C H A P T E R 3
Applications of Differentiation

CHAPTER 3
Applications of Differentiation

Section 3.1 Extrema on an Interval

1. A: neither
B: absolute maximum (and relative maximum)
C: neither
D: neither
E: relative maximum
F: relative minimum
G: neither

2. A: absolute minimum
B: relative maximum
C: neither
D: relative minimum
E: relative maximum
F: relative minimum
G: neither

3. $f(x) = \dfrac{x^2}{x^2 + 4}$

$$f'(x) = \frac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$$

$f'(0) = 0$

4. $f(x) = \cos\dfrac{\pi x}{2}$

$$f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2}$$

$f'(0) = 0$

$f'(2) = 0$

5. $f(x) = x + \dfrac{27}{2x^2} = x + \dfrac{27}{2}x^{-2}$

$$f'(x) = 1 - 27x^{-3} = 1 - \frac{27}{x^3}$$

$$f'(3) = 1 - \frac{27}{3^3} = 1 - 1 = 0$$

6. $f(x) = -3x\sqrt{x + 1}$

$$f'(x) = -3x\left[\frac{1}{2}(x + 1)^{-1/2}\right] + \sqrt{x + 1}\,(-3)$$

$$= -\frac{3}{2}(x + 1)^{-1/2}[x + 2(x + 1)]$$

$$= -\frac{3}{2}(x + 1)^{-1/2}(3x + 2)$$

$$f'\left(-\frac{2}{3}\right) = 0$$

7. $f(x) = (x + 2)^{2/3}$

$$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$$

$f'(-2)$ is undefined.

8. Using the limit definition of the derivative,

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{(4 - |x|) - 4}{x - 0} = -1$$

$f'(0)$ does not exist, since the one-sided derivatives are not equal.

9. Critical number: $x = 2$

$x = 2$: absolute maximum

10. Critical number: $x = 0$

$x = 0$: neither

11. Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maximum

$x = 2$: absolute minimum

12. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum

13. $f(x) = x^2(x - 3) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

Critical numbers: $x = 0, x = 2$

14. $g(x) = x^2(x^2 - 4) = x^4 - 4x^2$

$g'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

Critical numbers: $x = 0, x = \pm\sqrt{2}$

15. $g(t) = t\sqrt{4 - t},\ t < 3$

$$g'(t) = t\left[\frac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$$

$$= \frac{1}{2}(4 - t)^{-1/2}[-t + 2(4 - t)]$$

$$= \frac{8 - 3t}{2\sqrt{4 - t}}$$

Critical number: $t = \frac{8}{3}$.

16. $f(x) = \dfrac{4x}{x^2 + 1}$

$$f'(x) = \frac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \frac{4(1 - x^2)}{(x^2 + 1)^2}$$

Critical numbers: $x = \pm 1$

17. $h(x) = \sin^2 x + \cos x, 0 < x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

On $(0, 2\pi)$, critical numbers: $x = \dfrac{\pi}{3}, x = \pi, x = \dfrac{5\pi}{3}$

18. $f(\theta) = 2 \sec \theta + \tan \theta, 0 < \theta < 2\pi$

$$f'(\theta) = 2 \sec \theta \tan \theta + \sec^2 \theta$$

$$= \sec \theta(2 \tan \theta + \sec \theta)$$

$$= \sec \theta\left[2\left(\frac{\sin \theta}{\cos \theta}\right) + \frac{1}{\cos \theta}\right]$$

$$= \sec^2 \theta(2 \sin \theta + 1)$$

On $(0, 2\pi)$, critical numbers: $\theta = \dfrac{7\pi}{6}, \theta = \dfrac{11\pi}{6}$

19. $f(x) = 2(3 - x), [-1, 2]$

$f'(x) = -2 \Longrightarrow$ No critical numbers

Left endpoint: $(-1, 8)$ Maximum

Right endpoint: $(2, 2)$ Minimum

20. $f(x) = \dfrac{2x + 5}{3}, [0, 5]$

$f'(x) = \dfrac{2}{3} \Longrightarrow$ No critical numbers

Left endpoint: $\left(0, \dfrac{5}{3}\right)$ Minimum

Right endpoint: $(5, 5)$ Maximum

21. $f(x) = -x^2 + 3x, [0, 3]$

$f'(x) = -2x + 3$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3}{2}, \frac{9}{4}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

22. $f(x) = x^2 + 2x - 4, [-1, 1]$

$f'(x) = 2x + 2 = 2(x + 1)$

Left endpoint: $(-1, -5)$ Minimum

Right endpoint: $(1, -1)$ Maximum

23. $f(x) = x^3 - \frac{3}{2}x^2,\ [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

24. $f(x) = x^3 - 12x, [0, 4]$

$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$

Left endpoint: $(0, 0)$

Critical number: $(2, -16)$ Minimum

Right endpoint: $(4, 16)$ Maximum

Note: $x = -2$ is not in the interval.

25. $f(x) = 3x^{2/3} - 2x, [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \dfrac{2\left(1 - \sqrt[3]{x}\right)}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

26. $g(x) = \sqrt[3]{x}, [-1, 1]$

$g'(x) = \dfrac{1}{3x^{2/3}}$

Left endpoint: $(-1, -1)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(1, 1)$ Maximum

27. $g(t) = \dfrac{t^2}{t^2 + 3}, [-1, 1]$

$g'(t) = \dfrac{6t}{(t^2 + 3)^2}$

Left endpoint: $\left(-1, \frac{1}{4}\right)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $\left(1, \frac{1}{4}\right)$ Maximum

28. $f(x) = \dfrac{2x}{x^2 + 1}, \qquad [-2, 2]$

$f'(x) = \dfrac{(x^2 + 1)2 - 2x(2x)}{(x^2 + 1)^2} = \dfrac{2 - 2x^2}{(x^2 + 1)^2}$

$f'(x) = \dfrac{2(1 - x^2)}{(x^2 + 1)^2} = 0$

$x = -1, 1 \qquad$ Critical numbers

Left endpoint $f(-2) = -4/5$	Critical number $f(-1) = -1$	Critical number $f(1) = 1$	Right endpoint $f(2) = 4/5$
	Minimum	Maximum	

29. $h(s) = \dfrac{1}{s-2}, [0, 1]$

$h'(s) = \dfrac{-1}{(s-2)^2}$

Left endpoint: $\left(0, -\dfrac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

30. $h(t) = \dfrac{t}{t-2}, [3, 5]$

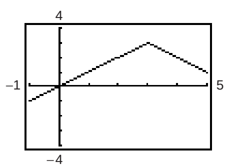
$h'(t) = \dfrac{-2}{(t-2)^2}$

Left endpoint: $(3, 3)$ Maximum

Right endpoint: $\left(5, \dfrac{5}{3}\right)$ Minimum

31. $y = 3 - |t - 3|, [-1, 5]$

From the graph, you see that $t = 3$ is a critical number.



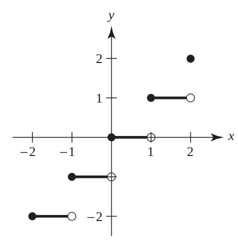
Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

32. $f(x) = [\![x]\!], \quad [-2, 2]$

From the graph of f, we see that the maximum value of f is 2 for $x = 2$, and the minimum value is -2 for $-2 \le x < -1$.



33. $f(x) = \cos \pi x, \left[0, \dfrac{1}{6}\right]$

$f'(x) = -\pi \sin \pi x$

Left endpoint: $(0, 1)$ Maximum

Right endpoint: $\left(\dfrac{1}{6}, \dfrac{\sqrt{3}}{2}\right)$ Minimum

34. $g(x) = \sec x, \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\dfrac{\pi}{6}, \dfrac{2}{\sqrt{3}}\right) \approx \left(-\dfrac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\dfrac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

35. $y = \dfrac{4}{x} + \tan\left(\dfrac{\pi x}{8}\right), [1, 2]$

$$y' = \frac{-4}{x^2} + \frac{\pi}{8}\sec^2\frac{\pi x}{8} = 0$$

$$\frac{\pi}{8}\sec^2\frac{\pi x}{8} = \frac{4}{x^2}$$

On the interval $[1, 2]$, this equation has no solutions. Thus, there are no critical numbers.

Left endpoint: $\left(1, \sqrt{2} + 3\right) \approx (1, 4.4142)$ Maximum

Right endpoint: $(2, 3)$ Minimum

36. $y = x^2 - 2 - \cos x, [-1, 3]$

$y' = 2x - \sin x$

Left endpoint: $(-1, -1.5403)$

Right endpoint: $(3, 7.99)$ Maximum

Critical number: $(0, -3)$ Minimum

37. (a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

38. (a) Minimum: $(4, 1)$

Maximum: $(1, 4)$

(b) Maximum: $(1, 4)$

(c) Minimum: $(4, 1)$

(d) No extrema

39. $f(x) = x^2 - 2x$

(a) Minimum: $(1, -1)$

Maximum: $(-1, 3)$

(b) Maximum: $(3, 3)$

(c) Minimum: $(1, -1)$

(d) Minimum: $(1, -1)$

40. (a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

(b) Minimum: $(-2, 0)$

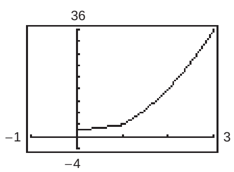
(c) Maximum: $(0, 2)$

(d) Maximum: $\left(1, \sqrt{3}\right)$

41. $f(x) = \begin{cases} 2x + 2, & 0 \le x \le 1 \\ 4x^2, & 1 < x \le 3 \end{cases}$

Left endpoint: $(0, 2)$ Minimum

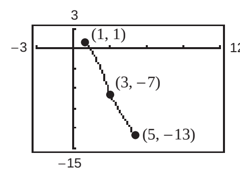
Right endpoint: $(3, 36)$ Maximum



42. $f(x) = \begin{cases} 2 - x^2, & 1 \le x < 3 \\ 2 - 3x, & 3 \le x \le 5 \end{cases}$

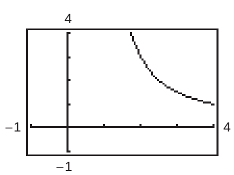
Left endpoint: $(1, 1)$ Maximum

Right endpoint: $(5, -13)$ Minimum



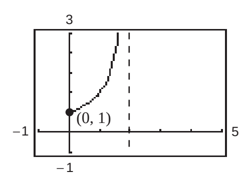
43. $f(x) = \dfrac{3}{x - 1}, (1, 4]$

Right endpoint: $(4, 1)$ Minimum

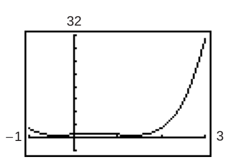


44. $f(x) = \dfrac{2}{2 - x}, [0, 2)$

Left endpoint: $(0, 1)$ Minimum



45. $f(x) = x^4 - 2x^3 + x + 1, \quad [-1, 3]$



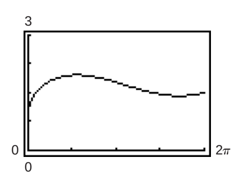
$f'(x) = 4x^3 - 6x^2 + 1 = (2x - 1)(2x^2 - 2x - 1) = 0$

$x = \dfrac{1}{2}, \dfrac{1 \pm \sqrt{3}}{2} \approx 0.5, -0.366, 1.366$

Maximum: $f(3) = 31$

Minimum: $f\left(\dfrac{1 \pm \sqrt{3}}{2}\right) = \dfrac{3}{4}$

46. $f(x) = \sqrt{x} + \cos \dfrac{x}{2}, \quad [0, 2\pi]$

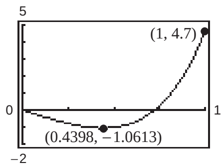


$f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2} \sin \dfrac{x}{2}$

Maximum: $(1.729, 1.964)$

Minimum: $f(0) = 1$

47. (a)



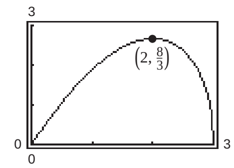
Maximum: $(1, 4.7)$ (endpoint)

Minimum: $(0.4398, -1.0613)$

(b)
$$f(x) = 3.2x^5 + 5x^3 - 3.5x,\ [0, 1]$$
$$f'(x) = 16x^4 + 15x^2 - 3.5$$
$$16x^4 + 15x^2 - 3.5 = 0$$
$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)}$$
$$= \frac{-15 \pm \sqrt{449}}{32}$$
$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$
$$f(0) = 0$$
$$f(1) = 4.7 \quad \text{Maximum (endpoint)}$$
$$f\left(\sqrt{\frac{-15 + \sqrt{449}}{32}}\right) \approx -1.0613$$

Minimum: $(0.4398, -1.0613)$

48. (a)



Maximum: $\left(2, \dfrac{8}{3}\right)$

Minimum: $(0, 0), (3, 0)$

(b) $f(x) = \dfrac{4}{3}x\sqrt{3 - x},\ [0, 3]$

$$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right]$$
$$= \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)[-x + 2(3 - x)]$$
$$= \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 - x)}{\sqrt{3 - x}}$$

Critical number: $x = 2$

$f(0) = 0$ Minimum

$f(3) = 0$ Minimum

$f(2) = \dfrac{8}{3}$

Maximum: $\left(2, \dfrac{8}{3}\right)$

49. $f(x) = (1 + x^3)^{1/2},\ [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$
$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$
$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, we have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$
$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 + \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

50. $f(x) = \dfrac{1}{x^2 + 1},\ \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$
$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$
$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, we have $x = 0, \pm 1$.

$|f''(1)| = \dfrac{1}{2}$ is the maximum value.

51. $f(x) = (x + 1)^{2/3}, [0, 2]$

$f'(x) = \frac{2}{3}(x + 1)^{-1/3}$

$f''(x) = -\frac{2}{9}(x + 1)^{-4/3}$

$f'''(x) = \frac{8}{27}(x + 1)^{-7/3}$

$f^{(4)}(x) = -\frac{56}{81}(x + 1)^{-10/3}$

$f^{(5)}(x) = \frac{560}{243}(x + 1)^{-13/3}$

$|f^{(4)}(0)| = \frac{56}{81}$ is the maximum value.

52. $f(x) = \frac{1}{x^2 + 1}, [-1, 1]$

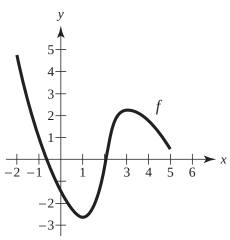
$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$

$f^{(4)}(x) = \frac{24(5x^4 - 10x^2 + 1)}{(x^2 + 1)^5}$

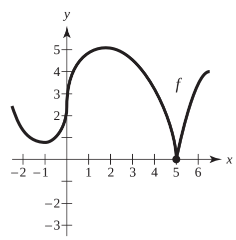
$f^{(5)}(x) = \frac{-240x(3x^4 - 10x^2 + 3)}{(x^2 + 1)^6}$

$|f^{(4)}(0)| = 24$ is the maximum value.

53.



54.



55. (a) Yes
(b) No

56. (a) No
(b) Yes

57. (a) No
(b) Yes

58. (a) No
(b) Yes

59. $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. P is decreasing for $I > 12$.

60. $C = 2x + \frac{300{,}000}{x}, 1 \le x \le 300$

$C(1) = 300{,}002$

$C(300) = 1600$

$C' = 2 - \frac{300{,}000}{x^2} = 0$

$2x^2 = 300{,}000$

$x^2 = 150{,}000$

$x = 100\sqrt{15} \approx 387 > 300$ (outside of interval)

C is minimized when $x = 300$ units.

Yes, if $1 \le x \le 400$, then $x = 387$ would minimize C.

61. $x = \dfrac{v^2 \sin 2\theta}{32}, \dfrac{\pi}{4} \le \theta \le \dfrac{3\pi}{4}$

$\dfrac{d\theta}{dt}$ is constant.

$$\frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} \text{ (by the Chain Rule)}$$

$$= \frac{v^2 \cos 2\theta}{16}\frac{d\theta}{dt}$$

In the interval $[\pi/4, 3\pi/4]$, $\theta = \pi/4, 3\pi/4$ indicate minimums for dx/dt and $\theta = \pi/2$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = \pi/4$ and $3\pi/4$. Thus, the lawn farthest from the sprinkler gets the most water.

62. $S = 6hs + \dfrac{3s^2}{2}\left(\dfrac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{2}$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\frac{\pi}{2}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{2}\right)$$

S is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radians.

63. True. See Exercise 27.

64. True. This is stated in the Extreme Value Theorem.

65. True

66. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$$g(x) = f(x - k)$$

$$= (x - k)^2$$

$x = k$ is a critical number of g.

67. If f has a maximum value at $x = c$, then $f(c) \ge f(x)$ for all x in I. Hence, $-f(c) \le -f(x)$ for all x in I. Thus, $-f$ has a minimum value at $x = c$.

68. $f(x) = ax^3 + bx^2 + cx + d, \quad a \ne 0$

$f'(x) = 3ax^2 + 2bx + c$

The quadratic polynomial can have zero, one, or two zeros.

$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Zero critical numbers: $b^2 < 3ac$.

Example: $(a = b = c = 1, d = 0)$ $f(x) = x^3 + x^2 + x$ has no critical numbers.

One critical number: $b^2 = 3ac$.

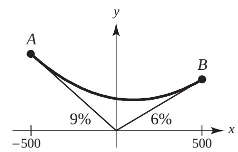
Example: $(a = 1, b = c = d = 0)$ $f(x) = x^3$ has one critical number, $x = 0$.

Two critical numbers: $b^2 > 3ac$.

Example: $(a = c = 1, b = 2, d = 0)$ $f(x) = x^3 + 2x^2 + x$ has two critical numbers: $x = -1, -\dfrac{1}{3}$.

69. (a) $y = ax^2 + bx + c$

$y' = 2ax + b$



The coordinates of B are $(500, 30)$, and those of A are $(-500, 45)$. From the slopes at A and B,

$$-1000a + b = -0.09$$

$$1000a + b = 0.06.$$

Solving these two equations, you obtain $a = 3/40{,}000$and $b = -3/200$. From the points $(500, 30)$ and $(-500, 45)$, you obtain

$$30 = \frac{3}{40{,}000}500^2 + 500\left(\frac{-3}{200}\right) + c$$

$$45 = \frac{3}{40{,}000}500^2 - 500\left(\frac{-3}{200}\right) + c.$$

In both cases, $c = 18.75 = \frac{75}{4}$. Thus,

$$y = \frac{3}{40{,}000}x^2 - \frac{3}{200}x + \frac{75}{4}.$$

(b)

x	-500	-400	-300	-200	-100	0	100	200	300	400	500
d	0	0.75	3	6.75	12	18.75	12	6.75	3	0.75	0

For $-500 \le x \le 0$, $d = (ax^2 + bx + c) - (-0.09x)$.

For $0 \le x \le 500$, $d = (ax^2 + bx + c) - (0.06x)$.

(c) The lowest point on the highway is $(100, 18)$, which is not directly over the point where the two hillsides come together.

Section 3.2 Rolle's Theorem and the Mean Value Theorem

1. Rolle's Theorem does not apply to $f(x) = 1 - |x - 1|$ over $[0, 2]$ since f is not differentiable at $x = 1$.

2. Rolle's Theorem does not apply to $f(x) = \cot(x/2)$ over $[\pi, 3\pi]$ since f is not continuous at $x = 2\pi$.

3. $f(x) = \left|\frac{1}{x}\right|$

$f(-1) = f(1) = 1$. But, f is not continuous on $[-1, 1]$.

4. $f(x) = \sqrt{(2 - x^{2/3})^3}$

$f(-1) = f(1) = 1$

$$f'(x) = \frac{-\sqrt{(2 - x^{2/3})}}{x^{1/3}}$$

f is not differentiable at $x = 0$.

5. $f(x) = x^2 - x - 2 = (x - 2)(x + 1)$

x-intercepts: $(-1, 0), (2, 0)$

$f'(x) = 2x - 1 = 0$ at $x = \frac{1}{2}$.

6. $f(x) = x(x - 3)$

x-intercepts: $(0, 0), (3, 0)$

$f'(x) = 2x - 3 = 0$ at $x = \frac{3}{2}$.

7. $f(x) = x\sqrt{x+4}$

x-intercepts: $(-4, 0), (0, 0)$

$$f'(x) = x\frac{1}{2}(x+4)^{-1/2} + (x+4)^{1/2}$$

$$= (x+4)^{-1/2}\left(\frac{x}{2} + (x+4)\right)$$

$$f'(x) = \left(\frac{3}{2}x + 4\right)(x+4)^{-1/2} = 0 \text{ at } x = -\frac{8}{3}$$

8. $f(x) = -3x\sqrt{x+1}$

x-intercepts: $(-1, 0), (0, 0)$

$$f'(x) = -3x\frac{1}{2}(x+1)^{-1/2} - 3(x+1)^{1/2}$$

$$= -3(x+1)^{-1/2}\left(\frac{x}{2} + (x+1)\right)$$

$$f'(x) = -3(x+1)^{-1/2}\left(\frac{3}{2}x + 1\right) = 0 \text{ at } x = -\frac{2}{3}$$

9. $f(x) = x^2 + 3x - 4$

$f(-4) = f(1) = 0$

$f'(x) = 2x + 3 = 0$ for $x = \frac{-3}{2}$

$c = \frac{-3}{2}$ and $f'\left(\frac{-3}{2}\right) = 0$

10. $f(x) = \sin 2x$

$f(0) = f\left(\frac{\pi}{2}\right) = 0$

$f'(x) = 2\cos 2x = 0$ for $x = \frac{\pi}{4}$

$c = \frac{\pi}{4}$ and $f'\left(\frac{\pi}{4}\right) = 0$

11. $f(x) = x^2 - 2x, [0, 2]$

$f(0) = f(2) = 0$

f is continuous on $[0, 2]$. f is differentiable on $(0, 2)$. Rolle's Theorem applies.

$$f'(x) = 2x - 2$$

$$2x - 2 = 0 \Rightarrow x = 1$$

c-value: 1

12. $f(x) = x^2 - 5x + 4, [1, 4]$

$f(1) = f(4) = 0$

f is continuous on $[1, 4]$. f is differentiable on $(1, 4)$. Rolle's Theorem applies.

$$f'(x) = 2x - 5$$

$$2x - 5 = 0 \Rightarrow x = \frac{5}{2}$$

c-value: $\frac{5}{2}$

13. $f(x) = (x-1)(x-2)(x-3), [1, 3]$

$f(1) = f(3) = 0$

f is continuous on $[1, 3]$. f is differentiable on $(1, 3)$. Rolle's Theorem applies.

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = \frac{6 \pm \sqrt{3}}{3}$$

$c = \frac{6 - \sqrt{3}}{3}, c = \frac{6 + \sqrt{3}}{3}$

14. $f(x) = (x-3)(x+1)^2, [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = (x-3)(2)(x+1) + (x+1)^2$$

$$= (x+1)[2x - 6 + x + 1]$$

$$= (x+1)(3x-5)$$

c-value: $\frac{5}{3}$

15. $f(x) = x^{2/3} - 1, [-8, 8]$

$f(-8) = f(8) = 3$

f is continuous on $[-8, 8]$. f is not differentiable on $(-8, 8)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.

16. $f(x) = 3 - |x - 3|, [0, 6]$

$f(0) = f(6) = 0$

f is continuous on $[0, 6]$. f is not differentiable on $(0, 6)$ since $f'(3)$ does not exist. Rolle's Theorem does not apply.

17. $f(x) = \dfrac{x^2 - 2x - 3}{x + 2}, \; [-1, 3]$

$f(-1) = f(3) = 0$

f is continuous on $[-1, 3]$. (**Note:** The discontinuity, $x = -2$, is not in the interval.) f is differentiable on $(-1, 3)$. Rolle's Theorem applies.

$$f'(x) = \frac{(x + 2)(2x - 2) - (x^2 - 2x - 3)(1)}{(x + 2)^2} = 0$$

$$\frac{x^2 + 4x - 1}{(x + 2)^2} = 0$$

$$x = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

c-value: $-2 + \sqrt{5}$

18. $f(x) = \dfrac{x^2 - 1}{x}, \; [-1, 1]$

$f(-1) = f(1) = 0$

f is not continuous on $[-1, 1]$ since $f(0)$ does not exist. Rolle's Theorem does not apply.

19. $f(x) = \sin x, \; [0, 2\pi]$

$f(0) = f(2\pi) = 0$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = \cos x$$

c-values: $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

20. $f(x) = \cos x, \; [0, 2\pi]$

$f(0) = f(2\pi) = 1$

f is continuous on $[0, 2\pi]$. f is differentiable on $(0, 2\pi)$. Rolle's Theorem applies.

$$f'(x) = -\sin x$$

c-value: π

21. $f(x) = \dfrac{6x}{\pi} - 4\sin^2 x, \; \left[0, \dfrac{\pi}{6}\right]$

$f(0) = f\left(\dfrac{\pi}{6}\right) = 0$

f is continuous on $[0, \pi/6]$. f is differentiable on $(0, \pi/6)$. Rolle's Theorem applies.

$$f'(x) = \frac{6}{\pi} - 8\sin x \cos x = 0$$

$$\frac{6}{\pi} = 8\sin x \cos x$$

$$\frac{3}{4\pi} = \frac{1}{2}\sin 2x$$

$$\frac{3}{2\pi} = \sin 2x$$

$$\frac{1}{2}\arcsin\left(\frac{3}{2\pi}\right) = x$$

$$x \approx 0.2489$$

c-value: 0.2489

22. $f(x) = \cos 2x, \ \left[-\frac{\pi}{12}, \frac{\pi}{6}\right]$

$$f\left(-\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$f\left(-\frac{\pi}{12}\right) \neq f\left(\frac{\pi}{0}\right)$$

Rolle's Theorem does not apply.

23. $f(x) = \tan x, \ [0, \pi]$

$f(0) = f(\pi) = 0$

f is not continuous on $[0, \pi]$ since $f(\pi/2)$ does not exist. Rolle's Theorem does not apply.

24. $f(x) = \sec x, \ \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

$$f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

f is continuous on $[-\pi/4, \pi/4]$. f is differentiable on $(-\pi/4, \pi/4)$. Rolle's Theorem applies.

$$f'(x) = \sec x \tan x$$

$$\sec x \tan x = 0$$

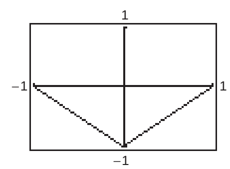
$$x = 0$$

c-value: 0

25. $f(x) = |x| - 1, \ [-1, 1]$

$f(-1) = f(1) = 0$

f is continuous on $[-1, 1]$. f is not differentiable on $(-1, 1)$ since $f'(0)$ does not exist. Rolle's Theorem does not apply.



26. $f(x) = x - x^{1/3}, \ [0, 1]$

$f(0) = f(1) = 0$

f is continuous on $[0, 1]$. f is differentiable on $(0, 1)$. (**Note:** f is not differentiable at $x = 0$.) Rolle's Theorem applies.

$$f'(x) = 1 - \frac{1}{3\sqrt[3]{x^2}} = 0$$

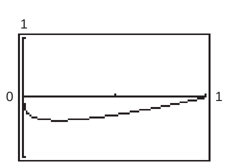
$$1 = \frac{1}{3\sqrt[3]{x^2}}$$

$$\sqrt[3]{x^2} = \frac{1}{3}$$

$$x^2 = \frac{1}{27}$$

$$x = \sqrt{\frac{1}{27}} = \frac{\sqrt{3}}{9}$$

c-value: $\frac{\sqrt{3}}{9} \approx 0.1925$



27. $f(x) = 4x - \tan \pi x, \ \left[-\frac{1}{4}, \frac{1}{4}\right]$

$$f\left(-\frac{1}{4}\right) = f\left(\frac{1}{4}\right) = 0$$

f is continuous on $[-1/4, 1/4]$. f is differentiable on $(-1/4, 1/4)$. Rolle's Theorem applies.

$$f'(x) = 4 - \pi \sec^2 \pi x = 0$$

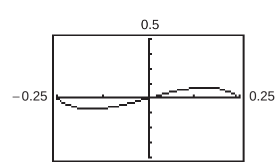
$$\sec^2 \pi x = \frac{4}{\pi}$$

$$\sec \pi x = \pm\frac{2}{\sqrt{\pi}}$$

$$x = \pm\frac{1}{\pi} \operatorname{arcsec} \frac{2}{\sqrt{\pi}} = \pm\frac{1}{\pi} \arccos \frac{\sqrt{\pi}}{2}$$

$$\approx \pm 0.1533 \text{ radian}$$

c-values: ± 0.1533 radian



28. $f(x) = \dfrac{x}{2} - \sin\dfrac{\pi x}{6},\ [-1, 0]$

$f(-1) = f(0) = 0$

f is continuous on $[-1, 0]$. f is differentiable on $(-1, 0)$. Rolle's Theorem applies.

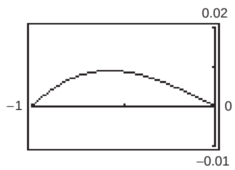
$$f'(x) = \frac{1}{2} - \frac{\pi}{6}\cos\frac{\pi x}{6} = 0$$

$$\cos\frac{\pi x}{6} = \frac{3}{\pi}$$

$$x = -\frac{6}{\pi}\arccos\frac{3}{\pi} \quad [\text{Value needed in } (-1, 0).]$$

$$\approx -0.5756 \text{ radian}$$

c-value: -0.5756



29. $f(t) = -16t^2 + 48t + 32$

(a) $f(1) = f(2) = 64$

(b) $v = f'(t)$ must be 0 at some time in $(1, 2)$.

$$f'(t) = -32t + 48 = 0$$

$$t = \frac{3}{2} \text{ seconds}$$

30. $C(x) = 10\left(\dfrac{1}{x} + \dfrac{x}{x + 3}\right)$

(a) $C(3) = C(6) = \dfrac{25}{3}$

(b) $$C'(x) = 10\left(-\frac{1}{x^2} + \frac{3}{(x + 3)^2}\right) = 0$$

$$\frac{3}{x^2 + 6x + 9} = \frac{1}{x^2}$$

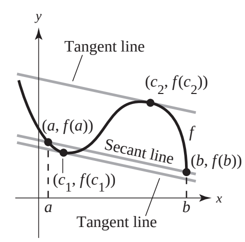
$$2x^2 - 6x - 9 = 0$$

$$x = \frac{6 \pm \sqrt{108}}{4}$$

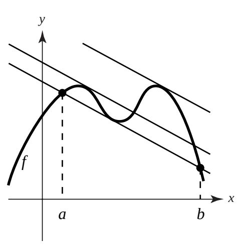
$$= \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

In the interval $(3, 6)$: $c = \dfrac{3 + 3\sqrt{3}}{2} \approx 4.098$

31.



32.



33. f is not continuous on the interval $[0, 6]$. (f is not continuous at $x = 2$.)

34. f is not differentiable at $x = 2$. The graph of f is not smooth at $x = 2$.

35. $f(x) = \dfrac{1}{x - 3},\ [0, 6]$

f has a discontinuity at $x = 3$.

36. $f(x) = |x - 3|$, $[0, 6]$

f is not differentiable at $x = 3$.

37. $f(x) = x^2 + 1$

(a) slope $= \dfrac{5 - 2}{2 + 1} = 1$

secant line: $y - 2 = 1(x + 1)$

$$y = x + 3$$

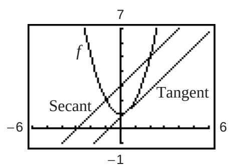
(b) $f'(x) = 2x = 1 \implies c = \dfrac{1}{2}$

$f'\left(\dfrac{1}{2}\right) = 1, \quad f\left(\dfrac{1}{2}\right) = \dfrac{5}{4}$

(c) Tangent line: $y - \dfrac{5}{4} = 1\left(x - \dfrac{1}{2}\right)$

$$y = x + \frac{3}{4}$$

(d)



38. $f(x) = -x^2 - x + 6$

(a) slope $= \dfrac{4 - 0}{-2 - 2} = -1$

secant line: $y - 0 = -1(x - 2)$

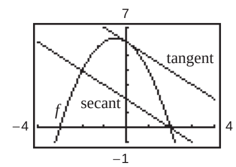
$$y = -x + 2$$

(b) $f'(x) = -2x - 1 = -1 \implies c = 0$ and $f(c) = 6$

(c) Tangent line: $y - 6 = -1(x - 0)$

$$y = -x + 6$$

(d)



39. $f(x) = x^2$ is continuous on $[-2, 1]$ and differentiable on $(-2, 1)$.

$$\frac{f(1) - f(-2)}{1 - (-2)} = \frac{1 - 4}{3} = -1$$

$f'(x) = 2x = -1$ when $x = -\dfrac{1}{2}$. Therefore,

$$c = -\frac{1}{2}.$$

40. $f(x) = x(x^2 - x - 2)$ is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$.

$$\frac{f(1) - f(-1)}{1 - (-1)} = -1$$

$$f'(x) = 3x^2 - 2x - 2 = -1$$

$$(3x + 1)(x - 1) = 0$$

$$c = -\frac{1}{3}$$

41. $f(x) = x^{2/3}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1) - f(0)}{1 - 0} = 1$$

$$f'(x) = \frac{2}{3}x^{-1/3} = 1$$

$$x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$c = \frac{8}{27}$$

42. $f(x) = (x + 1)/x$ is continuous on $[1/2, 2]$ and differentiable on $(1/2, 2)$.

$$\frac{f(2) - f(1/2)}{2 - (1/2)} = \frac{(3/2) - 3}{3/2} = -1$$

$$f'(x) = \frac{-1}{x^2} = -1$$

$$x^2 = 1$$

$$c = 1$$

43. $f(x) = \sqrt{2-x}$ is continuous on $[-7, 2]$ and differentiable on $(-7, 2)$.

$$\frac{f(2)-f(-7)}{2-(-7)} = \frac{0-3}{9} = -\frac{1}{3}$$
$$f'(x) = \frac{-1}{2\sqrt{2-x}} = -\frac{1}{3}$$
$$2\sqrt{2-x} = 3$$
$$\sqrt{2-x} = \frac{3}{2}$$
$$2-x = \frac{9}{4}$$
$$x = -\frac{1}{4}$$
$$c = -\frac{1}{4}$$

44. $f(x) = x^3$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

$$\frac{f(1)-f(0)}{1-0} = \frac{1-0}{1} = 1$$
$$f'(x) = 3x^2 = 1$$
$$x = \pm\frac{\sqrt{3}}{3}$$

In the interval $(0, 1)$: $c = \dfrac{\sqrt{3}}{3}$.

45. $f(x) = \sin x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

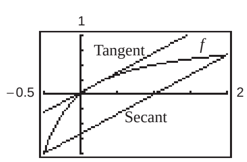
$$\frac{f(\pi)-f(0)}{\pi-0} = \frac{0-0}{\pi} = 0$$
$$f'(x) = \cos x = 0$$
$$c = \frac{\pi}{2}$$

46. $f(x) = 2\sin x + \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$.

$$\frac{f(\pi)-f(0)}{\pi-0} = \frac{0-0}{\pi} = 0$$
$$f'(x) = 2\cos x + 2\cos 2x = 0$$
$$2[\cos x + 2\cos^2 x - 1] = 0$$
$$2(2\cos x - 1)(\cos x + 1) = 0$$
$$\cos x = \frac{1}{2}$$
$$\cos x = -1$$
$$x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

In the interval $(0, \pi)$: $c = \dfrac{\pi}{3}$

47. $f(x) = \dfrac{x}{x+1}$, $\left[-\dfrac{1}{2}, 2\right]$

(a) 

(b) Secant line:

$$\text{slope} = \frac{f(2)-f(-1/2)}{2-(-1/2)} = \frac{2/3-(-1)}{5/2} = \frac{2}{3}$$
$$y - \frac{2}{3} = \frac{2}{3}(x-2)$$
$$3y - 2 = 2x - 4$$
$$3y - 2x + 2 = 0$$

(c) $f'(x) = \dfrac{1}{(x+1)^2} = \dfrac{2}{3}$

$$(x+1)^2 = \frac{3}{2}$$
$$x = -1 \pm \sqrt{\frac{3}{2}} = -1 \pm \frac{\sqrt{6}}{2}$$

In the interval $[-1/2, 2]$, $c = -1 + \left(\sqrt{6}/2\right)$.

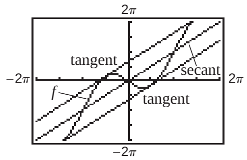
$$f(c) = \frac{-1+\left(\sqrt{6}/2\right)}{\left[-1+\left(\sqrt{6}/2\right)\right]+1} = \frac{-2+\sqrt{6}}{\sqrt{6}} = \frac{-2}{\sqrt{6}} + 1$$

Tangent line: $y - 1 + \dfrac{2}{\sqrt{6}} = \dfrac{2}{3}\left(x - \dfrac{\sqrt{6}}{2} + 1\right)$

$$y - 1 + \frac{\sqrt{6}}{3} = \frac{2}{3}x - \frac{\sqrt{6}}{3} + \frac{2}{3}$$
$$3y - 2x - 5 + 2\sqrt{6} = 0$$

48. $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

(a)



(b) Secant line:

$$\text{slope} = \frac{f(\pi) - f(-\pi)}{\pi - (-\pi)} = \frac{\pi - (-\pi)}{2\pi} = 1$$

$$y - \pi = 1(x - \pi)$$

$$y = x$$

(c) $f'(x) = 1 - 2\cos x = 1$

$\cos x = 0$

$$c = \pm\frac{\pi}{2}, \qquad f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2$$

$$f\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2} + 2$$

Tangent lines: $y - \left(\frac{\pi}{2} - 2\right) = 1\left(x - \frac{\pi}{2}\right)$

$$y = x - 2$$

$$y - \left(-\frac{\pi}{2} + 2\right) = 1\left(x + \frac{\pi}{2}\right)$$

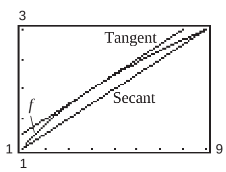
$$y = x + 2$$

49. $f(x) = \sqrt{x},\ [1, 9]$

$(1, 1), (9, 3)$

$$m = \frac{3 - 1}{9 - 1} = \frac{1}{4}$$

(a)



(b) Secant line: $y - 1 = \frac{1}{4}(x - 1)$

$$y = \frac{1}{4}x + \frac{3}{4}$$

$$0 = x - 4y + 3$$

(c) $$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\frac{f(9) - f(1)}{9 - 1} = \frac{1}{4}$$

$$\frac{1}{2\sqrt{c}} = \frac{1}{4}$$

$$\sqrt{c} = 2$$

$$c = 4$$

$$(c, f(c)) = (4, 2)$$

$$m = f'(4) = \frac{1}{4}$$

Tangent line: $y - 2 = \frac{1}{4}(x - 4)$

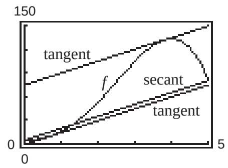
$$y = \frac{1}{4}x + 1$$

$$0 = x - 4y + 4$$

50. $f(x) = -x^4 + 4x^3 + 8x^2 + 5, \ (0, 5), (5, 80)$

$$m = \frac{80 - 5}{5 - 0} = 15$$

(a)



(b) Secant line: $y - 5 = 15(x - 0)$

$$0 = 15x - y + 5$$

$$f'(x) = -4x^3 + 12x^2 + 16x$$

$$\frac{f(5) - f(1)}{5 - 1} = 15$$

$$-4c^3 + 12c^2 + 16c = 15$$

$$0 = 4c^3 - 12c^2 - 16c + 15$$

$$c \approx 0.67 \text{ or } c \approx 3.79$$

(c) First tangent line: $y - f(c) = m(x - c)$

$$y - 9.59 = 15(x - 0.67)$$

$$0 = 15x - y - 0.46$$

Second tangent line: $y - f(c) = m(x - c)$

$$y - 131.35 = 15(x - 3.79)$$

$$0 = 15x - y + 74.5$$

51. $s(t) = -4.9t^2 + 500$

(a) $v_{\text{avg}} = \dfrac{s(3) - s(0)}{3 - 0} = \dfrac{455.9 - 500}{3} = -14.7 \text{ m/sec}$

(b) $s(t)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Therefore, the Mean Value Theorem applies.

$$v(t) = s'(t) = -9.8t = -14.7 \text{ m/sec}$$

$$t = \frac{-14.7}{-9.8} = 1.5 \text{ seconds}$$

52. $S(t) = 200\left(5 - \dfrac{9}{2 + t}\right)$

(a) $\dfrac{S(12) - S(0)}{12 - 0} = \dfrac{200[5 - (9/14)] - 200[5 - (9/2)]}{12}$

$$= \frac{450}{7}$$

(b) $S'(t) = 200\left(\dfrac{9}{(2 + t)^2}\right) = \dfrac{450}{7}$

$$\frac{1}{(2 + t)^2} = \frac{1}{28}$$

$$2 + t = 2\sqrt{7}$$

$$t = 2\sqrt{7} - 2 \approx 3.2915 \text{ months}$$

$S'(t)$ is equal to the average value in April.

53. No. Let $f(x) = x^2$ on $[-1, 2]$.

$$f'(x) = 2x$$

$f'(0) = 0$ and zero is in the interval $(-1, 2)$ but $f(-1) \neq f(2)$.

54. $f(a) = f(b)$ and $f'(c) = 0$ where c is in the interval (a, b).

(a) $g(x) = f(x) + k$

$g(a) = g(b) = f(a) + k$

$g'(x) = f'(x) \Longrightarrow g'(c) = 0$

Interval: $[a, b]$

Critical number of g: c

(b) $g(x) = f(x - k)$

$g(a + k) = g(b + k) = f(a)$

$g'(x) = f'(x - k)$

$g'(c + k) = f'(c) = 0$

Interval: $[a + k, b + k]$

Critical number of g: $c + k$

(c) $g(x) = f(kx)$

$g\left(\dfrac{a}{k}\right) = g\left(\dfrac{b}{k}\right) = f(a)$

$g'(x) = kf'(kx)$

$g'\left(\dfrac{c}{k}\right) = kf'(c) = 0$

Interval: $\left[\dfrac{a}{k}, \dfrac{b}{k}\right]$

Critical number of g: $\dfrac{c}{k}$

55. $f(x) = \begin{cases} 0, & x = 0 \\ 1 - x, & 0 < x \le 1 \end{cases}$

No, this does not contradict Rolle's Theorem. f is not continuous on $[0, 1]$.

56. No. If such a function existed, then the Mean Value Theorem would say that there exists $c \in (-2, 2)$ such that

$$f'(c) = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{6 + 2}{4} = 2.$$

But, $f'(x) < 1$ for all x.

57. Let $S(t)$ be the position function of the plane. If $t = 0$ corresponds to 2 P.M., $S(0) = 0$, $S(5.5) = 2500$ and the Mean Value Theorem says that there exists a time t_0, $0 < t_0 < 5.5$, such that

$$S'(t_0) = v(t_0) = \frac{2500 - 0}{5.5 - 0} \approx 454.54.$$

Applying the Intermediate Value Theorem to the velocity function on the intervals $[0, t_0]$ and $[t_0, 5.5]$, you see that there are at least two times during the flight when the speed was 400 miles per hour. $(0 < 400 < 454.54)$

58. Let $T(t)$ be the temperature of the object. Then $T(0) = 1500°$ and $T(5) = 390°$. The average temperature over the interval $[0, 5]$ is

$$\frac{390 - 1500}{5 - 0} = -222° \text{ F/hr}.$$

By the Mean Value Theorem, there exists a time to, $0 < t_0 < 5$, such that $T'(t_0) = -222°\text{F/hr}$.

59. Let $S(t)$ be the difference in the positions of the 2 bicyclists,

$$S(t) = S_1(t) - S_2(t).$$

Since $S(0) = S(2.25) = 0$, there must exist a time $t_0 \in (0, 2.25)$ such that

$$S'(t_0) = v(t_0) = 0.$$

At this time, $v_1(t_0) = v_2(t_0)$.

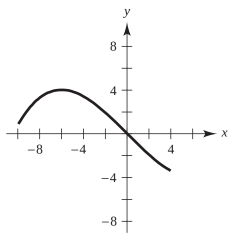
60. Let $t = 0$ correspond to 9:13 A.M. By the Mean Value Theorem, There exists t_0 in $\left(0, \frac{1}{30}\right)$ such that

$$v'(t_0) = a(t_0) = \frac{85 - 35}{1/30} = 1500 \text{ miles per hour}^2.$$

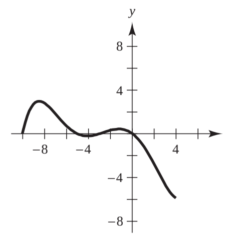
61. (a) f is continuous on $[-10, 4]$ and changes sign, $(f(-8) > 0,\ f(3) < 0)$. By the Intermediate Value Theorem, there exists at least one value of x in $[-10, 4]$ satisfying $f(x) = 0$.

(b) There exist real numbers a and b such that $-10 < a < b < 4$ and $f(a) = f(b) = 2$. Therefore, by Rolle's Theorem there exists at least one number c in $(-10, 4)$ such that $f'(c) = 0$. This is called a critical number.

(c)



(d)

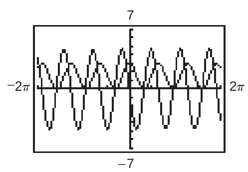


(e) No, f' did not have to be continuous on $[-10, 4]$.

62. $f(x) = 3\cos^2\left(\dfrac{\pi x}{2}\right), \quad f'(x) = 6\cos\left(\dfrac{\pi x}{2}\right)\left(-\sin\left(\dfrac{\pi x}{2}\right)\right)\left(\dfrac{\pi}{2}\right)$

$$= -3\pi\cos\left(\frac{\pi x}{2}\right)\sin\left(\frac{\pi x}{2}\right)$$

(a)



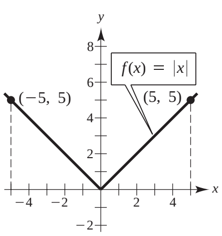
(b) f and f' are both continuous on the entire real line.

(c) Since $f(-1) = f(1) = 0$, Rolle's Theorem applies on $[-1, 1]$. Since $f(1) = 0$ and $f(2) = 3$, Rolle's Theorem does not apply on $[1, 2]$.

(d) $\lim\limits_{x \to 3^-} f'(x) = 0$

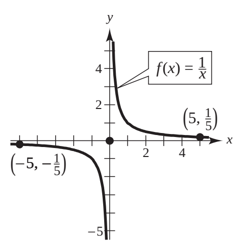
$\lim\limits_{x \to 3^+} f'(x) = 0$

63. f is continuous on $[-5, 5]$ and does not satisfy the conditions of the Mean Value Theorem. $\Longrightarrow f$ is not differentiable on $(-5, 5)$. Example: $f(x) = |x|$



64. f is not continuous on $[-5, 5]$.

Example: $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$



65. $f(x) = x^5 + x^3 + x + 1$

f is differentiable for all x.

$f(-1) = -2$ and $f(0) = 1$, so the Intermediate Value Theorem implies that f has at least one zero c in $[-1, 0]$, $f(c) = 0$.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$$f'(a) = f(c_2) - f(c_1) = 0.$$

But, $f'(x) = 5x^4 + 3x^2 + 1 > 0$ for all x. Hence, f has exactly one real zero.

66. $f(x) = 2x - 2 - \cos x$

$f(0) = -3$, $f(\pi) = 2\pi - 2 + 1 = 2\pi - 1 > 0$. By the Intermediate Value Theorem, f has at least one zero.

