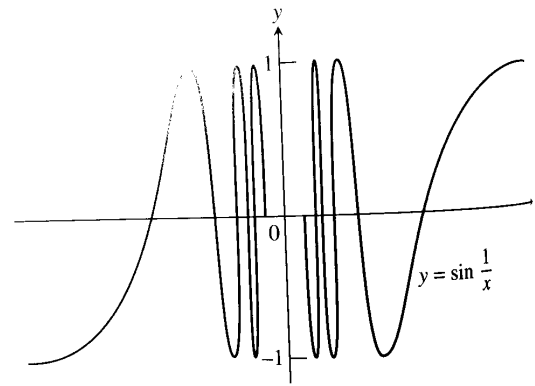
b.



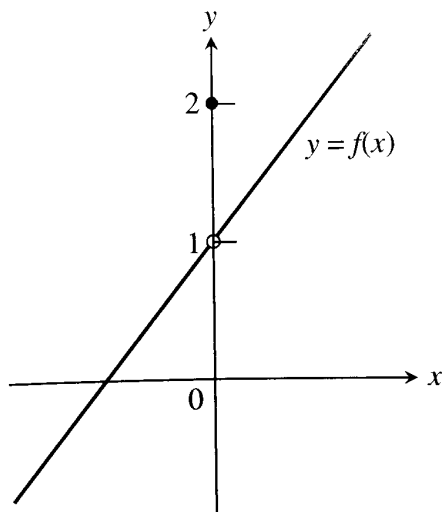
c.



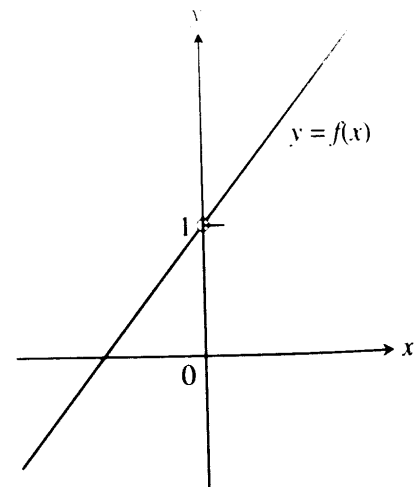
d.



e.



f.





**Example 5: Testing for Continuity**

Test for continuity at the indicated points. Clearly show the three things that make a function continuous at a point.

a.  $f(x) = 2x - 1$  at  $x = 2$

b.  $g(x) = \frac{x^2-4}{x-2}$  at  $x = 2$

c.  $h(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$

d.  $k(x) = \begin{cases} \frac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2 \end{cases}$

e.  $h(x) = \begin{cases} x, & x \geq 0 \\ x + 1, & x < 0 \end{cases}$

**Example 7: Using Continuity**

For each of the following, find the vertical asymptote, the removable discontinuity, the non-removable discontinuity, and the horizontal asymptote (when applicable).

a.  $f(x) = \frac{1}{2x+2}$

b.  $f(x) = \frac{x^2+2x-8}{x^2-4}$

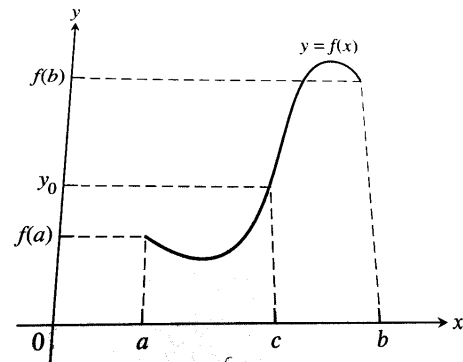
### ***Intermediate Value Theorem***

*Functions that are continuous on intervals have properties that make them particularly useful in mathematics and its applications. A function is said to have the intermediate value property if it never takes on two values without taking on all the values \_\_\_\_\_.*

*If you are 5ft on your 13<sup>th</sup> birthday and on your 14<sup>th</sup> birthday you are 5'6", then at some point you had to be 5'4"*

*A function  $y = f(x)$  that is continuous on a closed interval  $[a,b]$  takes on every value \_\_\_\_\_  $f(a)$  and  $f(b)$ .*

