CHAPTER 2
THE BASIC OF CONTROL THEORY

1. Laplace Transform Review.

Laplace Transform Review.

- ☑ **Laplace Transform** is defined as,

$$\boxed{\mathcal{L}\left[f(t)\right] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt}$$

Where s = σ+jω is a complex variable. By knowing *f(t)* we can find the function ***F(s)*** which is called Laplace transform of *f(t)*.

- ☑ **Inverse Laplace**

$$\boxed{\mathcal{L}^{-1}\left[F(s)\right] = f(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st}ds}$$

The inverse Laplace transform allows us to find ***f(t)*** given *F(s)*.

1) The Laplace Transform cont..

- Transform table:

	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$t\,u(t)$	$\frac{1}{s^2}$
4.	$t^n\,u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}\,u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t\,u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t\,u(t)$	$\frac{s}{s^2+\omega^2}$

Impulse function

Step function

Ramp function

	$f(t)$	$F(s)$
8.	$Ae^{-at}\cos \omega t\,u(t)$	$\frac{A(s+a)}{(s+a)^2+\omega^2}$
9.	$Be^{-at}\sin \omega t\,u(t)$	$\frac{B\omega}{(s+a)^2+\omega^2}$

1) The Laplace Transform cont..

- Transform Properties

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t - T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

Exercise 1: Laplace Transform.

Find the Laplace transform of

$$f(t) = Ae^{-at}u(t)$$

Solution:

$$F(s) = \mathcal{L}[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt$$

$$= \int_{0-}^{\infty} Ae^{-at}e^{-st}dt = A\int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= -\frac{A}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty}$$

$$= \frac{A}{s+a}$$

.

1) The Laplace Transform cont..

- Example: Find the Laplace Transform for the following.

i. Unit function:

$$\boxed{f(t) = 1}$$

ii. Ramp function:

$$\boxed{f(t) = t}$$

iii. Step function:

$$\boxed{f(t) = Ae^{-at}}$$

1) The Laplace Transform cont..

- Transform Theorem
  1. Differentiation Theorem

$$L\{\frac{df(t)}{dt}\} = sF(s) - f(0)$$

$$L\{\frac{d^2 f(t)}{dt^2}\} = s^2F(s) - sf(0) - \dot{f}(0)$$

  2. Integration Theorem:

$$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$$

  3. Initial Value Theorem: $f(0) = \lim_{t\to\infty} sF(s)$

  4. Final Value Theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

1) The Laplace Transform cont..

- The inverse Laplace Transform can be obtained using:

$$\boxed{f(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{+st}\,ds}$$

- Partial fraction method can be used to find the inverse Laplace Transform of a complicated function.
- We can convert the function to a sum of simpler terms for which we know the inverse Laplace Transform.

$$F(s) = F_1(s) + F_2(s) + \Lambda + F_n(s)$$

$$f(t) = L^{-1}[F_1(s)] + L^{-1}[F_2(s)] + \Lambda + L^{-1}[F_n(s)]$$

$$= f_1(t) + f_2(t) + \Lambda + f_n(t)$$

1) The Laplace Transform cont..

- We will consider three cases and show that $F(s)$ can be expanded into partial fraction:
  i. Case 1:

     Roots of denominator $A(s)$ are real and distinct.
  ii. Case 2:

     Roots of denominator $A(s)$ are real and repeated.
  iii. Case 3:

     Roots of denominator $A(s)$ are complex conjugate.

1) The Laplace Transform cont..

- Case 1: Roots of denominator A(s) are real and distinct.

Example 1:

$$\boxed{F(s) = \frac{2}{(s+1)(s+2)}}$$

Solution:

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$= \frac{2}{s+1} - \frac{2}{s+2}$$

It is found that:
A = 2 and B = -2

$$\boxed{f(t) = 2e^{-t} - 2e^{-2t}}$$

Example 2: $$Y(s) = \frac{s+3}{(s+1)(s+2)}$$

Problem: Find the Inverse Laplace Transform for the following.

Solution:

$$Y(s) = \frac{s+3}{(s+1)(s+2)}$$

$$Y(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1)\frac{s+3}{(s+1)(s+2)}\Big|_{s=-1} = 2$$

$$B = (s+2)\frac{s+3}{(s+1)(s+2)}\Big|_{s=-2} = -1$$

$$= \frac{2}{s+1} - \frac{1}{s+2}$$

$$y(t) = 2e^{-t} - e^{-2t}$$

1) The Laplace Transform cont..

- **Case 2:** **Roots of denominator F(s) are real and repeated.**

Example 1:

$$\boxed{F(s)=\frac{2}{(s+1)(s+2)^2}}$$

Solution:

$$F(s)=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}$$

It is found that:
A = 2, B = -2 and C = -2

$$=\frac{2}{s+1}-\frac{2}{s+2}-\frac{2}{(s+2)^2}$$

$$\boxed{f(t)=2e^{-t}-2e^{-2t}-2te^{-2t}}$$

Example 2: Find the Inverse Laplace Transform of

$$X(s)=\frac{3s+4}{(s+1)(s+2)^2}$$

Solution:

Step 1: Use the partial fraction expansion of X(s) to write

$$X(s)=\frac{A}{(s+1)}+\frac{B}{(s+2)}+\frac{C}{(s+2)^2}$$

Solving the A, B and C by the method of residues

$$\frac{(3s+4)}{(s+1)(s+2)^2}=\frac{A(s+2)^2}{(s+1)(s+2)^2}+\frac{B(s+1)(s+2)}{(s+2)^2(s+1)}+\frac{C(s+1)}{(s+2)^2(s+1)}$$

Cont'd…Example

$$(3s+4) = A(s+2)^2 + B(s+2)(s+1) + C(s+1)$$
$$= A(s^2+4s+4) + B(s^2+3s+2) + C(s+1)$$
$$= (A+B)s^2 + (4A+3B+C)s + (4A+2B+C)$$

$so, compare \quad coefficient,$

$$A + B = 0 \qquad --------(1)$$
$$4A + 3B + C = 3 ------(2)$$
$$4A + 2B + C = 4 ------(3)$$

$(3) - (2);$

$$-B = 1$$
$$B = -1.$$

$From (1)$

$$A + B = 0$$
$$A = 1$$

$Substitute B \quad and \quad A, \text{int}\, o (2)$

$$4(1) + 3(-1) + C = 3$$
$$C = 2.$$

A=1, B=-1 and C=2

$$X(s)=\frac{1}{(s+1)}-\frac{1}{(s+2)}+\frac{2}{(s+2)^2}$$

Step 2: Construct the Inverse Laplace transform from the above partial-fraction term above.

- The pole of the 1st term is at s = -1, so

$$e^{-t}u(t)\xleftrightarrow{L_u}\frac{1}{(s+1)}$$

- The pole of the 2nd term is at s = -2, so

$$e^{-2t}u(t)\xleftrightarrow{L_u}\frac{1}{(s+2)}$$

-The double pole of the 3rd term is at s = -1, so

$$2te^{-t}u(t)\xleftrightarrow{L_u}\frac{1}{(s+2)^2}$$

Step 3: Combining the terms.