Suppose f had 2 zeros, $f(c_1) = f(c_2) = 0$. Then Rolle's Theorem would guarantee the existence of a number a such that

$$f'(a) = f(c_2) - f(c_1) = 0.$$

But, $f'(x) = 2 + \sin x \geq 1$ for all x. Hence, f has exactly one real zero.

67. f continuous at $x = 0$: $1 = b$
f continuous at $x = 1$: $a + 1 = 5 + c$
f differentiable at $x = 1$: $a = 2 + 4 = 6$Hence, $c = 2$.

$$f(x) = \begin{cases} 1, & x = 0 \\ 6x + 1, & 0 < x \leq 1 \\ x^2 + 4x + 2, & 1 < x \leq 3 \end{cases}$$

$$= \begin{cases} 6x + 1, & 0 \leq x \leq 1 \\ x^2 + 4x + 2, & 1 < x \leq 3 \end{cases}$$

68. f continuous at $x = -1$: $a = 2$
f continuous at $x = 0$: $2 = c$
f continuous at $x = 1$: $b + 2 = d + 4 \implies b = d + 2$
f differentiable at $x = 0$: $0 = 0$
f differentiable at $x = 1$: $2b = d$

Thus, $b = -2$ and $d = -4$.

69. $f'(x) = 0$

$f(x) = c$

$f(2) = 5$

Hence, $f(x) = 5$.

70. $f'(x) = 4$

$f(x) = 4x + c$

$f(0) = 1 \implies c = 1$

Hence, $f(x) = 4x + 1$.

71. $f'(x) = 2x$

$f(x) = x^2 + c$

$f(1) = 0 \implies 0 = 1 + c \implies c = -1$

Hence, $f(x) = x^2 - 1$.

72. $f'(x) = 2x + 3$

$f(x) = x^2 + 3x + c$

$f(1) = 0 \implies 0 = 1 + 3 + c \implies c = -4$

Hence, $f(x) = x^2 + 3x - 4$.

73. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

74. False. f must also be continuous *and* differentiable on each interval. Let

$$f(x) = \frac{x^3 - 4x}{x^2 - 1}.$$

75. True. A polynomial is continuous and differentiable everywhere.

76. True

77. Suppose that $p(x) = x^{2n+1} + ax + b$ has two real roots x_1 and x_2. Then by Rolle's Theorem, since $p(x_1) = p(x_2) = 0$, there exists c in (x_1, x_2) such that $p'(c) = 0$. But $p'(x) = (2n + 1)x^{2n} + a \neq 0$, since $n > 0, a > 0$. Therefore, $p(x)$ cannot have two real roots.

78. Suppose $f(x)$ is not constant on (a, b). Then there exists x_1 and x_2 in (a, b) such that $f(x_1) \neq f(x_2)$. Then by the Mean Value Theorem, there exists c in (a, b) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0.$$

This contradicts the fact that $f'(x) = 0$ for all x in (a, b).

79. If $p(x) = Ax^2 + Bx + C$, then

$$p'(x) = 2Ax + B = \frac{f(b) - f(a)}{b - a} = \frac{(Ab^2 + Bb + C) - (Aa^2 + Ba + C)}{b - a}$$

$$= \frac{A(b^2 - a^2) + B(b - a)}{b - a}$$

$$= \frac{(b - a)[A(b + a) + B]}{b - a}$$

$$= A(b + a) + B.$$

Thus, $2Ax = A(b + a)$ and $x = (b + a)/2$ which is the midpoint of $[a, b]$.

80. (a) $f(x) = x^2$, $g(x) = -x^3 + x^2 + 3x + 2$

$f(-1) = g(-1) = 1$, $f(2) = g(2) = 4$

Let $h(x) = f(x) - g(x)$. Then, $h(-1) = h(2) = 0$. Thus, by Rolle's Theorem these exists $c \in (-1, 2)$ such that

$$h'(c) = f'(c) - g'(c) = 0.$$

Thus, at $x = c$, the tangent line to f is parallel to the tangent line to g.

$$h(x) = x^3 - 3x - 2,\ h'(x) = 3x^2 - 3 = 0 \implies x = c = 1$$

(b) Let $h(x) = f(x) - g(x)$. Then $h(a) = h(b) = 0$ by Rolle's Theorem, there exists c in (a, b) such that

$$h'(c) = f'(c) - g'(c) = 0.$$

Thus, at $x = c$, the tangent line to f is parallel to the tangent line to g.

81. Suppose $f(x)$ has two fixed points c_1 and c_2. Then, by the Mean Value Theorem, there exists c such that

$$f'(c) = \frac{f(c_2) - f(c_1)}{c_2 - c_1} = \frac{c_2 - c_1}{c_2 - c_1} = 1.$$

This contradicts the fact that $f'(x) < 1$ for all x.

82. $f(x) = \frac{1}{2}\cos x$ differentiable on $(-\infty, \infty)$.

$$f'(x) = -\tfrac{1}{2}\sin x$$

$$-\tfrac{1}{2} \le f'(x) \le \tfrac{1}{2} \implies f'(x) < 1 \quad \text{for all real numbers.}$$

Thus, from Exercise 60, f has, at most, one fixed point. $(x \approx 0.4502)$

83. Let $f(x) = \cos x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\frac{\cos b - \cos a}{b - a} = -\sin c$$

$$\cos b - \cos a = (-\sin c)(b - a)$$

$$|\cos b - \cos a| = |-\sin c||b - a|$$

$$|\cos b - \cos a| \le |b - a| \text{ since } |-\sin c| \le 1.$$

84. Let $f(x) = \sin x$. f is continuous and differentiable for all real numbers. By the Mean Value Theorem, for any interval $[a, b]$, there exists c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\sin(b) - \sin(a) = (b - a)\cos(c)$$

$$|\sin(b) - \sin(a)| = |b - a|\,|\cos(c)|$$

$$|\sin a - \sin b| \le |a - b|.$$

85. Let $0 < a < b$. $f(x) = \sqrt{x}$ satisfies the hypotheses of the Mean Value Theorem on $[a, b]$. Hence, there exists c in (a, b) such that

$$f'(c) = \frac{1}{2\sqrt{c}} = \frac{f(b) - f(a)}{b - a} = \frac{\sqrt{b} - \sqrt{a}}{b - a}.$$

Thus $\sqrt{b} - \sqrt{a} = (b - a)\dfrac{1}{2\sqrt{c}} < \dfrac{b - a}{2\sqrt{a}}$.

Section 3.3 Increasing and Decreasing Functions and the First Derivative Test

1. (a) Increasing: $(0, 6)$ and $(8, 9)$. Largest: $(0, 6)$

(b) Decreasing: $(6, 8)$ and $(9, 10)$. Largest: $(6, 8)$

2. (a) Increasing: $(4, 5)$, $(6, 7)$. Largest: $(4, 5)$, $(6, 7)$

(b) Decreasing: $(-3, 1)$, $(1, 4)$, $(5, 6)$. Largest: $(-3, 1)$

3. $f(x) = x^2 - 6x + 8$

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

4. $y = -(x + 1)^2$

Increasing on: $(-\infty, -1)$

Decreasing on: $(-1, \infty)$

5. $y = \dfrac{x^3}{4} - 3x$

Increasing on: $(-\infty, -2)$, $(2, \infty)$

Decreasing on: $(-2, 2)$

6. $f(x) = x^4 - 2x^2$

Increasing on: $(-1, 0)$, $(1, \infty)$

Decreasing on: $(-\infty, -1)$, $(0, 1)$

7. $f(x) = \sin x + 2, \quad 0 < x < 2\pi$

$f'(x) = \cos x$

Critical numbers: $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

8. $h(x) = \cos\dfrac{x}{2}, \quad 0 < x < 2\pi$

$h'(x) = -\dfrac{1}{2}\sin\dfrac{x}{2}$

Critical numbers: none

$h'(x) < 0$ for $0 < x < 2\pi$

h decreasing on $0 < x < 2\pi$

9. $f(x) = \dfrac{1}{x^2} = x^{-2}$

$f'(x) = \dfrac{-2}{x^3}$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on $(-\infty, 0)$

Decreasing on $(0, \infty)$

10. $y = \dfrac{x^2}{x + 1}$

$y' = \dfrac{x(x + 2)}{(x + 1)^2}$

Critical numbers: $x = 0, -2$ Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -2$	$-2 < x < -1$	$-1 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on $(-\infty, -2), (0, \infty)$

Decreasing on $(-2, -1), (-1, 0)$

11. $g(x) = x^2 - 2x - 8$

$g'(x) = 2x - 2$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $g'(x)$:	$g' < 0$	$g' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

12. $h(x) = 27x - x^3$

$h'(x) = 27 - 3x^2 = 3(3 - x)(3 + x)$

$h'(x) = 0$

Critical numbers: $x = \pm 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
Sign of $h'(x)$:	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $(-3, 3)$

Decreasing on $(-\infty, -3), (3, \infty)$

13. $y = x\sqrt{16 - x^2}$ Domain: $[-4, 4]$

$$y' = \frac{-2(x^2 - 8)}{\sqrt{16 - x^2}} = \frac{-2}{\sqrt{16 - x^2}}\left(x - 2\sqrt{2}\right)\left(x + 2\sqrt{2}\right)$$

Critical numbers: $x = \pm 2\sqrt{2}$

Test intervals:	$-4 < x < -2\sqrt{2}$	$-2\sqrt{2} < x < 2\sqrt{2}$	$2\sqrt{2} < x < 4$
Sign of y':	$y' < 0$	$y' > 0$	$y' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on $\left(-2\sqrt{2}, 2\sqrt{2}\right)$

Decreasing on $\left(-4, -2\sqrt{2}\right), \left(2\sqrt{2}, 4\right)$

14. $y = x + \dfrac{4}{x}$

$$y' = \frac{(x - 2)(x + 2)}{x^2}$$

Critical numbers: $x = \pm 2$ Discontinuity: 0

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of y':	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing: $(-\infty, -2), (2, \infty)$

Decreasing: $(-2, 0), (0, 2)$

15. $y = x - 2\cos x, \quad 0 < x < 2\pi$

$y' = 1 + 2\sin x$

$y' = 0$: $\sin x = -\dfrac{1}{2}$

Critical numbers: $x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Test intervals:	$0 < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of y'	$y' > 0$	$y' < 0$	$y' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \dfrac{7\pi}{6}\right), \left(\dfrac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\dfrac{7\pi}{6}, \dfrac{11\pi}{6}\right)$

16. $f(x) = \cos^2 x - \cos x, \quad 0 < x < 2\pi$

$f'(x) = -2\cos x \sin x + \sin x = \sin x(1 - 2\cos x)$

$\sin x = 0 \implies x = \pi$

$1 - 2\cos x = 0 \implies \cos x = \frac{1}{2} \implies x = \frac{\pi}{3}, \frac{5\pi}{3}$

Critical numbers: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

Test intervals:	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \pi$	$\pi < x < \frac{5\pi}{3}$	$\frac{5\pi}{3} < x < 2\pi$
Sign of $f'(x)$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(\frac{\pi}{3}, \pi\right), \left(\frac{5\pi}{3}, 2\pi\right)$

Decreasing on: $\left(0, \frac{\pi}{3}\right), \left(\pi, \frac{5\pi}{3}\right)$

17. $f(x) = x^2 - 6x$

$f'(x) = 2x - 6 = 0$

Critical number: $x = 3$

Test intervals:	$-\infty < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(3, \infty)$

Decreasing on: $(-\infty, 3)$

Relative minimum: $(3, -9)$

18. $f(x) = x^2 + 8x + 10$

$f'(x) = 2x + 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-4, \infty)$

Decreasing on: $(-\infty, -4)$

Relative minimum: $(-4, -6)$

19. $f(x) = -2x^2 + 4x + 3$

$f'(x) = -4x + 4 = 0$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 1)$

Decreasing on: $(1, \infty)$

Relative maximum: $(1, 5)$

20. $f(x) = -(x^2 + 8x + 12)$

$f'(x) = -2x - 8 = 0$

Critical number: $x = -4$

Test intervals:	$-\infty < x < -4$	$-4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, -4)$

Decreasing on: $(-4, \infty)$

Relative maximum: $(-4, 4)$

21. $f(x) = 2x^3 + 3x^2 - 12x$

$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0$

Critical numbers: $x = -2, 1$

Test intervals:	$-\infty < x < -2$	$-2 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (1, \infty)$

Decreasing on: $(-2, 1)$

Relative maximum: $(-2, 20)$

Relative minimum: $(1, -7)$

22. $f(x) = x^3 - 6x^2 + 15$

$f'(x) = 3x^2 - 12x = 3x(x - 4)$

Critical numbers: $x = 0, 4$

Test intervals:	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on $(-\infty, 0), (4, \infty)$

Decreasing on $(0, 4)$

Relative maximum: $(0, 15)$

Relative minimum: $(4, -17)$

23. $f(x) = x^2(3 - x) = 3x^2 - x^3$

$f'(x) = 6x - 3x^2 = 3x(2 - x)$

Critical numbers: $x = 0, 2$

Test intervals:	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(0, 2)$

Decreasing on: $(-\infty, 0), (2, \infty)$

Relative maximum: $(2, 4)$

Relative minimum: $(0, 0)$

24. $f(x) = (x + 2)^2(x - 1)$

$f'(x) = 3x(x + 2)$

Critical numbers: $x = -2, 0$

Test intervals:	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -2), (0, \infty)$

Decreasing on: $(-2, 0)$

Relative maximum: $(-2, 0)$

Relative minimum: $(0, -4)$

25. $f(x) = \dfrac{x^5 - 5x}{5}$

$f'(x) = x^4 - 1$

Critical numbers: $x = -1, 1$

Test intervals:	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 1)$

Relative maximum: $\left(-1, \frac{4}{5}\right)$

Relative minimum: $\left(1, -\frac{4}{5}\right)$

26. $f(x) = x^4 - 32x + 4$

$f'(x) = 4x^3 - 32 = 4(x^3 - 8)$

Critical number: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(2, \infty)$

Decreasing on: $(-\infty, 2)$

Relative minimum: $(2, -44)$

27. $f(x) = x^{1/3} + 1$

$f'(x) = \dfrac{1}{3}x^{-2/3} = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

28. $f(x) = x^{2/3} - 4$

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

Critical number: $x = 0$

Test intervals:	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(0, \infty)$

Decreasing on: $(-\infty, 0)$

Relative minimum: $(0, -4)$

29. $f(x) = (x - 1)^{2/3}$

$$f'(x) = \frac{2}{3(x-1)^{1/3}}$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(1, \infty)$

Decreasing on: $(-\infty, 1)$

Relative minimum: $(1, 0)$

30. $f(x) = (x - 1)^{1/3}$

$$f'(x) = \frac{1}{3(x-1)^{2/3}}$$

Critical number: $x = 1$

Test intervals:	$-\infty < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, \infty)$

No relative extrema

31. $f(x) = 5 - |x - 5|$

$$f'(x) = -\frac{x-5}{|x-5|} = \begin{cases} 1, & x < 5 \\ -1, & x > 5 \end{cases}$$

Critical number: $x = 5$

Test intervals:	$-\infty < x < 5$	$5 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$
Conclusion:	Increasing	Decreasing

Increasing on: $(-\infty, 5)$

Decreasing on: $(5, \infty)$

Relative maximum: $(5, 5)$

32. $f(x) = |x + 3| - 1$

$$f'(x) = \frac{x+3}{|x+3|} = \begin{cases} 1, & x > -3 \\ -1, & x < -3 \end{cases}$$

Critical number: $x = -3$

Test intervals:	$-\infty < x < -3$	$-3 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing

Increasing on: $(-3, \infty)$

Decreasing on: $(-\infty, -3)$

Relative minimum: $(-3, -1)$

33. $f(x) = x + \dfrac{1}{x}$

$f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2}$

Critical numbers: $x = -1, 1$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -1), (1, \infty)$

Decreasing on: $(-1, 0), (0, 1)$

Relative maximum: $(-1, -2)$

Relative minimum: $(1, 2)$

34. $f(x) = \dfrac{x}{x + 1}$

$f'(x) = \dfrac{(x + 1)(1) - (x)(1)}{(x + 1)^2} = \dfrac{1}{(x + 1)^2}$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, -1), (-1, \infty)$

No relative extrema

35. $f(x) = \dfrac{x^2}{x^2 - 9}$

$f'(x) = \dfrac{(x^2 - 9)(2x) - (x^2)(2x)}{(x^2 - 9)^2} = \dfrac{-18x}{(x^2 - 9)^2}$

Critical number: $x = 0$

Discontinuities: $x = -3, 3$

Test intervals:	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < 3$	$3 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion:	Increasing	Increasing	Decreasing	Decreasing

Increasing on: $(-\infty, -3), (-3, 0)$

Decreasing on: $(0, 3), (3, \infty)$

Relative maximum: $(0, 0)$

36. $f(x) = \dfrac{x+3}{x^2} = \dfrac{1}{x} + \dfrac{3}{x^2}$

$f'(x) = -\dfrac{1}{x^2} - \dfrac{6}{x^3} = \dfrac{-(x+6)}{x^3}$

Critical number: $x = -6$

Discontinuity: $x = 0$

Test intervals:	$-\infty < x < -6$	$-6 < x < 0$	$0 < x < \infty$
Sign of $f'(x)$:	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion:	Decreasing	Increasing	Decreasing

Increasing on: $(-6, 0)$

Decreasing on: $(-\infty, -6), (0, \infty)$

Relative minimum: $\left(-6, -\dfrac{1}{12}\right)$

37. $f(x) = \dfrac{x^2 - 2x + 1}{x + 1}$

$f'(x) = \dfrac{(x+1)(2x-2) - (x^2 - 2x + 1)(1)}{(x+1)^2} = \dfrac{x^2 + 2x - 3}{(x+1)^2} = \dfrac{(x+3)(x-1)}{(x+1)^2}$

Critical numbers: $x = -3, 1$

Discontinuity: $x = -1$

Test intervals:	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Decreasing	Increasing

Increasing on: $(-\infty, -3), (1, \infty)$

Decreasing on: $(-3, -1), (-1, 1)$

Relative maximum: $(-3, -8)$

Relative minimum: $(1, 0)$

38. $f(x) = \dfrac{x^2 - 3x - 4}{x - 2}$

$f'(x) = \dfrac{(x-2)(2x-3) - (x^2 - 3x - 4)(1)}{(x-2)^2} = \dfrac{x^2 - 4x + 10}{(x-2)^2}$

Discontinuity: $x = 2$

Test intervals:	$-\infty < x < 2$	$2 < x < \infty$
Sign of $f'(x)$:	$f' > 0$	$f' > 0$
Conclusion:	Increasing	Increasing

Increasing on: $(-\infty, 2), (2, \infty)$

No relative extrema

39. (a) $f(x) = \frac{x}{2} + \cos x,\ 0 < x < 2\pi$

$f'(x) = \frac{1}{2} - \sin x = 0$

Critical numbers: $x = \frac{\pi}{6}, \frac{5\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{6}\right), \left(\frac{5\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$

(b) Relative maximum: $\left(\frac{\pi}{6}, \frac{\pi + 6\sqrt{3}}{12}\right)$

Relative minimum: $\left(\frac{5\pi}{6}, \frac{5\pi - 6\sqrt{3}}{12}\right)$

40. (a) $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x,\ 0 < x < 2\pi$

$f'(x) = \cos 2x = 0$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right), \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\frac{\pi}{4}, \frac{1}{2}\right), \left(\frac{5\pi}{4}, \frac{1}{2}\right)$

Relative minima: $\left(\frac{3\pi}{4}, -\frac{1}{2}\right), \left(\frac{7\pi}{4}, -\frac{1}{2}\right)$

41. (a) $f(x) = \sin x + \cos x, \qquad 0 < x < 2\pi$

$f'(x) = \cos x - \sin x = 0 \implies \sin x = \cos x$

Critical numbers: $x = \frac{\pi}{4}, \frac{5\pi}{4}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

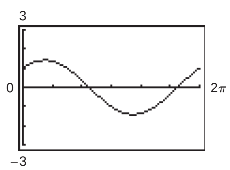
Increasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{5\pi}{4}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(b) Relative maximum: $\left(\frac{\pi}{4}, \sqrt{2}\right)$

Relative minimum: $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$

(c)



42. (a) $f(x) = x + 2\sin x, \qquad 0 < x < 2\pi$

$f'(x) = 1 + 2\cos x = 0 \implies \cos x = -\frac{1}{2}$

Critical numbers: $\frac{2\pi}{3}, \frac{4\pi}{3}$

Test intervals:	$0 < x < \frac{2\pi}{3}$	$\frac{2\pi}{3} < x < \frac{4\pi}{3}$	$\frac{4\pi}{3} < x < 2\pi$
Sign of $f'(x)$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

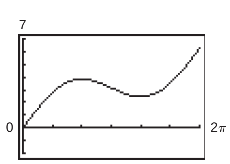
Increasing on: $\left(0, \frac{2\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Decreasing on: $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

(b) Relative maximum: $\left(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3}\right) \approx \left(\frac{2\pi}{3}, 3.826\right)$

Relative minimum: $\left(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3}\right) \approx \left(\frac{4\pi}{3}, 2.457\right)$

(c)



43. (a) $f(x) = \cos^2(2x), \quad 0 < x < 2\pi$

$f'(x) = -2\cos 2x \sin 2x = 0 \implies \cos 2x = 0$ or $\sin 2x = 0$

Critical numbers: $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \pi$
Sign of $f'(x)$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

Test intervals:	$\pi < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Decreasing	Increasing	Decreasing	Increasing

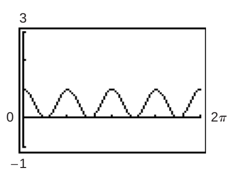
Increasing on: $\left(\frac{\pi}{4}, \frac{\pi}{2}\right), \left(\frac{3\pi}{4}, \pi\right), \left(\frac{5\pi}{4}, \frac{3\pi}{2}\right), \left(\frac{7\pi}{4}, 2\pi\right)$

Decreasing on: $\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{2}, \frac{3\pi}{4}\right), \left(\pi, \frac{5\pi}{4}\right), \left(\frac{3\pi}{2}, \frac{7\pi}{4}\right)$

(b) Relative maxima: $\left(\frac{\pi}{2}, 1\right), (\pi, 1), \left(\frac{3\pi}{2}, 1\right)$

Relative minima: $\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$

(c)



44. (a) $f(x) = \sqrt{3}\sin x + \cos x$

$f'(x) = \sqrt{3}\cos x - \sin x = 0 \implies \tan x = \sqrt{3}$

Critical numbers: $x = \frac{\pi}{3}, \frac{4\pi}{3}$

Test intervals:	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < x < \frac{4\pi}{3}$	$\frac{4\pi}{3} < x < 2\pi$
Sign of $f'(x)$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

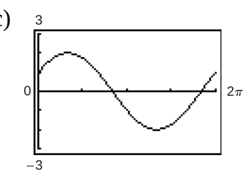
Increasing on: $\left(0, \frac{\pi}{3}\right), \left(\frac{4\pi}{3}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{3}, \frac{4\pi}{3}\right)$

(b) Relative maximum: $\left(\frac{\pi}{3}, 2\right)$

Relative minimum: $\left(\frac{4\pi}{3}, -2\right)$

(c)



45. (a) $f(x) = \sin^2 x + \sin x, \quad 0 < x < 2\pi$

$f'(x) = 2\sin x\cos x + \cos x = \cos x(2\sin x + 1) = 0$

Critical numbers: $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{7\pi}{6}$	$\frac{7\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < \frac{11\pi}{6}$	$\frac{11\pi}{6} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{7\pi}{6}, \frac{3\pi}{2}\right), \left(\frac{11\pi}{6}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{7\pi}{6}\right), \left(\frac{3\pi}{2}, \frac{11\pi}{6}\right)$

(b) Relative minima: $\left(\frac{7\pi}{6}, -\frac{1}{4}\right), \left(\frac{11\pi}{6}, -\frac{1}{4}\right)$

Relative maxima: $\left(\frac{\pi}{2}, 2\right), \left(\frac{3\pi}{2}, 0\right)$

46. (a) $f(x) = \dfrac{\sin x}{1 + \cos^2 x}, \; 0 < x < 2\pi$

$f'(x) = \dfrac{\cos x(2 + \sin^2 x)}{(1 + \cos^2 x)^2} = 0$

Critical numbers: $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f'(x)$:	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion:	Increasing	Decreasing	Increasing

Increasing on: $\left(0, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, 2\pi\right)$

Decreasing on: $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

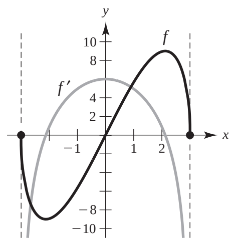
(b) Relative maximum: $\left(\frac{\pi}{2}, 1\right)$

Relative minimum: $\left(\frac{3\pi}{2}, -1\right)$

47. $f(x) = 2x\sqrt{9 - x^2}, [-3, 3]$

(a) $f'(x) = \dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}}$

(b)



(c) $\dfrac{2(9 - 2x^2)}{\sqrt{9 - x^2}} = 0$

Critical numbers: $x = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$

(d) Intervals:

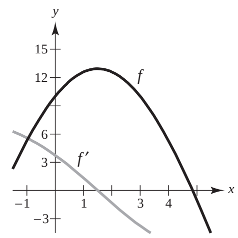
$\left(-3, -\dfrac{3\sqrt{2}}{2}\right)$	$\left(-\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$	$\left(\dfrac{3\sqrt{2}}{2}, 3\right)$
$f'(x) < 0$	$f'(x) > 0$	$f'(x) < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

48. $f(x) = 10\left(5 - \sqrt{x^2 - 3x + 16}\right), [0, 5]$

(a) $f'(x) = -\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}}$

(b)



(c) $-\dfrac{5(2x - 3)}{\sqrt{x^2 - 3x + 16}} = 0$

Critical number: $x = \dfrac{3}{2}$

(d) Intervals:

$\left(0, \dfrac{3}{2}\right)$	$\left(\dfrac{3}{2}, 5\right)$
$f'(x) > 0$	$f'(x) < 0$
Increasing	Decreasing

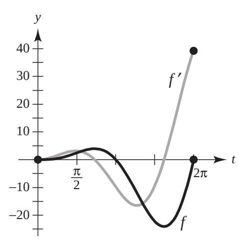
f is increasing when f' is positive and decreasing when f' is negative.

49. $f(t) = t^2 \sin t, [0, 2\pi]$

(a) $f'(t) = t^2 \cos t + 2t \sin t$

$= t(t \cos t + 2 \sin t)$

(b)



(c) $t(t \cos t + 2 \sin t) = 0$

$t = 0$ or $t = -2 \tan t$

$t \cot t = -2$

$t \approx 2.2889, 5.0870$ (graphing utility)

Critical numbers: $t = 2.2889, t = 5.0870$

(d) Intervals:

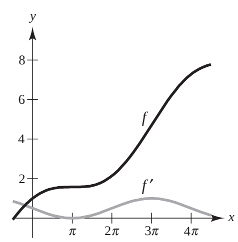
$(0, 2.2889)$	$(2.2889, 5.0870)$	$(5.0870, 2\pi)$
$f'(t) > 0$	$f'(t) < 0$	$f'(t) > 0$
Increasing	Decreasing	Increasing

f is increasing when f' is positive and decreasing when f' is negative.

50. $f(x) = \dfrac{x}{2} + \cos\dfrac{x}{2}, [0, 4\pi]$

(a) $f'(x) = \dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2}$

(b)



(c) $\dfrac{1}{2} - \dfrac{1}{2}\sin\dfrac{x}{2} = 0$

$\sin\dfrac{x}{2} = 1$

$\dfrac{x}{2} = \dfrac{\pi}{2}$

Critical number: $x = \pi$

(d) Intervals:

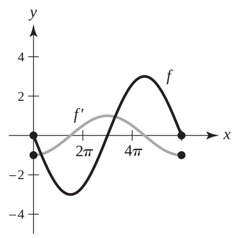
$(0, \pi)$	$(\pi, 4\pi)$
$f'(x) > 0$	$f'(x) > 0$
Increasing	Increasing

f is increasing when f' is positive.

51. (a) $f(x) = -3\sin\dfrac{x}{3},\ [0, 6\pi]$

$f'(x) = -\cos\dfrac{x}{3}$

(b)



(c) Critical numbers: $x = \dfrac{3\pi}{2}, \dfrac{9\pi}{2}$

(d) Intervals:

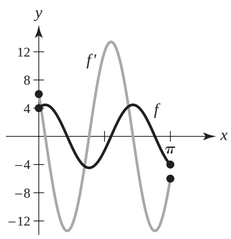
$\left(0, \frac{3\pi}{2}\right)$	$\left(\frac{3\pi}{2}, \frac{9\pi}{2}\right)$	$\left(\frac{9\pi}{2}, 6\pi\right)$
$f' < 0$	$f' > 0$	$f' < 0$
Decreasing	Increasing	Decreasing

f is increasing when f' is positive and decreasing when f' is negative.

52. (a) $f(x) = 2\sin 3x + 4\cos 3x,\ [0, \pi]$

$f'(x) = 6\cos 3x - 12\sin 3x$

(b)



(c) $f'(x) = 0 \implies \tan 3x = \dfrac{1}{2} \implies x \approx 0.1545, 1.2017, 2.2489$

(d) f' is positive on $(0, 0.1545), (1.2017, 2.2489)$

f' is negative on $(0.1545, 1.2017), (2.2489, \pi)$

f is increasing when f' is positive and decreasing when f' is negative.

53. $f(x) = \dfrac{x^5 - 4x^3 + 3x}{x^2 - 1} = \dfrac{(x^2 - 1)(x^3 - 3x)}{x^2 - 1} = x^3 - 3x,\ x \neq \pm 1$

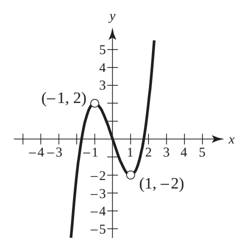
$f(x) = g(x) = x^3 - 3x$ for all $x \neq \pm 1$.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1),\ x \neq \pm 1 \implies f'(x) \neq 0$

f symmetric about origin

zeros of f: $(0, 0), (\pm\sqrt{3}, 0)$

No relative extrema



Holes at $(-1, 2)$ and $(1, -2)$

54. $f(t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = g(t),\ -2 < t < 2$

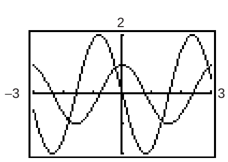
$f'(t) = -4\sin t\cos t = -2\sin 2t$

f symmetric with respect to y-axis

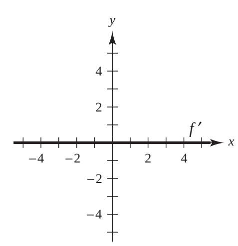
zeros of f: $\pm\dfrac{\pi}{4}$

Relative maximum: $(0, 1)$

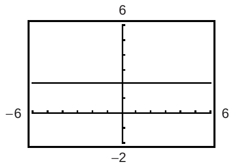
Relative minimum: $\left(-\dfrac{\pi}{2}, -1\right), \left(\dfrac{\pi}{2}, -1\right)$



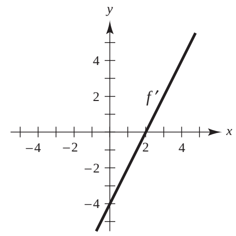
55. $f(x) = c$ is constant $\implies f'(x) = 0$.



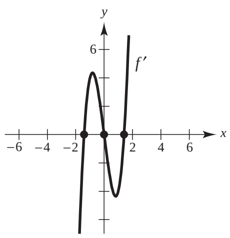
56. $f(x)$ is a line of slope $\approx 2 \Rightarrow f'(x) = 2$.



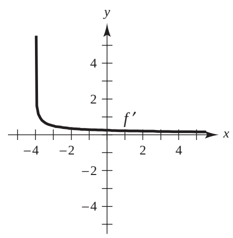
57. f is quadratic $\Rightarrow f'$ is a line.



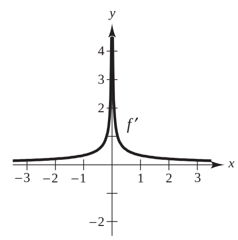
58. f is a 4th degree polynomial $\Rightarrow f'$ is a cubic polynomial.



59. f has positive, but decreasing slope



60. f has positive slope



61. (a) f increasing on $(2, \infty)$ because $f' > 0$ on $(2, \infty)$

f decreasing on $(-\infty, 2)$ because $f' < 0$ on $(-\infty, 2)$

(b) f has a relative minimum at $x = 2$.

62. (a) f increasing on $(-\infty, -1)$ because $f' > 0$ on $(-\infty, -1)$

f decreasing on $(-1, \infty)$ because $f' < 0$ on $(-1, \infty)$

(b) f has a relative maximum at $x = -1$.

63. (a) f increasing on $(-\infty, 0)$ and $(1, \infty)$ because $f' > 0$ there

f decreasing on $(0, 1)$ because $f' < 0$ there

(b) f has a relative maximum at $x = 0$, and a relative minimum at $x = 1$.

64. (a) f increasing on $(-\infty, -1)$ and $(0, 1)$ because $f' > 0$ there

f decreasing on $(-1, 0)$ and $(1, \infty)$ because $f' < 0$ there

(b) f has a relative maximum at $x = -1$ and $x = 1$.

f has a relative minimum at $x = 0$.

In Exercises 65–70, $f'(x) > 0$ on $(-\infty, -4)$, $f'(x) < 0$ on $(-4, 6)$ and $f'(x) > 0$ on $(6, \infty)$.

65. $g(x) = f(x) + 5$

$g'(x) = f'(x)$

$g'(0) = f'(0) < 0$

66. $g(x) = 3f(x) - 3$

$g'(x) = 3f'(x)$

$g'(-5) = 3f'(-5) > 0$

67. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(-6) = -f'(-6) < 0$

68. $g(x) = -f(x)$

$g'(x) = -f'(x)$

$g'(0) = -f'(0) > 0$

69. $g(x) = f(x - 10)$

$g'(x) = f'(x - 10)$

$g'(0) = f'(-10) > 0$

70. $g(x) = f(x - 10)$

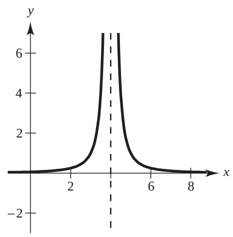
$g'(x) = f'(x - 10)$

$g'(8) = f'(-2) < 0$

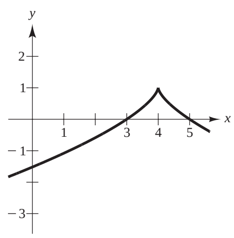
71. $f'(x)\begin{cases} > 0, & x < 4 \Rightarrow f \text{ is increasing on } (-\infty, 4). \\ \text{undefined}, & x = 4 \\ < 0, & x > 4 \Rightarrow f \text{ is decreasing on } (4, \infty). \end{cases}$

Two possibilities for $f(x)$ are given below.

(a)



(b)



72. Critical number: $x = 5$

$f'(4) = -2.5 \Rightarrow f$ is decreasing at $x = 4$.

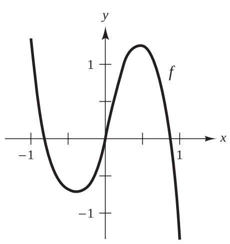
$f'(6) = 3 \Rightarrow f$ is increasing at $x = 6$.

$(5, f(5))$ is a relative minimum.

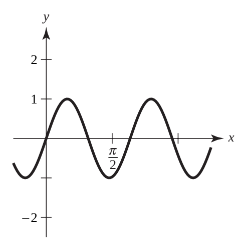
73. The critical numbers are in intervals $(-0.50, -0.25)$ and $(0.25, 0.50)$ since the sign of f' changes in these intervals. f is decreasing on approximately $(-1, -0.40)$, $(0.48, 1)$, and increasing on $(-0.40, 0.48)$.

Relative minimum when $x \approx -0.40$.

Relative maximum when $x \approx 0.48$.



74.



Answers will vary.

75. $s(t) = 4.9(\sin\theta)t^2$

(a) $s'(t) = 4.9(\sin\theta)(2t) = 9.8(\sin\theta)t$

speed $= |s'(t)| = |9.8(\sin\theta)t|$

(b)

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$\lvert s'(t)\rvert$	0	$4.9\sqrt{2}\,t$	$4.9\sqrt{3}\,t$	$9.8t$	$4.9\sqrt{3}\,t$	$4.9\sqrt{2}\,t$	0

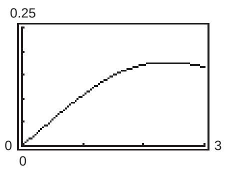
The speed is maximum for $\theta = \frac{\pi}{2}$.

76. $C = \dfrac{3t}{27 + t^3}, t \geq 0$

(a)

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration seems greater near $t = 2.5$ hours.

(b) 

The concentration is greatest when $t \approx 2.38$ hours.

(c) $C' = \dfrac{(27 + t^3)(3) - (3t)(3t^2)}{(27 + t^3)^2}$

$= \dfrac{3(27 - 2t^3)}{(27 + t^3)^2}$

$C' = 0$ when $t = 3/\sqrt[3]{2} \approx 2.38$ hours.

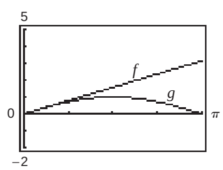
By the First Derivative Test, this is a maximum.

77. $f(x) = x, g(x) = \sin x, 0 < x < \pi$

(a)

x	0.5	1	1.5	2	2.5	3
$f(x)$	0.5	1	1.5	2	2.5	3
$g(x)$	0.479	0.841	0.997	0.909	0.598	0.141

$f(x)$ seems greater than $g(x)$ on $(0, \pi)$.

(b) 

$x > \sin x$ on $(0, \pi)$

(c) Let $h(x) = f(x) - g(x) = x - \sin x$

$h'(x) = 1 - \cos x > 0$ on $(0, \pi)$.

Therefore, $h(x)$ is increasing on $(0, \pi)$. Since $h(0) = 0$, $h(x) > 0$ on $(0, \pi)$. Thus,

$x - \sin x > 0$

$x > \sin x$

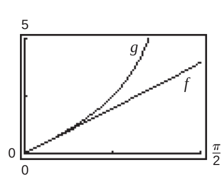
$f(x) > g(x)$ on $(0, \pi)$.

78. $f(x) = x, g(x) = \tan x$

(a)

x	0.25	0.5	0.75	1.0	1.25	1.5
$f(x)$	0.25	0.5	0.75	1.0	1.25	1.5
$g(x)$	0.2553	0.5463	0.9316	1.5574	3.0096	14.1014

On $\left(0, \dfrac{\pi}{2}\right)$, $\tan x > x$.

(b) 

On $\left(0, \dfrac{\pi}{2}\right)$, $\tan x$.

(c) Let $h(x) = \tan x - x$.

$h'(x) = \sec^2 x - 1 > 0$ on $\left(0, \dfrac{\pi}{2}\right)$

Because $h(0) = 0$ and $h'(x) > 0$ on $\left(0, \dfrac{\pi}{2}\right)$,

$h(x) > 0 \implies \tan x > x$ on $\left(0, \dfrac{\pi}{2}\right)$.

79. $v = k(R - r)r^2 = k(Rr^2 - r^3)$

$v' = k(2Rr - 3r^2)$

$= kr(2R - 3r) = 0$

$r = 0 \text{ or } \frac{2}{3}R$

Maximum when $r = \frac{2}{3}R$.

80. $P = 2.44x - \dfrac{x^2}{20{,}000} - 5000,\ 0 \le x \le 35{,}000$

$P' = 2.44 - \dfrac{x}{10{,}000} = 0$

$x = 24{,}400$

Test intervals:	$0 < x < 24{,}400$	$24{,}400 < x < 35{,}000$
Sign of P':	$P' > 0$	$P' < 0$

Increasing when $0 < x < 24{,}400$ hamburgers.

Decreasing when $24{,}400 < x < 35{,}000$ hamburgers.

81. $P = \dfrac{vR_1R_2}{(R_1 + R_2)^2}$, v and R_1 are constant

$$\frac{dP}{dR_2} = \frac{(R_1 + R_2)^2(vR_1) - vR_1R_2[2(R_1 + R_2)(1)]}{(R_1 + R_2)^4}$$

$$= \frac{vR_1(R_1 - R_2)}{(R_1 + R_2)^3} = 0 \Longrightarrow R_2 = R_1$$

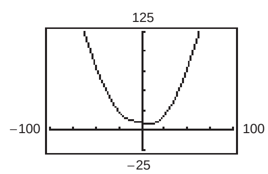
Maximum when $R_1 = R_2$.

82. $R = \sqrt{0.001T^4 - 4T + 100}$

(a) $R' = \dfrac{0.004T^3 - 4}{2\sqrt{0.001T^4 - 4T + 100}} = 0$

$T = 10°,\ R \approx 8.3666\Omega$

(b)

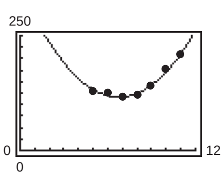


The minimum resistance is approximately $R \approx 8.37\Omega$ at $T = 10°$.

83. (a) Using a graphing utility, ($t = 5$ represents 1995)

$M = 5.267t^2 - 71.19t + 356.9.$

(b)



(c) $M'(t) = 10.534t - 71.19 = 0 \Longrightarrow t \approx 6.8$

By the First Derivative Test, $t \approx 6.8$ is a minimum.

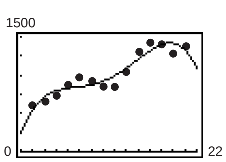
$M(6.8) \approx 116.3$

The minimum of the actual data is 115.6 at $t = 7$.

84. (a) Using a graphing utility, ($t = 8$ represents 1988)

$B = -0.33772t^4 + 19.2641t^3 - 398.081t^2 + 3585.41t - 11138.1.$

(b)



(c) Analytically,

$B' = -1.35088t^3 + 57.7923t^2 - 796.162t + 3585.41 = 0$ when $t \approx 19.7$

The maximum value of the data is 1429.5 at $t = 18$ (1998).

85. (a) $s(t) = 6t - t^2, t \geq 0$

$v(t) = 6 - 2t$

(b) $v(t) = 0$ when $t = 3$.

Moving in positive direction for $0 < t < 3$ because $v(t) > 0$ on $0 < t < 3$.

(c) Moving in negative direction when $t > 3$.

(d) The particle changes direction at $t = 3$.

86. (a) $s(t) = t^2 - 7t + 10, t \geq 0$

$v(t) = 2t - 7$

(b) $v(t) = 0$ when $t = \frac{7}{2}$

Particle moving in positive direction or $t > \frac{7}{2}$ because $v'(t) > 0$ on $\left(\frac{7}{2}, \infty\right)$.

(c) Particle moving in negative direction on $\left(0, \frac{7}{2}\right)$.

(d) The particle changes direction at $t = \frac{7}{2}$.

87. (a) $s(t) = t^3 - 5t^2 + 4t, t \geq 0$

$v(t) = 3t^2 - 10t + 4$

(b) $v(t) = 0$ for $t = \dfrac{10 \pm \sqrt{100 - 48}}{6} = \dfrac{5 \pm \sqrt{13}}{3}$

Particle is moving in a positive direction on $\left(0, \dfrac{5 - \sqrt{13}}{3}\right) \approx (0, 0.4648)$ and $\left(\dfrac{5 + \sqrt{13}}{3}, \infty\right) \approx (2.8685, \infty)$ because $v > 0$ on these intervals.

(c) Particle is moving in a negative direction on $\left(\dfrac{5 - \sqrt{13}}{3}, \dfrac{5 + \sqrt{13}}{3}\right) \approx (0.4648, 2.8685)$

(d) The particle changes direction at $t = \dfrac{5 \pm \sqrt{13}}{3}$.

88. (a) $s(t) = t^3 - 20t^2 + 128t - 280$

$v(t) = 3t^2 - 40t + 128$

(b) $v(t) = (3t - 16)(t - 8) = 0 \implies t = \frac{16}{3}, 8$

$v(t) > 0$ for $\left(0, \frac{16}{3}\right)$ and $(8, \infty)$

(c) $v(t) < 0$ for $\left(\frac{16}{3}, 8\right)$

(d) The particle changes direction at $t = \frac{16}{3}$ and 8.

89. Answers will vary.

90. Answers will vary.

91. (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

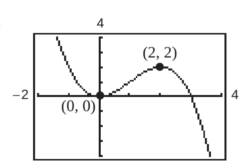
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$.

$(0, 0)$:	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
$(2, 2)$:	$2 = 8a_3 + 4a_2$	$(f(2) = 2)$
	$0 = 12a_3 + 4a_2$	$(f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0, a_2 = \dfrac{3}{2}, a_3 = -\dfrac{1}{2}$:

$$f(x) = -\frac{1}{2}x^3 + \frac{3}{2}x^2.$$

(d)



92. (a) Use a cubic polynomial $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

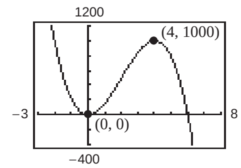
(b) $f'(x) = 3a_3x^2 + 2a_2x + a_1$

$(0, 0)$:	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
$(4, 1000)$:	$1000 = 64a_3 + 16a_2$	$(f(4) = 1000)$
	$0 = 48a_3 + 8a_2$	$(f'(4) = 0)$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = \dfrac{375}{2}$, $a_3 = \dfrac{-125}{4}$

$$f(x) = \frac{-125}{4}x^3 + \frac{375}{2}x^2.$$

(d)



93. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

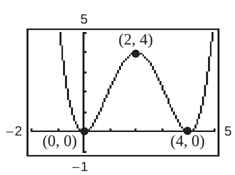
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(0, 0)$:	$0 = a_0$	$(f(0) = 0)$
	$0 = a_1$	$(f'(0) = 0)$
$(4, 0)$:	$0 = 256a_4 + 64a_3 + 16a_2$	$(f(4) = 0)$
	$0 = 256a_4 + 48a_3 + 8a_2$	$(f'(4) = 0)$
$(2, 4)$:	$4 = 16a_4 + 8a_3 + 4a_2$	$(f(2) = 4)$
	$0 = 32a_4 + 12a_3 + 4a_2$	$(f'(2) = 0)$

(c) The solution is $a_0 = a_1 = 0$, $a_2 = 4$, $a_3 = -2$, $a_4 = \dfrac{1}{4}$.

$$f(x) = \frac{1}{4}x^4 - 2x^3 + 4x^2$$

(d)



94. (a) Use a fourth degree polynomial $f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

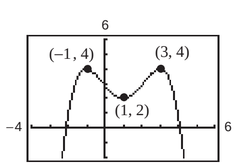
(b) $f'(x) = 4a_4x^3 + 3a_3x^2 + 2a_2x + a_1$

$(1, 2)$:	$2 = a_4 + a_3 + a_2 + a_1 + a_0$	$(f(1) = 2)$
	$0 = 4a_4 + 3a_3 + 2a_2 + a_1$	$(f'(1) = 0)$
$(-1, 4)$:	$4 = a_4 - a_3 + a_2 - a_1 + a_0$	$(f(-1) = 4)$
	$0 = -4a_4 + 3a_3 - 2a_2 + a_1$	$(f'(-1) = 0)$
$(3, 4)$:	$4 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$	$(f(3) = 4)$
	$0 = 108a_4 + 27a_3 + 6a_2 + a_1$	$(f'(3) = 0)$

(c) The solution is $a_0 = \frac{23}{8}$, $a_1 = -\frac{3}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{2}$, $a_4 = -\frac{1}{8}$

$$f(x) = -\tfrac{1}{8}x^4 + \tfrac{1}{2}x^3 + \tfrac{1}{4}x^2 - \tfrac{3}{2}x + \tfrac{23}{8}.$$

(d)



95. True.

Let $h(x) = f(x) + g(x)$ where f and g are increasing. Then $h'(x) = f'(x) + g'(x) > 0$ since $f'(x) > 0$ and $g'(x) > 0$.

