Chapter 2: Limits of functions:

In this chapter we will cover:

2.2 The limit of a function (intuitive approach) .

2.3 Limit laws. Limits of elementary functions and the squeeze theorem.

2.6 Limits at infinity and infinite limits . Horizontal and vertical asymptotes.

2.5 Continuity

2.7 Review .

2.2 The limit of a function (intuitive approach):

In many scientific problems, such as the instantaneous velocity problem or the tangent line to a curve, we need to calculate a limit quantity of the form $\lim_{x \to a} f(x)$.

Let us develop an understanding of the meaning of the expression $\lim_{x \to a} f(x)$.

You have already encountered the limit of a sequence : $\lim_{n \to \infty} y_n$. To find $\lim_{n \to \infty} y_n$ in general, we determine what value the sequence y_n approaches as n approaches ∞ (that is for increasingly larger values of n).

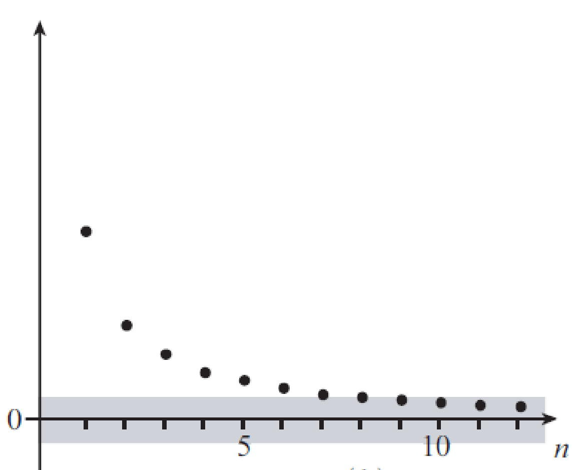
A more precise way to state this is: $\lim_{n \to \infty} y_n = L$ if the sequence y_n is arbitrarily close to L (as close as desired to L)

when n is large enough.

Examples:

1. Consider $y_n = \frac{1}{n}$. What is $\lim_{n \to \infty} y_n$?

Graphing the sequence $y_n = \frac{1}{n}$ produces the following graph:



the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$

We observe then that $\lim_{n \to \infty} \frac{1}{n} = 0$.

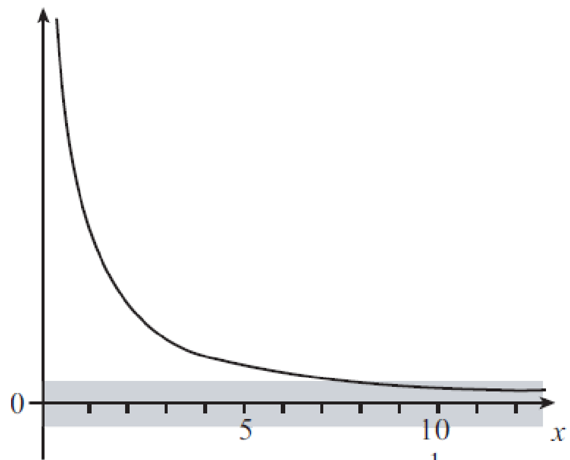
Are we confident that $y_n = \frac{1}{n}$ is as close as desired to 0 for n large enough ? How close to 0 do we want y_n to be?

(note that closer we want y_n to be to 0, larger n we need).

Similarly for functions:

2. What is $\lim_{x\to\infty}\frac{1}{x}$? (or what value (if any) the function $f(x)=\frac{1}{x}$ approaches when x approaches ∞?).

Consider a similar approach as in Example 1 above and graph $f(x)=\frac{1}{x}$, as shown below.



the function $f(x) = \frac{1}{x}$

We see that $\lim_{x\to\infty}\frac{1}{x}=0$

since the function $f(x)=\frac{1}{x}$ approaches 0 as

x approaches ∞ .

3. Similarly, deduce: $\lim_{x\to\infty}\left(3-\frac{1}{x^2}\right)$=?

A more rigorous way to say that $\lim_{x\to\infty} f(x)=L$ is to say that $f(x)$ is arbitrarily close to L (as close as desired to L) when x is large enough.

