Chapter 25

Forward-looking rational expectations

In this chapter we analyze models where current expectations of a future value of an endogenous variable have an influence on the current value of this variable. Under the hypothesis of rational expectations such models lead to expectational difference equations. These are important for many topics in macroeconomics, including the theory of asset price bubbles. Both at the formal level and in substance the framework departs from the simpler framework where only *past* expectations of current and future variables influence current variables, which was considered in the preceding chapter.

In order to put things clearly in relief, we start by comparing the two types of frameworks. Then, in Section 25.2 the method of repeated forward substitution is presented. The set of solutions to linear expectational difference equations, in the "normal case", is studied in Section 25.3 and illustrated by simple economic examples. The complementary case is briefly treated in Section 25.4. Finally, Section 25.5 concludes. Some of the technical aspects are dealt with in the appendix.

25.1 Expectational difference equations

In the preceding chapter we studied stochastic models where rational expectations entered in one of the following ways:

$$y_t = \alpha \ E(y_t|I_{t-1}) + \beta \ x_t,$$

or

$$y_t = \alpha \ E(y_t|I_{t-1}) + \gamma E(y_{t+1}|I_{t-1}) + \beta \ x_t, \qquad t = 0, 1, 2, ...$$

Here y_t is the endogenous variable, α, β, and γ are given coefficients, x_t is an exogenous stochastic variable, and $E(y_t|I_{t-1})$ and $E(y_{t+1}|I_{t-1})$ are the mathematical expectations conditional on the model and the information available at the end of period $t-1$. The distinctive feature is that y_t depends only on agents' expectations formed in the *preceding* period. These models are called models with *past expectations* affecting current endogenous variables.

In many macroeconomic problems, however, there is an important role for agents' *current* expectations of *future* values of the endogenous variables. Even in the previous chapter, where small stochastic AD-AS models were considered, we had to drastically simplify by assuming money demand is independent of the interest rate. In reality money demand depends on the nominal interest rate. When this is taken into account, output in an AD-AS model depends on the expected real interest rate, which in turn depends on expected inflation between the current and the *next* period. As a minimum this gives rise to an equation of the form

$$y_t = aE(y_{t+1}|I_t) + c\, x_t, \qquad t = 0, 1, 2, ..., \tag{25.1}$$

where $a \neq 0$ (otherwise the model is uninteresting) and $E(y_{t+1}|I_t)$ is the mathematical expectation of y_{t+1} conditional on information available at the end of period t.[1] Here the expectation of a future value of an endogenous variable has an impact on the current value of this variable. We call this a model with *current expectations of future variable values* or, for short, a model with *forward-looking expectations.* Thinking of an appropriate AD-AS model, it will have investment demand (and therefore also aggregate demand) depending on both the expected real interest rate and expected future aggregate demand, *two* forward-looking variables. Or we might think of equity shares. Their market value today will depend on the expectations, formed today, of the market value tomorrow.

The conditioning information, I_t, is assumed to contain knowledge of the realized values of y and x up until and including period t. The hypothesis is that the "generally held" subjective expectation conditional on I_t coincides with the objective conditional expectation based on the model (including knowledge of the exact values of the parameters a and c and knowledge of the stochastic process which x_t follows). As we discussed in Chapter 24, the assumption of rational expectations should generally be seen as just a simplification which may under certain conditions lead to useful approximative conclusions. Assuming rational expectations implies that the results which emerge from the model cannot depend on *systematic* expectation errors from the economic agents' side.

[1] We imagine that all agents have the same information, hence also the same rational expectation.

For ease of exposition we will use the notation $E_t y_{t+1} \equiv E(y_{t+1}|I_t)$ and thus write

$$y_t = aE_t y_{t+1} + c\, x_t, \quad t = 0, 1, 2, ... \tag{25.2}$$

A stochastic difference equation of this form is called a linear *expectation difference equation of first order* with constant coefficient a.[2] A *solution* is a specified stochastic process $\{y_t\}$ which satisfies (25.2), given the stochastic process followed by x_t. In the economic applications usually no initial value, y_0, is given. On the contrary, the interpretation is that y_t depends, for all t, on expectations about the future. Indeed, y_t is considered to be a *jump variable* and can immediately shift its value in response to the emergence of new information about the future x's. For example, a share price may immediately jump to a new value when the accounts of the firm become publicly known (often even before, due to sudden rumors).[3]

Owing to the lack of an initial condition for y_t, there can easily be infinitely many processes for y_t satisfying our expectation difference equation. We have an infinite forward-looking "regress", where a variable's value today depends on its expected value tomorrow, this value depending on the expected value the day after tomorrow and so on. Then usually there are infinitely many expected sequences which can be self-fulfilling in the sense that if only the agents expect a particular sequence, then the aggregate outcome of their behavior will be that the sequence is realized. It "bites its own tail" so to speak. Yet, when an equation like (25.2) is part of a larger model, there will often (but not always) be conditions that allow us to select *one* of the many solutions to (25.2) as the only *economically* relevant one. For example, an economy-wide transversality condition or another general equilibrium condition may rule out divergent solutions and leave a unique convergent solution as the final solution.

We assume $a \neq 0$, since otherwise (25.2) itself already is the unique solution. It turns out that the set of solutions to (25.2) takes a different form depending on whether $|a| < 1$ or $|a| > 1$:

The case $|a| < 1$. In general, there is a unique *fundamental solution* and infinitely many explosive *bubble solutions*.

The case $|a| > 1$. In general, there is no fundamental solution but infinitely many non-explosive solutions. (The case $|a| = 1$ resembles this.)

[2] Later we allow the coefficients a and c to be time-dependent.

[3] We said that y_t "depends on" the expectation of y_{t+1}. It would be inaccurate to say that y_t is *determined* (in a one-way-sense) by expectations about the future. Rather there is *mutual dependence*. In view of y_t being an element in the information I_t, the expectation of y_{t+1} in (25.2) may depend on y_t just as much as y_t depends on the expectation of y_{t+1}.

In the case $|a| < 1$ the expected future has modest influence on the present. Here we will concentrate on this case, since it is the case most frequently appearing in macroeconomic models with rational expectations.

25.2 Solutions when $|a| < 1$

Various solution methods are available. *Repeated forward substitution* is the most easily understood method.

25.2.1 Repeated forward substitution

Repeated forward substitution consists of the following steps. We first shift (25.2) one period ahead:

$$y_{t+1} = a\ E_{t+1}y_{t+2} + c\ x_{t+1}.$$

Then we take the conditional expectation on both sides to get

$$E_t y_{t+1} = a\ E_t(E_{t+1}y_{t+2}) + c\ E_t x_{t+1} = a\ E_t y_{t+2} + c\ E_t x_{t+1}, \tag{25.3}$$

where the second equality sign is due to the *law of iterated expectations*, which says that

$$E_t(E_{t+1}y_{t+2}) = E_t y_{t+2}. \tag{25.4}$$

see Box 25.1. Inserting (25.3) into (25.2) gives

$$y_t = a^2 E_t y_{t+2} + ac\ E_t x_{t+1} + c\ x_t. \tag{25.5}$$

The procedure is repeated by forwarding (25.2) two periods ahead; then taking the conditional expectation and inserting into (25.5), we get

$$y_t = a^3 E_t y_{t+3} + a^2 c\ E_t x_{t+2} + ac\ E_t x_{t+1} + c\ x_t.$$

We continue in this way and the general form (for $n = 0, 1, 2, ...$) becomes

$$y_{t+n} = a\ E_{t+n}(y_{t+n+1}) + c\ x_{t+n},$$

$$E_t y_{t+n} = a\ E_t y_{t+n+1} + c\ E_t x_{t+n},$$

$$y_t = a^{n+1} E_t y_{t+n+1} + c x_t + c \sum_{i=1}^{n} a^i E_t x_{t+i}. \tag{25.6}$$

Box 25.1. The law of iterated expectations

The method of repeated forward substitution is based on the law of iterated expectations which says that $E_t(E_{t+1}y_{t+2}) = E_t y_{t+2}$, as in (25.4). The logic is the following. Events in period $t+1$ are stochastic and so $E_{t+1}y_{t+2}$ (the expectation conditional on these events) is a stochastic variable. Then the law of iterated expectations says that the conditional expectation of this stochastic variable as seen from period t is the same as the conditional expectation of y_{t+2} itself as seen from period t. So, given that expectations are rational, then an earlier expectation of a later expectation of y is just the earlier expectation of y. Put differently: my best forecast today of how I am going to forecast tomorrow a share price the day after tomorrow, will be the same as my best forecast today of the share price the day after tomorrow. If beforehand we have good reasons to expect that we will revise our expectations upward, say, when next period's additional information arrives, the original expectation would be biased, hence not rational.[4]

25.2.2 The fundamental solution

PROPOSITION 1 If

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \text{ exists,} \tag{25.7}$$

then

$$y_t = c\sum_{i=0}^{\infty} a^i E_t x_{t+i} = cx_t + c\sum_{i=1}^{\infty} a^i E_t x_{t+i}, \qquad t = 0, 1, 2, ..., \tag{25.8}$$

is a solution to the expectation difference equation (25.2).

Proof Assume (25.7). Then the formula (25.8) is meaningful. In view of (25.6), it satisfies (25.2) if and only if $\lim_{n\to\infty} a^{n+1}E_t y_{t+n+1} = 0$. Hence, it is enough to show that the process (25.8) satisfies this latter condition.

In (25.8), replace t by $t+n+1$ to get $y_{t+n+1} = c\sum_{i=0}^{\infty} a^i E_{t+n+1}x_{t+n+1+i}$. Using the law of iterated expectations, this yields

$$\begin{aligned} E_t y_{t+n+1} &= c\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} \qquad \text{so that} \\ a^{n+1}E_t y_{t+n+1} &= c\, a^{n+1}\sum_{i=0}^{\infty} a^i E_t x_{t+n+1+i} = c\sum_{j=n+1}^{\infty} a^j E_t x_{t+j}. \end{aligned}$$

[4] A formal account of conditional expectations and the law of iterated expectations is given in Appendix B of Chapter 24.