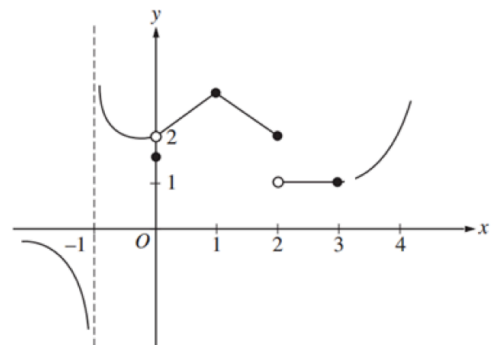
*In other words, if  $y_0$  is between  $f(a)$  and  $f(b)$ , then  $y_0 = f(c)$  for some  $c$  in  $[a,b]$ .*



### **Example 7: Using the Intermediate Value Theorem**

- a. Let  $f(x) = x^3 - x - 1$ . Use the IVT to show that there is at least one root on  $[1,2]$ .
- b. Let  $f(x) = x^2 + 2x - 1$ . Use the IVT to show that there is at least one root on  $[-1,1]$ .

- c. The graph of a function  $f$  is shown to the right. If  $\lim_{x \rightarrow b} f(x)$  exists and  $f$  is not continuous at  $b$ , then what is the value of  $b$ ?



d. What is the value of  $k$  to make  $h(x)$  a continuous function when

$$h(x) = \begin{cases} \frac{k}{x^2}, & x < -2 \\ 9 - x^2, & x \geq -2 \end{cases}$$

e. Let  $f$  be a continuous function. Selected values of  $f$  are given in the table below.

$x$	1	2	3	5	8
$f(x)$	-2	3	1	-5	7

What is the least number of solutions that the equation  $f(x) = \frac{1}{2}$  have on the closed interval  $1 \leq x \leq 8$ . [Hint: Draw a picture first]

f. Let  $f$  be a continuous function on  $[0,2]$ . Selected values of  $f$  are given in the table below.

$x$	0	1	2
$f(x)$	1	K	2

The equation  $f(x) = \frac{1}{2}$  has at least two solutions in the interval  $[0,2]$  for what value of  $k$ ?

## **Worksheet #1: Finding Limits Numerically & Graphically**

Use a graphing calculator to find the following limits either graphically or numerically. Show a sketch of the graph or reproduce part of the table as evidence.

1.  $\lim_{x \rightarrow 0^+} \left( \frac{x+1}{x} \right)$

2.  $\lim_{x \rightarrow -1^+} \left( \frac{x}{x+1} \right)$

3.  $\lim_{x \rightarrow 3^+} \left( \frac{x-4}{x-3} \right)$

4.  $\lim_{x \rightarrow 1} \left( \frac{x-3}{x^2-1} \right)$

5.  $\lim_{x \rightarrow 0} \left( \frac{|x|}{x} \right)$

6.  $\lim_{x \rightarrow 1^-} \left( \frac{x-3}{x^2-1} \right)$

$$7. \quad \lim_{x \rightarrow 2} \left( \frac{x-2}{\sqrt{x^2-4}} \right)$$

$$8. \quad \lim_{x \rightarrow 2} f(x), \text{ if } f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

$$9. \quad \lim_{\theta \rightarrow 0} \left( \frac{\tan \theta}{\theta} \right)$$

$$10. \quad \lim_{t \rightarrow 0} \left( \frac{1-\cos t}{t^2} \right)$$

$$11. \quad \lim_{x \rightarrow 0} \left( \frac{\sin 2x}{x} \right)$$

$$12. \quad \lim_{x \rightarrow 9} \left( \frac{\sqrt{x}-3}{x-9} \right)$$

## **Worksheet #2: Finding Limits Analytically**

Find the following limits *WITHOUT* using a graphing calculator. Show your work.

1.  $\lim_{x \rightarrow -1} \left( \frac{x^2 + 3x + 2}{x^2 + 1} \right)$

2.  $\lim_{x \rightarrow 1} \left( \frac{3x^3 - 4x^2 - 5x + 2}{x^2 - x - 2} \right)$

3.  $\lim_{x \rightarrow 1} f(x), \text{ if } f(x) = \begin{cases} x^2 + 4, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$

