## Derivatives of Trig Functions

In Part 3 we have introduced the idea of a derivative of a function, which 1 we defined in terms of a limit. Then we began the task of finding rules that compute derivatives without limits. Here is our list of rules so far.
Constant function rule: $D_{x}[c]=0$
Identity function rule: $\quad D_{x}[x]=1$
Power rule:

$$
D_{x}\left[x^{n}\right]=n x^{n-1}
$$

Exponential rule:
$D_{x}\left[e^{x}\right]=e^{x}$
Constant multiple rule: $D_{x}[c f(x)]=c f^{\prime}(x)$
Sum-difference rule:
$D_{x}[f(x) \pm g(x)]=f^{\prime}(x) \pm g^{\prime}(x)$
Product rule:
$D_{x}[f(x) g(x)]=f^{\prime}(x) g(x)+f(x) g^{\prime}(x)$
Quotient rule:

$$
D_{x}\left[\frac{f(x)}{g(x)}\right]=\frac{f^{\prime}(x) g(x)-f(x) g^{\prime}(x)}{\left.g^{( } x\right)^{2}}
$$

In this chapter we will expand this list by adding six new rules for the derivatives of the six trigonometric functions:
$D_{x}[\sin (x)] \quad D_{x}[\tan (x)] \quad D_{x}[\sec (x)] \quad D_{x}[\cos (x)] \quad D_{x}[\csc (x)] \quad D_{x}[\cot (x)]$
This will require a few ingredients. First, we will need the addition formulas for sine and cosine (Equations 3.12 and 3.13 on page 46):

$$
\begin{aligned}
& \sin (\alpha+\beta)=\sin (\alpha) \cos (\beta)+\cos (\alpha) \sin (\beta) \\
& \cos (\alpha+\beta)=\cos (\alpha) \cos (\beta)-\sin (\alpha) \sin (\beta)
\end{aligned}
$$

Recall also from Chapter 3 the fundamental identity $\quad \sin ^{2}(x)+\cos ^{2}(x)=1$.

And we will need these limits from Theorem 10.2 (Chapter 10, page 152):

$$
\lim _{h \rightarrow 0} \frac{\cos (h)-1}{h}=0 \quad \text { and } \quad \lim _{h \rightarrow 0} \frac{\sin (h)}{h}=1 .
$$

Let's start by computing the derivative of $f(x)=\sin (x)$.

$$
\begin{array}{rlr}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} & \text { (Definition 16.1) }  \tag{Definition16.1}\\
& =\lim _{h \rightarrow 0} \frac{\sin (x+h)-\sin (x)}{h} & (f(x)=\sin (x)) \\
& =\lim _{h \rightarrow 0} \frac{\sin (x) \cos (h)+\cos (x) \sin (h)-\sin (x)}{h} & \text { (addition formula for sin) } \\
& =\lim _{h \rightarrow 0} \frac{\sin (x) \cos (h)-\sin (x)+\cos (x) \sin (h)}{h} & \text { (regroup) } \\
& =\lim _{h \rightarrow 0} \frac{\sin (x)(\cos (h)-1)+\cos (x) \sin (h)}{h} & \text { (factor out } \sin (x)) \\
& =\lim _{h \rightarrow 0}\left(\frac{\sin (x)(\cos (h)-1)}{h}+\frac{\cos (x) \sin (h)}{h}\right) & \text { (break up fraction) } \\
& =\lim _{h \rightarrow 0}\left(\sin (x) \frac{\cos (h)-1}{h}+\cos (x) \frac{\sin (h)}{h}\right) \\
& =\lim _{h \rightarrow 0} \sin (x) \cdot \lim _{h \rightarrow 0} \frac{\cos (h)-1}{h}+\lim _{h \rightarrow 0} \cos (x) \cdot \lim _{h \rightarrow 0} \frac{\sin (h)}{h} & \text { (limit laws) } \\
& =\sin (x) \cdot 0+\cos (x) \cdot 1 \\
& =\cos (x) . & \text { (Theorem 10.2) }
\end{array}
$$

Therefore the derivative of $\sin (x)$ is $\cos (x)$. This is our latest derivative rule.
Rule $9 D_{x}[\sin (x)]=\cos (x)$
This rule makes sense when we compare the graph of $\sin (x)$ with its derivative $\cos (x)$. The tangent to $\sin (x)$ has slope 0 at integer multiples of $\frac{\pi}{2}$, and these are exactly the places that $\cos (x)=0$. And notice that where the tangent to $\sin (x)$ has positive slope, $\cos (x)$ is positive; where the tangent to $\sin (x)$ has negative slope, $\cos (x)$ is negative. As the derivative of $\sin (x), \cos (x)$ equals the slope of the tangent to $\sin (x)$ at $(x, \sin (x))$.