It remains to show that $\lim_{n\to\infty}\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = 0$. From the identity

$$\sum_{j=1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{n} a^j E_t x_{t+j} + \sum_{j=n+1}^{\infty} a^j E_t x_{t+j}$$

follows

$$\sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{n} a^j E_t x_{t+j}.$$

Letting $n \to \infty$, this gives

$$\lim_{n\to\infty} \sum_{j=n+1}^{\infty} a^j E_t x_{t+j} = \sum_{j=1}^{\infty} a^j E_t x_{t+j} - \sum_{j=1}^{\infty} a^j E_t x_{t+j} = 0,$$

which was to be proved. □

The solution (25.8) is called the *fundamental solution* of (25.2). It is (for $c \neq 0$) defined only when the condition (25.7) holds. In general this condition requires that $|a| < 1$. In addition, (25.7) requires that the absolute value of the expectation of the exogenous variable does not increase "too fast". More precisely, the requirement is that $|E_t x_{t+i}|$, when $i \to \infty$, has a growth factor less than $|a|^{-1}$. As an example, let $0 < a < 1$ and $g > 0$, and suppose that $E_t x_{t+i} > 0$ for $i = 0, 1, 2, ...$, and that $1+g$ is an upper bound for the growth factor of $E_t x_{t+i}$. Then

$$E_t x_{t+i} \leq (1+g) E_t x_{t+i-1} \leq (1+g)^i E_t x_t = (1+g)^i x_t,$$

so that $a^i E_t x_{t+i} \leq a^i (1+g)^i x_t$. By summing from $i = 1$ to n,

$$\sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \sum_{i=1}^{n} [a(1+g)]^i .$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} \sum_{i=1}^{n} a^i E_t x_{t+i} \leq x_t \lim_{n\to\infty} \sum_{i=1}^{n} [a(1+g)]^i = x_t \frac{a(1+g)}{1-a(1+g)} < \infty,$$

if $1+g < a^{-1}$, using the sum rule for an infinite geometric series.

As noted in the proof of Proposition 1, the fundamental solution, (25.8), has the property that

$$\lim_{n\to\infty} a^n E_t y_{t+n} = 0. \tag{25.9}$$

That is, the expected value of y is not "explosive": its absolute value has a growth factor less than $|a|^{-1}$. Given $|a| < 1$, the fundamental solution is the only solution of (25.2) with this property. Indeed, it is seen from (25.6) that whenever (25.9) holds, (25.8) must also hold. In Example 1 below, y_t is interpreted as the market price of a share and x_t as dividends. Then the fundamental solution gives the share price as the present value of the expected future flow of dividends.

EXAMPLE 1 *(the fundamental value of an equity share)* Consider arbitrage between shares of stock and a riskless asset paying the constant rate of return $r > 0$. Let period t be the current period. Let p_{t+i} be the market price of the share at the beginning of period $t+i$ and d_{t+i} the dividend paid out at the end of that period, $t+i$, $i = 0, 1, 2, ...$[5] As seen from period t there is uncertainty about p_{t+i} and d_{t+i} for $i = 1, 2, ...$ Suppose agents have rational expectations and care only about expected return. Then the no-arbitrage condition reads

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r. \tag{25.10}$$

This can be written

$$p_t = \frac{1}{1+r} E_t p_{t+1} + \frac{1}{1+r} d_t, \tag{25.11}$$

which is of the same form as (25.2) with $a = c = 1/(1+r) \in (0, 1)$. Assuming dividends do not grow "too fast", we find the fundamental solution, denoted p_t^*, as

$$p_t^* = \frac{1}{1+r} d_t + \frac{1}{1+r} \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} E_t d_{t+i} = \sum_{i=0}^{\infty} \frac{1}{(1+r)^{i+1}} E_t d_{t+i}. \tag{25.12}$$

The fundamental solution is simply the present value of expected future dividends, in finance theory denoted the *fundamental value.*

[5] An investor who buys n_t shares at time t (the beginning of period t) thus invests $V_t \equiv p_t n_t$ units of account at time t. At the end of the period the return comes out as the dividend d_t and the potential sales value of the share at time $t+1$. This is slightly unlike standard *finance* notation in discrete time, where V_t would be the-end-of-period-t market value of a share of stock that begins to yield dividends in period $t+1$. As noted earlier, in this text, V_t denotes the-beginning-of-period-t market value of the asset. Notice also that in our notation, a flow of consumption goods in period $t-1$ is payed for at the end of the period at a price P_{t-1} in monetary terms. We have consistency with this if we interpret the price p_t of the share at time t as an asset price in *real* terms, defined by $p_t \equiv 1/P_{t-1}$. In line with this, r in (25.10) is a *real* interest rate from the end of period $t-1$ to the end of period t.

If the dividend process is $d_{t+1} = d_t + \varepsilon_{t+1}$, where ε_{t+1} is white noise, then the process is known as a *random walk* and $E_t d_{t+i} = d_t$ for $i = 1, 2, \ldots$. Thus $p_t^* = d_t/r$, by the sum rule for an infinite geometric series. In this case the fundamental value is itself a random walk. More generally, the dividend process could be a *martingale,* that is, a sequence of stochastic variables with the property that the expected value next period exists and equals the current actual value, where the expectation is conditional on all information up to and including the current period. Again $E_t d_{t+1} = d_t$; but in a martingale ε_{t+1} need not be white noise; it is enough that $E_t \varepsilon_{t+1} = 0$.[6] Given the constant required return r, we still have $p_t^* = d_t/r$. So the fundamental value itself is in this case a martingale. □

As noted in the example, in finance theory the present value of the expected future flow of dividends on an equity share is referred to as the *fundamental value* of the share. It is by analogy with this that the general designation *fundamental solution* has been introduced for solutions of form (25.8). We could also think of p_t as the market price of a house rented out and d_t as the rent. Or p_t could be the market price of a mine and d_t the revenue (net of extraction costs) from the extracted oil in period t.

25.2.3 Bubble solutions

Other than the fundamental solution, the expectation difference equation (25.2) has infinitely many *bubble solutions.* In view of $|a| < 1$, these are characterized by violating the condition (25.9). That is, they are solutions whose expected value explodes over time.

It is convenient to first consider the *homogenous* expectation equation associated with (25.2). This is defined as that equation which emerges by setting $c = 0$ in (25.2):

$$y_t = aE_t y_{t+1}. \tag{25.13}$$

Every stochastic process $\{b_t\}$ of the form

$$b_{t+1} = a^{-1} b_t + u_{t+1}, \quad \text{where } E_t u_{t+1} = 0, \tag{25.14}$$

has the property that

$$b_t = aE_t b_{t+1}, \tag{25.15}$$

and is thus a solution to (25.13). The "disturbance" u_{t+1} represents "new information" which may be related to unexpected movements in "fundamentals", x_{t+1}. But it does not have to. In fact, u_{t+1} may be related to conditions that *per se* have no economic relevance whatsoever.

[6] A random walk is thus a special case of a martingale.

For ease of notation, from now we just write b_t even if we think of the whole process $\{b_t\}$ rather than the value taken by b in the specific period t. The meaning should be clear from the context. A solution to (25.13) is referred to as a *homogenous solution* associated with (25.2). Let b_t be a given homogenous solution and let K be an arbitrary constant. Then $B_t = Kb_t$ is also a homogenous solution (try it out for yourself). Conversely, any homogenous solution b_t associated with (25.2) can be written in the form (25.14). To see this, let b_t be a given homogenous solution, that is, $b_t = aE_t b_{t+1}$. Let $u_{t+1} = b_{t+1} - E_t b_{t+1}$. Then

$$b_{t+1} = E_t b_{t+1} + u_{t+1} = a^{-1} b_t + u_{t+1},$$

where $E_t u_{t+1} = E_t b_{t+1} - E_t b_{t+1} = 0$. Thus, b_t is of the form (25.14).

Returning to our original expectation difference equation (25.2), we have:

PROPOSITION 2 Let $\tilde{y}_t$ be a particular solution to (25.2). Then:
(i) every stochastic process of the form

$$y_t = \tilde{y}_t + b_t, \tag{25.16}$$

where b_t satisfies (25.14), is a solution to (25.2);
(ii) every solution to (25.2) can be written in the form (25.16) with b_t being an appropriately chosen homogenous solution associated with (25.2).

Proof. Let the particular solution $\tilde{y}_t$ be given. (i) Let $y_t = \tilde{y}_t + b_t$. Then $y_t = a\ E_t \tilde{y}_{t+1} + c\ x_t + b_t$, since $\tilde{y}_t$ satisfies (25.2). Consequently, by (25.13),

$$y_t = a\ E_t \tilde{y}_{t+1} + c\ x_t + a\ E_t b_{t+1} = a\ E_t(\tilde{y}_{t+1} + b_{t+1}) + c\ x_t = a\ E_t Y_{t+1} + c\ x_t,$$

saying that (25.16) satisfies (25.2). (ii) Let Y_t be an arbitrary solution to (25.2). Define $b_t = Y_t - \tilde{y}_t$. Then we have

$$\begin{aligned} b_t &= Y_t - \tilde{y}_t = aE_t Y_{t+1} + cx_t - (aE_t \tilde{y}_{t+1} + cx_t) \\ &= aE_t(Y_{t+1} - \tilde{y}_{t+1}) = aE_t b_{t+1}, \end{aligned}$$

where the second equality follows from the fact that both Y_t and $\tilde{y}_t$ are solutions to (25.2). This shows that b_t is a solution to the homogenous equation (25.13) associated with (25.2). Since $Y_t = \tilde{y}_t + b_t$, the proposition is hereby proved. □

Proposition 2 holds for any $a \neq 0$. In case the fundamental solution (25.8) exists and $|a| < 1$, it is convenient to choose this solution as the particular solution in (25.16). Thus, referring to the right-hand side of (25.8) as y_t^*, we can use the particular form,

$$y_t = y_t^* + b_t. \tag{25.17}$$

When the component b_t is different from zero, the solution (25.17) is called a *bubble solution* and b_t is called the *bubble component*. In the typical economic interpretation the bubble component shows up only because it is expected to show up next period, cf. (25.15). The name bubble springs from the fact that the expected value conditional on the information available in period t explodes over time when $|a| < 1$. To see this, as an example, let $0 < a < 1$. Then, from (25.13), by repeated forward substitution we get

$$b_t = a\ E_t(aE_{t+1}b_{t+2}) = a^2 E_t b_{t+2} = ... = a^i E_t b_{t+i}, \qquad i = 1, 2, ...$$

It follows that $E_t b_{t+i} = a^{-i} b_t$, that is, not only does it hold that

$$\lim_{i\to\infty} E_t b_{t+i} = \begin{cases} \infty, \text{ if } b_t > 0 \\ -\infty, \text{ if } b_t < 0 \end{cases},$$

but the absolute value of $E_t b_{t+i}$ will for rising i grow *geometrically* towards infinity with a growth factor equal to $1/a > 1$.

