CHAPTER 3:

Derivatives

3.1: Derivatives, Tangent Lines, and Rates of Change

3.2: Derivative Functions and Differentiability

3.3: Techniques of Differentiation

3.4: Derivatives of Trigonometric Functions

3.5: Differentials and Linearization of Functions

3.6: Chain Rule

3.7: Implicit Differentiation

3.8: Related Rates

- Derivatives represent slopes of tangent lines and rates of change (such as velocity).
- In this chapter, we will define derivatives and derivative functions using limits.
- We will develop short cut techniques for finding derivatives.
- Tangent lines correspond to local linear approximations of functions.
- Implicit differentiation is a technique used in applied related rates problems.

SECTION 3.1: DERIVATIVES, TANGENT LINES, AND RATES OF CHANGE

LEARNING OBJECTIVES

- Relate difference quotients to slopes of secant lines and average rates of change.
- Know, understand, and apply the Limit Definition of the Derivative at a Point.
- Relate derivatives to slopes of tangent lines and instantaneous rates of change.
- Relate opposite reciprocals of derivatives to slopes of normal lines.

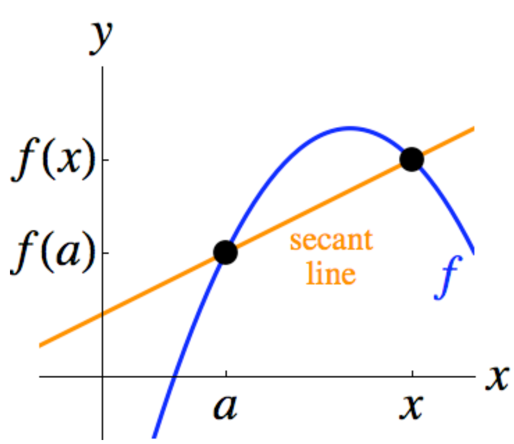
PART A: SECANT LINES

• For now, assume that f is a polynomial function of x. (We will relax this assumption in Part B.) Assume that a is a constant.

• Temporarily fix an arbitrary real value of x. (By "**arbitrary**," we mean that **any** real value will do). Later, instead of thinking of x as a **fixed** (or single) value, we will think of it as a "moving" or "varying" **variable** that can take on different values.

The secant line to the graph of f on the interval $[a, x]$, where $a < x$, is the line that passes through the points $(a, f(a))$ and $(x, f(x))$.

• *secare* is Latin for "to cut."



The **slope** of this secant line is given by: $\dfrac{\text{rise}}{\text{run}} = \dfrac{f(x) - f(a)}{x - a}$.

• We call this a difference quotient, because it has the form: $\dfrac{\text{difference of outputs}}{\text{difference of inputs}}$.

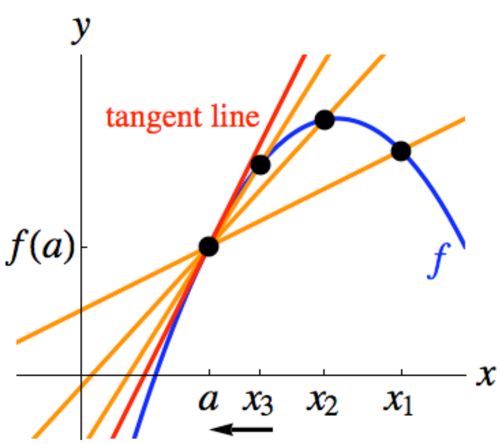
PART B: TANGENT LINES and DERIVATIVES

If we now treat x as a variable and let $x \to a$, the corresponding **secant lines** approach the red tangent line below.

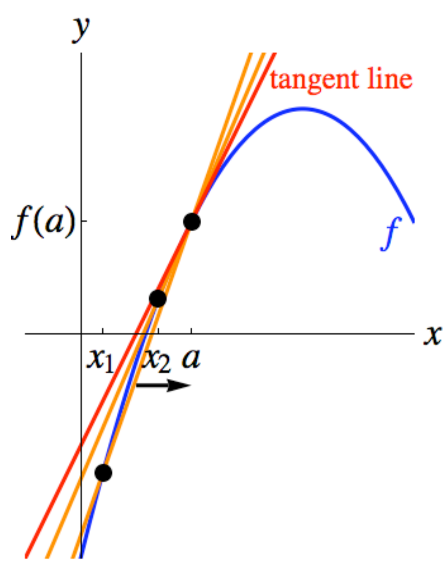
• *tangere* is Latin for "to touch." A **secant line** to the graph of f must intersect it in at least two distinct points. A **tangent line** only need intersect the graph in one point, where the line might "just touch" the graph. (There could be other intersection points).

• This "limiting process" makes the tangent line a creature of **calculus**, not just precalculus.

Below, we let x approach a from the **right** $\left(x \to a^+\right)$.



Below, we let x approach a from the **left** $\left(x \to a^-\right)$. (See Footnote 1.)



• We define the **slope** of the tangent line to be the (two-sided) **limit** of the difference quotient as $x \to a$, if that limit exists.

• We denote this slope by $f'(a)$, read as "***f* prime** of (or at) a."

$f'(a)$, the derivative of f at a, is the **slope of the tangent line** to the graph of f at the point $\left(a, f(a)\right)$, if that slope exists (as a real number).

f is differentiable at $a \Leftrightarrow f'(a)$ exists.

• **Polynomial** functions are **differentiable everywhere** on $\mathbb{R}$. (See Section 3.2.)

• The statements of this section apply to **any** function that is differentiable at a.

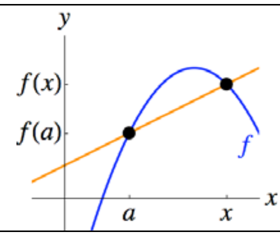
<u>Limit Definition of the Derivative at a Point *a* (Version 1)</u>

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \text{ if it exists}$$

• If f is continuous at *a*, we have the indeterminate Limit Form $\frac{0}{0}$.

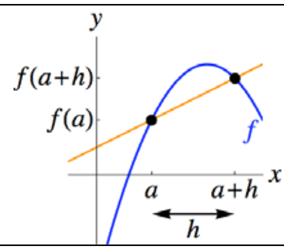
• **Continuity** involves limits of **function values**, while **differentiability** involves limits of **difference quotients**.

<u>Version 1: Variable endpoint (*x*)</u>

Slope of secant line: $\dfrac{f(x) - f(a)}{x - a}$		a is constant; x is variable

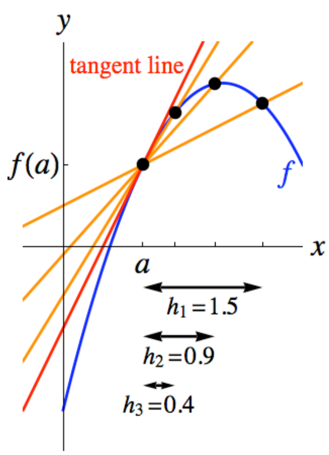
A second version, where *x* is replaced by $a + h$, is more commonly used.

<u>Version 2: Variable run (*h*)</u>

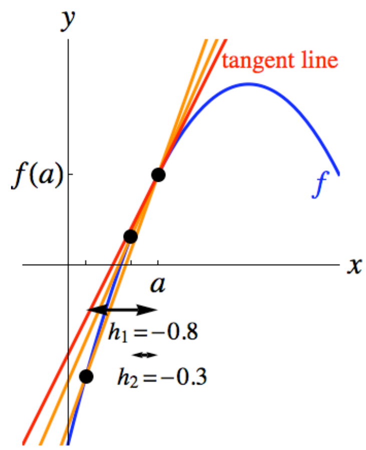
Slope of secant line: $\dfrac{f(a+h) - f(a)}{h}$		a is constant; h is variable

If we let the **run** $h \to 0$, the corresponding **secant lines** approach the red **tangent line** below.

Below, we let *h* approach 0 from the **right** $\left(h \to 0^+\right)$.



Below, we let *h* approach 0 from the **left** $\left(h \to 0^-\right)$. (Footnote 1.)



Limit Definition of the Derivative at a Point *a* (Version 2)

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \text{ if it exists}$$

Version 3: Two-Sided Approach

Limit Definition of the Derivative at a Point *a* (Version 3)

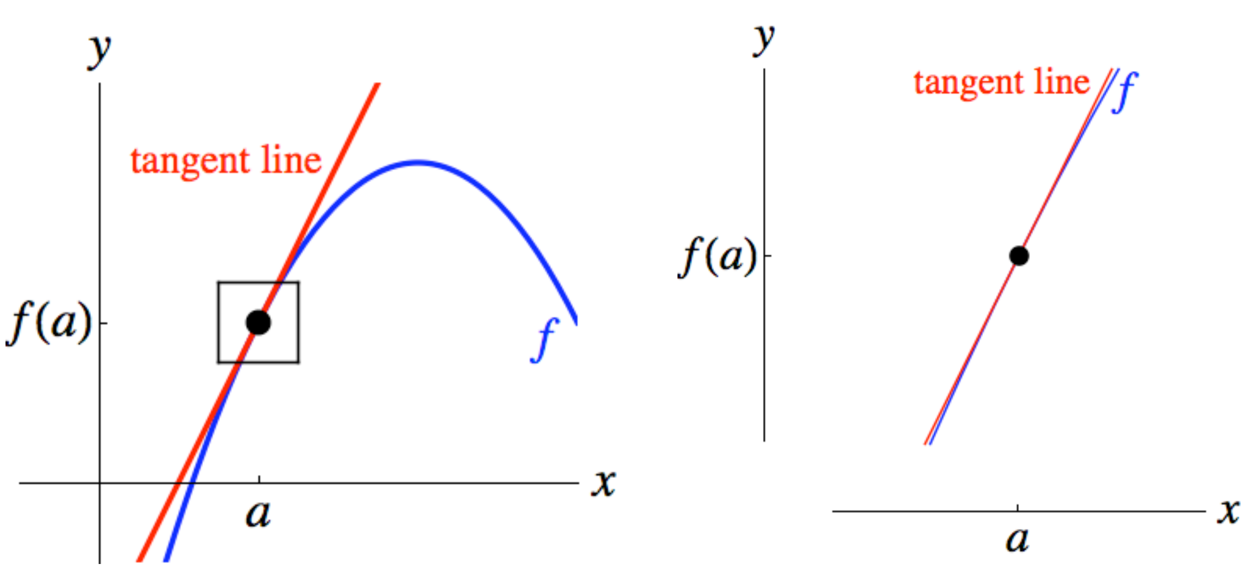
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}, \text{ if it exists}$$

• The reader is encouraged to draw a figure to understand this approach.

Principle of Local Linearity

The **tangent line** to the graph of f at the point $(a, f(a))$, if it exists, represents the **best local linear approximation** to the function close to a. The graph of f resembles this line if we "zoom in" on the point $(a, f(a))$.

• The **tangent line model** linearizes the function locally around a.
We will expand on this in Section 3.5.



(The figure on the right is a "zoom in" on the box in the figure on the left.)

PART C: FINDING DERIVATIVES USING THE LIMIT DEFINITIONS

Example 1 (Finding a Derivative at a Point Using Version 1 of the Limit Definition)

Let $f(x) = x^3$. Find $f'(1)$ using Version 1 of the Limit Definition of the Derivative at a Point.

§ Solution

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \quad (\text{Here, } a = 1.)$$

$$= \lim_{x \to 1} \frac{\left[x^3\right] - \left[(1)^3\right]}{x - 1}$$

TIP 1: The brackets here are unnecessary, but better safe than sorry.

$$= \lim_{x \to 1} \frac{x^3 - 1}{x - 1} \quad \left(\text{Limit Form } \frac{0}{0}\right)$$

We will factor the numerator using the **Difference of Two Cubes** template and then simplify. **Synthetic Division** can also be used. (See Chapter 2 in the Precalculus notes).

$$= \lim_{x \to 1} \frac{\overset{(1)}{\cancel{(x-1)}}\left(x^2 + x + 1\right)}{\underset{(1)}{\cancel{(x-1)}}}$$

$$= \lim_{x \to 1} \left(x^2 + x + 1\right)$$

$$= (1)^2 + (1) + 1$$

$$= 3$$

§

Example 2 (Finding a Derivative at a Point Using Version 2 of the Limit Definition; Revisiting Example 1)

Let $f(x) = x^3$, as in Example 1. Find $f'(1)$ using Version 2 of the Limit Definition of the Derivative at a Point.

§ Solution

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \quad (\text{Here, } a = 1.)$$

$$= \lim_{h \to 0} \frac{\left[(1+h)^3\right] - \left[(1)^3\right]}{h}$$

We will use the **Binomial Theorem** to expand $(1+h)^3$.

(See Chapter 9 in the Precalculus notes.)

$$= \lim_{h \to 0} \frac{\left[(1)^3 + 3(1)^2(h) + 3(1)(h)^2 + (h)^3\right] - [1]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{1} + 3h + 3h^2 + h^3 \cancel{-1}}{h}$$

$$= \lim_{h \to 0} \frac{3h + 3h^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\overset{(1)}{\cancel{h}}\left(3 + 3h + h^2\right)}{\underset{(1)}{\cancel{h}}}$$

$$= \lim_{h \to 0} \left(3 + 3h + h^2\right)$$

$$= 3 + 3(0) + (0)^2$$

$$= 3$$

We obtain the same result as in Example 1: $f'(1) = 3$. §

PART D: FINDING EQUATIONS OF TANGENT LINES

Example 3 (Finding Equations of Tangent Lines; Revisiting Examples 1 and 2)

Find an equation of the **tangent line** to the graph of $y = x^3$ at the point where $x = 1$. (Review Section 0.14: Lines in the Precalculus notes.)

§ Solution

• Let $f(x) = x^3$, as in Examples 1 and 2.

• Find $f(1)$, the ***y*-coordinate** of the **point** of interest.

$$\begin{aligned} f(1) &= (1)^3 \\ &= 1 \end{aligned}$$

• The **point** of interest is then: $(1, f(1)) = (1, 1)$.

• Find $f'(1)$, the **slope** (*m*) of the desired tangent line.

In Part C, we showed (twice) that: $f'(1) = 3$.

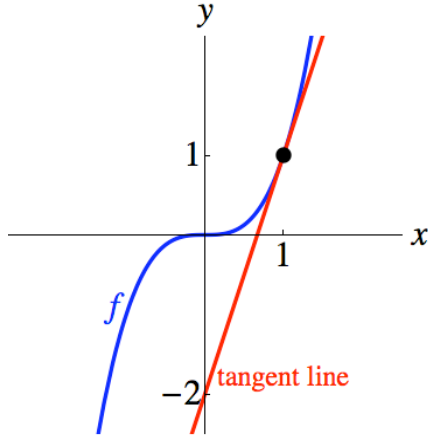
• Find a **Point-Slope Form** for the equation of the tangent line.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 3(x - 1) \end{aligned}$$

• Find the **Slope-Intercept Form** for the equation of the tangent line.

$$\begin{aligned} y - 1 &= 3x - 3 \\ y &= 3x - 2 \end{aligned}$$

• Observe how the **red tangent line** below is consistent with the equation above.



• The **Slope-Intercept Form** can also be obtained directly.

Remember the **Basic Principle of Graphing**: The graph of an equation consists of all points (such as $(1,1)$ here) whose coordinates satisfy the equation.

$$y = mx + b \quad \Rightarrow$$

$$(1) = (3)(1) + b \quad \Rightarrow$$

$$(\text{Solve for } b.)$$

$$b = -2 \quad \Rightarrow$$

$$y = 3x - 2$$

§

PART E: NORMAL LINES

Assume that P is a point on a graph where a tangent line exists.

The normal line to the graph at P is the line that contains P and that is **perpendicular** to the **tangent line** at P.

Example 4 (Finding Equations of Normal Lines; Revisiting Example 3)

Find an equation of the **normal line** to the graph of $y = x^3$ at $P(1,1)$.

§ Solution

• In Examples 1 and 2, we let $f(x) = x^3$, and we found that the **slope** of the **tangent line** at $(1,1)$ was given by: $f'(1) = 3$.

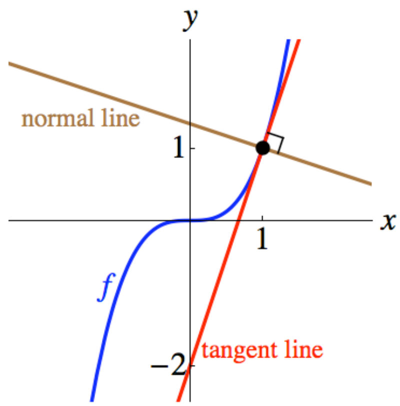
• The **slope** of the **normal line** at $(1,1)$ is then $-\frac{1}{3}$, the **opposite reciprocal** of the slope of the tangent line.

• A **Point-Slope Form** for the equation of the **normal line** is given by:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -\frac{1}{3}(x - 1)$$

• The **Slope-Intercept Form** is given by: $y = -\frac{1}{3}x + \frac{4}{3}$.



WARNING 1: The Slope-Intercept Form for the equation of the **normal line** at *P* **cannot** be obtained by taking the Slope-Intercept Form for the equation of the **tangent line** at *P* and replacing the slope with its opposite reciprocal, **unless** *P* lies on the *y*-axis. In this Example, the normal line is **not** given by: $y = -\frac{1}{3}x - 2$. §

PART F: NUMERICAL APPROXIMATION OF DERIVATIVES

The **Principle of Local Linearity** implies that the slope of the **tangent line** at the point $(a, f(a))$ can be "well approximated" by the slope of the **secant line** on a "small" interval containing *a*.