So what is the derivative of $\cos (x)$ ? Since $D_{x}[\sin (x)]=\cos (x)$, you might first guess that $D_{x}[\cos (x)]=\sin (x)$. But this is not quite right because for $0<x<\pi$ the tangents to $\cos (x)$ have negative slope, while $\sin (x)$ is positive. However, the graphs below suggest $D_{x}[\cos (x)]=-\sin (x)$.


In fact this turns out to be exactly right. This chapter's Exercise 19 asks you to adapt the computation on the previous page to get the following rule.

Rule $10 \quad D_{x}[\cos (x)]=-\sin (x)$
We now have derivative rules for sin and cos. Next, let's compute the derivative of $\tan (x)$. We will use the rules we just derived in conjunction with the quotient rule and familiar identities.

$$
\begin{aligned}
D_{x}[\tan (x)] & =D_{x}\left[\frac{\sin (x)}{\cos (x)}\right] \\
& =\frac{D_{x}[\sin (x)] \cos (x)-\sin (x) D_{x}[\cos (x)]}{\cos ^{2}(x)} \\
& =\frac{\cos (x) \cos (x)-\sin (x)(-\sin (x))}{\cos ^{2}(x)}=\frac{\cos ^{2}(x)+\sin ^{2}(x)}{\cos ^{2}(x)} \\
& =\frac{1}{\cos ^{2}(x)}=\left(\frac{1}{\cos (x)}\right)^{2}=\sec ^{2}(x)
\end{aligned}
$$

We have a new rule: Rule $11 D_{x}[\tan (x)]=\sec ^{2}(x)$
Exercise 20 asks you to do a similar computation to show that
Rule $12 D_{x}[\cot (x)]=-\csc ^{2}(x)$
These two latest formulas fit the shapes of the graphs of tan and cot as suggested by Figure 21.1.



Figure 21.1. Any tangent line to the graph of $y=\tan (x)$ has positive slope. Indeed the slope of the tangent at $x$ is the positive number $y^{\prime}=\sec ^{2}(x)$. Any tangent line to the graph of $y=\cot (x)$ has negative slope; the slope of the tangent at $x$ is the negative number $y^{\prime}=-\csc ^{2}(x)$.

There are just two more trig functions to consider: sec and csc. We have

$$
\begin{aligned}
D_{x}[\sec (x)]=D_{x}\left[\frac{1}{\cos (x)}\right] & =\frac{D_{x}[1] \cdot \cos (x)-1 \cdot D_{x}[\cos (x)]}{\cos ^{2}(x)} \\
& =\frac{0 \cdot \cos (x)-1 \cdot(-\sin (x))}{\cos ^{2}(x)} \\
& =\frac{\sin (x)}{\cos ^{2}(x)}=\frac{1}{\cos (x)} \cdot \frac{\sin (x)}{\cos (x)}=\sec (x) \tan (x) .
\end{aligned}
$$

This is our latest rule. Rule $13 D_{x}[\sec (x)]=\sec (x) \tan (x)$

This chapter's Exercise 21 asks you to do a similar computation to prove

$$
\text { Rule } 14 \quad D_{x}[\csc (x)]=-\csc (x) \cot (x) \text {. }
$$

We now have derivative rules for all six trig functions, which was this chapter's goal. Here is a summary of what we've discovered.

## Derivatives of Trig Functions

$$
\begin{array}{lll}
D_{x}[\sin (x)]=\cos (x) & D_{x}[\tan (x)]=\sec ^{2}(x) & D_{x}[\sec (x)]=\sec (x) \tan (x) \\
D_{x}[\cos (x)]=-\sin (x) & D_{x}[\cot (x)]=-\csc ^{2}(x) & D_{x}[\csc (x)]=-\csc (x) \cot (x)
\end{array}
$$

Example 21.1 Find the derivative of $y=\frac{\sin (x)}{x^{2}+1}$.
This is a quotient, so we use the quotient rule combined with our new rule for the derivative of sin.

$$
\begin{aligned}
D_{x}\left[\frac{\sin (x)}{x^{2}+1}\right] & =\frac{D_{x}[\sin (x)]\left(x^{2}+1\right)-\sin (x) D_{x}\left[x^{2}+1\right]}{\left(x^{2}+1\right)^{2}} \\
& =\frac{\cos (x)\left(x^{2}+1\right)-\sin (x) 2 x}{\left(x^{2}+1\right)^{2}}
\end{aligned}
$$