96. False.

Let $h(x) = f(x)g(x)$ where $f(x) = g(x) = x$. Then $h(x) = x^2$ is decreasing on $(-\infty, 0)$.

97. False.

Let $f(x) = x^3$, then $f'(x) = 3x^2$ and f only has one critical number. Or, let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3(x^2 + 1)$ has no critical numbers.

98. True.

If $f(x)$ is an nth-degree polynomial, then the degree of $f'(x)$ is $n - 1$.

99. False. For example, $f(x) = x^3$ does not have a relative extrema at the critical number $x = 0$.

100. False.

The function might not be continuous.

101. Assume that $f'(x) < 0$ for all x in the interval (a, b) and let $x_1 < x_2$ be any two points in the interval. By the Mean Value Theorem, we know there exists a number c such that $x_1 < c < x_2$, and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$ and $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) < 0$, which implies that $f(x_2) < f(x_1)$. Thus, f is decreasing on the interval.

102. Suppose $f'(x)$ changes from positive to negative at c. Then there exists a and b in I such that $f'(x) > 0$ for all x in (a, c) and $f'(x) < 0$ for all x in (c, b). By Theorem 3.5, f is increasing on (a, c) and decreasing on (c, b). Therefore, $f(c)$ is a maximum of f on (a, b) and thus, a relative maximum of f.

103. Let $f(x) = (1 + x)^n - nx - 1$. Then

$$\begin{aligned} f'(x) &= n(1 + x)^{n-1} - n \\ &= n\left[(1 + x)^{n-1} - 1\right] > 0 \text{ since } x > 0 \text{ and } n > 1. \end{aligned}$$

Thus, $f(x)$ is increasing on $(0, \infty)$. Since $f(0) = 0 \Longrightarrow f(x) > 0$ on $(0, \infty)$

$$(1 + x)^n - nx - 1 > 0 \Longrightarrow (1 + x)^n > 1 + nx.$$

104. Let x_1 and x_2 be two real numbers, $x_1 < x_2$. Then $x_1^3 < x_2^3 \implies f(x_1) < f(x_2)$. Thus f is increasing on $(-\infty, \infty)$.

105. Let x_1 and x_2 be two positive real numbers, $0 < x_1 < x_2$. Then

$$\frac{1}{x_1} > \frac{1}{x_2}$$

$$f(x_1) > f(x_2)$$

Thus, f is decreasing on $(0, \infty)$.

Section 3.4 Concavity and the Second Derivative Test

1. $y = x^2 - x - 2, y'' = 2$

Concave upward: $(-\infty, \infty)$

2. $y = -x^3 + 3x^2 - 2, y'' = -6x + 6$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

3. $f(x) = \dfrac{24}{x^2 + 12}, y'' = \dfrac{-144(4 - x^2)}{(x^2 + 12)^3}$

Concave upward: $(-\infty, -2), (2, \infty)$

Concave downward: $(-2, 2)$

4. $f(x) = \dfrac{x^2 - 1}{2x + 1}, y'' = \dfrac{-6}{(2x + 1)^3}$

Concave upward: $\left(-\infty, -\frac{1}{2}\right)$

Concave downward: $\left(-\frac{1}{2}, \infty\right)$

5. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}, y'' = \dfrac{4(3x^2 + 1)}{(x^2 - 1)^3}$

Concave upward: $(-\infty, -1), (1, \infty)$

Concave downward: $(-1, 1)$

6. $y = \dfrac{1}{270}(-3x^5 + 40x^3 + 135x), y'' = \dfrac{-2}{9}x(x - 2)(x + 2)$

Concave upward: $(-\infty, -2), (0, 2)$

Concave downward: $(-2, 0), (2, \infty)$

7. $g(x) = 3x^2 - x^3$

$g'(x) = 6x - 3x^2$

$g''(x) = 6 - 6x$

Concave upward: $(-\infty, 1)$

Concave downward: $(1, \infty)$

8. $h(x) = x^5 - 5x + 2$

$h'(x) = 5x^4 - 5$

$h''(x) = 20x^3$

Concave upward: $(0, \infty)$

Concave downward: $(-\infty, 0)$

9. $y = 2x - \tan x, \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$y' = 2 - \sec^2 x$

$y'' = -2\sec^2 x \tan x$

Concave upward: $\left(-\dfrac{\pi}{2}, 0\right)$

Concave downward: $\left(0, \dfrac{\pi}{2}\right)$

10. $y = x + 2\csc x, \quad (-\pi, \pi)$

$y' = 1 - 2\csc x \cot x$

$y'' = -2\csc x(-\csc^2 x) - 2\cot x(-\csc x \cot x)$

$= 2(\csc^3 x + \csc x \cot^2 x)$

Concave upward: $(0, \pi)$

Concave downward: $(-\pi, 0)$

11. $f(x) = x^3 - 6x^2 + 12x$

$f'(x) = 3x^2 - 12x + 12$

$f''(x) = 6(x - 2) = 0$ when $x = 2$.

The concavity changes at $x = 2$. $(2, 8)$ is a point of inflection.

Concave upward: $(2, \infty)$

Concave downward: $(-\infty, 2)$

12. $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

$f''(x) = 12x - 6 = 0$ when $x = \frac{1}{2}$.

Test interval	$-\infty < x < \frac{1}{2}$	$\frac{1}{2} < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

Point of inflection: $\left(\frac{1}{2}, -\frac{13}{2}\right)$

13. $f(x) = \frac{1}{4}x^4 - 2x^2$

$f'(x) = x^3 - 4x$

$f''(x) = 3x^2 - 4$

$f''(x) = 3x^2 - 4 = 0$ when $x = \pm\frac{2}{\sqrt{3}}$.

Test interval:	$-\infty < x < -\frac{2}{\sqrt{3}}$	$-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$	$\frac{2}{\sqrt{3}} < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(\pm\frac{2}{\sqrt{3}}, -\frac{20}{9}\right)$

14. $f(x) = 2x^4 - 8x + 3$

$f'(x) = 8x^3 - 8$

$f''(x) = 24x^2 = 0$ when $x = 0$.

However, $(0, 3)$ is not a point of inflection since $f''(x) \geq 0$ for all x.

Concave upward on $(-\infty, \infty)$

15. $f(x) = x(x - 4)^3$

$f'(x) = x[3(x - 4)^2] + (x - 4)^3$

$= (x - 4)^2(4x - 4)$

$f''(x) = 4(x - 1)[2(x - 4)] + 4(x - 4)^2$

$= 4(x - 4)[2(x - 1) + (x - 4)]$

$= 4(x - 4)(3x - 6) = 12(x - 4)(x - 2)$

$f''(x) = 12(x - 4)(x - 2) = 0$ when $x = 2, 4$.

Test interval:	$-\infty < x < 2$	$2 < x < 4$	$4 < x < \infty$
Sign of $f''(x)$:	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion:	Concave upward	Concave downward	Concave upward

Points of inflection: $(2, -16)$, $(4, 0)$

16. $f(x) = x^3(x - 4)$

$f'(x) = x^3 + 3x^2(x - 4)$

$= x^2[x + 3(x - 4)] = 4x^2(x - 3)$

$f''(x) = 4x^2 + 8x(x - 3) = 4x[x + 2(x - 3)] = 12x(x - 2) = 0$

$f''(x) = 12x(x - 2) = 0$ when $x = 0, 2$.

Test interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
Sign of $f''(x)$	$f''(x) > 0$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave upward	Concave downward	Concave upward

Points of inflection: $(0, 0)$, $(2, -16)$

17. $f(x) = x\sqrt{x+3}$, Domain: $[-3, \infty)$

$$f'(x) = x\left(\frac{1}{2}\right)(x+3)^{-1/2} + \sqrt{x+3} = \frac{3(x+2)}{2\sqrt{x+3}}$$

$$f''(x) = \frac{6\sqrt{x+3} - 3(x+2)(x+3)^{-1/2}}{4(x+3)} = \frac{3(x+4)}{4(x+3)^{3/2}}$$

$f''(x) > 0$ on the entire domain of f (except for $x = -3$, for which $f''(x)$ is undefined). There are no points of inflection.

Concave upward on $(-3, \infty)$

18. $f(x) = x\sqrt{x+1}$, Domain: $[-1, \infty)$

$$f'(x) = (x)\frac{1}{2}(x+1)^{-1/2} + \sqrt{x+1} = \frac{3x+2}{2\sqrt{x+1}}$$

$$f''(x) = \frac{6\sqrt{x+1} - (3x+2)(x+1)^{-1/2}}{4(x+1)} = \frac{3x+4}{4(x+1)^{3/2}}$$

$f''(x) > 0$ on the entire domain of f (except for $x = -1$, for which $f''(x)$ is undefined).
There are no points of inflection.

Concave upward on $(-1, \infty)$

19. $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

Test intervals:	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < 0$	$0 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $\left(-\sqrt{3}, -\dfrac{\sqrt{3}}{4}\right), (0, 0), \left(\sqrt{3}, \dfrac{\sqrt{3}}{4}\right)$

20. $f(x) = \dfrac{x+1}{\sqrt{x}}$, Domain: $x > 0$

$$f'(x) = \frac{x-1}{2x^{3/2}}$$

$$f''(x) = \frac{3-x}{4x^{5/2}}$$

Point of inflection: $\left(3, \dfrac{4}{\sqrt{3}}\right) = \left(3, \dfrac{4\sqrt{3}}{3}\right)$

Test intervals	$0 < x < 3$	$3 < x < \infty$
Sign of $f''(x)$	$f'' > 0$	$f'' < 0$
Conclusion	Concave upward	Concave downward

21. $f(x) = \sin\dfrac{x}{2}, 0 \le x \le 4\pi$

$$f'(x) = \frac{1}{2}\cos\left(\frac{x}{2}\right)$$

$$f''(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right)$$

$f''(x) = 0$ when $x = 0, 2\pi, 4\pi$.

Point of inflection: $(2\pi, 0)$

Test interval:	$0 < x < 2\pi$	$2\pi < x < 4\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward

22. $f(x) = 2\csc\dfrac{3x}{2}, 0 < x < 2\pi$

$f'(x) = -3\csc\dfrac{3x}{2}\cot\dfrac{3x}{2}$

$f''(x) = \dfrac{9}{2}\left(\csc^3\dfrac{3x}{2} + \csc\dfrac{3x}{2}\cot^2\dfrac{3x}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $\left(0, \dfrac{2\pi}{3}\right), \left(\dfrac{4\pi}{3}, 2\pi\right)$

Concave downward: $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

No points of inflection

23. $f(x) = \sec\left(x - \dfrac{\pi}{2}\right), 0 < x < 4\pi$

$f'(x) = \sec\left(x - \dfrac{\pi}{2}\right)\tan\left(x - \dfrac{\pi}{2}\right)$

$f''(x) = \sec^3\left(x - \dfrac{\pi}{2}\right) + \sec\left(x - \dfrac{\pi}{2}\right)\tan^2\left(x - \dfrac{\pi}{2}\right) \neq 0$ for any x in the domain of f.

Concave upward: $(0, \pi), (2\pi, 3\pi)$

Concave downward: $(\pi, 2\pi), (3\pi, 4\pi)$

No points of inflection

24. $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

$f''(x) = \sin x - \cos x$

$f''(x) = 0$ when $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$.

Test interval:	$0 < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\dfrac{3\pi}{4}, 0\right), \left(\dfrac{7\pi}{4}, 0\right)$

25. $f(x) = 2\sin x + \sin 2x, 0 \le x \le 2\pi$

$f'(x) = 2\cos x + 2\cos 2x$

$f''(x) = -2\sin x - 4\sin 2x = -2\sin x(1 + 4\cos x)$

$f''(x) = 0$ when $x = 0, 1.823, \pi, 4.460$.

Test interval:	$0 < x < 1.823$	$1.823 < x < \pi$	$\pi < x < 4.460$	$4.460 < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$	$f'' > 0$
Conclusion:	Concave downward	Concave upward	Concave downward	Concave upward

Points of inflection: $(1.823, 1.452), (\pi, 0), (4.46, -1.452)$

26. $f(x) = x + 2\cos x,\ [0, 2\pi]$

$f'(x) = 1 - 2\sin x$

$f''(x) = -2\cos x$

$f''(x) = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Test intervals:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f'' < 0$	$f'' > 0$	$f'' < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

27. $f(x) = x^4 - 4x^3 + 2$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

$f''(x) = 12x^2 - 24x = 12x(x - 2)$

Critical numbers: $x = 0, x = 3$

However, $f''(0) = 0$, so we must use the First Derivative Test. $f'(x) < 0$ on the intervals $(-\infty, 0)$ and $(0, 3)$; hence, $(0, 2)$ is not an extremum. $f''(3) > 0$ so $(3, -25)$ is a relative minimum.

28. $f(x) = x^2 + 3x - 8$

$f'(x) = 2x + 3$

$f''(x) = 2$

Critical number: $x = -\frac{3}{2}$

$f''\left(-\frac{3}{2}\right) > 0$

Therefore, $\left(-\frac{3}{2}, -\frac{41}{4}\right)$ is a relative minimum.

29. $f(x) = (x - 5)^2$

$f'(x) = 2(x - 5)$

$f''(x) = 2$

Critical number: $x = 5$

$f''(5) > 0$

Therefore, $(5, 0)$ is a relative minimum.

30. $f(x) = -(x - 5)^2$

$f'(x) = -2(x - 5)$

$f''(x) = -2$

Critical number: $x = 5$

$f''(5) < 0$

Therefore, $(5, 0)$ is a relative maximum.

31. $f(x) = x^3 - 3x^2 + 3$

$f'(x) = 3x^2 - 6x = 3x(x - 2)$

$f''(x) = 6x - 6 = 6(x - 1)$

Critical numbers: $x = 0, x = 2$

$f''(0) = -6 < 0$

Therefore, $(0, 3)$ is a relative maximum.

$f''(2) = 6 > 0$

Therefore, $(2, -1)$ is a relative minimum.

32. $f(x) = x^3 - 9x^2 + 27x$

$f'(x) = 3x^2 - 18x + 27 = 3(x - 3)^2$

$f''(x) = 6(x - 3)$

Critical number: $x = 3$

However, $f''(3) = 0$, so we must use the First Derivative Test. $f'(x) \geq 0$ for all x and, therefore, there are no relative extrema.

33. $g(x) = x^2(6 - x)^3$

$g'(x) = x(x - 6)^2(12 - 5x)$

$g''(x) = 4(6 - x)(5x^2 - 24x + 18)$

Critical numbers: $x = 0, \frac{12}{5}, 6$

$g''(0) = 432 > 0$

Therefore, $(0, 0)$ is a relative minimum.

$g''\left(\frac{12}{5}\right) = -155.52 < 0$

Therefore, $\left(\frac{12}{5}, 268.7\right)$ is a relative maximum.

$g''(6) = 0$

Test fails. By the First Derivative Test, $(6, 0)$ is not an extremum.

34. $g(x) = -\frac{1}{8}(x + 2)^2(x - 4)^2$

$g'(x) = \dfrac{-(x - 4)(x - 1)(x + 2)}{2}$

$g''(x) = 3 + 3x - \frac{3}{2}x^2$

Critical numbers: $x = -2, 1, 4$

$g''(-2) = -9 < 0$

$(-2, 0)$ is a relative maximum.

$g''(1) = 9/2 > 0$

$(1, -10.125)$ is a relative minimum.

$g''(4) = -9 < 0$

$(4, 0)$ is a relative maximum.

35. $f(x) = x^{2/3} - 3$

$f'(x) = \dfrac{2}{3x^{1/3}}$

$f''(x) = \dfrac{-2}{9x^{4/3}}$

Critical number: $x = 0$

However, $f''(0)$ is undefined, so we must use the First Derivative Test. Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, $(0, -3)$ is a relative minimum.

36. $f(x) = \sqrt{x^2 + 1}$

$f'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

Critical number: $x = 0$

$f''(x) = \dfrac{1}{(x^2 + 1)^{3/2}}$

$f''(0) = 1 > 0$

Therefore, $(0, 1)$ is a relative minimum.

37. $f(x) = x + \dfrac{4}{x}$

$f'(x) = 1 - \dfrac{4}{x^2} = \dfrac{x^2 - 4}{x^2}$

$f''(x) = \dfrac{8}{x^3}$

Critical numbers: $x = \pm 2$

$f''(-2) < 0$

Therefore, $(-2, -4)$ is a relative maximum.

$f''(2) > 0$

Therefore, $(2, 4)$ is a relative minimum.

38. $f(x) = \dfrac{x}{x - 1}$

$f'(x) = \dfrac{-1}{(x - 1)^2}$

There are no critical numbers and $x = 1$ is not in the domain. There are no relative extrema.

39. $f(x) = \cos x - x, \ 0 \le x \le 4\pi$

$f'(x) = -\sin x - 1 \le 0$

Therefore, f is non-increasing and there are no relative extrema.

40. $f(x) = 2\sin x + \cos 2x,\ 0 \le x \le 2\pi$

$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x\cos x = 2\cos x(1 - 2\sin x) = 0$ when $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

$f''(x) = -2\sin x - 4\cos 2x$

$f''\left(\frac{\pi}{6}\right) < 0$

$f''\left(\frac{\pi}{2}\right) > 0$

$f''\left(\frac{5\pi}{6}\right) < 0$

$f''\left(\frac{3\pi}{2}\right) > 0$

Relative maxima: $\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$

Relative minima: $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$

41. $f(x) = 0.2x^2(x - 3)^3, [-1, 4]$

(a) $f'(x) = 0.2x(5x - 6)(x - 3)^2$

$f''(x) = (x - 3)(4x^2 - 9.6x + 3.6)$

$= 0.4(x - 3)(10x^2 - 24x + 9)$

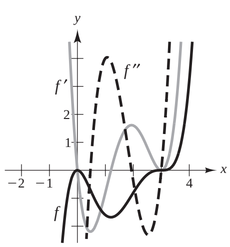
(b) $f''(0) < 0 \Longrightarrow (0, 0)$ is a relative maximum.

$f''\left(\frac{6}{5}\right) > 0 \Longrightarrow (1.2, -1.6796)$ is a relative minimum.

Points of inflection:

$(3, 0), (0.4652, -0.7049), (1.9348, -0.9049)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

42. $f(x) = x^2\sqrt{6 - x^2}, [-\sqrt{6}, \sqrt{6}]$

(a) $f'(x) = \dfrac{3x(4 - x^2)}{\sqrt{6 - x^2}}$

$f'(x) = 0$ when $x = 0, x = \pm 2$.

$f''(x) = \dfrac{6(x^4 - 9x^2 + 12)}{(6 - x^2)^{3/2}}$

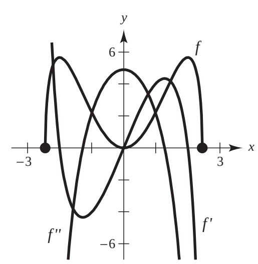
$f''(x) = 0$ when $x = \pm\sqrt{\dfrac{9 - \sqrt{33}}{2}}$.

(b) $f''(0) > 0 \Longrightarrow (0, 0)$ is a relative minimum.

$f''(\pm 2) < 0 \Longrightarrow \left(\pm 2, 4\sqrt{2}\right)$ are relative maxima.

Points of inflection: $(\pm 1.2758, 3.4035)$

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

43. $f(x) = \sin x - \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x, [0, \pi]$

(a) $f'(x) = \cos x - \cos 3x + \cos 5x$

$f'(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}$.

$f''(x) = -\sin x + 3\sin 3x - 5\sin 5x$

$f''(x) = 0$ when $x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x \approx 1.1731, x \approx 1.9685$

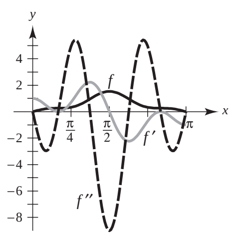
(b) $f''\left(\frac{\pi}{2}\right) < 0 \Rightarrow \left(\frac{\pi}{2}, 1.53333\right)$ is a relative maximum.

Points of inflection: $\left(\frac{\pi}{6}, 0.2667\right)$, $(1.1731, 0.9638)$,

$(1.9685, 0.9637)$, $\left(\frac{5\pi}{6}, 0.2667\right)$

Note: $(0, 0)$ and $(\pi, 0)$ are not points of inflection since they are endpoints.

(c)



The graph of f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

44. $f(x) = \sqrt{2x}\sin x, [0, 2\pi]$

(a) $f'(x) = \sqrt{2x}\cos x + \dfrac{\sin x}{\sqrt{2x}}$

Critical numbers: $x \approx 1.84, 4.82$

$$f''(x) = -\sqrt{2x}\sin x + \frac{\cos x}{\sqrt{2x}} + \frac{\cos x}{\sqrt{2x}} - \frac{\sin x}{2x\sqrt{2x}}$$

$$= \frac{2\cos x}{\sqrt{2x}} - \frac{(4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

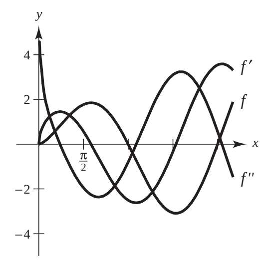
$$= \frac{4x\cos x - (4x^2 + 1)\sin x}{2x\sqrt{2x}}$$

(b) Relative maximum: $(1.84, 1.85)$

Relative minimum: $(4.82, -3.09)$

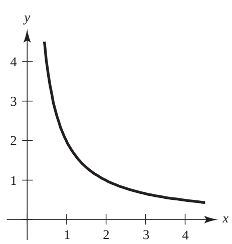
Points of inflection: $(0.75, 0.83)$, $(3.42, -0.72)$

(c)



f is increasing when $f' > 0$ and decreasing when $f' < 0$. f is concave upward when $f'' > 0$ and concave downward when $f'' < 0$.

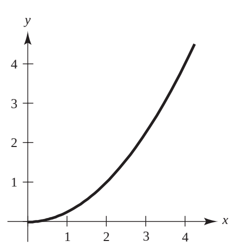
45. (a)



$f' < 0$ means f decreasing

f' increasing means concave upward

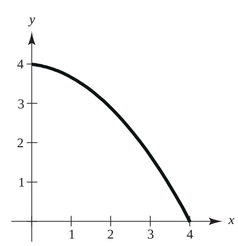
(b)



$f' > 0$ means f increasing

f' increasing means concave upward

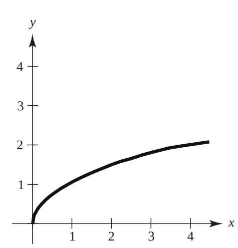
46. (a)



$f' < 0$ means f decreasing

f' decreasing means concave downward

(b)



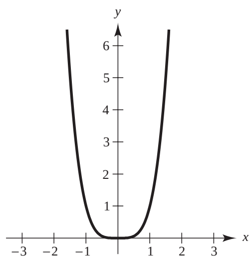
$f' > 0$ means f increasing

f' decreasing means concave downward

47. Let $f(x) = x^4$.

$f''(x) = 12x^2$

$f''(0) = 0$, but $(0, 0)$ is not a point of inflection.



48. (a) The rate of change of sales is increasing.
$S'' > 0$

(b) The rate of change of sales is decreasing.
$S' > 0, S'' < 0$

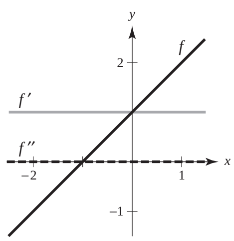
(c) The rate of change of sales is constant.
$S' = C, S'' = 0$

(d) Sales are steady.
$S = C, S' = 0, S'' = 0$

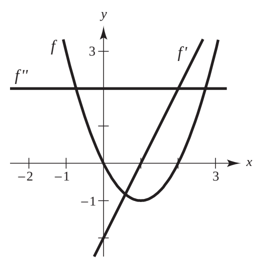
(e) Sales are declining, but at a lower rate.
$S' < 0, S'' > 0$

(f) Sales have bottomed out and have started to rise.
$S' > 0$

49.



50.



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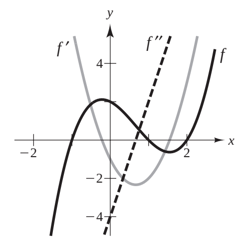
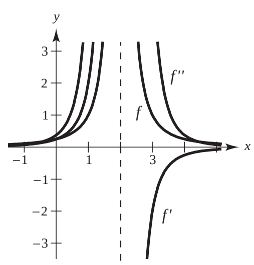
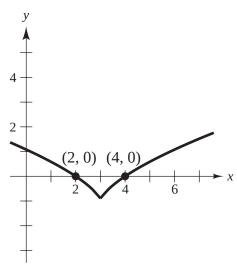


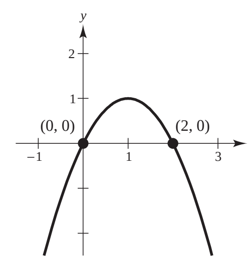
52.



53.



54.



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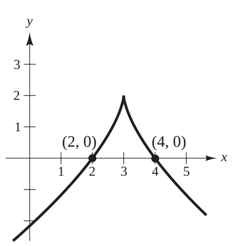
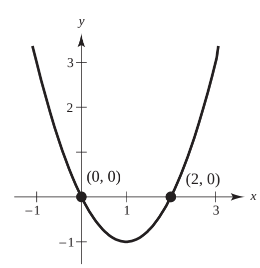
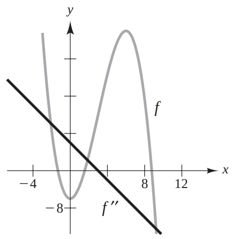


56.



57.



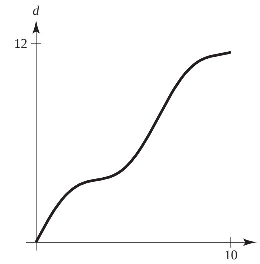
f'' is linear.

f' is quadratic.

f is cubic.

f concave upwards on $(-\infty, 3)$, downward on $(3, \infty)$.

58. (a)



(b) Since the depth d is always increasing, there are no relative extrema. $f'(x) > 0$

(c) The rate of change of d is decreasing until you reach the widest point of the jug, then the rate increases until you reach the narrowest part of the jug's neck, then the rate decreases until you reach the top of the jug.

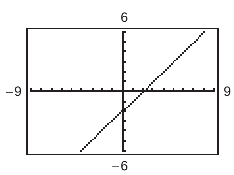
59. (a) $n = 1$:

$f(x) = x - 2$

$f'(x) = 1$

$f''(x) = 0$

No inflection points



$n = 2$:

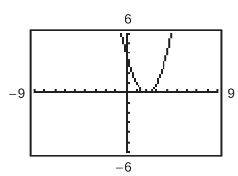
$f(x) = (x - 2)^2$

$f'(x) = 2(x - 2)$

$f''(x) = 2$

No inflection points

Relative minimum: $(2, 0)$



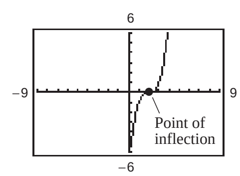
$n = 3$:

$f(x) = (x - 2)^3$

$f'(x) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

Inflection point: $(2, 0)$



$n = 4$:

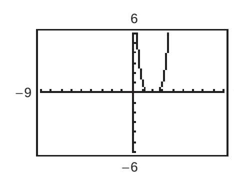
$f(x) = (x - 2)^4$

$f'(x) = 4(x - 2)^3$

$f''(x) = 12(x - 2)^2$

No inflection points:

Relative minimum: $(2, 0)$



Conclusion: If $n \ge 3$ and n is odd, then $(2, 0)$ is an inflection point. If $n \ge 2$ and n is even, then $(2, 0)$ is a relative minimum.

(b) Let $f(x) = (x - 2)^n$, $f'(x) = n(x - 2)^{n-1}$, $f''(x) = n(n - 1)(x - 2)^{n-2}$.

For $n \ge 3$ and odd, $n - 2$ is also odd and the concavity changes at $x = 2$.

For $n \ge 4$ and even, $n - 2$ is also even and the concavity does not change at $x = 2$.

Thus, $x = 2$ is an inflection point if and only if $n \ge 3$ is odd.

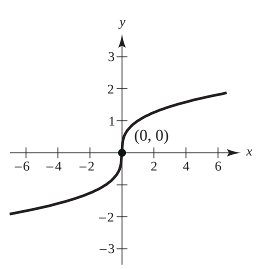
60. (a) $f(x) = \sqrt[3]{x}$

$f'(x) = \frac{1}{3}x^{-2/3}$

$f''(x) = -\frac{2}{9}x^{-5/3}$

Inflection point: $(0, 0)$

(b) $f''(x)$ does not exist at $x = 0$.



61. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(3, 3)$

Relative minimum: $(5, 1)$

Point of inflection: $(4, 2)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(3) &= 27a + 9b + 3c + d = 3 \\ f(5) &= 125a + 25b + 5c + d = 1 \end{aligned}\right\} 98a + 16b + 2c = -2 \Longrightarrow 49a + 8b + c = -1$$

$f'(3) = 27a + 6b + c = 0,\ f''(4) = 24a + 2b = 0$

$$\begin{array}{rlcrl} 49a + 8b + c &= -1 & \qquad & 24a + 2b &= 0 \\ \underline{27a + 6b + c} &\underline{= \ \ 0} & & \underline{22a + 2b} &\underline{= -1} \\ 22a + 2b &= -1 & & 2a &= 1 \end{array}$$

$a = \frac{1}{2}, b = -6, c = \frac{45}{2}, d = -24$

$f(x) = \frac{1}{2}x^3 - 6x^2 + \frac{45}{2}x - 24$

62. $f(x) = ax^3 + bx^2 + cx + d$

Relative maximum: $(2, 4)$

Relative minimum: $(4, 2)$

Point of inflection: $(3, 3)$

$f'(x) = 3ax^2 + 2bx + c, f''(x) = 6ax + 2b$

$$\left.\begin{aligned} f(2) &= 8a + 4b + 2c + d = 4 \\ f(4) &= 64a + 16b + 4c + d = 2 \end{aligned}\right\} 56a + 12b + 2c = -2 \Longrightarrow 28a + 6b + c = -1$$

$f'(2) = 12a + 4b + c = 0,\ f'(4) = 48a + 8b + c = 0,\ f''(3) = 18a + 2b = 0$

$$\begin{array}{rlcrl} 28a + 6b + c &= -1 & \qquad & 18a + 2b &= 0 \\ 12a + 4b + c &= \ \ 0 & & 16a + 2b &= -1 \\ 16a + 2b &= -1 & & 2a &= 1 \end{array}$$

$a = \frac{1}{2}, b = -\frac{9}{2}, c = 12, d = -6$

$f(x) = \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6$

63. $f(x) = ax^3 + bx^2 + cx + d$

Maximum: $(-4, 1)$

Minimum: $(0, 0)$

(a) $f'(x) = 3ax^2 + 2bx + c, \quad f''(x) = 6ax + 2b$

$$\begin{aligned} f(0) = 0 &\Longrightarrow d = 0 \\ f(-4) = 1 &\Longrightarrow -64a + 16b - 4c = 1 \\ f'(-4) = 0 &\Longrightarrow \quad 48a - 8b + c = 0 \\ f'(0) = 0 &\Longrightarrow \qquad\qquad\qquad c = 0 \end{aligned}$$

Solving this system yields $a = \frac{1}{32}$ and $b = 6a = \frac{3}{16}$.

$f(x) = \frac{1}{32}x^3 + \frac{3}{16}x^2$

(b) The plane would be descending at the greatest rate at the point of inflection.

$$f''(x) = 6ax + 2b = \frac{3}{16}x + \frac{3}{8} = 0 \implies x = -2.$$

Two miles from touchdown.

64. (a) line OA: $y = -0.06x$ slope: -0.06

line CB: $y = 0.04x + 50$ slope: 0.04

$f(x) = ax^3 + bx^2 + cx + d$

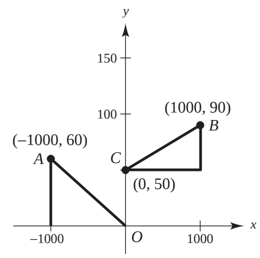
$f'(x) = 3ax^2 + 2bx + c$

$(-1000, 60)$: $60 = (-1000)^3 a + (1000)^2 b - 1000c + d$

$-0.06 = (1000)^2\, 3a - 2000b + c$

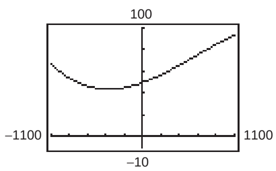
$(1000, 90)$: $90 = (1000)^3 a + (1000)^2 b + 1000c + d$

$0.04 = (1000)^2\, 3a + 2000b + c$

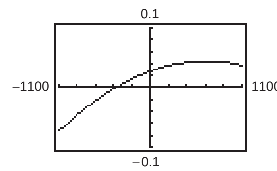


The solution to this system of 4 equations is $a = -1.25 \times 10^{-8}$, $b = 0.000025$, $c = 0.0275$, and $d = 50$.

(b) $y = -1.25 \times 10^{-8}x^3 + 0.000025x^2 + 0.0275x + 50$



(c)



(d) The steepest part of the road is 6% at the point A.

65. $D = 2x^4 - 5Lx^3 + 3L^2x^2$

$D' = 8x^3 - 15Lx^2 + 6L^2x = x(8x^2 - 15Lx + 6L^2) = 0$

$$x = 0 \text{ or } x = \frac{15L \pm \sqrt{33}L}{16} = \left(\frac{15 \pm \sqrt{33}}{16}\right)L$$

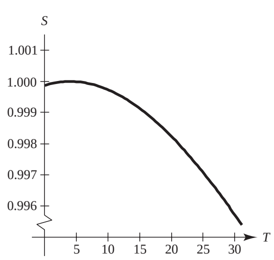
By the Second Derivative Test, the deflection is maximum when

$$x = \left(\frac{15 - \sqrt{33}}{16}\right)L \approx 0.578L.$$

66. $S = \dfrac{5.755T^3}{10^8} - \dfrac{8.521T^2}{10^6} + \dfrac{0.654T}{10^4} + 0.99987, \quad 0 < T < 25$

(a) The maximum occurs when $T \approx 4°$ and $S \approx 0.999999$.

(b)



(c) $S(20°) \approx 0.9982$

67. $C = 0.5x^2 + 15x + 5000$

$$\overline{C} = \frac{C}{x} = 0.5x + 15 + \frac{5000}{x}$$

$\overline{C}$ = average cost per unit

$$\frac{d\overline{C}}{dx} = 0.5 - \frac{5000}{x^2} = 0 \text{ when } x = 100$$

By the First Derivative Test, $\overline{C}$ is minimized when $x = 100$ units.

68. $C = 2x + \dfrac{300{,}000}{x}$

$$C' = 2 - \frac{300{,}000}{x^2} = 0 \text{ when } x = 100\sqrt{15} \approx 387$$

By the First Derivative Test, C is minimized when $x \approx 387$ units.

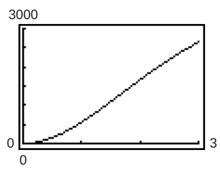
69. $S = \dfrac{5000t^2}{8 + t^2},\ 0 \le t \le 3$

(a)

t	0.5	1	1.5	2	2.5	3
S	151.5	555.6	1097.6	1666.7	2193.0	2647.1

Increasing at greatest rate when $t \approx 1.5$.

(b)



Increasing at greatest rate when $t \approx 1.5$.

(c) $S = \dfrac{5000t^2}{8 + t^2}$

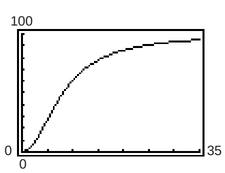
$$S'(t) = \frac{80{,}000t}{(8 + t^2)^2}$$

$$S''(t) = \frac{80{,}000(8 - 3t^2)}{(8 + t^2)^3}$$

$S''(t) = 0$ for $t = \pm\sqrt{\dfrac{8}{3}}$. Hence, $t = \dfrac{2\sqrt{6}}{3} \approx 1.633$ yrs.

70. $S = \dfrac{100t^2}{65 + t^2},\ t > 0$

(a)



(b) $S'(t) = \dfrac{13{,}000t}{(65 + t^2)^2}$

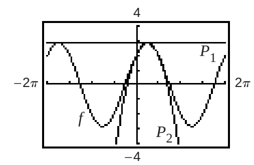
$$S''(t) = \frac{13{,}000(65 - 3t^2)}{(65 + t^2)^3} = 0 \implies t = 4.65$$

S is concave upwards on $(0, 4.65)$, concave downwards on $(4.65, 30)$.

(c) $S'(t) > 0$ for $t > 0$.

As t increases, the speed increases, but at a slower rate.

71. $f(x) = 2(\sin x + \cos x)$, $\quad f\left(\frac{\pi}{4}\right) = 2\sqrt{2}$



$f'(x) = 2(\cos x - \sin x)$, $\quad f'\left(\frac{\pi}{4}\right) = 0$

$f''(x) = 2(-\sin x - \cos x)$, $\quad f''\left(\frac{\pi}{4}\right) = -2\sqrt{2}$

$$P_1(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) = 2\sqrt{2}$$

$$P_1'(x) = 0$$

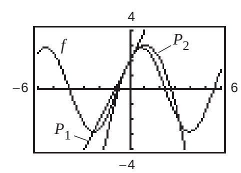
$$P_2(x) = 2\sqrt{2} + 0\left(x - \frac{\pi}{4}\right) + \frac{1}{2}(-2\sqrt{2})\left(x - \frac{\pi}{4}\right)^2 = 2\sqrt{2} - \sqrt{2}\left(x - \frac{\pi}{4}\right)^2$$

$$P_2'(x) = -2\sqrt{2}\left(x - \frac{\pi}{4}\right)$$

$$P_2''(x) = -2\sqrt{2}$$

The values of f, P_1, P_2, and their first derivatives are equal at $x = \pi/4$. The values of the second derivatives of f and P_2 are equal at $x = \pi/4$. The approximations worsen as you move away from $x = \pi/4$.

72. $f(x) = 2(\sin x + \cos x)$, $\quad f(0) = 2$



$f'(x) = 2(\cos x - \sin x)$, $\quad f'(0) = 2$

$f''(x) = 2(-\sin x - \cos x)$, $\quad f''(0) = -2$

$$P_1(x) = 2 + 2(x - 0) = 2(1 + x)$$

$$P_1'(x) = 2$$

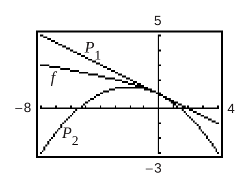
$$P_2(x) = 2 + 2(x - 0) + \tfrac{1}{2}(-2)(x - 0)^2 = 2 + 2x - x^2$$

$$P_2'(x) = 2 - 2x$$

$$P_2''(x) = -2$$

The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

73. $f(x) = \sqrt{1 - x}$, $\quad f(0) = 1$



$f'(x) = -\dfrac{1}{2\sqrt{1 - x}}$, $\quad f'(0) = -\dfrac{1}{2}$

$f''(x) = -\dfrac{1}{4(1 - x)^{3/2}}$, $\quad f''(0) = -\dfrac{1}{4}$

$$P_1(x) = 1 + \left(-\frac{1}{2}\right)(x - 0) = 1 - \frac{x}{2}$$

$$P_1'(x) = -\frac{1}{2}$$

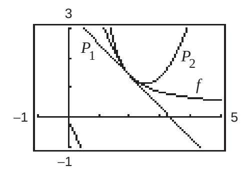
$$P_2(x) = 1 + \left(-\frac{1}{2}\right)(x - 0) + \frac{1}{2}\left(-\frac{1}{4}\right)(x - 0)^2 = 1 - \frac{x}{2} - \frac{x^2}{8}$$

$$P_2'(x) = -\frac{1}{2} - \frac{x}{4}$$

$$P_2''(x) = -\frac{1}{4}$$

The values of f, P_1, P_2, and their first derivatives are equal at $x = 0$. The values of the second derivatives of f and P_2 are equal at $x = 0$. The approximations worsen as you move away from $x = 0$.

74. $f(x) = \dfrac{\sqrt{x}}{x-1}$, $\quad f(2) = \sqrt{2}$



$f'(x) = \dfrac{-(x+1)}{2\sqrt{x}(x-1)^2}$, $\quad f'(2) = -\dfrac{3}{2\sqrt{2}} = -\dfrac{3\sqrt{2}}{4}$

$f''(x) = \dfrac{3x^2+6x-1}{4x^{3/2}(x-1)^3}$, $\quad f''(2) = \dfrac{23}{8\sqrt{2}} = \dfrac{23\sqrt{2}}{16}$

$P_1(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) = -\dfrac{3\sqrt{2}}{4}x + \dfrac{5\sqrt{2}}{2}$

$P_1'(x) = -\dfrac{3\sqrt{2}}{4}$

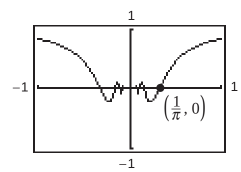
$P_2(x) = \sqrt{2} + \left(-\dfrac{3\sqrt{2}}{4}\right)(x-2) + \dfrac{1}{2}\left(\dfrac{23\sqrt{2}}{16}\right)(x-2)^2 = \sqrt{2} - \dfrac{3\sqrt{2}}{4}(x-2) + \dfrac{23\sqrt{2}}{32}(x-2)^2$

$P_2'(x) = -\dfrac{3\sqrt{2}}{4} + \dfrac{23\sqrt{2}}{16}(x-2)$

$P_2''(x) = \dfrac{23\sqrt{2}}{16}$

The values of f, P_1, P_2 and their first derivatives are equal at $x = 2$. The values of the second derivatives of f and P_2 are equal at $x = 2$. The approximations worsen as you move away from $x = 2$.

75. $f(x) = x\sin\left(\dfrac{1}{x}\right)$



$f'(x) = x\left[-\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)\right] + \sin\left(\dfrac{1}{x}\right) = -\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right) + \sin\left(\dfrac{1}{x}\right)$

$f''(x) = -\dfrac{1}{x}\left[\dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)\right] + \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) - \dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^3}\sin\left(\dfrac{1}{x}\right) = 0$

$x = \dfrac{1}{\pi}$

Point of inflection: $\left(\dfrac{1}{\pi}, 0\right)$

When $x > 1/\pi$, $f'' < 0$, so the graph is concave downward.

76. $f(x) = x(x-6)^2 = x^3 - 12x^2 + 36x$

$f'(x) = 3x^2 - 24x + 36 = 3(x-2)(x-6) = 0$

$f''(x) = 6x - 24 = 6(x-4) = 0$

Relative extrema: $(2, 32)$ and $(6, 0)$

Point of inflection $(4, 16)$ is midway between the relative extrema of f.

77. Assume the zeros of f are all real. Then express the function as $f(x) = a(x-r_1)(x-r_2)(x-r_3)$ where r_1, r_2, and r_3 are the distinct zeros of f. From the Product Rule for a function involving three factors, we have

$$f'(x) = a[(x-r_1)(x-r_2) + (x-r_1)(x-r_3) + (x-r_2)(x-r_3)]$$

$$f''(x) = a[(x-r_1) + (x-r_2) + (x-r_1) + (x-r_3) + (x-r_2) + (x-r_3)]$$
$$= a[6x - 2(r_1 + r_2 + r_3)].$$

Consequently, $f''(x) = 0$ if

$$x = \frac{2(r_1+r_2+r_3)}{6} = \frac{r_1+r_2+r_3}{3} = \text{(Average of } r_1, r_2, \text{ and } r_3).$$

78.
$$p(x) = ax^3 + bx^2 + cx + d$$
$$p'(x) = 3ax^2 + 2bx + c$$
$$p''(x) = 6ax + 2b$$
$$6ax + 2b = 0$$
$$x = -\frac{b}{3a}$$

The sign of $p''(x)$ changes at $x = -b/3a$. Therefore, $(-b/3a, p(-b/3a))$ is a point of inflection.

$$p\left(-\frac{b}{3a}\right) = a\left(-\frac{b^3}{27a^3}\right) + b\left(\frac{b^2}{9a^2}\right) + c\left(-\frac{b}{3a}\right) + d = \frac{2b^3}{27a^2} - \frac{bc}{3a} + d$$

When $p(x) = x^3 - 3x^2 + 2$, $a = 1$, $b = -3$, $c = 0$, and $d = 2$.

$$x_0 = \frac{-(-3)}{3(1)} = 1$$
$$y_0 = \frac{2(-3)^3}{27(1)^2} - \frac{(-3)(0)}{3(1)} + 2 = -2 - 0 + 2 = 0$$

The point of inflection of $p(x) = x^3 - 3x^2 + 2$ is $(x_0, y_0) = (1, 0)$.

79. True. Let $y = ax^3 + bx^2 + cx + d$, $a \neq 0$. Then $y'' = 6ax + 2b = 0$ when $x = -(b/3a)$, and the concavity changes at this point.

80. False. $f(x) = 1/x$ has a discontinuity at $x = 0$.

81. False. Concavity is determined by f''. For example, let $f(x) = x$ and $c = 2$. $f'(c) = f'(2) > 0$, but f is not concave upward at $c = 2$.

82. False. For example, let $f(x) = (x - 2)^4$.

83. f and g are concave upward on (a, b) implies that f' and g' are increasing on (a, b), and hence $f'' > 0$ and $g'' > 0$. Thus, $(f + g)'' > 0 \implies f + g$ is concave upward on (a, b) by Theorem 3.7.

84. f, g are positive, increasing, and concave upward on $(a, b) \implies f(x) > 0$, $f'(x) \geq 0$ and $f''(x) > 0$, and $g(x) > 0$, $g'(x) \geq 0$ and $g''(x) > 0$ on (a, b). For $x \in (a, b)$,

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$
$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x) > 0$$

Thus, fg is concave upward on (a, b).

Section 3.5 Limits at Infinity

1. $\lim_{x\to\infty} f(x) = 4$ means that $f(x)$ approaches 4 as x becomes large.

2. $\lim_{x\to-\infty} f(x) = 2$ means that $f(x)$ approaches 2 as x becomes very large (in absolute value) and negative.