When using Version 2 of the Limit Definition of the Derivative, this implies that:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \text{ when } h \approx 0.$$

Example 5 (Numerically Approximating a Derivative; Revisiting Example 2)

Let $f(x) = x^3$, as in Example 2. We will find approximations of $f'(1)$. (See Example 8 in Part H.)

h	$\frac{f(1+h) - f(1)}{h}$, or $3 + 3h + h^2$ $(h \neq 0)$ (See Example 2.)
0.1	3.31
0.01	3.0301
0.001	3.003001
$\to 0$	$\to 3$
-0.001	2.997001
-0.01	2.9701
-0.1	2.71

• If we only have a **table of values** for a function f instead of a rule for $f(x)$, we may have to resort to numerically approximating derivatives. §

PART G: AVERAGE RATE OF CHANGE

The average rate of change of f on $[a, b]$ is equal to the **slope** of the **secant line** on $[a, b]$, which is given by: $\dfrac{\text{rise}}{\text{run}} = \dfrac{f(b) - f(a)}{b - a}$. (See Footnotes 2 and 3.)

Example 6 (Average Velocity)

Average velocity is a common example of an average rate of change.

Let's say a car is driven due north 100 miles during a two-hour trip.
What is the average velocity of the car?

• Let t = the time (in hours) elapsed since the beginning of the trip.

• Let $y = s(t)$, where s is the position function for the car (in miles).
s gives the **signed** distance of the car from the starting position.

•• The position (s) values would be **negative** if the car were **south** of the starting position.

• Let $s(0) = 0$, meaning that $y = 0$ corresponds to the starting position.
Therefore, $s(2) = 100$ (miles).

The average velocity on the time-interval $[a, b]$ is the **average rate of change of position with respect to time**. That is,

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

where Δ (uppercase delta) denotes "change in"

$$= \frac{s(b) - s(a)}{b - a}, \text{ a } \textbf{difference quotient}$$

Here, the average velocity of the car on $[0, 2]$ is:

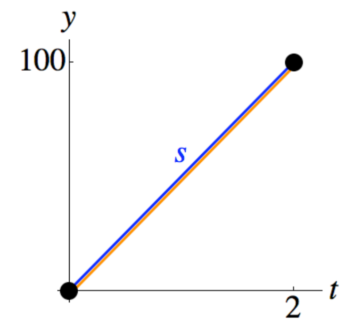
$$\frac{s(2) - s(0)}{2 - 0} = \frac{100 - 0}{2}$$

$$= 50 \ \frac{\text{miles}}{\text{hour}} \left(\text{or } \frac{\text{mi}}{\text{hr}} \text{ or mph} \right)$$

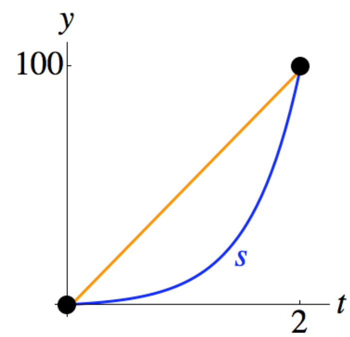
TIP 2: The unit of velocity is the unit of **slope** given by: $\dfrac{\text{unit of } s}{\text{unit of } t}$.

The average velocity is 50 mph on $[0, 2]$ in the three scenarios below. It is the slope of the orange secant line. (Axes are scaled differently.)

• Below, the velocity is constant (50 mph).
(We are not requiring the car to slow down to a stop at the end.)

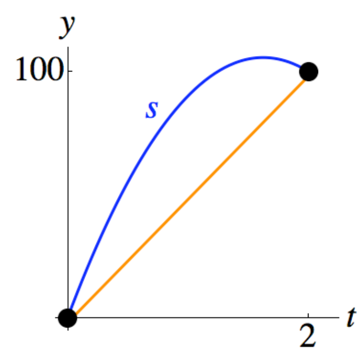


• Below, the velocity is increasing; the car is accelerating.



• Below, the car "breaks the rules," backtracks, and goes south.

WARNING 2: The car's velocity is **negative** in value when it is backtracking; this happens when the graph falls.



Note: The Mean Value Theorem for Derivatives in Section 4.2 will imply that the car must be going **exactly** 50 mph at some time value t in $(0, 2)$. The theorem applies in all three scenarios above, because s is **continuous** on $[0, 2]$ and is **differentiable** on $(0, 2)$. §

PART H: INSTANTANEOUS RATE OF CHANGE

The instantaneous rate of change of f at a is equal to $f'(a)$, if it exists.

Example 7 (Instantaneous Velocity)

Instantaneous velocity (or simply velocity) is a common example of an instantaneous rate of change.

Let's say a car is driven due north for two hours, beginning at noon. How can we find the instantaneous velocity of the car at 1pm?

(If this is positive, this can be thought of as the **speedometer** reading at 1pm.)

- Let t = the time (in hours) elapsed since noon.
- Let $y = s(t)$, where s is the position function for the car (in miles).

Consider **average velocities** on **variable** time intervals of the form $[a, a+h]$, if $h > 0$, or the form $[a+h, a]$, if $h < 0$, where h is a variable run. (We can let $h = \Delta t$.)

The average velocity on the time-interval $[a, a+h]$, if $h > 0$, or $[a+h, a]$, if $h < 0$, is given by:

$$\frac{s(a+h) - s(a)}{h}$$

- This equals the **slope of the secant line** to the graph of s on the interval.

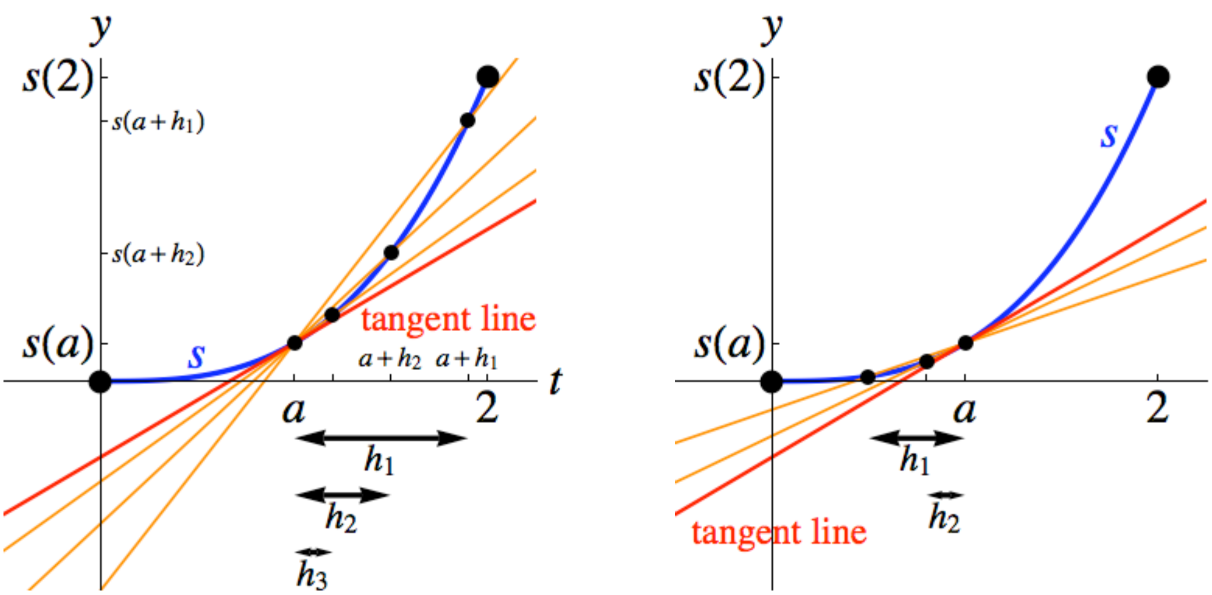
- (See Footnote 1 on the $h < 0$ case.)

Let's assume there exists a **non-vertical tangent line** to the graph of s at the point $(a, s(a))$.

- Then, as $h \to 0$, the slopes of the **secant** lines will approach the slope of this **tangent** line, which is $s'(a)$.

- Likewise, as $h \to 0$, the **average** velocities will approach the **instantaneous** velocity at a.

Below, we let $h \to 0^+$. Below, we let $h \to 0^-$.



The <u>instantaneous velocity</u> (or simply <u>velocity</u>) at a is given by:

$$s'(a), \text{ or } v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}, \text{ if it exists}$$

In our Example, the instantaneous velocity of the car at 1pm is given by:

$$s'(1), \text{ or } v(1) = \lim_{h \to 0} \frac{s(1+h) - s(1)}{h}$$

Let's say $s(t) = t^3$. Example 2 then implies that $s'(1) = v(1) = 3$ mph. §

Example 8 (Numerically Approximating an Instantaneous Velocity; Revisiting Examples 5 and 7)

Again, let's say the position function s is defined by: $s(t) = t^3$ on $[0, 2]$.

We will approximate $v(1)$, the **instantaneous velocity** of the car at 1pm.

We will first compute **average velocities** on intervals of the form $[1, 1+h]$.

Here, we let $h \to 0^+$.

Interval	Value of h (in hours)	Average velocity, $\dfrac{s(1+h)-s(1)}{h}$
$[1, 2]$	1	$\dfrac{s(2)-s(1)}{1} = 7$ mph
$[1, 1.1]$	0.1	$\dfrac{s(1.1)-s(1)}{0.1} = 3.31$ mph
$[1, 1.01]$	0.01	$\dfrac{s(1.01)-s(1)}{0.01} = 3.0301$ mph
$[1, 1.001]$	0.001	$\dfrac{s(1.001)-s(1)}{0.001} = 3.003001$ mph
	$\to 0^+$	$\to 3$ mph

• These average velocities approach 3 mph, which is $v(1)$.

• **WARNING 3: Tables can sometimes be misleading.** The table here does **not** represent a rigorous evaluation of $v(1)$. Answers are not always integer-valued.

We could also consider this approach:

Interval	Value of h	Average velocity, $\frac{s(1+h)-s(1)}{h}$ (rounded off to six significant digits)
$[1, 2]$	1 hour	7.00000 mph
$\left[1, 1\frac{1}{60}\right]$	1 minute	3.05028 mph
$\left[1, 1\frac{1}{3600}\right]$	1 second	3.00083 mph
	$\to 0^+$	$\to 3$ mph

Here, we let $h \to 0^-$.

Interval	Value of h (in hours)	Average velocity, $\frac{s(1+h)-s(1)}{h}$
$[0, 1]$	-1	$\frac{s(0)-s(1)}{-1} = 1$ mph
$[0.9, 1]$	-0.1	$\frac{s(0.9)-s(1)}{-0.1} = 2.71$ mph
$[0.99, 1]$	-0.01	$\frac{s(0.99)-s(1)}{-0.01} = 2.9701$ mph
$[0.999, 1]$	-0.001	$\frac{s(0.999)-s(1)}{-0.001} = 2.997001$ mph
	$\to 0^-$	$\to 3$ mph

• Because of the way we normally look at slopes, we may prefer to rewrite the first difference quotient $\frac{s(0)-s(1)}{-1}$ as $\frac{s(1)-s(0)}{1}$, and so forth. (See Footnote 1.) §

Example 9 (Rate of Change of a Profit Function)

A company sells widgets. Assume that all widgets produced are sold. $P(x)$, the profit (in dollars) if x widgets are produced and sold, is modeled by: $P(x) = -x^2 + 200x - 5000$. Find the **instantaneous** rate of change of profit at 60 widgets. (In economics, this is referred to as marginal profit.)

WARNING 4: We will treat the **domain** of P as $[0, \infty)$, even though one could argue that the domain should only consist of **integers**. Be aware of this issue with applications such as these.

§ Solution

We want to find $P'(60)$.

$$P'(60)$$

$$= \lim_{h \to 0} \frac{P(60+h) - P(60)}{h}$$

$$= \lim_{h \to 0} \frac{\left[-(60+h)^2 + 200(60+h) - 5000\right] - \left[-(60)^2 + 200(60) - 5000\right]}{h}$$

WARNING 5: Grouping symbols are essential when expanding $P(60)$ here, because we are subtracting an expression with more than one term.

$$= \lim_{h \to 0} \frac{\left[-(3600 + 120h + h^2) + 12{,}000 + 200h - 5000\right] - \left[-3600 + 12{,}000 - 5000\right]}{h}$$

TIP 3: Instead of simplifying within the brackets immediately, we will take advantage of **cancellations**.

$$= \lim_{h \to 0} \frac{\cancel{-3600} - 120h - h^2 \cancel{+12{,}000} + 200h \cancel{-5000} \cancel{+3600} \cancel{-12{,}000} \cancel{+5000}}{h}$$

$$= \lim_{h \to 0} \frac{80h - h^2}{h}$$

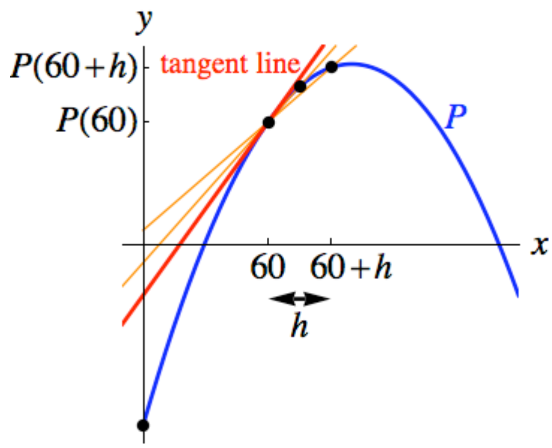
$$= \lim_{h \to 0} \frac{\overset{(1)}{\cancel{h}}(80 - h)}{\underset{(1)}{\cancel{h}}}$$

$$= \lim_{h \to 0} (80 - h)$$

$$= 80 - (0)$$

$$= 80 \ \frac{\text{dollars}}{\text{widget}}$$

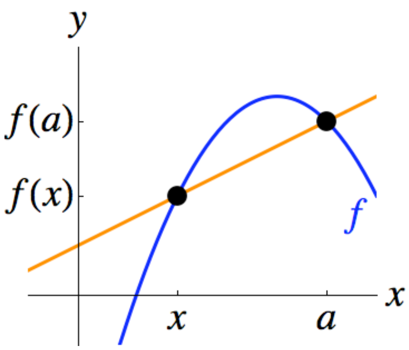
This is the slope of the **red tangent line** below.

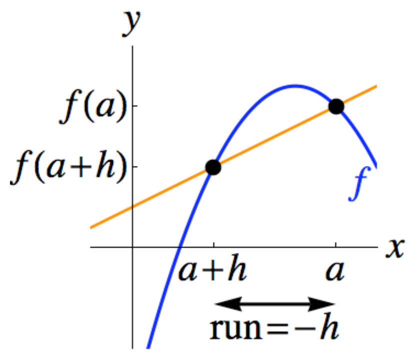


• If we produce and sell one more widget (from 60 to 61), we expect to make about \$80 more in profit. What would be your business strategy if **marginal profit** is positive? §

FOOTNOTES

1. **Difference quotients with negative denominators.** Our forms of difference quotients allow negative denominators ("runs"), as well. They still represent slopes of secant lines.





Left figure $(x < a)$: $\text{slope} = \dfrac{\text{rise}}{\text{run}} = \dfrac{f(a) - f(x)}{a - x} = \dfrac{-\left[f(x) - f(a)\right]}{-(x - a)} = \dfrac{f(x) - f(a)}{x - a}$

Right figure $(h < 0)$:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(a) - f(a+h)}{a - (a+h)} = \frac{-\left[f(a+h) - f(a)\right]}{-h} = \frac{f(a+h) - f(a)}{h}$$

2. **Average rate of change and assumptions made about a function.** When defining the average rate of change of a function f on an interval $[a, b]$, where $a < b$, sources typically do not state the assumptions made about f. The formula $\dfrac{f(b) - f(a)}{b - a}$ seems only to require the existence of $f(a)$ and $f(b)$, but we typically assume more than just that.

   • Although the slope of the secant line on $[a, b]$ can still be defined, we need more for the existence of derivatives (i.e., the differentiability of f) and the existence of non-vertical tangent lines.

   • We ordinarily assume that f is **continuous** on $[a, b]$. Then, there are no holes, jumps, or vertical asymptotes on $[a, b]$ when f is graphed. (See Section 2.8.)

   • We may also assume that f is **differentiable** on $[a, b]$. Then, the graph of f makes no sharp turns and does not exhibit "infinite steepness" (corresponding to vertical tangent lines). However, this assumption may lead to circular reasoning, because the ideas of secant lines and average rate of change are used to develop the ideas of derivatives, tangent lines, and instantaneous rate of change. Differentiability is defined in terms of the existence of derivatives.

   • We may also need to assume that f' is continuous on $[a, b]$. (See Footnote 3.)

3. **Average rate of change of f as the average value of f'.** Assume that f' is continuous on $\left[a, b\right]$. Then, the **average rate of change of f** on $\left[a, b\right]$ is equal to the **average value of f'** on $\left[a, b\right]$. In Chapter 5, we will assume that a function (say, g) is continuous on $\left[a, b\right]$ and then define the **average value of g** on $\left[a, b\right]$ to be $\dfrac{\int_a^b g(x)\,dx}{b-a}$; the numerator is a <u>definite integral</u>, which will be defined as a limit of sums. Then, the **average value of f'** on $\left[a, b\right]$ is given by: $\dfrac{\int_a^b f'(x)\,dx}{b-a}$, which is equal to $\dfrac{f(b)-f(a)}{b-a}$ by the <u>Fundamental Theorem of Calculus</u>. The theorem assumes that the integrand [function], f', is continuous on $\left[a, b\right]$.