Example 21.2 Find the derivative of $x^{2}+x^{3} \tan (x)+\pi$.
This is the sum of a power, a product and a constant, so we begin with the sum-difference rule, breaking the problem into three separate derivatives, then using applicable rules.

$$
\begin{aligned}
D_{x}\left[x^{2}+x^{3} \tan (x)+\pi\right] & =D_{x}\left[x^{2}\right]+D_{x}\left[x^{3} \tan (x)\right]+D_{x}[\pi] \\
& =2 x+\underbrace{D_{x}\left[x^{3}\right] \tan (x)+x^{3} D_{x}[\tan (x)]}_{\text {product rule }}+0 \\
& =2 x+3 x^{2} \tan (x)+x^{3} \sec ^{2}(x)
\end{aligned}
$$

With practice you will quickly reach the point where you will do such a problem in your head, in one step. (You may already be there.)
Example 21.3 If $z=\frac{e^{w} \sec (w)}{w}$, find the derivative $\frac{d z}{d w}$.
This is a quotient, so our first step is to apply the quotient rule.

$$
\frac{d z}{d w}=\frac{D_{w}\left[e^{w} \sec (w)\right] \cdot w+e^{w} \sec (w) \cdot D_{w}[w]}{w^{2}}
$$

This now involves $D_{w}\left[e^{w} \sec (w)\right]$, and that requires the product rule.

$$
\begin{aligned}
& =\frac{\left(D_{w}\left[e^{w}\right] \sec (w)+e^{w} D_{w}[\sec (w)]\right) \cdot w+e^{w} \sec (w) \cdot 1}{w^{2}} \\
& =\frac{\left(e^{w} \sec (w)+e^{w} \sec (w) \tan (w)\right) \cdot w+e^{w} \sec (w)}{w^{2}} \\
& =\frac{e^{w} \sec (w)(w+w \tan (w)+1)}{w^{2}}
\end{aligned}
$$

Example 21.4 Find the equation of the tangent line to the graph of $y=$ $\cos (x)$ at the point $\left(\frac{\pi}{6}, \cos \left(\frac{\pi}{6}\right)\right)$.

The slope of the tangent line at the point $(x, \cos (x))$ is given by the derivative $\frac{d y}{d x}=-\sin (x)$. In this problem we are interested in the tangent line at the exact point $\left(\frac{\pi}{6}, \cos \left(\frac{\pi}{6}\right)\right)=\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$, so that tangent line has slope $-\sin \left(\frac{\pi}{6}\right)=-\frac{1}{2}$.


So we are looking for the equation of the line through the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$, with slope $-\frac{1}{2}$. We can get this with the point-slope formula for a line.

$$
\begin{aligned}
y-y_{0} & =m\left(x-x_{0}\right) \\
y-\frac{\sqrt{3}}{2} & =-\frac{1}{2}\left(x-\frac{\pi}{6}\right) \\
y & =-\frac{1}{2} x+\frac{\pi}{12}+\frac{\sqrt{3}}{2}
\end{aligned}
$$

Answer: The equation of the tangent line is $y=-\frac{1}{2} x+\frac{\pi+6 \sqrt{3}}{12}$.

## Exercises for Chapter 21

In exercises $1-14$ find the derivative of the indicated function.