3. $f(x) = \dfrac{3x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 3$

Matches (f)

4. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c)

5. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) < 1$

Matches (d)

6. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a)

7. $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) > 1$

Matches (b)

8. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

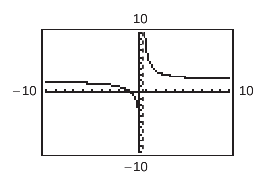
No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e)

9. $f(x) = \dfrac{4x + 3}{2x - 1}$

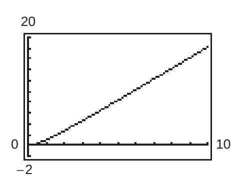
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2



$\lim_{x\to\infty} f(x) = 2$

10. $f(x) = \dfrac{2x^2}{x + 1}$

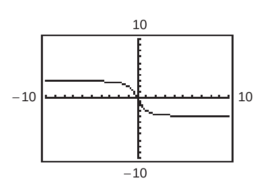
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998



$\lim_{x\to\infty} f(x) = \infty$ (Limit does not exist)

11. $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	-2	-2.98	-2.9998	-3	-3	-3	-3

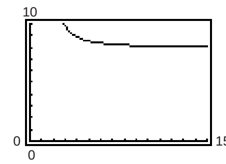


$\lim_{x\to\infty} f(x) = -3$

12. $f(x) = \dfrac{8x}{\sqrt{x^2 - 3}}$

x	10^1	10^2	10^3	10^4	10^5	10^6	10^7
$f(x)$	8.12	8.001	8	8	8	8	8

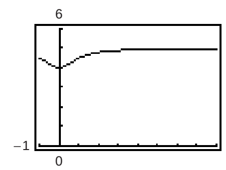
$\lim_{x\to\infty} f(x) = 8$



13. $f(x) = 5 - \dfrac{1}{x^2 + 1}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5

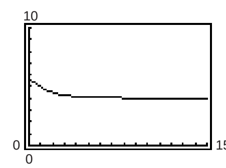
$\lim_{x\to\infty} f(x) = 5$



14. $f(x) = 4 + \dfrac{3}{x^2 + 2}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4.0	4.0	4	4

$\lim_{x\to\infty} f(x) = 4$



15. (a) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^3 - 3x^2 + 10}{x^2} = 5x - 3 + \dfrac{10}{x^2}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^3 - 3x^2 + 10}{x^3} = 5 - \dfrac{3}{x} + \dfrac{10}{x^3}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^4} = \dfrac{5x^3 - 3x^2 + 10}{x^4} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{10}{x^4}$

$\lim_{x\to\infty} h(x) = 0$

16. (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{5x^2 - 3x + 7}{x} = 5x - 3 + \dfrac{7}{x}$

$\lim_{x\to\infty} h(x) = \infty$ (Limit does not exist)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^2 - 3x + 7}{x^2} = 5 - \dfrac{3}{x} + \dfrac{7}{x^2}$

$\lim_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^2 - 3x + 7}{x^3} = \dfrac{5}{x} - \dfrac{3}{x^2} + \dfrac{7}{x^3}$

$\lim_{x\to\infty} h(x) = 0$

17. (a) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x^2 - 1} = 1$

(c) $\lim_{x\to\infty} \dfrac{x^2 + 2}{x - 1} = \infty$ (Limit does not exist)

18. (a) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x^3 - 1} = 0$

(b) $\lim_{x\to\infty} \dfrac{3 - 2x}{3x - 1} = -\dfrac{2}{3}$

(c) $\lim_{x\to\infty} \dfrac{3 - 2x^2}{3x - 1} = -\infty$ (Limit does not exist)

19. (a) $\lim_{x\to\infty} \frac{5 - 2x^{3/2}}{3x^2 - 4} = 0$

(b) $\lim_{x\to\infty} \frac{5 - 2x^{3/2}}{3x^{3/2} - 4} = -\frac{2}{3}$

(c) $\lim_{x\to\infty} \frac{5 - 2x^{3/2}}{3x - 4} = -\infty$ (Limit does not exist)

20. (a) $\lim_{x\to\infty} \frac{5x^{3/2}}{4x^2 + 1} = 0$

(b) $\lim_{x\to\infty} \frac{5x^{3/2}}{4x^{3/2} + 1} = \frac{5}{4}$

(c) $\lim_{x\to\infty} \frac{5x^{3/2}}{4\sqrt{x} + 1} = \infty$ (Limit does not exist)

21. $\lim_{x\to\infty} \frac{2x - 1}{3x + 2} = \lim_{x\to\infty} \frac{2 - (1/x)}{3 + (2/x)} = \frac{2 - 0}{3 + 0} = \frac{2}{3}$

22. $\lim_{x\to\infty} \frac{3x^3 + 2}{9x^3 - 2x^2 + 7} = \frac{3}{9} = \frac{1}{3}$

23. $\lim_{x\to\infty} \frac{x}{x^2 - 1} = \lim_{x\to\infty} \frac{1/x}{1 - (1/x^2)} = \frac{0}{1} = 0$

24. $\lim_{x\to\infty} \left(4 + \frac{3}{x}\right) = 4 + 0 = 4$

25. $\lim_{x\to-\infty} \frac{5x^2}{x + 3} = \lim_{x\to-\infty} \frac{5x}{1 + (3/x)} = -\infty$

Limit does not exist

26. $\lim_{x\to\infty} \left(\frac{1}{2}x - \frac{4}{x^2}\right) = -\infty$ (Limit does not exist)

27. $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2 - x}} = \lim_{x\to-\infty} \frac{1}{\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } x = -\sqrt{x^2}\right)$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1 - (1/x)}} = -1$$

28. $\lim_{x\to\infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to-\infty} \frac{1}{\left(\frac{\sqrt{x^2 + 1}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{x + (1/x)}} = -1$$

29. $\lim_{x\to-\infty} \frac{2x + 1}{\sqrt{x^2 - x}} = \lim_{x\to-\infty} \frac{2 + \frac{1}{x}}{\left(\frac{\sqrt{x^2 - x}}{-\sqrt{x^2}}\right)}$ $\left(\text{for } x < 0, x = -\sqrt{x^2}\right)$

$$= \lim_{x\to-\infty} \frac{-2 - \left(\frac{1}{x}\right)}{\sqrt{1 - \frac{1}{x}}} = -2$$

30. $\lim_{x\to-\infty} \frac{-3x + 1}{\sqrt{x^2 + x}} = \lim_{x\to-\infty} \frac{-3 + (1/x)}{\frac{\sqrt{x^2 + x}}{-\sqrt{x^2}}}$, $\left(\text{for } x < 0 \text{ we have } -\sqrt{x^2} = x\right)$

$$= \lim_{x\to-\infty} \frac{3 - (1/x)}{\sqrt{1 + (1/x)}} = 3$$

31. Since $(-1/x) \le (\sin 2x)/x \le (1/x)$ for all $x \ne 0$, we have by the Squeeze Theorem,

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0.$$

Therefore, $\lim_{x\to\infty} \frac{\sin 2x}{x} = 0.$

32. $$\lim_{x\to\infty} \frac{x - \cos x}{x} = \lim_{x\to\infty} \left(1 - \frac{\cos x}{x}\right)$$

$$= 1 - 0 = 1$$

Note:

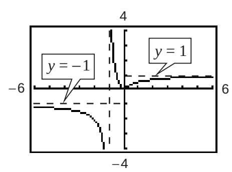
$\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by the Squeeze Theorem since

$$-\frac{1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}.$$

33. $$\lim_{x\to\infty} \frac{1}{2x + \sin x} = 0$$

34. $$\lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos 0 = 1$$

35. $f(x) = \frac{|x|}{x+1}$

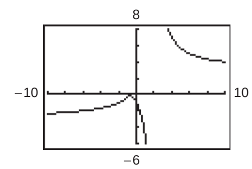


$$\lim_{x\to\infty} \frac{|x|}{x+1} = 1$$

$$\lim_{x\to-\infty} \frac{|x|}{x+1} = -1$$

Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.

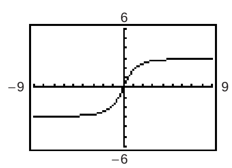
36. $f(x) = \frac{|3x + 2|}{x - 2}$



$y = 3$ is a horizontal asymptote (to the right).

$y = -3$ is a horizontal asymptote (to the left).

37. $f(x) = \frac{3x}{\sqrt{x^2 + 2}}$

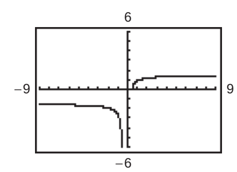


$$\lim_{x\to\infty} f(x) = 3$$

$$\lim_{x\to-\infty} f(x) = -3$$

Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.

38. $f(x) = \frac{\sqrt{9x^2 - 2}}{2x + 1}$



$y = \frac{3}{2}$ is a horizontal asymptote (to the right).

$y = -\frac{3}{2}$ is a horizontal asymptote (to the left).

39. $$\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1$$

(Let $x = 1/t$.)

40. $$\lim_{x\to\infty} x \tan\frac{1}{x} = \lim_{t\to 0^+} \frac{\tan t}{t} = \lim_{t\to 0^+} \left[\frac{\sin t}{t} \cdot \frac{1}{\cos t}\right]$$

$$= (1)(1) = 1$$

(Let $x = 1/t$.)

41. $$\lim_{x\to-\infty} \left(x + \sqrt{x^2 + 3}\right) = \lim_{x\to-\infty} \left[\left(x + \sqrt{x^2 + 3}\right) \cdot \frac{x - \sqrt{x^2 + 3}}{x - \sqrt{x^2 + 3}}\right] = \lim_{x\to-\infty} \frac{-3}{x - \sqrt{x^2 + 3}} = 0$$

42. $$\lim_{x\to\infty} \left(2x - \sqrt{4x^2 + 1}\right) = \lim_{x\to\infty} \left[\left(2x - \sqrt{4x^2 + 1}\right) \cdot \frac{2x + \sqrt{4x^2 + 1}}{2x + \sqrt{4x^2 + 1}}\right] = \lim_{x\to\infty} \frac{-1}{2x + \sqrt{4x^2 + 1}} = 0$$

43. $$\lim_{x\to\infty} \left(x - \sqrt{x^2 + x}\right) = \lim_{x\to\infty} \left[\left(x - \sqrt{x^2 + x}\right) \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}}\right]$$

$$= \lim_{x\to\infty} \frac{-x}{x + \sqrt{x^2 + x}} = \lim_{x\to\infty} \frac{-1}{1 + \sqrt{1 + (1/x)}} = -\frac{1}{2}$$

44. $$\lim_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right)=\lim_{x\to-\infty}\left[\left(3x+\sqrt{9x^2-x}\right)\cdot\frac{3x-\sqrt{9x^2-x}}{3x-\sqrt{9x^2-x}}\right]$$
$$=\lim_{x\to-\infty}\frac{x}{3x-\sqrt{9x^2-x}}$$
$$=\lim_{x\to-\infty}\frac{1}{3-\dfrac{\sqrt{9x^2-x}}{-\sqrt{x^2}}}\quad\left(\text{for } x<0 \text{ we have } x=-\sqrt{x^2}\right)$$
$$=\lim_{x\to-\infty}\frac{1}{3+\sqrt{9-(1/x)}}=\frac{1}{6}$$

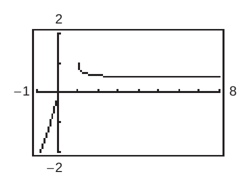
45. $$\lim_{x\to\infty}\left(4x-\sqrt{16x^2-x}\right)\frac{4x+\sqrt{16x^2-x}}{4x+\sqrt{16x^2-x}}=\lim_{x\to\infty}\frac{16x^2-(16x^2-x)}{4x+\sqrt{(16x^2-x)}}$$
$$=\lim_{x\to\infty}\frac{x}{4x+\sqrt{16x^2-x}}$$
$$=\lim_{x\to\infty}\frac{1}{4+\sqrt{16-1/x}}$$
$$=\frac{1}{4+4}=\frac{1}{8}$$

46. $$\lim_{x\to-\infty}\left(\frac{x}{2}+\sqrt{\frac{1}{4}x^2+x}\right)\frac{\left(\dfrac{x}{2}-\sqrt{\dfrac{1}{4}x^2+x}\right)}{\left(\dfrac{x}{2}-\sqrt{\dfrac{1}{4}x^2+x}\right)}=\lim_{x\to-\infty}\frac{\dfrac{x^2}{4}-\left(\dfrac{1}{4}x^2+x\right)}{\dfrac{x}{2}-\sqrt{\dfrac{1}{4}x^2+x}}$$
$$=\lim_{x\to-\infty}\frac{-x}{\dfrac{x}{2}-\sqrt{\dfrac{1}{4}x^2+x}}\cdot\frac{\left(\dfrac{-1}{x}\right)}{\left(\dfrac{-1}{x}\right)}$$
$$=\lim_{x\to-\infty}\frac{1}{-\dfrac{1}{2}-\sqrt{\dfrac{1}{4}-\dfrac{1}{x}}}$$
$$=\frac{1}{-\dfrac{1}{2}-\dfrac{1}{2}}=-1$$

Note: You must use $-1/x$ because $x<0$.

47.

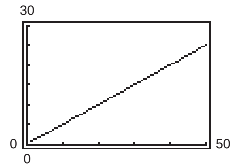
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500



$$\lim_{x\to\infty}\left(x-\sqrt{x(x-1)}\right)=\lim_{x\to\infty}\frac{x-\sqrt{x^2-x}}{1}\cdot\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}}$$
$$=\lim_{x\to\infty}\frac{x}{x+\sqrt{x^2-x}}$$
$$=\lim_{x\to\infty}\frac{1}{1+\sqrt{1-(1/x)}}$$
$$=\frac{1}{2}$$

48.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

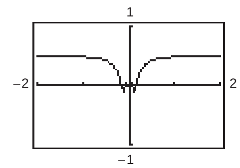


$$\lim_{x\to\infty} \frac{x^2 - x\sqrt{x^2 - x}}{1} \cdot \frac{x^2 + x\sqrt{x^2 - x}}{x^2 + x\sqrt{x^2 - x}} = \lim_{x\to\infty} \frac{x^3}{x^2 + x\sqrt{x^2 - x}} = \infty$$

Limit does not exist.

49.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500

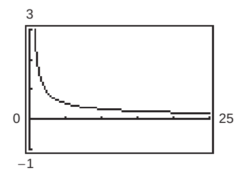


Let $x = 1/t$.

$$\lim_{x\to\infty} x \sin\left(\frac{1}{2x}\right) = \lim_{t\to 0^+} \frac{\sin(t/2)}{t} = \lim_{t\to 0^+} \frac{1}{2}\,\frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

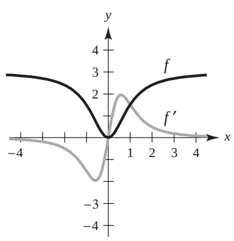
50.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001



$$\lim_{x\to\infty} \frac{x + 1}{x\sqrt{x}} = 0$$

51. (a)



(b) $\lim\limits_{x\to\infty} f(x) = 3$ $\quad \lim\limits_{x\to\infty} f'(x) = 0$

(c) Since $\lim\limits_{x\to\infty} f(x) = 3$, the graph approaches that of a horizontal line, $\lim\limits_{x\to\infty} f'(x) = 0$.

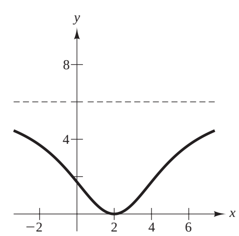
52. $x = 2$ is a critical number.

$f'(x) < 0$ for $x < 2$.

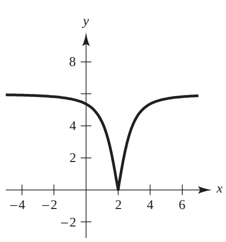
$f'(x) > 0$ for $x > 2$.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 6$

For example, let $f(x) = \dfrac{-6}{0.1(x - 2)^2 + 1} + 6$.



53. Yes. For example, let $f(x) = \dfrac{6|x - 2|}{\sqrt{(x - 2)^2 + 1}}$.



54. (a) The function is even: $\lim\limits_{x\to-\infty} f(x) = 5$

(b) The function is odd: $\lim\limits_{x\to-\infty} f(x) = -5$

55. $y = \dfrac{2 + x}{1 - x}$

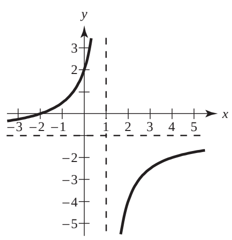
Intercepts: $(-2, 0), (0, 2)$

Symmetry: none

Horizontal asymptote: $y = -1$ since

$$\lim_{x \to -\infty} \frac{2 + x}{1 - x} = -1 = \lim_{x \to \infty} \frac{2 + x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



56. $y = \dfrac{x - 3}{x - 2}$

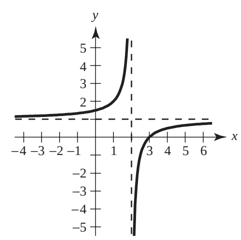
Intercepts: $(3, 0), \left(0, \dfrac{3}{2}\right)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x - 3}{x - 2} = 1 = \lim_{x \to \infty} \frac{x - 3}{x - 2}.$$

Discontinuity: $x = 2$ (Vertical asymptote)



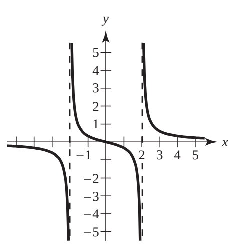
57. $y = \dfrac{x}{x^2 - 4}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 2$



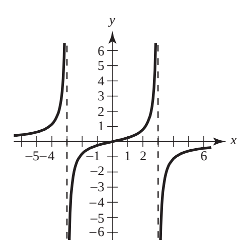
58. $y = \dfrac{2x}{9 - x^2}$

Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = \pm 3$



59. $y = \dfrac{x^2}{x^2 + 9}$

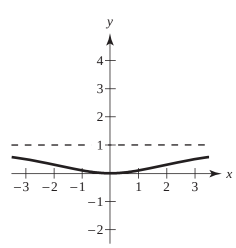
Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x^2}{x^2 + 9} = 1 = \lim_{x \to \infty} \frac{x^2}{x^2 + 9}.$$

Relative minimum: $(0, 0)$



60. $y = \dfrac{x^2}{x^2 - 9}$

Intercept: $(0, 0)$

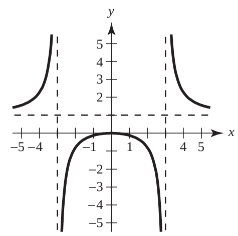
Symmetry: y-axis

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \frac{x^2}{x^2 - 9} = 1 = \lim_{x \to \infty} \frac{x^2}{x^2 - 9}.$$

Discontinuities: $x = \pm 3$ (Vertical asymptotes)

Relative maximum: $(0, 0)$



61. $y = \dfrac{2x^2}{x^2 - 4}$

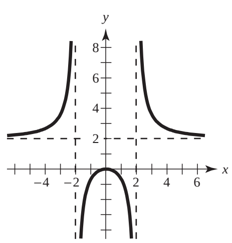
Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Vertical asymptotes: $x = \pm 2$

Relative maximum: $(0, 0)$



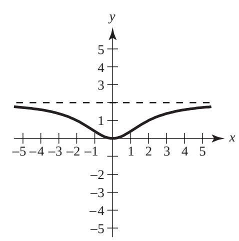
62. $y = \dfrac{2x^2}{x^2 + 4}$

Intercept: $(0, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$

Relative minimum: $(0, 0)$



63. $xy^2 = 4$

Domain: $x > 0$

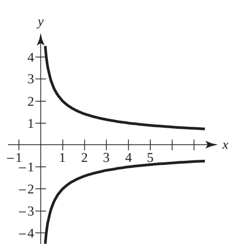
Intercepts: none

Symmetry: x-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0 = \lim_{x \to \infty} -\frac{2}{\sqrt{x}}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



64. $x^2y = 4$

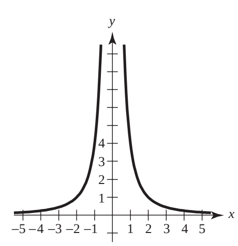
Intercepts: none

Symmetry: y-axis

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{4}{x^2} = 0 = \lim_{x \to \infty} \frac{4}{x^2}.$$

Discontinuity: $x = 0$ (Vertical asymptote)



65. $y = \dfrac{2x}{1 - x}$

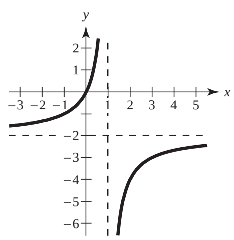
Intercept: $(0, 0)$

Symmetry: none

Horizontal asymptote: $y = -2$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x} = -2 = \lim_{x \to \infty} \frac{2x}{1 - x}.$$

Discontinuity: $x = 1$ (Vertical asymptote)



66. $y = \dfrac{2x}{1 - x^2}$

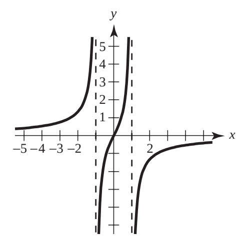
Intercept: $(0, 0)$

Symmetry: origin

Horizontal asymptote: $y = 0$ since

$$\lim_{x \to -\infty} \frac{2x}{1 - x^2} = 0 = \lim_{x \to \infty} \frac{2x}{1 - x^2}.$$

Discontinuities: $x = \pm 1$ (Vertical asymptotes)



67. $y = 2 - \dfrac{3}{x^2}$

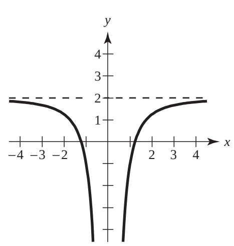
Intercepts: $\left(\pm\sqrt{3/2}, 0\right)$

Symmetry: y-axis

Horizontal asymptote: $y = 2$ since

$$\lim_{x \to -\infty} \left(2 - \frac{3}{x^2}\right) = 2 = \lim_{x \to \infty} \left(2 - \frac{3}{x^2}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



68. $y = 1 + \dfrac{1}{x}$

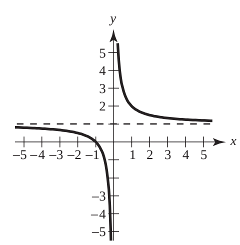
Intercept: $(-1, 0)$

Symmetry: none

Horizontal asymptote: $y = 1$ since

$$\lim_{x \to -\infty} \left(1 + \frac{1}{x}\right) = 1 = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right).$$

Discontinuity: $x = 0$ (Vertical asymptote)



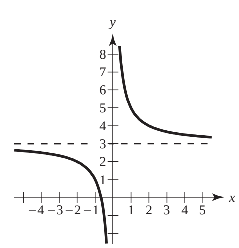
69. $y = 3 + \dfrac{2}{x}$

Intercept: $y = 0 = 3 + \dfrac{2}{x} \Longrightarrow \dfrac{2}{x} = -3 \Longrightarrow x = -\dfrac{2}{3};\ \left(-\dfrac{2}{3}, 0\right)$

Symmetry: none

Horizontal asymptote: $y = 3$

Vertical asymptote: $x = 0$



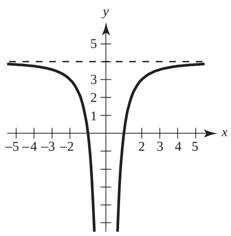
70. $y = 4\left(1 - \dfrac{1}{x^2}\right)$

Intercepts: $(\pm 1, 0)$

Symmetry: y-axis

Horizontal asymptote: $y = 4$

Vertical asymptote: $x = 0$



71. $y = \dfrac{x^3}{\sqrt{x^2 - 4}}$

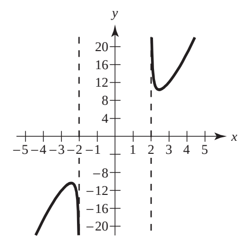
Domain: $(-\infty, -2), (2, \infty)$

Intercepts: none

Symmetry: origin

Horizontal asymptote: none

Vertical asymptotes: $x = \pm 2$ (discontinuities)



72. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

Domain: $(-\infty, -2), (2, \infty)$

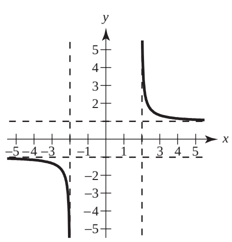
Intercepts: none

Symmetry: origin

Horizontal asymptotes: $y = \pm 1$ since

$$\lim_{x \to \infty} \frac{x}{\sqrt{x^2 - 4}} = 1, \quad \lim_{x \to -\infty} \frac{x}{\sqrt{x^2 - 4}} = -1.$$

Vertical asymptotes: $x = \pm 2$ (discontinuities)



73. $f(x) = 5 - \dfrac{1}{x^2} = \dfrac{5x^2 - 1}{x^2}$

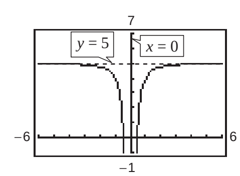
Domain: $(-\infty, 0), (0, \infty)$

$f'(x) = \dfrac{2}{x^3} \Longrightarrow$ No relative extrema

$f''(x) = -\dfrac{6}{x^4} \Longrightarrow$ No points of inflection

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 5$



74. $f(x) = \dfrac{x^2}{x^2 - 1}$

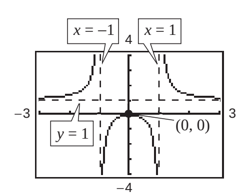
$$f'(x) = \frac{(x^2 - 1)(2x) - x^2(2x)}{(x^2 - 1)^2} = \frac{-2x}{(x^2 - 1)^2} = 0 \text{ when } x = 0.$$

$$f''(x) = \frac{(x^2 - 1)^2(-2) + 2x(2)(x^2 - 1)(2x)}{(x^2 - 1)^4} = \frac{2(3x^2 + 1)}{(x^2 - 1)^3}$$

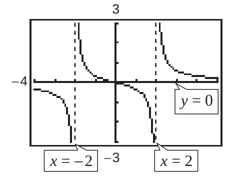
Since $f''(0) < 0$, then $(0, 0)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 1$



75. $f(x) = \dfrac{x}{x^2 - 4}$



$$f'(x) = \frac{(x^2 - 4) - x(2x)}{(x^2 - 4)^2}$$

$$= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \neq 0 \text{ for any } x \text{ in the domain of } f.$$

$$f''(x) = \frac{(x^2 - 4)^2(-2x) + (x^2 + 4)(2)(x^2 - 4)(2x)}{(x^2 - 4)^2}$$

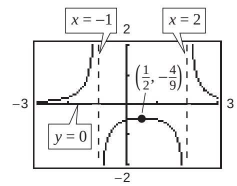
$$= \frac{2x(x^2 + 12)}{(x^2 - 4)^3} = 0 \text{ when } x = 0.$$

Since $f''(x) > 0$ on $(-2, 0)$ and $f''(x) < 0$ on $(0, 2)$, then $(0, 0)$ is a point of inflection.

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 0$

76. $f(x) = \dfrac{1}{x^2 - x - 2} = \dfrac{1}{(x + 1)(x - 2)}$



$$f'(x) = \frac{-(2x - 1)}{(x^2 - x - 2)^2} = 0 \text{ when } x = \frac{1}{2}.$$

$$f''(x) = \frac{(x^2 - x - 2)^2(-2) + (2x - 1)(2)(x^2 - x - 2)(2x - 1)}{(x^2 - x - 2)^4}$$

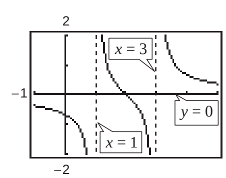
$$= \frac{6(x^2 - x + 1)}{(x^2 - x - 2)^3}$$

Since $f''\left(\frac{1}{2}\right) < 0$, then $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a relative maximum. Since $f''(x) \neq 0$, nor is it undefined in the domain of f, there are no points of inflection.

Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = 0$

77. $f(x) = \dfrac{x - 2}{x^2 - 4x + 3} = \dfrac{x - 2}{(x - 1)(x - 3)}$



$$f'(x) = \frac{(x^2 - 4x + 3) - (x - 2)(2x - 4)}{(x^2 - 4x + 3)^2} = \frac{-x^2 + 4x - 5}{(x^2 - 4x + 3)^2} \neq 0$$

$$f''(x) = \frac{(x^2 - 4x + 3)^2(-2x + 4) - (-x^2 + 4x - 5)(2)(x^2 - 4x + 3)(2x - 4)}{(x^2 - 4x + 3)^4}$$

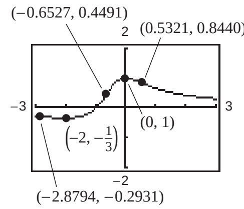
$$= \frac{2(x^3 - 6x^2 + 15x - 14)}{(x^2 - 4x + 3)^3} = \frac{2(x - 2)(x^2 - 4x + 7)}{(x^2 - 4x + 3)^3} = 0 \text{ when } x = 2.$$

Since $f''(x) > 0$ on $(1, 2)$ and $f''(x) < 0$ on $(2, 3)$, then $(2, 0)$ is a point of inflection.

Vertical asymptotes: $x = 1, x = 3$

Horizontal asymptote: $y = 0$

78. $f(x) = \dfrac{x+1}{x^2+x+1}$



$$f'(x) = \frac{-x(x+2)}{(x^2+x+1)^2} = 0 \text{ when } x = 0, -2.$$

$$f''(x) = \frac{2(x^3+3x^2-1)}{(x^2+x+1)^3} = 0 \text{ when } x \approx 0.5321, -0.6527, -2.8794.$$

$f''(0) < 0$

Therefore, $(0, 1)$ is a relative maximum.

$f''(-2) > 0$

Therefore,

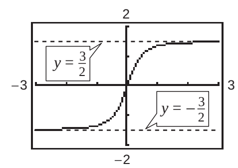
$$\left(-2, -\frac{1}{3}\right)$$

is a relative minimum.

Points of inflection: $(0.5321, 0.8440)$, $(-0.6527, 0.4491)$ and $(-2.8794, -0.2931)$

Horizontal asymptote: $y = 0$

79. $f(x) = \dfrac{3x}{\sqrt{4x^2+1}}$



$$f'(x) = \frac{3}{(4x^2+1)^{3/2}} \Longrightarrow \text{No relative extrema}$$

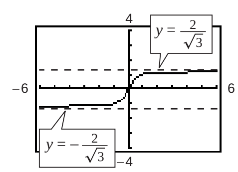
$$f''(x) = \frac{-36x}{(4x^2+1)^{5/2}} = 0 \text{ when } x = 0.$$

Point of inflection: $(0, 0)$

Horizontal asymptotes: $y = \pm\dfrac{3}{2}$

No vertical asymptotes

80. $g(x) = \dfrac{2x}{\sqrt{3x^2+1}}$



$$g'(x) = \frac{2}{(3x^2+1)^{3/2}}$$

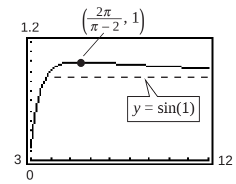
$$g''(x) = \frac{-18x}{(3x^2+1)^{5/2}}$$

No relative extrema. Point of inflection: $(0, 0)$.

Horizontal asymptotes: $y = \pm\dfrac{2}{\sqrt{3}}$

No vertical asymptotes

81. $g(x) = \sin\left(\dfrac{x}{x-2}\right)$, $3 < x < \infty$



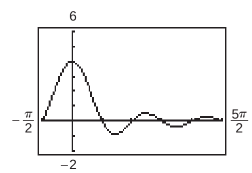
$$g'(x) = \frac{-2\cos\left(\dfrac{x}{x-2}\right)}{(x-2)^2}$$

Horizontal asymptote: $y = \sin(1)$

Relative maximum: $\dfrac{x}{x-2} = \dfrac{\pi}{2} \implies x = \dfrac{2\pi}{\pi - 2} \approx 5.5039$

No vertical asymptotes

82. $f(x) = \dfrac{2\sin 2x}{x}$; Hole at $(0, 4)$



$$f'(x) = \frac{4x\cos 2x - 2\sin 2x}{x^2}$$

There are an infinite number of relative extrema. In the interval $(-2\pi, 2\pi)$, you obtain the following.

Relative minima: $(\pm 2.25, -0.869)$, $(\pm 5.45, -0.365)$

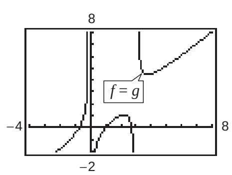
Relative maxima: $(\pm 3.87, 0.513)$

Horizontal asymptote: $y = 0$

No vertical asymptotes

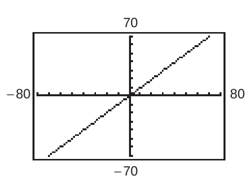
83. $f(x) = \dfrac{x^3 - 3x^2 + 2}{x(x-3)}$, $g(x) = x + \dfrac{2}{x(x-3)}$

(a)



(b) $$f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)} = \frac{x^2(x-3)}{x(x-3)} + \frac{2}{x(x-3)} = x + \frac{2}{x(x-3)} = g(x)$$

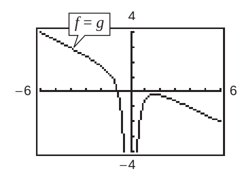
(c)



The graph appears as the slant asymptote $y = x$.

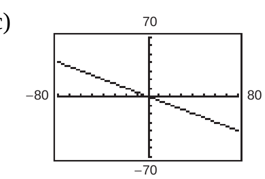
84. $f(x) = -\dfrac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\dfrac{1}{2}x + 1 - \dfrac{1}{x^2}$

(a)



(b) $$f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2} = -\left[\frac{x^3}{2x^2} - \frac{2x^2}{2x^2} + \frac{2}{2x^2}\right] = -\frac{1}{2}x + 1 - \frac{1}{x^2} = g(x)$$

(c)



The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

85. $C = 0.5x + 500$

$$\overline{C} = \frac{C}{x}$$

$$\overline{C} = 0.5 + \frac{500}{x}$$

$$\lim_{x \to \infty} \left(0.5 + \frac{500}{x}\right) = 0.5$$

86. $\displaystyle\lim_{v_1/v_2 \to \infty} 100\left[1 - \frac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

87. $\displaystyle\lim_{t \to \infty} N(t) = \infty$

$\displaystyle\lim_{t \to \infty} E(t) = c$

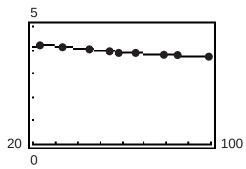
88. (a) $\displaystyle\lim_{t \to 0^+} T = 425°$

This is the temperature of the oven.

(b) $\displaystyle\lim_{t \to \infty} T = 72°$, the temperature of the room.

89. $y = \dfrac{3.351t^2 + 42.461t - 543.730}{t^2}$

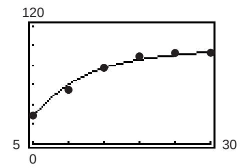
(a)



(b) Yes. $\displaystyle\lim_{t \to \infty} y = 3.351$

90. $S = \dfrac{100t^2}{65 + t^2}$, $t > 0$

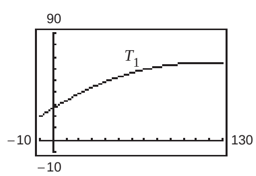
(a)



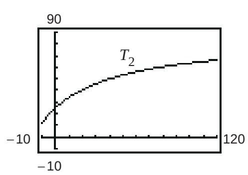
(b) Yes. $\displaystyle\lim_{t \to \infty} S = \frac{100}{1} = 100$

91. (a) $T_1(t) = -0.003t^2 + 0.677t + 26.564$

(b)



(c)



$$T_2 = \frac{1451 + 86t}{58 + t}$$

(d) $T_1(0) \approx 26.6$

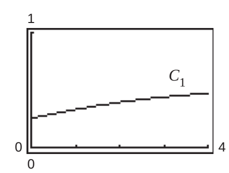
$T_2(0) \approx 25.0$

(e) $\displaystyle\lim_{t \to \infty} T_2 = \frac{86}{1} = 86$

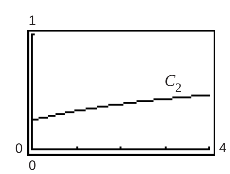
(f) No. The limiting temperature is 86.
T_1 has no horizontal asymptote.

92. (a) Using a graphing utility,
$C_1 = -0.00800x^2 + 0.0865x + 0.252.$

(b)



(c) $C_2 = \dfrac{5 + 3x}{20 + 4x}$



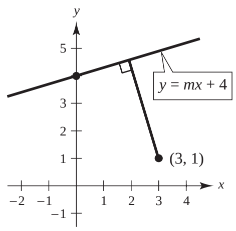
(d) $\displaystyle\lim_{x \to \infty} C_1 = -\infty$

$\displaystyle\lim_{x \to \infty} C_2 = \frac{3}{4}$

Model C_1 is unrealistic as $x \to \infty$. Model C_2 is better.

(e) The limiting concentration is $3/4 = 75\%$.

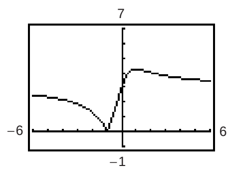
93. line: $mx - y + 4 = 0$



(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$

$= \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

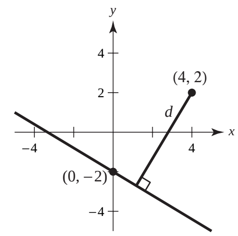
(b)



(c) $\lim_{m \to \infty} d(m) = 3 = \lim_{m \to -\infty} d(m)$

The line approaches the vertical line $x = 0$. Hence, the distance approaches 3.

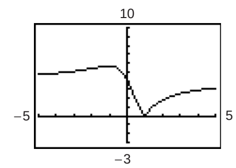
94. line: $y + 2 = m(x - 0) \Rightarrow mx - y - 2 = 0$



(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(4) - 1(2) - 2|}{\sqrt{m^2 + 1}}$

$= \dfrac{|4m - 4|}{\sqrt{m^2 + 1}}$

(b)



(c) $\lim_{m \to \infty} d(m) = 4; \lim_{m \to -\infty} d(m) = 4$

As the line becomes closer to the y-axis, the distance approaches 4.

95. $f(x) = \dfrac{2x^2}{x^2 + 2}$

(a) $\lim_{x \to \infty} f(x) = 2 = L$

(b)

$$f(x_1) + \varepsilon = \frac{2x_1^2}{x_1^2 + 2} + \varepsilon = 2$$

$$2x_1^2 + \varepsilon x_1^2 + 2\varepsilon = 2x_1^2 + 4$$

$$x_1^2 \varepsilon = 4 - 2\varepsilon$$

$$x_1 = \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$

$x_2 = -x_1$ by symmetry

(c) Let $M = \sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}} > 0$. For $x > M$:

$$x > \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$

$$x^2 \varepsilon > 4 - 2\varepsilon$$

$$2x^2 + x^2\varepsilon + 2\varepsilon > 2x^2 + 4$$

$$\frac{2x^2}{x^2 + 2} + \varepsilon > 2$$

$$\left|\frac{2x^2}{x^2 + 2} - 2\right| > |-\varepsilon| = \varepsilon$$

$$|f(x) - L| > \varepsilon$$

(d) Similarly, $N = -\sqrt{\dfrac{4 - 2\varepsilon}{\varepsilon}}$.

96. $f(x) = \dfrac{6x}{\sqrt{x^2 + 2}}$

(a) $\lim_{x \to \infty} f(x) = 6 = L$

$\lim_{x \to -\infty} f(x) = -6 = K$

(b)

$$f(x_1) + \varepsilon = \frac{6x_1}{\sqrt{x_1^2 + 2}} + \varepsilon = 6$$

$$6x_1 = (6 - \varepsilon)\sqrt{x_1^2 + 2}$$

$$36x_1^2 = (x_1^2 + 2)(6 - \varepsilon)^2$$

$$36x_1^2 - (6 - \varepsilon)^2 x_1^2 = 2(6 - \varepsilon)^2$$

$$x_1^2[36 - 36 + 12\varepsilon - \varepsilon^2] = 2(6 - \varepsilon)^2$$

$$x_1^2 = \frac{2(6 - \varepsilon)^2}{12\varepsilon - \varepsilon^2}$$

$$x_1 = (6 - \varepsilon)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}}$$

$x_2 = -x_1$ by symmetry

(c) $M = x_1 = (6 - \varepsilon)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

(d) $N = x_2 = (\varepsilon - 6)\sqrt{\dfrac{2}{12\varepsilon - \varepsilon^2}}$

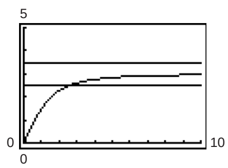
97. $\lim_{x\to\infty} \dfrac{3x}{\sqrt{x^2+3}} = 3$

(a) For $\varepsilon = 0.5$, we need $M > 0$ such that

$$|f(x) - L| = \left|\frac{3x}{\sqrt{x^2+3}} - 3\right| < \varepsilon = 0.5 \text{ whenever}$$

$$x > M \Longrightarrow 2.5 < \frac{3x}{\sqrt{x^2+3}} < 3.5.$$

By graphing $y_1 = \dfrac{3x}{\sqrt{x^2+3}}$, $y_2 = 2.5$ and $y_3 = 3.5$,



you see that the inequality is satisfied for $x > 2.7$, so let $M = 3$.

(b) For $\varepsilon = 0.1$, the inequality becomes

$$2.9 < \frac{3x}{\sqrt{x^2+3}} < 3.1.$$

From the corresponding graph you obtain $M = 7$.

98. $\lim_{x\to-\infty} \dfrac{3x}{\sqrt{x^2+3}} = -3$

(a) For $\varepsilon = 0.5$, we need $N < 0$ such that

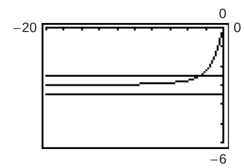
$$|f(x) - L| = \left|\frac{3x}{\sqrt{x^2+3}} + 3\right| < \varepsilon = 0.5 \text{ whenever}$$

$x < N$.

$$-0.5 < \frac{3x}{\sqrt{x^2+3}} + 3 < 0.5$$

$$-3.5 < \frac{3x}{\sqrt{x^2+3}} < -2.5$$

By graphing $y_1 = \dfrac{3x}{\sqrt{x^2+3}}$, $y_2 = -2.5$ and $y_3 = -3.5$,



you see that the inequality is satisfied for $x < -2.7$, so let $N = -3$.

(b) For $\varepsilon = 0.1$, the inequality becomes

$$-3.1 < \frac{3x}{\sqrt{x^2+3}} < -2.9.$$

From the corresponding graph you obtain $N = -7$.

99. $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$. Let $\varepsilon > 0$ be given. We need $M > 0$ such that

$$|f(x) - L| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \varepsilon \text{ whenever } x > M.$$

$$x^2 > \frac{1}{\varepsilon},\ x > \frac{1}{\sqrt{\varepsilon}},\ \text{let } M = \frac{1}{\sqrt{\varepsilon}}.$$

Hence, for $x > M$, we have

$$x > \frac{1}{\sqrt{\varepsilon}} \Longrightarrow x^2 > \frac{1}{\varepsilon} \Longrightarrow \frac{1}{x^2} < \varepsilon \Longrightarrow |f(x) - L| < \varepsilon.$$

100. $\lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$. Let $\varepsilon > 0$ be given. We need to find $M > 0$ such that

$$|f(x) - L| = \left|\frac{2}{\sqrt{x}} - 0\right| = \frac{2}{\sqrt{x}} < \varepsilon \text{ whenever } x > M.$$

$$\frac{2}{\sqrt{x}} < \varepsilon \Longrightarrow \frac{\sqrt{x}}{2} > \frac{1}{\varepsilon} \Longrightarrow x > \frac{4}{\varepsilon^2}.$$

Let $M = 4/\varepsilon^2$.

For $x > M = 4/\varepsilon^2$, we have

$$\sqrt{x} > 2/\varepsilon \Longrightarrow \frac{2}{\sqrt{x}} < \varepsilon \Longrightarrow |f(x) - L| < \varepsilon.$$

101. $\lim\limits_{x\to-\infty} \dfrac{1}{x^3} = 0$. Let $\varepsilon > 0$. We need to find $N < 0$ such that

$|f(x) - L| = \left|\dfrac{1}{x^3} - 0\right| = \dfrac{-1}{x^3} < \varepsilon$ whenever $x < N$.

$\dfrac{-1}{x^3} < \varepsilon \Longrightarrow -x^3 > \dfrac{1}{\varepsilon} \Longrightarrow x < \dfrac{-1}{\varepsilon^{1/3}}$.

Let $N = \dfrac{-1}{\sqrt[3]{\varepsilon}}$.

Hence, for $x < N < \dfrac{-1}{\sqrt[3]{\varepsilon}}$,

$\dfrac{1}{x} > -\sqrt[3]{\varepsilon}$

$-\dfrac{1}{x} < \sqrt[3]{\varepsilon}$

$-\dfrac{1}{x^3} < \varepsilon$

$\Longrightarrow |f(x) - L| < \varepsilon$.

102. $\lim\limits_{x\to-\infty} \dfrac{1}{x-2} = 0$. Let $\varepsilon > 0$ be given. We need $N < 0$ such that

$|f(x) - L| = \left|\dfrac{1}{x-2} - 0\right| = \dfrac{-1}{x-2} < \varepsilon$

whenever $x < N$.

$\dfrac{-1}{x-2} < \varepsilon \Longrightarrow x - 2 < \dfrac{-1}{\varepsilon} \Longrightarrow x < 2 - \dfrac{1}{\varepsilon}$.

Let $N = 2 - \dfrac{1}{\varepsilon}$.

Hence, for $x < N < 2 - \dfrac{1}{\varepsilon}$,

$x - 2 < \dfrac{-1}{\varepsilon}$

$\dfrac{-1}{x-2} < \varepsilon$

$\Longrightarrow |f(x) - L| < \varepsilon$.

103. $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n x^n + \cdots + a_1 x + a_0}{b_m x^m + \cdots + b_1 x + b_0}$

Divide $p(x)$ and $q(x)$ by x^m.

Case 1: If $n < m$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{\dfrac{a_n}{x^{m-n}} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{0 + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{0}{b_m} = 0$.

Case 2: If $m = n$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{a_n + \cdots + 0 + 0}{b_m + \cdots + 0 + 0} = \dfrac{a_n}{b_m}$.

Case 3: If $n > m$: $\lim\limits_{x\to\infty} \dfrac{p(x)}{q(x)} = \lim\limits_{x\to\infty} \dfrac{a_n x^{n-m} + \cdots + \dfrac{a_1}{x^{m-1}} + \dfrac{a_0}{x^m}}{b_m + \cdots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}} = \dfrac{\pm\infty + \cdots + 0}{b_m + \cdots + 0} = \pm\infty$.

104. $\lim\limits_{x\to\infty} x^3 = \infty$. Let $M > 0$ be given. We need to find $N > 0$ such that $f(x) = x^3 > M$ whenever $x > N$.

$x^3 > M \Longrightarrow x > M^{1/3}$. Let $N = M^{1/3}$. Hence, for $x > N = M^{1/3}$, $x > M^{1/3} \Longrightarrow x^3 > M \Longrightarrow f(x) > M$.

105. False. Let $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$. (See Exercise 2.)

106. False. Let $y_1 = \sqrt{x+1}$, then $y_1(0) = 1$. Thus, $y_1' = 1/\left(2\sqrt{x+1}\right)$ and $y_1'(0) = 1/2$. Finally,

$$y_1'' = -\frac{1}{4(x+1)^{3/2}} \text{ and } y_1''(0) = -\frac{1}{4}.$$

Let $p = ax^2 + bx + 1$, then $p(0) = 1$. Thus, $p' = 2ax + b$ and $p'(0) = \frac{1}{2} \Rightarrow b = \frac{1}{2}$.
Finally, $p'' = 2a$ and $p''(0) = -\frac{1}{4} \Rightarrow a = -\frac{1}{8}$. Therefore,

$$f(x) = \begin{cases} (-1/8)x^2 + (1/2)x + 1, & x < 0 \\ \sqrt{x+1}, & x \geq 0 \end{cases} \text{ and } f(0) = 1,$$

$$f'(x) = \begin{cases} (1/2) - (1/4)x, & x < 0 \\ 1/\left(2\sqrt{x+1}\right), & x > 0 \end{cases} \text{ and } f'(0) = \frac{1}{2}, \text{ and}$$

$$f''(x) = \begin{cases} (-1/4), & x < 0 \\ -1/(4(x+1)^{3/2}), & x > 0 \end{cases} \text{ and } f''(0) = -\frac{1}{4}.$$

$f''(x) < 0$ for all real x, but $f(x)$ *increases* without bound.

Section 3.6 A Summary of Curve Sketching

1. f has constant negative slope. Matches (d)

2. The slope of f approaches ∞ as $x \to 0^-$, and approaches $-\infty$ as $x \to 0^+$. Matches (c)

3. The slope is periodic, and zero at $x = 0$. Matches (a)

4. The slope is positive up to approximately $x = 1.5$. Matches (b)

5. (a) $f'(x) = 0$ for $x = -2$ and $x = 2$

f' is negative for $-2 < x < 2$ (decreasing function).

f' is positive for $x > 2$ and $x < -2$ (increasing function).