SECTION 3.2: DERIVATIVE FUNCTIONS and DIFFERENTIABILITY

LEARNING OBJECTIVES

- Know, understand, and apply the Limit Definition of the Derivative Function.
- Know short cuts for differentiation, including the Power Rule.
- Evaluate derivative functions and relate their values to slopes of tangent lines and rates of change.
- Be able to find higher-order derivatives, and recognize notations for various orders.
- Understand the relationships between position, velocity, and acceleration in rectilinear motion.
- Recognize differentiability of functions on open and closed intervals.
- Recognize possible behaviors of functions and their graphs where they are not differentiable.

PART A: DERIVATIVE FUNCTIONS

Let f be a function. f may have a derivative function, called f', whose rule may be given by either of the following.

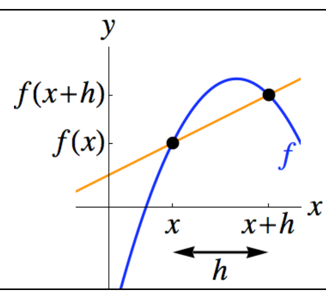
Limit Definition of the Derivative Function (Version 1)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}, \text{ if it exists}$$

Limit Definition of the Derivative Function (Version 2; "Two-Sided" Approach)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}, \text{ if it exists}$$

We have taken Limit Definitions from Section 3.1 and replaced the constant *a* with the variable *x*.

$\frac{f(x+h) - f(x)}{h}$		*x*, *h* are variable

Take either version. The **domain** of f' consists of all real values of x for which the indicated limit exists. We say that f is differentiable at those values.

The process of finding $f'(x)$ is called differentiation. We may say that f' is the **derivative** of the function f, or that $f'(x)$ is the **derivative** of the expression $f(x)$ with respect to x. x is the variable of differentiation.

(See Footnote 1.)

PART B: SHORT CUTS FOR DIFFERENTIATION

We may think of **derivative functions** as **slope functions**.

Some Short Cuts for Differentiation

Assumptions:

- c, m, b, and n are real constants.
- f is a function that is differentiable "where we care."

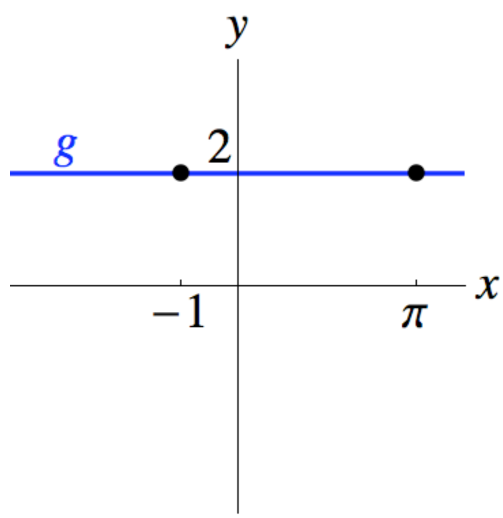
	If $g(x) =$	then $g'(x) =$	Comments
1.	c	0	The derivative of a constant is 0.
2.	$mx + b$	m	The derivative of a linear function is the slope.
3.	x^n	nx^{n-1}	Power Rule
4.	$c \cdot f(x)$	$c \cdot f'(x)$	Constant Multiple Rule

• **Proofs.** The Limit Definition of the Derivative can be used to prove these short cuts. (See Footnotes 2 and 3.)

Example 1 (Rule 1: Differentiating a Constant [Function])

Let $g(x) = 2$. Then, $g'(x) = 0$ (for all real values of *x*; this goes without saying). For instance, $g'(-1) = 0$ and $g'(\pi) = 0$.

Observe that, for each real value of *x*, the corresponding point on the graph of *g* below has a **horizontal tangent line**, namely the graph itself.



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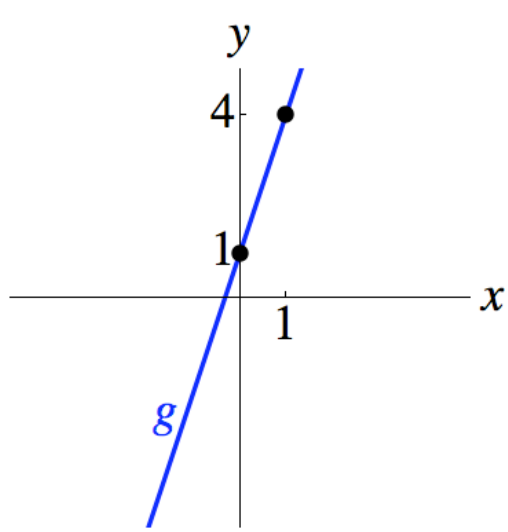
Example 2 (Rule 2: Differentiating a Linear Function)

Let $g(x) = 3x + 1$. Then, $g'(x) = 3$ (for all real values of *x*).
For instance, $g'(0) = 3$ and $g'(1) = 3$.

Observe that, for each real value of *x*, the corresponding point on the graph of *g* below has a **tangent line** of **slope** 3, namely the graph itself.



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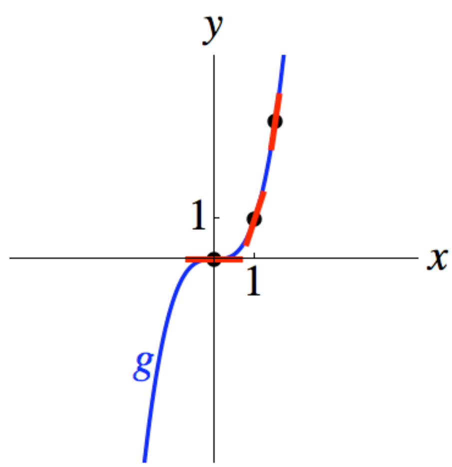
Example 3 (Motivating Rule 3: Differentiating a Power Function; Evaluating Derivatives and Velocities)

Let $g(x) = x^3$. Unlike in Examples 1 and 2, the derivative function g' will **not** be a constant function. Different tangent lines to the graph of g can have **different slopes**.



We will use the Limit Definition of the Derivative to find $g'(x)$.
This will parallel our work in Example 2 in Section 3.1.

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(x+h)^3\right] - \left[x^3\right]}{h}$$

We will use the **Binomial Theorem** to expand $(x+h)^3$.
(See Chapter 9 in the Precalculus notes.)

$$= \lim_{h \to 0} \frac{\left[(x)^3 + 3(x)^2(h) + 3(x)(h)^2 + (h)^3\right] - \left[x^3\right]}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 \cancel{- x^3}}{h}$$

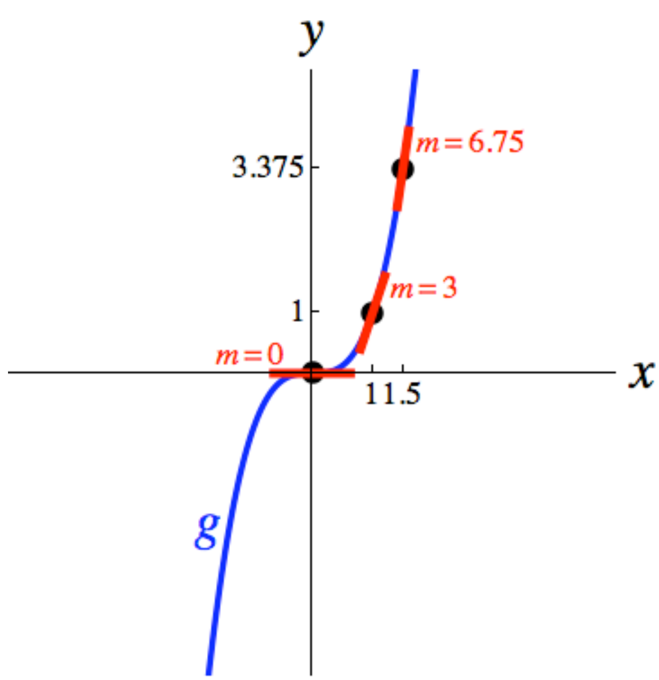
$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\overset{(1)}{\cancel{h}}\left(3x^2 + 3xh + h^2\right)}{\underset{(1)}{\cancel{h}}}$$

$$= \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right)$$
$$= 3x^2 + 3x(0) + (0)^2$$
$$= 3x^2$$

We now have the **derivative function rule** $g'(x) = 3x^2$ (for all real values of x). For instance, $g'(1) = 3(1)^2 = 3$. We already knew this from Section 3.1, Part C, but we can now quickly find derivatives for **other** values of x. For example, $g'(0) = 3(0)^2 = 0$, and $g'(1.5) = 3(1.5)^2 = 6.75$.

WARNING 1: Do not confuse **the original function's values**, which correspond to ***y*-coordinates** of points, with **derivative values**, which correspond to **slopes** of tangent lines. For example, $g(1.5) = (1.5)^3 = 3.375$, which means that the **point** $(1.5,\ 3.375)$ lies on the graph of g. Also, $g'(1.5) = 3(1.5)^2 = 6.75$, which is the **slope** of the tangent line at that point.



In Section 3.1, we saw that derivatives can be related to **rates of change**, such as **velocity**. In Section 3.1, Example 7, if the **position** function is given by $s(t) = t^3$, then the **velocity** function is given by $v(t) = s'(t) = 3t^2$. For instance, $v(1) = 3$ mph, and $v(1.5) = 6.75$ mph. §

Example Set 4 (Rule 3: Power Rule)

Unless we are instructed to use the Limit Definition of the Derivative, as in Example 3, we will use the **Power Rule of Differentiation** as a short cut to differentiate a power of x such as x^3. We "**bring down**" the exponent (3) as a coefficient, and we then **subtract one** to get the new exponent, resulting in $3x^2$.

TIP 1: Be prepared to rewrite a variety of expressions as **powers of *x*** so that the Power Rule may be readily applied.

WARNING 2: We will differentiate expressions such as 3^x and x^x in Chapter 7. They do **not** represent power functions, and the Power Rule here does **not** apply.

If $g(x) =$	then $g'(x) =$
x^2	$2x^1 = 2x$
x^3	$3x^2$
x^4	$4x^3$
$x^{\sqrt{17}}$	$\sqrt{17}x^{\sqrt{17}-1}$
$\frac{1}{x} = x^{-1}$	$-1x^{-2} = -\frac{1}{x^2}$
$\frac{1}{x^2} = x^{-2}$	$-2x^{-3} = -\frac{2}{x^3}$
$\sqrt{x} = x^{1/2}$	$\frac{1}{2}x^{-1/2} = \frac{1}{2x^{1/2}}$, or $\frac{1}{2\sqrt{x}}$
$\sqrt[3]{x} = x^{1/3}$	$\frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$, or $\frac{1}{3\left(\sqrt[3]{x^2}\right)}$
$\frac{1}{\sqrt[4]{x^5}} = \frac{1}{x^{5/4}} = x^{-5/4}$	$-\frac{5}{4}x^{-9/4} = -\frac{5}{4x^{9/4}}$, or $-\frac{5}{4\left(\sqrt[4]{x^9}\right)}$

- In an algebra class, $\sqrt[4]{x^9}$ may be rewritten as $x^2\left(\sqrt[4]{x}\right)$.
- We will discuss domain issues later in this section. §

The above table demonstrates the following:

The derivative of an **even** function is **odd**.
The derivative of an **odd** function is **even**.

Example 5 (Rule 4: Constant Multiple Rule)

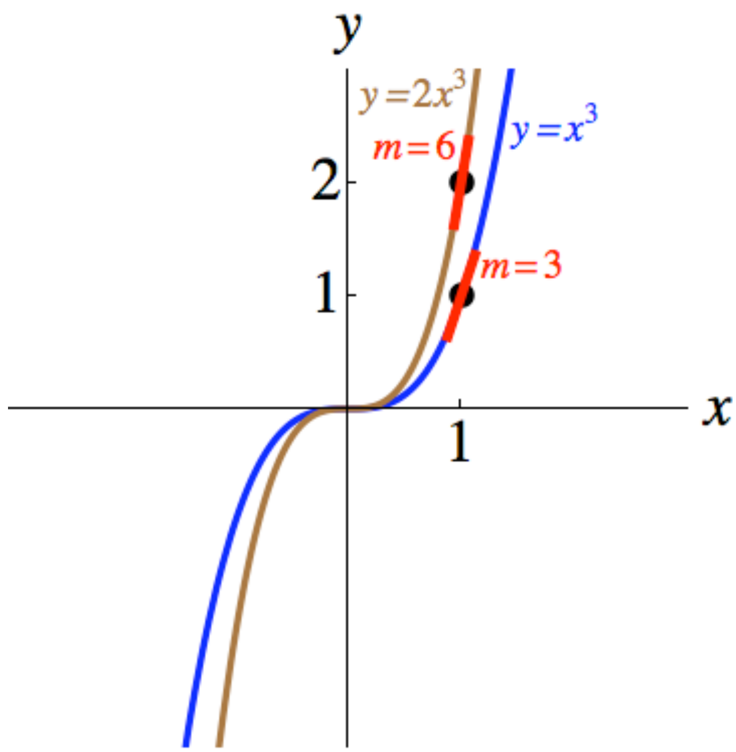
Informally, the **Constant Multiple Rule** states that the derivative of a **constant multiple** equals the **constant multiple** of the derivative.

• For example, the derivative of **twice** x^3 is **twice** the derivative of x^3.

• That is, if $g(x) = 2x^3$, then $g'(x) = 2\left(3x^2\right) = 6x^2$.

TIP 2: Basically, we **multiply** the coefficient by the exponent, and we then **subtract one** from the old exponent to get the new exponent.

• For instance, $g'(1) = 6(1)^2 = 6$.



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PART C: HIGHER-ORDER DERIVATIVES

Indeed, we can take the derivative of the derivative of …, and so on.

> Higher-Order Derivatives
>
> $f''(x)$, read as "***f*** **double prime** of (or at) x," is the second derivative (or the second-order derivative) of f with respect to x.
>
> • It is the [first] derivative of $f'(x)$ with respect to x.
>
> $f'''(x)$, read as "***f*** **triple prime** of (or at) x," is the third derivative (or the third-order derivative) of f with respect to x.
>
> • It is the [first] derivative of $f''(x)$ with respect to x.
>
> Higher-order derivatives are denoted by $f^{(4)}(x)$, $f^{(5)}(x)$, etc.
> Roman numerals might also be used: $f^{\text{IV}}(x)$, $f^{\text{V}}(x)$, etc.

• (See Footnote 4. See Chapter 4 for graphical interpretations of f''.)

Example 6 (Higher-Order Derivatives)

Let $f(x) = x^3$. Then, $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, and $f^{(4)}(x) = 0$. §

PART D: RECTILINEAR MOTION: POSITION, VELOCITY, and ACCELERATION

In Section 3.1, Parts G and H, we discussed the motion of a car being driven due north. This is an example of rectilinear motion, or motion along a **coordinate line**. The **starting position** of the car corresponds to "0" on the line. The real numbers on the line correspond to **position values**, which are **signed distances** of the car from the starting position.

• For instance, "3" corresponds to the position three miles due **north** of the starting point. Positive directions are typically associated with north, up, east, right, or forward.

• Also, "-3" corresponds to the position three miles due **south** of the starting point. Negative directions are typically associated with south, down, west, left, or backward.

$$\begin{array}{c} + \\ \uparrow \\ 0 \\ \downarrow \\ - \end{array} \qquad\qquad - \leftarrow 0 \rightarrow +$$

In our car examples, we let s be the position function for the car.

$s(t)$ gives the **position value** of the car (in miles) t hours after the trip begins.

The **independent variable** t represents "time" or "time elapsed." It will be our **variable of differentiation**.

Let v be the velocity function for the car. Then, $v(t) = s'(t)$, because velocity is the **rate of change of position** with respect to time. The unit of velocity here is miles per hour, or $\frac{\text{mi}}{\text{hr}}$, or mph.

Let a be the acceleration function for the car. Then, $a(t) = v'(t) = s''(t)$, because acceleration is the **rate of change of velocity** with respect to time. The unit of acceleration here is miles per hour per hour, or $\frac{\text{mi}}{\text{hr}^2}$.

Example 7 (Average Acceleration)

A commercial says that a car can go from 0 [mph] to 60 [mph] in 5 seconds. The **average acceleration** of the car on that five-second interval $[0, 5]$ is given by:

$$\frac{v(5) - v(0)}{5 - 0} = \frac{60 - 0}{5} = 12\ \frac{\text{mph}}{\text{sec}}$$

We can convert to a more "internally consistent" unit:

$$12\ \frac{\text{mph}}{\text{sec}} = \left(12\ \frac{\text{mi/hr}}{\cancel{\text{sec}}}\right)\left(\frac{3600\ \cancel{\text{sec}}}{1\ \text{hr}}\right) = 43{,}200\ \frac{\text{mi}}{\text{hr}^2}$$

§

Example 8 (Position, Velocity, and Acceleration)

Let $s(t) = t^3$, as in Example 3. Then, $v(t) = s'(t) = 3t^2$, and $a(t) = v'(t) = 6t$. For instance, $a(1) = 6(1) = 6\ \frac{\text{mi}}{\text{hr}^2}$. This means that the car's acceleration is 6 miles per hour per hour when one hour has elapsed. §

PART E: NOTATIONS FOR DERIVATIVES

Let $y = f(x)$, where $f(x) = x^3$, say. The various orders of derivatives can be denoted in a variety of ways.