Let us consider a special case of (25.2) that allows a simple graphical illustration of both the fundamental solution and some bubble solutions.

When x_t has constant mean

Suppose the stochastic process x_t (the "fundamentals") takes the form $x_t = \bar{x} + \varepsilon_t$, where $\bar{x}$ is a constant and ε_t is white noise. Then

$$y_t = a\ E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad 0 < |a| < 1. \tag{25.18}$$

The fundamental solution is

$$y_t^* = c\ x_t + c\sum_{i=1}^{\infty} a^i \bar{x} = c\bar{x} + c\varepsilon_t + c\frac{a\bar{x}}{1-a} = \frac{c\bar{x}}{1-a} + c\varepsilon_t.$$

Referring to (i) of Proposition 2,

$$y_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t + b_t \tag{25.19}$$

is thus also a solution of (25.18) if b_t is of the form (25.14).

It may be instructive to consider the case where all stochastic features are eliminated. So we assume $u_t \equiv \varepsilon_t \equiv 0$. Then we have a model with perfect foresight; the solution (25.19) simplifies to

$$y_t = \frac{c\bar{x}}{1-a} + b_0 a^{-t}, \tag{25.20}$$

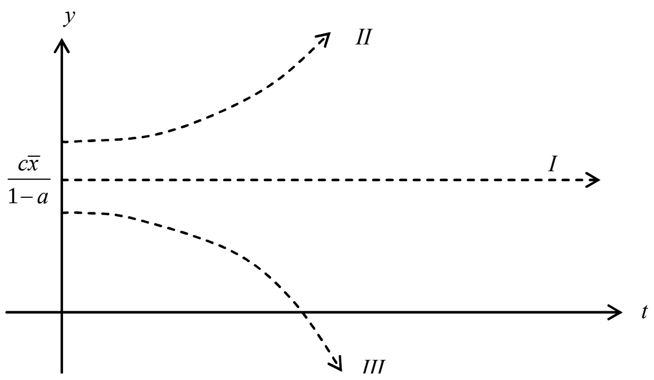

Figure 25.1: Deterministic bubbles (the case $0 < a < 1$, $c > 0$, and $x_t = \bar{x}$).

where we have used repeated *backward* substitution in (25.14). By setting $t = 0$ we see that $y_0 = \frac{c\bar{x}}{1-a} + b_0$. Inserting this into (25.20) gives

$$y_t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})a^{-t}. \tag{25.21}$$

In Fig. 1 we have drawn three trajectories for the case $0 < a < 1$, $c > 0$. Trajectory I has $y_0 = c\bar{x}/(1-a)$ and represents the fundamental solution, whereas trajectory II, with $y_0 > c\bar{x}/(1-a)$, and trajectory III, with $y_0 < c\bar{x}/(1-a)$, are bubble solutions. Since we have imposed no apriori boundary condition, one y_0 is as good as any other. The interpretation is that there are infinitely many trajectories with the property that if only the economic agents expect the economy will follow that particular trajectory, the aggregate outcome of their behavior will be that this trajectory is realized. This is the potential indeterminacy arising when y_t is not a predetermined variable. However, as alluded to above, in a complete economic model there will often be restrictions on the endogenous variable(s) not immediately visible in the basic expectation difference equation(s), here (25.18). It may be that the economic meaning of y_t precludes negative values and, hence, no-one can rationally expect a path such as III in Fig. 1. And/or it may be that for some reason there is an upper bound on y_t (which could be the full-employment ceiling for output in a situation where the natural growth rate of output is smaller than $a^{-1} - 1$). Then no one can rationally expect a trajectory like II in the figure.

Our conclusion so far is that in order for a solution of a first-order linear expectation difference equation with constant coefficient a, where $|a| < 1$, to differ from the fundamental solution, the solution must have the form (25.17) where b_t has the form described in (25.14). This provides a clue as to what asset price bubbles might look like.

Asset price bubbles

A stylized fact of stock markets is that stock price indices are volatile on a month-to-month, year-to-year, and especially decade-to-decade scale, cf. Fig. 25.2. There are different views about how these swings should be understood. According to the Efficient Market Hypothesis the swings just reflect unpredictable changes in the "fundamentals", that is, changes in the present value of rationally expected future dividends. This is for instance the view of Nobel Laureate Eugene Fama () from University of Chicago.

Nobel laureate Robert Shiller (1981, 2003) from Yale University, and others, have pointed to the phenomenon of "excess volatility", however. The view is that asset prices tend to fluctuate more than can be rationalized by shifts in information about fundamentals (present values of dividends). Although in no way a verification, graphs like those in Fig. 25.2 and Fig. 25.3 are suggestive. Fig. 25.2 shows the monthly real S&P composite stock prices and real S&P composite earnings for the period 1871-2008. The unusually large increase in real stock prices since the mid-90's, which ended with the collapse in 2000, is known as the "dot-com bubble". Fig. 25.3 shows, on a monthly basis, the ratio of real S&P stock prices to an average of the previous ten years' real S&P earnings along with the long-term real interest rate. It is seen that this ratio reached an all-time high in 2000, by many observers considered as "the year the dot-com bubble burst".

Shiller's interpretation of the large stock market swings is that they are due to shifts in fads or fashions and "animal spirits" (a notion from Keynes).

A third possible source of large stock market swings was pointed out by Blanchard (1979) and Blanchard and Watson (1982). They argued that bubble phenomena need not be due to irrational behavior and absence of rational expectations. This lead to the theory of *rational bubbles* − the idea that excess volatility can be explained as speculative bubbles arising solely from self-fulfilling *rational* expectations.

Consider an asset which yields either dividends or services in production or consumption in every period in the future. The fundamental value of the asset is, at the theoretical level, defined as the present value of the future flow of dividends or services.[7] An *asset price bubble* (or a *speculative bubble)* is then defined as the deviation of the market price, p_t, of the asset from its fundamental value, p_t^*:

$$b_t = p_t - p_t^*.$$

[7] In practice there are many ambiguities involved in this definition of the fundamental value because it relates to an unknown future.

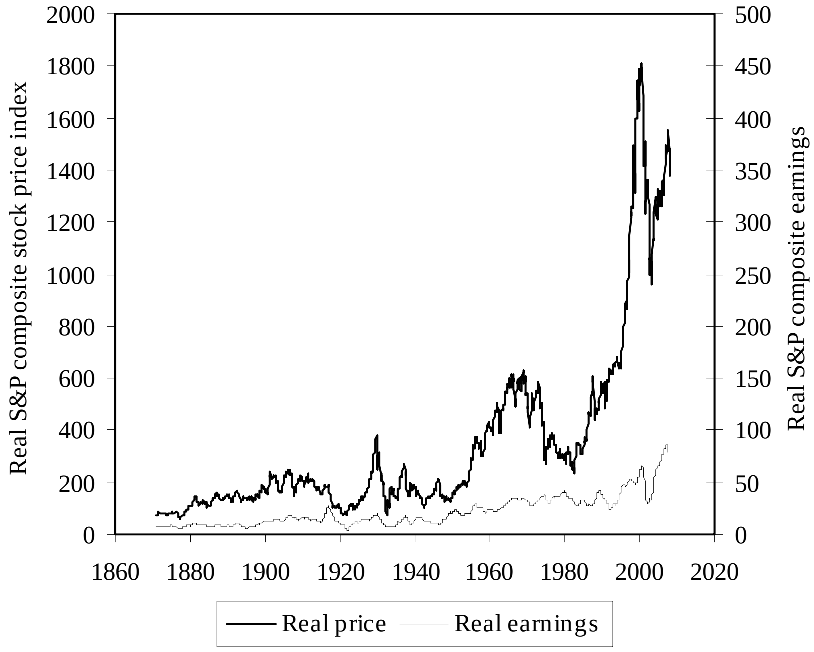

Figure 25.2: Monthly real S&P composite stock prices from January 1871 to January 2008 (left) and monthly real S&P composite earnings from January 1871 to September 2007 (right). Source: http://www.econ.yale.edu/~shiller/data.htm.