4.  $\lim_{x \rightarrow \pi} \left( \frac{1 - \cos x}{x} \right)$

5.  $\lim_{x \rightarrow \pi} (\tan 5x)$

6. If  $\lim_{x \rightarrow c} f(x) = -\frac{1}{2}$ ,  $\lim_{x \rightarrow c} g(x) = \frac{2}{3}$ , find  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$

$$7. \quad \lim_{x \rightarrow 2} \left( \frac{x-2}{x^2-4} \right)$$

$$8. \quad \lim_{x \rightarrow -9} \left( \frac{x^2+6x-27}{x+9} \right)$$

$$9. \quad \lim_{x \rightarrow 2} \left( \frac{x-2}{|x-2|} \right)$$

$$10. \quad \lim_{x \rightarrow 4} \left( \frac{x^2-5x+4}{x-4} \right)$$

$$11. \quad \lim_{x \rightarrow 0} \left( \frac{\sqrt{x+4}-2}{x} \right)$$

$$12. \quad \lim_{\Delta x \rightarrow 0} \left( \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \right)$$

$$13. \quad \lim_{x \rightarrow 0} \left( \frac{\sin 5x}{4 \sin 6x} \right)$$

$$14. \quad \lim_{x \rightarrow 0} \left( \frac{\sin x^2}{3x^2} \right)$$

## **Worksheet #3: Review for Limits Quiz**

For numbers 1-10, evaluate the following limits *WITHOUT* using a graphing calculator. Show your work.

1.  $\lim_{x \rightarrow \frac{5}{2}} [x]$

2.  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + 5x - 3}{3x + 2} \right)$

3.  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + 5x - 3}{3x^2 + 2} \right)$

4.  $\lim_{x \rightarrow \infty} \left( \frac{x^2 + 5x - 3}{3x^3 + 2} \right)$

5.  $\lim_{x \rightarrow 0} \left( \frac{x}{\sin(2x)} \right)$

6.  $\lim_{x \rightarrow \infty} \left( \frac{\sin x}{2x} \right)$

7.  $\lim_{x \rightarrow 0} \left( \frac{\tan(5x)}{\sin(3x)} \right)$

8.  $\lim_{x \rightarrow \infty} \left( \frac{4x^2 + 5x}{-x - 3} \right)$



9. 
$$\lim_{x \rightarrow \infty} \left( \frac{5x - 7x^2}{4x^2 + 1} \right)$$

10. 
$$\lim_{x \rightarrow -3} \frac{|x+3|}{x+3}$$

11. Use a table of values to evaluate the following limit: 
$$\lim_{x \rightarrow \infty} \left( 1 + \frac{1}{x} \right)^x$$

12. Make a table of values (4 of them would work) to evaluate 
$$\lim_{x \rightarrow 2} \left( \frac{x+3}{x-2} \right)$$

**For questions 13 & 14, find ALL asymptotes (vertical, horizontal, and oblique) and justify your response.**

13.  $y = \ln(x)$

14. 
$$f(x) = \frac{(x+2)(x-3)}{(x+2)(x-1)}$$

15. Let  $h(x) = \frac{(x-1)(x+3)}{(x+3)(x-2)}$ . Identify all values of  $c$  where the  $\lim_{x \rightarrow c} h(x)$  EXISTS.

16. Let  $g(x) = \frac{x^2+5x+6}{x^2+3x+2}$ ,

a. Find the domain of  $g(x)$

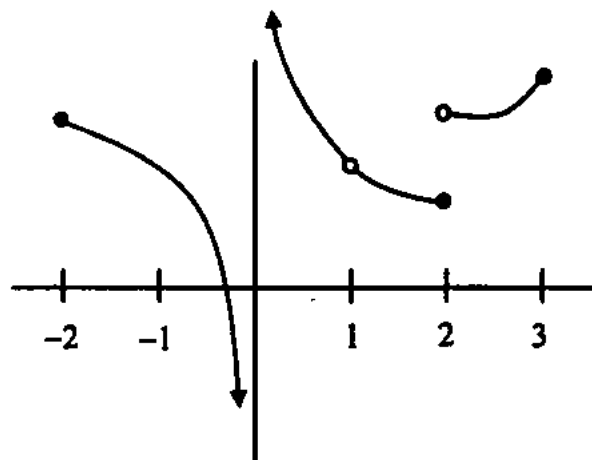
b. Find the  $\lim_{x \rightarrow c} g(x)$  for all values of  $c$  where  $g(x)$  is not defined.

c. Find any horizontal asymptotes and justify your response.

d. Find any vertical asymptotes and justify your response.

e. Write an extension to the function so that  $g(x)$  is continuous for all  $x < -1$ .