4. What is $\lim_{x\to\infty} x^2 =$

But $\lim_{x\to\infty} x\sin(x) =$

NOTE:

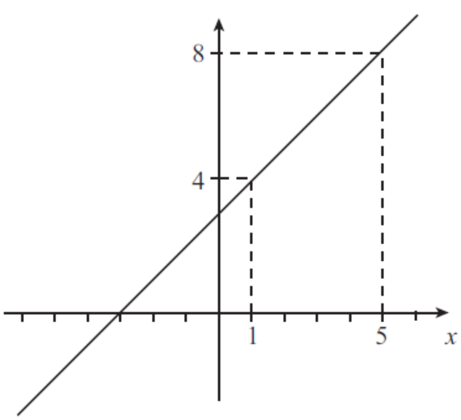
In a more careful approach, in order to convince oneself that $\lim_{x\to\infty} f(x) = L$ for a specific function $f(x)$, that is that $f(x)$ is indeed arbitrarily close to L (as close as desired to L) when x is large enough ,one will often consider a table of values for increasingly larger values of x to study the behavior of $f(x)$ for those values, consider different challenges of "closeness" and determine if $f(x)$ satisfies those challenges, or undertake a more rigorous $\varepsilon-\delta$ limit approach. Due to time restrictions, we will not delve into the $\varepsilon-\delta$ limit approach in this class, and will usually rely on the graph of $f(x)$ (and later on algebraic methods) to find $\lim_{x\to\infty} f(x)$. We should remember though that

$\lim_{x\to\infty} f(x) = L$ is to say that $f(x)$ is arbitrarily close to L (as close as desired to L) when x is large enough .

5. Let us now turn to the more general case: $\lim_{x\to a} f(x)$ in which we want to determine the value that $f(x)$ approaches when x approaches a.

The main difference now is that x approaches a finite value a, and that it can approach this value from the left $(x \to a^-)$ or from the right $(x \to a^+)$.

For example, what is $\lim_{x\to 1}(x+3)$? We may well guess (and are correct in saying) that $\lim_{x\to 1}(x+3) = 4$.



the function $f(x) = x + 3$

This follows easily from a graph of $f(x)$ as shown here (or from our intuition as well) .

Note that this time we look at both $\lim_{x\to 1^-}(x+3) = 4$ and at $\lim_{x\to 1^+}(x+3) = 4$, which implies that indeed $\lim_{x\to 1}(x+3) = 4$.

This may not seem very useful , since in this case (as is actually the case for many functions) , if $f(x) = x+3$: $\lim_{x\to 1} f(x) = f(1) = 4$. But keep in mind that $\lim_{x\to a} f(x)$ represents actually the value that $f(x)$ approaches when x approaches a (which in this case agrees with $f(a)$). There are many instances in which this needs not to be the case, and calculating the limit is the only way to determine what value $f(x)$ approaches when x approaches a.

For example, calculate:

6.

a) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ what is $f(1)$?

b) $\lim_{x \to 0} \frac{\sin(x)}{x}$ what is $f(0)$?

c) $\lim_{x \to 0} \frac{\sqrt{x^2 + 9} - 3}{x^2}$ what is $f(0)$?

d) $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$ what is $f(0)$?

e) The function $f(x) = \begin{cases} 0, \text{ for } x \le 0 \\ and \\ 1, \text{ for } x > 0 \end{cases}$ is graphed below (this is usually called the step function):

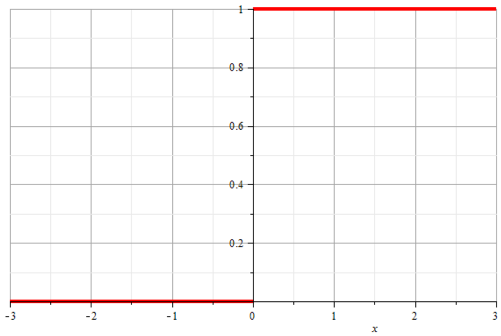


Figure 4: Graph of the function $f(x)$ given above;

What is $\lim_{x \to 0} f(x)$? We see that :

$\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 1$; hence $\lim_{x \to 0} f(x)$ does not exist in this case, since there is no single number L such that $f(x)$ is arbitrarily close to L (or which $f(x)$ approaches) when x is close enough to 0 .

Definition 3 :

a) We write $\lim_{x \to a^-} f(x)$=L if $f(x)$ approaches L when x approaches a, but $x < a$.

b) We write $\lim_{x \to a^+} f(x)$=L if $f(x)$ approaches L when x approaches a, but $x > a$.

Theorem 1: $\lim_{x \to a} f(x) = L \leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$=L .

Therefore, if **(1)** $\boxed{\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)}$, then $\lim_{x \to a} f(x)$ does not exist.