First derivative	Second derivative	*n*th derivative
$f'(x) = 3x^2$ (Lagrange's notation)	$f''(x) = 6x$ (Lagrange's notation)	$f^{(n)}(x)$ (Lagrange's notation)
$y' = 3x^2$ (See Warning 3.)	$y'' = 6x$ (See Warning 3.)	(See Warning 4.)
$\frac{dy}{dx} = 3x^2$ (Leibniz's notation; see note on next page)	$\frac{d^2y}{dx^2} = 6x$ (Leibniz's notation; see "Differential operators" note)	$\frac{d^ny}{dx^n}$ (Leibniz's notation; see "Differential operators" note)
$\frac{d}{dx}y = 3x^2$, or $\frac{d}{dx}\left(x^3\right) = 3x^2$ (See "Differential operators" note.)	$\frac{d^2}{dx^2}y = 6x$, or $\frac{d^2}{dx^2}\left(x^3\right) = 6x$ (See "Differential operators" note.)	$\frac{d^n}{dx^n}y$, or $\frac{d^n}{dx^n}\left(x^3\right)$ (See "Differential operators" note.)
$D_x\left(x^3\right) = 3x^2$ (Euler's notation; see "Differential operators" note.)	$D_x^2\left(x^3\right) = 6x$, or $D_{xx}\left(x^3\right) = 6x$ (Euler's notation; see "Differential operators" note.)	$D_x^n\left(x^3\right)$ (Euler's notation; see "Differential operators" note.)

WARNING 3: The y' notation suffers the critical drawback of not indicating the **variable of differentiation** (here, *x*). In this work, we will assume that $y' = \frac{dy}{dx}$. Other derivatives such as $\frac{dy}{dt}$, $\frac{dy}{d\theta}$, etc. will **not** be denoted by y'.

WARNING 4: It is not recommended to use $y^{(n)}$, since it is easily confused with y^n, the *n*th **power** of *y*. (See Footnote 4.)

• **Leibniz's notation.** The notation $\frac{dy}{dx}$ evokes the idea of slope. It may be thought of as a quotient of <u>differentials</u> (*dy* and *dx*), which represent "infinitesimal" (arbitrarily small) changes in *y* and *x*. (See Section 3.5.) Separating the differentials is frequently done in practice, although many think of $\frac{dy}{dx}$ as an inseparable entity rather than a quotient in rigorous work.

• **Differential operators.**

•• $\frac{d}{dx}$ and D_x "operate" on the following expression by differentiating it with respect to *x*.

•• Some sources simply use *D*. Generically, if f is the cubing function, then Df is three times the squaring function.

•• $\frac{d^2}{dx^2}$ indicates repeated (or "iterated") differentiation.
For example, $\frac{d^2}{dx^2} y = \frac{d}{dx}\left(\frac{d}{dx} y\right)$.

• **Newton's notation (obsolete).** Sir Isaac Newton referred to <u>fluxions</u>, where derivatives were taken with respect to time: $\dot{x} = \frac{dx}{dt}$.

PART F: DIFFERENTIABILITY ON INTERVALS; RIGHT-HAND and LEFT-HAND DERIVATIVES and TANGENT LINES

Assume that f is a function and a and b are real constants such that $a < b$.

Differentiability on an Open Interval

f is differentiable on the open interval (a, b) $\Leftrightarrow$
f is differentiable at all real numbers in (a, b)

• This extends to unbounded open intervals of the form (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$.

Right-Hand Derivative at a Point a; Right-Hand Tangent Lines

The right-hand derivative at a is defined as:

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}, \text{ if it exists}$$

We define the right-hand tangent line at the point $(a, f(a))$ to be the line passing through this point whose slope is equal to $f'_+(a)$.

• If the above limit can be said to be ∞ or $-\infty$, and if f is continuous from the right at a, then the right-hand tangent line is **vertical**. Informally, a **vertical tangent line** indicates where a graph is becoming "infinitely steep."

• (See Footnote 5 on notation.)

Left-Hand Derivative at a Point b; Left-Hand Tangent Lines

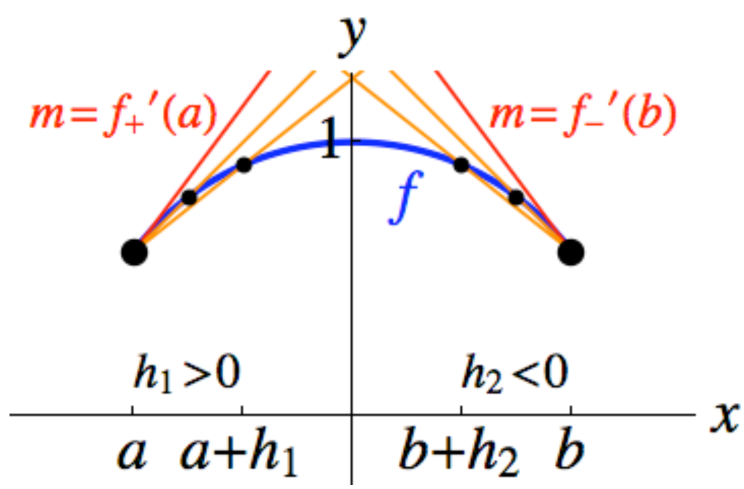
The left-hand derivative at b is defined as:

$$f'_-(b) = \lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}, \text{ if it exists}$$

We define the left-hand tangent line at the point $(b, f(b))$ to be the line passing through this point whose slope is equal to $f'_-(b)$.

• If the above limit can be said to be ∞ or $-\infty$, and if f is continuous from the left at a, then the left-hand tangent line is **vertical**.

• (See Section 3.1, Footnote 1 on sign issues.)



<u>Relating One-Sided and Two-Sided Derivatives</u>

As with limits, $f'(a)$ exists $\Leftrightarrow$ $f'_+(a)$ and $f'_-(a)$ both exist, and $f'_+(a) = f'_-(a)$.

If c is a real constant, then $f'(a) = c \Leftrightarrow f'_+(a) = c$ and $f'_-(a) = c$.

<u>Differentiability on a Closed Interval</u>

f is differentiable on the closed interval $[a, b]$ $\Leftrightarrow$

1) f is defined on $[a, b]$,

2) f is differentiable on (a, b),

3) $f'_+(a)$ exists, and

4) $f'_-(b)$ exists

• 3) and 4) weaken the differentiability requirements at the endpoints, *a* and *b*. Imagine taking limits as we "push outwards" towards the endpoints. Observe the similarity with the idea of **continuity** on a closed interval.

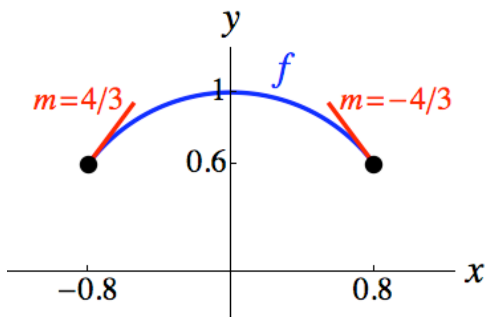
• Differentiability on **half-open, half-closed intervals** such as $[a, b)$ can be similarly defined. In the case of $[a, b)$, we would replace $[a, b]$ with $[a, b)$ in 1), and we would delete 4).

<u>WARNING 5</u>: Differentiability of f on an interval such as $[a, b]$ or $[a, b)$ does **not** imply differentiability [in a two-sided sense] at *a*. That is, *a* might **not** be in $\text{Dom}(f')$. (Many sources avoid this issue.)

Example 9 (Differentiability on a Closed Interval)

Let $f(x)=\sqrt{1-x^2}$ on the restricted x-interval $[-0.8, 0.8]$.
Then, f is differentiable on that interval.

Parts of the **one-sided tangent lines** at the endpoints of the graph of f are drawn in red below.



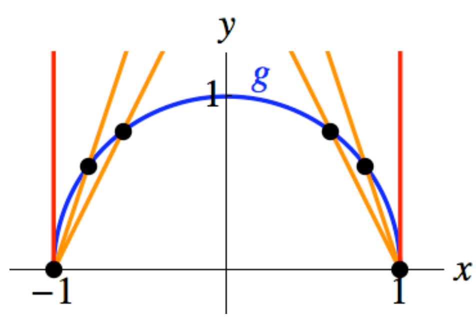
• Methods from Section 3.6 will allow us to find $f'(x)$.

• It turns out that $f'_+(-0.8)=\frac{4}{3}$ and $f'_-(0.8)=-\frac{4}{3}$. §

Example 10 (Vertical Tangent Lines)

Let $g(x)=\sqrt{1-x^2}$ on the implied domain, $[-1,1]$.

Then, g is differentiable on the open interval $(-1,1)$ but is **not** differentiable on the closed interval $[-1,1]$.



• There is a **right-hand vertical tangent line** (in red) at the point $(-1,0)$, because the limit of the slopes of the secant lines (in orange) "coming in" from the **right** can be said to be ∞. Informally, we will write $g'_+(-1):\infty$,

because $\lim_{h\to 0^+}\frac{g(-1+h)-g(-1)}{h}=\infty$. (It is **not** $-\infty$, because the secant lines **rise** from left to right, and we still look at slopes "left-to-right.")
Also, g is **continuous from the right** at -1.

• There is a **left-hand vertical tangent line** (in red) at the point $(1,0)$, because the limit of the slopes of the secant lines (in orange) "coming in" from the **left** can be said to be $-\infty$. Informally, we will write $g'_{-}(1):-\infty$, because $\lim_{h\to 0^-}\frac{g(1+h)-g(1)}{h}=-\infty$. Also, g is **continuous from the left** at 1. §

PART G: NON-DIFFERENTIABILITY

We will examine a variety of situations in which a function f is **not** differentiable at a real constant a. That is, $f'(a)$ **does not exist (DNE)**.

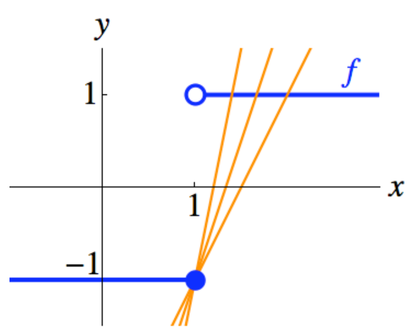
> Differentiability Implies (and therefore Requires) Continuity
>
> 1) If f is differentiable at a, then f is continuous at a.
>
> 2) If f is **not** continuous at a, then f is **not** differentiable at a.
>
> • 1) and 2) form a pair of **contrapositive** statements. Therefore, they are logically equivalent. Since 1) is true, 2) must also hold true.

• Footnote 6 has a proof.

Example 11 (Differentiability Requires Continuity)

Let $f(x)=\begin{cases}1, & x>1\\ -1, & x\le 1\end{cases}$. Then, f is **not continuous** at $x=1$.

Therefore, f is **not differentiable** at $x=1$.



The slopes of the secant lines "coming in" from the right at $x=1$ approach ∞, so $\lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\infty$, and $f'(1)$ does not exist (DNE). However, because f is **not** continuous from the right at $x=1$, its graph does **not** have a right-hand vertical tangent line at the point $(1,-1)$. §

We now consider situations where a function is **continuous** at *a*, but it is **not differentiable** there. This typically means that its graph makes a **sharp turn** at $x = a$ (indicating two tangent lines) or it has a **vertical tangent line** there.

<u>Losing Differentiability at a Corner</u>

Assume that f is **continuous** at a. f is **not differentiable** at a, and its graph has a <u>corner</u> at the point $(a, f(a))$ ⇔ 1), 2), or 3) below holds:

1) $f'_+(a)$ and $f'_-(a)$ both exist, but $f'_+(a) \neq f'_-(a)$,
2) $f'_+(a)$ exists and $f'_-(a) : \pm\infty$ (left-hand tangent line is vertical), or
3) $f'_+(a) : \pm\infty$ (right-hand tangent line is vertical) and $f'_-(a)$ exists

• A point on a graph is a **corner** ⇔ there are **two distinct tangent lines** there, one from each side.

<u>Example 12 (Losing Differentiability at a Corner; Derivatives of Piecewise-Defined Functions)</u>

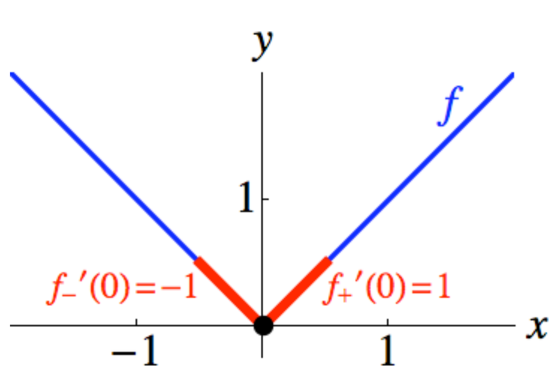
Let $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.

We can use our **basic rules** to differentiate the different rules for $f(x)$ on their different subdomains (indicated by "$x \geq 0$" and "$x < 0$"), although we must investigate values of *x* where the rule for $f(x)$ **changes** (here, at $x = 0$).

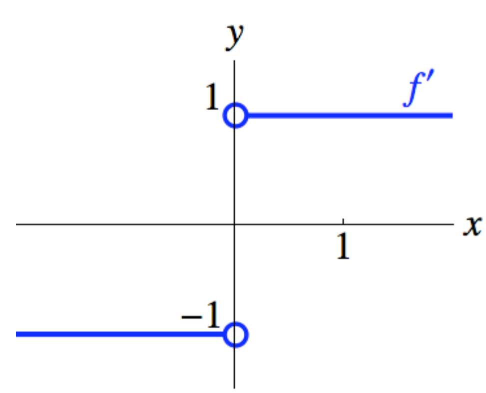
$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Although f is continuous at $x = 0$, we can see from the graph of f below that f is **not** differentiable there, and there is a **corner** at the origin. This is because $f'_+(0) = 1$, while $f'_-(0) = -1$.

Graph of f



Graph of f'



• Here is the proof that $f'_+(0) = 1$:

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - 0}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

• Here is the proof that $f'_-(0) = -1$:

$$f'_-(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

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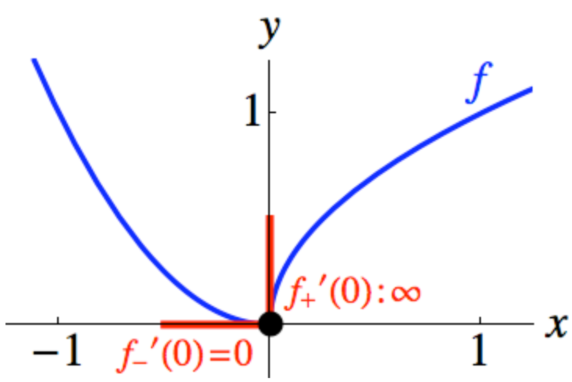
Example 13 (Losing Differentiability at a Corner with a Vertical Tangent Line)

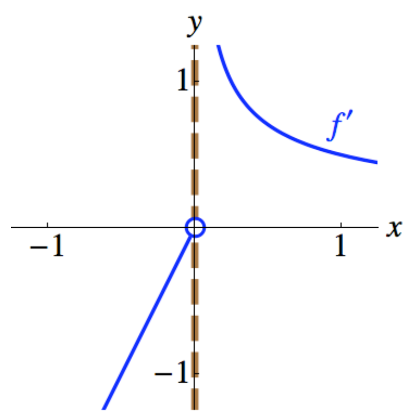
Let $f(x) = \begin{cases} x^2, & x < 0 \\ \sqrt{x}, & x \geq 0 \end{cases}$. Then, $f'(x) = \begin{cases} 2x, & x < 0 \\ \dfrac{1}{2\sqrt{x}}, & x > 0 \end{cases}$.

Although f is continuous at $x = 0$, we can see from the graph of f below that f is **not** differentiable there, and there is a **corner** at the origin. This is because $f'_+(0) : \infty$, while $f'_-(0) = 0$.