17. Using the function below, over what intervals does  $\lim_{x \rightarrow c} f(x)$  exist?



## **Worksheet #4: Continuity Practice**

*All work must be shown in this course for full credit.*

1. What is the definition of continuity?
  
2. Sketch a possible graph for each function described
  - a.  $f(5)$  exists, but  $\lim_{x \rightarrow 5} f(x)$  does not exist.
  
  - b. The  $\lim_{x \rightarrow 5} f(x)$  exists, but  $f(5)$  does not exist.

3. If  $f(x) = \left\{ \begin{array}{ll} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{array} \right\}$  and if  $f$  is continuous at  $x = 2$ , then what is  $k$ ?

4. Let  $f(x) = \begin{cases} x^2 - a^2x, & x < 2 \\ 4 - 2x^2, & x \geq 2 \end{cases}$ , find all values of  $a$  that make  $f$  continuous at 2.

5. Let  $f(x) = \begin{cases} \sin(x), & x < 0 \\ x^2, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2 \\ x - 3, & x \geq 2 \end{cases}$  For what values of  $x$  is  $f$  NOT continuous?

6. If the function  $f$  is continuous for all real numbers and if  $f(x) = \frac{x^2-4}{x+2}$ , when  $x \neq -2$ , then find  $f(-2)$ .

7. Let  $f$  be the function given by  $f(x) = \frac{(x-1)(x^2-4)}{x^2-a}$ . For what positive values of  $a$  is  $f$  continuous for all real numbers?

- a. None                      b. 1 only                      c. 2 only                      d. 4 only                      e. 1 and 4

8. Let  $g(x) = \frac{x^2+5x+6}{x^2+7x+10}$ .

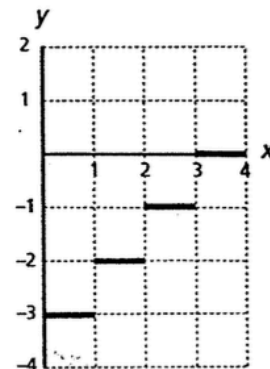
- a. Find the domain of  $g(x)$
  - b. Find the  $\lim_{x \rightarrow c} g(x)$  for all values of  $c$  where  $g(x)$  is not defined.
  - c. Find any horizontal asymptotes and justify your response.
  - d. Find any vertical asymptotes and justify your response.
  - e. Write an extension to the function so that  $g(x)$  is continuous at  $x = -2$ .
9. Without using a picture, give a written explanation of why the function  $f(x) = x^2 - 4x + 3$  has a zero in the interval  $[2, 4]$

## **Worksheet #5: Continuity Review**

*All work must be shown in this course for full credit.*

1. Is  $f(x) = x^3 + 2x + 1$  continuous at  $x = 2$  ?
2. Is  $f(x) = \frac{x^2-16}{x-4}$  continuous at  $x = 4$  ?
3. Is the following function continuous?  $f(x) = \begin{cases} x + 3, & x < 2 \\ 5, & x = 2 \\ x^4 - 11, & x > 2 \end{cases}$
4. Is the following function continuous?  $f(x) = \begin{cases} x^3 - 1, & x < 3 \\ x^2 + 14, & x \geq 3 \end{cases}$
5. Is there a way to define  $f(c)$  for the  $f(x) = \frac{x^2-16}{x-4}$ , at  $x = 4$ , so that  $f(x)$  is continuous at  $x = c$  ?
6. Is there a way to define  $f(c)$  for the  $f(x) = \frac{8x}{x-1}$ , at  $x = 1$ , so that  $f(x)$  is continuous at  $x = c$  ?
7. How can we be sure that the function  $f(x) = 3x^3 - 2x^2 - 31$  has a root on the interval  $[0, 3]$  ?

5. Estimate the limit, if it exists:  $\lim_{x \rightarrow 3} f(x)$ , where  $f(x)$  is represented by the given graph:
- (A) 0  
 (B) -1  
 (C) 3  
 (D) 1  
 (E) The limit does not exist.



6. Given the function:

$$f(x) = \begin{cases} \sin 2x, & x \leq \pi \\ 2x + k, & x > \pi \end{cases}$$

what value of  $k$  will make this piecewise function continuous?