(b) $f''(x) = 0$ at $x = 0$ (Inflection point).

f'' is positive for $x > 0$ (Concave upwards).

f'' is negative for $x < 0$ (Concave downward).

(c) f' is increasing on $(0, \infty)$. $(f'' > 0)$

(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x.

6. (a) x_0, x_2, x_4

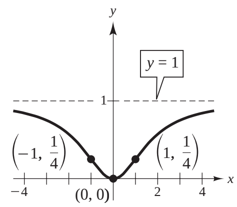
(b) x_2, x_3

(c) x_1

(d) x_1

(e) x_2, x_3

7. $y = \dfrac{x^2}{x^2 + 3}$



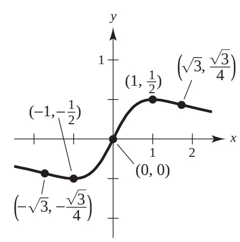
$y' = \dfrac{6x}{(x^2 + 3)^2} = 0$ when $x = 0$.

$y'' = \dfrac{18(1 - x^2)}{(x^2 + 3)^3} = 0$ when $x = \pm 1$.

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$x = -1$	$\frac{1}{4}$	$-$	0	Point of inflection
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < 1$		$+$	$+$	Increasing, concave up
$x = 1$	$\frac{1}{4}$	$+$	0	Point of inflection
$1 < x < \infty$		$+$	$-$	Increasing, concave down

8. $y = \dfrac{x}{x^2 + 1}$



$y' = \dfrac{1 - x^2}{(x^2 + 1)^2} = \dfrac{(1 - x)(x + 1)}{(x^2 + 1)^2} = 0$ when $x = \pm 1$.

$y'' = -\dfrac{2x(3 - x^2)}{(x^2 + 1)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\dfrac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\dfrac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\dfrac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\dfrac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up

9. $y = \dfrac{1}{x-2} - 3$

$y' = -\dfrac{1}{(x-2)^2} < 0$ when $x \neq 2$.

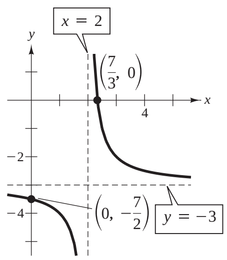
$y'' = \dfrac{2}{(x-2)^3}$

No relative extrema, no points of inflection

Intercepts: $\left(\dfrac{7}{3}, 0\right), \left(0, -\dfrac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$



10. $y = \dfrac{x^2+1}{x^2-9}$

$y' = \dfrac{-20x}{(x^2-9)^2} = 0$ when $x = 0$.

$y'' = \dfrac{60(x^2+3)}{(x^2-9)^3} < 0$ when $x = 0$.

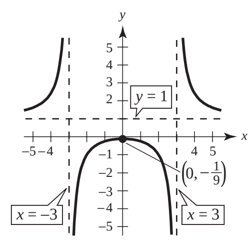
Therefore, $\left(0, -\dfrac{1}{9}\right)$ is a relative maximum.

Intercept: $\left(0, -\dfrac{1}{9}\right)$

Vertical asymptotes: $x = \pm 3$

Horizontal asymptote: $y = 1$

Symmetric about y-axis



11. $y = \dfrac{2x}{x^2-1}$

$y' = \dfrac{-2(x^2+1)}{(x^2-1)^2} < 0$ if $x \neq \pm 1$.

$y'' = \dfrac{4x(x^2+3)}{(x^2-1)^3} = 0$ if $x = 0$.

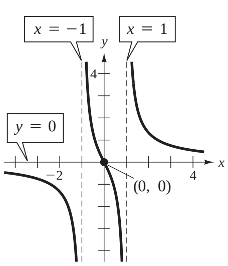
Inflection point: $(0, 0)$

Intercept: $(0, 0)$

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 0$

Symmetry with respect to the origin



12. $f(x) = \dfrac{x+2}{x} = 1 + \dfrac{2}{x}$

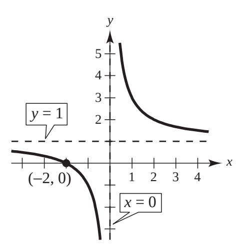
$f'(x) = \dfrac{-2}{x^2} < 0$ when $x \neq 0$.

$f''(x) = \dfrac{4}{x^3} \neq 0$

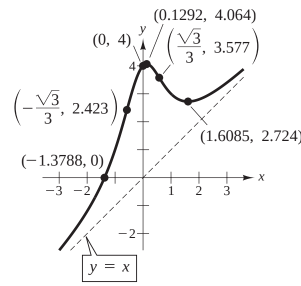
Intercept: $(-2, 0)$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$



13. $g(x) = x + \dfrac{4}{x^2 + 1}$



$g'(x) = 1 - \dfrac{8x}{(x^2 + 1)^2} = \dfrac{x^4 + 2x^2 - 8x + 1}{(x^2 + 1)^2} = 0$ when $x \approx 0.1292, 1.6085$

$g''(x) = \dfrac{8(3x^2 - 1)}{(x^2 + 1)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$

$g''(0.1292) < 0$, therefore, $(0.1292, 4.064)$ is relative maximum.

$g''(1.6085) > 0$, therefore, $(1.6085, 2.724)$ is a relative minimum.

Points of inflection: $\left(-\dfrac{\sqrt{3}}{3}, 2.423\right), \left(\dfrac{\sqrt{3}}{3}, 3.577\right)$

Intercepts: $(0, 4), (-1.3788, 0)$

Slant asymptote: $y = x$

14. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x - 4)(x^2 + 4x + 16)}{x^3} = 0$ when $x = 4$.

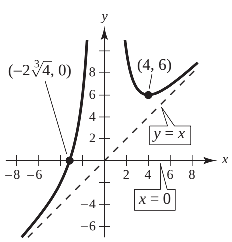
$f''(x) = \dfrac{192}{x^4} > 0$ if $x \neq 0$.

Therefore, $(4, 6)$ is a relative minimum.

Intercept; $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



15. $f(x) = \dfrac{x^2 + 1}{x} = x + \dfrac{1}{x}$

$f'(x) = 1 - \dfrac{1}{x^2} = 0$ when $x = \pm 1$.

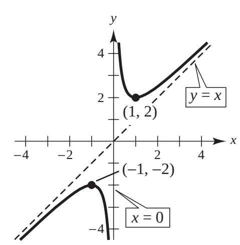
$f''(x) = \dfrac{2}{x^3} \neq 0$

Relative maximum: $(-1, -2)$

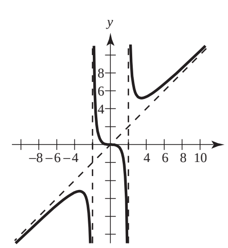
Relative minimum: $(1, 2)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$



16. $f(x) = \dfrac{x^3}{x^2 - 4} = x + \dfrac{4x}{x^2 - 4}$



$f'(x) = \dfrac{x^2(x^2 - 12)}{(x^2 - 4)^2} = 0$ when $x = 0, \pm 2\sqrt{3}$.

$f''(x) = \dfrac{8x(x^2 + 12)}{(x^2 - 4)^3} = 0$ when $x = 0$.

Intercept: $(0, 0)$

Relative maximum: $\left(-2\sqrt{3}, -3\sqrt{3}\right)$

Relative minimum: $\left(2\sqrt{3}, 3\sqrt{3}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm 2$

Slant asymptote: $y = x$

17. $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$

$y' = 1 - \dfrac{4}{(x - 4)^2}$

$= \dfrac{(x - 2)(x - 6)}{(x - 4)^2} = 0$ when $x = 2, 6$.

$y'' = \dfrac{8}{(x - 4)^3}$

$y'' < 0$ when $x = 2$.

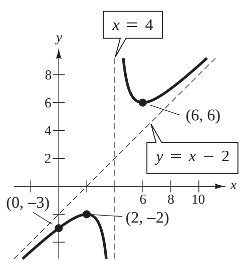
Therefore, $(2, -2)$ is a relative maximum.

$y'' > 0$ when $x = 6$.

Therefore, $(6, 6)$ is a relative minimum.

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$



18. $y = \dfrac{2x^2 - 5x + 5}{x - 2} = 2x - 1 + \dfrac{3}{x - 2}$

$y' = 2 - \dfrac{3}{(x - 2)^2} = \dfrac{2x^2 - 8x + 5}{(x - 2)^2} = 0$ when $x = \dfrac{4 \pm \sqrt{6}}{2}$.

$y'' = \dfrac{6}{(x - 2)^3} \neq 0$

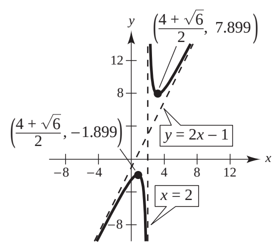
Relative maximum: $\left(\dfrac{4 - \sqrt{6}}{2}, -1.8990\right)$

Relative minimum: $\left(\dfrac{4 + \sqrt{6}}{2}, 7.8990\right)$

Intercept: $(0, -5/2)$

Vertical asymptote: $x = 2$

Slant asymptote: $y = 2x - 1$



19. $y = x\sqrt{4 - x}$,

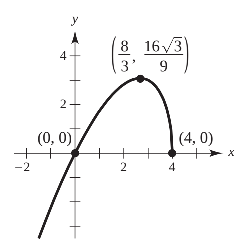
Domain: $(-\infty, 4]$

$y' = \dfrac{8 - 3x}{2\sqrt{4 - x}} = 0$ when $x = \dfrac{8}{3}$ and undefined when $x = 4$.

$y'' = \dfrac{3x - 16}{4(4 - x)^{3/2}} = 0$ when $x = \dfrac{16}{3}$ and undefined when $x = 4$.

Note: $x = \frac{16}{3}$ is not in the domain.

	y	y'	y''	Conclusion
$-\infty < x < \dfrac{8}{3}$		+	−	Increasing, concave down
$x = \dfrac{8}{3}$	$\dfrac{16}{3\sqrt{3}}$	0	−	Relative maximum
$\dfrac{8}{3} < x < 4$		−	−	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint



20. $g(x) = x\sqrt{9-x}$ Domain: $x \le 9$

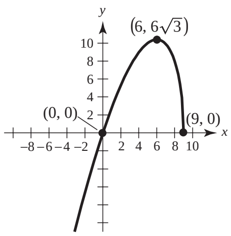
$$g'(x) = \frac{3(6-x)}{2\sqrt{9-x}} = 0 \text{ when } x = 6.$$

$$g''(x) = \frac{3(x-12)}{4(9-x)^{3/2}} < 0 \text{ when } x = 6.$$

Relative maximum: $(6, 6\sqrt{3})$

Intercepts: $(0, 0)$, $(9, 0)$

Concave downward on $(-\infty, 9)$



21. $h(x) = x\sqrt{9-x^2}$ Domain: $-3 \le x \le 3$

$$h'(x) = \frac{9-2x^2}{\sqrt{9-x^2}} = 0 \text{ when } x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2}.$$

$$h''(x) = \frac{x(2x^2-27)}{(9-x^2)^{3/2}} = 0 \text{ when } x = 0.$$

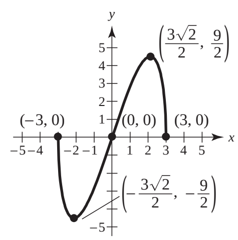
Relative maximum: $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{9}{2}\right)$

Relative minimum: $\left(-\dfrac{3\sqrt{2}}{2}, -\dfrac{9}{2}\right)$

Intercepts: $(0, 0)$, $(\pm 3, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$



22. $y = x\sqrt{16-x^2}$ Domain: $-4 \le x \le 4$

$$y' = \frac{2(8-x^2)}{\sqrt{16-x^2}} = 0 \text{ when } x = \pm 2\sqrt{2}.$$

$$y'' = \frac{2x(x^2-24)}{(16-x^2)^{3/2}} = 0 \text{ when } x = 0.$$

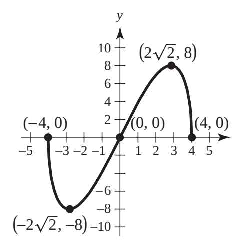
Relative maximum: $(2\sqrt{2}, 8)$

Relative minimum: $(-2\sqrt{2}, -8)$

Intercepts: $(0, 0)$, $(\pm 4, 0)$

Symmetric with respect to the origin

Point of inflection: $(0, 0)$

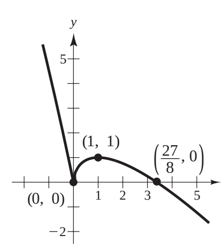


23. $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1-x^{1/3})}{x^{1/3}}$$

$= 0$ when $x = 1$ and undefined when $x = 0$.

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \ne 0.$$



	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	1	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

24. $y = 3(x-1)^{2/3} - (x-1)^2$

$$y' = \frac{2}{(x-1)^{1/3}} - 2(x-1) = \frac{2 - 2(x-1)^{4/3}}{(x-1)^{1/3}} = 0 \text{ when } x = 0, 2$$

$(y'$ undefined for $x = 1)$.

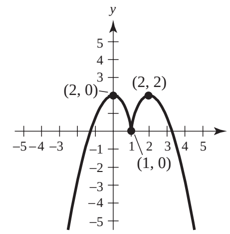
$$y'' = \frac{-2}{3(x-1)^{4/3}} - 2 < 0 \text{ for all } x \neq 1.$$

Concave downward on $(-\infty, 1)$ and $(1, \infty)$

Relative maximum: $(0, 2)$, $(2, 2)$

Relative minimum: $(1, 0)$

Intercepts: $(0, 2)$, $(1, 0)$, $(-1.280, 0)$, $(3.280, 0)$

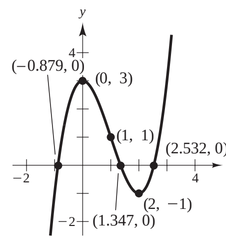


25. $y = x^3 - 3x^2 + 3$

$y' = 3x^2 - 6x = 3x(x-2) = 0$ when $x = 0$, $x = 2$.

$y'' = 6x - 6 = 6(x-1) = 0$ when $x = 1$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	3	0	$-$	Relative maximum
$0 < x < 1$		$-$	$-$	Decreasing, concave down
$x = 1$	1	$-$	0	Point of inflection
$1 < x < 2$		$-$	$+$	Decreasing, concave up
$x = 2$	-1	0	$+$	Relative minimum
$2 < x < \infty$		$+$	$+$	Increasing, concave up

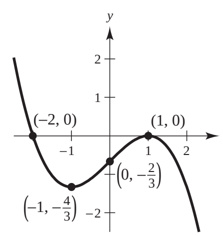


26. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$.

$y'' = -2x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

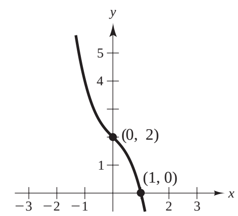


27. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

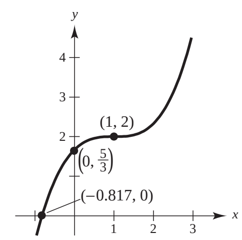


	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	2	$-$	0	Point of inflection
$0 < x < \infty$		$-$	$-$	Decreasing, concave down

28. $f(x) = \frac{1}{3}(x - 1)^3 + 2$

$f'(x) = (x - 1)^2 = 0$ when $x = 1$.

$f''(x) = 2(x - 1) = 0$ when $x = 1$.

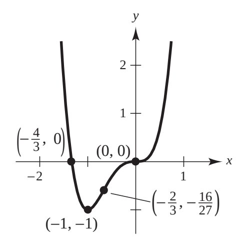


	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	2	0	0	Point of inflection
$1 < x < \infty$		$+$	$+$	Increasing, concave up

29. $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

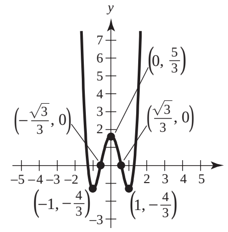


	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-1	0	$+$	Relative minimum
$-1 < x < -\frac{2}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	$+$	0	Point of inflection
$-\frac{2}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		$+$	$+$	Increasing, concave up

30. $y = 3x^4 - 6x^2 + \frac{5}{3}$

$y' = 12x^3 - 12x = 12x(x^2 - 1) = 0$ when $x = 0, x = \pm 1$.

$y'' = 36x^2 - 12 = 12(3x^2 - 1) = 0$ when $x = \pm\frac{\sqrt{3}}{3}$.

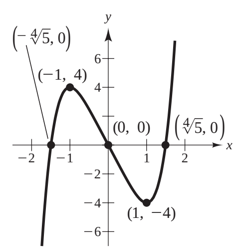


	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-4/3$	0	$+$	Relative minimum
$-1 < x < -\frac{\sqrt{3}}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{\sqrt{3}}{3}$	0	$+$	0	Point of inflection
$-\frac{\sqrt{3}}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	$5/3$	0	$-$	Relative maximum
$0 < x < \frac{\sqrt{3}}{3}$		$-$	$-$	Decreasing, concave down
$x = \frac{\sqrt{3}}{3}$	0	$-$	0	Point of inflection
$\frac{\sqrt{3}}{3} < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	$-4/3$	0	$+$	Relative minimum
$1 < x < \infty$		$+$	$+$	Increasing, concave up

31. $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1$.

$y'' = 20x^3 = 0$ when $x = 0$.

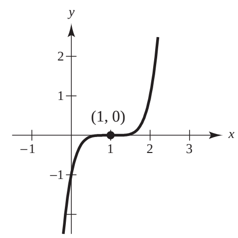


	y	y'	y''	Conclusion
$-\infty < x < -1$		$+$	$-$	Increasing, concave down
$x = -1$	4	0	$-$	Relative maximum
$-1 < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	$-$	0	Point of inflection
$0 < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	-4	0	$+$	Relative minimum
$1 < x < \infty$		$+$	$+$	Increasing, concave up

32. $y = (x - 1)^5$

$y' = 5(x - 1)^4 = 0$ when $x = 1$.

$y'' = 20(x - 1)^3 = 0$ when $x = 1$.

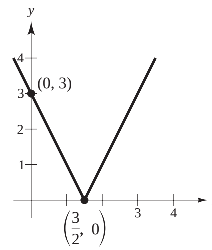


	y	y'	y''	Conclusion
$-\infty < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		$+$	$+$	Increasing, concave up

33. $y = \lvert 2x - 3 \rvert$

$y' = \dfrac{2(2x - 3)}{\lvert 2x - 3 \rvert}$ undefined at $x = \dfrac{3}{2}$.

$y'' = 0$



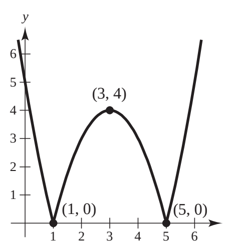
	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		$-$	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		$+$	Increasing

34. $y = \lvert x^2 - 6x + 5 \rvert$

$$y' = \frac{2(x - 3)(x^2 - 6x + 5)}{\lvert x^2 - 6x + 5 \rvert} = \frac{2(x - 3)(x - 5)(x - 1)}{\lvert (x - 5)(x - 1) \rvert}$$

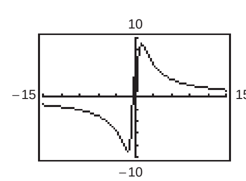
$= 0$ when $x = 3$ and undefined when $x = 1$, $x = 5$.

$$y'' = \frac{2(x^2 - 6x + 5)}{\lvert x^2 - 6x + 5 \rvert} = \frac{2(x - 5)(x - 1)}{\lvert (x - 5)(x - 1) \rvert}$$ undefined when $x = 1$, $x = 5$.



	y	y'	y''	Conclusion
$-\infty < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		$+$	$-$	Increasing, concave down
$x = 3$	4	0	$-$	Relative maximum
$3 < x < 5$		$-$	$-$	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		$+$	$+$	Increasing, concave up

35. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x(x^2 + 1)}$



$x = 0$ vertical asymptote

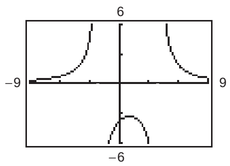
$y = 0$ horizontal asymptote

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

Points of inflection: $(-1.84, -7.86)$, $(1.84, 7.86)$

36. $f(x) = 5\left(\dfrac{1}{x-4} - \dfrac{1}{x+2}\right)$

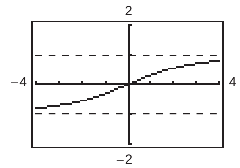


$x = -2, 4$ vertical asymptote

$y = 0$ horizontal asymptote

$\left(1, -\frac{10}{3}\right)$ relative maximum

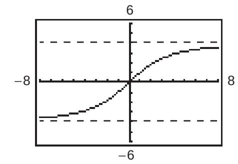
37. $y = \dfrac{x}{\sqrt{x^2+7}}$



$(0, 0)$ point of inflection

$y = \pm 1$ horizontal asymptotes

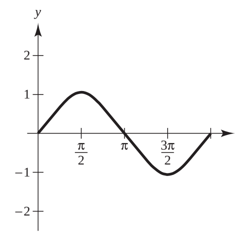
38. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$



$y = \pm 4$ horizontal asymptotes

$(0, 0)$ point of inflection

39. $y = \sin x - \dfrac{1}{18}\sin 3x,\ 0 \le x \le 2\pi$



$$\begin{aligned}
y' &= \cos x - \frac{1}{6}\cos 3x \\
&= \cos x - \frac{1}{6}[\cos 2x \cos x - \sin 2x \sin x] \\
&= \cos x - \frac{1}{6}[(1 - 2\sin^2 x)\cos x - 2\sin^2 x \cos x] \\
&= \cos x\left[1 - \frac{1}{6}(1 - 2\sin^2 x - 2\sin^2 x)\right] \\
&= \cos x\left[\frac{5}{6} + \frac{2}{3}\sin^2 x\right]
\end{aligned}$$

$y' = 0$: $\quad \cos x = 0 \Rightarrow x = \pi/2, 3\pi/2$

$\dfrac{5}{6} + \dfrac{2}{3}\sin^2 x = 0 \Rightarrow \sin^2 x = -5/4$, impossible

$$\begin{aligned}
y'' = -\sin x + \frac{1}{2}\sin 3x = 0 \Rightarrow 2\sin x &= \sin 3x \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= 2\sin x \cos^2 x + (2\cos^2 x - 1)\sin x \\
&= \sin x(2\cos^2 x + 2\cos^2 x - 1) \\
&= \sin x(4\cos^2 x - 1)
\end{aligned}$$

$\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

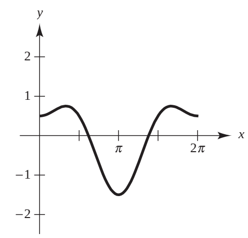
$2 = 4\cos^2 x - 1 \Rightarrow \cos x = \pm\sqrt{3}/2 \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Relative maximum: $\left(\dfrac{\pi}{2}, \dfrac{19}{18}\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -\dfrac{19}{18}\right)$

Inflection points: $\left(\dfrac{\pi}{6}, \dfrac{4}{9}\right), \left(\dfrac{5\pi}{6}, \dfrac{4}{9}\right), (\pi, 0), \left(\dfrac{7\pi}{6}, -\dfrac{4}{9}\right), \left(\dfrac{11\pi}{6}, -\dfrac{4}{9}\right)$

40. $y = \cos x - \dfrac{1}{2}\cos 2x,\ 0 \le x \le 2\pi$



$y' = -\sin x + \sin 2x = -\sin x(1 - 2\cos x) = 0$ when $x = 0, \pi, \dfrac{\pi}{3}, \dfrac{5\pi}{3}$.

$y'' = -\cos x + 2\cos 2x = -\cos x + 2(2\cos^2 x - 1)$

$= 4\cos^2 x - \cos x - 2 = 0$ when $\cos x = \dfrac{1 \pm \sqrt{33}}{8} \approx 0.8431, -0.5931$.

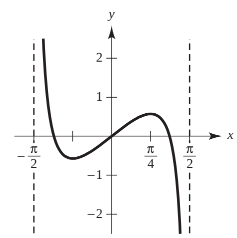
Therefore, $x \approx 0.5678$ or 5.7154, $x \approx 2.2057$ or 4.0775.

Relative maxima: $\left(\dfrac{\pi}{3}, \dfrac{3}{4}\right), \left(\dfrac{5\pi}{3}, \dfrac{3}{4}\right)$

Relative minimum: $\left(\pi, -\dfrac{3}{2}\right)$

Inflection points: $(0.5678, 0.6323), (2.2057, -0.4449), (5.7154, 0.6323), (4.0775, -0.4449)$

41. $y = 2x - \tan x,\ -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$



$y' = 2 - \sec^2 x = 0$ when $x = \pm\dfrac{\pi}{4}$.

$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

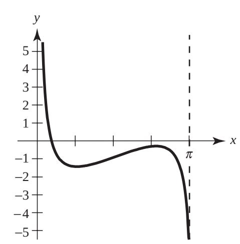
Relative maximum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\dfrac{\pi}{4}, 1 - \dfrac{\pi}{2}\right)$

Inflection point: $(0, 0)$

Vertical asymptotes: $x = \pm\dfrac{\pi}{2}$

42. $y = 2(x - 2) + \cot x,\ 0 < x < \pi$



$y' = 2 - \csc^2 x = 0$ when $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

$y'' = 2\csc^2 x \cot x = 0$ when $x = \dfrac{\pi}{2}$.

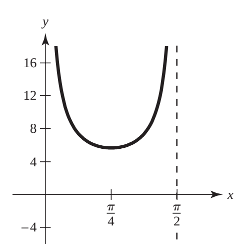
Relative maximum: $\left(\dfrac{3\pi}{4}, \dfrac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 3\right)$

Point of inflection: $\left(\dfrac{\pi}{2}, \pi - 4\right)$

Vertical asymptotes: $x = 0, \pi$

43. $y = 2(\csc x + \sec x),\ 0 < x < \dfrac{\pi}{2}$



$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \implies x = \dfrac{\pi}{4}$

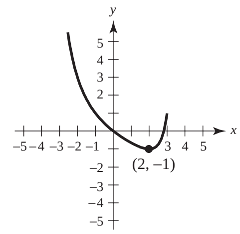
Relative minimum: $\left(\dfrac{\pi}{4}, 4\sqrt{2}\right)$

Vertical asymptotes: $x = 0, x = \dfrac{\pi}{2}$

44. $y = \sec^2\left(\dfrac{\pi x}{8}\right) - 2\tan\left(\dfrac{\pi x}{8}\right) - 1, \ -3 < x < 3$

$y' = 2\sec^2\left(\dfrac{\pi x}{8}\right)\tan\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) - 2\sec^2\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) = 0 \Rightarrow x = 2$

Relative minimum: $(2, -1)$



45. $g(x) = x\tan x, \ -\dfrac{3\pi}{2} < x < \dfrac{3\pi}{2}$

$g'(x) = \dfrac{x + \sin x\cos x}{\cos^2 x} = 0$ when $x = 0$.

$g''(x) = \dfrac{2(\cos x + x\sin x)}{\cos^3 x}$

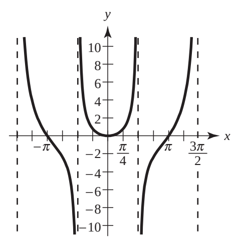
Vertical asymptotes: $x = -\dfrac{3\pi}{2}, -\dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$

Intercepts: $(-\pi, 0), (0, 0), (\pi, 0)$

Symmetric with respect to y-axis.

Increasing on $\left(0, \dfrac{\pi}{2}\right)$ and $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$

Points of inflection: $(\pm 2.80, -1)$



46. $g(x) = x\cot x, \ -2\pi < x < 2\pi$

$g'(x) = \dfrac{\sin x\cos x - x}{\sin^2 x}$

$g'(0)$ does not exist. But $\lim\limits_{x\to 0} x\cot x = \lim\limits_{x\to 0}\dfrac{x}{\tan x} = 1.$

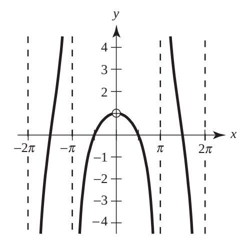
Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

Intercepts: $\left(-\dfrac{3\pi}{2}, 0\right), \left(-\dfrac{\pi}{2}, 0\right), \left(\dfrac{\pi}{2}, 0\right), \left(\dfrac{3\pi}{2}, 0\right)$

Symmetric with respect to y-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$

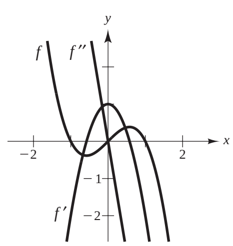
Points of inflection: $(\pm 4.49, 1)$



47. f is cubic.

f' is quadratic.

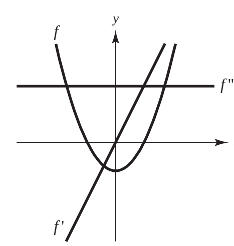
f'' is linear.



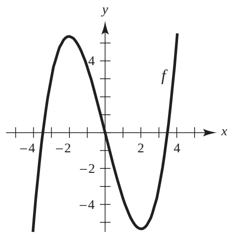
48. f'' is constant.

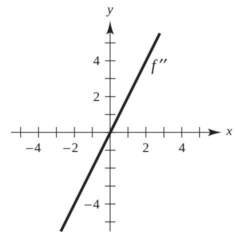
f' is linear.

f is quadratic.



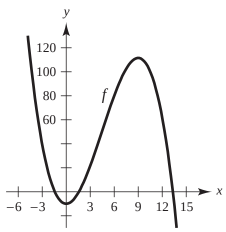
49.

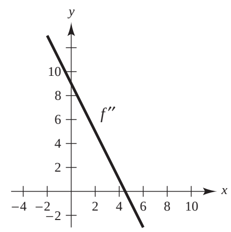




(any vertical translate of f will do)

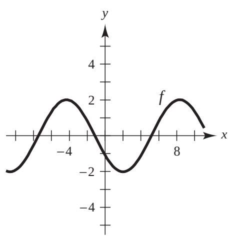
50.

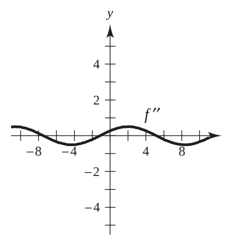




(any vertical translate of f will do)

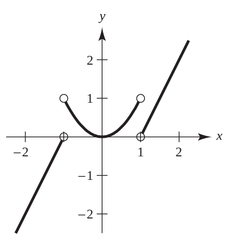
51.

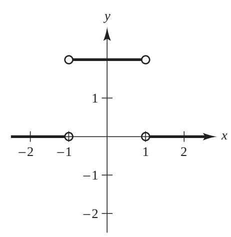




(any vertical translate of f will do)

52.





(any vertical translate of the 3 segments of f will do)

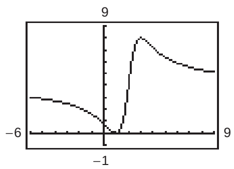
53. Since the slope is negative, the function is decreasing on $(2, 8)$, and hence $f(3) > f(5)$.

54. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

55. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

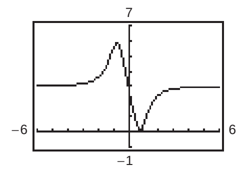


The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

56. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

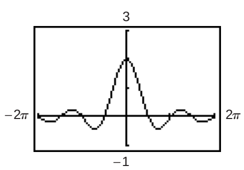


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it since $f(c)$ is undefined.

57. $h(x) = \dfrac{\sin 2x}{x}$

Vertical asymptote: none

Horizontal asymptote: $y = 0$



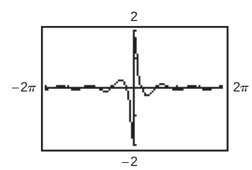
Yes, it is possible for a graph to cross its horizontal asymptote.

It is not possible to cross a vertical asymptote because the function is not continuous there.

58. $f(x) = \dfrac{\cos 3x}{4x}$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$



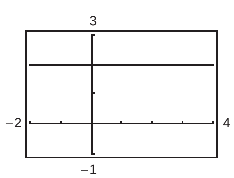
Yes, it is possible for a graph to cross its horizontal asymptote.

It is not possible to cross a vertical asymptote because the function is not continuous there.

59. $h(x) = \dfrac{6 - 2x}{3 - x}$

$$= \frac{2(3-x)}{3-x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

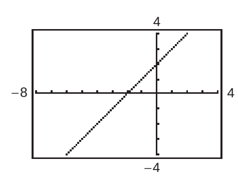


hole at $(3, 2)$

60. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

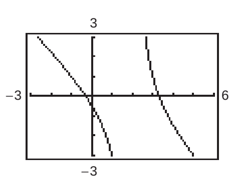
$$= \frac{(x+2)(x-1)}{x-1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



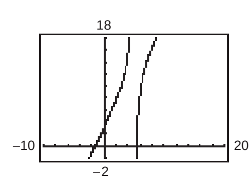
hole at $(1, 3)$

61. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$



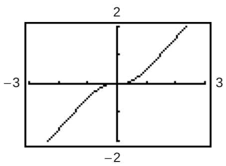
The graph appears to approach the slant asymptote $y = -x + 1$.

62. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



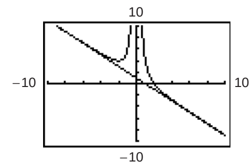
The graph appears to approach the slant asymptote $y = 2x + 2$.

63. $f(x) = \dfrac{x^3}{x^2 + 1} = x - \dfrac{x}{x^2 + 1}$



The graph appears to approach the slant asymptote $y = x$.

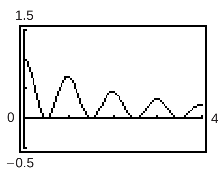
64. $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2} = -x + 1 + \dfrac{4}{x^2}$



The graph appears to approach the slant asymptote $y = -x + 1$.

65. $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2 + 1}}$, $(0, 4)$

(a)



On $(0, 4)$ there seem to be 7 critical numbers:

0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

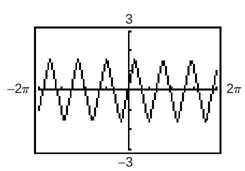
(b) $f'(x) = \dfrac{-\cos \pi x(x \cos \pi x + 2\pi(x^2 + 1) \sin \pi x)}{(x^2 + 1)^{3/2}} = 0$

Critical numbers $\approx \dfrac{1}{2}, 0.97, \dfrac{3}{2}, 1.98, \dfrac{5}{2}, 2.98, \dfrac{7}{2}$.

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

66. $f(x) = \tan(\sin \pi x)$

(a)



(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x)$

$= -\tan(\sin \pi x) = -f(x)$

Symmetry with respect to the origin

(c) Periodic with period 2

(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of f is concave downward.

67. Vertical asymptote: $x = 5$

Horizontal asymptote: $y = 0$

$$y = \frac{1}{x - 5}$$

68. Vertical asymptote: $x = -3$

Horizontal asymptote: none

$$y = \frac{x^2}{x + 3}$$

69. Vertical asymptote: $x = 5$

Slant asymptote: $y = 3x + 2$

$$y = 3x + 2 + \frac{1}{x - 5} = \frac{3x^2 - 13x - 9}{x - 5}$$

70. Vertical asymptote: $x = 0$

Slant asymptote: $y = -x$

$$y = -x + \frac{1}{x} = \frac{1 - x^2}{x}$$

71. $f(x) = \dfrac{ax}{(x - b)^2}$

(a) The graph has a vertical asymptote at $x = b$. If $a > 0$, the graph approaches ∞ as $x \to b$. If $a < 0$, the graph approaches $-\infty$ as $x \to b$. The graph approaches its vertical asymptote faster as $|a| \to 0$.

(b) As b varies, the position of the vertical asymptote changes: $x = b$. Also, the coordinates of the minimum $(a > 0)$ or maximum $(a < 0)$ are changed.

72. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2),\ a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \frac{1}{a}$.

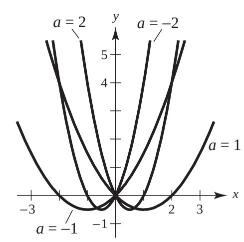
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\frac{2}{a}, 0\right)$

Relative minimum: $\left(\frac{1}{a}, -\frac{1}{2}\right)$

Points of inflection: none

(b)



73. $f(x) = \frac{3x^n}{x^4 + 1}$

(a) For n even, f is symmetric about the y-axis. For n odd, f is symmetric about the origin.

(b) The x-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 3$ as the horizontal asymptote.

(d) There is a slant asymptote $y = 3x$ if $n = 5$:

$$\frac{3x^5}{x^4 + 1} = 3x - \frac{3x}{x^4 + 1}.$$

(e)

n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

74. Tangent line at P: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $(0, f(x_0) - x_0 f'(x_0))$

(c) Normal line: $y - y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\frac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0) f'(x_0)$$

x-intercept: $(x_0 + f(x_0) f'(x_0), 0)$

(d) Let $x = 0$: $y - y_0 = \frac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \frac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \frac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\frac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = y_0^2 + \left(\frac{f(x_0)}{f'(x_0)}\right) = \frac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

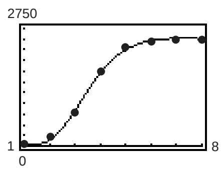
$$|PC| = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = |x_0 - (x_0 + f(x_0) f'(x_0))| = |f(x_0) f'(x_0)|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

75. (a)



(b) When $t = 10$, $N = 2434$ bacteria.

(c) N is greatest (≈ 2518) at $t \approx 7.2$.

(d) $N'(t)$ is greatest when $t \approx 3.2$.

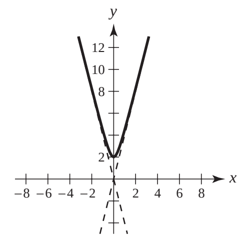
(Find the t-value of the point of inflection.)

(e) $\lim_{t\to\infty} N(t) = \dfrac{13{,}250}{7} \approx 1893$bacteria

76. $y = \sqrt{4 + 16x^2}$

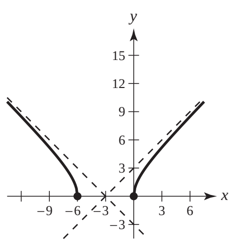
As $x \to \infty$, $y \to 4x$. As $x \to -\infty$, $y \to -4x$.

Slant asymptotes: $y = \pm 4x$



77. $y = \sqrt{x^2 + 6x} = \sqrt{(x + 3)^2 - 9}$

$y \to x + 3$ as $x \to \infty$, and $y \to -x - 3$ as $x \to -\infty$.



78. Let $\lambda = \dfrac{\dfrac{f(x) - f(a)}{x - a} - \dfrac{f(b) - f(a)}{b - a}}{x - b}$, $a < x < b$.

$$\lambda(x - b) = \frac{f(x) - f(a)}{x - a} - \frac{f(b) - f(a)}{b - a}$$

$$\lambda(x - b)(x - a) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

$$f(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) + \lambda(x - b)(x - a)$$

Let $h(t) = f(t) - \left\{ f(a) + \dfrac{f(b) - f(a)}{b - a}(t - a) + \lambda(t - a)(t - b) \right\}$.

$h(a) = 0$, $h(b) = 0$, $h(x) = 0$

By Rolle's Theorem, there exist numbers α_1 and α_2 such that $a < \alpha_1 < x < \alpha_2 < b$ and $h'(\alpha_1) = h'(\alpha_2) = 0$. By Rolle's Theorem, there exists β in (a, b) such that $h''(\beta) = 0$. Finally,

$$0 = h''(\beta) = f''(\beta) - \{2\lambda\} \Longrightarrow \lambda = \frac{1}{2}f''(\beta).$$

Section 3.7 Optimization Problems

1. (a)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

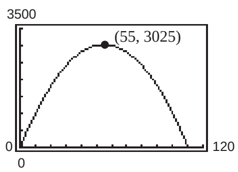
(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d)



The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$$\frac{d^2P}{dx^2} = -2 < 0$$

P is a maximum when $x = 110 - x = 55$.
The two numbers are 55 and 55.

2. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

(b) $V = x(24 - 2x)^2,\ 0 < x < 12$

(c) $\dfrac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

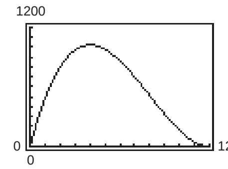
$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\dfrac{d^2V}{dx^2} = 12(2x - 16)$

$\dfrac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4$, $V = 1024$ is maximum.

(d)



The maximum volume seems to be 1024.

3. Let x and y be two positive numbers such that $x + y = S$.

$P = xy = x(S - x) = Sx - x^2$

$\dfrac{dP}{dx} = S - 2x = 0$ when $x = \dfrac{S}{2}$.

$\dfrac{d^2P}{dx^2} = -2 < 0$ when $x = \dfrac{S}{2}$.

P is a maximum when $x = y = S/2$.

4. Let x and y be two positive numbers such that $xy = 192$.

$S = x + y = x + \dfrac{192}{x}$

$\dfrac{dS}{dx} = 1 - \dfrac{192}{x^2} = 0$ when $x = \sqrt{192}$.

$\dfrac{d^2S}{dx^2} = \dfrac{384}{x^3} > 0$ when $x = \sqrt{192}$.

S is a minimum when $x = y = \sqrt{192}$.

5. Let x and y be two positive numbers such that $xy = 192$.

$S = x + 3y = \dfrac{192}{y} + 3y$

$\dfrac{dS}{dy} = 3 - \dfrac{192}{y^2} = 0$ when $y = 8$.

$\dfrac{d^2S}{dy^2} = \dfrac{384}{y^3} > 0$ when $y = 8$.

S is minimum when $y = 8$ and $x = 24$.

6. Let x be a positive number.

$S = x + \dfrac{1}{x}$

$\dfrac{dS}{dx} = 1 - \dfrac{1}{x^2} = 0$ when $x = 1$.

$\dfrac{d^2S}{dx^2} = \dfrac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

7. Let x and y be two positive numbers such that $x + 2y = 100$.

$P = xy = y(100 - 2y) = 100y - 2y^2$

$\dfrac{dP}{dy} = 100 - 4y = 0$ when $y = 25$.

$\dfrac{d^2P}{dy^2} = -4 < 0$ when $y = 25$.

P is a maximum when $x = 50$ and $y = 25$.

8. Let x and y be two positive numbers such that $x^2 + y = 27$.

$P = xy = x(27 - x^2) = 27x - x^3$

$\dfrac{dP}{dx} = 27 - 3x^2 = 0$ when $x = 3$.

$\dfrac{d^2P}{dx^2} = -6x < 0$ when $x = 3$.

The product is a maximum when $x = 3$ and $y = 18$.

9. Let x be the length and y the width of the rectangle.

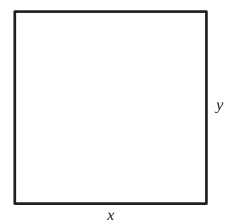
$$2x + 2y = 100$$
$$y = 50 - x$$
$$A = xy = x(50 - x)$$
$$\frac{dA}{dx} = 50 - 2x = 0 \text{ when } x = 25.$$
$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 25.$$

A is maximum when $x = y = 25$ meters.

10. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$
$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$
$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$
$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)



11. Let x be the length and y the width of the rectangle.

$$xy = 64$$
$$y = \frac{64}{x}$$
$$P = 2x + 2y = 2x + 2\left(\frac{64}{x}\right) = 2x + \frac{128}{x}$$
$$\frac{dP}{dx} = 2 - \frac{128}{x^2} = 0 \text{ when } x = 8.$$
$$\frac{d^2P}{dx^2} = \frac{256}{x^3} > 0 \text{ when } x = 8.$$

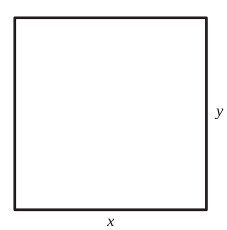
P is minimum when $x = y = 8$ feet.

12. Let x be the length and y the width of the rectangle.

$$xy = A$$
$$y = \frac{A}{x}$$
$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$
$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$
$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

P is minimum when $x = y = \sqrt{A}$ centimeters.

(A square!)



13. $d = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$= \sqrt{x^2 - 7x + 16}$

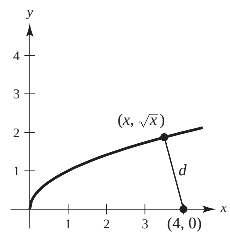
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$

$$f'(x) = 2x - 7 = 0$$

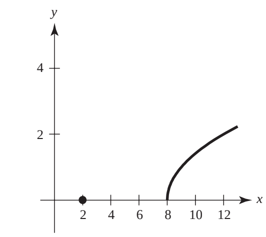
$$x = \frac{7}{2}$$

By the First Derivative Test, the point nearest to $(4, 0)$ is $(7/2, \sqrt{7/2})$.



14. $f(x) = \sqrt{x-8}$, $(2, 0)$

From the graph, it is clear that $(8, 0)$ is the closest point on the graph of f to $(2, 0)$.



15. $d = \sqrt{(x-2)^2 + [x^2 - (1/2)]^2}$

$= \sqrt{x^4 - 4x + (17/4)}$

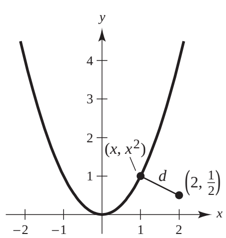
Since d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \frac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.



16. $f(x) = (x+1)^2$, $(5, 3)$

$d = \sqrt{(x-5)^2 + [(x+1)^2 - 3]^2}$

$= \sqrt{x^2 - 10x + 25 + (x^2 + 2x - 2)^2}$

$= \sqrt{x^2 - 10x + 25 + x^4 + 4x^3 - 8x + 4}$

$= \sqrt{x^4 + 4x^3 + x^2 - 18x + 29}$

Since d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 + 4x^3 + x^2 - 18x + 29$$

$$g'(x) = 4x^3 + 12x^2 + 2x - 18$$

$$= 2(x-1)(2x^2 + 8x + 9)$$

By the First Derivative Test, $x = 1$ yields a minimum. Hence, $(1, 4)$ is closest to $(5, 3)$.

17. $\dfrac{dQ}{dx} = kx(Q_0 - x) = kQ_0x - kx^2$

$\dfrac{d^2Q}{dx^2} = kQ_0 - 2kx$

$= k(Q_0 - 2x) = 0$ when $x = \dfrac{Q_0}{2}$.

$\dfrac{d^3Q}{dx^3} = -2k < 0$ when $x = \dfrac{Q_0}{2}$.

dQ/dx is maximum when $x = Q_0/2$.

18. $F = \dfrac{v}{22 + 0.02v^2}$

$\dfrac{dF}{dv} = \dfrac{22 - 0.02v^2}{(22 + 0.02v^2)^2}$

$= 0$ when $v = \sqrt{1100} \approx 33.166$.

By the First Derivative Test, the flow rate on the road is maximized when $v \approx 33$ mph.

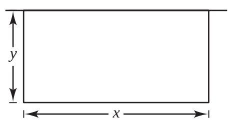
19. $xy = 180{,}000$ (see figure)

$S = x + 2y = \left(x + \dfrac{360{,}000}{x}\right)$ where S is the length of fence needed.

$$\frac{dS}{dx} = 1 - \frac{360{,}000}{x^2} = 0 \text{ when } x = 600.$$

$$\frac{d^2S}{dx^2} = \frac{720{,}000}{x^3} > 0 \text{ when } x = 600.$$

S is a minimum when $x = 600$ meters and $y = 300$ meters.