.

$$x(t)=e^{-t}u(t)-e^{-2t}u(t)+2te^{-2t}u(t).$$

1) The Laplace Transform cont..

- **Case 3:** **Roots of denominator F(s) are complex conjugate.**
- Example:

$$\boxed{F(s)=\frac{3}{s(s^2+2s+5)}}$$

Solution:

$$F(s)=\frac{A}{s}+\frac{Bs+C}{s^2+2s+5}$$

$$=\frac{3/5}{s}-\frac{3}{5}\left(\frac{s+2}{s^2+2s+5}\right)$$

$$=\frac{3/5}{s}-\frac{3}{5}\left[\frac{(s+1)+(1/2)(2)}{(s+1)^2+2^2}\right]$$

$$\boxed{f(t)=\frac{3}{5}-\frac{3}{5}e^{-t}(\cos 2t+\frac{1}{2}\sin 2t)}$$

It is found that:
A = 3/5, B = -3/5
and C = -6/5

Exercise 2: Laplace Transform Function ~ Differential Equation.

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 2y = 6e^{-4t} \qquad y(0^-) = 2 \quad \frac{dy}{dt}(0^-) = -4$$

Solution:

$$\left[s^2Y(s) - 2s + 4\right] + 9\left[sY(s) - 2\right] + 2Y(s) = \frac{6}{s+4}$$

$$Y(s) = \frac{6}{(s+4)(s^2+9s+2)} + \frac{2s+14}{s^2+9s+2}$$

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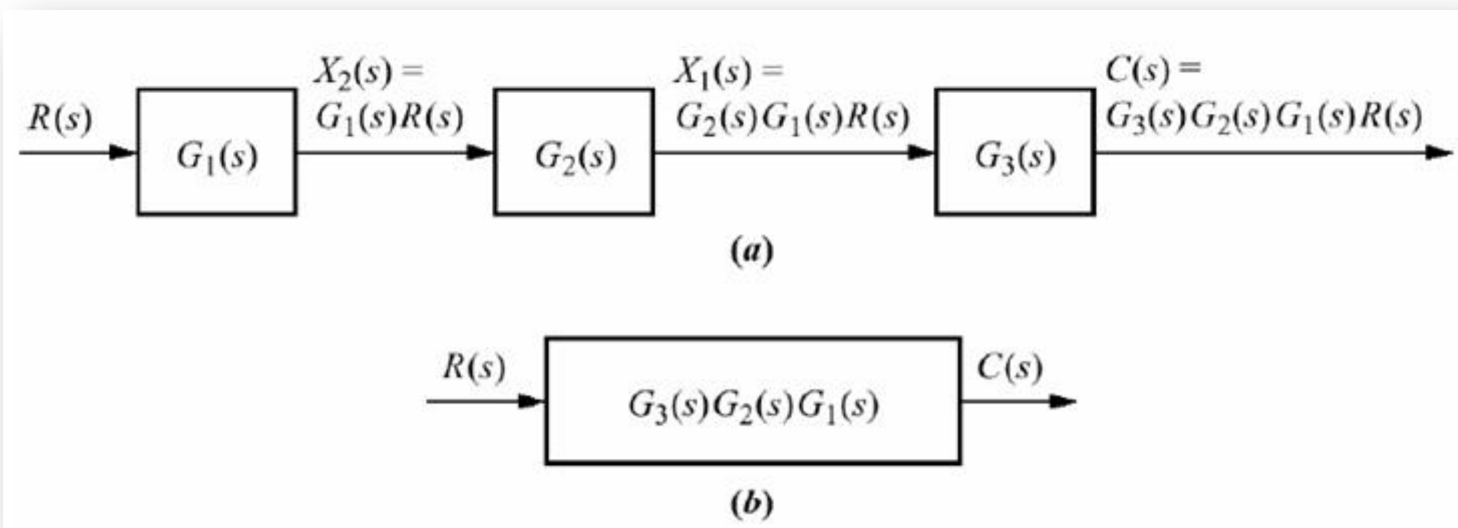
2. Block Diagram

2) Block Diagram

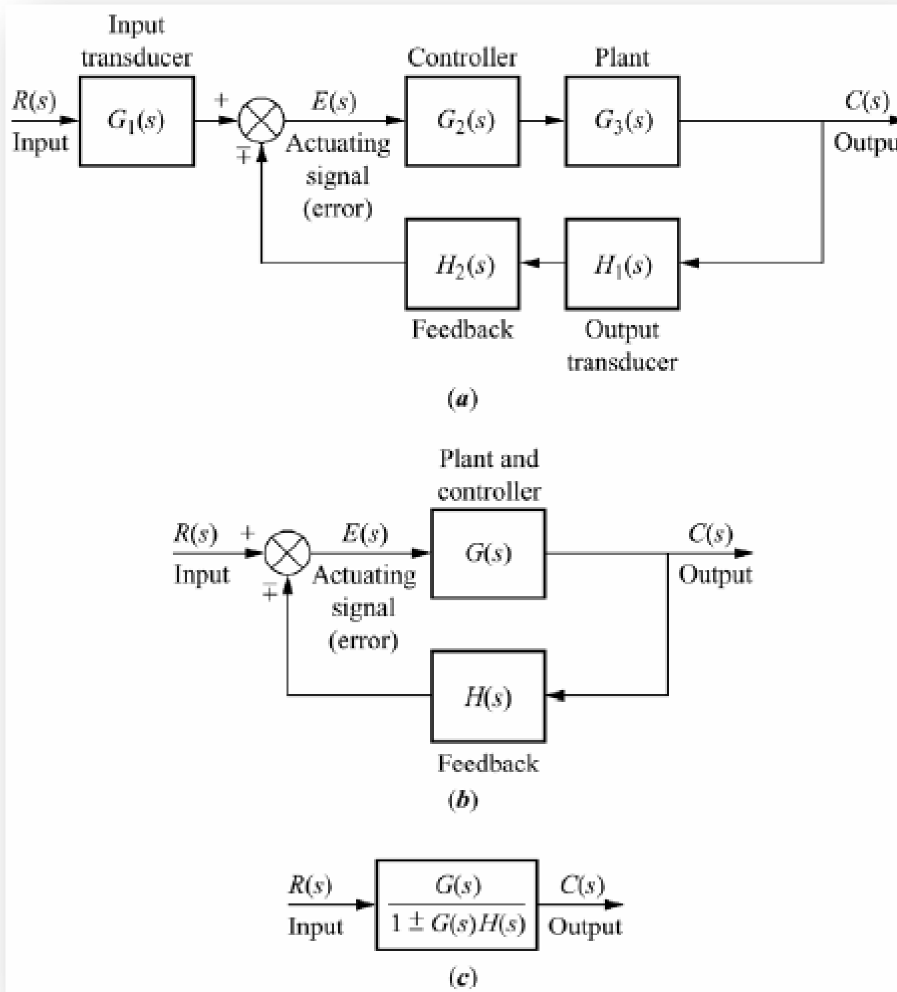
- A block diagram of a system is a practical representation of the functions performed by each component and of the flow of signals.