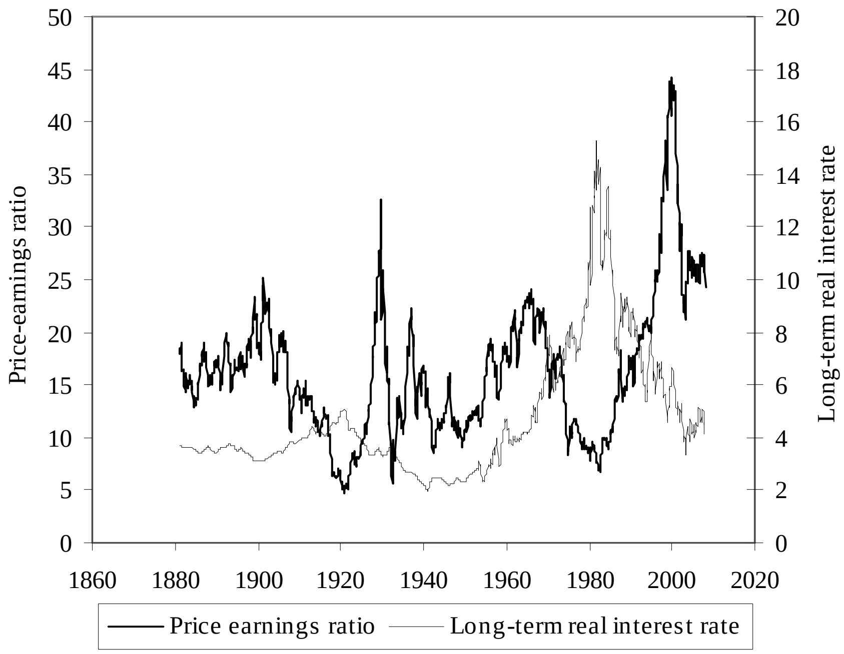

Figure 25.3: S&P price-earnings ratio and long-term real interest rates from January 1881 to January 2008. The earnings are calculated as a moving average over the preceding ten years. The long-term real interest rate is the 10-year Treasury rate from 1953 and government bond yields from Sidney Homer, "A History of Interest Rates" from before 1953. Source: http://www.econ.yale.edu/~shiller/data.htm.

An asset price bubble that emerges in a setting where the no-arbitrage condition (25.10) holds under rational expectations, is called a *rational bubble*.

EXAMPLE 2 *(an ever-expanding rational bubble)* Consider again an equity share for which the no-arbitrage condition is

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r, \tag{25.22}$$

as in Example 1. Let the price of the share be $p_t = p_t^* + b_t$, where the bubble component follows the deterministic process, $b_{t+1} = (1+r)b_t$, $b_0 > 0$, so that $b_t = b_0(1+r)^t$. This is called a *deterministic rational bubble*. Agents may be ready to pay a price over and above the fundamental value if they expect they can sell at a higher price later; trading with such motivation is called *speculative behavior*. If generally held, this expectation may be self-fulfilling. Yet we are not acquainted with such *ever*-expanding incidents in real world situations. So a deterministic rational bubble seems implausible. □

A *stochastic* rational bubble which sooner or later *bursts* seems less implausible.

EXAMPLE 3 *(a bursting bubble)* The no-arbitrage condition is once more (25.22) where for simplicity we still assume the required rate of return is constant, though possibly including a risk premium. The implied expectation difference equation is $p_t = aE_t p_{t+1} + cd_t$, with $a = c = 1/(1+r) \in (0,1)$. Following Blanchard (1979), we assume that the market price, p_t, of the share contains a stochastic bubble of the following form:

$$b_{t+1} = \begin{cases} \frac{1+r}{q_t} b_t & \text{with probability } q_t, \\ 0 & \text{with probability } 1 - q_t, \end{cases} \tag{25.23}$$

where $t = 0, 1, 2, ...$ and $b_0 > 0$. In addition we may assume that $q_t = f(b_t, p_t^*)$, $f_b \leq 0$, $f_{p^*} \geq 0$. If $f_b < 0$, the probability that the bubble persists at least one period ahead is less, the greater the bubble has already become. If $f_{p^*} > 0$, the probability that the bubble persists at least one period ahead is higher the greater the fundamental value has become. In this way the probability of a crash becomes greater and greater as the share price comes further and further away from fundamentals. As a compensation, the longer time the bubble has lasted, the higher is the expected growth rate of the bubble in the absence of a collapse.

This bubble satisfies the criterion for a rational bubble. Indeed, (**??**) implies

$$E_t\ b_{t+1} = (\frac{1+r}{q_{t+1}} b_t) q_{t+1} + 0 \cdot (1 - q_{t+1}) = (1+r)b_t.$$

This is of the form (25.14) with $a^{-1} = 1 + r$, and the bubble is therefore a rational stochastic bubble. The stochastic component is $u_{t+1} = b_{t+1} - E_t b_{t+1}$ $= b_{t+1} - (1 + r)b_t$ and has conditional expectation equal to zero. Although u_{t+1} must have zero conditional expectation, it need not be white noise (it can for instance have varying variance).

The market price of the share is $p_t = p_t^* + b_t$, where p_t^* is the fundamental value of the share, which depends only on the expected dividends. Suppose dividends are known to follow the process: $d_t = (1 + g)^t \bar{d} + \varepsilon_t$, where g is a constant satisfying $0 < g < r$ and ε_t is white noise. Thus, $E_t d_{t+i} = (1+g)^{t+i}\bar{d}$ for $i = 1, 2, \ldots$. Applying the general formula, (25.12), we get in this case,

$$p_t^* = (1 + r)^{-1} \sum_{i=0}^{\infty} (1 + r)^{-i} E_t \; d_{t+i} = (1 + r)^{-1} \left(d_t + \frac{(1 + g)^{t+1}\bar{d}}{r - g} \right),$$

in view of $(1 + g)/(1 + r) \in (0, 1)$. Essentially, the fundamental value grows at the same rate as dividends, the rate g. If the noise term in dividends is ignored, we have $d_{t+1} = (1 + g)^{t+1}\bar{d} = (1 + g)d_t$ so that $p_t^* = d_t/(r - g)$. The market price of the share is then $d_t/(r - g) + b_t$, where b_t follows the process (**??**). As long as the bubble has not yet crashed, it is growing faster than the fundamental value, namely with the growth factor $(1 + r)/q_{t+1} > 1 + r$ $> 1 + g$, and so the ratio p_t^*/p_t tends to zero as time proceeds. The major motive for buying and holding the asset must be the expected capital gains. That is, the asset is held primarily with a view to selling at a higher price later.

In spite of the presence of a bubble, if all dividend payments are reinvested in the stock, then the present value of the portfolio has the *martingale* property (in line with Example 1). To see this, let the number of shares of the stock at the beginning of period t be N_t. Define $M_t \equiv (1 + r)^{-t} p_t N_t$. Then reinvestment of the dividends implies $N_{t+1} = N_t + N_t d_t / p_{t+1}$. Hence, $p_{t+1}N_{t+1} = (p_{t+1} + d_t)N_t$ so that

$$E_t p_{t+1} N_{t+1} = (E_t p_{t+1} + d_t)N_t = (1 + r)p_t N_t,$$

since, by (25.22), $E_t p_{t+1} + d_t = (1 + r)p_t$. It follows that $E_t M_{t+1} \equiv E_t(1 + r)^{-(t+1)} p_{t+1} N_{t+1} = (1 + r)^{-t} p_t N_t \equiv M_t$, showing that M is a martingale. Having found that a time series for an asset price looks like a martingale does therefore not rule out that a bubble can be present. □

In this example the bubble did not have the implausible ever-expanding form considered in Example 2. Yet, under certain conditions even a bursting rational bubble can be ruled out or is at least implausible. Below we review some of the arguments.

25.2.4 When rational bubbles in asset prices can or can not be ruled out

We here consider different cases where rational asset price bubbles seem unlikely to arise. We concentrate on assets whose services are valued independently of the price.[8] Let p_t be the market price and p_t^* the fundamental value of the asset as of time t. Even if the asset yields services rather than dividends, p_t^* is in principle the same for all agents. This is because a user who, in a given period, values the service flow of the asset relatively low can hire it out to the one who values it highest (the one with the highest willingness to pay).

Partial equilibrium arguments

The principle of reasoning to be used is called *backward induction:* If we know something about an asset price in the future, we can conclude something about the asset price today.

(a) Assets which can be freely disposed of ("free disposal") In a market with self-interested rational agents, an object which can be freely disposed of can never have a negative price. Nobody will be willing to pay for getting rid of the object if it can just be thrown away. Consequently such assets (share certificates for instance) can not have *negative* rational bubbles; if they had, the expected asset price at some point in the future would be negative, which can not be a rational expectation. In fact, if $p_t < p_t^*$, then everyone will *buy* the asset and hold it forever, which by own use or by hiring out will imply a discounted value equal to p_t^*. Hence, there is excess demand until p_t has risen to p_t^*.

When a negative rational bubble can be ruled out, then, if at the first date of trading of the asset there were no positive bubble, neither can a positive bubble arise later. Let us make this precise:

PROPOSITION 3 Assume free disposal of a given asset. Then, if a rational bubble in the asset price is present today, it must be positive and must have been present also yesterday and so on back to the first date of trading the asset. And if a rational bubble bursts, it will not restart later.

Proof As argued above, in view of free disposal, a negative rational bubble in the asset price can be ruled out. It follows that $b_t = p_t - p_t^* \geq 0$ for $t = 0, 1, 2, ...$, where $t = 0$ is the first date of trading the asset. That is, any rational bubble in the asset price must be a positive bubble. We now show

[8] This is in contrast to assets that serve as means of payment.

by contradiction that if for $t = 1, 2, ..., b_t > 0$, then $b_{t-1} > 0$. Let $b_t > 0$. Then, if $b_{t-1} = 0$, we have $E_{t-1}b_t = E_{t-1}u_t = 0$ (from (25.14) with t replaced by $t-1$), implying, since $b_t < 0$ is not possible, that $b_t = 0$ with probability *one* as seen from period $t-1$. Ignoring zero probability events, this rules out $b_t > 0$ and we have arrived at a contradiction. Thus $b_{t-1} > 0$. Replacing t by $t-1$ and so on backward in time, we end up with $b_0 > 0$. This reasoning also implies that if a bubble bursts in period t, it can not restart in period $t+1$, nor, by extension, in any subsequent period. □

This proposition (due to Diba and Grossman, 1988) informs us that a rational bubble in an asset price must have been there since trading of the asset began. Yet such a conclusion is not without ambiguities. If radically new information or new technology comes up at some point in time, is a share in the firm then the same asset as before? Even if an earlier bubble has crashed, cannot a new rational bubble arise later in case of an utterly new situation?