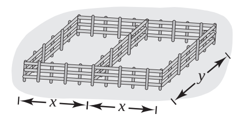
20. $4x + 3y = 200$ is the perimeter. (see figure)

$$A = 2xy = 2x\left(\frac{200 - 4x}{3}\right) = \frac{8}{3}(50x - x^2)$$

$$\frac{dA}{dx} = \frac{8}{3}(50 - 2x) = 0 \text{ when } x = 25.$$

$$\frac{d^2A}{dx^2} = -\frac{16}{3} < 0 \text{ when } x = 25.$$

A is a maximum when $x = 25$ feet and $y = \frac{100}{3}$ feet.



21. (a) $A = 4(\text{area of side}) + 2(\text{area of Top})$

(1) $A = 4(3)(11) + 2(3)(3) = 150$ square inches

(2) $A = 4(5)(5) + 2(5)(5) = 150$ square inches

(3) $A = 4(3.25)(6) + 2(6)(6) = 150$ square inches

(b) $V = (\text{length})(\text{width})(\text{height})$

(1) $V = (3)(3)(11) = 99$ cubic inches

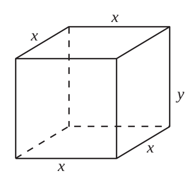
(2) $V = (5)(5)(5) = 125$ cubic inches

(3) $V = (6)(6)(3.25) = 117$ cubic inches

(c) $S = 4xy + 2x^2 = 150 \Longrightarrow y = \dfrac{150 - 2x^2}{4x}$

$$V = x^2y = x^2\left(\frac{150 - 2x^2}{4x}\right) = \frac{75}{2}x - \frac{1}{2}x^3$$

$$V' = \frac{75}{2} - \frac{3}{2}x^2 = 0 \Longrightarrow x = \pm 5$$



By the First Derivative Test, $x = 5$ yields the maximum volume. Dimensions: $5 \times 5 \times 5$. (A cube!)

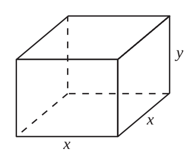
22. $S = 2x^2 + 4xy = 337.5$

$$y = \frac{337.5 - 2x^2}{4x}$$

$$V = x^2y = x^2\left[\frac{337.5 - 2x^2}{4x}\right] = 84.375x - \frac{1}{2}x^3$$

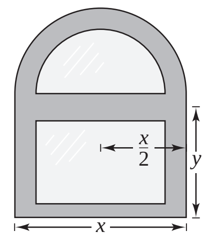
$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Longrightarrow x^2 = 56.25 \Longrightarrow x = 7.5 \text{ and } y = 7.5.$$

$$\frac{d^2V}{dx^2} = -3x < 0 \text{ for } x = 7.5.$$



The maximum value occurs when $x = y = 7.5$ cm.

23. $16 = 2y + x + \pi\left(\frac{x}{2}\right)$



$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8}$$

$$= 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right)$$

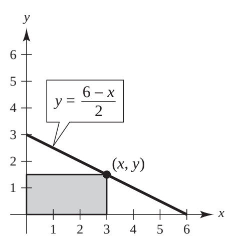
$$= 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}.$$

$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \frac{16}{4 + \pi}$ feet and $x = \frac{32}{4 + \pi}$ feet.

24. You can see from the figure that $A = xy$ and $y = \frac{6 - x}{2}$.



$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

A is a maximum when $x = 3$ and $y = 3/2$.

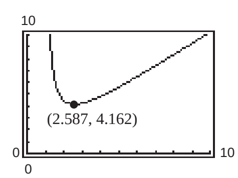
25. (a) $\frac{y - 2}{0 - 1} = \frac{0 - 2}{x - 1}$

$$y = 2 + \frac{2}{x - 1}$$

$$L = \sqrt{x^2 + y^2} = \sqrt{x^2 + \left(2 + \frac{2}{x - 1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x - 1} + \frac{4}{(x - 1)^2}}, \quad x > 1$$

(b)



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) $\text{Area} = A(x) = \frac{1}{2}xy = \frac{1}{2}x\left(2 + \frac{2}{x - 1}\right) = x + \frac{x}{x - 1}$

$$A'(x) = 1 + \frac{(x - 1) - x}{(x - 1)^2} = 1 - \frac{1}{(x - 1)^2} = 0$$

$$(x - 1)^2 = 1$$

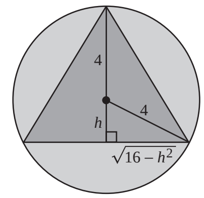
$$x - 1 = \pm 1$$

$$x = 0, 2 \quad (\text{select } x = 2)$$

Then $y = 4$ and $A = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

26. (a)



$$A = \frac{1}{2}\text{base} \times \text{height}$$

$$= \frac{1}{2}\left(2\sqrt{16-h^2}\right)(4+h)$$

$$= \sqrt{16-h^2}(4+h)$$

$$\frac{dA}{dh} = \frac{1}{2}(16-h^2)^{-1/2}(-2h)(4+h) + (16-h^2)^{1/2}$$

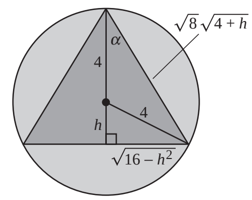
$$= (16-h^2)^{-1/2}\left[-h(4+h) + (16-h^2)\right]$$

$$= \frac{-2(h^2+2h-8)}{\sqrt{16-h^2}} = \frac{-2(h+4)(h-2)}{\sqrt{16-h^2}}$$

$\dfrac{dA}{dh} = 0$ when $h = 2$, which is a maximum by the First Derivative Test.

Hence, the sides are $2\sqrt{16-h^2} = 4\sqrt{3}$, an equilateral triangle. Area $= 12\sqrt{3}$ sq. units.

(b)



$$\cos\alpha = \frac{4+h}{\sqrt{8}\sqrt{4+h}} = \frac{\sqrt{4+h}}{\sqrt{8}}$$

$$\tan\alpha = \frac{\sqrt{16-h^2}}{4+h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{16-h^2}\right)(4+h)$$

$$= (4+h)^2\tan\alpha$$

$$= 64\cos^4\alpha\tan\alpha$$

$$A'(\alpha) = 64\left[\cos^4\alpha\sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$$

$$\Longrightarrow \quad \cos^4\alpha\sec^2\alpha = 4\cos^3\alpha\sin\alpha\tan\alpha$$

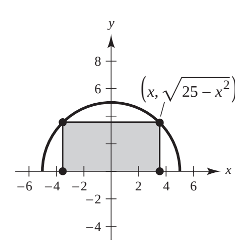
$$1 = 4\cos\alpha\sin\alpha\tan\alpha$$

$$\frac{1}{4} = \sin^2\alpha$$

$$\sin\alpha = \frac{1}{2} \Longrightarrow \alpha = 30^\circ \text{ and } A = 12\sqrt{3}.$$

(c) Equilateral triangle

27. $A = 2xy = 2x\sqrt{25-x^2}$ (see figure)



$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25-x^2}}\right) + 2\sqrt{25-x^2}$$

$$= 2\left(\frac{25-2x^2}{\sqrt{25-x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

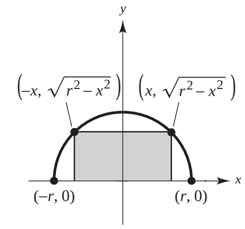
$$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

Width: $\dfrac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

28. $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.



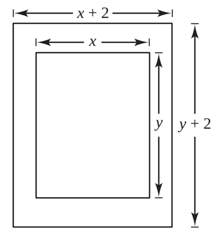
29. $xy = 30 \Rightarrow y = \dfrac{30}{x}$

$$A = (x + 2)\left(\frac{30}{x} + 2\right) \quad \text{(see figure)}$$

$$\frac{dA}{dx} = (x + 2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right) = \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$

$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$

By the First Derivative Test, the dimensions $(x + 2)$ by $(y + 2)$ are $(2 + \sqrt{30})$ by $(2 + \sqrt{30})$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.



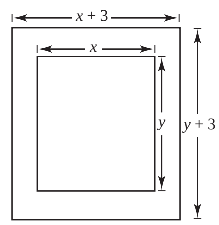
30. $xy = 36 \Rightarrow y = \dfrac{36}{x}$

$$A = (x + 3)(y + 3) = (x + 3)\left(\frac{36}{x} + 3\right)$$

$$= 36 + \frac{108}{x} + 3x + 9$$

$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

Dimensions: 9×9

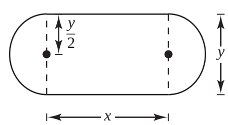


31. (a) $P = 2x + 2\pi r$

$$= 2x + 2\pi\left(\frac{y}{2}\right)$$

$$= 2x + \pi y = 200$$

$$\Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$$



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31. —CONTINUED—

(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) = 1528$

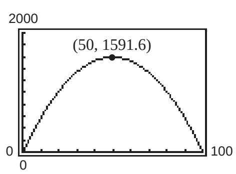
The maximum area of the rectangle is approximately 1592 m^2.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

(d) $A' = \frac{2}{\pi}(100 - 2x)$. $A' = 0$ when $x = 50$.

Maximum when length $= 50$ m and width $= \frac{100}{\pi}$.

(e)



Maximum area is approximately

1591.55 m^2 ($x = 50$ m).

32. $V = \pi r^2 h = 22$ cubic inches or $h = \frac{22}{\pi r^2}$

(a)

Radius, r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \frac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \frac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \frac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \frac{22}{\pi(0.8)^2}\right] \approx 59.0$

—CONTINUED—

32. —CONTINUED—

(b)

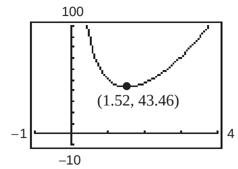
Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\dfrac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\dfrac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\dfrac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\dfrac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\dfrac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\dfrac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to be about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d)



The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\dfrac{dS}{dr} = 4\pi r - \dfrac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$$h = \frac{22}{\pi r^2} \approx 3.04 \text{ in.}$$

Note: Notice that

$$h = \frac{22}{\pi r^2} = \frac{22}{\pi(11/\pi)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r.$$

33. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \implies y = 108 - 4x$$
$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

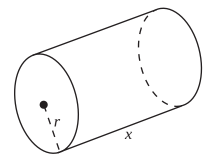
$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

34. $V = \pi r^2 x$

$x + 2\pi r = 108 \Longrightarrow x = 108 - 2\pi r$ (see figure)



$$V = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$$

$$\frac{dV}{dr} = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r)$$

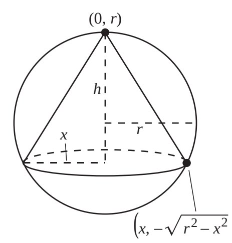
$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$

Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.

35. $V = \frac{1}{3}\pi x^2 h = \frac{1}{3}\pi x^2\left(r + \sqrt{r^2 - x^2}\right)$ (see figure)

$$\frac{dV}{dx} = \frac{1}{3}\pi\left[\frac{-x^3}{\sqrt{r^2 - x^2}} + 2x\left(r + \sqrt{r^2 - x^2}\right)\right] = \frac{\pi x}{3\sqrt{r^2 - x^2}}\left(2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2\right) = 0$$



$$2r^2 + 2r\sqrt{r^2 - x^2} - 3x^2 = 0$$

$$2r\sqrt{r^2 - x^2} = 3x^2 - 2r^2$$

$$4r^2(r^2 - x^2) = 9x^4 - 12x^2r^2 + 4r^4$$

$$0 = 9x^4 - 8x^2r^2 = x^2(9x^2 - 8r^2)$$

$$x = 0, \frac{2\sqrt{2}r}{3}$$

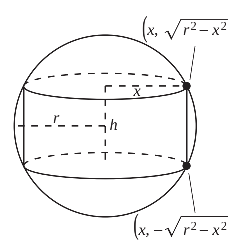
By the First Derivative Test, the volume is a maximum when

$$x = \frac{2\sqrt{2}r}{3} \text{ and } h = r + \sqrt{r^2 - x^2} = \frac{4r}{3}.$$

Thus, the maximum volume is

$$V = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81} \text{ cubic units.}$$

36. $V = \pi x^2 h = \pi x^2\left(2\sqrt{r^2 - x^2}\right) = 2\pi x^2\sqrt{r^2 - x^2}$ (see figure)



$$\frac{dV}{dx} = 2\pi\left[x^2\left(\frac{1}{2}\right)(r^2 - x^2)^{-1/2}(-2x) + 2x\sqrt{r^2 - x^2}\right]$$

$$= \frac{2\pi x}{\sqrt{r^2 - x^2}}(2r^2 - 3x^2)$$

$$= 0 \text{ when } x = 0 \text{ and } x^2 = \frac{2r^2}{3} \Longrightarrow x = \frac{\sqrt{6}r}{3}.$$

By the First Derivative Test, the volume is a maximum when

$$x = \frac{\sqrt{6}r}{3} \text{ and } h = \frac{2r}{\sqrt{3}}.$$

Thus, the maximum volume is

$$V = \pi\left(\frac{2}{3}r^2\right)\left(\frac{2r}{\sqrt{3}}\right) = \frac{4\pi r^3}{3\sqrt{3}}.$$

37. No, there is no minimum area. If the sides are x and y, then $2x + 2y = 20 \Longrightarrow y = 10 - x$.
The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

38. No. The volume will change because the shape of the container changes when squeezed.

39. $V = 12 = \frac{4}{3}\pi r^3 + \pi r^2 h$

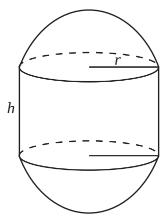
$$h = \frac{12 - (4/3)\pi r^3}{\pi r^2} = \frac{12}{\pi r^2} - \frac{4}{3}r$$

$$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{12}{\pi r^2} - \frac{4}{3}r\right) = 4\pi r^2 + \frac{24}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{24}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{24}{r^2} = 0 \text{ when } r = \sqrt[3]{9/\pi} \approx 1.42 \text{ cm.}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{48}{r^3} > 0 \text{ when } r = \sqrt[3]{9/\pi} \text{ cm.}$$

The surface area is minimum when $r = \sqrt[3]{9/\pi}$ cm and $h = 0$. The resulting solid is a sphere of radius $r \approx 1.42$ cm.



40. $V = 3000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{3000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{3000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{6000}{r}\right]$$

$$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{6000}{r^2}\right] = 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}} \approx 5.636 \text{ feet and } h \approx 22.545 \text{ feet.}$$

By the Second Derivative Test, we have

$$\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0 \text{ when } r = \sqrt[3]{\frac{1125}{2\pi}}.$$

Therefore, these dimensions will produce a minimum cost.

41. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when

$$y = \frac{30}{9 + 4\sqrt{3}} \text{ and } x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}.$$

42. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2,\ 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$, when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$, and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2,\ 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$, when $x \approx 2.62$, $A \approx 13.102$, and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20 - 5x}{6}\right)^2, 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$, when $x \approx 2.0475$, $A \approx 14.091$, and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \le x \le \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$, when $x \approx 1.748$, $A \approx 15.138$, and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

43. Let S be the strength and k the constant of proportionality.
Given $h^2 + w^2 = 24^2$, $h^2 = 24^2 - w^2$,

$$S = kwh^2$$

$$S = kw(576 - w^2) = k(576w - w^3)$$

$$\frac{dS}{dw} = k(576 - 3w^2) = 0 \text{ when } w = 8\sqrt{3}, h = 8\sqrt{6}.$$

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = 8\sqrt{3}.$$

These values yield a maximum.

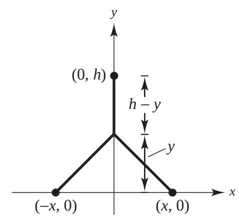
44. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{(x^2 + y^2)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



45. $$R = \frac{v_0^2}{g}\sin 2\theta$$

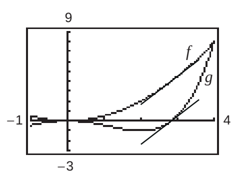
$$\frac{dR}{d\theta} = \frac{2v_0^2}{g}\cos 2\theta = 0 \text{ when } \theta = \frac{\pi}{4}, \frac{3\pi}{4}.$$

$$\frac{d^2R}{d\theta^2} = -\frac{4v_0^2}{g}\sin 2\theta < 0 \text{ when } \theta = \frac{\pi}{4}.$$

By the Second Derivative Test, R is maximum when $\theta = \pi/4$.

46. $f(x) = \frac{1}{2}x^2$ $\qquad g(x) = \frac{1}{16}x^4 - \frac{1}{2}x^2$ on $[0, 4]$

(a)



(b) $d(x) = f(x) - g(x) = \frac{1}{2}x^2 - \left(\frac{1}{16}x^4 - \frac{1}{2}x^2\right) = x^2 - \frac{1}{16}x^4$

$$d'(x) = 2x - \frac{1}{4}x^3 = 0 \Longrightarrow 8x = x^3$$

$$\Longrightarrow x = 0, 2\sqrt{2} \text{ (in } [0, 4])$$

The maximum distance is $d = 4$ when $x = 2\sqrt{2}$.

(c) $f'(x) = x$, Tangent line at $\left(2\sqrt{2}, 4\right)$ is

$$y - 4 = 2\sqrt{2}\left(x - 2\sqrt{2}\right)$$

$$y = 2\sqrt{2}x - 4.$$

$g'(x) = \frac{1}{4}x^3 - x$, Tangent line at $\left(2\sqrt{2}, 0\right)$ is

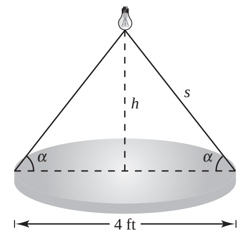
$$y - 0 = \left(\tfrac{1}{4}\left(2\sqrt{2}\right)^3 - 2\sqrt{2}\right)\left(x - 2\sqrt{2}\right)$$

$$y = 2\sqrt{2}x - 8.$$

The tangent lines are parallel and 4 vertical units apart.

(d) The tangent lines will be parallel. If $d(x) = f(x) - g(x)$, then $d'(x) = 0 = f'(x) - g'(x)$ implies that $f'(x) = g'(x)$ at the point x where the distance is maximum.

47. $\sin\alpha = \dfrac{h}{s} \Longrightarrow s = \dfrac{h}{\sin\alpha},\ 0 < \alpha < \dfrac{\pi}{2}$



$\tan\alpha = \dfrac{h}{2} \Longrightarrow h = 2\tan\alpha \Longrightarrow s = \dfrac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$

$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)]$$

$$= \frac{k}{4}\cos\alpha[\cos^2\alpha - 2\sin^2\alpha]$$

$$= \frac{k}{4}\cos\alpha[1 - 3\sin^2\alpha]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$

Since α is acute, we have

$$\sin\alpha = \frac{1}{\sqrt{3}} \Longrightarrow h = 2\tan\alpha = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2} \text{ feet.}$$

Since $(d^2I)/(d\alpha^2) = (k/4)\sin\alpha(9\sin^2\alpha - 7) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.

48. Let F be the illumination at point P which is x units from source 1.

$$F = \frac{kI_1}{x^2} + \frac{kI_2}{(d-x)^2}$$

$$\frac{dF}{dx} = \frac{-2kI_1}{x^3} + \frac{2kI_2}{(d-x)^3} = 0 \text{ when } \frac{2kI_1}{x^3} = \frac{2kI_2}{(d-x)^3}.$$

$$\frac{\sqrt[3]{I_1}}{\sqrt[3]{I_2}} = \frac{x}{d-x}$$

$$(d-x)\sqrt[3]{I_1} = x\sqrt[3]{I_2}$$

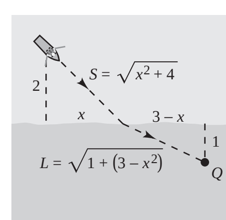
$$d\sqrt[3]{I_1} = x\left(\sqrt[3]{I_1} + \sqrt[3]{I_2}\right)$$

$$x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}$$

$$\frac{d^2F}{dx^2} = \frac{6kI_1}{x^4} + \frac{6kI_2}{(d-x)^4} > 0 \text{ when } x = \frac{d\sqrt[3]{I_1}}{\sqrt[3]{I_1} + \sqrt[3]{I_2}}.$$

This is the minimum point.

49. $S = \sqrt{x^2+4},\ L = \sqrt{1 + (3-x)^2}$



$$\text{Time} = T = \frac{\sqrt{x^2+4}}{2} + \frac{\sqrt{x^2-6x+10}}{4}$$

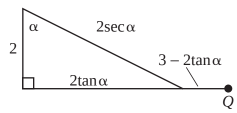
$$\frac{dT}{dx} = \frac{x}{2\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$$

$$\frac{x^2}{x^2+4} = \frac{9-6x+x^2}{4(x^2-6x+10)}$$

$$x^4 - 6x^3 + 9x^2 + 8x - 12 = 0$$

You need to find the roots of this equation in the interval $[0, 3]$. By using a computer or graphics calculator, you can determine that this equation has only one root in this interval $(x = 1)$. Testing at this value and at the endpoints, you see that $x = 1$ yields the minimum time. Thus, the man should row to a point 1 mile from the nearest point on the coast.

50. (a)



$$T = \frac{2 \sec \alpha}{2} + \frac{3 - 2 \tan \alpha}{4} = \sec \alpha + \frac{3 - 2 \tan \alpha}{4}$$

(b) $\dfrac{dT}{d\alpha} = \sec \alpha \tan \alpha - \dfrac{1}{2} \sec^2 \alpha = 0$

$$\tan \alpha = \frac{1}{2} \sec \alpha$$

$$\sin \alpha = \frac{1}{2}$$

$$\alpha = \frac{\pi}{6}$$

$$T\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{3}{4}$$

$$T(0) = \frac{7}{4} \qquad \text{Endpoint}$$

$$T\left(\arctan\left(\frac{3}{2}\right)\right) = \frac{\sqrt{13}}{2} \qquad \text{Endpoint}$$

Minimum time: $\dfrac{\sqrt{3}}{2} + \dfrac{3}{4} \approx 1.616$ hours

(c) $T = \dfrac{2 \sec \alpha}{v_1} + \dfrac{3 - 2 \tan \alpha}{v_2}$

$$\frac{dT}{d\alpha} = \frac{2}{v_1} \sec \alpha \tan \alpha - \frac{2}{v_2} \sec^2 \alpha = 0$$

$$\frac{\tan \alpha}{v_1} = \frac{\sec \alpha}{v_2}$$

$$\sin \alpha = \frac{v_1}{v_2}$$

α depends on $\dfrac{v_1}{v_2}$.

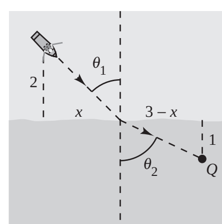
(d) Cost $= (2 \sec \alpha)c_1 + (3 - 2 \tan \alpha)c_2$

$$= \frac{2 \sec \alpha}{\left(\dfrac{1}{c_1}\right)} + \frac{(3 - 2 \tan \alpha)}{\left(\dfrac{1}{c_2}\right)}$$

From above part (c), minimum cost when

$$\sin \alpha = \frac{1/c_1}{1/c_2} = \frac{c_2}{c_1}.$$

51. $T = \dfrac{\sqrt{x^2 + 4}}{v_1} + \dfrac{\sqrt{x^2 - 6x + 10}}{v_2}$



$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + 4}} + \frac{x - 3}{v_2\sqrt{x^2 - 6x + 10}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + 4}} = \sin \theta_1 \text{ and } \frac{x - 3}{\sqrt{x^2 - 6x + 10}} = -\sin \theta_2$$

we have

$$\frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2} = 0 \Longrightarrow \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{4}{v_1(x^2 + 4)^{3/2}} + \frac{1}{v_2(x^2 - 6x + 10)^{3/2}} > 0$$

this condition yields a minimum time.

52. $T = \dfrac{\sqrt{x^2 + d_1^2}}{v_1} + \dfrac{\sqrt{d_2^2 + (a - x)^2}}{v_2}$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + d_1^2}} + \frac{x - a}{v_2\sqrt{d_2^2 + (a - x)^2}} = 0$$

Since

$$\frac{x}{\sqrt{x^2 + d_1^2}} = \sin\theta_1 \text{ and } \frac{x - a}{\sqrt{d_2^2 + (a - x)^2}} = -\sin\theta_2$$

we have

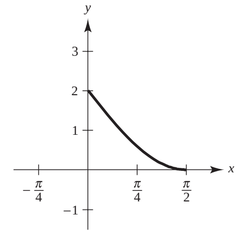
$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0 \Longrightarrow \frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}.$$

Since

$$\frac{d^2T}{dx^2} = \frac{d_1^2}{v_1(x^2 + d_1^2)^{3/2}} + \frac{d_2^2}{v_2[d_2^2 + (a - x)^2]^{3/2}} > 0$$

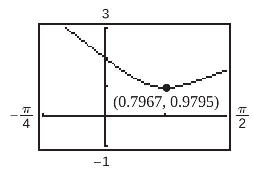
this condition yields a minimum time.

53. $f(x) = 2 - 2\sin x$



(a) Distance from origin to y-intercept is 2.
Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$



Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

54. $C(x) = 2k\sqrt{x^2 + 4} + k(4 - x)$

$$C'(x) = \frac{2xk}{\sqrt{x^2 + 4}} - k = 0$$

$$2x = \sqrt{x^2 + 4}$$

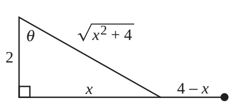
$$4x^2 = x^2 + 4$$

$$3x^2 = 4$$

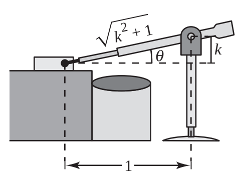
$$x = \frac{2}{\sqrt{3}}$$

Or, use Exercise 50(d): $\sin\theta = \dfrac{C_2}{C_1} = \dfrac{1}{2} \Longrightarrow \theta = 30°$.

Thus, $x = \dfrac{2}{\sqrt{3}}$.



55. $F\cos\theta = k(W - F\sin\theta)$



$$F = \frac{kW}{\cos\theta + k\sin\theta}$$

$$\frac{dF}{d\theta} = \frac{-kW(k\cos\theta - \sin\theta)}{(\cos\theta + k\sin\theta)^2} = 0$$

$k\cos\theta = \sin\theta \Longrightarrow k = \tan\theta \Longrightarrow \theta = \arctan k$

Since

$$\cos\theta + k\sin\theta = \frac{1}{\sqrt{k^2 + 1}} + \frac{k^2}{\sqrt{k^2 + 1}} = \sqrt{k^2 + 1},$$

the minimum force is

$$F = \frac{kW}{\cos\theta + k\sin\theta} = \frac{kW}{\sqrt{k^2 + 1}}.$$

56. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144 - r^2}$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144 - r^2)^{-1/2}(-2r) + 2r\sqrt{144 - r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r - 3r^3}{\sqrt{144 - r^2}}\right] = \pi\left[\frac{r(96 - r^2)}{\sqrt{144 - r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone: $S = \pi\left(4\sqrt{6}\right)\sqrt{\left(4\sqrt{6}\right)^2 + \left(4\sqrt{3}\right)^2} = 48\sqrt{6}\pi$

Area of sector: $144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72} = \frac{2\pi}{3}\left(3 - \sqrt{6}\right) \approx 1.153 \text{ radians or } 66^\circ$$

57. (a)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10^\circ$	$8\sin 10^\circ$	≈ 22.1
8	$8 + 16\cos 20^\circ$	$8\sin 20^\circ$	≈ 42.5
8	$8 + 16\cos 30^\circ$	$8\sin 30^\circ$	≈ 59.7
8	$8 + 16\cos 40^\circ$	$8\sin 40^\circ$	≈ 72.7
8	$8 + 16\cos 50^\circ$	$8\sin 50^\circ$	≈ 80.5
8	$8 + 16\cos 60^\circ$	$8\sin 60^\circ$	≈ 83.1

(b)

Base 1	Base 2	Altitude	Area
8	$8 + 16\cos 10^\circ$	$8\sin 10^\circ$	≈ 22.1
8	$8 + 16\cos 20^\circ$	$8\sin 20^\circ$	≈ 42.5
8	$8 + 16\cos 30^\circ$	$8\sin 30^\circ$	≈ 59.7
8	$8 + 16\cos 40^\circ$	$8\sin 40^\circ$	≈ 72.7
8	$8 + 16\cos 50^\circ$	$8\sin 50^\circ$	≈ 80.5
8	$8 + 16\cos 60^\circ$	$8\sin 60^\circ$	≈ 83.1
8	$8 + 16\cos 70^\circ$	$8\sin 70^\circ$	≈ 80.7
8	$8 + 16\cos 80^\circ$	$8\sin 80^\circ$	≈ 74.0
8	$8 + 16\cos 90^\circ$	$8\sin 90^\circ$	$= 64.0$

The maximum cross-sectional area is approximately 83.1 square feet.

(c) $A = (a + b)\frac{h}{2}$

$= [8 + (8 + 16\cos\theta)]\frac{8\sin\theta}{2}$

$= 64(1 + \cos\theta)\sin\theta,\ 0^\circ < \theta < 90^\circ$

(d) $\frac{dA}{d\theta} = 64(1 + \cos\theta)\cos\theta + (-64\sin\theta)\sin\theta$

$= 64(\cos\theta + \cos^2\theta - \sin^2\theta)$

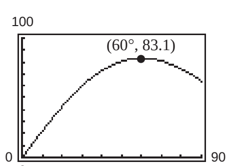
$= 64(2\cos^2\theta + \cos\theta - 1)$

$= 64(2\cos\theta - 1)(\cos\theta + 1)$

$= 0$ when $\theta = 60^\circ, 180^\circ, 300^\circ$.

The maximum occurs when $\theta = 60^\circ$.

(e)



58. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$$P = (0.12)d - id$$

$$d = ki^2 \text{ (since } d \text{ is proportional to } i^2)$$

$$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$$

$$\frac{dP}{di} = k(0.24i - 3i^2) = 0 \text{ when } i = \frac{0.24}{3} = 0.08.$$

$$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0 \text{ when } i = 0.08 \text{ (**Note:** } k > 0).$$

The profit is a maximum when $i = 8\%$.

59. $C = 100\left(\frac{200}{x^2} + \frac{x}{x+30}\right),\ 1 \le x$

$$C' = 100\left(-\frac{400}{x^3} + \frac{30}{(x+30)^2}\right)$$

Approximation: $x \approx 40.45$ units, or 4045 units

60. $P = -\frac{1}{10}s^3 + 6s^2 + 400$

(a) $\frac{dP}{ds} = -\frac{3}{10}s^2 + 12s = -\frac{3}{10}s(s - 40) = 0$ when $x = 0,\ s = 40$.

$$\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12$$

$$\frac{d^2P}{ds^2}(0) > 0 \Longrightarrow s = 0 \text{ yields a minimum.}$$

$$\frac{d^2P}{ds^2}(40) < 0 \Longrightarrow s = 40 \text{ yields a maximum.}$$

The maximum profit occurs when $s = 40$, which corresponds to \$40,000 ($P = \$3{,}600{,}000$).

(b) $\frac{d^2P}{ds^2} = -\frac{3}{5}s + 12 = 0$ when $s = 20$.

The point of diminishing returns occurs when $s = 20$, which corresponds to \$20,000 being spent on advertising.

61. $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10) = 282m - 128 = 0 \text{ when } m = \frac{64}{141}.$$

Line: $y = \frac{64}{141}x$

$$S = \left|4\left(\frac{64}{141}\right) - 1\right| + \left|5\left(\frac{64}{141}\right) - 6\right| + \left|10\left(\frac{64}{141}\right) - 3\right|$$

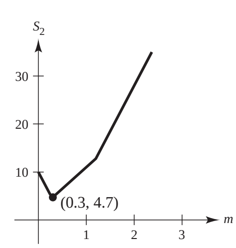
$$= \left|\frac{256}{141} - 1\right| + \left|\frac{320}{141} - 6\right| + \left|\frac{640}{141} - 3\right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

62. $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line $y = 0.3x$

$$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7 \text{ mi.}$$

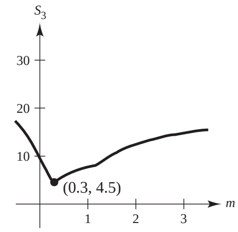


63. $S_3 = \frac{|4m - 1|}{\sqrt{m^2 + 1}} + \frac{|5m - 6|}{\sqrt{m^2 + 1}} + \frac{|10m - 3|}{\sqrt{m^2 + 1}}$

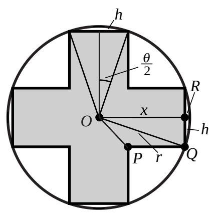
Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi.}$$



64. (a) Label the figure so that $r^2 = x^2 + h^2$.
Then, the area A is 8 times the area of the region given by $OPQR$:



$$A = 8\left[\frac{1}{2}h^2 + (x - h)h\right]$$

$$= 8\left[\frac{1}{2}(r^2 - x^2) + \left(x - \sqrt{r^2 - x^2}\right)\sqrt{r^2 - x^2}\right]$$

$$= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2$$

$$A'(x) = 8\sqrt{r^2 - x^2} - \frac{8x^2}{\sqrt{r^2 - x^2}} + 8x = 0$$

$$\frac{8x^2}{\sqrt{r^2 - x^2}} = 8x + 8\sqrt{r^2 - x^2}$$

$$x^2 = x\sqrt{r^2 - x^2} + (r^2 - x^2)$$

$$2x^2 - r^2 = x\sqrt{r^2 - x^2}$$

$$4x^4 - 4x^2r^2 + r^4 = x^2(r^2 - x^2)$$

$$5x^4 - 5x^2r^2 + r^4 = 0 \quad \text{Quadratic in } x^2.$$

$$x^2 = \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} = \frac{r^2}{10}\left[5 \pm \sqrt{5}\right].$$

Take positive value.

$$x = r\sqrt{\frac{5 + \sqrt{5}}{10}} \approx 0.85065r \quad \text{Critical number}$$

—CONTINUED—

(b) Note that $\sin\frac{\theta}{2} = \frac{h}{r}$ and $\cos\frac{\theta}{2} = \frac{x}{r}$. The area A of the cross equals the sum of two large rectangles minus the common square in the middle.

$$A = 2(2x)(2h) - 4h^2 = 8xh - 4h^2$$

$$= 8r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} - 4r^2\sin^2\frac{\theta}{2}$$

$$= 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right)$$

$$A'(\theta) = 4r^2\left(\cos\theta - \sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = 0$$

$$\cos\theta = \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\theta$$

$$\tan\theta = 2$$

$$\theta = \arctan(2) \approx 1.10715 \quad \text{or} \quad 63.4°$$

64. —CONTINUED—

(c) Note that $x^2 = \frac{r^2}{10}\left(5 + \sqrt{5}\right)$ and $r^2 - x^2 = \frac{r^2}{10}\left(5 - \sqrt{5}\right)$.

$$\begin{aligned} A(x) &= 8x\sqrt{r^2 - x^2} + 4x^2 - 4r^2 \\ &= 8\left[\frac{r^2}{10}\left(5 + \sqrt{5}\right)\frac{r^2}{10}\left(5 - \sqrt{5}\right)\right]^{1/2} + 4\frac{r^2}{10}\left(5 + \sqrt{5}\right) - 4r^2 \\ &= 8\left[\frac{r^4}{10}(20)\right]^{1/2} + 2r^2 + \frac{2}{5}\sqrt{5}r^2 - 4r^2 \\ &= \frac{8}{5}r^2\sqrt{5} - 2r^2 + \frac{2\sqrt{5}}{5}r^2 \\ &= 2r^2\left[\frac{4}{5}\sqrt{5} - 1 + \frac{\sqrt{5}}{5}\right] = 2r^2\left(\sqrt{5} - 1\right) \end{aligned}$$

Using the angle approach, note that $\tan\theta = 2$, $\sin\theta = \frac{2}{\sqrt{5}}$ and $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1 - \cos\theta) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)$.

$$\begin{aligned} \text{Thus, } A(\theta) &= 4r^2\left(\sin\theta - \sin^2\frac{\theta}{2}\right) \\ &= 4r^2\left(\frac{2}{\sqrt{5}} - \frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right) \\ &= \frac{4r^2\left(\sqrt{5} - 1\right)}{2} = 2r^2\left(\sqrt{5} - 1\right) \end{aligned}$$

65. $f(x) = x^3 - 3x;\ x^4 + 36 \le 13x^2$

$$\begin{aligned} x^4 - 13x^2 + 36 &= (x^2 - 9)(x^2 - 4) \\ &= (x - 3)(x - 2)(x + 2)(x + 3) \le 0 \end{aligned}$$

Hence, $-3 \le x \le -2$ or $2 \le x \le 3$.

$f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$

f is increasing on $(-\infty, -1)$ and $(1, \infty)$.

Hence, f is increasing on $[-3, -2]$ and $[2, 3]$.

$f(-2) = -2$, $f(3) = 18$. The maximum value of f is 18.

66. Let $a = \left(x + \frac{1}{x}\right)^3$ and $b = x^3 + \frac{1}{x^3}$, $x > 0$.

$$\begin{aligned} a^2 - b^2 &= \left(x + \frac{1}{x}\right)^6 - \left(x^3 + \frac{1}{x^3}\right)^2 \\ &= \left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6} + 2\right) \end{aligned}$$

$$\begin{aligned} \text{Let } f(x) &= \frac{(x + 1/x)^6 - (x^6 + 1/x^6 + 2)}{(x + 1/x)^3 + (x^3 + 1/x^3)} \\ &= \frac{a^2 - b^2}{a + b} = a - b \\ &= \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) - \left(x^3 + \frac{1}{x^3}\right) \\ &= 3x + \frac{3}{x} = 3\left(x + \frac{1}{x}\right). \end{aligned}$$

Let $g(x) = x + \frac{1}{x}$, $g'(x) = 1 - \frac{1}{x^2} = 0 \Longrightarrow x = 1$.

$g''(x) = \frac{2}{x^3}$ and $g''(1) = 2 > 0$. Hence g is a minimum at $x = 1$: $g(1) = 2$.

Finally, f is a minimum of $3(2) = 6$.

Section 3.8 Newton's Method

1. $f(x) = x^2 - 3$

$f'(x) = 2x$

$x_1 = 1.7$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.7000	−0.1100	3.4000	−0.0324	1.7324
2	1.7324	0.0012	3.4648	0.0003	1.7321

2. $f(x) = 2x^2 - 3$

$f'(x) = 4x$

$x_1 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1	−1	4	$-\frac{1}{4}$	$\frac{5}{4}$
2	$\frac{5}{4} = 1.25$	0.125	5.0	0.025	1.225

3. $f(x) = \sin x$

$f'(x) = \cos x$

$x_1 = 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	3.0000	0.1411	−0.9900	−0.1425	3.1425
2	3.1425	−0.0009	−1.0000	0.0009	3.1416

4. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

5. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

Approximation of the zero of f is 0.682.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.3750	1.7500	−0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

6. $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

Approximation of the zero of f is 0.755.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.4688	1.3125	−0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

7. $f(x) = 3\sqrt{x-1} - x$

$f'(x) = \dfrac{3}{2\sqrt{x-1}} - 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.2000	0.1416	2.3541	0.0602	1.1398
2	1.1398	−0.0181	3.0118	−0.0060	1.1458
3	1.1458	−0.0003	2.9284	−0.0001	1.1459

Approximation of the zero of f is 1.146.

Similarly, the other zero is approximately 7.854.

8. $f(x) = x - 2\sqrt{x+1}$

$f'(x) = 1 - \dfrac{1}{\sqrt{x+1}}$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	5	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	0.00085	4.8284

Approximation of the zero of f is 4.8284.

9. $f(x) = x^3 + 3$

$f'(x) = 3x^2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	−1.5000	−0.3750	6.7500	−0.0556	−1.4444
2	−1.4444	−0.0134	6.2589	−0.0021	−1.4423
3	−1.4423	−0.00003	6.2407	−0.0000033	−1.4422

Approximation of the zero of f is −1.44224957.

10. $f(x) = 1 - 2x^3$

$f'(x) = -6x^2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1	−1	−6	0.1667	0.8333
2	0.8333	−0.1573	−4.1663	0.0378	0.7955
3	0.7955	−0.0068	−3.7969	0.0018	0.7937
4	0.7937	0.0000	−3.7798	0.0000	0.7937

Approximation of the zero of f is 0.7937.

11. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	0.5000	−0.3360	1.6400	−0.2049	0.7049
2	0.7049	−0.0921	0.7824	−0.1177	0.8226
3	0.8226	−0.0231	0.4037	−0.0573	0.8799
4	0.8799	−0.0045	0.2495	−0.0181	0.8980
5	0.8980	−0.0004	0.2048	−0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

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11. —CONTINUED—

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	-0.1600	-0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

12. $f(x) = \frac{1}{2}x^4 - 3x - 3$

$f'(x) = 2x^3 - 3$

Approximation of the zero of f is -0.8937.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1	0.5	-5	-0.1	-0.9
2	-0.9	0.0281	-4.458	-0.0063	-0.8937
3	-0.8937	0.0001	-4.4276	0.0000	-0.8937

Approximation of the zero of f is 2.0720.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2	-1	13	-0.0769	2.0769
2	2.0769	0.0725	14.9175	0.0049	2.0720
3	2.0720	-0.0003	14.7910	0.0000	2.0720

13. $f(x) = x + \sin(x + 1)$

$f'(x) = 1 + \cos(x + 1)$

Approximation of the zero of f is -0.489.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.0206	1.8776	-0.0110	-0.4890
2	-0.4890	0.0000	1.8723	0.0000	-0.4890

14. $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

Approximation of the zero of f is 0.866.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0000	3.0087	0.0000	0.8655

15. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \frac{1}{2\sqrt{x + 4}}$

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	0.0000	1.7661	0.0000	0.5687

16. $h(x) = f(x) - g(x) = 3 - x - \dfrac{1}{x^2 + 1}$

$h'(x) = -1 + \dfrac{2x}{(x^2 + 1)^2}$

Point of intersection of the graphs of f and g occurs when $x \approx 2.893$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	2.9000	−0.0063	−0.9345	0.0067	2.8933
2	2.8933	0.0000	−0.9341	0.0000	2.8933

17. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

Note: $f(x) = x$ and $g(x) = \tan x$ intersect infinitely often.

n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	4.5000	−0.1373	−21.5048	0.0064	4.4936
2	4.4936	−0.0039	−20.2271	0.0002	4.4934

18. $h(x) = f(x) - g(x) = x^2 - \cos x$

$h'(x) = 2x + \sin x$

One point of intersection of the graphs of f and g occurs when $x \approx 0.824$. Since $f(x) = x^2$ and $g(x) = \cos x$ are both symmetric with respect to the y-axis, the other point of intersection occurs when $x \approx -0.824$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	0.8000	−0.0567	2.3174	−0.0245	0.8245
2	0.8245	0.0009	2.3832	0.0004	0.8241

19. (a) $f(x) = x^2 - a,\ a > 0$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) $\sqrt{5}$: $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{5}{x_n}\right)$. $x_1 = 2$

n	1	2	3	4
x_n	2	2.25	2.2361	2.2361

For example, given $x_1 = 2$,

$$x_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{9}{4} = 2.25.$$

$\sqrt{5} \approx 2.236$

$\sqrt{7}$: $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{7}{x_n}\right)$, $x_1 = 2$

n	1	2	3	4	5
x_n	2	2.75	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

20. (a) $f(x) = x^n - a,\ a > 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^n - a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

(b) $\sqrt[4]{6}$: $x_{i+1} = \dfrac{3x_i^4 + 6}{4x_i^3}$, $x_1 = 1.5$

i	1	2	3	4
x_i	1.5	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

$\sqrt[3]{15}$: $x_{i+1} = \dfrac{2x_i^3 + 15}{3x_i^2}$, $x_1 = 2.5$

i	1	2	3	4
x_i	2.5	2.4667	2.4662	2.4662

$\sqrt[3]{15} \approx 2.466$

21. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

22. $y = 4x^3 - 12x^2 + 12x - 3 = f(x)$

$y' = 12x^2 - 24x + 12 = f'(x)$

$x_1 = \frac{3}{2}$

$f'(x_2) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	$\frac{3}{2}$	$\frac{3}{2}$	3	$\frac{1}{2}$	1
2	1	1	0	—	—

23. $y = -x^3 + 6x^2 - 10x + 6 = f(x)$

$y' = -3x^2 + 12x - 10 = f'(x)$

$x_1 = 2$

$x_2 = 1$

$x_3 = 2$

$x_4 = 1$ and so on.

Fails to converge

24. $f(x) = 2 \sin x + \cos 2x$

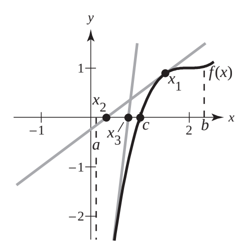
$f'(x) = 2 \cos x - 2 \sin 2x$

$x_1 = \frac{3\pi}{2}$

Fails because $f'(x_1) = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
1	$\frac{3\pi}{2}$	-3	0

25. Answers will vary. See page 222.



Newton's Method uses tangent lines to approximate c such that $f(c) = 0$.

First, estimate an initial x_1 close to c (see graph).

Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let x_{n+1} be the final approximation of c.

26. Newton's Method could fail if $f'(c) \approx 0$, or if the initial value x_1 is far from c.

27. Let $g(x) = f(x) - x = \cos x - x$

$g'(x) = -\sin x - 1.$

The fixed point is approximately 0.74.

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.4597	-1.8415	0.2496	0.7504
2	0.7504	-0.0190	-1.6819	0.0113	0.7391
3	0.7391	0.0000	-1.6736	0.0000	0.7391

28. Let $g(x) = f(x) - x = \cot x - x$

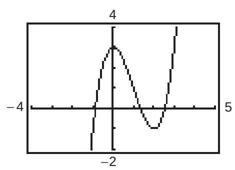
$g'(x) = -\csc^2 x - 1.$

The fixed point is approximately 0.86.

n	x_n	$g(x_n)$	$g'(x_n)$	$\dfrac{g(x_n)}{g'(x_n)}$	$x_n - \dfrac{g(x_n)}{g'(x_n)}$
1	1.0000	-0.3579	-2.4123	0.1484	0.8516
2	0.8516	0.0240	-2.7668	-0.0087	0.8603
3	0.8603	0.0001	-2.7403	0.0000	0.8603

29. $f(x) = x^3 - 3x^2 + 3,\ \ f'(x) = 3x^2 - 6x$

(a)



(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

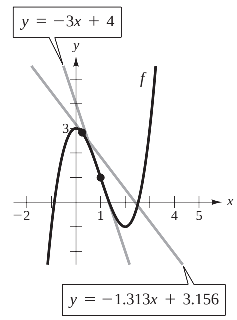
Continuing, the zero is 1.347.

(c) $x_1 = \dfrac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

(d)

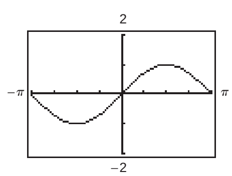


The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

30. $f(x) = \sin x,\ f'(x) = \cos x$

(a)



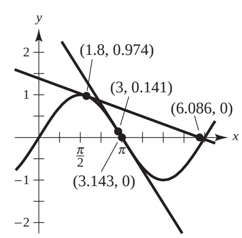
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)



The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

31. $f(x) = \dfrac{1}{x} - a = 0$

$f'(x) = -\dfrac{1}{x^2}$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

32. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$\frac{1}{3} \approx 0.333$

(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$\frac{1}{11} \approx 0.091$

33. $f(x) = x \cos x,\ (0, \pi)$

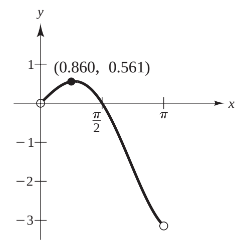
$f'(x) = -x \sin x + \cos x = 0$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$[F'(x) = -2 \sin x + x \cos x]$

n	x_n	$F(x_n)$	$F'(x_n)$	$\dfrac{F(x_n)}{F'(x_n)}$	$x_n - \dfrac{F(x_n)}{F'(x_n)}$
1	0.9000	−0.0834	−2.1261	0.0392	0.8608
2	0.8608	−0.0010	−2.0778	0.0005	0.8603

Approximation to the critical number: 0.860



34. $f(x) = x \sin x,\ (0, \pi)$

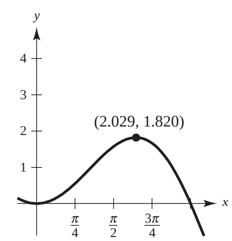
$f'(x) = x \cos x + \sin x = 0$

Letting $F(x) = f'(x)$, we can use Newton's Method as follows.