- (A)  $-2\pi$   
 (B)  $-\pi$   
 (C) 0  
 (D)  $\pi$   
 (E)  $2\pi$

7. Find the limit:  $\lim_{x \rightarrow 0} x \left( e^x + \frac{1}{x} \right)$ .

- (A) 0  
 (B) 1  
 (C) 2  
 (D) The limit does not exist.  
 (E) None of these

8. Identify the vertical asymptotes for  $f(x) = \frac{x^2 + 3x - 4}{x^2 + x - 2}$ .

- (A)  $x = -2, x = 1$   
 (B)  $x = -2$   
 (C)  $x = 1$   
 (D)  $y = -2, y = 1$   
 (E)  $y = -2$

9. If  $p(x)$  is a continuous function on the closed interval  $[1, 3]$ , with  $p(1) \leq K \leq p(3)$  and  $c$  is in the closed interval  $[1, 3]$ , then which of the following statements must be true?

- (A)  $p(c) = \frac{p(3) + p(1)}{2}$   
 (B)  $p(c) = \frac{p(3) - p(1)}{2}$   
 (C) There is at least one value  $c$ , such that  $p(c) = K$ .  
 (D) There is only one value  $c$ , such that  $p(c) = K$ .  
 (E)  $c = 2$

10. How many vertical asymptotes exist for the function

$$f(x) = \frac{1}{2 \sin^2 x - \sin x - 1} \text{ in the open interval } 0 < x < 2\pi?$$

- (A) 0  
 (B) 1  
 (C) 2  
 (D) 3  
 (E) 4

FREE RESPONSE

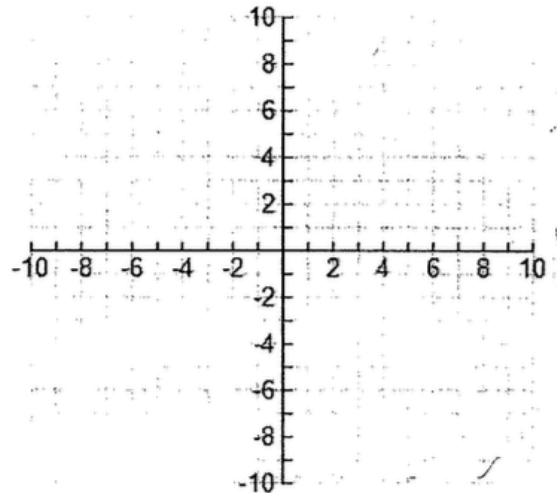
Show all work in the space provided. All steps must be shown. Indicate clearly the methods you use, because you will be graded on the correctness of your methods as well as the accuracy of your final answers. Correct answers without **supporting work** ["bald" answers] will NOT receive credit. Justifications require mathematical [non-calculator] reasons. Your work must be expressed in standard mathematical notation. Unless otherwise specified, answers [numeric or algebraic] need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.

FR1. On the axes provided below, sketch a graph of a function that has all of the following attributes listed below:

I.  $\lim_{x \rightarrow 3} f(x) = 4$

II.  $f(3) = -2$

III.  $\lim_{x \rightarrow -3} f(x) = \infty$



FR2.  $f(x)$  and  $g(x)$  are continuous functions for all  $x \in \text{Reals}$ . The table below has values for the functions for selected values of  $x$ . The function  $h(x) = g(f(x)) + 2$ .

$x$	$f(x)$	$g(x)$
1	3	4
3	9	-10
5	7	5
7	11	25

Explain why there must be a value  $c$  for  $1 < c < 5$  such that  $h(c) = 0$ .

FR3. Find  $a$  such that the function  $f(x) = \begin{cases} \frac{4\sin x}{x}, & x < 0 \\ a + 15x, & x \geq 0 \end{cases}$  is continuous for all real numbers.



## Worksheet #6: AP Multiple Choice Limits & Continuity

All work must be shown in this course for full credit.

### 2016 No Calculator

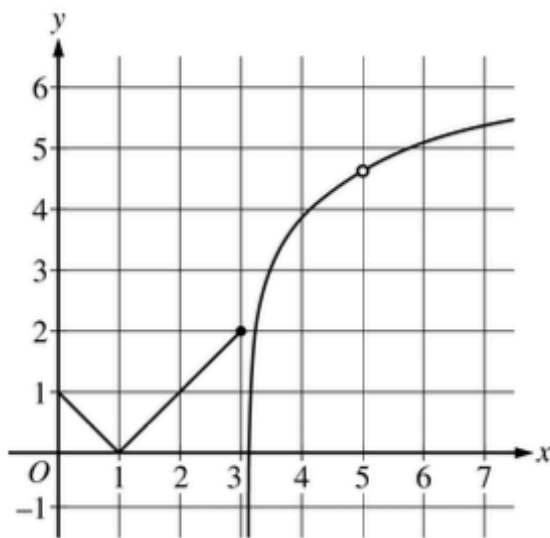
3.  $\lim_{x \rightarrow \infty} \frac{\sqrt{9x^4 + 1}}{4x^2 + 3}$  is

- (A)  $\frac{1}{3}$       (B)  $\frac{3}{4}$       (C)  $\frac{3}{2}$       (D)  $\frac{9}{4}$       (E) infinite

7. For which of the following pairs of functions  $f$  and  $g$  is  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$  infinite?