These ambiguities reflect the difficulty involved in the concepts of rational expectations and rational bubbles when we are dealing with uncertainties about future developments of the economy. The present value of many economic assets of macroeconomic importance, not the least shares in firms, depends on vague beliefs about future preferences and technologies and can not be determined in any objective way. There is no well-defined probability distribution over the potential future outcomes.

(b) Bonds with finite maturity The finite maturity ensures that the value of the bond is given at some finite future date. Therefore, if there were a positive bubble in the market price of the bond, no rational agent would buy just before that date. Anticipating this, no one would buy the date before, and so on ... nobody will buy in the first place. By this backward-induction argument follows that a positive bubble cannot get started. And since there also is "free disposal", *all* rational bubbles can be precluded. This argument in itself does not, however, rule out positive bubbles on *perpetuities* ("consols") of unique historical origin available in a limited amount.

In the remaining cases we assume that negative rational bubbles are ruled out. So, the discussion is about whether *positive* rational asset price bubbles may exist or not.

(c) Assets whose supply is elastic Real capital goods (including buildings) can be reproduced and have clearly defined costs of reproduction. This precludes rational bubbles on this kind of assets, since a potential buyer can avoid the overcharge by producing instead. Notice, however, that building sites with a specific amenity value and apartments in attractive quarters of a city are not easily reproducible. Therefore, rational bubbles on such assets are more difficult to rule out.

What about shares of stock in a firm? The price evolution of these will be limited to the extent that market participants expect that the firm will issue more shares if there is a bubble. On the other hand, it is not obvious that the firm would always do this. The firm might anticipate that the bubble would burst. To "fool" the market the firm just enjoy its solid equity and continues to behave as if no bubble is present. Thus, it is hard to completely rule out rational bubbles on shares of stock by this kind of argument.

(d) Assets for which there exists a "backstop-technology" For some articles of trade there exists substitutes in elastic supply which will be demanded if the price of the article becomes sufficiently high. Such a substitute is called a "backstop-technology". For example oil and other fossil fuels will, when their prices become sufficiently high, be subject to intense competition from substitutes (renewable energy sources). This precludes the unbounded bubble process in the price of oil.

On account of the arguments (c) and (d) bubbles seem less unlikely when it comes to assets which are not reproducible or substitutable and whose "fundamentals" are difficult to ascertain. By fundamentals we mean any information relating to the payoff capacity of an asset: a firm's technology, resources, market conditions etc. For some assets the fundamentals are not easily ascertained. Examples are paintings of past great artists, rare stamps, diamonds, gold etc. Also new firms that introduce novel products and technologies are potential candidates (cf. the IT bubble in the late 1990s).

Adding general equilibrium arguments

The above considerations are of a partial equilibrium nature. On top of this, *general equilibrium* arguments can be put forward to limit the possibility of rational bubbles. We may briefly give a flavour of two of such general equilibrium arguments. We still consider assets whose services are valued independently of the price and which, as in (a) above, can be freely disposed of. A house, a machine, or a share in a firm yields a service in consumption or production or in the form of a dividend stream. Since such an asset has an intrinsic value, p_t^*, equal to the present value of the flow of services, one might believe that positive rational bubbles on such assets can be ruled out in general equilibrium. As we shall see, this is indeed true for an economy with a finite number of "neoclassical" households (to be defined below), but not necessarily in an overlapping generations model. Yet even there, rational bubbles can under certain conditions be ruled out.

(e) An economy with a finite number of infinitely-lived households
Assume that the economy consists of a finite number of infinitely-lived agents − here called households − indexed $i = 1, 2, ..., N$. The households are "neoclassical" in the sense that they save only with a view to future consumption.

Under point (a) we saw that $p_t < p_t^*$ can not be an equilibrium. We now consider the case of a positive bubble, i.e., $p_t > p_t^*$. All owners of the bubble asset who are users will in this case prefer to *sell* and then *rent*; this would imply excess supply and could thus not be an equilibrium. Hence, we turn to households that are not users, but speculators. These may pursue "short selling", that is, rent the asset (for a contracted interval of time) and immediately sell it at p_t. This results in excess supply and so the asset price falls to p_t^*. Then the speculators buy the asset back and return it to the original owner in accordance with the loan accord. So $p_t > p_t^*$ can not be an equilibrium.

Even ruling out "short selling" (which has sometimes been outright forbidden), we can exclude positive bubbles in the present setup with a finite number of households. Presuming that owners who are not users would want to hold the bubble asset forever as a permanent investment will contradict that these owners are "neoclassical". Indeed, their transversality condition would be violated because the value of their wealth would grow at a rate asymptotically equal to the rate of interest. This would allow them to increase their consumption now without decreasing it later and without violating their No-Ponzi-Game condition.

We have to instead imagine that the households owning the bubble asset hold it against future sale. This could on the face of it seem rational enough if there were some probability that not only would the bubble continue to exist, but it would also grow so that the return would be at least as high as that yielded on an alternative investment. Owners holding the asset against expecting a capital gain will thus plan to sell at some later point in time. Let t_i be the point in time where household i wishes to sell and let

$$T = \max\{t_1, t_2, ..., t_N\}.$$

Then nobody will plan to hold the asset after T. The household speculator, i, having $t_i = T$ will thus not have anyone to sell to (other than people who will only pay p_T^*). Anticipating this, no-one would buy or hold the asset the period before, and so on. So no-one will want to buy or hold the asset in the first place.

The conclusion is that $p_t > p_t^*$ cannot be a rational expectations equilibrium in a setup with a finite number of "neoclassical" households.

The same line of reasoning does not, however, go through in an overlapping generations model where *new* households – that is, new traders – enter the economy every period.

(f) An economy with interest rate above the output growth rate In an overlapping generations (OLG) model with an infinite sequence of new decision makers, rational bubbles are under certain conditions theoretically possible. The argument is that with $N \to \infty$, T as defined above is not bounded. Although this unboundedness is a necessary condition for rational bubbles, it is not sufficient, however.

To see why, let us return to the arbitrage examples 1, 2, and 3 where we have $a^{-1} = 1 + r$ so that a hypothetical rational bubble has the form $b_{t+1} = (1 + r)b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. So in expected value the hypothetical bubble is growing at a rate equal to the interest rate, r. If at the same time r is higher than the long-run output growth rate, the value of the expanding bubble asset would sooner or later be larger than GDP and aggregate saving would not suffice to back its continued growth. Agents with rational expectations anticipate this and so the bubble never gets started.

This point is valid when the interest rate in the OLG economy is higher than the growth rate of the economy – which is normally considered the realistic case. Yet, the opposite case *is* possible and in that situation it is less easy to rule out rational asset price bubbles. Similarly in situations with imperfect credit markets. It turns out that the presence of segmented financial markets or externalities that create a wedge between private and social returns on productive investment may increase the scope for rational bubbles (Blanchard, 2008).

25.2.5 Time-dependent coefficients

In the theory above we assumed that the coefficient a is constant. But the concepts can easily be extended to the case with a time-dependent. Consider the expectational difference equation

$$y_t = a_t E_t y_{t+1} + c_t x_t, \tag{25.24}$$

where $0 < |a_t| < 1$ for all t. We also allow the coefficient, c, to x_t to be time-dependent. This is less crucial, however, because $c_t x_t$ could always be replaced by $c\tilde{x}_t$, where $\tilde{x}_t$ is a new exogenous variable defined by $\tilde{x}_t \equiv c_t x_t / c$.

Repeated forward substitution in (25.24) and use of the law of iterated expectations give, in analogy with (25.6),

$$y_t = (\Pi_{j=0}^{n} a_{t+j}) E_t y_{t+n+1} + c_t x_t + \sum_{i=1}^{n} (\Pi_{j=0}^{i-1} a_{t+j}) c_{t+i} E_t x_{t+i}. \tag{25.25}$$

In analogy with Proposition 1 one can show (see Appendix A) that:

> if $\lim_{n\to\infty} \sum_{i=1}^{n} (\Pi_{j=0}^{i-1} a_{t+j}) c_{t+i} E_t x_{t+i}$ exists, then (25.24) has a solution with the property $\lim_{n\to\infty} \left[(\Pi_{j=0}^{n} a_{t+j}) E_t y_{t+n+1} \right] = 0$, namely
>
> $$y_t^* = c_t x_t + \sum_{i=1}^{\infty} (\Pi_{j=0}^{i-1} a_{t+j}) c_{t+i} E_t x_{t+i}. \tag{25.26}$$

This is the *fundamental solution* of (25.24).

In addition, (25.24) has infinitely many bubble solutions of the form $y_t = y_t^* + b_t$, where b_t satisfies $b_{t+1} = a_t^{-1} b_t + u_{t+1}$ with $E_t u_{t+1} = 0$.

EXAMPLE 4 *(time-dependent required rate of return)* We modify the no-arbitrage condition from Example 1 to

$$\frac{d_t + E_t p_{t+1} - p_t}{p_t} = r_t, \tag{25.27}$$

where r_t is the required rate of return. The corresponding expectational difference equation is $p_t = a_t E_t p_{t+1} + c_t d_t$, with $a_t = c_t = 1/(1 + r_t) \in (0, 1)$. Assuming dividends do not grow "too fast", we find the fundamental solution

$$p_t^* = \frac{1}{1 + r_t} d_t + \sum_{i=1}^{\infty} \frac{1}{\Pi_{j=0}^{i}(1 + r_{t+j})} E_t d_{t+i} = \sum_{i=0}^{\infty} \frac{1}{\Pi_{j=0}^{i}(1 + r_{t+j})} E_t d_{t+i}.$$

A bubble solution is of the form $p_t = p_t^* + b_t$, where b_t could be a bursting bubble like in Example 3 (replace r in (**??**) by r_t); if the probability of a crash is increasing with the size of the bubble, then also the required rate of return is likely to be increasing when agents are risk-averse. □

For now, we shall return to the simpler case with constant coefficients, a and c.