1. $y=\sqrt[3]{x} \sin (x)$
2. $f(r)=5 r-\cos (r)+\frac{1}{r}$
3. $f(z)=\sin ^{2}(z)$
4. $y=x^{4} \tan (x)$
5. $y=x^{2} \sec (x)$
6. $f(r)=3 e^{r}-\frac{1}{r^{2}}+\sin (r)$
7. $y=\tan (x)+\frac{1}{x^{2}}+e^{2}+3$
8. $f(\theta)=5 \theta-\cot (\theta)+\sqrt{\theta}$
9. $f(s)=\tan (s)-\frac{3}{s^{2}}+2 e^{s}$
10. $y=\tan ^{2}(x)$
11. $y=\frac{\sqrt{x} \cos (x)}{x^{3}+1}$
12. $y=\frac{x \sin (x)}{e^{x}}$
13. $y=\frac{x^{2}+5}{x+\sec (x)}$
14. $y=\frac{x \cos (x)}{\sin (x)+1}$
15. Find $\frac{d z}{d w}$ if $z=\frac{5}{w}+\frac{\tan (w)}{w+1}$.
16. Find $\frac{d z}{d w}$ if $z=\sqrt{w}+5(w+1) \sec (w)$.
17. Find $\lim _{h \rightarrow 0} \frac{\sin \left(\frac{\pi}{3}+h\right)-\sin \left(\frac{\pi}{3}\right)}{h}$
18. Find $\lim _{h \rightarrow 0} \frac{\tan \left(\frac{\pi}{4}+h\right)-\tan \left(\frac{\pi}{4}\right)}{h}$
19. Adapt this chapter's derivation of the rule $D_{x}[\sin (x)]=\cos (x)$ to show that $D_{x}[\cos (x)]=$ $-\sin (x)$.
20. Adapt this chapter's derivation of the rule $D_{x}[\tan (x)]=\sec ^{2}(x)$ to show that $D_{x}[\cot (x)]=-\csc ^{2}(x)$.
21. Adapt this chapter's derivation of the rule $D_{x}[\sec (x)]=\sec (x) \tan (x)$ to show that $D_{x}[\csc (x)]=-\csc (x) \cot (x)$.
22. Suppose $f(x)=\left(x^{2}-\pi^{2}\right) \cos (x)$. Find the equation of the tangent line to the graph of $f(x)$ at the point $(\pi, f(\pi))$.
23. Suppose $f(x)=\frac{\sin (x)}{x}$. Find the equation of the tangent line to the graph of $f(x)$ at the point $(\pi, f(\pi))$.
24. Suppose $f(x)=x^{3}-x+2$. Find the equation of the line tangent to the graph of $f(x)$ at the point $(2,8)$.
25. Find the equation of the tangent line to the graph of $y=\sin (x)$ at the point where $x=\pi$.
26. Find all values of $x$ for which the tangents the graphs of $y=\sqrt{3} \sin (x)$ and $y=$ $-\cos (x)$ are parallel.
27. Find all values of $x$ for which the tangents the graphs of $y=x \sin (x)$ and $y=$ $x^{2} / 4-\cos (x)$ are parallel.

## Exercise Solutions for Chapter 21

1. $y=\sqrt[3]{x} \sin (x)=x^{1 / 3} \sin (x) \quad y^{\prime}=\frac{1}{3} x^{1 / 3-1} \sin (x)+x^{1 / 3} \cos (x)=\frac{\sin (x)}{\sqrt[3]{x}^{2}}+\sqrt[3]{x} \cos (x)$
2. $f(z)=\sin ^{2}(z)=\sin (z) \sin (z) \quad f^{\prime}(z)=\cos (z) \sin (z)+\sin (z) \cos (z)=2 \sin (z) \cos (z)$
3. $y=x^{2} \sec (x) \quad$ By product rule: $y^{\prime}=2 x \sec (x)+x^{2} \sec (x) \tan (x)$
4. $y=\tan (x)+\frac{1}{x^{2}}+e^{2}+3=\tan (x)+x^{-2}+e^{2}+3 \quad y^{\prime}=\sec ^{2}(x)-2 x^{-3}+0+0=\sec ^{2}(x)-\frac{2}{x^{3}}$
5. $f(s)=\tan (s)-\frac{3}{s^{2}}+2 e^{s} \quad f^{\prime}(s)=\sec ^{2}(s)+\frac{6}{x^{3}}+2 e^{s}$
6. $y=\frac{\sqrt{x} \cos (x)}{x^{3}+1}=\frac{x^{1 / 2} \cos (x)}{x^{3}+1}$

$$
y^{\prime}=\frac{D_{x}\left[x^{1 / 2} \cos (x)\right]\left(x^{3}+1\right)-x^{1 / 2} \cos (x) \cdot D_{x}\left[x^{3}+1\right]}{\left(x^{3}+1\right)^{2}}
$$

$$
\begin{array}{r}
=\frac{\left(\frac{1}{2} x^{-1 / 2} \cos (x)+x^{1 / 2}(-\sin (x))\right)\left(x^{3}+1\right)-x^{1 / 2} \cos (x) 3 x^{2}}{\left(x^{3}+1\right)^{2}} \\
=\frac{\left(\frac{\cos (x)}{2 \sqrt{x}}-\sqrt{x} \sin (x)\right)\left(x^{3}+1\right)-\sqrt{x} \cos (x) 3 x^{2}}{\left(x^{3}+1\right)^{2}}
\end{array}
$$

13. $y=\frac{x^{2}+5}{x+\sec (x)} \quad y^{\prime}=\frac{2 x(x+\sec (x))-\left(x^{2}+5\right) \sec (x) \tan (x)}{(x+\sec (x))^{2}}$
14. $z=\frac{5}{w}+\frac{\tan (w)}{w+1}=5 w^{-1}+\frac{\tan (w)}{w+1}$.