- (A)  $f(x) = x^2 + 2x$  and  $g(x) = x^2 + \ln x$   
(B)  $f(x) = 3x^3$  and  $g(x) = x^4$   
(C)  $f(x) = 3^x$  and  $g(x) = x^3$   
(D)  $f(x) = 3e^x + x^3$  and  $g(x) = 2e^x + x^2$   
(E)  $f(x) = \ln(3x)$  and  $g(x) = \ln(2x)$

### 2016 Calculator



Graph of  $f$

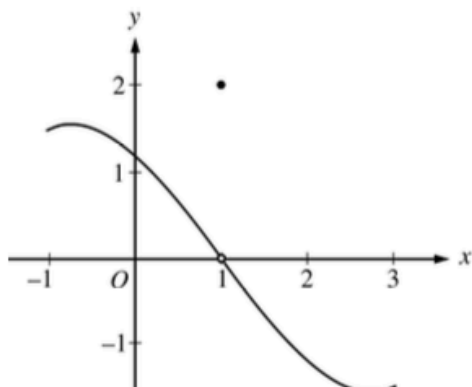
76. The graph of a function  $f$  is shown above. Which of the following limits does not exist?

- (A)  $\lim_{x \rightarrow 1^-} f(x)$       (B)  $\lim_{x \rightarrow 1} f(x)$       (C)  $\lim_{x \rightarrow 3^-} f(x)$       (D)  $\lim_{x \rightarrow 3} f(x)$       (E)  $\lim_{x \rightarrow 5} f(x)$

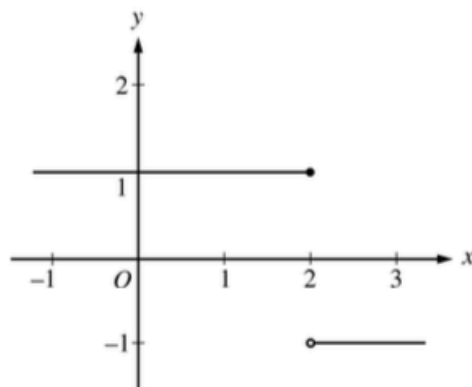
77. Let  $f$  be a function that is continuous on the closed interval  $[1, 3]$  with  $f(1) = 10$  and  $f(3) = 18$ . Which of the following statements must be true?
- (A)  $10 \leq f(2) \leq 18$
- (B)  $f$  is increasing on the interval  $[1, 3]$ .
- (C)  $f(x) = 17$  has at least one solution in the interval  $[1, 3]$ .
- (D)  $f'(x) = 8$  has at least one solution in the interval  $(1, 3)$ .
- (E)  $\int_1^3 f(x) dx > 20$

82. If  $f$  is a continuous function such that  $f(2) = 6$ , which of the following statements must be true?
- (A)  $\lim_{x \rightarrow 1} f(2x) = 3$
- (B)  $\lim_{x \rightarrow 2} f(2x) = 12$
- (C)  $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = 6$
- (D)  $\lim_{x \rightarrow 2} f(x^2) = 36$
- (E)  $\lim_{x \rightarrow 2} (f(x))^2 = 36$

**2017 No Calculator**



Graph of  $f$



Graph of  $g$

15. The graphs of the functions  $f$  and  $g$  are shown in the figures above. Which of the following statements is false?
- (A)  $\lim_{x \rightarrow 1} f(x) = 0$
- (B)  $\lim_{x \rightarrow 2} g(x)$  does not exist.
- (C)  $\lim_{x \rightarrow 1} (f(x)g(x+1))$  does not exist.
- (D)  $\lim_{x \rightarrow 1} (f(x+1)g(x))$  exists.

20. Let  $f$  be the function given by  $f(x) = \frac{x-2}{2|x-2|}$ . Which of the following is true?

(A)  $\lim_{x \rightarrow 2} f(x) = \frac{1}{2}$

(B)  $f$  has a removable discontinuity at  $x = 2$ .