- Cascaded sub-systems:



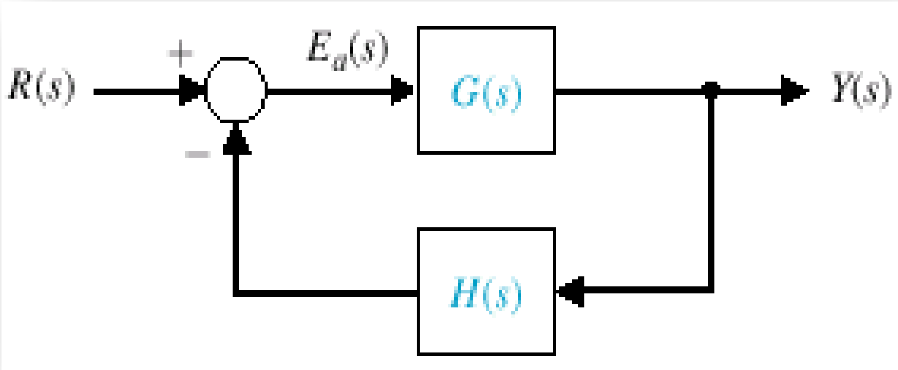
2) Block Diagram cont..

- Feedback Control System



2) Block Diagram cont..

- Feedback Control System



The negative feedback of the control system is given by:

$E_a(s) = R(s) - H(s)Y(s)$

$Y(s) = G(s)E_a(s)$

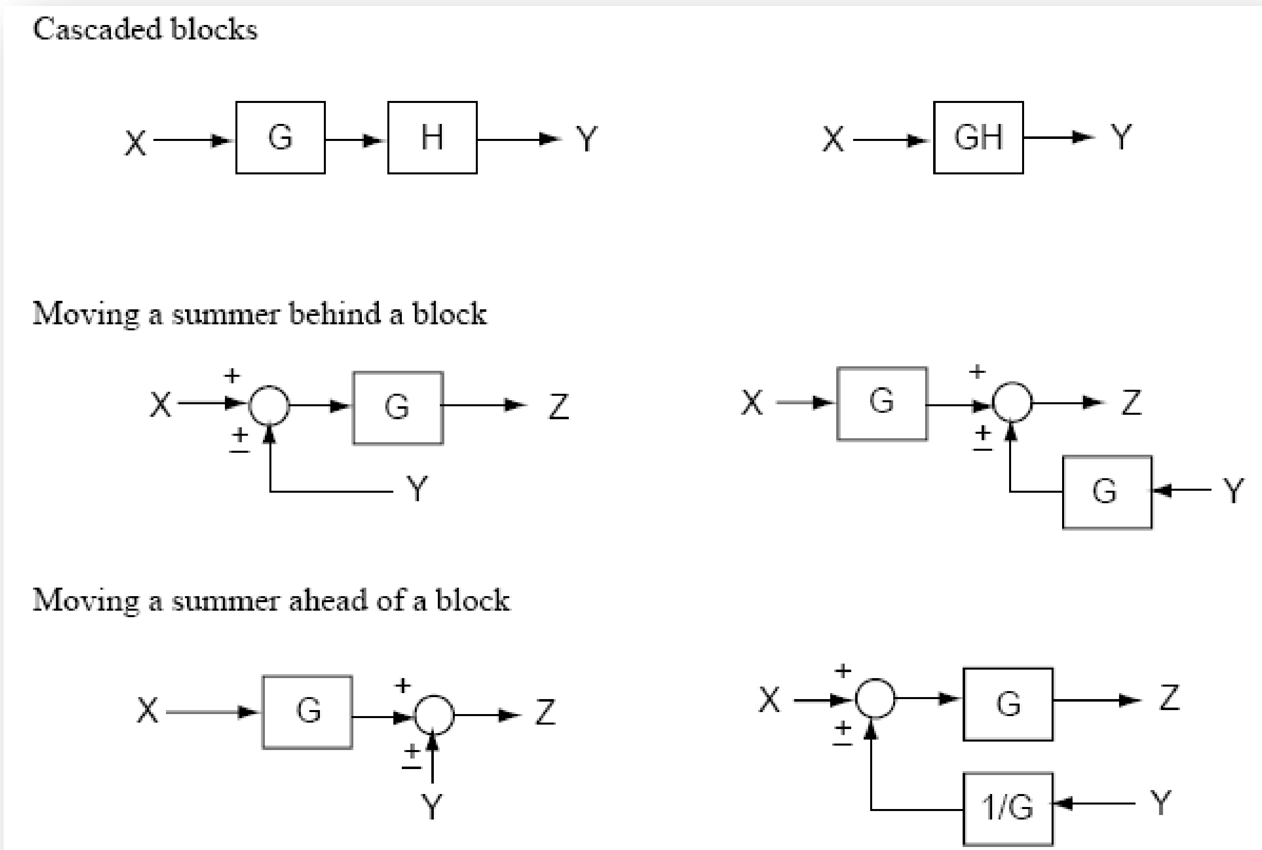
Therefore,

$$Y(s) = G(s)[R(s) - H(s)Y(s)]$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}}$$

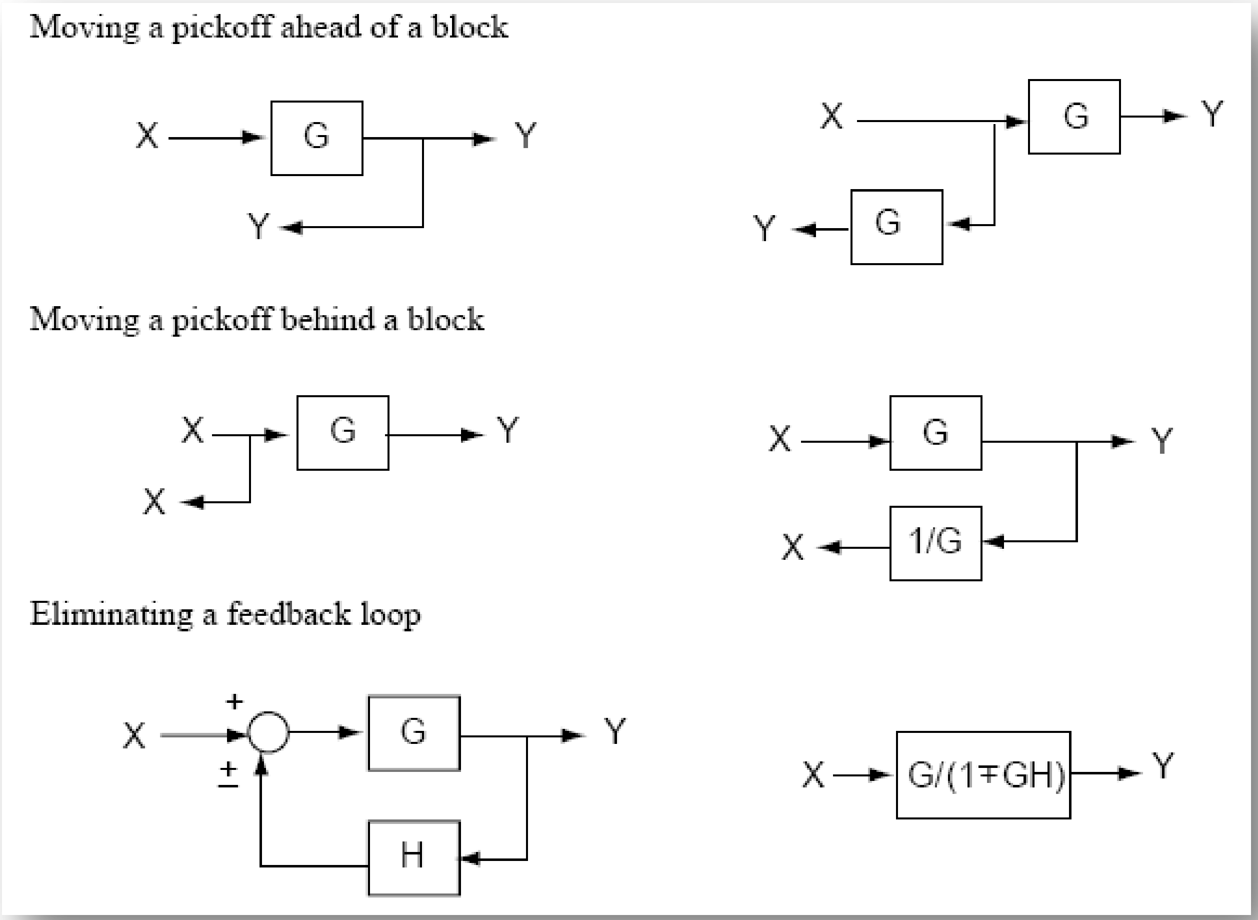
2) Block Diagram cont..