Answer: $\frac{d z}{d w}=-5 w^{-2}+\frac{\sec ^{2}(w)(w+1)-\tan (w) \cdot 1}{(w+1)^{2}}=-\frac{5}{w^{2}}+\frac{\sec ^{2}(w)(w+1)-\tan (w)}{(w+1)^{2}}$
17. Find $\lim _{h \rightarrow 0} \frac{\sin (\pi / 3+h)-\sin (\pi / 3)}{h}$. Let $f(x)=\sin (x)$. Then by the definition of the derivative, this limit equals $f^{\prime}(\pi / 3)=\cos (\pi / 3)=\frac{1}{2}$.
19. Adapt the derivation of the rule $D_{x}[\sin (x)]=\cos (x)$ to prove $D_{x}[\cos (x)]=-\sin (x)$.

$$
\begin{array}{rlr}
D_{x}[\cos (x)] & =\lim _{h \rightarrow 0} \frac{\cos (x+h)-\cos (x)}{h} & \text { (Definition 16.1) } \\
& =\lim _{h \rightarrow 0} \frac{\cos (x) \cos (h)-\sin (x) \sin (h)-\cos (x)}{h} & \text { (addition formula for } \cos \text { ) } \\
& =\lim _{h \rightarrow 0} \frac{\cos (x) \cos (h)-\cos (x)-\sin (x) \sin (h)}{h} & \text { (regroup) } \\
& =\lim _{h \rightarrow 0} \frac{\cos (x)(\cos (h)-1)+\sin (x) \sin (h)}{h} & \text { (factor out } \sin (x)) \\
& =\lim _{h \rightarrow 0}\left(\cos (x) \frac{\cos (h)-1}{h}-\sin (x) \frac{\sin (h)}{h}\right) & \text { (break up fraction) } \\
& =\lim _{h \rightarrow 0} \cos (x) \cdot \lim _{h \rightarrow 0} \frac{\cos (h)-1}{h}-\lim _{h \rightarrow 0} \sin (x) \cdot \lim _{h \rightarrow 0} \frac{\sin (h)}{h} & \text { (limit laws) } \\
& =\cos (x) \cdot 0-\sin (x) \cdot 1=-\sin (x) . & \text { (Theorem 10.2) } \tag{Theorem10.2}
\end{array}
$$

21. Adapt this chapter's derivation of the rule $D_{x}[\sec (x)]=\sec (x) \tan (x)$ to show that $D_{x}[\csc (x)]=-\csc (x) \cot (x)$.

$$
\begin{array}{r}
D_{x}[\csc (x)]=D_{x}\left[\frac{1}{\sin (x)}\right]=\frac{D_{x}[1] \cdot \sin (x)-1 \cdot D_{x}[\sin (x)]}{\sin ^{2}(x)}=\frac{0 \cdot \sin (x)-1 \cdot(\cos (x))}{\sin ^{2}(x)} \\
=\frac{-\cos (x)}{\sin ^{2}(x)}=-\frac{1}{\sin (x)} \cdot \frac{\cos (x)}{\sin (x)}=-\csc (x) \cot (x) .
\end{array}
$$

23. Find the equation of the tangent line to the graph of $f(x)=\frac{\sin (x)}{x}$ at $(\pi, f(\pi))$. The slope of the line is given by the derivative $f^{\prime}(x)=\frac{\cos (x) x-\sin (x) \cdot 1}{x^{2}}$. The line passes through the point $(\pi, f(\pi))=(\pi, 0)$. At this point the slope of the tangent line is $m=f^{\prime}(\pi)=\frac{\cos (\pi) \pi-\sin (\pi)}{\pi^{2}}=\frac{-1 \cdot \pi-0}{\pi^{2}}=-\frac{1}{\pi}$. We can get the equation of the line by the point-slope formula:

$$
\begin{aligned}
y-y_{0} & =m\left(x-x_{0}\right) \\
y-0 & =-\frac{1}{\pi}(x-\pi) \\
\text { Answer: } y & =-\frac{1}{\pi} x+1 .
\end{aligned}
$$

25. Find the equation of the tangent line to the graph of $y=\sin (x)$ at the point $x=\pi$. We are looking for the tangent at the point $(\pi, \sin (\pi))=(\pi, 0)$. The slope of the line is given by the derivative $\frac{d y}{d x}=\cos (x)$. At $(\pi, 0)$ the slope of the tangent line is $m=\left.\frac{d y}{d x}\right|_{x=\pi}=\cos (\pi)=-1$. The point-slope formula gives the line's equation:

$$
\begin{aligned}
y-y_{0} & =m\left(x-x_{0}\right) \\
y-0 & =--1 \cdot(x-\pi)
\end{aligned}
$$

Answer: $\quad y=-x+\pi$.