$[F'(x) = 2 \cos x - x \sin x]$

n	x_n	$F(x_n)$	$F'(x_n)$	$\dfrac{F(x_n)}{F'(x_n)}$	$x_n - \dfrac{F(x_n)}{F'(x_n)}$
1	2.0000	0.0770	−2.6509	−0.0290	2.0290
2	2.0290	−0.0007	−2.7044	0.0002	2.0288

Approximation to the critical number: 2.029



35. $y = f(x) = 4 - x^2,\ (1, 0)$

$d = \sqrt{(x-1)^2 + (y-0)^2} = \sqrt{(x-1)^2 + (4-x^2)^2} = \sqrt{x^4 - 7x^2 - 2x + 17}$

d is minimized when $D = x^4 - 7x^2 - 2x + 17$ is a minimum.

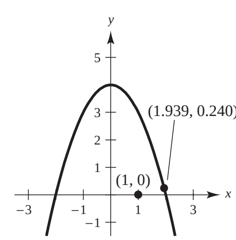
$g(x) = D' = 4x^3 - 14x - 2$

$g'(x) = 12x^2 - 14$

n	x_n	$g(x_n)$	$g'(x_n)$	$\dfrac{g(x_n)}{g'(x_n)}$	$x_n - \dfrac{g(x_n)}{g'(x_n)}$
1	2.0000	2.0000	34.0000	0.0588	1.9412
2	1.9412	0.0830	31.2191	0.0027	1.9385
3	1.9385	−0.0012	31.0934	0.0000	1.9385

$x \approx 1.939$

Point closest to $(1, 0)$ is $\approx (1.939, 0.240)$.

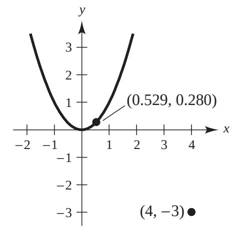


36. $y = f(x) = x^2, \ (4, -3)$

$d = \sqrt{(x-4)^2 + (y+3)^2} = \sqrt{(x-4)^2 + (x^2+3)^2} = \sqrt{x^4 + 7x^2 - 8x + 25}$

d is minimum when $D = x^4 + 7x^2 - 8x + 25$ is minimum.

$$g(x) = D' = 4x^3 + 14x - 8$$
$$g'(x) = 12x^2 + 14$$



n	x_n	$g(x_n)$	$g'(x_n)$	$\dfrac{g(x_n)}{g'(x_n)}$	$x_n - \dfrac{g(x_n)}{g'(x_n)}$
1	0.5000	−0.5000	17.0000	−0.0294	0.5294
2	0.5294	0.0051	17.3632	0.0003	0.5291
3	0.5291	−0.0000	17.3594	0.0000	0.5291

$x \approx 0.529$

Point closest to $(4, -3)$ is approximately $(0.529, 0.280)$.

37. Minimize: $T = \dfrac{\text{Distance rowed}}{\text{Rate rowed}} + \dfrac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2+4}}{3} + \frac{\sqrt{x^2-6x+10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2+4}} + \frac{x-3}{4\sqrt{x^2-6x+10}} = 0$$

$$4x\sqrt{x^2-6x+10} = -3(x-3)\sqrt{x^2+4}$$

$$16x^2(x^2-6x+10) = 9(x-3)^2(x^2+4)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216$. Since $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	−1.0480	150.2780	−0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ miles

38. Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{(50+t^3)^2} = 0$$

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

Since $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

Approximation: $t \approx 4.486$ hours

39. $2{,}500{,}000 = -76x^3 + 4830x^2 - 320{,}000$

$76x^3 - 4830x^2 + 2{,}820{,}000 = 0$

Let $f(x) = 76x^3 - 4830x^2 + 2{,}820{,}000$

$f'(x) = 228x^2 - 9660x.$

From the graph, choose $x_1 = 40$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	40.0000	−44000.0000	−21600.0000	2.0370	37.9630
2	37.9630	17157.6209	−38131.4039	−0.4500	38.4130
3	38.4130	780.0914	−34642.2263	−0.0225	38.4355
4	38.4355	2.6308	−34465.3435	−0.0001	38.4356

The zero occurs when $x \approx 38.4356$ which corresponds to \$384,356. (The larger zero is $x \approx 46.071$)

40. $170 = 0.808x^3 - 17.974x^2 + 71.248x + 110.843,\ 1 \le x \le 5$

Let $f(x) = 0.808x^3 - 17.974x^2 + 71.248x - 59.157$

$f'(x) = 2.424x^2 - 35.948x + 71.248.$

From the graph, choose $x_1 = 1$ and $x_1 = 3.5$. Apply Newton's Method.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.0000	−5.0750	37.7240	−0.1345	1.1345
2	1.1345	−0.2805	33.5849	−0.0084	1.1429
3	1.1429	0.0006	33.3293	0.0000	1.1429

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	3.5000	4.6725	−24.8760	−0.1878	3.6878
2	3.6878	−0.3286	−28.3550	0.0116	3.6762
3	3.6762	−0.0009	−28.1450	0.0000	3.6762

The zeros occur when $x \approx 1.1429$ and $x \approx 3.6762$. These approximately correspond to engine speeds of 1143 rev/min and 3676 rev/min.

41. False. Let $f(x) = (x^2 - 1)/(x - 1)$. $x = 1$ is a discontinuity. It is not a zero of $f(x)$. This statement would be true if $f(x) = p(x)/q(x)$ is given in **reduced** form.

42. True

43. True

44. True

45. $f(x) = -\sin x$

$f'(x) = -\cos x$

Let $(x_0, y_1) = (x_0, -\sin(x_0))$ be a point on the graph of f. If (x_0, y_0) is a point of tangency, then

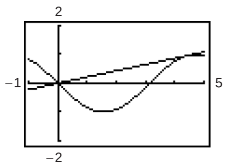
$$-\cos(x_0) = \frac{y_0 - 0}{x_0 - 0} = \frac{y_0}{x_0} = \frac{-\sin(x_0)}{x_0}.$$

Thus, $x_0 = \tan(x_0)$.

$x_0 \approx 4.4934$

Slope $= -\cos(x_0) \approx 0.217$

You can verify this answer by graphing $y_1 = -\sin x$ and the tangent line $y_2 = 0.217x$.



46. $f(x) = 2x^3 - 20x^2 - 12x - 24$

(a) There is one real root between 10 and 11.

(b) Using Newton's Method and $x_1 = 2$, the first few approximations are very poor.

n	1	2	3	$\cdots$	14	15	16	17
x_n	2	0.3529	-0.8547	$\cdots$	10.8270	10.6723	10.6679	10.6679

zero ≈ 10.6679

(c) Using $x_1 = 10$,

n	1	2	3	4	5
x_n	10	10.7660	10.6696	10.6679	10.6679

zero ≈ 10.6679

Using $x_1 = 100$, $x_{11} \approx 10.6679$.

(d) The convergence is faster if you select a starting value x_1 close to the actual zero.

Section 3.9 Differentials

1. $f(x) = x^2$

$f'(x) = 2x$

Tangent line at $(2, 4)$: $y - f(2) = f'(2)(x - 2)$

$$y - 4 = 4(x - 2)$$

$$y = 4x - 4$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

2. $f(x) = \dfrac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \dfrac{-12}{x^3}$

Tangent line at $\left(2, \dfrac{3}{2}\right)$:

$$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$$

$$y = -\frac{3}{2}x + \frac{9}{2}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \dfrac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\dfrac{3}{2}x + \dfrac{9}{2}$	1.65	1.515	1.5	1.485	1.35

3. $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at $(2, 32)$: $y - f(2) = f'(2)(x - 2)$

$$y - 32 = 80(x - 2)$$

$$y = 80x - 128$$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

4. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $\left(2, \sqrt{2}\right)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sqrt{2} = \frac{1}{2\sqrt{2}}(x - 2)$$

$$y = \frac{x}{2\sqrt{2}} + \frac{1}{\sqrt{2}}$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \dfrac{x}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

5. $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at $(2, \sin 2)$:

$$y - f(2) = f'(2)(x - 2)$$

$$y - \sin 2 = (\cos 2)(x - 2)$$

$$y = (\cos 2)(x - 2) + \sin 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8632
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8677

6. $f(x) = \csc x$

$f'(x) = -\csc x \cot x$

Tangent line at $(2, \csc 2)$: $y - f(2) = f'(2)(x - 2)$

$$y - \csc 2 = (-\csc 2 \cot 2)(x - 2)$$

$$y = (-\csc 2 \cot 2)(x - 2) + \csc 2$$

x	1.9	1.99	2	2.01	2.1
$f(x) = \csc x$	1.0567	1.0948	1.0998	1.1049	1.1585
$T(x) = (-\csc 2 \cot 2)(x - 2) + \csc 2$	1.0494	1.0947	1.0998	1.1048	1.1501

7. $y = f(x) = \frac{1}{2}x^3, f'(x) = \frac{3}{2}x^2, x = 2, \Delta x = dx = 0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(2.1) - f(2)\\ &= 0.6305\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx\\ &= f'(2)(0.1)\\ &= 6(0.1) = 0.6\end{aligned}$$

8. $y = f(x) = 1 - 2x^2, f'(x) = -4x, x = 0, \Delta x = dx = -0.1$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(-0.1) - f(0)\\ &= [1 - 2(-0.1)^2] - [1 - 2(0)^2] = -0.02\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx\\ &= f'(0)(-0.1)\\ &= (0)(-0.1) = 0\end{aligned}$$

9. $y = f(x) = x^4 + 1, f'(x) = 4x^3, x = -1, \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(-0.99) - f(-1)\\ &= [(-0.99)^4 + 1] - [(-1)^4 + 1] \approx -0.0394\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx\\ &= f'(-1)(0.01)\\ &= (-4)(0.01) = -0.04\end{aligned}$$

10. $y = f(x) = 2x + 1, f'(x) = 2, x = 2, \Delta x = dx = 0.01$

$$\begin{aligned}\Delta y &= f(x + \Delta x) - f(x)\\ &= f(2.01) - f(2)\\ &= [2(2.01) + 1] - [2(2) + 1] = 0.02\end{aligned}$$

$$\begin{aligned}dy &= f'(x)\,dx\\ &= f'(2)(0.01)\\ &= 2(0.01) = 0.02\end{aligned}$$

11. $y = 3x^2 - 4$

$dy = 6x\,dx$

12. $y = 3x^{2/3}$

$dy = 2x^{-1/3}\,dx = \dfrac{2}{x^{1/3}}\,dx$

13. $y = \dfrac{x + 1}{2x - 1}$

$dy = \dfrac{-3}{(2x - 1)^2}\,dx$

14. $y = \sqrt{9 - x^2}$

$dy = \dfrac{1}{2}(9 - x^2)^{-1/2}(-2x)\,dx = \dfrac{-x}{\sqrt{9 - x^2}}\,dx$

15. $y = x\sqrt{1 - x^2}$

$dy = \left(x\dfrac{-x}{\sqrt{1 - x^2}} + \sqrt{1 - x^2}\right)dx = \dfrac{1 - 2x^2}{\sqrt{1 - x^2}}\,dx$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$dy = \left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}\right)dx = \dfrac{x - 1}{2x\sqrt{x}}\,dx$

17. $y = 2x - \cot^2 x$

$$\begin{aligned}dy &= (2 + 2\cot x\csc^2 x)\,dx\\ &= (2 + 2\cot x + 2\cot^3 x)\,dx\end{aligned}$$

18. $y = x\sin x$

$dy = (x\cos x + \sin x)\,dx$

19. $y = \frac{1}{3}\cos\left(\frac{6\pi x - 1}{2}\right)$

$dy = -\pi \sin\left(\frac{6\pi x - 1}{2}\right) dx$

20. $y = \frac{\sec^2 x}{x^2 + 1}$

$dy = \left[\frac{(x^2 + 1)2\sec^2 x \tan x - \sec^2 x(2x)}{(x^2 + 1)^2}\right] dx$

$= \left[\frac{2\sec^2 x(x^2 \tan x + \tan x - x)}{(x^2 + 1)^2}\right] dx$

21. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + (1)(0.04) = 1.04$

22. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + (-1)(-0.1) = 1.1$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + (-1)(0.04) = 0.96$

23. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + \left(-\frac{1}{2}\right)(-0.1) = 1.05$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + \left(-\frac{1}{2}\right)(0.04) = 0.98$

24. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + 0(-0.1) = 1$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$\approx 1 + 0(0.04) = 1$

25. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$\approx 8 + \left(-\frac{1}{2}\right)(-0.07) = 8.035$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$\approx 8 + \left(-\frac{1}{2}\right)(0.1) = 7.95$

26. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$\approx 8 + (3)(-0.07) = 7.79$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$\approx 8 + (3)(0.1) = 8.3$

27. $A = x^2$

$x = 12$

$\Delta x = dx = \pm\frac{1}{64}$

$dA = 2x\,dx$

$\Delta A \approx dA = 2(12)\left(\pm\frac{1}{64}\right)$

$= \pm\frac{3}{8}$ square inches

28. $A = \frac{1}{2}bh$, $b = 36$, $h = 50$

$db = dh = \pm 0.25$

$dA = \frac{1}{2}b\,dh + \frac{1}{2}h\,db$

$\Delta A \approx dA = \frac{1}{2}(36)(\pm 0.25) + \frac{1}{2}(50)(\pm 0.25)$

$= \pm 10.75$ square centimeters

29. $A = \pi r^2$

$r = 14$

$\Delta r = dr = \pm\frac{1}{4}$

$\Delta A \approx dA = 2\pi r\,dr = \pi(28)\left(\pm\frac{1}{4}\right)$

$= \pm 7\pi$ square inches

30. $x = 12$ inches

$\Delta x = dx = \pm 0.03$ inch

(a) $V = x^3$

$dV = 3x^2\,dx = 3(12)^2(\pm 0.03)$

$= \pm 12.96$ cubic inches

(b) $S = 6x^2$

$dS = 12x\,dx = 12(12)(\pm 0.03)$

$= \pm 4.32$ square inches

31. (a) $x = 15$ centimeters

$\Delta x = dx = \pm 0.05$ centimeters

$A = x^2$

$dA = 2x\,dx = 2(15)(\pm 0.05)$

$= \pm 1.5$ square centimeters

Percentage error:

$$\frac{dA}{A} = \frac{1.5}{(15)^2} = 0.00666\ldots = \frac{2}{3}\%$$

(b) $\dfrac{dA}{A} = \dfrac{2x\,dx}{x^2} = \dfrac{2\,dx}{x} \le 0.025$

$$\frac{dx}{x} \le \frac{0.025}{2} = 0.0125 = 1.25\%$$

32. (a) $C = 56$ centimeters

$\Delta C = dC = \pm 1.2$ centimeters

$$C = 2\pi r \Longrightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\,dC = \frac{1}{2\pi}(56)(\pm 1.2) = \frac{33.6}{\pi}$$

$$\frac{dA}{A} = \frac{33.6/\pi}{[1/(4\pi)](56)^2} \approx 0.042857 = 4.2857\%$$

(b) $\dfrac{dA}{A} = \dfrac{(1/2\pi)C\,dC}{(1/4\pi)C^2} = \dfrac{2\,dC}{C} \le 0.03$

$$\frac{dC}{C} \le \frac{0.03}{2} = 0.015 = 1.5\%$$

33. $r = 6$ inches

$\Delta r = dr = \pm 0.02$ inches

(a) $V = \dfrac{4}{3}\pi r^3$

$dV = 4\pi r^2\,dr = 4\pi(6)^2(\pm 0.02) = \pm 2.88\pi$ cubic inches

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr = 8\pi(6)(\pm 0.02) = \pm 0.96\pi$ square inches

(c) Relative error: $\dfrac{dV}{V} = \dfrac{4\pi r^2\,dr}{(4/3)\pi r^3} = \dfrac{3\,dr}{r}$

$$= \frac{3}{6}(0.02) = 0.01 = 1\%$$

Relative error: $\dfrac{dS}{S} = \dfrac{8\pi r\,dr}{4\pi r^2} = \dfrac{2\,dr}{r}$

$$= \frac{2(0.02)}{6} = 0.00666\ldots = \frac{2}{3}\%$$

34. $P = (500x - x^2) - \left(\frac{1}{2}x^2 - 77x + 3000\right)$, x changes from 115 to 120

$dP = (500 - 2x - x + 77)\,dx = (577 - 3x)\,dx = [577 - 3(115)](120 - 115) = 1160$

Approximate percentage change: $\dfrac{dP}{P}(100) = \dfrac{1160}{43517.50}(100) \approx 2.7\%$

35. $V = \pi r^2 h = 40\pi r^2$, $r = 5$ cm, $h = 40$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 80\pi r\,dr = 80\pi(5)(0.2) = 80\pi\text{ cm}^3$

36. $V = \frac{4}{3}\pi r^3$, $r = 100$ cm, $dr = 0.2$ cm

$\Delta V \approx dV = 4\pi r^2\,dr = 4\pi(100)^2(0.2) = 8000\pi\text{ cm}^3$

37. (a) $T = 2\pi\sqrt{L/g}$

$$dT = \frac{\pi}{g\sqrt{L/g}}\,dL$$

Relative error:

$$\frac{dT}{T} = \frac{(\pi\,dL)/\left(g\sqrt{L/g}\right)}{2\pi\sqrt{L/g}}$$

$$= \frac{dL}{2L}$$

$$= \frac{1}{2}(\text{relative error in } L)$$

$$= \frac{1}{2}(0.005) = 0.0025$$

Percentage error: $\frac{dT}{T}(100) = 0.25\% = \frac{1}{4}\%$

(b) $(0.0025)(3600)(24) = 216$ seconds

$= 3.6$ minutes

38. $E = IR$

$$R = \frac{E}{I}$$

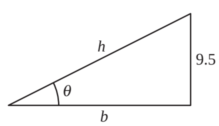
$$dR = -\frac{E}{I^2}dI$$

$$\frac{dR}{R} = \frac{-(E/I^2)dI}{E/I} = -\frac{dI}{I}$$

$$\left|\frac{dR}{R}\right| = \left|-\frac{dI}{I}\right| = \left|\frac{dI}{I}\right|$$

39. $\theta = 26°45' = 26.75°$

$d\theta = \pm 15' = \pm 0.25°$



(a) $h = 9.5 \csc\theta$

$$dh = -9.5 \csc\theta \cot\theta\, d\theta$$

$$\frac{dh}{h} = -\cot\theta\, d\theta$$

$$\left|\frac{dh}{h}\right| = (\cot 26.75°)(0.25°)$$

Converting to radians,
$(\cot 0.4669)(0.0044) \approx 0.0087 = 0.87\%$ (in radians).

(b) $\left|\frac{dh}{h}\right| = \cot\theta\, d\theta \le 0.02$

$$\frac{d\theta}{\theta} \le \frac{0.02}{\theta(\cot\theta)} = \frac{0.02\tan\theta}{\theta}$$

$$\frac{d\theta}{\theta} \le \frac{0.02\tan 26.75°}{26.75°} \approx \frac{0.02\tan 0.4669}{0.4669}$$

$\approx 0.0216 = 2.16\%$ (in radians)

40. See Exercise 41.

$$A = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(9.5\cot\theta)(9.5) = 45.125\cot\theta$$

$$dA = -45.125\csc^2\theta\, d\theta$$

$$\left|\frac{dA}{A}\right| = \frac{\csc^2\theta\, d\theta}{\cot\theta} = \frac{d\theta}{\sin\theta\cos\theta}$$

$$= \frac{0.25°}{(\sin 26.75°)(\cos 26.75°)}$$

$$\approx \frac{0.0044}{(\sin 0.4669)(\cos 0.4669)}$$

$\approx 0.0109 = 1.09\%$ (in radians)

41. $r = \frac{v_0^{\,2}}{32}(\sin 2\theta)$

$v_0 = 2200$ ft/sec

θ changes from $10°$ to $11°$.

$$dr = \frac{(2200)^2}{16}(\cos 2\theta)\, d\theta$$

$$\theta = 10\left(\frac{\pi}{180}\right)$$

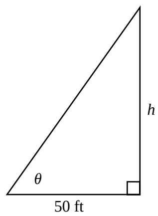
$$d\theta = (11 - 10)\frac{\pi}{180}$$

$$\Delta r \approx dr$$

$$= \frac{(2200)^2}{16}\cos\left(\frac{20\pi}{180}\right)\left(\frac{\pi}{180}\right)$$

≈ 4961 feet

42. $h = 50 \tan \theta$



$$\theta = 71.5° = 1.2479 \text{ radians}$$

$$dh = 50 \sec^2 \theta \cdot d\theta$$

$$\left|\frac{dh}{x}\right| = \left|\frac{50 \sec^2(1.2479)}{50 \tan(1.2479)} d\theta\right| \le 0.06$$

$$\left|\frac{9.9316}{2.9886} d\theta\right| \le 0.06$$

$$|d\theta| \le 0.018$$

43. Let $f(x) = \sqrt{x}$, $x = 100$, $dx = -0.6$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}\,dx$$

$$f(x + \Delta x) = \sqrt{99.4}$$

$$\approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

44. Let $f(x) = \sqrt[3]{x}$, $x = 27$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}\,dx$$

$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

45. Let $f(x) = \sqrt[4]{x}$, $x = 625$, $dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}\,dx$$

$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1)$$

$$= 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

46. Let $f(x) = x^3$, $x = 3$, $dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2\,dx$$

$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01)$$

$$= 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

47. Let $f(x) = \sqrt{x}$, $x = 4$, $dx = 0.02$, $f'(x) = 1/\left(2\sqrt{x}\right)$.

Then

$$f(4.02) \approx f(4) + f'(4)\,dx$$

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$$

48. Let $f(x) = \tan x$, $x = 0$, $dx = 0.05$, $f'(x) = \sec^2 x$.

Then

$$f(0.05) \approx f(0) + f'(0)\,dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

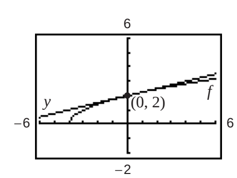
49. $f(x) = \sqrt{x + 4}$

$$f'(x) = \frac{1}{2\sqrt{x + 4}}$$

At $(0, 2)$, $f(0) = 2$, $f'(0) = \dfrac{1}{4}$

Tangent line: $y - 2 = \dfrac{1}{4}(x - 0)$

$$y = \frac{1}{4}x + 2$$



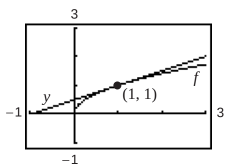
50. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

$f(1) = 1,\ f(1) = \dfrac{1}{2}$

Tangent line at $(1, 1)$: $y - 1 = \dfrac{1}{2}(x - 1)$

$$y = \frac{1}{2}x + \frac{1}{2}$$



51. $f(x) = \tan x$

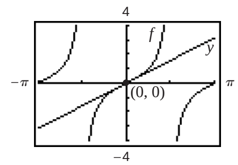
$f'(x) = \sec^2 x$

$f(0) = 0$

$f'(0) = 1$

Tangent line at $(0, 0)$: $y - 0 = (x - 0)$

$$y = x$$



52. $f(x) = \dfrac{1}{1 - x} = (1 - x)^{-1}$

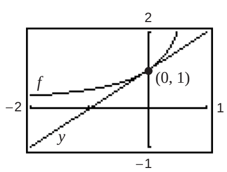
$f'(x) = \dfrac{1}{(1 - x)^2}$

$f(0) = 1$

$f'(0) = 1$

Tangent line at $(0, 1)$: $y - 1 = 1(x - 0)$

$$y = x + 1$$



53. In general, when $\Delta x \to 0$, dy approaches Δy.

54. Propagated error $= f(x + \Delta x) - f(x)$,

relative error $= \left|\dfrac{dy}{y}\right|$, and the percent error $= \left|\dfrac{dy}{y}\right| \times 100$.

55. True

56. True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

57. True

58. False

Let $f(x) = \sqrt{x}$, $x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

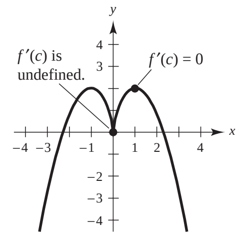
and

$$dy = f'(x)\,dx = \frac{1}{2\sqrt{1}}(3) = \frac{3}{2}.$$

Thus, $dy > \Delta y$ in this example.

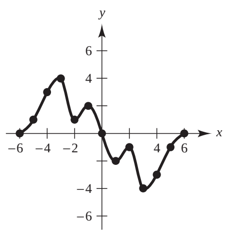
Review Exercises for Chapter 3

1. A number c in the domain of f is a critical number if $f'(c) = 0$ or f' is undefined at c.



2. (a) $f(4) = -f(-4) = -3$

(c)



At least six critical numbers on $(-6, 6)$.

(b) $f(-3) = -f(3) = -(-4) = 4$

(d) Yes. Since $f(-2) = -f(2) = -(-1) = 1$ and $f(1) = -f(-1) = -2$, the Mean Value says that there exists at least one value c in $(-2, 1)$ such that

$$f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{-2 - 1}{1 + 2} = -1.$$

(e) No, $\lim_{x \to 0} f(x)$ exists because f is continuous at $(0, 0)$.

(f) Yes, f is differentiable at $x = 2$.

3. $g(x) = 2x + 5\cos x, [0, 2\pi]$

$g'(x) = 2 - 5\sin x$

$\quad = 0$ when $\sin x = \frac{2}{5}$.

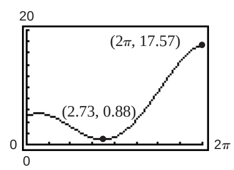
Critical numbers: $x \approx 0.41, x \approx 2.73$

Left endpoint: $(0, 5)$

Critical number: $(0.41, 5.41)$

Critical number: $(2.73, 0.88)$ Minimum

Right endpoint: $(2\pi, 17.57)$ Maximum



4. $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}, [0, 2]$

$$f'(x) = x\left[-\frac{1}{2}(x^2 + 1)^{-3/2}(2x)\right] + (x^2 + 1)^{-1/2}$$

$$= \frac{1}{(x^2 + 1)^{3/2}}$$

No critical numbers

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $\left(2, 2/\sqrt{5}\right)$ Maximum

5. Yes. $f(-3) = f(2) = 0$. f is continuous on $[-3, 2]$, differentiable on $(-3, 2)$.

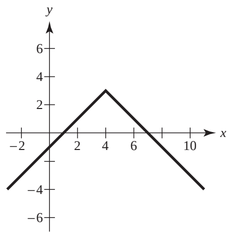
$f'(x) = (x + 3)(3x - 1) = 0$ for $x = \frac{1}{3}$.

$c = \frac{1}{3}$ satisfies $f'(c) = 0$.

6. No. f is not differentiable at $x = 2$.

7. $f(x) = 3 - |x - 4|$

(a)



$f(1) = f(7) = 0$

(b) f is not differentiable at $x = 4$.

8. No; the function is discontinuous at $x = 0$ which is in the interval $[-2, 1]$.

9.
$$f(x) = x^{2/3},\ 1 \le x \le 8$$
$$f'(x) = \frac{2}{3}x^{-1/3}$$
$$\frac{f(b) - f(a)}{b - a} = \frac{4 - 1}{8 - 1} = \frac{3}{7}$$
$$f'(c) = \frac{2}{3}c^{-1/3} = \frac{3}{7}$$
$$c = \left(\frac{14}{9}\right)^3 = \frac{2744}{729} \approx 3.764$$

10.
$$f(x) = \frac{1}{x},\ 1 \le x \le 4$$
$$f'(x) = -\frac{1}{x^2}$$
$$\frac{f(b) - f(a)}{b - a} = \frac{(1/4) - 1}{4 - 1} = \frac{-3/4}{3} = -\frac{1}{4}$$
$$f'(c) = \frac{-1}{c^2} = -\frac{1}{4}$$
$$c = 2$$

11.
$$f(x) = x - \cos x,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$
$$f'(x) = 1 + \sin x$$
$$\frac{f(b) - f(a)}{b - a} = \frac{(\pi/2) - (-\pi/2)}{(\pi/2) - (-\pi/2)} = 1$$
$$f'(c) = 1 + \sin c = 1$$
$$c = 0$$

12.
$$f(x) = \sqrt{x} - 2x,\ 0 \le x \le 4$$
$$f'(x) = \frac{1}{2\sqrt{x}} - 2$$
$$\frac{f(b) - f(a)}{b - a} = \frac{-6 - 0}{4 - 0} = -\frac{3}{2}$$
$$f'(c) = \frac{1}{2\sqrt{c}} - 2 = -\frac{3}{2}$$
$$c = 1$$

13.
$$f(x) = Ax^2 + Bx + C$$
$$f'(x) = 2Ax + B$$
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{A(x_2{}^2 - x_1{}^2) + B(x_2 - x_1)}{x_2 - x_1}$$
$$= A(x_1 + x_2) + B$$
$$f'(c) = 2Ac + B = A(x_1 + x_2) + B$$
$$2Ac = A(x_1 + x_2)$$
$$c = \frac{x_1 + x_2}{2} = \text{Midpoint of } [x_1, x_2]$$

14.
$$f(x) = 2x^2 - 3x + 1$$
$$f'(x) = 4x - 3$$
$$\frac{f(b) - f(a)}{b - a} = \frac{21 - 1}{4 - 0} = 5$$
$$f'(c) = 4c - 3 = 5$$
$$c = 2 = \text{Midpoint of } [0, 4]$$

15. $f(x) = (x - 1)^2(x - 3)$

$f'(x) = (x - 1)^2(1) + (x - 3)(2)(x - 1)$

$= (x - 1)(3x - 7)$

Critical numbers: $x = 1$ and $x = \frac{7}{3}$

Interval:	$-\infty < x < 1$	$1 < x < \frac{7}{3}$	$\frac{7}{3} < x < \infty$
Sign of $f'(x)$:	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion:	Increasing	Decreasing	Increasing

16. $g(x) = (x + 1)^3$

$g'(x) = 3(x + 1)^2$

Critical number: $x = -1$

Interval	$-\infty < x < -1$	$-1 < x < \infty$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) > 0$
Conclusion	Increasing	Increasing

17. $h(x) = \sqrt{x}(x - 3) = x^{3/2} - 3x^{1/2}$

Domain: $(0, \infty)$

$$h'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2}$$

$$= \frac{3}{2}x^{-1/2}(x - 1) = \frac{3(x - 1)}{2\sqrt{x}}$$

Critical number: $x = 1$

Interval:	$0 < x < 1$	$1 < x < \infty$
Sign of $h'(x)$:	$h'(x) < 0$	$h'(x) > 0$
Conclusion:	Decreasing	Increasing

18. $f(x) = \sin x + \cos x, \quad 0 \le x \le 2\pi$

$f'(x) = \cos x - \sin x$

Critical numbers: $x = \dfrac{\pi}{4}, x = \dfrac{5\pi}{4}$

Interval	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < 2\pi$
Sign of $f'(x)$	$f'(x) > 0$	$f'(x) < 0$	$f'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

19. $h(t) = \frac{1}{4}t^4 - 8t$

$h'(t) = t^3 - 8 = 0$ when $t = 2$.

Relative minimum: $(2, -12)$

Test Interval:	$-\infty < t < 2$	$2 < t < \infty$
Sign of $h'(t)$:	$h'(t) < 0$	$h'(t) > 0$
Conclusion:	Decreasing	Increasing

20. $g(x) = \dfrac{3}{2}\sin\left(\dfrac{\pi x}{2} - 1\right), \ [0, 4]$

$$g'(x) = \frac{3}{2}\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi x}{2} - 1\right)$$

$$= 0 \text{ when } x = 1 + \frac{2}{\pi}, 3 + \frac{2}{\pi}$$

Relative maximum: $\left(1 + \dfrac{2}{\pi}, \dfrac{3}{2}\right)$

Relative minimum: $\left(3 + \dfrac{2}{\pi}, -\dfrac{3}{2}\right)$

Test Interval	$0 < x < 1 + \frac{2}{\pi}$	$1 + \frac{2}{\pi} < x < 3 + \frac{2}{\pi}$	$3 + \frac{2}{\pi} < x < 4$
Sign of $g'(x)$	$g'(x) > 0$	$g'(x) < 0$	$g'(x) > 0$
Conclusion	Increasing	Decreasing	Increasing

21. $y = \frac{1}{3}\cos(12t) - \frac{1}{4}\sin(12t)$

$v = y' = -4\sin(12t) - 3\cos(12t)$

(a) When $t = \frac{\pi}{8}$, $y = \frac{1}{4}$ inch and $v = y' = 4$ inches/second.

(b) $y' = -4\sin(12t) - 3\cos(12t) = 0$ when $\frac{\sin(12t)}{\cos(12t)} = -\frac{3}{4} \Rightarrow \tan(12t) = -\frac{3}{4}$.

Therefore, $\sin(12t) = -\frac{3}{5}$ and $\cos(12t) = \frac{4}{5}$. The maximum displacement is

$$y = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right) - \frac{1}{4}\left(-\frac{3}{5}\right) = \frac{5}{12} \text{ inch.}$$

(c) Period: $\frac{2\pi}{12} = \frac{\pi}{6}$

Frequency: $\frac{1}{\pi/6} = \frac{6}{\pi}$

22. (a) $y = A\sin\left(\sqrt{k/m}\,t\right) + B\cos\left(\sqrt{k/m}\,t\right)$

$y' = A\sqrt{k/m}\cos\left(\sqrt{k/m}\,t\right) - B\sqrt{k/m}\sin\left(\sqrt{k/m}\,t\right)$

$= 0$ when $\frac{\sin\sqrt{k/m}\,t}{\cos\sqrt{k/m}\,t} = \frac{A}{B} \Rightarrow \tan\left(\sqrt{k/m}\,t\right) = \frac{A}{B}$.

Therefore,

$$\sin\left(\sqrt{k/m}\,t\right) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\cos\left(\sqrt{k/m}\,t\right) = \frac{B}{\sqrt{A^2 + B^2}}.$$

When $v = y' = 0$,

$$y = A\left(\frac{A}{\sqrt{A^2 + B^2}}\right) + B\left(\frac{B}{\sqrt{A^2 + B^2}}\right) = \sqrt{A^2 + B^2}.$$

(b) Period: $\frac{2\pi}{\sqrt{k/m}}$

Frequency: $\frac{1}{2\pi/\sqrt{k/m}} = \frac{1}{2\pi}\sqrt{k/m}$

23. $f(x) = x + \cos x,\ 0 \le x \le 2\pi$

$f'(x) = 1 - \sin x$

$f''(x) = -\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

Points of inflection: $\left(\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$

Test Interval:	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
Sign of $f''(x)$:	$f''(x) < 0$	$f''(x) > 0$	$f''(x) < 0$
Conclusion:	Concave downward	Concave upward	Concave downward

24. $f(x) = (x + 2)^2(x - 4) = x^3 - 12x - 16$

$f'(x) = 3x^2 - 12$

$f''(x) = 6x = 0$ when $x = 0$.

Point of inflection: $(0, -16)$

Test Interval	$-\infty < x < 0$	$0 < x < \infty$
Sign of $f''(x)$	$f''(x) < 0$	$f''(x) > 0$
Conclusion	Concave downward	Concave upward

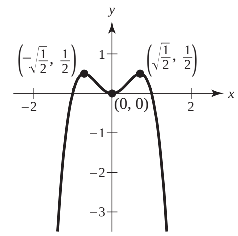
25. $g(x) = 2x^2(1 - x^2)$

$g'(x) = -4x(2x^2 - 1)$ Critical numbers: $x = 0, \pm\dfrac{1}{\sqrt{2}}$

$g''(x) = 4 - 24x^2$

$g''(0) = 4 > 0$ Relative minimum at $(0, 0)$

$g''\left(\pm\dfrac{1}{\sqrt{2}}\right) = -8 < 0$ Relative maximums at $\left(\pm\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$



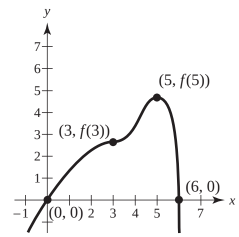
26. $h(t) = t - 4\sqrt{t + 1}$ Domain: $[-1, \infty)$

$h'(t) = 1 - \dfrac{2}{\sqrt{t + 1}} = 0 \implies t = 3$

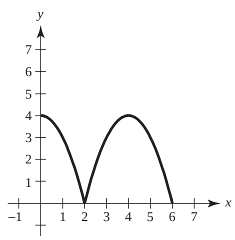
$h''(t) = \dfrac{1}{(t + 1)^{3/2}}$

$h''(3) = \dfrac{1}{8} > 0$ $(3, -5)$ is a relative minimum.

27.



28.



29. The first derivative is positive and the second derivative is negative. The graph is increasing and is concave down.

30. $C = \left(\dfrac{Q}{x}\right)s + \left(\dfrac{x}{2}\right)r$

$$\frac{dC}{dx} = -\frac{Qs}{x^2} + \frac{r}{2} = 0$$

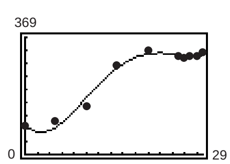
$$\frac{Qs}{x^2} = \frac{r}{2}$$

$$x^2 = \frac{2Qs}{r}$$

$$x = \sqrt{\frac{2Qs}{r}}$$

31. (a) $D = 0.0034t^4 - 0.2352t^3 + 4.9423t^2 - 20.8641t + 94.4025$

(b)



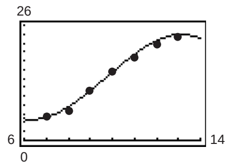
(c) Maximum at $(21.9, 319.5)$ (≈ 1992)

Minimum at $(2.6, 69.6)$ (≈ 1972)

(d) Outlays increasing at greatest rate at the point of inflection $(9.8, 173.7)$ (≈ 1979)

32. (a) $S = -0.1222t^3 + 3.565t^2 - 30.49t + 85.8,\ 7 \le t \le 13$

(b)



(c) $S'(t) = -0.3666t^2 + 7.13t - 30.49$

$S''(t) = -0.7332t + 7.13$

$S''(t) = 0 \Longrightarrow t \approx 9.7$ (1999)

(d) No, the coefficient of t^3 is negative.

33. $\displaystyle\lim_{x\to\infty} \frac{2x^2}{3x^2+5} = \lim_{x\to\infty} \frac{2}{3+5/x^2} = \frac{2}{3}$

34. $\displaystyle\lim_{x\to\infty} \frac{2x}{3x^2+5} = \lim_{x\to\infty} \frac{2/x}{3+5/x^2} = 0$

35. $\displaystyle\lim_{x\to-\infty} \frac{3x^2}{x+5} = -\infty$

36. $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2+x}}{-2x} = 1/2$

37. $\displaystyle\lim_{x\to\infty} \frac{5\cos x}{x} = 0$, since $|5\cos x| \le 5$.

38. $\displaystyle\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+4}} = \lim_{x\to\infty} \frac{3}{\sqrt{1+4/x^2}} = 3$

39. $\displaystyle\lim_{x\to-\infty} \frac{6x}{x+\cos x} = 6$

40. $\displaystyle\lim_{x\to-\infty} \frac{x}{2\sin x}$ does not exist.

41. $h(x) = \dfrac{2x+3}{x-4}$

Discontinuity: $x = 4$

$$\lim_{x\to\infty} \frac{2x+3}{x-4} = \lim_{x\to\infty} \frac{2+(3/x)}{1-(4/x)} = 2$$

Vertical asymptote: $x = 4$

Horizontal asymptote: $y = 2$

42. $g(x) = \dfrac{5x^2}{x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2}{x^2+2} = \lim_{x\to\infty} \frac{5}{1+(2/x^2)} = 5$$

Horizontal asymptote: $y = 5$

43. $f(x) = \dfrac{3}{x} - 2$

Discontinuity: $x = 0$

$$\lim_{x\to\infty} \left(\frac{3}{x} - 2\right) = -2$$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = -2$

44. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

$$\lim_{x\to\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{x\to\infty} \frac{3x/x}{\sqrt{x^2+2}/\sqrt{x^2}}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1+(2/x^2)}} = 3$$

$$\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2+2}} = \lim_{x\to-\infty} \frac{3x/x}{\sqrt{x^2+2}/\left(-\sqrt{x^2}\right)}$$

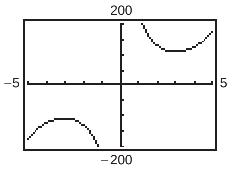
$$= \lim_{x\to-\infty} \frac{3}{-\sqrt{1+(2/x^2)}} = -3$$

Horizontal asymptotes: $y = \pm 3$

45. $f(x) = x^3 + \dfrac{243}{x}$

Relative minimum: $(3, 108)$

Relative maximum: $(-3, -108)$

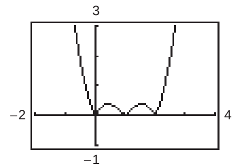


Vertical asymptote: $x = 0$

46. $f(x) = |x^3 - 3x^2 + 2x| = |x(x - 1)(x - 2)|$

Relative minima: $(0, 0), (1, 0), (2, 0)$

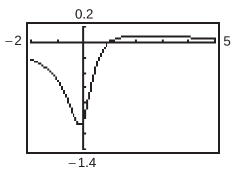
Relative maxima: $(1.577, 0.38), (0.423, 0.38)$



47. $f(x) = \dfrac{x - 1}{1 + 3x^2}$

Relative minimum: $(-0.155, -1.077)$

Relative maximum: $(2.155, 0.077)$

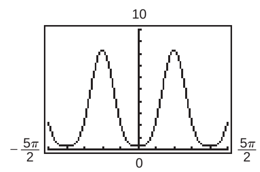


Horizontal asymptote: $y = 0$

48. $g(x) = \dfrac{\pi^2}{3} - 4\cos x + \cos 2x$

Relative minima: $(2\pi k, 0.29)$ where k is any integer.

Relative maxima: $((2k - 1)\pi, 8.29)$ where k is any integer.



49. $f(x) = 4x - x^2 = x(4 - x)$

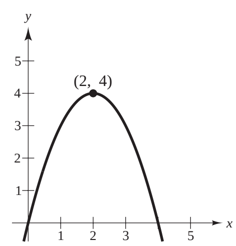
Domain: $(-\infty, \infty)$; Range: $(-\infty, 4]$

$f'(x) = 4 - 2x = 0$ when $x = 2$.

$f''(x) = -2$

Therefore, $(2, 4)$ is a relative maximum.

Intercepts: $(0, 0), (4, 0)$



50. $f(x) = 4x^3 - x^4 = x^3(4 - x)$

Domain: $(-\infty, \infty)$; Range: $(-\infty, 27)$

$f'(x) = 12x^2 - 4x^3 = 4x^2(3 - x) = 0$ when $x = 0, 3$.

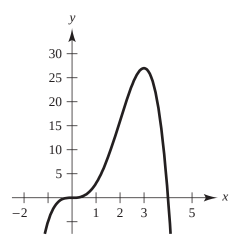
$f''(x) = 24x - 12x^2 = 12x(2 - x) = 0$ when $x = 0, 2$.

$f''(3) < 0$

Therefore, $(3, 27)$ is a relative maximum.

Points of inflection: $(0, 0), (2, 16)$

Intercepts: $(0, 0), (4, 0)$



51. $f(x) = x\sqrt{16 - x^2}$

Domain: $[-4, 4]$; Range: $[-8, 8]$

$f'(x) = \dfrac{16 - 2x^2}{\sqrt{16 - x^2}} = 0$ when $x = \pm 2\sqrt{2}$ and undefined when $x = \pm 4$.

$f''(x) = \dfrac{2x(x^2 - 24)}{(16 - x^2)^{3/2}}$

$f''(-2\sqrt{2}) > 0$

Therefore, $(-2\sqrt{2}, -8)$ is a relative minimum.

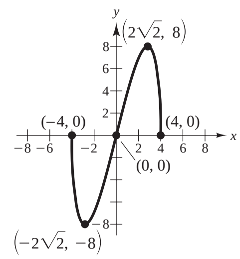
$f''(2\sqrt{2}) < 0$

Therefore, $(2\sqrt{2}, 8)$ is a relative maximum.

Point of inflection: $(0, 0)$

Intercepts: $(-4, 0), (0, 0), (4, 0)$

Symmetry with respect to origin



52. $f(x) = (x^2 - 4)^2$

Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$f'(x) = 4x(x^2 - 4) = 0$ when $x = 0, \pm 2$.

$f''(x) = 4(3x^2 - 4) = 0$ when $x = \pm\dfrac{2\sqrt{3}}{3}$.

$f''(0) < 0$

Therefore, $(0, 16)$ is a relative maximum.

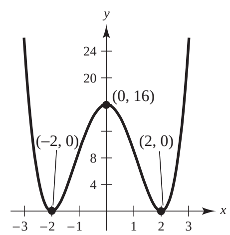
$f''(\pm 2) > 0$

Therefore, $(\pm 2, 0)$ are relative minima.

Points of inflection: $(\pm 2\sqrt{3}/3, 64/9)$

Intercepts: $(-2, 0), (0, 16), (2, 0)$

Symmetry with respect to y-axis



53. $f(x) = (x - 1)^3(x - 3)^2$

Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = (x - 1)^2(x - 3)(5x - 11) = 0$ when $x = 1, \dfrac{11}{5}, 3$.

$f''(x) = 4(x - 1)(5x^2 - 22x + 23) = 0$ when $x = 1, \dfrac{11 \pm \sqrt{6}}{5}$.

$f''(3) > 0$

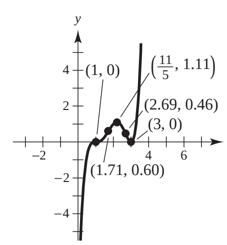
Therefore, $(3, 0)$ is a relative minimum.

$f''\left(\dfrac{11}{5}\right) < 0$

Therefore, $\left(\dfrac{11}{5}, \dfrac{3456}{3125}\right)$ is a relative maximum.

Points of inflection: $(1, 0), \left(\dfrac{11 - \sqrt{6}}{5}, 0.60\right), \left(\dfrac{11 + \sqrt{6}}{5}, 0.46\right)$

Intercepts: $(0, -9), (1, 0), (3, 0)$



54. $f(x) = (x - 3)(x + 2)^3$

Domain: $(-\infty, \infty)$; Range: $\left[-\frac{16,875}{256}, \infty\right)$

$f'(x) = (x - 3)(3)(x + 2)^2 + (x + 2)^3$

$= (4x - 7)(x + 2)^2 = 0$ when $x = -2, \frac{7}{4}$.

$f''(x) = (4x - 7)(2)(x + 2) + (x + 2)^2(4)$

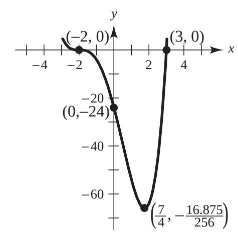
$= 6(2x - 1)(x + 2) = 0$ when $x = -2, \frac{1}{2}$.