Condition (1) is one of the easiest and most useful ways to show that $\lim_{x \to a} f(x)$ does not exist.

7.

Find each of the following real limits, if they exist. If the real limit does not exist, state whether the function tends to infinity, tends to minus infinity, or has no limit at all.

(a) $\lim_{x \to 0} \sin x$ (b) $\lim_{x \to 0} \sin \frac{1}{x}$ (c) $\lim_{x \to 2} \frac{1}{x-2}$ (d) $\lim_{x \to 2} \frac{1}{(x-2)^2}$

(e) $\lim_{x \to 1} \frac{x^2 - 3x + 2}{1 - x}$ (f) $\lim_{x \to \pi/2} \tan x$

Homework: Do problems 1,2,4,6,8,10,11,20,23,25 from Exercise Set 2.2

2.3. Limit laws. Limits of elementary functions and the squeeze theorem.

A. Limits Laws. Limits of elementary functions:

Goals:

- Show an efficient yet rigorous method to calculate limits: show that $\lim_{x\to a} f(x) = f(a)$ if $f(x)$ is an elementary function and if $f(a)$ exists .
- Solve limits of the form: $\left[\frac{0}{0}\right]$;

Lesson:

Theorem 1 (Limit Laws):

Consider $f : D_1 \to R_1$ and $g : D_2 \to R_2$ and $a \in D_1 \cap D_2$ such that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then the following limits exist and are equal to the indicated values:

$$(1)\quad \begin{cases} \lim_{x\to a}(k \cdot f(x)) = k \cdot L \text{ for any } k \in R \\ \lim_{x\to a}(f(x) + g(x)) = L + M \\ \lim_{x\to a}(f(x) - g(x)) = L - M \\ \lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M \\ \lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M} \text{ for } M \neq 0 \\ \lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M \\ \lim_{x\to a}(f(x))^n = L^n \text{ for any } n \in N \\ \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} \text{ for any } n \in N \text{ and } L > 0 \text{ (when n is even)} \end{cases}$$

Proof:

These laws can be shown using the definition 1 in 2.4, and the methods of section 2.4. Some of these proofs are harder than others, and are shown in Appendix F of the textbook.

From now on, we consider properties (1) true and we use them in problems. Using these, calculate:

Example 1:

a) $\lim_{x\to 5}(x^2 + 2x + 3)$ b) $\lim_{x\to 3} \frac{x^2 - 2}{x^2 - 1}$ c) $\lim_{x\to 2} \sqrt{\frac{x-1}{x+1}}$

We see that the method to calculate limits ultimately amounts now to substituting x with a in $f(x)$ when $f(a)$ exists.

We state this clearly in general:

Theorem 2 (substitution):

a) If $f(x)=P_n(x)$ is a polynomial of degree n then $\lim\limits_{x\to a} P_n(x)=P_n(a)$.

b) If $f(x)=\dfrac{P_n(x)}{Q_m(x)}$ is a rational function (ratio of two polynomials) then $\lim\limits_{x\to a}\dfrac{P_n(x)}{Q_m(x)}=\dfrac{P_n(a)}{Q_m(a)}$ if $Q_m(a)\neq 0$.

c) If $f(x)$ is an algebraic function (a combination of $\sqrt{\ }$, polynomials and rational functions by using addition, subtraction, multiplication or composition), then $\lim\limits_{x\to a} f(x)= f(a)$ if $f(a)$ exists (or, in other words, if a is in the domain of f(x)) .

Proof: The proof of a) and b) are direct consequences of Theorem 1. The proof of part c) is given in Appendix F in the textbook (listed as Theorem 8).

Example 2:

Using Theorem 2, calculate the limits:

a) $\lim\limits_{x\to 2}\sqrt{\dfrac{2x^2+1}{3x-2}}$

b) $\lim\limits_{x\to 3}\left(5x^3-3x^2+x-6\right)$.

Do also Example 1 (Page 99, of Section 2.3) in the textbook.

The next theorem will help us solve cases of study of the form $\left[\dfrac{0}{0}\right]$.

Theorem 3: If $f(x)=g(x)$ for all $x\in I$, except for $a\in I$, then $\lim\limits_{x\to a} f(x)=\lim\limits_{x\to a} g(x)$.

This theorem allows us to solve limits the form $\left[\dfrac{0}{0}\right]$, by first algebraically simplifying the function (usually by cancelling an $(x-a)^p$ factor from both top and bottom of the fraction) and then calculating the limit of the simplified function.