Graph of f



Graph of f'



• Here is the proof that $f'_+(0) : \infty$:

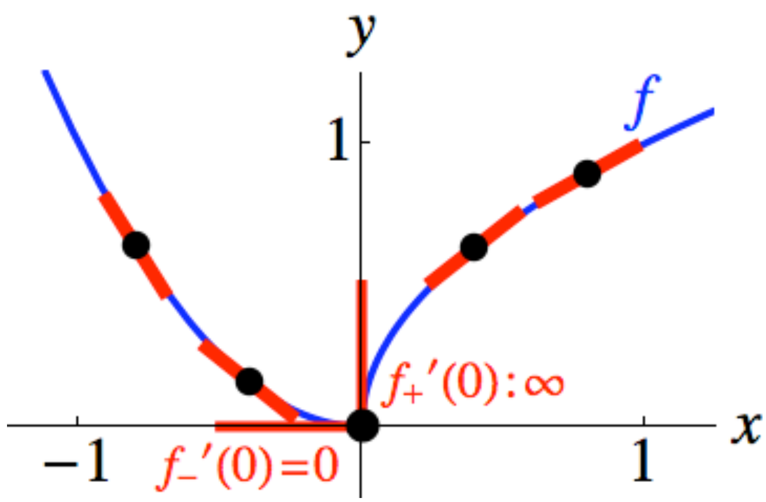
$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - 0}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h}$$

$$= \lim_{h \to 0^+} \frac{1}{\sqrt{h}} \quad \left(\text{Limit Form: } \frac{1}{0^+} \right) = \infty$$

• Here is the proof that $f'_{-}(0) = 0$:

$$f'_{-}(0) = \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{f(h) - 0}{h} = \lim_{h \to 0^-} \frac{h^2}{h}$$
$$= \lim_{h \to 0^-} h = 0$$

• **An informal short cut.** If f is a "simple" function that is continuous at a, a **one-sided derivative** (even informally as $\pm\infty$) can be guessed at by taking the corresponding one-sided **limit of the derivative** as $x \to a$. That is, we tentatively guess that $f'_{+}(a) = \lim_{x \to a^+} f'(x)$, and $f'_{-}(a) = \lim_{x \to a^-} f'(x)$, where ∞ and $-\infty$ are possible informal results. Instead of taking the limit of slopes of **secant lines**, we are taking the limit of slopes of **tangent lines**.



Here:

•• Guess: $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{1}{2\sqrt{x}}$ $\left(\text{Limit Form: } \frac{1}{0^+} \right) = \infty$,

which suggests that $f'_{+}(0) : \infty$, and

•• Guess: $\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} (2x) = 0$,

which suggests that $f'_{-}(0) = 0$.

However, this "trick" does **not** work for more complicated functions! (See Footnote 7.) §

Losing Differentiability at a Cusp

Assume that f is **continuous** at a. f is **not differentiable** at a, and its graph has a cusp at the point $(a, f(a))$ $\Leftrightarrow$ 1) or 2) below holds:

1) $f'_+(a): \infty$ and $f'_-(a): -\infty$, or

2) $f'_+(a): -\infty$ and $f'_-(a): \infty$

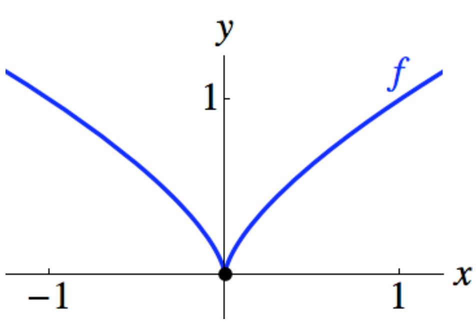
• The right-hand and left-hand tangent lines at a cusp are **both vertical**, but the secant lines "coming in" from one side **fall**, while the secant lines "coming in" from the other side **rise**.

Example 14 (Losing Differentiability at a Cusp)

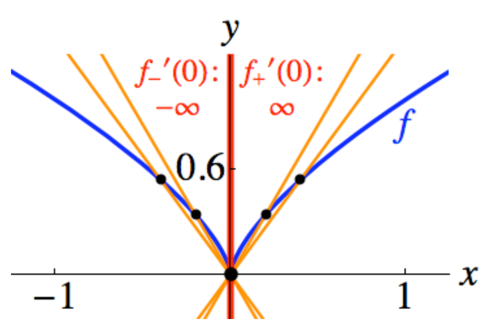
Let $f(x) = x^{2/3}$, or $\sqrt[3]{x^2}$. For **graphing** purposes, observe that f is an even, nonnegative function with domain $\mathbb{R}$.

Although f is continuous at $x = 0$, we can see from the graph of f below that f is **not** differentiable there, and there is a **cusp** at the origin. This is because $f'_+(0): \infty$, while $f'_-(0): -\infty$.

Graph of f



With secant lines (in orange) and tangent line (in red)



• Here is the proof that $f'_+(0): \infty$:

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{f(h) - 0}{h} = \lim_{h \to 0^+} \frac{h^{2/3}}{h}$$

$$= \lim_{h \to 0^+} \frac{1}{h^{1/3}} = \lim_{h \to 0^+} \frac{1}{\sqrt[3]{h}} \quad \left(\text{Limit Form: } \frac{1}{0^+} \right) = \infty$$

• Here is the proof that $f'_-(0):-\infty$:

$$f'_-(0)=\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{f(h)-0}{h}=\lim_{h\to 0^-}\frac{h^{2/3}}{h}$$

$$=\lim_{h\to 0^-}\frac{1}{h^{1/3}}=\lim_{h\to 0^-}\frac{1}{\sqrt[3]{h}}\quad\left(\text{Limit Form: }\frac{1}{0^-}\right)=-\infty$$

• **Using the informal short cut.** $f'(x)=\dfrac{2}{3}x^{-1/3}=\dfrac{2}{3\left(\sqrt[3]{x}\right)}$.

•• Guess: $\displaystyle\lim_{x\to 0^+}f'(x)=\lim_{x\to 0^+}\frac{2}{3\left(\sqrt[3]{x}\right)}\quad\left(\text{Limit Form: }\frac{2}{0^+}\right)=\infty$,

which suggests that $f'_+(0):\infty$, and

•• Guess: $\displaystyle\lim_{x\to 0^-}f'(x)=\lim_{x\to 0^-}\frac{2}{3\left(\sqrt[3]{x}\right)}\quad\left(\text{Limit Form: }\frac{2}{0^-}\right)=-\infty$,

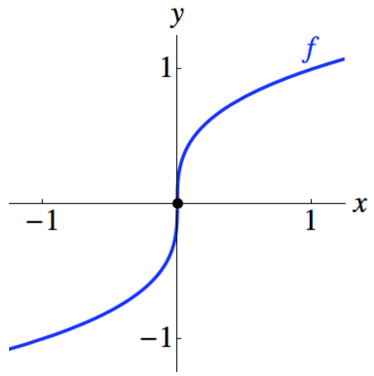
which suggests that $f'_-(0):-\infty$. §

Example 15 (Losing Differentiability at a Point with a Vertical Tangent Line)

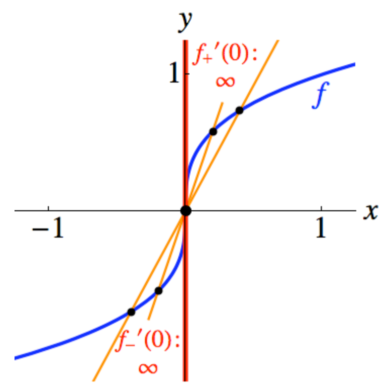
Let $f(x)=x^{1/3}$, or $\sqrt[3]{x}$. For **graphing** purposes, observe that f is an odd function with domain $\mathbb{R}$.

Although f is continuous at $x=0$, we can see from the graph of f below that f is **not** differentiable there, and there is a **vertical tangent line** at the origin, even though there is **neither** a corner nor a cusp there. This is because $f'_+(0):\infty$ and $f'_-(0):\infty$.

Graph of f



With secant lines (in orange) and tangent line (in red)



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FOOTNOTES

1. **Functions that are nowhere differentiable.** Functions that are nowhere continuous are also nowhere differentiable. See Footnote 3 in Section 2.8.

2. **Proof of the Power Rule of Differentiation.** An elegant proof of the Power Rule for **all real** constants *n* will be found in the Footnotes for Section 7.5; it will employ Logarithmic Differentiation. Some sources use the Binomial Theorem to first prove it for **positive integers** *n*. $n = 0$ corresponds to the special case of differentiating 1; 0^0 is often defined to be 1 for this purpose. For **positive rational** values of *n*, let $n = \frac{p}{q}$, where *p* and *q* are positive integers. Let $y = x^n$. Then, $y = x^{p/q}$, and $y^q = x^p$. The Implicit Differentiation technique from Section 3.7 can be used to prove the Power Rule in this case. For **negative rational** values of *n*, let $m = -n$. Then, $x^n = x^{-m} = \frac{1}{x^m}$, and the Reciprocal (or Quotient) Rule of Differentiation in Section 3.3 can be applied. In these last two cases, the domain of the derivative might not be $\mathbb{R}$.

3. **Proof of the Constant Multiple Rule of Differentiation.** Assume that f is a function that is differentiable "where we care," and *c* is a real constant. Let $g(x) = c \cdot f(x)$.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = \lim_{h\to 0} \frac{c \cdot [f(x+h) - f(x)]}{h}$$

(We will exploit the Constant Multiple Rule of Limits.)

$$= c \cdot \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = c \cdot f'(x)$$

4. $\boldsymbol{f^n}$ **notation.** We use $f^{(n)}$ to denote an *n*th-order derivative, as opposed to f^n. This is because *n* often represents an **exponent** in the notation f^n, **except** when $n = -1$ (in which case we have a function inverse). For example, f^2 is often taken to mean ff; that is, $f^2(x) = [f(x)][f(x)]$. For example, we will accept that $\sin^2 x = (\sin x)(\sin x)$, which is the standard interpretation. <u>Note</u>: $f''(x)$ is typically **not** equivalent to $[f'(x)][f'(x)]$.

   • On the other hand (and this compounds the confusion), some sources use *n* to indicate the **number of applications** of f in compositions of f with itself; the result is called an <u>iterated function</u>. For example, they would let $f^2 = f \circ f$, and they would use the rule: $f^2(x) = f(f(x))$. This is typically different from the rule $f^2(x) = [f(x)][f(x)]$. However, our use of the notation f^{-1} for "f inverse" is more consistent with this second interpretation, since $f^{-1} \circ f$ is an identity function, which could be construed as f^0 in this context. <u>Note</u>: $f''(x)$ is typically **not** equivalent to $f'(f'(x))$.

5. **Notation for right-hand and left-hand derivatives.** There appears to be no standard notation for right-hand and left-hand derivatives. In fact, $f'_+(a)$ sometimes denotes the upper right Dini derivative at *a*, which is a bit different from what we are calling a right-hand derivative. If the upper right Dini derivative at *a* and the lower right Dini derivative at *a* exist and are equal, then their common value is the right-hand (or right-hand Dini) derivative at *a*. Likewise, if the upper left and lower left Dini derivatives at *a* are equal, then their common value is the left-hand derivative at *a*. See T.P. Lukashenko, "Dini derivative," *SpringerLink, Encyclopedia of Mathematics*, Web, 4 July 2011, <http://eom.springer.de/>.

6. **Proving that differentiability implies (and requires) continuity.** f is differentiable at *a* $\Leftrightarrow \lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists; see Version 1 of the Limit Definition of $f'(a)$ in Section 3.1.

   • **A rigorous approach:** Assume that f is differentiable at *a*. This implies that $f(a)$ exists; in fact, it implies that f is defined on an open interval containing *a*.

   $$\lim_{x\to a} f(x) = \lim_{x\to a}\left[f(x)-f(a)+f(a)\right] = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right]+f(a)$$
   $$=\left[\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right]\right]\cdot\left[\lim_{x\to a}(x-a)\right]+f(a)=\left[f'(a)\right]\cdot 0+f(a)=f(a).$$

   Therefore, $\lim\limits_{x\to a} f(x)=f(a)$, which defines continuity of f at *a*.

   • **An intuitive approach:** As $x\to a$, $x-a\to 0$. The only way the limit can exist as a real number is if it has the Limit Form $\dfrac{0}{0}$. (The Limit Form $\dfrac{\text{DNE}}{0}$ cannot yield a real number *c* as a limit, basically because $c\cdot 0=0$.) This requires that $f(x)-f(a)\to 0$ as $x\to a$. This, in turn, requires that $f(x)\to f(a)$ as $x\to a$, or $\lim\limits_{x\to a} f(x)=f(a)$, which defines continuity of f at *a*.

7. **A function with a derivative that is defined but discontinuous at 0; failure of the "limit of the derivative" short cut for one-sided derivatives.** See Gelbaum and Olmsted, *Counterexamples in Analysis* (Dover), p.36. Let $f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & x\neq 0 \\ 0, & x=0\end{cases}$.

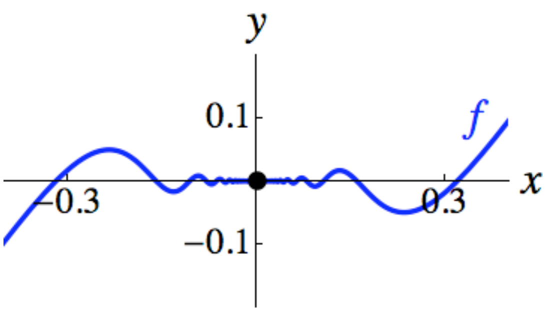
   Then, $f'(x)=\begin{cases} 2x\sin\left(\dfrac{1}{x}\right)-\cos\left(\dfrac{1}{x}\right), & x\neq 0 \\ 0, & x=0\end{cases}$.

   The methods of Sections 3.2-3.6 can be used to work out the top rule. Why is $f'(0)=0$?

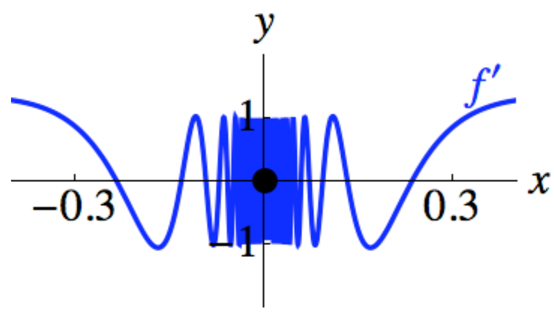
   $$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{h}\right)-0}{h}=\lim_{h\to 0}\left[h\sin\left(\frac{1}{h}\right)\right]=0$$ by the Squeeze (Sandwich) Theorem from Section 2.6. f' is **discontinuous** at 0, because $\lim\limits_{x\to 0} f'(x)$

does not exist (DNE). Also, because $\lim_{x \to 0^+} f'(x)$ does not exist (DNE) and $\lim_{x \to 0^-} f'(x)$ does not exist (DNE), the "limit of the derivative" short cut for guessing one-sided derivatives (even informally as $\pm\infty$), as described in Example 13, fails for this example. (See also Section 3.6, Footnote 4.)

Graph of f



Graph of f'



(Axes are scaled differently.)

SECTION 3.3: TECHNIQUES OF DIFFERENTIATION

LEARNING OBJECTIVES

• Learn how to differentiate using short cuts, including:
the Linearity Properties, the Product Rule, the Quotient Rule, and (perhaps) the Reciprocal Rule.

PART A: BASIC RULES OF DIFFERENTIATION

In Section 3.2, we discussed Rules 1 through 4 below.

Basic Short Cuts for Differentiation

Assumptions:
- c, m, b, and n are real constants.
- f and g are functions that are differentiable "where we care."

	If $h(x)=$	then $h'(x)=$	Comments
1.	c	0	The derivative of a constant is 0.
2.	$mx+b$	m	The derivative of a linear function is the slope.
3.	x^n	nx^{n-1}	Power Rule
4.	$c \cdot f(x)$	$c \cdot f'(x)$	Constant Multiple Rule (Linearity)
5.	$f(x)+g(x)$	$f'(x)+g'(x)$	Sum Rule (Linearity)
6.	$f(x)-g(x)$	$f'(x)-g'(x)$	Difference Rule (Linearity)

• **Linearity.** Because of Rules 4, 5, and 6, the differentiation operator D_x is called a linear operator. (The operations of taking limits (Ch.2) and integrating (Ch.5) are also linear.) The Sum Rule, for instance, may be thought of as "the derivative of a **sum** equals the **sum** of the derivatives, if they exist." Linearity allows us to take derivatives **term-by-term** and then to **"pop out" constant factors**.

• **Proofs.** The Limit Definition of the Derivative can be used to prove these short cuts. The Linearity Properties of **Limits** are crucial to proving the Linearity Properties of **Derivatives**. (See Footnote 1.)

Armed with these short cuts, we may now differentiate **all polynomial functions**.

Example 1 (Differentiating a Polynomial Using Short Cuts)

Let $f(x) = -4x^3 + 6x - 5$. Find $f'(x)$.

§ Solution

$$\begin{aligned} f'(x) &= D_x\left(-4x^3 + 6x - 5\right) \\ &= D_x\left(-4x^3\right) + D_x(6x) - D_x(5) && \text{(Sum and Difference Rules)} \\ &= -4 \cdot D_x\left(x^3\right) + D_x(6x) - D_x(5) && \text{(Constant Multiple Rule)} \end{aligned}$$

TIP 1: Students get used to applying the Linearity Properties, skip all of this work, and give the "answer only."

$$\begin{aligned} &= -4\left(3x^2\right) + 6 - 0 \\ &= -12x^2 + 6 \end{aligned}$$

Challenge to the Reader: Observe that the "-5" term has no impact on the derivative. Why does this make sense graphically? Hint: How would the graphs of $y = -4x^3 + 6x$ and $y = -4x^3 + 6x - 5$ be different? Consider the **slopes** of corresponding tangent lines to those graphs. §

Example 2 (Equation of a Tangent Line; Revisiting Example 1)

Find an equation of the **tangent line** to the graph of $y = -4x^3 + 6x - 5$ at the point $(1, -3)$.

§ Solution

• Let $f(x) = -4x^3 + 6x - 5$, as in Example 1.

• Just to be safe, we can **verify** that the **point** $(1, -3)$ lies on the graph by verifying that $f(1) = -3$. (Remember that function values correspond to y-coordinates here.)