$f''\left(\frac{7}{4}\right) > 0$

Therefore, $\left(\frac{7}{4}, -\frac{16,875}{256}\right)$ is a relative minimum.

Points of inflection: $(-2, 0), \left(\frac{1}{2}, -\frac{625}{16}\right)$

Intercepts: $(-2, 0), (0, -24), (3, 0)$



55. $f(x) = x^{1/3}(x + 3)^{2/3}$

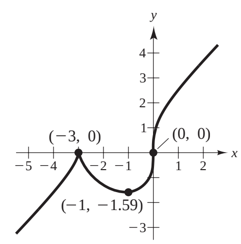
Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

$f'(x) = \dfrac{x + 1}{(x + 3)^{1/3}x^{2/3}} = 0$ when $x = -1$ and undefined when $x = -3, 0$.

$f''(x) = \dfrac{-2}{x^{5/3}(x + 3)^{4/3}}$ is undefined when $x = 0, -3$.

By the First Derivative Test $(-3, 0)$ is a relative maximum and $\left(-1, -\sqrt[3]{4}\right)$ is a relative minimum. $(0, 0)$ is a point of inflection.

Intercepts: $(-3, 0), (0, 0)$



56. $f(x) = (x - 2)^{1/3}(x + 1)^{2/3}$

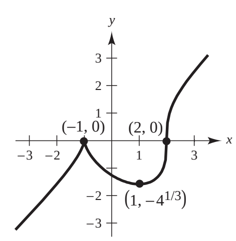
Graph of Exercise 39 translated 2 units to the right (x replaces by $x - 2$).

$(-1, 0)$ is a relative maximum.

$\left(1, -\sqrt[3]{4}\right)$ is a relative minimum.

$(2, 0)$ is a point of inflection.

Intercepts: $(-1, 0), (2, 0), (0, -2^{1/3})$



57. $f(x) = \dfrac{x + 1}{x - 1}$

Domain: $(-\infty, 1), (1, \infty)$; Range: $(-\infty, 1), (1, \infty)$

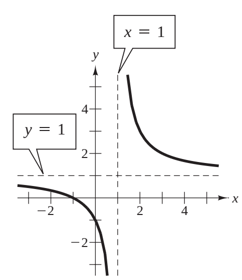
$f'(x) = \dfrac{-2}{(x - 1)^2} < 0$ if $x \neq 1$.

$f''(x) = \dfrac{4}{(x - 1)^3}$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 1$

Intercepts: $(-1, 0), (0, -1)$



58. $f(x) = \dfrac{2x}{1 + x^2}$

Domain: $(-\infty, \infty)$; Range: $[-1, 1]$

$f'(x) = \dfrac{2(1 - x)(1 + x)}{(1 + x^2)^2} = 0$ when $x = \pm 1$.

$f''(x) = \dfrac{-4x(3 - x^2)}{(1 + x^2)^3} = 0$ when $x = 0, \pm\sqrt{3}$.

$f''(1) < 0$

Therefore, $(1, 1)$ is a relative maximum.

$f''(-1) > 0$

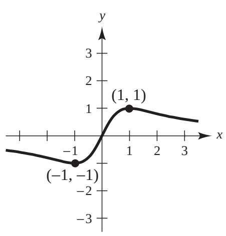
Therefore, $(-1, -1)$ is a relative minimum.

Points of inflection: $\left(-\sqrt{3}, -\sqrt{3}/2\right)$, $(0, 0)$, $\left(\sqrt{3}, \sqrt{3}/2\right)$

Intercept: $(0, 0)$

Symmetric with respect to the origin

Horizontal asymptote: $y = 0$



59. $f(x) = \dfrac{4}{1 + x^2}$

Domain: $(-\infty, \infty)$; Range: $(0, 4]$

$f'(x) = \dfrac{-8x}{(1 + x^2)^2} = 0$ when $x = 0$.

$f''(x) = \dfrac{-8(1 - 3x^2)}{(1 + x^2)^3} = 0$ when $x = \pm\dfrac{\sqrt{3}}{3}$.

$f''(0) < 0$

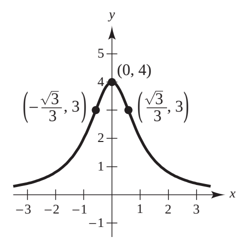
Therefore, $(0, 4)$ is a relative maximum.

Points of inflection: $\left(\pm\sqrt{3}/3, 3\right)$

Intercept: $(0, 4)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



60. $f(x) = \dfrac{x^2}{1 + x^4}$

Domain: $(-\infty, \infty)$; Range: $\left[0, \dfrac{1}{2}\right]$

$f'(x) = \dfrac{(1 + x^4)(2x) - x^2(4x^3)}{(1 + x^4)^2} = \dfrac{2x(1 - x)(1 + x)(1 + x^2)}{(1 + x^4)^2} = 0$ when $x = 0, \pm 1$.

$f''(x) = \dfrac{(1 + x^4)^2(2 - 10x^4) - (2x - 2x^5)(2)(1 + x^4)(4x^3)}{(1 + x^4)^4} = \dfrac{2(1 - 12x^4 + 3x^8)}{(1 + x^4)^3} = 0$ when $x = \pm\sqrt[4]{\dfrac{6 \pm \sqrt{33}}{3}}$.

$f''(\pm 1) < 0$

Therefore, $\left(\pm 1, \dfrac{1}{2}\right)$ are relative maxima.

$f''(0) > 0$

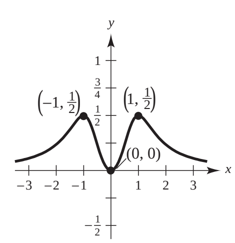
Therefore, $(0, 0)$ is a relative minimum.

Points of inflection: $\left(\pm\sqrt[4]{\dfrac{6 - \sqrt{33}}{3}}, 0.29\right)$, $\left(\pm\sqrt[4]{\dfrac{6 + \sqrt{33}}{3}}, 0.40\right)$

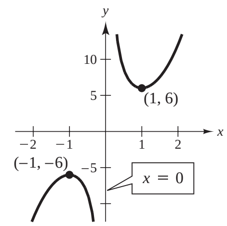
Intercept: $(0, 0)$

Symmetric to the y-axis

Horizontal asymptote: $y = 0$



61. $f(x) = x^3 + x + \dfrac{4}{x}$



Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, -6], [6, \infty)$

$$f'(x) = 3x^2 + 1 - \frac{4}{x^2} = \frac{3x^4 + x^2 - 4}{x^2} = \frac{(3x^2 + 4)(x^2 - 1)}{x^2} = 0 \text{ when } x = \pm 1.$$

$$f''(x) = 6x + \frac{8}{x^3} = \frac{6x^4 + 8}{x^3} \neq 0$$

$f''(-1) < 0$

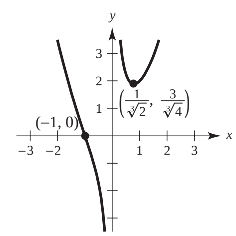
Therefore, $(-1, -6)$ is a relative maximum.

$f''(1) > 0$

Therefore, $(1, 6)$ is a relative minimum.

Vertical asymptote: $x = 0$
Symmetric with respect to origin

62. $f(x) = x^2 + \dfrac{1}{x} = \dfrac{x^3 + 1}{x}$



Domain: $(-\infty, 0), (0, \infty)$; Range: $(-\infty, \infty)$

$$f'(x) = 2x - \frac{1}{x^2} = \frac{2x^3 - 1}{x^2} = 0 \text{ when } x = \frac{1}{\sqrt[3]{2}}.$$

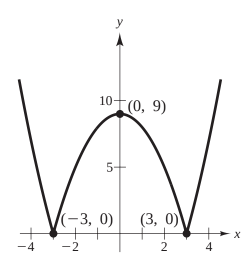
$$f''(x) = 2 + \frac{2}{x^3} = \frac{2(x^3 + 1)}{x^3} = 0 \text{ when } x = -1.$$

$$f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$$

Therefore, $\left(\dfrac{1}{\sqrt[3]{2}}, \dfrac{3}{\sqrt[3]{4}}\right)$ is a relative minimum.

Point of inflection: $(-1, 0)$
Intercept: $(-1, 0)$
Vertical asymptote: $x = 0$

63. $f(x) = |x^2 - 9|$



Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

$$f'(x) = \frac{2x(x^2 - 9)}{|x^2 - 9|} = 0 \text{ when } x = 0 \text{ and is undefined when } x = \pm 3.$$

$$f''(x) = \frac{2(x^2 - 9)}{|x^2 - 9|} \text{ is undefined at } x = \pm 3.$$

$f''(0) < 0$

Therefore, $(0, 9)$ is a relative maximum.

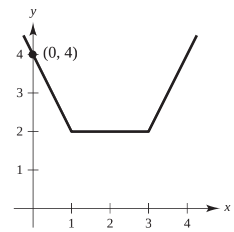
Relative minima: $(\pm 3, 0)$
Intercepts: $(\pm 3, 0), (0, 9)$
Symmetric to the y-axis

64. $f(x) = |x-1| + |x-3| = \begin{cases} -2x+4, & x \le 1 \\ 2, & 1 < x \le 3 \\ 2x-4, & x > 3 \end{cases}$

Domain: $(-\infty, \infty)$

Range: $[2, \infty)$

Intercept: $(0, 4)$



65. $f(x) = x + \cos x$

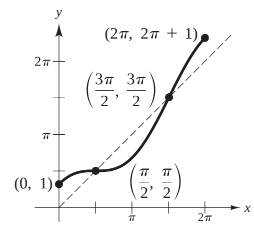
Domain: $[0, 2\pi]$; Range: $[1, 1+2\pi]$

$f'(x) = 1 - \sin x \ge 0$, f is increasing.

$f''(x) = -\cos x = 0$ when $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Points of inflection: $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right), \left(\dfrac{3\pi}{2}, \dfrac{3\pi}{2}\right)$

Intercept: $(0, 1)$



66. $f(x) = \dfrac{1}{\pi}(2\sin \pi x - \sin 2\pi x)$

Domain: $[-1, 1]$; Range: $\left[\dfrac{-3\sqrt{3}}{2\pi}, \dfrac{3\sqrt{3}}{2\pi}\right]$

$f'(x) = 2(\cos \pi x - \cos 2\pi x) = -2(2\cos \pi x + 1)(\cos \pi x - 1) = 0$

Critical Numbers: $x = \pm\dfrac{2}{3}, 0$

$f''(x) = 2\pi(-\sin \pi x + 2\sin 2\pi x) = 2\pi \sin \pi x(-1 + 4\cos \pi x) = 0$ when $x = 0, \pm 1, \pm 0.420$.

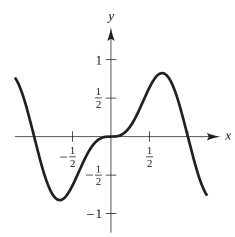
By the First Derivative Test: $\left(-\dfrac{2}{3}, \dfrac{-3\sqrt{3}}{2\pi}\right)$ is a relative minimum.

$\left(\dfrac{2}{3}, \dfrac{3\sqrt{3}}{2\pi}\right)$ is a relative maximum.

Points of inflection: $(-0.420, -0.462)$, $(0.420, 0.462)$, $(\pm 1, 0)$, $(0, 0)$

Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$

Symmetric with respect to the origin



67. $x^2 + 4y^2 - 2x - 16y + 13 = 0$

(a) $(x^2 - 2x + 1) + 4(y^2 - 4y + 4) = -13 + 1 + 16$

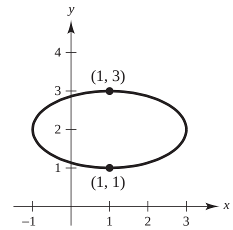
$$(x-1)^2 + 4(y-2)^2 = 4$$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{1} = 1$$

The graph is an ellipse:

Maximum: $(1, 3)$

Minimum: $(1, 1)$



(b) $x^2 + 4y^2 - 2x - 16y + 13 = 0$

$$2x + 8y\frac{dy}{dx} - 2 - 16\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(8y - 16) = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2 - 2x}{8y - 16} = \frac{1 - x}{4y - 8}$$

The critical numbers are $x = 1$ and $y = 2$. These correspond to the points $(1, 1)$, $(1, 3)$, $(2, -1)$, and $(2, 3)$. Hence, the maximum is $(1, 3)$ and the minimum is $(1, 1)$.

68. $f(x) = x^n$, n is a positive integer.

(a) $f'(x) = nx^{n-1}$

The function has a relative minimum at $(0, 0)$ when n is even.

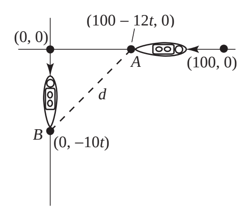
(b) $f''(x) = n(n-1)x^{n-2}$

The function has a point of inflection at $(0, 0)$ when n is odd and $n \geq 3$.

69. Let $t = 0$ at noon.

$$L = d^2 = (100 - 12t)^2 + (-10t)^2 = 10{,}000 - 2400t + 244t^2$$

$$\frac{dL}{dt} = -2400 + 488t = 0 \text{ when } t = \frac{300}{61} \approx 4.92 \text{ hr.}$$



Ship A at $(40.98, 0)$; Ship B at $(0, -49.18)$

$$d^2 = 10{,}000 - 2400t + 244t^2$$

$$\approx 4098.36 \text{ when } t \approx 4.92 \approx 4{:}55 \text{ P.M.}$$

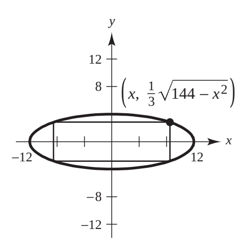
$$d \approx 64 \text{ km}$$

70. Ellipse: $\dfrac{x^2}{144} + \dfrac{y^2}{16} = 1$, $y = \dfrac{1}{3}\sqrt{144 - x^2}$

$$A = (2x)\left(\frac{2}{3}\sqrt{144 - x^2}\right) = \frac{4}{3}x\sqrt{144 - x^2}$$

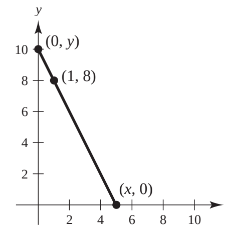
$$\frac{dA}{dx} = \frac{4}{3}\left[\frac{-x^2}{\sqrt{144 - x^2}} + \sqrt{144 - x^2}\right]$$

$$= \frac{4}{3}\left[\frac{144 - 2x^2}{\sqrt{144 - x^2}}\right] = 0 \text{ when } x = \sqrt{72} = 6\sqrt{2}.$$



The dimensions of the rectangle are $2x = 12\sqrt{2}$ by $y = \dfrac{2}{3}\sqrt{144 - 72} = 4\sqrt{2}$.

71. We have points $(0, y)$, $(x, 0)$, and $(1, 8)$. Thus,



$$m = \frac{y-8}{0-1} = \frac{0-8}{x-1} \text{ or } y = \frac{8x}{x-1}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{8x}{x-1}\right)^2$.

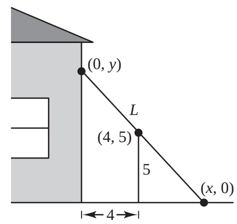
$$f'(x) = 2x + 128\left(\frac{x}{x-1}\right)\left[\frac{(x-1)-x}{(x-1)^2}\right] = 0$$

$$x - \frac{64x}{(x-1)^3} = 0$$

$$x[(x-1)^3 - 64] = 0 \text{ when } x = 0, 5 \text{ (minimum)}.$$

Vertices of triangle: $(0, 0)$, $(5, 0)$, $(0, 10)$

72. We have points $(0, y)$, $(x, 0)$, and $(4, 5)$. Thus,



$$m = \frac{y-5}{0-4} = \frac{5-0}{4-x} \text{ or } y = \frac{5x}{x-4}.$$

Let $f(x) = L^2 = x^2 + \left(\frac{5x}{x-4}\right)^2$

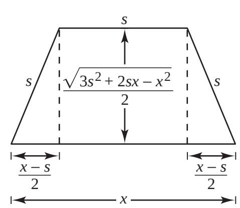
$$f'(x) = 2x + 50\left(\frac{x}{x-4}\right)\left[\frac{x-4-x}{(x-4)^2}\right] = 0$$

$$x - \frac{100x}{(x-4)^3} = 0$$

$$x[(x-4)^3 - 100] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{100}.$$

$$L = \sqrt{x^2 + \frac{25x^2}{(x-4)^2}} = \frac{x}{x-4}\sqrt{(x-4)^2 + 25} = \frac{\sqrt[3]{100}+4}{\sqrt[3]{100}}\sqrt{100^{2/3} + 25} \approx 12.7 \text{ feet}$$

73. $A = \text{(Average of bases)(Height)}$



$$= \left(\frac{x+s}{2}\right)\frac{\sqrt{3s^2+2sx-x^2}}{2} \text{ (see figure)}$$

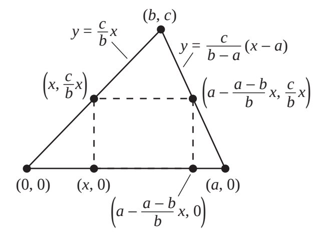
$$\frac{dA}{dx} = \frac{1}{4}\left[\frac{(s-x)(s+x)}{\sqrt{3s^2+2sx-x^2}} + \sqrt{3s^2+2sx-x^2}\right]$$

$$= \frac{2(2s-x)(s+x)}{4\sqrt{3s^2+2sx-x^2}} = 0 \text{ when } x = 2s.$$

A is a maximum when $x = 2s$.

74. Label triangle with vertices $(0, 0)$, $(a, 0)$, and (b, c). The equations of the sides of the triangle are $y = (c/b)x$ and $y = [c/(b - a)](x - a)$. Let $(x, 0)$ be a vertex of the inscribed rectangle. The coordinates of the upper left vertex are $(x, (c/b)x)$. The y-coordinate of the upper right vertex of the rectangle is $(c/b)x$. Solving for the x-coordinate $\bar{x}$ of the rectangle's upper right vertex, you get

$$\frac{c}{b}x = \frac{c}{b-a}(\bar{x} - a)$$
$$(b - a)x = b(\bar{x} - a)$$
$$\bar{x} = \frac{b-a}{b}x + a = a - \frac{a-b}{b}x.$$



Finally, the lower right vertex is

$$\left(a - \frac{a-b}{b}x, 0\right).$$

Width of rectangle: $a - \dfrac{a-b}{b}x - x$

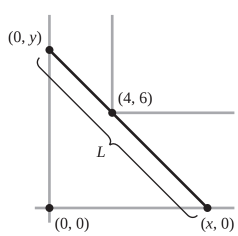
Height of rectangle: $\dfrac{c}{b}x$ (see figure)

$$A = (\text{Width})(\text{Height}) = \left(a - \frac{a-b}{b}x - x\right)\left(\frac{c}{b}x\right) = \left(a - \frac{a}{b}x\right)\frac{c}{b}x$$

$$\frac{dA}{dx} = \left(a - \frac{a}{b}x\right)\frac{c}{b} + \left(\frac{c}{b}x\right)\left(-\frac{a}{b}\right) = \frac{ac}{b} - \frac{2ac}{b^2}x = 0 \text{ when } x = \frac{b}{2}.$$

$$A\left(\frac{b}{2}\right) = \left(a - \frac{a}{b}\frac{b}{2}\right)\left(\frac{c}{b}\frac{b}{2}\right) = \left(\frac{a}{2}\right)\left(\frac{c}{2}\right) = \frac{1}{4}ac = \frac{1}{2}\left(\frac{1}{2}ac\right) = \frac{1}{2}(\text{Area of triangle})$$

75. You can form a right triangle with vertices $(0, 0)$, $(x, 0)$ and $(0, y)$. Assume that the hypotenuse of length L passes through $(4, 6)$.



$$m = \frac{y-6}{0-4} = \frac{6-0}{4-x} \text{ or } y = \frac{6x}{x-4}$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\dfrac{6x}{x-4}\right)^2$.

$$f'(x) = 2x + 72\left(\frac{x}{x-4}\right)\left[\frac{-4}{(x-4)^2}\right] = 0$$

$$x[(x-4)^3 - 144] = 0 \text{ when } x = 0 \text{ or } x = 4 + \sqrt[3]{144}.$$

$$L \approx 14.05 \text{ feet}$$

76. You can form a right triangle with vertices $(0, y)$, $(0, 0)$, and $(x, 0)$. Choosing a point (a, b) on the hypotenuse (assuming the triangle is in the first quadrant), the slope is

$$m = \frac{y-b}{0-a} = \frac{b-0}{a-x} \Rightarrow y = \frac{-bx}{a-x}.$$

Let $f(x) = L^2 = x^2 + y^2 = x^2 + \left(\dfrac{-bx}{a-x}\right)^2$.

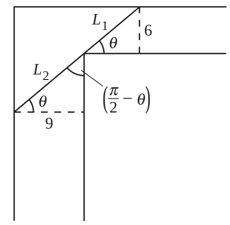
$$f'(x) = 2x + 2\left(\frac{-bx}{a-x}\right)\left[\frac{-ab}{(a-x)^2}\right]$$

$$\frac{2x[(a-x)^3 + ab^2]}{(a-x)^3} = 0 \text{ when } x = 0, a + \sqrt[3]{ab^2}.$$

Choosing the nonzero value, we have $y = b + \sqrt[3]{a^2b}$.

$$L = \sqrt{\left(a + \sqrt[3]{ab^2}\right)^2 + \left(b + \sqrt[3]{a^2b}\right)^2}$$
$$= (a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2)^{1/2}$$
$$= (a^{2/3} + b^{2/3})^{3/2} \text{ meters}$$

77. $\csc\theta = \dfrac{L_1}{6}$ or $L_1 = 6\csc\theta$ (see figure)



$\sec\theta = \dfrac{L_2}{9}$ or $L_2 = 9\sec\theta$

$$L = L_1 + L_2 = 6\csc\theta + 9\sec\theta$$

$$\frac{dL}{d\theta} = -6\csc\theta\cot\theta + 9\sec\theta\tan\theta = 0$$

$$\tan^3\theta = \frac{2}{3} \Longrightarrow \tan\theta = \frac{\sqrt[3]{2}}{\sqrt[3]{3}}$$

$$\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1+\left(\frac{2}{3}\right)^{2/3}} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{3^{1/3}}$$

$$\csc\theta = \frac{\sec\theta}{\tan\theta} = \frac{\sqrt{3^{2/3}+2^{2/3}}}{2^{1/3}}$$

$L = 6\dfrac{(3^{2/3}+2^{2/3})^{1/2}}{2^{1/3}} + 9\dfrac{(3^{2/3}+2^{2/3})^{1/2}}{3^{1/3}} = 3(3^{2/3}+2^{2/3})^{3/2}$ ft ≈ 21.07 ft (Compare to Exercise 72 using $a = 9$ and $b = 6$.)

78. Using Exercise 73 as a guide we have $L_1 = a\csc\theta$ and $L_2 = b\sec\theta$. Then $dL/d\theta = -a\csc\theta\cot\theta + b\sec\theta\tan\theta = 0$ when

$\tan\theta = \sqrt[3]{a/b}$, $\sec\theta = \dfrac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}$, $\csc\theta = \dfrac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}}$ and

$$L = L_1 + L_2 = a\csc\theta + b\sec\theta = a\frac{(a^{2/3}+b^{2/3})^{1/2}}{a^{1/3}} + b\frac{(a^{2/3}+b^{2/3})^{1/2}}{b^{1/3}} = (a^{2/3}+b^{2/3})^{3/2}.$$

This matches the result of Exercise 72.

79. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{600}+5\right)\left(\frac{110}{v}\right) = \frac{11v}{60} + \frac{550}{v}$$

$$\frac{dT}{dv} = \frac{11}{60} - \frac{550}{v^2} = \frac{11v^2 - 33{,}000}{60v^2}$$

$= 0$ when $v = \sqrt{3000} = 10\sqrt{30} \approx 54.8$ mph.

$\dfrac{d^2T}{dv^2} = \dfrac{1100}{v^3} > 0$ when $v = 10\sqrt{30}$ so this value yields a minimum.

80. Total cost = (Cost per hour)(Number of hours)

$$T = \left(\frac{v^2}{500}+7.50\right)\left(\frac{110}{v}\right) = \frac{11v}{50} + \frac{825}{v}$$

$$\frac{dT}{dv} = \frac{11}{50} - \frac{825}{v^2} = \frac{11v^2 - 41{,}250}{50v^2}$$

$= 0$ when $v = \sqrt{3750} = 25\sqrt{6} \approx 61.2$ mph.

$\dfrac{d^2T}{dv^2} = \dfrac{1650}{v^3} > 0$ when $v = 25\sqrt{6}$ so this value yields a minimum.

81. $f(x) = x^3 - 3x - 1$

From the graph you can see that $f(x)$ has three real zeros.

$f'(x) = 3x^2 - 3$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.5000	0.1250	3.7500	0.0333	-1.5333
2	-1.5333	-0.0049	4.0530	-0.0012	-1.5321

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	0.3750	-2.2500	-0.1667	-0.3333
2	-0.3333	-0.0371	-2.6667	0.0139	-0.3472
3	-0.3472	-0.0003	-2.6384	0.0001	-0.3473

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.9000	0.1590	7.8300	0.0203	1.8797
2	1.8797	0.0024	7.5998	0.0003	1.8794

The three real zeros of $f(x)$ are $x \approx -1.532$, $x \approx -0.347$, and $x \approx 1.879$.

82. $f(x) = x^3 + 2x + 1$

From the graph, you can see that $f(x)$ has one real zero.

$f'(x) = 3x^2 + 2$

f changes sign in $[-1, 0]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-0.5000	-0.1250	2.7500	-0.0455	-0.4545
2	-0.4545	-0.0029	2.6197	-0.0011	-0.4534

On the interval $[-1, 0]$: $x \approx -0.453$.

83. Find the zeros of $f(x) = x^4 - x - 3$.

$f'(x) = 4x^3 - 1$

From the graph you can see that $f(x)$ has two real zeros.

f changes sign in $[-2, -1]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	-1.2000	0.2736	-7.9120	-0.0346	-1.1654
2	-1.1654	0.0100	-7.3312	-0.0014	-1.1640

On the interval $[-2, -1]$: $x \approx -1.164$.

f changes sign in $[1, 2]$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.5000	0.5625	12.5000	0.0450	1.4550
2	1.4550	0.0268	11.3211	0.0024	1.4526
3	1.4526	-0.0003	11.2602	0.0000	1.4526

On the interval $[1, 2]$; $x \approx 1.453$.

84. Find the zeros of $f(x) = \sin \pi x + x - 1$.

$f'(x) = \pi \cos \pi x + 1$

From the graph you can see that $f(x)$ has three real zeros.

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.2000	-0.2122	3.5416	-0.0599	0.2599
2	0.2599	-0.0113	3.1513	-0.0036	0.2635
3	0.2635	0.0000	3.1253	0.0000	0.2635

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	0.0000	-2.1416	0.0000	1.0000

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.8000	0.2122	3.5416	0.0599	1.7401
2	1.7401	0.0113	3.1513	0.0036	1.7365
3	1.7365	0.0000	3.1253	0.0000	1.7365

The three real zeros of $f(x)$ are $x \approx 0.264$, $x = 1$, and $x \approx 1.737$.

85. $y = x(1 - \cos x) = x - x\cos x$

$$\frac{dy}{dx} = 1 + x\sin x - \cos x$$

$$dy = (1 + x\sin x - \cos x)\,dx$$

86. $y = \sqrt{36 - x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(36 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{36 - x^2}}$$

$$dy = \frac{-x}{\sqrt{36 - x^2}}dx$$

87. $S = 4\pi r^2 \quad dr = \Delta r = \pm 0.025$

$$dS = 8\pi r\,dr = 8\pi(9)(\pm 0.025)$$
$$= \pm 1.8\pi \text{ square cm}$$

$$\frac{dS}{S}(100) = \frac{8\pi r\,dr}{4\pi r^2}(100) = \frac{2\,dr}{r}(100)$$
$$= \frac{2(\pm 0.025)}{9}(100) \approx \pm 0.56\%$$

$$V = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\,dr = 4\pi(9)^2(\pm 0.025)$$
$$= \pm 8.1\pi \text{ cubic cm}$$

$$\frac{dV}{V}(100) = \frac{4\pi r^2\,dr}{(4/3)\pi r^3}(100) = \frac{3\,dr}{r}(100)$$
$$= \frac{3(\pm 0.025)}{9}(100) \approx \pm 0.83\%$$

88. $p = 75 - \frac{1}{4}x$

$$\Delta p = p(8) - p(7)$$
$$= \left(75 - \frac{8}{4}\right) - \left(75 - \frac{7}{4}\right) = -\frac{1}{4}$$

$$dp = -\frac{1}{4}dx = -\frac{1}{4}(1) = -\frac{1}{4}$$

$[\Delta p = dp$ because p is linear$]$

Problem Solving for Chapter 3

1. $p(x) = x^4 + ax^2 + 1$

(a) $p'(x) = 4x^3 + 2ax = 2x(2x^2 + a)$

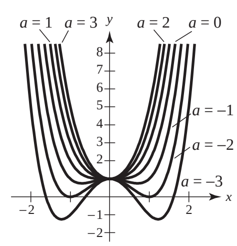
$p''(x) = 12x^2 + 2a$

For $a \geq 0$, there is one relative minimum at $(0, 1)$.

(b) For $a < 0$, there is a relative maximum at $(0, 1)$.

(c) For $a < 0$, there are two relative minima at $x = \pm\sqrt{-\frac{a}{2}}$.

(d) There are either 1 or 3 critical points. The above analysis shows that there cannot be exactly two relative extrema.



2. (a) For $a = -3, -2, -1, 0$, p has a relative maximum at $(0, 0)$.

For $a = 1, 2, 3$, p has a relative maximum at $(0, 0)$ and 2 relative minima.

(b) $p'(x) = 4ax^3 - 12x = 4x(ax^2 - 3) = 0 \implies x = 0, \pm\sqrt{\dfrac{3}{a}}$

$p''(x) = 12ax^2 - 12 = 12(ax^2 - 1)$

For $x = 0$, $p''(0) = -12 < 0 \implies p$ has a relative maximum at $(0, 0)$.

(c) If $a > 0$, $x = \pm\sqrt{\dfrac{3}{a}}$ are the remaining critical numbers.

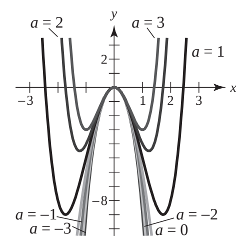
$p''\left(\pm\sqrt{\dfrac{3}{a}}\right) = 12a\left(\dfrac{3}{a}\right) - 12 = 24 > 0 \implies p$ has relative minima for $a > 0$.

(d) $(0, 0)$ lies on $y = -3x^2$.

Let $x = \pm\sqrt{\dfrac{3}{a}}$. Then

$$p(x) = a\left(\frac{3}{a}\right)^2 - 6\left(\frac{3}{a}\right) = \frac{9}{a} - \frac{18}{a} = -\frac{9}{a}.$$

Thus, $y = -\dfrac{9}{a} = -3\left(\pm\sqrt{\dfrac{3}{a}}\right)^2 = -3x^2$ is satisfied by all the relative extrema of p.



3. $f(x) = \dfrac{c}{x} + x^2$

$f'(x) = -\dfrac{c}{x^2} + 2x = 0 \implies \dfrac{c}{x^2} = 2x \implies x^3 = \dfrac{c}{2} \implies x = \sqrt[3]{\dfrac{c}{2}}$

$f''(x) = \dfrac{2c}{x^3} + 2$

If $c = 0$, $f(x) = x^2$ has a relative minimum, but no relative maximum.

If $c > 0$, $x = \sqrt[3]{\dfrac{c}{2}}$ is a relative minimum, because $f''\left(\sqrt[3]{\dfrac{c}{2}}\right) > 0$.

If $c < 0$, $x = \sqrt[3]{\dfrac{c}{2}}$ is a relative minimum too.

Answer: all c.

4. (a) $f(x) = ax^2 + bx + c$, $a \neq 0$

$f'(x) = 2ax$

$f''(x) = 2a \neq 0$

No point of inflection

(b) $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$

$f'(x) = 3ax^2 + 2bx + c$

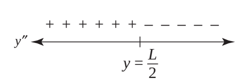
$f''(x) = 6ax + 2b = 0 \implies x = \dfrac{-b}{3a}$

One point of inflection

(c) $y' = ky\left(1 - \dfrac{y}{L}\right) = ky - \dfrac{k}{L}y^2$

$y'' = ky' - \dfrac{2k}{L}yy' = ky'\left(1 - \dfrac{2}{L}y\right)$

If $y = \dfrac{L}{2}$, then $y'' = 0$, and this is a point of inflection because of the analysis below.



5. Assume $y_1 < d < y_2$. Let $g(x) = f(x) - d(x - a)$. g is continuous on $[a, b]$ and therefore has a minimum $(c, g(c))$ on $[a, b]$. The point c cannot be an endpoint of $[a, b]$ because

$$g'(a) = f'(a) - d = y_1 - d < 0$$

$$g'(b) = f'(b) - d = y_2 - d > 0.$$

Hence, $a < c < b$ and $g'(c) = 0 \implies f'(c) = d$.

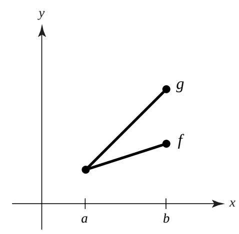
6. Let $h(x) = g(x) - f(x)$, which is continuous on $[a, b]$ and differentiable on (a, b). $h(a) = 0$ and $h(b) = g(b) - f(b)$.

By the Mean Value Theorem, there exists c in (a, b) such that

$$h'(c) = \frac{h(b) - h(a)}{b - a} = \frac{g(b) - f(b)}{b - a}.$$

Since $h'(c) = g'(c) - f'(c) > 0$ and $b - a > 0$,

$$g(b) - f(b) > 0 \implies g(b) > f(b).$$



7. Set $\dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = k.$

Define $F(x) = f(x) - f(a) - f'(a)(x - a) - k(x - a)^2$.

$F(a) = 0, F(b) = f(b) - f(a) - f'(a)(b - a) - k(b - a)^2 = 0$

F is continuous on $[a, b]$ and differentiable on (a, b).

There exists $c_1, a < c_1 < b$, satisfying $F'(c_1) = 0$.

$F'(x) = f'(x) - f'(a) - 2k(x - a)$ satisfies the hypothesis of Rolle's Theorem on $[a, c_1]$:

$$F'(a) = 0, F'(c_1) = 0.$$

There exists $c_2, a < c_2 < c_1$ satisfying $F''(c_2) = 0$.

Finally, $F''(x) = f''(x) - 2k$ and $F''(c_2) = 0$ implies that

$$k = \frac{f''(c_2)}{2}.$$

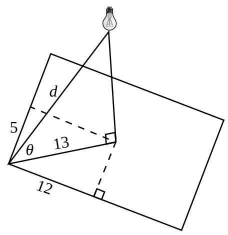
Thus, $k = \dfrac{f(b) - f(a) - f'(a)(b - a)}{(b - a)^2} = \dfrac{f''(c_2)}{2} \implies f(b) = f(a) + f'(a)(b - a) + \dfrac{1}{2}f''(c_2)(b - a)^2.$

8. (a)

$$\begin{aligned} dV &= 3x^2 dx = 3x^2\,\Delta x \\ \Delta V &= (x + \Delta x)^3 - x^3 = 3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ \Delta V - dV &= 3x(\Delta x)^2 + (\Delta x)^3 = [\underbrace{3x\,\Delta x + (\Delta x)^2}_{\varepsilon}]\,\Delta x \\ &= \varepsilon\Delta x, \text{ where } \varepsilon \to 0 \text{ as } \Delta x \to 0. \end{aligned}$$

(b) Let $\varepsilon = \dfrac{\Delta y}{\Delta x} - f'(x)$. Then $\varepsilon \to 0$ as $\Delta x \to 0$.

Furthermore, $\Delta y - dy = \Delta y - f'(x)\,dx = \varepsilon\,\Delta x$.

9. 

$d = \sqrt{13^2 + x^2}$, $\sin\theta = \dfrac{x}{d}$.

Let A be the amount of illumination at one of the corners, as indicated in the figure. Then

$$A = \frac{kI}{(13^2 + x^2)}\sin\theta = \frac{kIx}{(13^2 + x^2)^{3/2}}$$

$$A'(x) = kI\frac{(x^2 + 169)^{3/2}(1) - x\left(\frac{3}{2}\right)(x^2 + 169)^{1/2}(2x)}{(169 + x^2)^3} = 0$$

$$\Longrightarrow \quad (x^2 + 169)^{3/2} = 3x^2(x^2 + 169)^{1/2}$$

$$x^2 + 169 = 3x^2$$

$$2x^2 = 169$$

$$x = \frac{13}{\sqrt{2}} \approx 9.19 \text{ feet.}$$

By the First Derivative Test, this is a maximum.

10. Distance $= \sqrt{4^2 + x^2} + \sqrt{(4 - x)^2 + 4^2} = f(x)$

$$f'(x) = \frac{x}{\sqrt{4^2 + x^2}} - \frac{4 - x}{\sqrt{(4 - x)^2 + 4^2}} = 0$$

$$x\sqrt{(4 - x)^2 + 4^2} = -(x - 4)\sqrt{4^2 + x^2}$$

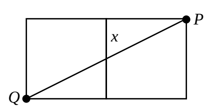
$$x^2[16 - 8x + x^2 + 16] = (x^2 - 8x + 16)(16 + x^2)$$

$$32x^2 - 8x^3 + x^4 = x^4 - 8x^3 + 32x^2 - 128x + 256$$

$$128x = 256$$

$$x = 2$$

The bug should head towards the midpoint of the opposite side.

Without Calculus: Imagine opening up the cube: 

The shortest distance is the line PQ, passing through the midpoint.

11. Let T be the intersection of PQ and RS. Let MN be the perpendicular to SQ and PR passing through T.

Let $TM = x$ and $TN = b - x$.

$$\frac{SN}{b-x} = \frac{MR}{x} \implies SN = \frac{b-x}{x}MR$$

$$\frac{NQ}{b-x} = \frac{PM}{x} \implies NQ = \frac{b-x}{x}PM$$

$$SQ = \frac{b-x}{x}(MR + PM) = \frac{b-x}{x}d$$

$$A(x) = \text{Area} = \frac{1}{2}dx + \frac{1}{2}\left(\frac{b-x}{x}d\right)(b-x) = \frac{1}{2}d\left[x + \frac{(b-x)^2}{x}\right] = \frac{1}{2}d\left[\frac{2x^2 - 2bx + b^2}{x}\right]$$

$$A'(x) = \frac{1}{2}d\left[\frac{x(4x-2b) - (2x^2 - 2bx + b^2)}{x^2}\right]$$

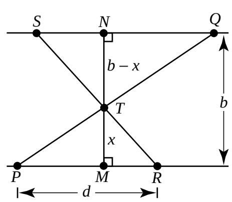
$$A'(x) = 0 \implies 4x^2 - 2xb = 2x^2 - 2bx + b^2$$

$$2x^2 = b^2$$

$$x = \frac{b}{\sqrt{2}}$$

Hence, we have $SQ = \dfrac{b-x}{x}d = \dfrac{b - \left(b/\sqrt{2}\right)}{b/\sqrt{2}}d = \left(\sqrt{2} - 1\right)d.$

Using the Second Derivative Test, this is a minimum. There is no maximum.



12. The line has equation $\dfrac{x}{3} + \dfrac{y}{4} = 1$ or $y = -\dfrac{4}{3}x + 4.$

Rectangle:

$$\text{Area} = A = xy = x\left(-\frac{4}{3}x + 4\right) = -\frac{4}{3}x^2 + 4x.$$

$$A'(x) = -\frac{8}{3}x + 4 = 0 \implies \frac{8}{3}x = 4 \implies x = \frac{3}{2}$$

Dimensions: $\dfrac{3}{2} \times 2$ Calculus was helpful.

Circle: The distance from the center (r, r) to the line $\dfrac{x}{3} + \dfrac{y}{4} - 1 = 0$ must be r:

$$r = \frac{\left|\frac{r}{3} + \frac{r}{4} - 1\right|}{\sqrt{\frac{1}{9} + \frac{1}{16}}} = \frac{12}{5}\left|\frac{7r - 12}{12}\right| = \frac{|7r - 12|}{5}$$

$$5r = |7r - 12| \implies r = 1 \text{ or } r = 6.$$

Clearly, $r = 1$.

Semicircle: The center lies on the line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ and satisfies $x = y = r$.

Thus $\dfrac{r}{3} + \dfrac{r}{4} = 1 \implies \dfrac{7}{12}r = 1 \implies r = \dfrac{12}{7}.$ No calculus necessary.

13. (a) Let $M > 0$ be given. Take $N = \sqrt{M}$. Then whenever $x > N = \sqrt{M}$, you have

$$f(x) = x^2 > M.$$

(b) Let $\varepsilon > 0$ be given. Let $M = \sqrt{\dfrac{1}{\varepsilon}}$. Then whenever $x > M = \sqrt{\dfrac{1}{\varepsilon}}$, you have

$$x^2 > \frac{1}{\varepsilon} \implies \frac{1}{x^2} < \varepsilon \implies \left|\frac{1}{x^2} - 0\right| < \varepsilon.$$

(c) Let $\varepsilon > 0$ be given. There exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.
Let $\delta = \dfrac{1}{N}$. Let $x = \dfrac{1}{y}$.

If $0 < y < \delta = \dfrac{1}{N}$, then $\dfrac{1}{x} < \dfrac{1}{N} \implies x > N$ and

$$|f(x) - L| = \left|f\left(\frac{1}{y}\right) - L\right| < \varepsilon.$$

14. $y = (1 + x^2)^{-1}$

$$y' = \frac{-2x}{(1 + x^2)^2}$$

$$y'' = \frac{2(3x^2 - 1)}{(x^2 + 1)^3} = 0 \implies x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

y'': $+++$ at $x < -\frac{\sqrt{3}}{3}$; $----$ between $-\frac{\sqrt{3}}{3}$ and 0; $----$ between 0 and $\frac{\sqrt{3}}{3}$; $+++$ at $x > \frac{\sqrt{3}}{3}$

The tangent line has greatest slope at $\left(-\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$ and least slope at $\left(\dfrac{\sqrt{3}}{3}, \dfrac{3}{4}\right)$.

15. (a)

x	0	0.5	1	2
$\sqrt{1 + x}$	1	1.2247	1.4142	1.7321
$\frac{1}{2}x + 1$	1	1.25	1.5	2

(b) Let $f(x) = \sqrt{1 + x}$. Using the Mean Value Theorem on the interval $[0, x]$, there exists c, $0 < c < x$, satisfying

$$f'(c) = \frac{1}{2\sqrt{1 + c}} = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1 + x} - 1}{x}.$$

Thus $\sqrt{1 + x} = \dfrac{x}{2\sqrt{1 + c}} + 1 < \dfrac{x}{2} + 1$ $\left(\text{because } \sqrt{1 + c} > 1\right)$.

16. (a)

x	0.1	0.2	0.3	0.4	0.5	1.0
$\sin x$	0.09983	0.19867	0.29552	0.38942	0.47943	0.84147

$\sin x < x$

(b) Let $f(x) = \sin x$. For $x > 1$, $\sin x < x$ is obvious. So assume $0 < x \le 1$. Then $f'(x) = \cos x$ and on $[0, x]$ you have by the Mean Value Theorem,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}, \quad 0 < c < x$$

$$\cos(c) = \frac{\sin x}{x}$$

Hence, $\left|\dfrac{\sin x}{x}\right| = |\cos(c)| < 1$

$$\implies |\sin x| < |x|$$

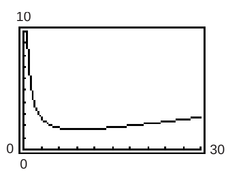
$$\implies \sin x < x.$$

17. (a) $s = \dfrac{v\,\dfrac{\text{km}}{\text{hr}}\left(1000\,\dfrac{\text{m}}{\text{km}}\right)}{\left(3600\,\dfrac{\text{sec}}{\text{hr}}\right)} = \dfrac{5}{18}v$

v	20	40	60	80	100
s	5.56	11.11	16.67	22.22	27.78
d	5.1	13.7	27.2	44.2	66.4

$d(t) = 0.071s^2 + 0.389s + 0.727$

(c)



$$T = \frac{1}{s}(0.071s^2 + 0.389s + 0.727) + \frac{5.5}{s}$$

The minimum is attained when $s \approx 9.365$ m/sec.

(b) The distance between the back of the first vehicle and the front of the second vehicle is $d(t)$, the safe stopping distance. The first vehicle passes the given point in $5.5/s$ seconds, and the second vehicle takes $d(s)/s$ more seconds. Hence,

$$T = \frac{d(s)}{s} + \frac{5.5}{s}.$$

(d) $$T(s) = 0.071s + 0.389 + \frac{6.227}{s}$$

$$T'(s) = 0.071 - \frac{6.227}{s^2} \implies s^2 = \frac{6.227}{0.071}$$

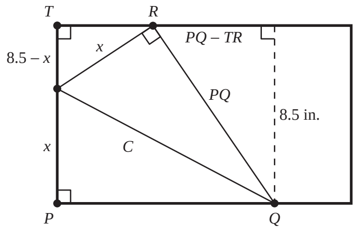
$$\implies s \approx 9.365 \text{ m/sec}$$

$T(9.365) \approx 1.719$ seconds

$$9.365 \text{ m/sec} \cdot \frac{3600}{1000} = 33.7 \text{ km/hr}$$

(e) $d(9.365) = 10.597$ m

18. (a)



$$x^2 + PQ^2 = C^2 \Longrightarrow PQ^2 = C^2 - x^2$$

$$TR^2 + (8.5 - x)^2 = x^2 \Longrightarrow TR^2 = 17x - 8.5^2$$

$$(PQ - TR)^2 + 8.5^2 = PQ^2 \Longrightarrow 2(PQ)(TR) = TR^2 + 8.5^2$$

Hence, $2(PQ)(TR) = 17x - 8.5^2 + 8.5^2$.

$$8.5x = (PQ)(TR) = \sqrt{C^2 - x^2}\sqrt{17x - 8.5^2}$$

$$\frac{(8.5x)^2}{17x - 8.5^2} = C^2 - x^2$$

$$C^2 = x^2 + \frac{(8.5x)^2}{17x - 8.5^2}$$

$$= \frac{17x^3}{17x - 8.5^2}$$

$$C^2 = \frac{2x^3}{2x - 8.5}$$

(b) Domain: $4.25 < x < 8.5$

(c) To minimize C, minimize $f(x) = C^2$:

$$f'(x) = \frac{(2x - 8.5)(6x^2) - 2x^3(2)}{(2x - 8.5)^2} = \frac{8x^3 - 51x^2}{(2x - 8.5)^2} = 0$$

$$x = \frac{51}{8} = 6.375$$

By the First Derivative Test, $x = 6.375$ is a minimum.

(d) For $x = 6.375$, $C \approx 11.0418$ inches.

19. $f(x) = \dfrac{x}{x + 1}$, $f'(x) = \dfrac{1}{(x + 1)^2}$, $f''(x) = \dfrac{-2}{(x + 1)^3}$

$P(0) = f(0)$: $c_0 = 0$

$P'(0) = f'(0)$: $c_1 = 1$

$P''(0) = f''(0)$: $2c_2 = -2 \Longrightarrow c_2 = -1$

$P(x) = x - x^2$