- Reduction Rules



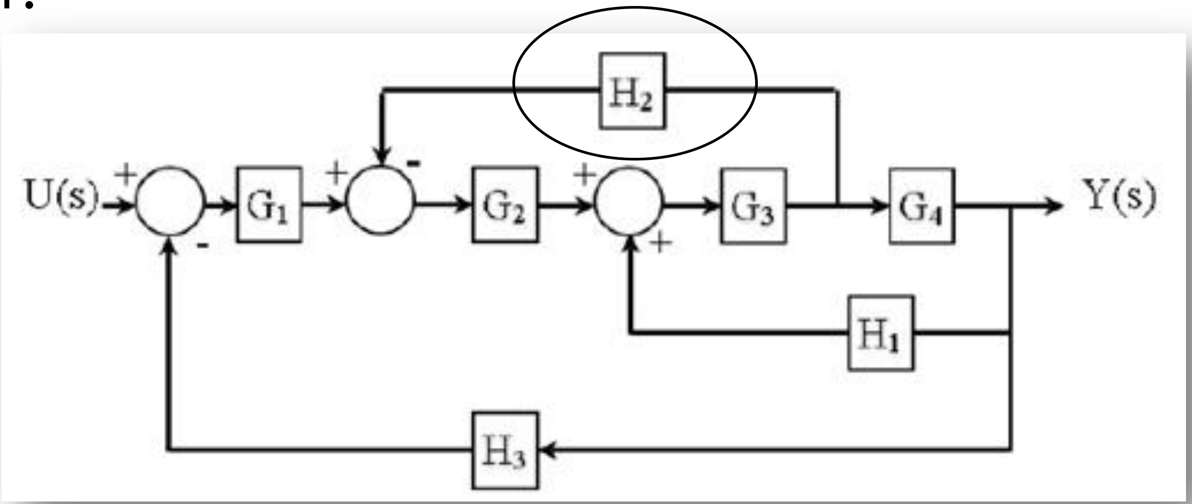
2) Block Diagram cont..

- Reduction Rules



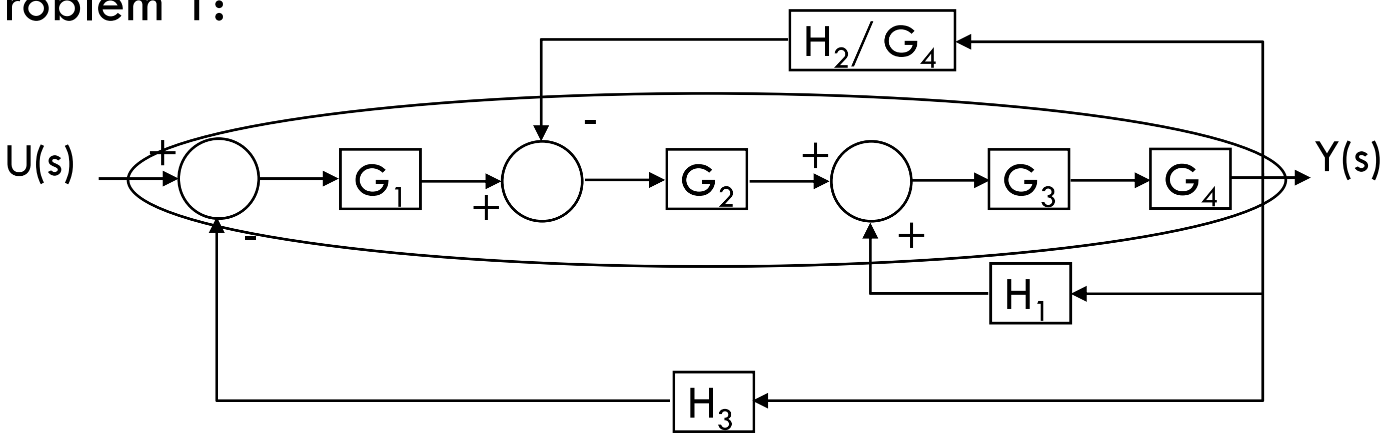
2) Block Diagram cont..

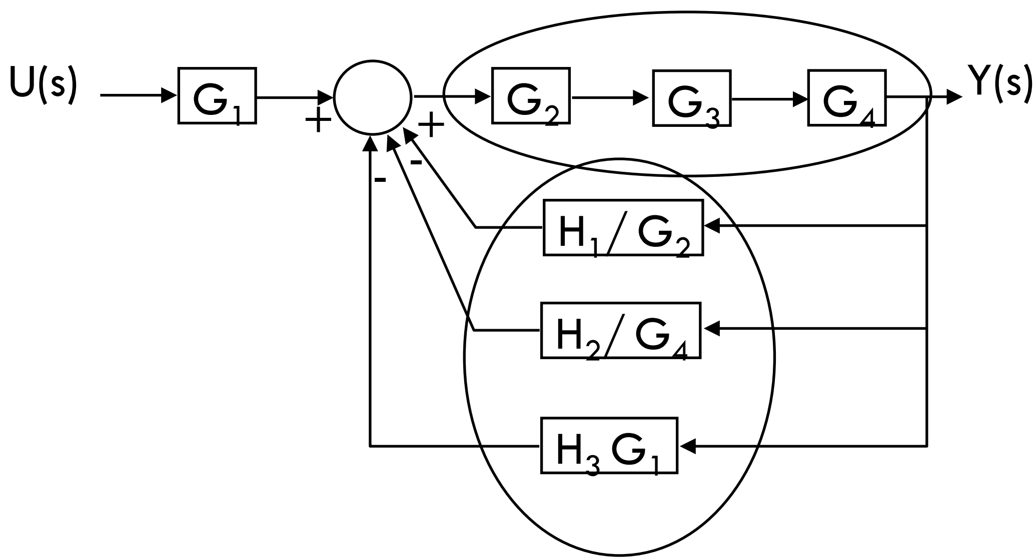
- Problem 1:



2) Block Diagram cont..