(C)  $f$  has a jump discontinuity at  $x = 2$ .

(D)  $f$  has a discontinuity due to a vertical asymptote at  $x = 2$ .

### 2017 Calculator

78. The continuous function  $f$  is positive and has domain  $x > 0$ . If the asymptotes of the graph of  $f$  are  $x = 0$  and  $y = 2$ , which of the following statements must be true?

(A)  $\lim_{x \rightarrow 0^+} f(x) = \infty$  and  $\lim_{x \rightarrow 2} f(x) = \infty$

(B)  $\lim_{x \rightarrow 0^+} f(x) = 2$  and  $\lim_{x \rightarrow \infty} f(x) = 0$

(C)  $\lim_{x \rightarrow 0^+} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} f(x) = 2$

(D)  $\lim_{x \rightarrow 2} f(x) = \infty$  and  $\lim_{x \rightarrow \infty} f(x) = 2$

### 2003 No Calculator

3. For  $x \geq 0$ , the horizontal line  $y = 2$  is an asymptote for the graph of the function  $f$ . Which of the following statements must be true?

(A)  $f(0) = 2$

(B)  $f(x) \neq 2$  for all  $x \geq 0$

(C)  $f(2)$  is undefined.

(D)  $\lim_{x \rightarrow 2} f(x) = \infty$

(E)  $\lim_{x \rightarrow \infty} f(x) = 2$

6.  $\lim_{x \rightarrow \infty} \frac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} =$

(A) 4

(B) 1

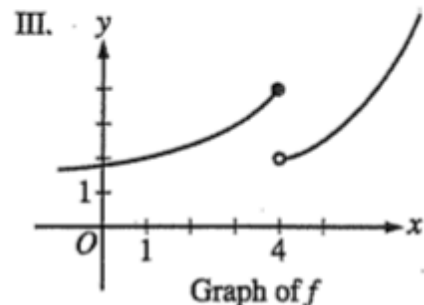
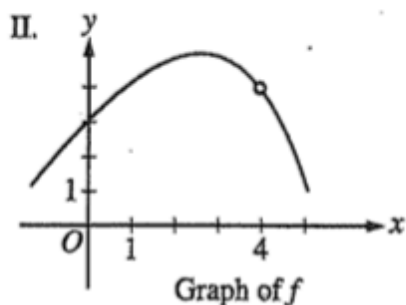
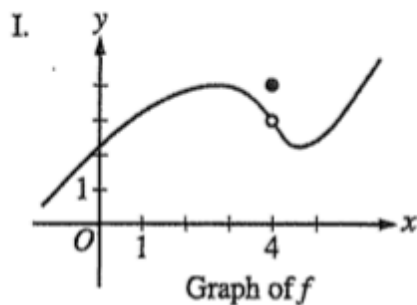
(C)  $\frac{1}{4}$

(D) 0

(E) -1

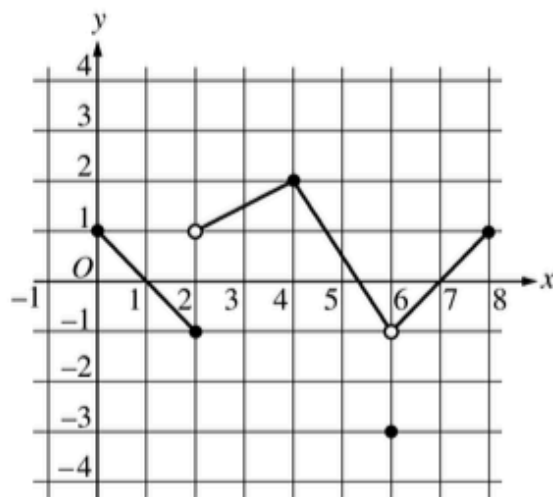
## 2003 Calculator

79. For which of the following does  $\lim_{x \rightarrow 4} f(x)$  exist?



- (A) I only  
 (B) II only  
 (C) III only  
 (D) I and II only  
 (E) I and III only

## 2014 No Calculator



5. The figure above shows the graph of the function  $f$ . Which of the following statements are true?

- I.  $\lim_{x \rightarrow 2^-} f(x) = f(2)$   
 II.  $\lim_{x \rightarrow 6^-} f(x) = \lim_{x \rightarrow 6^+} f(x)$   
 III.  $\lim_{x \rightarrow 6} f(x) = f(6)$