• Find *m*, the **slope** of the tangent line at the point where $x = 1$.
This is given by $f'(1)$, the value of the **derivative** function at $x = 1$.

$$m = f'(1)$$

From Example 1, remember that $f'(x) = -12x^2 + 6$.

$$= \left[-12x^2 + 6\right]_{x=1}$$
$$= -12(1)^2 + 6$$
$$= -6$$

• We can find a **Point-Slope Form** for the equation of the desired tangent line.

The line contains the **point**: $(x_1, y_1) = (1, -3)$.
It has **slope**: $m = -6$.

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = -6(x - 1)$$

• If we wish, we can rewrite the equation in **Slope-Intercept Form**.

$$y + 3 = -6x + 6$$
$$y = -6x + 3$$

• We can also obtain the **Slope-Intercept Form** directly.

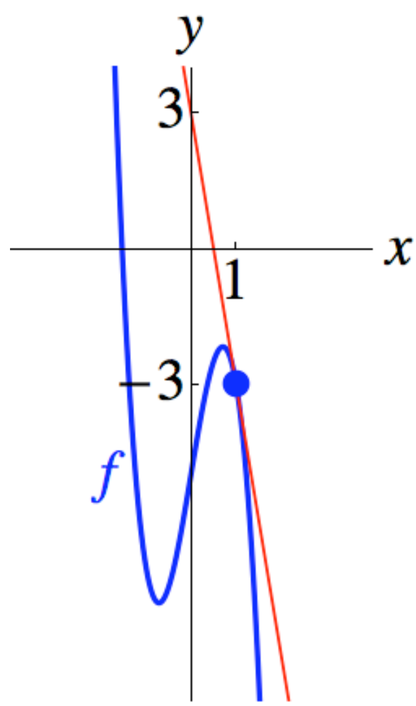
$$y = mx + b \quad \Rightarrow$$

$$(-3) = (-6)(1) + b$$

$$b = 3 \quad \Rightarrow$$

$$y = -6x + 3$$

• Observe how the **red tangent line** below is consistent with the equation above.



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Example 3 (Finding Horizontal Tangent Lines; Revisiting Example 1)

Find the *x*-coordinates of all points on the graph of $y = -4x^3 + 6x - 5$ where the **tangent line** is **horizontal**.

§ Solution

• Let $f(x) = -4x^3 + 6x - 5$, as in Example 1.

• We must find where the **slope** of the tangent line to the graph is 0.
We must solve the equation:

$$\begin{aligned} f'(x) &= 0 \\ -12x^2 + 6 &= 0 \qquad \text{(See Example 1.)} \\ -12x^2 &= -6 \\ x^2 &= \frac{1}{2} \\ x &= \pm\sqrt{\frac{1}{2}} \\ x &= \pm\frac{\sqrt{2}}{2} \end{aligned}$$

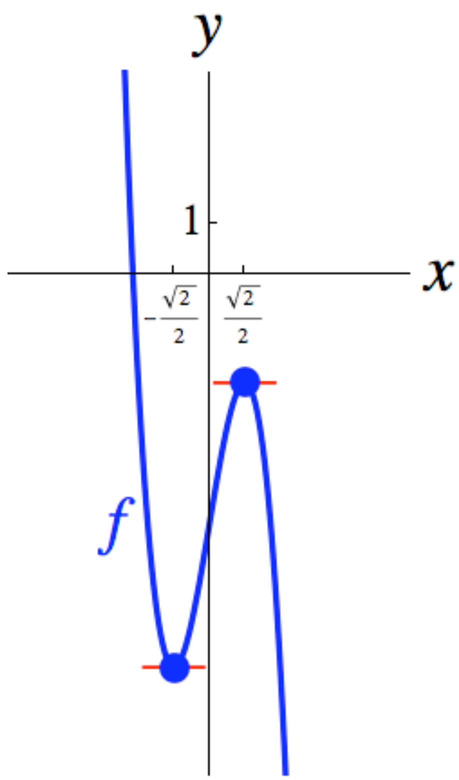
The desired *x*-coordinates are $\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$.

• The corresponding **points** on the graph are:

$\left(\frac{\sqrt{2}}{2},\ f\left(\frac{\sqrt{2}}{2}\right)\right)$, which is $\left(\frac{\sqrt{2}}{2},\ 2\sqrt{2} - 5\right)$, and

$\left(-\frac{\sqrt{2}}{2},\ f\left(-\frac{\sqrt{2}}{2}\right)\right)$, which is $\left(-\frac{\sqrt{2}}{2},\ -2\sqrt{2} - 5\right)$.

• The **red tangent lines** below are truncated.



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PART B: PRODUCT RULE OF DIFFERENTIATION

WARNING 1: The derivative of a product is typically **not** the product of the derivatives.

> Product Rule of Differentiation
>
> Assumptions:
>
> • f and g are functions that are differentiable "where we care."
>
> If $h(x) = f(x)g(x)$,
> then $h'(x) = f'(x)g(x) + f(x)g'(x)$.

• Footnote 2 uses the Limit Definition of the Derivative to prove this.

• Many sources switch terms and write: $h'(x) = f(x)g'(x) + f'(x)g(x)$, but our form is easier to extend to three or more factors.

Example 4 (Differentiating a Product)

Find $D_x\left[\left(x^4+1\right)\left(x^2+4x-5\right)\right]$.

§ Solution

TIP 2: Clearly **break** the product up into factors, as has already been done here. The **number of factors** (here, two) will equal the **number of terms** in the **derivative** when we use the Product Rule to "expand it out."

TIP 3: Pointer method. Imagine a **pointer** being moved from **factor to factor** as we write the derivative **term-by-term**. The **pointer** indicates which factor we **differentiate**, and then we **copy** the other factors to form the corresponding term in the derivative.

$\left(x^4+1\right)$	$\left(x^2+4x-5\right)$	
$\wedge\left(D_x\right)$	copy	$+$
copy	$\wedge\left(D_x\right)$	

$$D_x\left[\left(x^4+1\right)\left(x^2+4x-5\right)\right] = \left[D_x\left(x^4+1\right)\right]\cdot\left(x^2+4x-5\right) + \left(x^4+1\right)\cdot\left[D_x\left(x^2+4x-5\right)\right]$$

$$= \left[4x^3\right]\cdot\left(x^2+4x-5\right) + \left(x^4+1\right)\cdot\left[2x+4\right]$$

The Product Rule is especially convenient here if we do not have to simplify our result. Here, we will simplify.

$$= 6x^5 + 20x^4 - 20x^3 + 2x + 4$$

Challenge to the Reader: Find the derivative by first multiplying out the product and then differentiating term-by-term. §

The Product Rule can be extended to three or more factors.

- The Exercises include a related proof.

Example 5 (Differentiating a Product of Three Factors)

Find $\frac{d}{dt}\left[(t+4)(t^2+2)\left(\sqrt[3]{t}-t\right)\right]$. The result does not have to be simplified, and negative exponents are acceptable here. (Your instructor may object!)

§ Solution

$(t+4)$	(t^2+2)	$(t^{1/3}-t)$	
$\wedge(D_t)$	copy	copy	+
copy	$\wedge(D_t)$	copy	+
copy	copy	$\wedge(D_t)$	

$$\begin{aligned}\frac{d}{dt}\left[(t+4)(t^2+2)\left(\sqrt[3]{t}-t\right)\right] &= \left[D_t(t+4)\right]\cdot(t^2+2)\cdot\left(\sqrt[3]{t}-t\right) \quad + \\ &\quad (t+4)\cdot\left[D_t(t^2+2)\right]\cdot\left(\sqrt[3]{t}-t\right) \quad + \\ &\quad (t+4)\cdot(t^2+2)\cdot\left[D_t\left(t^{1/3}-t\right)\right] \\ &= [1]\cdot(t^2+2)\cdot\left(\sqrt[3]{t}-t\right) \quad + \\ &\quad (t+4)\cdot[2t]\cdot\left(\sqrt[3]{t}-t\right) \quad + \\ &\quad (t+4)\cdot(t^2+2)\cdot\left[\frac{1}{3}t^{-2/3}-1\right]\end{aligned}$$

§

TIP 4: Apply the **Constant Multiple Rule**, not the Product Rule, to something like $D_x\left(2x^3\right)$. While the Product Rule would work, it would be inefficient here.

PART C: QUOTIENT RULE (and RECIPROCAL RULE) OF DIFFERENTIATION

WARNING 2: The derivative of a quotient is typically **not** the quotient of the derivatives.

Quotient Rule of Differentiation

Assumptions:

- f and g are functions that are differentiable "where we care."
- g is nonzero "where we care."

If $h(x) = \dfrac{f(x)}{g(x)}$,

then $h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$.

- Footnote 3 proves this using the Limit Definition of the Derivative.
- Footnote 4 more elegantly proves this using the Product Rule.

TIP 5: Memorizing. The Quotient Rule can be memorized as:

$$D\left(\frac{\text{Hi}}{\text{Lo}}\right) = \frac{\text{Lo} \cdot D(\text{Hi}) - \text{Hi} \cdot D(\text{Lo})}{(\text{Lo})^2 \text{, the square of what's below}}$$

Observe that the numerator and the denominator on the right-hand side **rhyme**.

- At this point, we can differentiate **all rational functions**.

Reciprocal Rule of Differentiation

If $h(x) = \dfrac{1}{g(x)}$,

then $h'(x) = -\dfrac{g'(x)}{\left[g(x)\right]^2}$.

• This is a special case of the Quotient Rule where $f(x) = 1$.

Think: $-\dfrac{\mathrm{D}(\mathrm{Lo})}{(\mathrm{Lo})^2}$

TIP 6: While the **Reciprocal Rule** is useful, it is not all that necessary to memorize if the **Quotient Rule** has been memorized.

Example 6 (Differentiating a Quotient)

Find $D_x\left(\dfrac{7x-3}{3x^2+1}\right)$.

§ Solution

$$D_x\left(\frac{7x-3}{3x^2+1}\right) = \frac{\mathrm{Lo}\cdot D(\mathrm{Hi}) - \mathrm{Hi}\cdot D(\mathrm{Lo})}{(\mathrm{Lo})^2\text{, the square of what's below}}$$

$$= \frac{(3x^2+1)\cdot\left[D_x(7x-3)\right] - (7x-3)\cdot\left[D_x(3x^2+1)\right]}{(3x^2+1)^2}$$

$$= \frac{(3x^2+1)\cdot[7] - (7x-3)\cdot[6x]}{(3x^2+1)^2}$$

$$= \frac{-21x^2+18x+7}{(3x^2+1)^2}, \text{ or } \frac{7-21x^2+18x}{(3x^2+1)^2}, \text{ or } -\frac{21x^2-18x-7}{(3x^2+1)^2}$$

§

TIP 7: Rewriting. Instead of running with the first technique that comes to mind, examine the problem, **think**, and see if **rewriting or simplifying** first can help.

Example 7 (Rewriting Before Differentiating)

Let $s(w) = \dfrac{6w^2 - \sqrt{w}}{3w}$. Find $s'(w)$.

§ Solution

Rewriting $s(w)$ by splitting the fraction yields a simpler solution than applying the Quotient Rule directly would have.

$$\begin{aligned} s(w) &= \frac{6w^2}{3w} - \frac{\sqrt{w}}{3w} \\ &= 2w - \frac{1}{3} w^{-1/2} \quad \Rightarrow \\ s'(w) &= 2 + \frac{1}{6} w^{-3/2} \\ &= 2 + \frac{1}{6w^{3/2}}, \text{ or } \frac{12w^{3/2} + 1}{6w^{3/2}}, \text{ or } \frac{12w^2 + \sqrt{w}}{6w^2} \end{aligned}$$

§

FOOTNOTES

1. **Proof of the Sum Rule of Differentiation.** Throughout the Footnotes, we assume that f and g are functions that are differentiable "where we care." Let $p = f + g$. (We will use h for "run" in the Limit Definition of the Derivative.)

$$p'(x) = \lim_{h\to 0} \frac{p(x+h)-p(x)}{h} = \lim_{h\to 0} \frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)+g(x+h)-f(x)-g(x)}{h} = \lim_{h\to 0} \frac{\left[f(x+h)-f(x)\right]+\left[g(x+h)-g(x)\right]}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right] = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

(Observe that we have exploited the Sum Rule (linearity) of Limits.)

$$= f'(x)+g'(x)$$

The Difference Rule can be similarly proven, or, if we accept the Constant Multiple Rule, we can use: $f - g = f + (-g)$. Sec. 2.2, Footnote 1 extends to derivatives of linear combinations.

2. **Proof of the Product Rule of Differentiation.** Let $p = fg$.

$$p'(x) = \lim_{h\to 0} \frac{p(x+h)-p(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left[f(x+h)g(x+h)\right]-\left[f(x)g(x)\right]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \frac{f(x+h)g(x)-f(x)g(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot\left[g(x+h)-g(x)\right]}{h} + \lim_{h\to 0} \frac{\left[f(x+h)-f(x)\right]\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \left[f(x+h)\cdot\frac{g(x+h)-g(x)}{h}\right] + \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h}\cdot g(x)\right]$$

$$= \left[\lim_{h\to 0} f(x+h)\right]\cdot\left[\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}\right] + \left[\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\right]\left[\lim_{h\to 0} g(x)\right]$$

$$= \left[f(x)\right]\cdot\left[g'(x)\right] + \left[f'(x)\right]\cdot\left[g(x)\right], \text{ or}$$

$$f'(x)g(x)+f(x)g'(x)$$

Note: We have: $\lim_{h\to 0} f(x+h) = f(x)$ by continuity, because differentiability implies continuity. We have something similar for g in Footnote 3.

3. **Proof of the Quotient Rule of Differentiation, I.** Let $p = f / g$, where $g(x) \neq 0$.

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \left(\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \right] \cdot \frac{1}{h} \right)$$

$$= \lim_{h \to 0} \left(\left[\frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \right] \cdot \frac{1}{h} \right)$$

$$= \lim_{h \to 0} \left[\frac{f(x+h)g(x) - f(x)g(x+h)}{h} \cdot \frac{1}{g(x+h)g(x)} \right]$$

$$= \lim_{h \to 0} \left[\frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{h} \cdot \frac{1}{g(x+h)g(x)} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left[f(x+h)g(x) - f(x)g(x) \right] + \left[f(x)g(x) - f(x)g(x+h) \right]}{h} \cdot \frac{1}{g(x+h)g(x)} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left[f(x+h) - f(x) \right] \cdot g(x) + f(x) \cdot \left[g(x) - g(x+h) \right]}{h} \cdot \frac{1}{g(x+h)g(x)} \right]$$

$$= \lim_{h \to 0} \left[\frac{\left[f(x+h) - f(x) \right] \cdot g(x) - f(x) \cdot \left[g(x+h) - g(x) \right]}{h} \cdot \frac{1}{g(x+h)g(x)} \right]$$

$$= \left[\lim_{h \to 0} \frac{\left[f(x+h) - f(x) \right] \cdot g(x)}{h} - \lim_{h \to 0} \frac{f(x) \cdot \left[g(x+h) - g(x) \right]}{h} \right] \cdot \left[\lim_{h \to 0} \frac{1}{g(x+h)g(x)} \right]$$

$$= \left[\left[g(x) \right] \cdot \left[\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right] - \left[f(x) \right] \cdot \left[\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \right] \right] \cdot \left[\frac{1}{g(x)g(x)} \right]$$

(See Footnote 2, Note.)

$$= \left(\left[g(x) \right] \cdot \left[f'(x) \right] - \left[f(x) \right] \cdot \left[g'(x) \right] \right) \cdot \frac{1}{\left[g(x) \right]^2} = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x) \right]^2}$$

4. **Proof of the Quotient Rule of Differentiation, II, using the Product Rule.**

Let $h(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$.

Then, $g(x)h(x) = f(x)$.

Differentiate both sides with respect to *x*. Apply the **Product Rule** to the left-hand side. We obtain: $g'(x)h(x) + g(x)h'(x) = f'(x)$. Solving for $h'(x)$, we obtain:

$h'(x) = \dfrac{f'(x) - g'(x)h(x)}{g(x)}$. Remember that $h(x) = \dfrac{f(x)}{g(x)}$. Then,

$$h'(x) = \frac{f'(x) - g'(x)\left[\dfrac{f(x)}{g(x)}\right]}{g(x)}$$

$$= \frac{\left(f'(x) - g'(x)\left[\dfrac{f(x)}{g(x)}\right]\right)}{\left[g(x)\right]} \cdot \frac{\left[g(x)\right]}{\left[g(x)\right]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

This approach is attributed to Marie Agnessi (1748); see *The AMATYC Review*, Fall 2002 (Vol. 24, No. 1), p.2, Letter to the Editor by Joe Browne.

• See also “Quotient Rule Quibbles” by Eugene Boman in the Fall 2001 edition (vol.23, No.1) of *The AMATYC Review*, pp.55-58. The article suggests that the **Reciprocal Rule** for $D_x\left[\dfrac{1}{g(x)}\right]$ can be proven directly by using the Limit Definition of the Derivative, and then the **Product Rule** can be used in conjunction with the Reciprocal Rule to differentiate $\left[f(x)\right]\left[\dfrac{1}{g(x)}\right]$; the Spivak and Apostol calculus texts take this approach. The article presents another proof, as well.