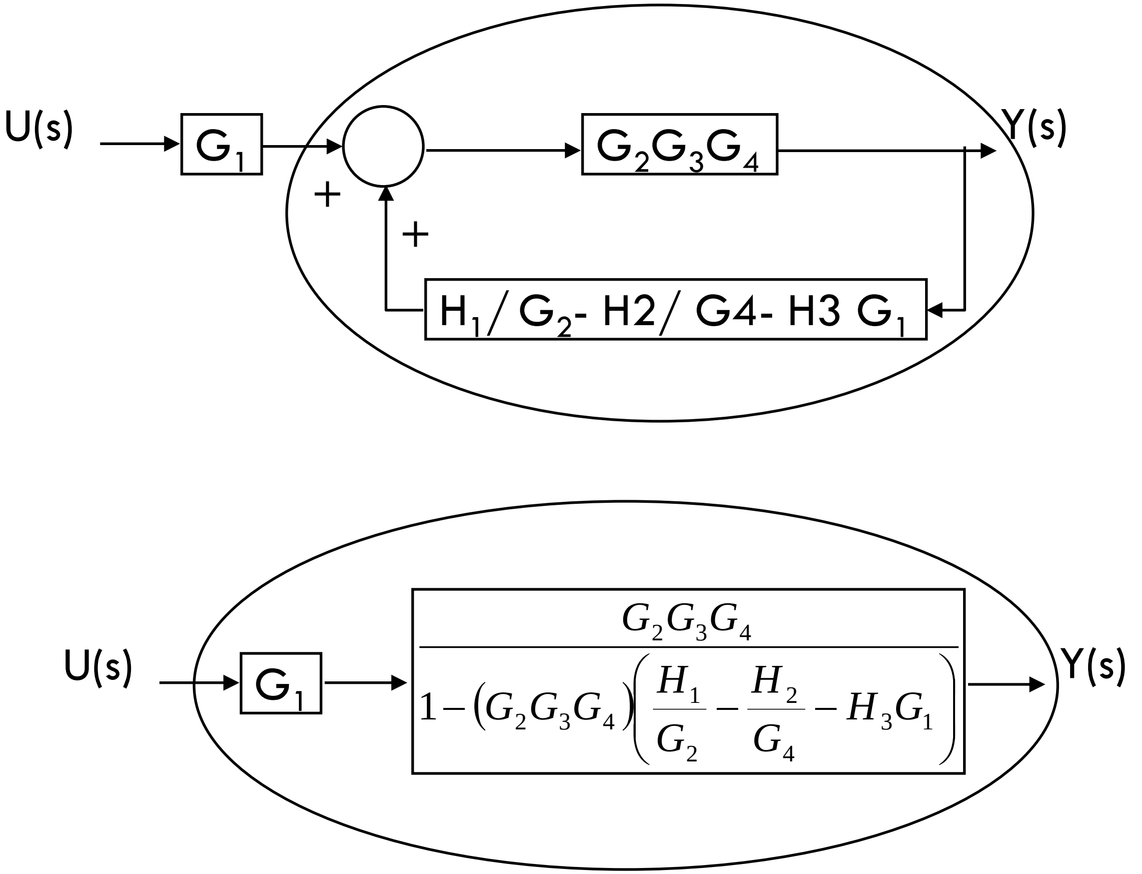
- Problem 1:





2) Block Diagram cont..

- Problem 1:



2) Block Diagram cont..

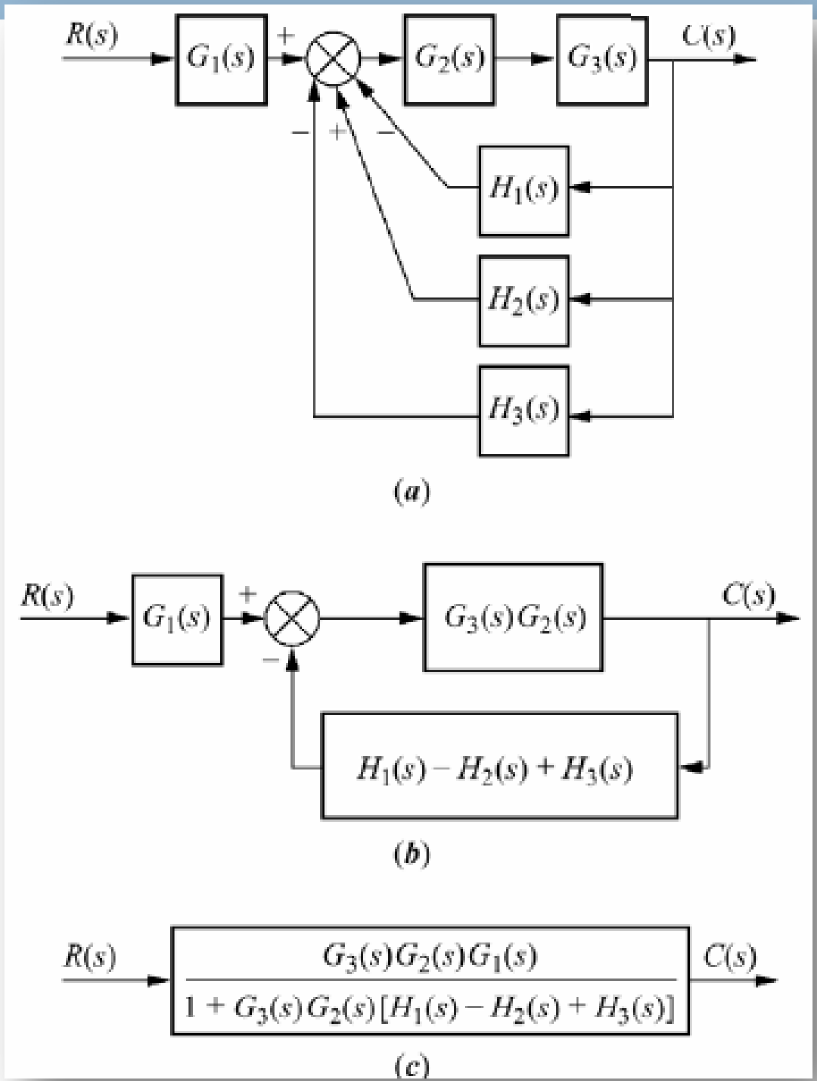
- Problem 1:

$$U(s) \longrightarrow \boxed{\dfrac{G_1G_2G_3G_4}{1-(G_2G_3G_4)\left(\dfrac{H_1}{G_2}-\dfrac{H_2}{G_4}-H_3G_1\right)}} \longrightarrow Y(s)$$

$$U(s) \longrightarrow \boxed{\dfrac{G_1G_2G_3G_4}{1-(G_3G_4H_1-G_2G_3H_2-G_1G_2G_3G_4H_3)}} \longrightarrow Y(s)$$

2) Block Diagram cont..