- (A) II only  
 (B) III only  
 (C) I and II only  
 (D) II and III only  
 (E) I, II, and III

7.  $\lim_{x \rightarrow \infty} \frac{x^3}{e^{3x}}$  is

- (A) 0      (B)  $\frac{2}{9}$       (C)  $\frac{2}{3}$       (D) 1      (E) infinite

$$f(x) = \begin{cases} \frac{x^2 - 7x + 10}{b(x - 2)} & \text{for } x \neq 2 \\ b & \text{for } x = 2 \end{cases}$$

10. Let  $f$  be the function defined above. For what value of  $b$  is  $f$  continuous at  $x = 2$ ?

- (A) -3      (B)  $\sqrt{2}$       (C) 3      (D) 5      (E) There is no such value of  $b$ .

16.  $\lim_{x \rightarrow 3^-} \frac{|x - 3|}{x - 3}$  is

- (A) -3      (B) -1      (C) 1      (D) 3      (E) nonexistent

24. Let  $f$  be the function defined by  $f(x) = \frac{(3x + 8)(5 - 4x)}{(2x + 1)^2}$ . Which of the following is a horizontal asymptote to the graph of  $f$ ?

- (A)  $y = -6$   
(B)  $y = -3$   
(C)  $y = -\frac{1}{2}$   
(D)  $y = 0$   
(E)  $y = \frac{3}{2}$

1. If  $f(x) = \begin{cases} -x^2 - 4x - 2, & x \leq -2 \\ x^2 + 4x + 6, & x > -2 \end{cases}$  find the  $\lim_{x \rightarrow -2} f(x)$ .

2. Find the intervals on which each of the function is continuous:

a)  $y = \frac{1}{(x-2)^2}$

b)  $f(x) = \begin{cases} 5-x, & x \leq 2 \\ 2x-3, & x > 2 \end{cases}$

c) For each function above give the points of discontinuity and the types. Are they removable or non-removable?

3. Find each limit:

a)  $\lim_{x \rightarrow 4} \sqrt{x^2 - 3}$

b)  $\lim_{x \rightarrow \infty} e^{-x} \cos x$

c)  $\lim_{x \rightarrow 0} e^x \sin x$

d)  $\lim_{x \rightarrow -\infty} \frac{2x^2 + 3}{5x^2 + 7}$

e)  $\lim_{x \rightarrow \infty} \frac{2x^3 + 31}{5x^2 + 7}$

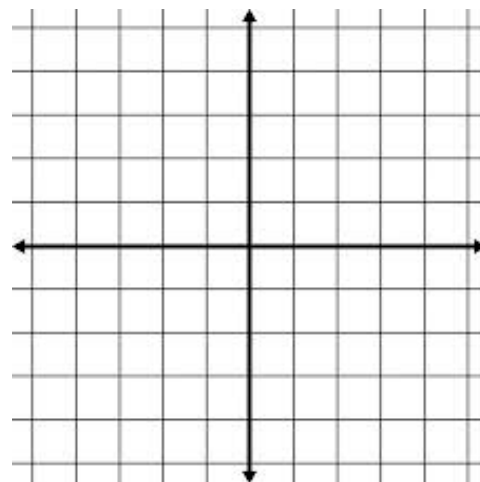
f)  $\lim_{x \rightarrow \infty} \frac{2x^2 + 9x}{5x^8 + 7x^4}$

g)  $\lim_{x \rightarrow 0} \frac{\sin 2x}{4x}$

h)  $\lim_{x \rightarrow 0} \frac{1/(x+1) - 1}{x}$

6. Sketch a graph of a function  $f$  that satisfies the given conditions.

$$\lim_{x \rightarrow 2} f(x) \text{ does not exist, } \lim_{x \rightarrow 2^+} f(x) = f(2) = 3$$



7. Determine the value of  $c$  such that the function is continuous on the entire real line.

$$f(x) = \begin{cases} x+3, & x \leq 2 \\ cx+6, & x > 2 \end{cases}$$

8. Write an extended function that would make  $f(x) = \frac{5x-5}{x^2-1}$  at  $x=1$ .

9. Use the intermediate value theorem to show that  $f(x) = 2x^3 - 3$  has a zero in the interval  $[1, 2]$ .

10. Find all of the vertical and horizontal asymptotes for  $y = \frac{2x-3}{2x^2-x-3}$