Example 3: Using Theorems 3 and 2 above, as well as Theorem 1 in 1.2 calculate the limits:

a) $\lim\limits_{x\to 1}\dfrac{x-1}{x^2-1}$

b) $\lim\limits_{h\to 0}\dfrac{(h+3)^2-9}{h}$

c) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+16}-4}{x}$

d) If $f(x)=\begin{cases}\sqrt{x-4} \text{ if } x>4\\ 8-2x \text{ if } x\le 4\end{cases}$ calculate $\lim\limits_{x\to 4} f(x)$. Then sketch a graph of this function to confirm your result.

e) Calculate $\lim\limits_{x\to 0}|x|$

f) Calculate $\lim\limits_{x\to 0}\dfrac{|x|}{x}$

g) $\lim_{x\to -1} \frac{2x^2+3x+1}{x^2-2x-3}$ h) $\lim_{t\to 1} \frac{t^4-1}{t^3-1}$ i) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{x^2+x}\right)$

j) $\lim_{x\to 3} \frac{x^2-9}{x^2-2x-3}$ k) $\lim_{x\to 0} \frac{x^3+x^2-x}{x^2+x}$ l) $\lim_{x\to 2}\left(\frac{2}{x\cdot(x-2)}-\frac{1}{x^2-3x+2}\right)$

m) $\lim_{x\to 0} \frac{\sqrt{x+3}-\sqrt{3-x}}{x}$ n) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{\sqrt{x+4}-2}$

B. The squeeze theorem:

New types of limits (including limits of trigonometric functions and combinations of these) can be calculated using an inequality result involving limits called the squeeze theorem.

This theorem is based on:

Theorem 4: If $f(x)\le g(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$, then :

(2) $L\le M$.

Proof: See the proof of Theorem 2 in Appendix F.

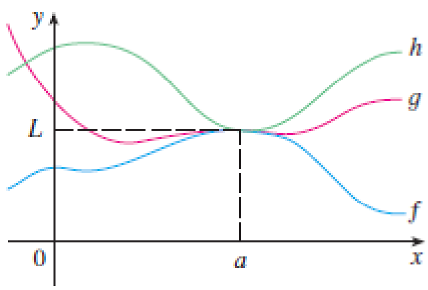
Based on this theorem, we can easily prove the following:

Theorem 5 (the squeeze theorem):

If $f(x)\le g(x)\le h(x)$ when x is near a (except possibly at a) and if $\lim_{x\to a} f(x)=\lim_{x\to a} h(x)=L$, then : $\lim_{x\to a} g(x)=L$.

Proof: Based on above, if $\lim_{x\to a} g(x)=M$, then $L\le M\le L$, which shows that $L=M$. Q.E.D.

The squeeze theorem can be illustrated graphically and in a diagram as below



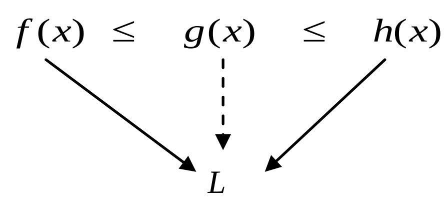


Figure 1: Illustration of the squeeze theorem. ***Figure 2: Diagram of the squeeze theorem***

This theorem is especially useful in calculating limits of trigonometric functions.

First, based on it, we can prove:

Theorem 6 (limits of trigonometric functions):

If $f(x)$ is a trigonometric function (one of the functions $\sin(x), \cos(x), \tan(x)$ or $\cot(x)$) or a combination (using addition, subtraction, multiplication or composition of these functions or a combination of these functions and the algebraic functions mention in Theorem 2) then

(3) $\lim_{x\to a} f(x) = f(a)$ if $f(a)$ exists (or, in other words, if a is in the domain of f(x)) .

Proof: see the proofs on page 123 of the textbook.

Note: This theorem, together with Theorem 2 above, allows us to do substitution for all "elementary functions" (algebraic, trigonometric or combinations of these) as long as a is in the domain of the function.

Example 4: Calculate the following limits:

a) $\lim_{x\to 0} \sin(x^2)$ b) $\lim_{x\to 0} \sin\left(\frac{x^2-\pi^2}{x-\pi}\right)$

However, the squeeze theorem is used directly for a different type of problems, where direct evaluation of the function is impossible:

Example 5: Using the squeeze theorem 5, show that $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$ (see Example 11 on page 105).