SECTION 3.4: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

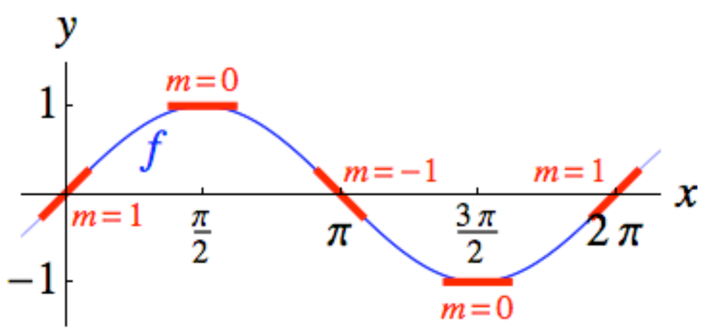
LEARNING OBJECTIVES

• Use the Limit Definition of the Derivative to find the derivatives of the basic sine and cosine functions. Then, apply differentiation rules to obtain the derivatives of the other four basic trigonometric functions.

• Memorize the derivatives of the six basic trigonometric functions and be able to apply them in conjunction with other differentiation rules.

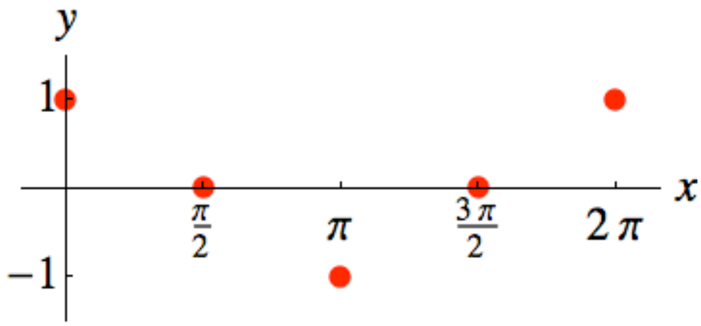
PART A: CONJECTURING THE DERIVATIVE OF THE BASIC SINE FUNCTION

Let $f(x) = \sin x$. The sine function is **periodic** with period 2π. One cycle of its graph is in bold below. Selected [truncated] **tangent lines** and their **slopes** (*m*) are indicated in red. (The leftmost tangent line and slope will be discussed in Part C.)

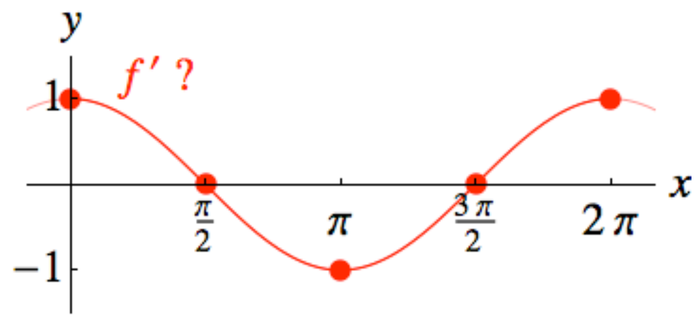


Remember that **slopes** of tangent lines correspond to **derivative** values (that is, values of f').

The graph of f' must then contain the five indicated points below, since their *y*-coordinates correspond to values of f'.



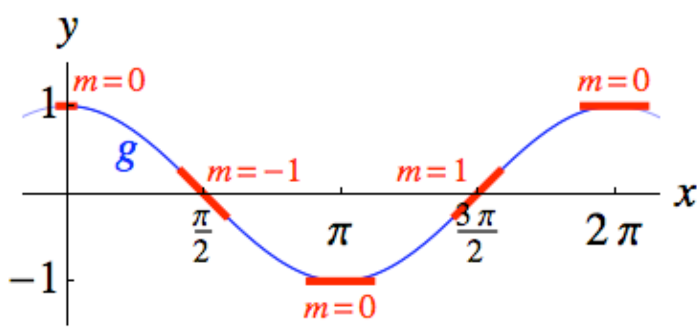
Do you know of a basic periodic function whose graph contains these points?



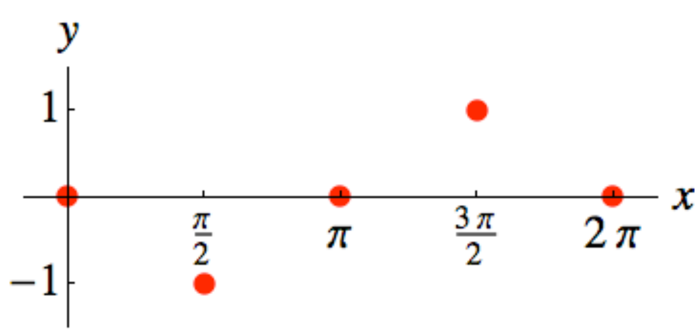
We conjecture that $f'(x) = \cos x$. We will prove this in Parts D and E.

PART B: CONJECTURING THE DERIVATIVE OF THE BASIC COSINE FUNCTION

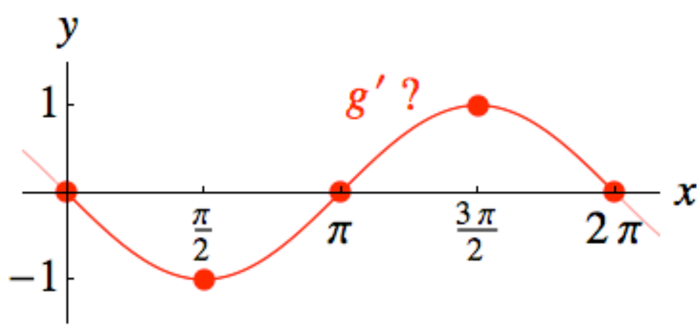
Let $g(x) = \cos x$. The cosine function is also periodic with period 2π.



The graph of g' must then contain the five indicated points below.



Do you know of a (fairly) basic periodic function whose graph contains these points?



We conjecture that $g'(x) = -\sin x$. If f is the sine function from Part A, then we also believe that $f''(x) = g'(x) = -\sin x$. We will prove these in Parts D and E.

PART C: TWO HELPFUL LIMIT STATEMENTS

Helpful Limit Statement #1

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

Helpful Limit Statement #2

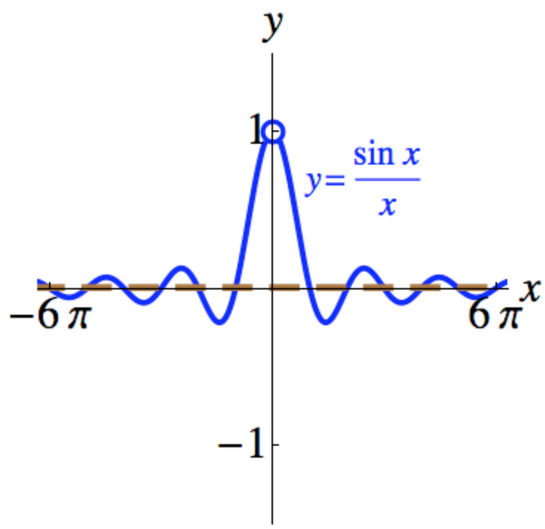
$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \quad \left(\text{or, equivalently, } \lim_{h\to 0} \frac{1-\cos h}{h} = 0 \right)$$

These limit statements, which are proven in Footnotes 1 and 2, will help us **prove** our conjectures from Parts A and B. In fact, only the **first** statement is needed for the proofs in Part E.

Statement #1 helps us graph $y = \frac{\sin x}{x}$.

- In Section 2.6, we proved that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ by the Sandwich (Squeeze) Theorem. Also, $\lim_{x\to -\infty} \frac{\sin x}{x} = 0$.

- Now, Statement #1 implies that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, where we replace h with x. Because $\frac{\sin x}{x}$ is undefined at $x = 0$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the graph has a **hole** at the point $(0,1)$.

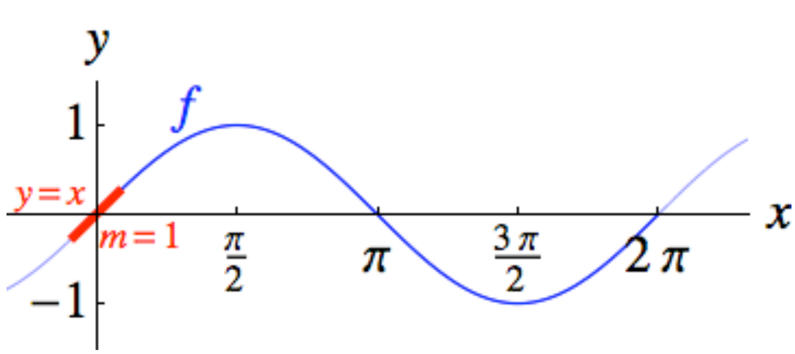


(Axes are scaled differently.)

Statement #1 also implies that, if $f(x) = \sin x$, then $f'(0) = 1$.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{\sin(0+h) - \sin(0)}{h} \\ &= \lim_{h \to 0} \frac{\sin h - 0}{h} \\ &= \lim_{h \to 0} \frac{\sin h}{h} \\ &= 1 \end{aligned}$$

This verifies that the **tangent line** to the graph of $y = \sin x$ at the **origin** does, in fact, have **slope** 1. Therefore, the tangent line is given by the **equation** $y = x$. By the **Principle of Local Linearity** from Section 3.1, we can say that $\sin x \approx x$ when $x \approx 0$. That is, the tangent line closely **approximates** the sine graph close to the origin.



PART D: "STANDARD" PROOFS OF OUR CONJECTURES

Derivatives of the Basic Sine and Cosine Functions

1) $D_x(\sin x) = \cos x$
2) $D_x(\cos x) = -\sin x$

§ Proof of 1)

Let $f(x) = \sin x$. Prove that $f'(x) = \cos x$.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \to 0} \frac{\overbrace{\sin x \cos h + \cos x \sin h}^{\text{by Sum Identity for sine}} - \sin x}{h} \\
&= \lim_{h \to 0} \frac{\overbrace{(\sin x \cos h - \sin x)}^{\text{Group terms with } \sin x.} + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h}
\end{aligned}
$$

(Now, group expressions containing *h*.)

$$
\begin{aligned}
&= \lim_{h \to 0} \left[(\sin x) \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} + (\cos x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= \cos x
\end{aligned}
$$

Q.E.D. §

§ Proof of 2)

Let $g(x) = \cos x$. Prove that $g'(x) = -\sin x$.
(This proof parallels the previous proof.)

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\overbrace{\cos x \cos h - \sin x \sin h}^{\text{by Sum Identity for cosine}} - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\overbrace{(\cos x \cos h - \cos x)}^{\text{Group terms with } \cos x.} - \sin x \sin h}{h} \\
&= \lim_{h \to 0} \frac{(\cos x)(\cos h - 1) - \sin x \sin h}{h}
\end{aligned}$$

$\left(\text{Now, group expressions containing } h.\right)$

$$\begin{aligned}
&= \lim_{h \to 0} \left[(\cos x) \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} - (\sin x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= -\sin x
\end{aligned}$$

Q.E.D.

• Do you see where the "$-$" sign in $-\sin x$ arose in this proof? §

PART E: MORE ELEGANT PROOFS OF OUR CONJECTURES

Derivatives of the Basic Sine and Cosine Functions

1) $D_x(\sin x) = \cos x$
2) $D_x(\cos x) = -\sin x$

Version 2 of the Limit Definition of the Derivative Function in Section 3.2, Part A, provides us with more elegant proofs. In fact, they do not even use Limit Statement #2 in Part C.

§ Proof of 1)

Let $f(x) = \sin x$. Prove that $f'(x) = \cos x$.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{\overbrace{(\sin x\cos h + \cos x \sin h)}^{\text{by Sum Identity for sine}} - \overbrace{(\sin x \cos h - \cos x \sin h)}^{\text{by Difference Identity for sine}}}{2h} \\
&= \lim_{h\to 0} \frac{\cancel{2}\cos x \sin h}{\cancel{2}h} \\
&= \lim_{h\to 0} \left[(\cos x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= \cos x
\end{aligned}
$$

Q.E.D. §

§ Proof of 2)

Let $g(x) = \cos x$. Prove that $g'(x) = -\sin x$.

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x-h)}{2h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x-h)}{2h}$$

$$= \lim_{h\to 0} \frac{\overbrace{(\cos x \cos h - \sin x \sin h)}^{\text{from Sum Identity for cosine}} - \overbrace{(\cos x \cos h + \sin x \sin h)}^{\text{from Difference Identity for cosine}}}{2h}$$

$$= \lim_{h\to 0} \frac{-\cancel{2} \sin x \sin h}{\cancel{2} h}$$

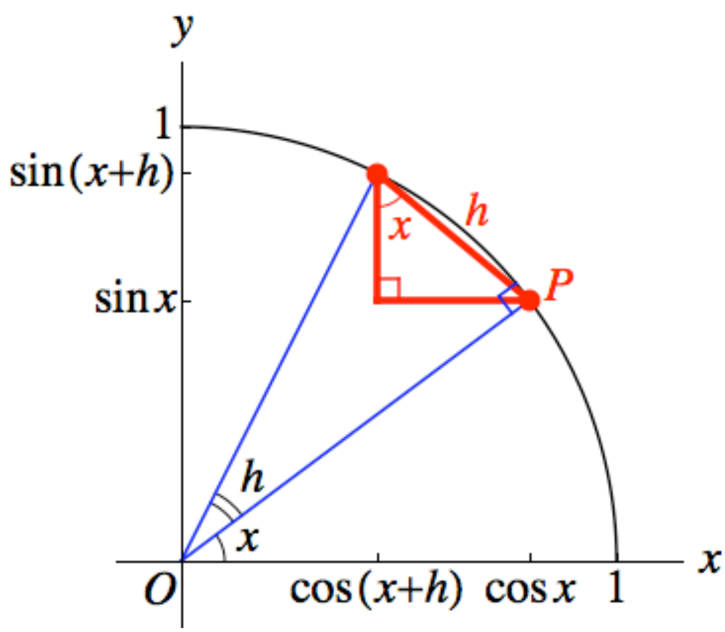
$$= \lim_{h\to 0} \left[(-\sin x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right]$$

$$= -\sin x$$

Q.E.D. §

§ A Geometric Approach

Jon Rogawski has recommended a more geometric approach, one that stresses the concept of the derivative. Examine the figure below.



• Observe that: $\sin(x+h)-\sin x \approx h\cos x$, which demonstrates that the **change** in a differentiable function on a small interval *h* is related to its derivative. (We will exploit this idea when we discuss **differentials** in Section 3.5.)

• Consequently, $\dfrac{\sin(x+h)-\sin x}{h} \approx \cos x$.

• In fact, $D_x(\sin x) = \lim\limits_{h\to 0} \dfrac{\sin(x+h)-\sin(x)}{h} = \cos x$.

• A similar argument shows: $D_x(\cos x) = \lim\limits_{h\to 0} \dfrac{\cos(x+h)-\cos(x)}{h} = -\sin x$.

• Some angle and length measures in the figure are **approximate**, though they become more accurate as $h \to 0$. (For clarity, the figure does not employ a small value of *h*.)

• Exercises in Sections 3.6 and 3.7 will show that the **tangent line** to any point *P* on a **circle** with center *O* is **perpendicular** to the line segment $\overline{OP}$.

§

PART F: DERIVATIVES OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

Basic Trigonometric Rules of Differentiation

1) $D_x(\sin x) = \cos x$	2) $D_x(\cos x) = -\sin x$
3) $D_x(\tan x) = \sec^2 x$	4) $D_x(\cot x) = -\csc^2 x$
5) $D_x(\sec x) = \sec x \tan x$	6) $D_x(\csc x) = -\csc x \cot x$

WARNING 1: Radians. We assume that *x*, *h*, etc. are measured in **radians** (corresponding to **real numbers**). If they are measured in degrees, the rules of this section and beyond would have to be modified. (Footnote 3 in Section 3.6 will discuss this.)

TIP 1: Memorizing.

- The sine and cosine functions are a pair of **cofunctions**, as are the tangent and cotangent functions and the secant and cosecant functions.
- Let's say you know Rule 5) on the derivative of the **secant** function. You can quickly modify that rule to find Rule 6) on the derivative of the **cosecant** function.
- You take $\sec x \tan x$, multiply it by -1 (that is, do a "**sign flip**"), and take the **cofunction** of each factor. We then obtain: $-\csc x \cot x$, which is $D_x(\csc x)$.
- This method also applies to Rules 1) and 2) and to Rules 3) and 4).
- The Exercises in Section 3.6 will demonstrate why this works.

TIP 2: Domains. In Rule 3), observe that $\tan x$ and $\sec^2 x$ share the same domain. In fact, all six rules exhibit the same property.

Rules 1) and 2) can be used to **prove** Rules 3) through 6). The proofs for Rules 4) and 6) are left to the reader in the Exercises for Sections 3.4 and 3.6 (where the **Cofunction Identities** will be applied).

§ Proof of 3)

$$D_x(\tan x) = D_x\left(\frac{\sin x}{\cos x}\right) \quad \text{(Quotient Identities)}$$

$$= \frac{\text{Lo}\cdot \text{D}(\text{Hi}) - \text{Hi}\cdot \text{D}(\text{Lo})}{(\text{Lo})^2} \quad \text{(Quotient Rule of Differentiation)}$$

$$= \frac{[\cos x]\cdot[D_x(\sin x)] - [\sin x]\cdot[D_x(\cos x)]}{(\cos x)^2}$$

$$= \frac{[\cos x]\cdot[\cos x] - [\sin x]\cdot[-\sin x]}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \quad \left(\text{Can: } = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x = \sec^2 x\right)$$

$$= \frac{1}{\cos^2 x} \quad \text{(Pythagorean Identities)}$$

$$= \sec^2 x \quad \text{(Reciprocal Identities)}$$

Q.E.D.