25.2.6 Three classes of bubble processes

Consider again the stochastic difference equation $y_t = aE_t y_{t+1} + cx_t$, where the exogenous stochastic variable x_t reflects the economic environment ("fundamentals"). As we saw, the defining characteristic of a bubble associated with this equation is: $b_{t+1} = a^{-1}b_t + u_{t+1}$, where $E_t u_{t+1} = 0$. We classified bubbles according to their deterministic or stochastic nature. But bubbles may also be distinguished according to which variables in the economic system they are related to. This leads to the following taxonomy:

1. *Markovian bubbles.* A Markovian bubble is a bubble that depends only on its own realization in the preceding period. That is, the probability distribution for b_{t+1} is a function only of time and the previous realization, b_t. A deterministic bubble, $b_{t+1} = a^{-1}b_t$, is an example. Another example is the bursting bubble in Example 3 above.

2. *Intrinsic bubbles.* An intrinsic bubble is a bubble that depends on the stochastic variable x_t, which in turn reflects "fundamentals". As an example, consider the stochastic process

$$b_t = a^{-t}x_t, \tag{25.28}$$

where x_{t+1} is a martingale, i.e., $x_{t+1} = x_t + \varepsilon_{t+1}$ with $E_t\varepsilon_{t+1} = 0$. Then $b_{t+1} = a^{-t-1}x_{t+1}$ so that

$$E_t b_{t+1} = a^{-t-1}E_t x_{t+1} = a^{-1}a^{-t}x_t = a^{-1}b_t. \tag{25.29}$$

We see that the process (25.28) satisfies the criterion for a rational bubble. For a financial asset this shows that a rational bubble can be closely related to the dividend process. This is one of the reasons why it is difficult to empirically disentangle rational bubbles from movements in market fundamentals (see Froot and Obstfeld, 1991).

3. *Extrinsic bubbles.* An extrinsic bubble on an asset is a bubble that depends on a stochastic variable which has no connection whatsoever with fundamentals in the economy. This kind of stochastic variables was termed "sunspots" by Cass and Shell (1983), using a metaphorical expression. Let z_t be an example of such a variable and assume z_t is a martingale. Then the process

$$b_t = a^{-t}z_t, \tag{25.30}$$

satisfies the criterion of a rational bubble in that (25.29) holds with x replaced by z. So stochastic variables which are basically irrelevant from a strict economic point of view may still have an impact on the economy if only people believe they do or if only every individual believes that most others believe it. The actual level of sunspot activity can be thought of as an economically irrelevant stochastic variable that nevertheless ends up affecting economic behavior.[9] If people believe that this variable has an impact on the course of the economy, this belief may be self-fulfilling.

The hypothesis of extrinsic bubbles has been applied to cases where multiple rational expectations equilibria may exist (like in Diamond's OLG model). In such cases it is possible that agents condition their expectations on some extrinsic phenomenon like the sunspot cycle. In this way expectations may become coordinated such that the resulting aggregate behavior validates the expectations. Since these notions have proved useful in particular in the case $|a| > 1$, we shall briefly return to them at the end of the next section, which deals with this case.

25.3 Solutions when $|a| > 1$

Although $|a| < 1$ is the most common case in economic applications, there exist economic examples where $|a| > 1$.[10] In this case the expected future has "large influence". Generally, there will then be no fundamental solution because the right-hand side of (25.8) will normally equal $\pm\infty$. On the other hand, there are infinitely many non-explosive solutions. Indeed, Proposition 2 still holds, since they were derived independently of the size of a. Any possible bubble component b_t will still satisfy $E_t b_{t+1} = a^{-1} b_t$, but now we get $\lim_{i\to\infty} E_t b_{t+i} = 0$, in view of $|a| > 1$. Consequently, instead of an explosive bubble component we have an implosive one (which is therefore not usually termed a bubble any longer).

Let us consider the case where x_t has constant mean, i.e.,

$$y_t = a\ E_t y_{t+1} + c(\bar{x} + \varepsilon_t), \qquad |a| > 1, \tag{25.31}$$

where $E_t \varepsilon_{t+i} = 0$ for $i = 1, 2, ...$ An educated guess (cf. Appendix B) is that the process

$$\tilde{y}_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t \tag{25.32}$$

satisfies (25.31). That this is indeed a solution is seen by shifting (25.32) one period ahead and taking the conditional expectation: $E_t \tilde{y}_{t+1} = c\bar{x}/(1-a)$. Multiplying by a and adding $c(\bar{x} + \varepsilon_t)$ gives

$$aE_t\tilde{y}_{t+1} + c(\bar{x} + \varepsilon_t) = \frac{ac\bar{x} + (1-a)c\bar{x}}{1-a} + c\varepsilon_t = \frac{c\bar{x}}{1-a} + c\varepsilon_t = \tilde{y}_t,$$

[9] In fact, since the level of actual sunspot activity may influence the temperature at the Earth and thereby economic conditions, the sunspot metaphor chosen by Cass and Shell was not particularly felicitous.

[10] See, e.g., Taylor (1986, p. 2009) and Blanchard and Fischer (1989, p. 217).

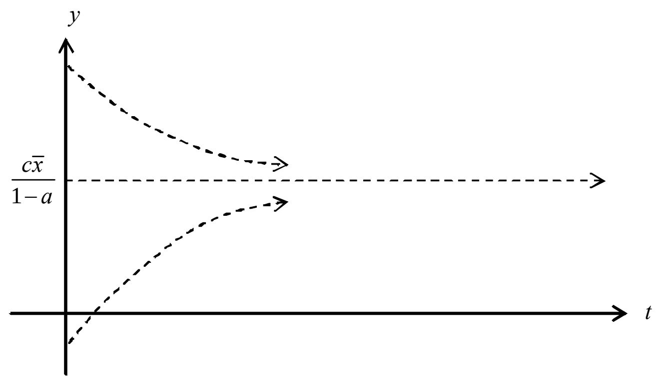

Figure 25.4: Deterministic implosive "bubbles" (the case $a > 1$, $c < 0$, and $x_t = \bar{x}$).

which shows that $\tilde{y}_t$ satisfies (25.31).

With this $\tilde{y}_t$ and the process b_t given by (25.14) we have from Proposition 2 that

$$y_t = \frac{c\bar{x}}{1-a} + b_t + c\varepsilon_t \tag{25.33}$$

is also a solution of (25.31). By backward substitution in (25.14) the bubble component b_t can be written as

$$b_t = \sum_{i=0}^{t-1} a^{-i} u_{t-i} + a^{-t} b_0. \tag{25.34}$$

If for example u_t is white noise, this shows that the bubble will gradually die out over time. And if also ε_t is white noise, we see that, as $t \to \infty$, y_t converges towards $c\bar{x}/(1-a)$ except for white noise. If $u_t \equiv 0$ and $\varepsilon_t \equiv 0$, we get again the formula (25.21) which now implies converging paths as illustrated in Fig. 20.2 (for the case $a > 1$, $c < 0$).[11]

Two theoretical implications should be mentioned. On the one hand, the lack of uniqueness (which follows from the fact that y_0 is a forward-looking variable) is much more "troublesome" in this case than in the case $|a| < 1$. When $|a| < 1$, imposing the restriction that the solution be non-explosive (say because of a transversality condition or some other restriction) removes the ambiguity. But when $|a| > 1$, this is no longer so. As Fig. 20.4 indicates, when $|a| > 1$, there are infinitely many non-explosive solutions.

[11] The fact that $|a| > 1$ is associated with convergence may seem confusing if one is more accustomed to difference equations on *backward*-looking form. Appendix C relates our forward-looking form to the backward-looking form, common in natural science and math textbooks. The relationship to the associated concepts of *characteristic equation* and *stable* and *unstable roots* is exposed.

On the other hand, exactly this feature opens up for the existence of non-explosive equilibrium paths with stochastic fluctuations driven by random events that *per se* have no connection whatsoever with fundamentals in the economy. The theory of *extrinsic bubbles* ("sunspot equilibria") has mainly been applied to this case ($|a| > 1$). The hypothesis is that in situations with multiple rational expectations equilibria it may happen that some extraneous stochastic phenomenon *de facto* becomes a coordination device. If people believe that this particular phenomenon has an impact on the economy, then it may end up having an impact due to the behavior induced by the associated conditional expectations. It turns out that when strong nonlinearities are present, cases like $|a| > 1$ may arise. These mechanisms have relevance for business cycle theory and have affinity with themes from Keynes like "animal spirits", "self-justifying beliefs", and "expectations volatility".

25.4 Concluding remarks

This chapter has studied forward-looking rational expectations giving rise to expectation difference equations of the form $y_t = aE_t y_{t+1} + cx_t$. The case $|a| < 1$ is the most common in macroeconomics. In that case there is only one solution which in expected value n periods ahead does not explode for n going to infinity. This is the *fundamental solution.* On the other hand there are infinitely many solutions which in expected value n periods ahead explode for n going to infinity, the *bubble solutions.* When conditions in the model as a whole allow us to rule out the latter, we are left with the fundamental solution. In the next chapter, we will apply the fundamental solution to a series of New Classical and Keynesian models with forward-looking expectations.

We have considered cases where, if not already from a partial equilibrium point of view, then at least from an general equilibrium point of view, rational asset price bubbles seem unlikely to occur. The latter theme is further explored in Chapter 27.

The empirical evidence concerning asset price bubbles in general and rational asset price bubbles in particular seems inconclusive. It is very difficult to statistically distinguish between bubbles and mis-specified fundamentals. Rational bubbles can also have quite complicated forms. For example Evans (1991) and Hall et al. (1999) study "regime-switching" rational bubbles.