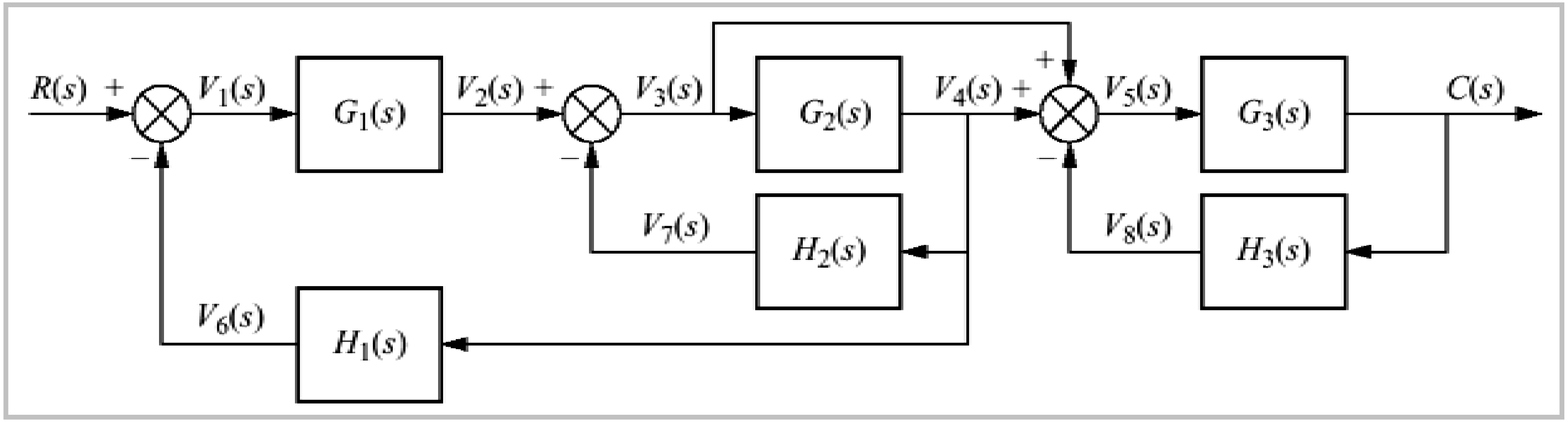
- **Problem 2:**



2) Block Diagram cont..

- **Problem 3:**

Reduce the system to a single transfer function



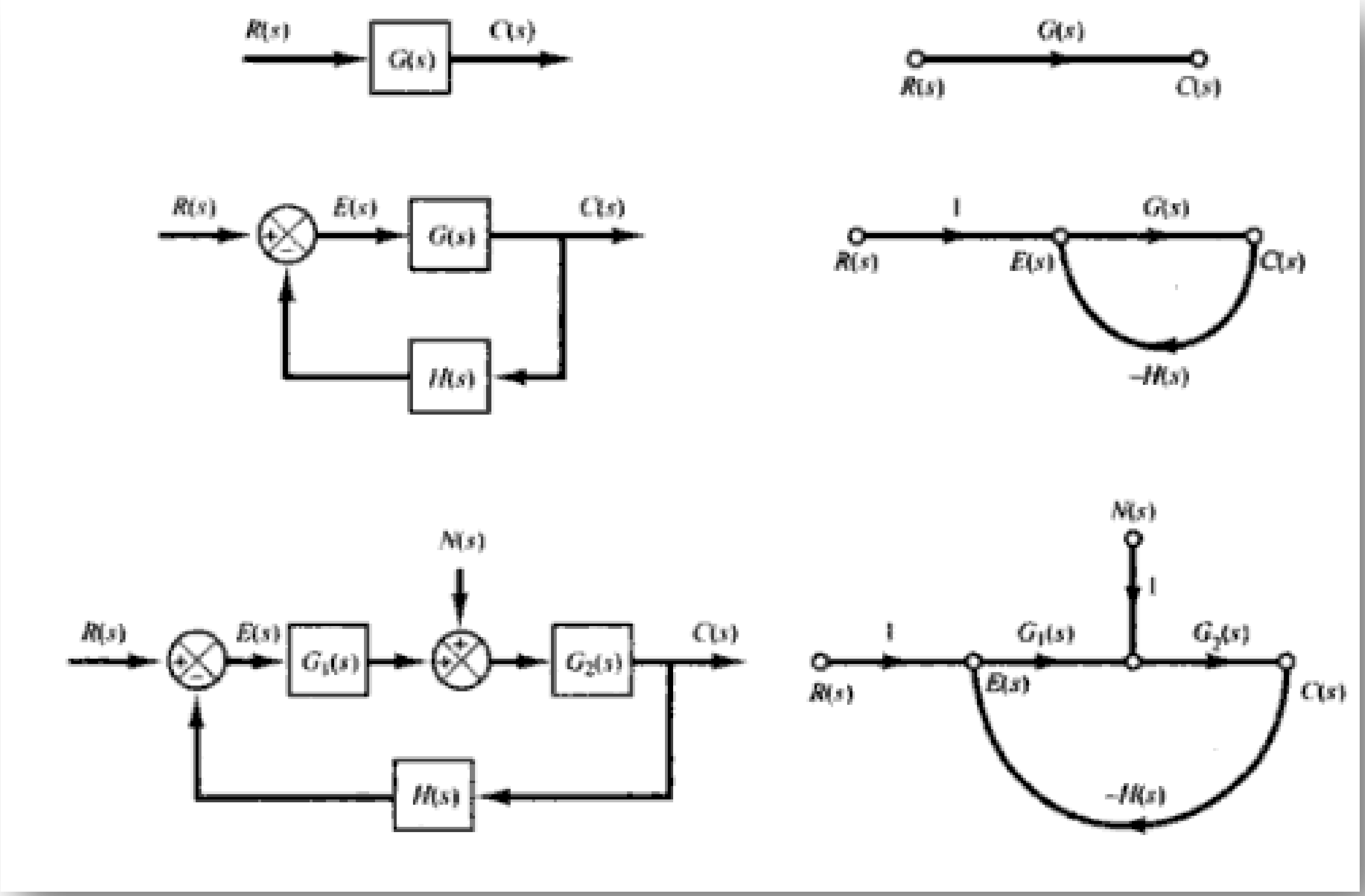
3. Signal-flow graph

3) Signal Flow Graph

- A signal flow graph is a graphical representation of the relationships between the variables of a set of linear algebraic equations.
- The basic element of a signal flow graph is a unidirectional path segments called branch.
- The input and output points or junctions are called nodes.
- A path is a branch or continuous sequence or branches that can be traversed from one signal node to another signal node.
- A loop is a closed path that originates and terminates on the same node, and along the path no node is met twice.
- Two loops are said to be non-touching if they do not have a same common node.