Homework: Problems 1, 2, 4,9,13,15,17,19,22,25,27,28,29,32,33,36,40,41,42,48,50 and 63 from Exercise Set 2.3 .

2.6. Limits at infinity and infinite limits:

In section 2.3 we were able to calculate $\lim_{x\to a} f(x) = f(a)$ if a is in the domain of $f(x)$, or for a case of study of the form $\left[\frac{0}{0}\right]$.

In this section we will study new important types of limits involving ∞ .

A. Infinite limits:

Definition 1:

a) $\lim_{x\to a} f(x) = +\infty$ if the values of $f(x)$ can be made arbitrarily large when x is close enough to a.

b) $\lim_{x\to a} f(x) = -\infty$ if the values of $f(x)$ can be made arbitrarily large negative when x is close enough to a.

Example 1:

a) Graph $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$ and infer infinite limit results from the graph.

b) Using the definition above, show that

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right) = +\infty,\ \lim_{x\to 0}\left(-\frac{1}{x^2}\right) = -\infty,\ \lim_{x\to 0^+}\left(\frac{1}{x}\right) = +\infty \text{ and } \lim_{x\to 0^-}\left(\frac{1}{x}\right) = -\infty$$

c) What is $\lim_{x\to 0}\left(\frac{1}{x}\right)$?

Note: We generalize the results from the example above and state:

(1) $\boxed{\frac{1}{0^+} = +\infty}$ **and** $\boxed{\frac{1}{0^-} = -\infty}$. These two results mean that $\lim_{x\to a} \frac{1}{f(x)} = +\infty$ if $\lim_{x\to a} f(x) = 0^+$ and $\lim_{x\to a} \frac{1}{f(x)} = -\infty$ if $\lim_{x\to a} f(x) = 0^-$ (of course, in these results a could be replaced by a^+ or a^-, as needed).

Also $\lim_{x\to a} f(x) = 0^+$ means that as x approaches a, f(x) approaches 0 but only with positive values (f(x) is positive as x approaches a), and $\lim_{x\to a} f(x) = 0^-$ means that as x approaches a, f(x) approaches 0 but only with negative values (f(x) is negative as x approaches a) .

The result (1) is the most important tool to calculate infinite limits. Therefore, in order to evaluate a limit of the form $\left[\frac{1}{0}\right]$, we evaluate the function, and determine the sign of the denominator (best by using a sign table for it !!), then use (1).

Example 2:

Calculate the limits:

a) $\lim_{x\to 3^+}\frac{2x}{x-3}$ and $\lim_{x\to 3^-}\frac{2x}{x-3}$, therefore $\lim_{x\to 3}\frac{2x}{x-3}=?$

b) $\lim_{x\to 3}\frac{2x}{(x-3)^2}$

c) $\lim_{x\to \frac{\pi}{2}^-}\tan(x)$ and $\lim_{x\to \frac{\pi}{2}^+}\tan(x)$, therefore $\lim_{x\to \frac{\pi}{2}}\tan(x)=?$

d) Do problems 29, 31 and 35 from Exercise Set 2.2 in the textbook.

In the previous examples, if $\lim_{x\to a} f(x)=+\infty\,(\text{or}-\infty)$, or if $\lim_{x\to a^-} f(x)=+\infty\,(\text{or}-\infty)$ or if $\lim_{x\to a^+} f(x)=+\infty\,(\text{or}-\infty)$, then the line $x=a$ is called a Vertical Asymptote (V.A.) for the graph of $f(x)$ (see, for example, the limits in Example 1). Therefore:

Definition 2: The line $x=a$ is called a Vertical Asymptote (V.A.) for the graph of $f(x)$ if at least one of the following six cases takes place:

$\lim_{x\to a} f(x)=+\infty\,(\text{or}-\infty)$, or $\lim_{x\to a^-} f(x)=+\infty\,(\text{or}-\infty)$ or $\lim_{x\to a^+} f(x)=+\infty\,(\text{or}-\infty)$.

Practically, to calculate the V.A. for a given function, we calculate first the zeros of the denominator (if the function is a fraction) and then calculate the limit of the function at these points.

Example 3:

a) Problem 38 from problem set 2.2 in the textbook.

b) Find all the vertical asymptotes of :

i) $f(x)=\frac{-8}{(x+5)(x-9)}$ **ii)** $f(x)=\frac{7x}{(10-3x)^4}$ **iii)** $f(x)=\frac{\cos(x)}{x}$

iv) $f(x)=\frac{x}{\cos(x)}$ **v)** $f(x)=\frac{\sin(x)}{x}$ **vi)** $f(x)=\frac{\tan(x)}{\sin(2x)}$

B. Limits at infinity:

We turn now to the (last main) case when $x\to\infty$.