Whatever the possible limits to the emergence of rational bubbles in asset prices, it is useful to be aware of their logical structure and the variety of forms they can take as logical possibilities. Rational bubbles may serve as a benchmark for the analytically harder cases of "irrational asset price bubbles", i.e., bubbles arising when a significant fraction of the market participants do not behave in accordance with the efficient market hypothesis. This would take us to *behavioral finance* theory.

Some of the economic models considered in the next chapter lead to more complicated expectational difference equations than above. An example is the equation $y_t = a_1 E_{t-1} y_t + a_2 E_t y_{t+1} + c\, x_t$. Here forward-looking expectations as well as past expectations of current variables enter into the determination of y_t. As we will see, however, a solution method based on repeated forward substitution can still be used. Sometimes in the economic literature appear complex stochastic difference equations where more elaborate methods are required.

25.5 Literature notes

(preliminary)

The exposition in sections 25.3.1-4 is much in debt to Blanchard and Fischer (1989, Ch. 5, Section 1).

Sometimes foreign exchange is added to the list of assets on which rational bubbles are possible; for a collection of theoretical and empirical studies of this candidate, see ...

Flood and Garber (1994).

Tirole, 1982, 1985.

Shleifer, A., 2000, *Efficient Markets: An Introduction to Behavioral Finance*, OUP.

Shleifer, A., and R.W. Vishny, 1997, The limits to arbitrage, Journal of Finance 52 (1), 35-55.

For surveys on the theory of rational bubbles and econometric bubble tests, see Salge (1997) and Gürkaynak (2008). For discussions of famous historical bubble episodes, see the symposium in *Journal of Economic Perspectives* 4, No. 2, 1990, and Shiller (2005).

LeRoy (2004) gives a survey and concludes in favor of the tenet that rational bubbles help explain what appears as excess volatility in asset prices.

Critics of RE: Shiller in The New Palgrave, ...

Hendry, Dynamic Econometrics.

For discussions of "animal spirits", "self-justifying beliefs", and "expectations volatility", see Keynes (1936, Ch. 12), Farmer (1993), Guesnerie (2001), and Akerlof and Shiller ().

Shiller (2003) gives an introduction to behavioral finance theory.

For solution methods for more intricate stochastic difference equations the reader may be referred to Blanchard and Fischer (1989, Chapter 5, Appendix), Obstfeld and Rogoff (1996), and Gourieroux and Monfort (1997).

25.6 Appendix

A. Proof of (25.26)

We shall show that if $\lim_{n\to\infty}\sum_{i=1}^{n}(\Pi_{j=0}^{i-1}a_{t+j})c_{t+i}E_t x_{t+i}$ exists, then (25.24) has the solution (25.26). Replace t by $t+n+1$ in (25.26) to get

$$\begin{aligned} y_{t+n+1} &= c_{t+n+1}x_{t+n+1} + \sum_{i=1}^{\infty}(\Pi_{j=0}^{i-1}a_{t+n+1+j})c_{t+n+1+i}E_{t+n+1}x_{t+n+1+i} \Rightarrow \\ E_t y_{t+n+1} &= c_{t+n+1}E_t x_{t+n+1} + \sum_{i=1}^{\infty}(\Pi_{j=0}^{i-1}a_{t+n+1+j})c_{t+n+1+i}E_t x_{t+n+1+i}. \qquad (25.35) \end{aligned}$$

Define the "discount factor" D_k by

$$D_k = \Pi_{j=0}^{k-1}a_{t+j}, \quad \text{for } k = 1, 2, \ldots.$$

Multiplying by $D_{n+1} = \Pi_{j=0}^{n}a_{t+j}$ on both sides in (25.35) gives

$$\begin{aligned} D_{n+1}E_t y_{t+n+1} &= \Pi_{j=0}^{n}a_{t+j}\left(c_{t+n+1}E_t x_{t+n+1} + \sum_{i=1}^{\infty}(\Pi_{j=0}^{i-1}a_{t+n+1+j})c_{t+n+1+i}E_t x_{t+n+1+i}\right) \\ &= D_{n+1}c_{t+n+1}E_t x_{t+n+1} + \sum_{i=1}^{\infty}(\Pi_{j=0}^{n+i}a_{t+n+1+j})c_{t+n+1+i}E_t x_{t+n+1+i} \\ &= D_{n+1}c_{t+n+1}E_t x_{t+n+1} + \sum_{i=1}^{\infty}D_{n+i+1}c_{t+n+1+i}E_t x_{t+n+1+i} \\ &= \sum_{k=n+1}^{\infty}D_k c_{t+k}E_t x_{t+k}. \qquad (25.36) \end{aligned}$$

In view of (25.25) it is enough to show that (25.26) implies $\lim_{n\to\infty}D_{n+1}E_t y_{t+n+1} = 0$. By (25.36) this is equivalent to showing that $\lim_{n\to\infty}\sum_{k=n+1}^{\infty}D_k c_{t+k}E_t x_{t+k} = 0$. We have

$$\begin{aligned} \sum_{k=1}^{\infty}D_k c_{t+k}E_t x_{t+k} &= \sum_{k=1}^{n}D_k c_{t+k}E_t x_{t+k} + \sum_{k=n+1}^{\infty}D_k c_{t+k}E_t x_{t+k} \Rightarrow \\ \sum_{k=n+1}^{\infty}D_k c_{t+k}E_t x_{t+k} &= \sum_{k=1}^{\infty}D_k c_{t+k}E_t x_{t+k} - \sum_{k=1}^{n}D_k c_{t+k}E_t x_{t+k} \Rightarrow \\ \lim_{n\to\infty}\sum_{k=n+1}^{\infty}D_k c_{t+k}E_t x_{t+k} &= \sum_{k=1}^{\infty}D_k c_{t+k}E_t x_{t+k} - \sum_{k=1}^{\infty}D_k c_{t+k}E_t x_{t+k} = 0, \end{aligned}$$

which was to be proved.

B. Repeated backward substitution

When $|a| > 1$, a *particular* solution, $\tilde{y}_t$, of our basic equation

$$y_t = aE_t y_{t+1} + cx_t, \qquad t = 0, 1, 2, ... \tag{25.37}$$

can often be found as a perfect-foresight solution constructed by repeated backward substitution. We will examine whether (25.37) has a solution with perfect foresight. We substitute $y_{t+1} = E_t y_{t+1}$ into (25.37) and write the resulting equation on backward-looking form:

$$y_{t+1} = \frac{1}{a} y_t - \frac{c}{a} x_t. \tag{25.38}$$

Repeated backward substitution gives

$$y_{t+1} = \left(\frac{1}{a}\right)^n y_{t+1-n} - c\left[\left(\frac{1}{a}\right)^n x_{t-n+1} + \left(\frac{1}{a}\right)^{n-1} x_{t-n+2} + ... + \frac{1}{a} x_t\right],$$

for $n = 1, 2, ...$. By letting $n \to \infty$ in this expression we see that a reasonable *guess* of a particular solution of (25.37) is

$$\tilde{y}_{t+1} = -c \sum_{i=1}^{\infty} (\frac{1}{a})^i x_{t+1-i}, \tag{25.39}$$

if this sum converges (by replacing t by $t-1$, we get the corresponding formula for $\tilde{y}_t$). By (25.39) follows that $E_t \tilde{y}_{t+1} = -c \sum_{i=1}^{\infty} (\frac{1}{a})^i x_{t+1-i} = \tilde{y}_{t+1}$, which corresponds to perfect foresight (reflecting that, by (25.39), $\tilde{y}_{t+1}$ is completely determined by past events which are included in the information on the basis of which expectation is formed in the preceding period). Hence, (25.39) implies

$$\begin{aligned}
\tilde{y}_{t+1} &= E_t \tilde{y}_{t+1} = -c\frac{1}{a} x_t - c\sum_{i=2}^{\infty} (\frac{1}{a})^i x_{t+1-i} = \frac{1}{a}\left(-cx_t - c\sum_{i=2}^{\infty} (\frac{1}{a})^{i-1} x_{t+1-i}\right) \\
&= \frac{1}{a}\left(-cx_t - c\sum_{i=1}^{\infty} (\frac{1}{a})^i x_{t-i}\right) = \frac{1}{a}\left(-cx_t + \tilde{y}_t\right), \qquad \text{so that} \\
\tilde{y}_t &= aE_t \tilde{y}_{t+1} + cx_t.
\end{aligned}$$

The process (25.39) therefore satisfies (25.37) and our guess is correct.

Consider the special case (25.31). Here (25.39) takes the form $\tilde{y}_{t+1} = -c\left[\frac{\bar{x}}{a-1} + \sum_{i=1}^{\infty} (\frac{1}{a})^i \varepsilon_{t+1-i}\right]$, where we can replace t by $t-1$. This is the background for the "educated guess", made in the main text, that also the simpler process (25.32) is a (particular) solution of (25.31).

C. The relationship between unstable roots and uniqueness of a converging solution

In the main text we considered stochastic first-order difference equations written on a *forward*-looking form. In math textbooks difference equations are usually written on a *backward*-looking form, suitable for the natural sciences. Concepts such as the *characteristic equation* and *stable* and *unstable roots* are associated with this backward-looking form. There is a link between these concepts and the question of uniqueness or non-uniqueness of a convergent solution to a forward-looking difference equation.