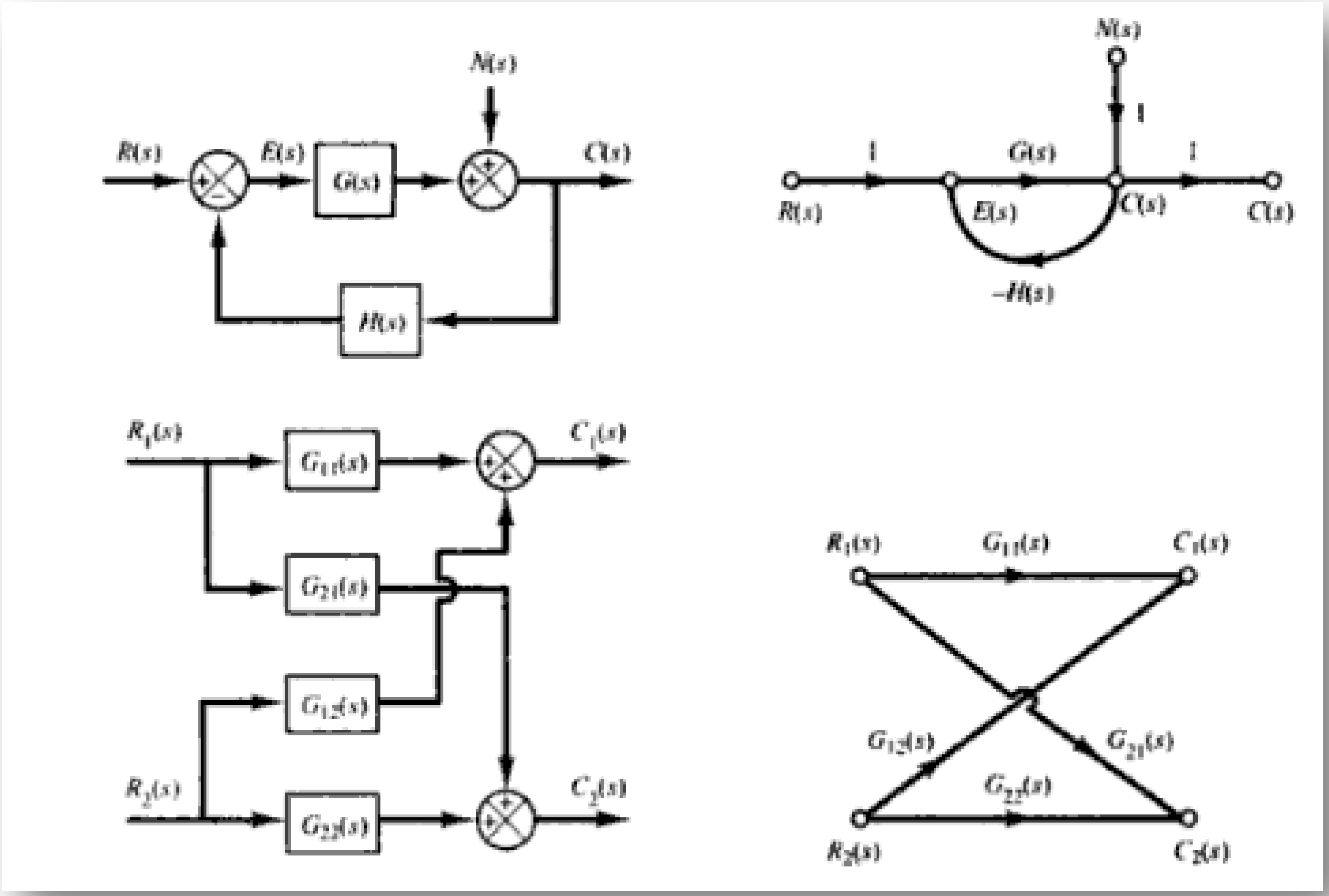
3) Signal Flow Graph cont..

- Signal flow graph of control systems



3) Signal Flow Graph cont..

- Signal flow graph of control systems



3) Signal Flow Graph cont..

- Mason's Gain Formula for Signal Flow Graph

$$\boxed{T_{ij} = \frac{\sum_k P_{ijk}\Delta_{ijk}}{\Delta}}$$

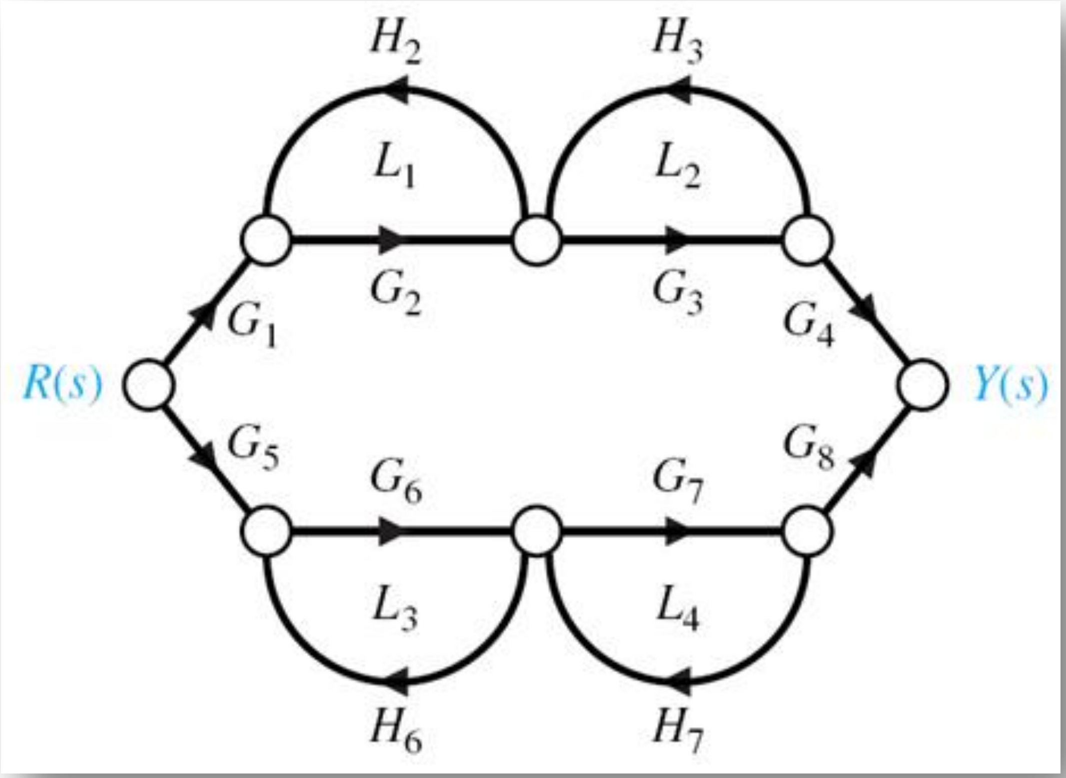
Where,

P_{ijk}	: k^{th} path from variable x_i to x_j
Δ	: Determinant of the graph
Δ_{ijk}	: Cofactor of the path P_{ijk}

$$\boxed{\begin{aligned}\Delta = 1 &- (\text{sum of all different loop gains}) \\ &+ (\text{sum of the gain products of all combinations of 2 nontouching loops}) \\ &- (\text{sum of the gain products of all combinations of 3 nontouching loops}) \\ &+ \ldots\end{aligned}}$$

3) Signal Flow Graph cont..

- Example 1: Transfer function of interacting system



3) Signal Flow Graph cont..