Definition 3:

a) $\lim_{x\to\infty} f(x)=L$ if $f(x)$ is arbitrarily close to L when x is large enough. Also:

b) $\lim_{x\to-\infty} f(x)=L$ if $f(x)$ is arbitrarily close to L when x is large enough and negative.

c) Definition 1 can be combined with Definition 3 (**Exercise**) to produce:
$\lim_{x\to\infty} f(x)=+\infty\,(\text{or}-\infty)$ and $\lim_{x\to-\infty} f(x)=+\infty\,(\text{or}-\infty)$.

Example 4:

a) Graph the corresponding functions or use a table of values to find : $\lim_{x\to\infty}\frac{1}{x}$ and $\lim_{x\to-\infty}\frac{1}{x}$

b) Calculate $\lim_{x\to\infty}\frac{1}{x^2}$.

Note: We generalize the results from the examples above and state:

(2) $\boxed{\frac{1}{\infty}=0}$ This means that $\lim_{x\to a}\frac{1}{f(x)}=0$ if $\lim_{x\to a} f(x)=\infty$ (which is relatively easy to prove using the Definitions 1. a from this section and Definition 1 in 1.4.

From now on, we will use the result (2) as a main tool to calculate limits at ∞.

Example 5: Calculate the following limits by using (2) above:

a) $\lim_{x\to\infty}\frac{x}{x+2}$ b) $\lim_{x\to\infty}\frac{x^2}{x^2-3x+2}$ c) $\lim_{x\to\infty}\frac{1}{2x+3}$

d) $\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$ e) $\lim_{x\to-\infty}\left(x+\sqrt{x^2+2x}\right)$

The following general results involving $\pm\infty$ will facilitate the direct substitution of $\pm\infty$ in limits:

(3) $$\infty+a=\begin{cases}\infty, \text{ if } a\in R\\ \infty+\infty=\infty, \text{ if } a=\lim_{x\to\pm\infty} g(x)=\infty\\ [\infty-\infty], \text{ if } a=\lim_{x\to\pm\infty} g(x)=-\infty\end{cases}$$

(4) $$\infty\cdot a=\begin{cases}\infty, \text{ if } a>0\\ -\infty, \text{ if } a<0\\ [\infty\cdot 0], \text{ if } a=\lim_{x\to\pm\infty} g(x)=0\end{cases}$$

(5) $$\infty^n=\begin{cases}\infty, \text{ if } n>0\\ 0, \text{ if } n<0\\ [\infty^0], \text{ if } n=\lim_{x\to\pm\infty} g(x)=0\end{cases}$$

(6) $$\frac{\infty}{a}=\begin{cases}\infty, \text{ if } a>0\\ -\infty, \text{ if } a<0\\ \pm\infty, \text{ if } a=0\\ \left[\frac{\infty}{\infty}\right], \text{ if } a=\lim_{x\to\pm\infty} g(x)=\infty\end{cases}$$

These results can be proved rigorously with a similar approach as described in (1) and (2). Understand them by thinking of ∞ as a quantity larger than any real number, memorize especially the formulas (1) to (6) , and solve the following problems by substitution:

Example 6: Calculate the following limits at ∞:

a) $\lim\limits_{x\to\infty}\frac{3x+5}{x-4}$

b) $\lim\limits_{x\to\infty}\frac{x-x\sqrt{x}}{2x^{\frac{3}{2}}+3x-5}$

c) $\lim\limits_{x\to-\infty}\frac{\sqrt{9x^6-x}}{x^3+1}$

d) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+ax}-\sqrt{x^2+bx}\right)$

e) $\lim\limits_{x\to\infty}\left(\sqrt{x^2+1}\right)$

f) $\lim\limits_{x\to\infty}\frac{x^4-3x^2+x}{x^3-x+2}$

g) $\lim\limits_{x\to\infty}\arctan\left(e^x\right)$

h) $\lim\limits_{x\to\infty}\frac{e^{3x}+e^{-3x}}{e^{3x}-e^{-3x}}$

i) $\lim\limits_{x\to0^+}\frac{e^{3x}+e^{-3x}}{e^{3x}-e^{-3x}}$

j) $\lim\limits_{x\to\infty}\frac{\sin^2(x)}{x^2+1}$

k) $\lim\limits_{x\to\infty}e^{-2x}\cos(x)$

l) $\lim\limits_{x\to3}\left(2x+|x-3|\right)$

m) $\lim\limits_{x\to-3^+}\frac{x+2}{x+3}$

n) $\lim\limits_{x\to-5^-}\frac{e^x}{(x-5)^3}$

o) $\lim\limits_{x\to2^+}\frac{x^2-2x-8}{x^2-5x+6}$

p) $\lim\limits_{x\to\pi^-}\cot(x)$

Definition 4:

a) The line $y=b$ is called a Horizontal Asymptote (H.A.) at $-\infty$ for the graph of $f(x)$ if

$\lim\limits_{x\to-\infty}f(x)=b$

b) The line $y=b$ is called a Horizontal Asymptote (H.A.) at $+\infty$ for the graph of $f(x)$ if

$\lim\limits_{x\to\infty}f(x)=b$

Therefore, a V.A. involve a finite x value and an infinite f(x), while a H.A. involve an infinite x value and a finite f(x) value there.

In problems, to calculate all H.A. for a given function, calculate $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$. If one of these limits produce a finite value, then the function $f(x)$ has a H.A. there, otherwise it doesn't.

Example 7:

Find the Horizontal Asymptotes at $+\infty$ and at $-\infty$ for the following functions. Then graph these functions to visualize your result:

a) $f(x)=\dfrac{x^2-1}{x^2+1}$ b) $f(x)=\dfrac{2x^2+x-1}{x^2+x-2}$ c) $f(x)=\dfrac{x^3-x}{x^2-6x+5}$

Homework:

What problems are left from above, and problems 19, 21,22,38,41,44 and 45 from the Exercise Set 2.6 in the textbook.

2.5 Continuity of functions:

Functions which have the property that $\lim_{x\to a} f(x)=f(a)$ are called continuous at a. Continuity of functions is a very important property which some functions verify on an entire interval or on their entire domain of definition. Functions which are continuous on an interval will satisfy other theorems (such as the intermediate value theorem) which allows us to find their zeros.

In this section we will learn tools (definitions and theorems) to establish the continuity of functions at a point and on an entire interval, as well as the intermediate value theorem for continuous functions.

Definition 1: A function $f:D\to R$ is continuous at a number $a\in D$ if **(1)** $\boxed{\lim_{x\to a} f(x)=f(a)}$.

Note that (1) actually says 3 things about the function $f(x)$. It says that:

a) $f(a)$ exists ($f(x)$ is defined at a) ;
b) $\lim_{x\to a} f(x)$ exists and
c) $\lim_{x\to a} f(x)=f(a)$.

Remembering the meaning of $\lim_{x\to a} f(x)$, we realize that (1) represents the intuitive fact that $f(x)$ does not have a jump (or a break – discontinuity) at a. Therefore, when we trace the graph of $f(x)$ close to a, the graph passes continuously through the point $(a, f(a))$.

Example 1: See Example 1 on page 119 in the textbook.

Example 2: Determine the continuity of the following functions at the indicated point:

a) $f(x)=\dfrac{x^2-x-2}{x-2}$ at $a=2$ b) $f(x)=\begin{cases} x^2 \text{ if } x\neq 0 \\ 1 \text{ for } x=0 \end{cases}$ at $a=0$

c) $f(x)=\begin{cases} \dfrac{x}{|x|} \text{ if } x\neq 0 \\ 0 \text{ for } x=0 \end{cases}$ at $a=0$. Then graph these functions to visualize your claim.

Note that some functions (such as $f(x)=\sqrt{x}$) are not defined to the left of $a=0$, and therefore only a ***side continuity*** can be defined there. Side continuity is also useful to define continuity on an entire interval.

Definition 2:

a) We say that $f: D \to R$ is **continuous from the right** at a number $a \in D$ if **(2)** $\boxed{\lim_{x \to a^+} f(x) = f(a)}$

b) **)** We say that $f: D \to R$ is **continuous from the left** at a number $a \in D$ if **(3)** $\boxed{\lim_{x \to a^-} f(x) = f(a)}$

Example 3: Show that $f(x) = \sqrt{x}$ is continuous from the right at $a = 0$. Can this function be continuous from the left at $a = 0$?

Definition 3:

a) A function $f: [a,b] \to R$ is continuous on the entire interval $[a,b]$ is it is continuous at each number $c \in (a,b)$ and it is right continuous at a and left continuous at b ;
b) A function $f: (a,b) \to R$ is continuous on the entire interval (a,b) is it is continuous at each number $c \in (a,b)$;

Example 4:

Show that $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on $(-1,1)$.