To clarify, we will for simplicity ignore uncertainty. That is, we assume expected values are always realized. Then the forward-looking form (25.37) reads $y_t = a\ y_{t+1} + c\ x_t$. The corresponding backward-looking form is

$$y_{t+1} - \frac{1}{a}y_t = -\frac{c}{a}x_t, \quad (a \neq 0),$$

or

$$y_{t+1} + ky_t = mx_t. \tag{25.40}$$

This is the standard form for a linear first-order difference equation with constant coefficient $k = -1/a$ and time-dependent right-hand side equal to mx_t, where $m \equiv -c/a$. The *homogeneous* difference equation corresponding to (25.40) is $y_{t+1} + ky_t = 0$, to which corresponds the characteristic equation $\rho + k = 0$. The characteristic root is $\rho = -k$ $(= 1/a)$. Any solution to this difference equation can be written

$$y_t = \tilde{y}_t + C\rho^t, \tag{25.41}$$

where $\tilde{y}_t$ is a particular solution of (25.40) and C is a constant depending on the initial value, y_0. If, for example $x_t = \bar{x}$ for all t, then (25.40) becomes

$$y_{t+1} + ky_t = m\bar{x},$$

and a particular solution is the stationary state

$$\tilde{y}_t = \frac{m\bar{x}}{1+k}. \qquad (k \neq -1)$$

By substitution into (25.41) we get $C = y_0 - m\bar{x}/(1+k)$. Hence, the general solution is

$$y_t = \frac{m\bar{x}}{1+k} + (y_0 - \frac{m\bar{x}}{1+k})\rho^t = \frac{c\bar{x}}{1-a} + (y_0 - \frac{c\bar{x}}{1-a})(\frac{1}{a})^t. \tag{25.42}$$

This is the same as (25.21).

Now define case A and case B in the following way:

Case A: $|\rho| < 1$, that is, $|a| > 1$.
Case B: $|\rho| > 1$, that is, $|a| < 1$.

The solution formula (25.42) shows that in case A all solutions converge. In this case the characteristic root is called a *stable root*. In case B the solution diverges unless $y_0 = c\bar{x}/(1-a)$. The characteristic root ρ is in case B called an *unstable root*.[12] Which of the two cases the researcher typically finds most "convenient" depends on whether y_0 is a predetermined or a jump variable:

I. y_0 being predetermined.
Case A: $|\rho| < 1$. The solution for y_t is unique and converges for every y_0.
Case B: $|\rho| > 1$. The solution for y_t is unique but does not converge when $y_0 \neq \frac{c\bar{x}}{1-a}$.

II. y_0 being a jump variable.
Case A: $|\rho| < 1$. Even if we can impose the restriction that y_t must converge, y_0 is not uniquely determined.
Case B: $|\rho| > 1$. If we can impose the restriction that y must converge, y_0 is uniquely determined as $y_0 = \frac{c\bar{x}}{1-a}$.

Hence, the cases I.A and II.B are the more "convenient" ones from the point of view of a researcher preferring unique solutions.

The question of multiplicity of solutions is harder in the case of a *non-linear* expectational difference equation. In this case, even if a condition corresponding to $|a| < 1$ is satisfied close to the steady state, there may be more than one non-explosive solution (for an example, see Blanchard and Fischer, 1989, Ch. 5, and the references therein).

In the appendix to Chapter 27 these matters are generalized to *systems* of first-order difference equations.

25.7 Exercises

25.1 *The housing market in an old city quarter (partial equilibrium analysis)* Consider the housing market in an old city quarter with unique amenity value (for convenience we will speak of "houses" although perhaps "apartments" would fit real world situations better). Let H be the aggregate stock of houses (apartments), measured in terms of some basic unit (a house of "normal size", somehow adjusted for quality) existing at a given point in time. No

[12] In the case $|\rho| = 1$ we have: if $\rho = -1$, the conclusion is as in case B; if $\rho = 1$, then $y_t = y_0 - c\bar{x}t$. Being a "knife-edge case", however, $|\rho| = 1$ is usually less interesting.

new construction is allowed, but repair and maintenance is required by law and so H is constant through time. Notation:

$$\begin{aligned} p_t &= \text{ the real price of a house (stock) at the beginning of period } t, \\ m &= \text{ real maintenance costs of a house (assumed constant over time)}, \\ \tilde{R}_t &= \text{ the real rental rate, i.e., the price of housing services (flow), in period } t, \\ R_t &= \tilde{R}_t - m = \text{ the } net \text{ rental rate} = \text{net revenue to the owner per unit} \\ & \quad \text{of housing services in period } t \end{aligned}$$

Let the housing services in period t be called S_t. Note that S_t is a *flow:* so and so many square meter-months are at the disposal for utilization (accommodation) for the owner or tenant during period t. We assume the rate of utilization of the house stock is constant over time. By choosing proper measurement units the rate of utilization is normalized to 1, and so $S_t = 1 \cdot H$. The prices p_t, m, and R_t are measured in *real* terms, that is, deflated by the consumer price index. We assume perfect competition in both the market for houses and the market for housing services.

Suppose the aggregate demand for housing services in period t is

$$D(\tilde{R}_t, X_t), \qquad D_1 < 0, D_2 > 0, \tag{*}$$

where the stochastic variable X_t reflects factors that in our partial equilibrium framework are exogenous (for example present value of expected future labor income in the region).

a) Set up an equation expressing equilibrium at the market for housing services. In a diagram in $(H, \tilde{R})$ space, for given X_t, illustrate how $\tilde{R}_t$ is determined.

b) Show that the equilibrium *net* rental rate at time t can be expressed as an implicit function of H, X_t, and m, written $R_t = \mathcal{R}(H, X_t, m)$. Sign the partial derivatives w.r.t. H and m of this function. Comment.

Suppose a constant tax rate $\tau_R \in [0, 1)$ is applied to rental income, after allowance for maintenance costs. In case of an owner-occupied house the owner still has to pay the tax $\tau_R R_t$ out of the implicit income, R_t, per house per year. Assume further there is a constant property tax rate $\tau_p \geq 0$ applied to the market value of houses. Finally, suppose a constant tax rate $\tau_r \in [0, 1)$ applies to interest income, whether positive or negative. We assume capital gains are not taxed and we ignore all complications arising from the fact that most countries have tax systems based on nominal income rather than real income. In a low-inflation world this limitation may not be serious.

We assume housing services are valued independently of whether the occupant owns or rents. We further assume that the market participants are risk-neutral and that transaction costs can be ignored. Then in equilibrium,

$$\frac{(1-\tau_R)R_t - \tau_p p_t + p^e_{t+1} - p_t}{p_t} = (1-\tau_r)r, \tag{**}$$

where p^e_{t+1} denotes the expected house price next period as seen from period t, and r is the real interest rate in the loan market. We assume $r > 0$ and all tax rates are constant over time.

c) Interpret (**).

Assume from now the market participants have rational expectations (and know the stochastic process which R_t follows as a consequence of the process of X_t).

d) Derive the expectational difference equation in p_t implied by (**).

e) Find the fundamental value of a house, assuming R_t does not grow "too fast". *Hint:* write (**) on the standard form for an expectational difference equation and use the formula for the fundamental solution.

Denote the fundamental value p^*_t. Assume R_t follows the process

$$R_t = \bar{R} + \varepsilon_t, \tag{***}$$

where $\bar{R}$ is a positive constant and ε_t is white noise with variance σ^2.

f) Find p^*_t under these conditions.

g) How does $E_{t-1}p^*_t$ (the conditional expectation one period beforehand of p^*_t) depend on each of the three tax rates? Comment.

h) How does $Var_{t-1}(p^*_t)$ (the conditional variance one period beforehand of p^*_t) depend on each of the three tax rates? Comment.

25.2 *A housing market with bubbles (partial equilibrium analysis)* We consider the same setup as in Exercise 25.1, including the equations (*), (**), and (***).

Suppose that until period 0 the houses were owned by the municipality. But in period 0 the houses are sold to the public at market prices. Suppose that by coincidence a large positive realization of ε_0 occurs and that this triggers a stochastic bubble of the form

$$\begin{aligned} b_{t+1} &= [1+\tau_p+(1-\tau_r)r]\, b_t+\varepsilon_{t+1}, \qquad t=0,1,2,..., \quad (\hat{}) \\ \text{where} \quad E_t\varepsilon_{t+1} &= 0 \text{ and } b_0=\varepsilon_0>0. \end{aligned}$$

Until further notice we assume b_0 is large enough relative to the stochastic process $\{\varepsilon_t\}$ to make the probability that b_{t+1} becomes non-positive negligible.

a) Can (^) be a rational bubble? You should answer this in two ways: 1) by using a short argument based on theoretical knowledge, and 2) by directly testing whether the price path $p_t = p_t^* + b_t$ is arbitrage free. Comment.

b) Determine the value of the bubble in period t, assuming ε_{t-i} known for $i = 0, 1, ..., t$.

c) Determine the market price, p_t, and the conditional expectation $E_t p_{t+1}$. Both results will reflect a kind of "overreaction" of the market price to the shock ε_t. In what sense?

d) It may be argued that a stochastic bubble of the described ever-lasting kind does not seem plausible. What kind of arguments could be used to support this view?

e) Still assuming $b_0 > 0$, construct a rational bubble which has a constant probability of bursting in each period $t = 1, 2, ...$

f) What is the expected further duration of the bubble as seen from any period $t = 0, 1, 2, ...,$ given $b_t > 0$? *Hint:* $\sum_{i=0}^{\infty} iq^i(1-q) = q/(1-q)$.[13]

g) If the bubble is alive in period t, what is the probability that the bubble is still alive in period $t + s$, where $s = 1, 2, ...$? What is the limit of this probability for $s \to \infty$?

h) Assess this last bubble model.

i) Housing prices are generally considered to be a good indicator of the turning points in business cycles in the sense that house prices tend

[13]Here is a proof of this formula. $\sum_{i=0}^{\infty} iq^i(1-q) = (1-q)q\sum_{i=0}^{\infty} iq^{i-1} = (1-q)q\sum_{i=0}^{\infty} dq^i/dq = (1-q)qd\left(\sum_{i=0}^{\infty} q^i\right)/dq = (1-q)qd\,(1-q)^{-1}/dq = (1-q)q\,(1-q)^{-2} = q\,(1-q)^{-1}$. □

to move in advance of aggregate economic activity, in the same direction. In the language of business cycle analysts housing prices are a *procyclical leading indicator*. Do you think this last bubble model fit this observation? *Hint:* consider how a rise in p affects residential investment and how this affects the economy as a whole.