- Example 1: Transfer function of interacting system

a) The paths connecting input R(s) to output Y(s) are:

$$P_1 = G_1 G_2 G_3 G_4$$
$$P_2 = G_5 G_6 G_7 G_8$$

b) There are four individual loops:

$$L_1 = G_2 H_2$$
$$L_2 = G_3 H_3$$
$$L_3 = G_6 H_6$$
$$L_4 = G_7 H_7$$

3) Signal Flow Graph cont..

- Example 1: Transfer function of interacting system

c) Loops L_1 and L_2 does not touch loops L_3 and L_4. Therefore, the determinant is:

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4)$$

d) The cofactor of the determinant along path 1 is evaluated by removing the loops that touch path 1 from Δ. Therefore have:

$$L_1 = L_2 = 0$$

and,

$$\Delta_1 = 1 - (L_3 + L_4)$$

Similarly, the cofactor for path 2 is:

$$\Delta_2 = 1 - (L_1 + L_2)$$

3) Signal Flow Graph cont..

- Example 1: Transfer function of interacting system

e) Therefore, the transfer function of the system is:

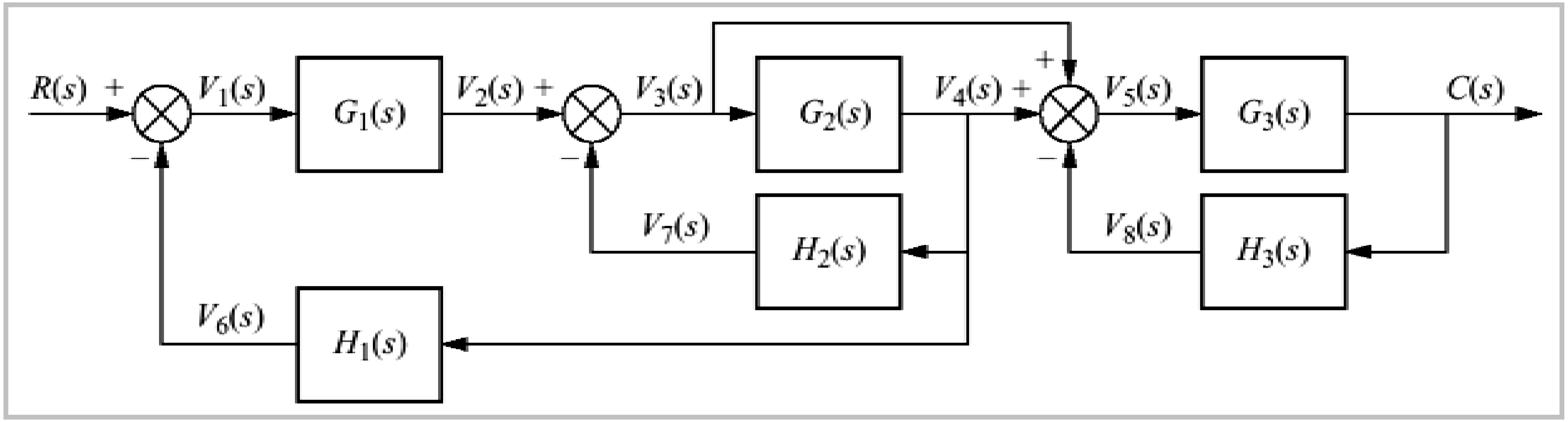
$$\frac{Y(s)}{R(s)} = T(s) = \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3G_4(1 - L_3 - L_4) + G_5G_6G_7G_8(1 - L_1 - L_2)}{1 - L_1 - L_2 - L_3 - L_4 + L_1L_3 + L_1L_4 + L_2L_3 + L_2L_4}$$

3) Signal Flow Graph cont..

- **Problem 1:**

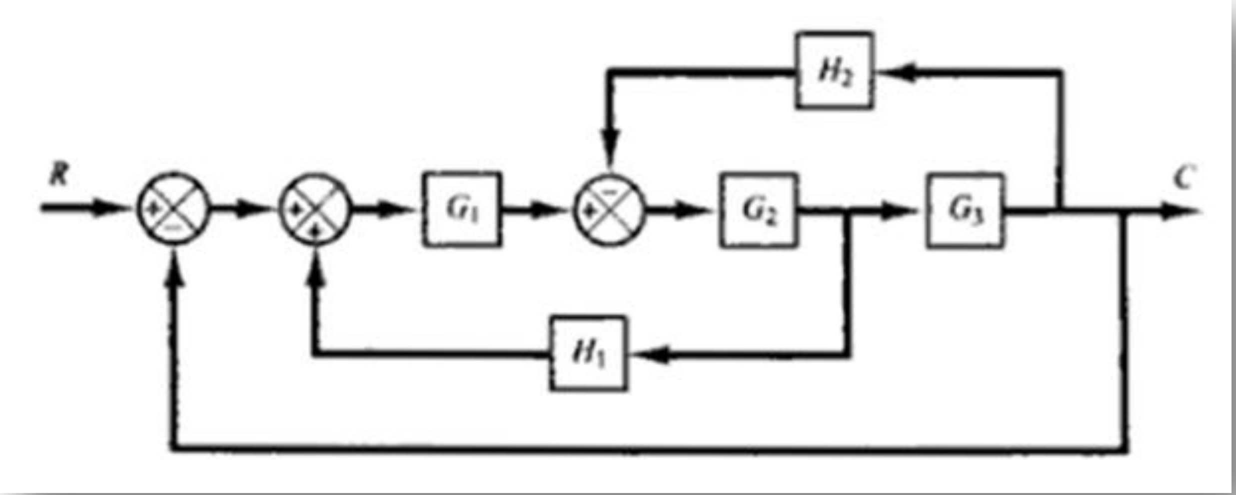
  Obtain the closed-loop transfer function by use of Mason's Gain Formula



3) Signal Flow Graph cont..

- **Problem 2:**

  Obtain the closed-loop transfer function by use of Mason's Gain Formula



4. Review state space variable

Introduction

- The basic questions that will be addressed in state-space approach include:
  1. What are state-space models?
  2. Why should we use them?
  3. How are they related to the transfer function used in classical control system?
  4. How do we develop a space-state model?

4) State-Space Model

A representation of the dynamics of N^{th}-order system as a first-order equation in an N-vector, which is called the state.

Convert the Nth-order differential equation that governs the dynamics of the system into N first-order differential equation.

4) State-Space Model

- The state of a system is described by a set of first-order differential equations written in terms of the state variable.

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + b_{11}u_1 + \ldots + b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + b_{21}u_1 + \ldots + b_{2m}u_m \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + b_{n1}u_1 + \ldots + b_{nm}u_m
\end{aligned}
$$

where $\dot{x} = dx/dt$.

4) State-Space Model

- In a matrix form, we have:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} . \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} . \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

- State vector:

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad u = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}$$

4) State Space Model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

Input equation

Output equation

$\mathbf{x}$ = state vector
$\mathbf{y}$ = output vector
$\mathbf{u}$ = input or control vector
$\mathbf{A}$ = system matrix
$\mathbf{B}$ = input matrix
$\mathbf{C}$ = output matrix
$\mathbf{D}$ = feedforward matrix