Following the model of this exercise (and remembering Theorems 2 and 6 in 1.3), we realize that all elementary functions (functions which are combinations of $\sqrt{\ }$, polynomials, rational and trigonometric functions) are continuous at each number in their domain. We state this clearly in the next theorem (which uses similar results for the other types of functions mentioned):

Theorem 1: The following types of functions are continuous at every number in their corresponding domains:

Polynomials, Rational Functions, Root Functions (Irrational Functions), Trigonometric Functions, Inverse Trigonometric Functions, Exponential Functions, Logarithmic Functions and combinations of these functions (using + , - , multiplication, division or composition).

Example 5:

a) Where is $f(x) = \dfrac{\ln(x)}{x^2 - 1}$ continuous and where is it discontinuous?

Hint: Find the domain of $f(x)$ first, then use the result from Theorem 1.

b) Do problems 15, 17 and 22 from Exercise Set 2.5 in the textbook.

Theorem 2 (The Intermediate Function Theorem) (optional):

Let $f: [a,b] \to R$ be a continuous function on the entire interval $[a,b]$, and let u be an arbitrary number strictly between $f(a)$ and $f(b)$, with $f(a) \neq f(b)$. Then there exists (at least) a number $c \in (a,b)$ such that $f(c) = u$.

Note: What Theorem 2 suggests is that continuous functions achieve (with a particular $f(c)$) all values in their range (if their range is an interval) . The proof of this theorem can be found in more advanced Calculus texts, and it is based on Cauchy's completion axiom (see a model of the proof on Wikipedia).

We Use Theorem 2 to find zeros of a continuous function, or to prove that such zeros exist.

Example 6:

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2. Can we reduce the length of the interval which contains this root? Can you suggest an algorithm to find the root to a desired precision?

Homework:

What left from above and problems 3,4,5,6,7, 16, 18, 19, 20, 21,4042,45,46,51 and 54.

2.7 Review:

In this chapter, we learnt

- The notion $\lim_{x \to a} f(x) = L$ at an intuitive level (Definition 1, how to use graphs and tables to determine limits: Section 2.2) ;
- The rigorous notion of $\lim_{x \to a} f(x) = L$ (Definition 2 and 3 (the $\varepsilon - \delta$ definition): Section 2.4);
- Practical methods to calculate limits: substitution and cases of study, infinite limits and limits at infinity (Sections 2.3 and 2.6)
- Continuity of functions based on limits (Section 2.5) .

Learn the main results from each section (definitions, theorems and formulas) (best by writing a brief review guide which include these) , then solve the following problems:

- In the textbook, do the Review problems from pages 165 to 168, **with the exception of** the problems involving the derivative of a function or rates of change (problems 11,12,13,14 from Concept Check, problems 19,20,21 from True or False, and problems 38 to 50 from Exercises).
- Do the problems below:

1. Calculate $\lim_{x \to 0.5} \cos(\pi x)$ by building a table of values $(x, \cos(\pi x))$ around 0.5 (the intuitive approach of guessing the limit using a table (used in 1.2)).

2. Calculate the following limits (using methods from the sections 2.3 and 2.6):

a) $\lim_{x \to 5} \frac{x^2 - 6x + 5}{x - 5}$ b) $\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$ c) $\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$

d) $\lim_{x \to 1} \frac{x^4 - 1}{x^3 - 1}$ e) $\lim_{x \to -2} \frac{x + 2}{x^3 + 8}$ f) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2 + x} \right)$

g) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

h) $\lim_{x\to 0}\frac{4-\sqrt{x}}{16x-x^2}$

i) $\lim_{x\to 0}\frac{(3+x)^{-1}-3^{-1}}{x}$

j) $\lim_{x\to \infty}\frac{1-x^2}{x^3-x-1}$

k) $\lim_{x\to \infty}\frac{3x-2}{2x+1}$

l) $\lim_{x\to -\infty}\frac{4x^3+6x^2-2}{2x^3-4x+5}$

m) $\lim_{x\to \infty}\frac{\sqrt{9x^6-x}}{x^3+1}$

n) $\lim_{x\to -\infty}\frac{\sqrt{9x^6-x}}{x^3+1}$

o) $\lim_{x\to \infty}\left(\sqrt{9x^2+x}-3x\right)$