• Footnote 3 gives a proof using the Limit Definition of the Derivative. §

§ Proof of 5)

$$D_x(\sec x) = D_x\left(\frac{1}{\cos x}\right)$$

Quotient Rule of Differentiation — Reciprocal Rule

$$= \frac{\text{Lo}\cdot \text{D}(\text{Hi}) - \text{Hi}\cdot \text{D}(\text{Lo})}{(\text{Lo})^2} \quad \text{or} \quad -\frac{\text{D}(\text{Lo})}{(\text{Lo})^2}$$

$$= \frac{[\cos x]\cdot[D_x(1)] - [1]\cdot[D_x(\cos x)]}{(\cos x)^2} \quad \text{or} \quad -\frac{D_x(\cos x)}{(\cos x)^2}$$

$$= \frac{[\cos x]\cdot[0] - [1]\cdot[-\sin x]}{(\cos x)^2} \quad \text{or} \quad -\frac{-\sin x}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} \quad \text{(Factoring or "Peeling")}$$

$$= \sec x \tan x \quad \text{(Reciprocal and Quotient Identities)}$$

Q.E.D. §

Example 1 (Finding a Derivative Using Several Rules)

Find $D_x\left(x^2 \sec x + 3\cos x\right)$.

§ Solution

We apply the **Product Rule of Differentiation** to the first term and the **Constant Multiple Rule** to the second term. (The Product Rule can be used for the second term, but it is inefficient.)

$$D_x\left(x^2 \sec x + 3\cos x\right) = D_x\left(x^2 \sec x\right) + D_x(3\cos x) \quad \text{(Sum Rule of Diff'n)}$$

$$= \left(\left[D_x\left(x^2\right)\right][\sec x] + \left[x^2\right]\left[D_x(\sec x)\right]\right) + 3\cdot\left[D_x(\cos x)\right]$$

$$= \left([2x][\sec x] + \left[x^2\right][\sec x \tan x]\right) + 3\cdot[-\sin x]$$

$$= 2x\sec x + x^2 \sec x \tan x - 3\sin x$$

§

Example 2 (Finding and Simplifying a Derivative)

Let $g(\theta) = \dfrac{\cos\theta}{1-\sin\theta}$. Find $g'(\theta)$.

§ Solution

Note: If $g(\theta)$ were $\dfrac{\cos\theta}{1-\sin^2\theta}$, we would be able to simplify considerably before we differentiate. Alas, we cannot here. Observe that we cannot "split" the fraction through its denominator.

$$g'(\theta) = \frac{\text{Lo}\cdot \text{D}(\text{Hi}) - \text{Hi}\cdot \text{D}(\text{Lo})}{(\text{Lo})^2} \quad (\text{Quotient Rule of Diff'n})$$

$$= \frac{[1-\sin\theta]\cdot\left[D_\theta(\cos\theta)\right] - [\cos\theta]\cdot\left[D_\theta(1-\sin\theta)\right]}{(1-\sin\theta)^2}$$

Note: $(1-\sin\theta)^2$ is **not** equivalent to $1-\sin^2\theta$.

$$= \frac{[1-\sin\theta]\cdot[-\sin\theta] - [\cos\theta]\cdot[-\cos\theta]}{(1-\sin\theta)^2}$$

TIP 3: Signs. Many students don't see why $D_\theta(-\sin\theta) = -\cos\theta$. Remember that differentiating the basic **sine** function does **not** lead to a "**sign flip**," while differentiating the basic **cosine** function **does**.

$$= \frac{-\sin\theta + \sin^2\theta + \cos^2\theta}{(1-\sin\theta)^2}$$

WARNING 2: Simplify.

$$= \frac{-\sin\theta + 1}{(1-\sin\theta)^2} \quad (\text{Pythagorean Identities})$$

$$= \frac{1-\sin\theta}{(1-\sin\theta)^2} \quad (\text{Rewriting})$$

$$= \frac{1}{1-\sin\theta}$$

§

Example 3 (Simplifying Before Differentiating)

Let $f(x) = \sin x \csc x$. Find $f'(x)$.

§ Solution

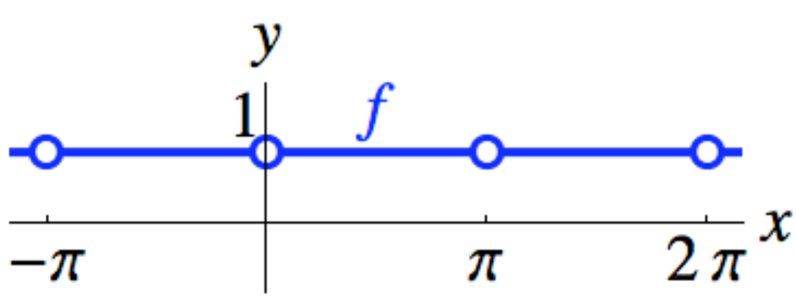
Simplifying $f(x)$ first is preferable to applying the Product Rule directly.

$$
\begin{aligned}
f(x) &= \sin x \csc x \\
&= (\sin x)\left(\frac{1}{\sin x}\right) \qquad \text{(Reciprocal Identities)} \\
&= 1, \quad (\sin x \neq 0) \quad \Rightarrow
\end{aligned}
$$

$$f'(x) = 0, \quad (\sin x \neq 0)$$

TIP 4: Domain issues.

$\text{Dom}(f) = \text{Dom}(f') = \{x \in \mathbb{R} \mid \sin x \neq 0\} = \{x \in \mathbb{R} \mid x \neq \pi n, \ (n \in \mathbb{Z})\}$.
In routine differentiation exercises, domain issues are often ignored. Restrictions such as $(\sin x \neq 0)$ here are rarely written.



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PART G: TANGENT LINES

Example 4 (Finding Horizontal Tangent Lines to a Trigonometric Graph)

Let $f(x) = 2\sin x - x$. Find the *x*-coordinates of all points on the graph of $y = f(x)$ where the **tangent line** is **horizontal**.

§ Solution

• We must find where the **slope** of the tangent line to the graph is 0.
We must solve the equation:

$$f'(x) = 0$$
$$D_x(2\sin x - x) = 0$$
$$2\cos x - 1 = 0$$
$$\cos x = \frac{1}{2}$$
$$x = \pm\frac{\pi}{3} + 2\pi n, \; (n \in \mathbb{Z})$$

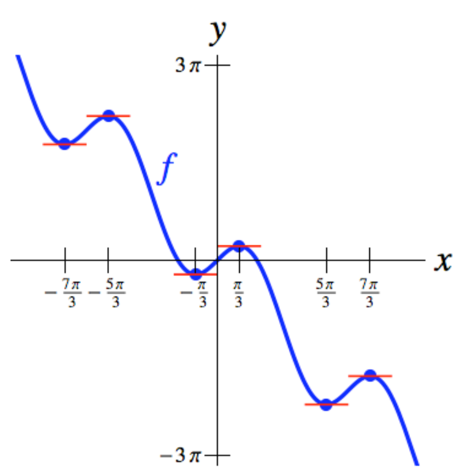
The desired *x*-coordinates are given by:

$$\left\{ x \in \mathbb{R} \,\middle|\, x = \pm\frac{\pi}{3} + 2\pi n, \; (n \in \mathbb{Z}) \right\}.$$

• Observe that there are **infinitely many** points on the graph where the tangent line is horizontal.

• Why does the graph of $y = 2\sin x - x$ below make sense? Observe that f is an **odd** function. Also, the "$-x$" term leads to downward drift; the graph oscillates about the line $y = -x$.

• The **red tangent lines** below are truncated.



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Example 5 (Equation of a Tangent Line; Revisiting Example 4)

Let $f(x) = 2\sin x - x$, as in Example 4. Find an equation of the **tangent line** to the graph of $y = f(x)$ at the point $\left(\frac{\pi}{6}, f\left(\frac{\pi}{6}\right)\right)$.

§ Solution

• $f\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{6}\right) - \frac{\pi}{6} = 2\left(\frac{1}{2}\right) - \frac{\pi}{6} = 1 - \frac{\pi}{6}$, so the **point** is at $\left(\frac{\pi}{6}, 1 - \frac{\pi}{6}\right)$.

• Find *m*, the **slope** of the tangent line there. This is given by $f'\left(\frac{\pi}{6}\right)$.

$$m = f'\left(\frac{\pi}{6}\right)$$

Now, $f'(x) = 2\cos x - 1$ (see Example 4).

$$= 2\cos\left(\frac{\pi}{6}\right) - 1$$

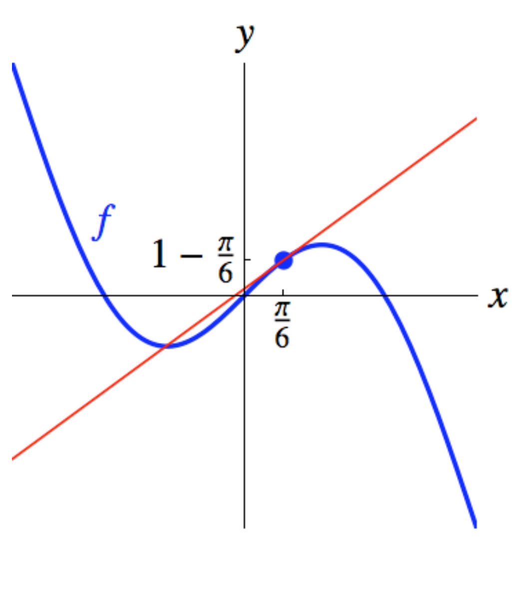
$$= 2\left(\frac{\sqrt{3}}{2}\right) - 1$$

$$= \sqrt{3} - 1$$

• Find a **Point-Slope Form** for the equation of the desired tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - \left(1 - \frac{\pi}{6}\right) = \left(\sqrt{3} - 1\right)\left(x - \frac{\pi}{6}\right), \text{ or } y - \frac{6 - \pi}{6} = \left(\sqrt{3} - 1\right)\left(x - \frac{\pi}{6}\right)$$



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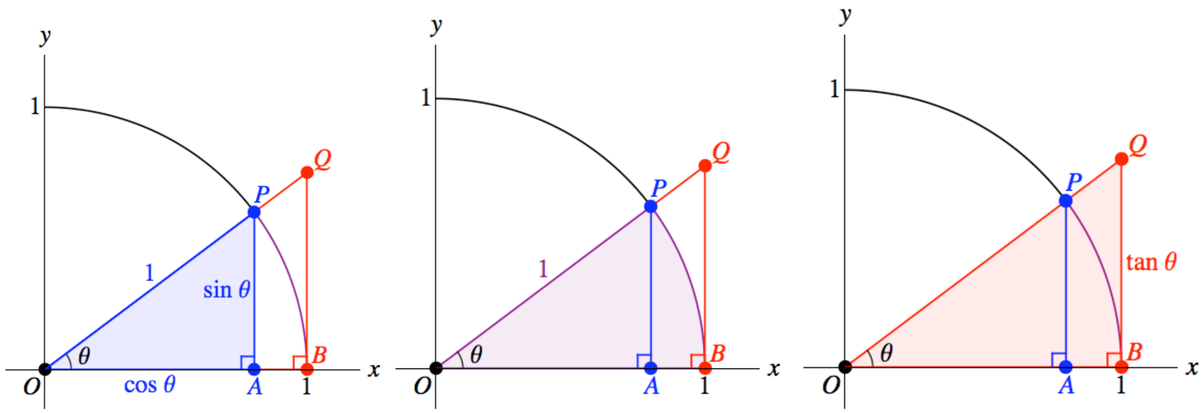
FOOTNOTES

1. **Proof of Limit Statement #1 in Part C.** First prove that $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$, where we use θ to represent angle measures instead of *h*.

<u>Side note</u>: The area of a circular sector such as *POB* below is given by:

$$\begin{pmatrix} \text{Ratio of } \theta \text{ to a} \\ \text{full revolution} \end{pmatrix} \begin{pmatrix} \text{Area of} \\ \text{the circle} \end{pmatrix} = \left(\frac{\theta}{2\pi}\right)\left(\pi r^2\right) = \frac{1}{2}\theta r^2 = \frac{1}{2}\theta \quad (\text{if } r = 1), \text{ where } \theta \in [0, 2\pi].$$

Area of Triangle *POA* $\leq$ Area of Sector *POB* $\leq$ Area of Triangle *QOB*



$$\begin{aligned}
\frac{1}{2}\sin\theta\cos\theta \quad &\leq \quad \frac{1}{2}\theta \quad \leq \quad \frac{1}{2}\tan\theta, \quad \forall\theta \in \left(0, \frac{\pi}{2}\right) \\
\sin\theta\cos\theta \quad &\leq \quad \theta \quad \leq \quad \tan\theta, \quad \text{"} \quad \text{"} \\
\cos\theta \quad &\leq \quad \frac{\theta}{\sin\theta} \quad \leq \quad \frac{1}{\cos\theta}, \quad \text{"} \quad \text{"}
\end{aligned}$$

(Now, we take reciprocals and reverse the inequality symbols.)

$$\begin{aligned}
\frac{1}{\cos\theta} \quad &\geq \quad \frac{\sin\theta}{\theta} \quad \geq \quad \cos\theta, \quad \text{"} \quad \text{"} \\
\underbrace{\cos\theta}_{\to 1} \quad &\leq \quad \frac{\sin\theta}{\theta} \quad \leq \quad \frac{1}{\underbrace{\cos\theta}_{\to 1}}, \quad \forall\theta \in \left(0, \frac{\pi}{2}\right)
\end{aligned}$$

as $\theta \to 0^+$. Therefore, $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$ by a one-sided variation of the Squeeze (Sandwich) Theorem from Section 2.6.

Now, prove that $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$. Let $\alpha = -\theta$.

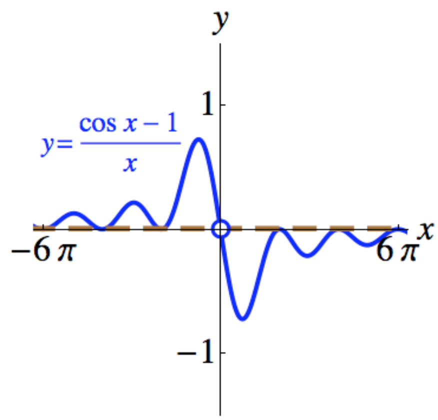
$$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = \lim_{\alpha \to 0^+} \frac{\sin(-\alpha)}{-\alpha} = \lim_{\alpha \to 0^+} \frac{-\sin(\alpha)}{-\alpha} = \lim_{\alpha \to 0^+} \frac{\sin(\alpha)}{\alpha} = 1. \text{ Therefore, } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$

2. **Proof of Limit Statement #2 in Part C.**

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\left(-\frac{1-\cos h}{h}\right)=\lim_{h\to 0}\left[-\frac{(1-\cos h)}{h}\cdot\frac{(1+\cos h)}{(1+\cos h)}\right]=\lim_{h\to 0}\left[-\frac{1-\cos^2 h}{h(1+\cos h)}\right]$$

$$=\lim_{h\to 0}\left[-\frac{\sin^2 h}{h(1+\cos h)}\right]=\lim_{h\to 0}\left(-\frac{\sin h}{h}\cdot\frac{\sin h}{1+\cos h}\right)=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{\sin h}{1+\cos h}\right)$$

$$=-(1)\cdot(0)=0\,.$$



3. **Proof of Rule 3) $D_x(\tan x)=\sec^2 x$, using the Limit Definition of the Derivative.**

Let $f(x)=\tan x$. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$

$$=\lim_{h\to 0}\frac{\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x}{h}=\lim_{h\to 0}\left[\frac{\left(\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x\right)}{h}\cdot\frac{(1-\tan x\tan h)}{(1-\tan x\tan h)}\right]$$

$$=\lim_{h\to 0}\frac{\tan x+\tan h-(\tan x)(1-\tan x\tan h)}{h(1-\tan x\tan h)}=\lim_{h\to 0}\frac{\tan h+\tan^2 x\tan h}{h(1-\tan x\tan h)}$$

$$=\lim_{h\to 0}\frac{(\tan h)(1+\tan^2 x)}{h(1-\tan x\tan h)}=\lim_{h\to 0}\frac{(\tan h)(\sec^2 x)}{h(1-\tan x\tan h)}$$ (Assume the limits in the next step exist.)

$$=\left(\lim_{h\to 0}\frac{\tan h}{h}\right)\cdot(\sec^2 x)\cdot\left(\lim_{h\to 0}\frac{1}{1-\tan x\tan h}\right)=\left[\lim_{h\to 0}\left(\frac{\sin h}{\cos h}\cdot\frac{1}{h}\right)\right]\cdot(\sec^2 x)\cdot(1)$$

$$=\left[\lim_{h\to 0}\left(\frac{\sin h}{h}\cdot\frac{1}{\cos h}\right)\right]\cdot(\sec^2 x)=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos h}\right)\cdot(\sec^2 x)=(1)(1)(\sec^2 x)$$

$$=\sec^2 x$$

4. **A joke.** The following is a “sin”: $\frac{\sin \not{x}}{\not{x}}=\sin$. Of course, this is ridiculous!