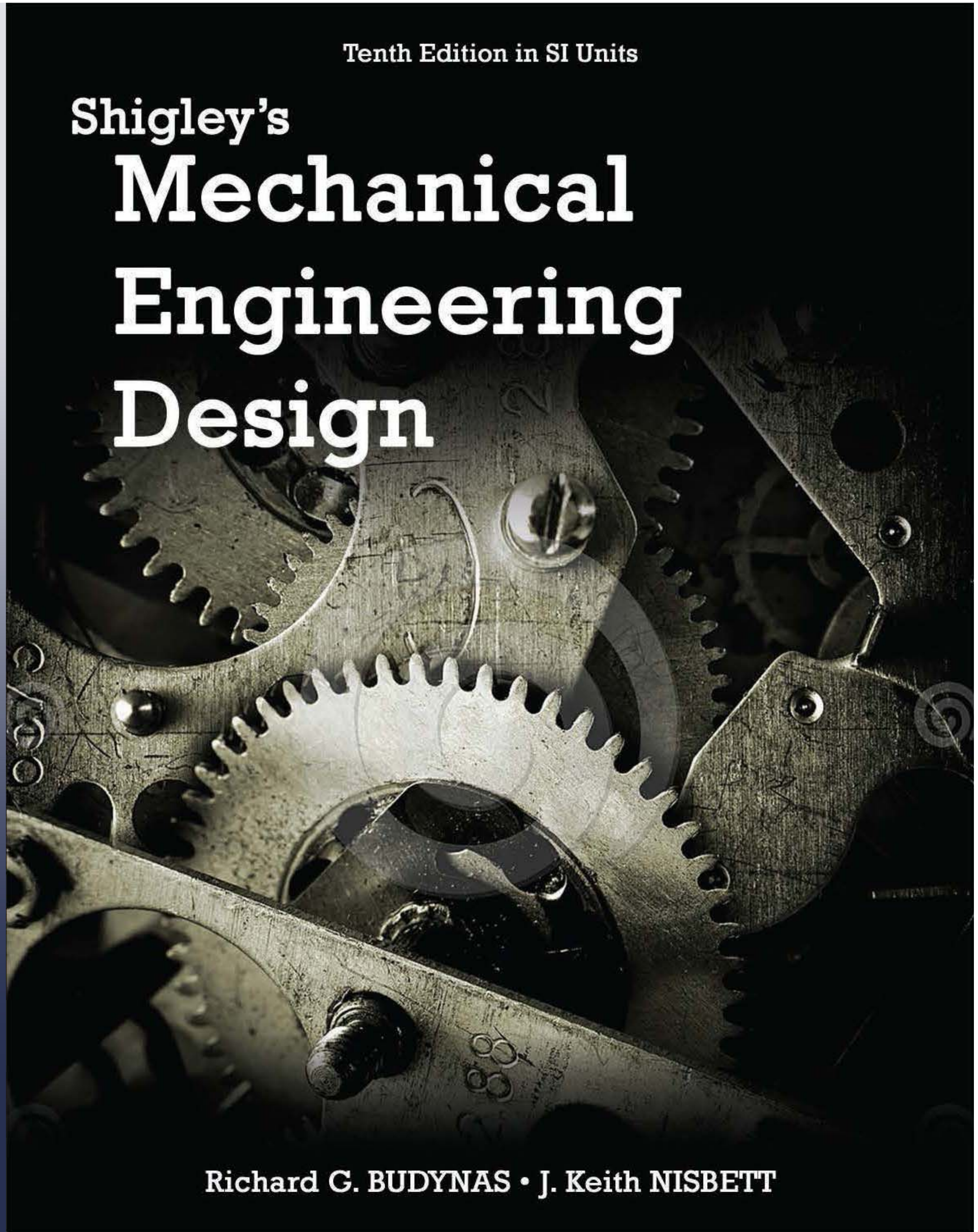
Lecture Slides

Chapter 3

Load and Stress Analysis



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Chapter Outline

Equilibrium and Free-Body Diagrams

- **Equilibrium**

  A system with zero acceleration is said to be in **equilibrium**, if that system is motionless or, at most, has constant velocity.

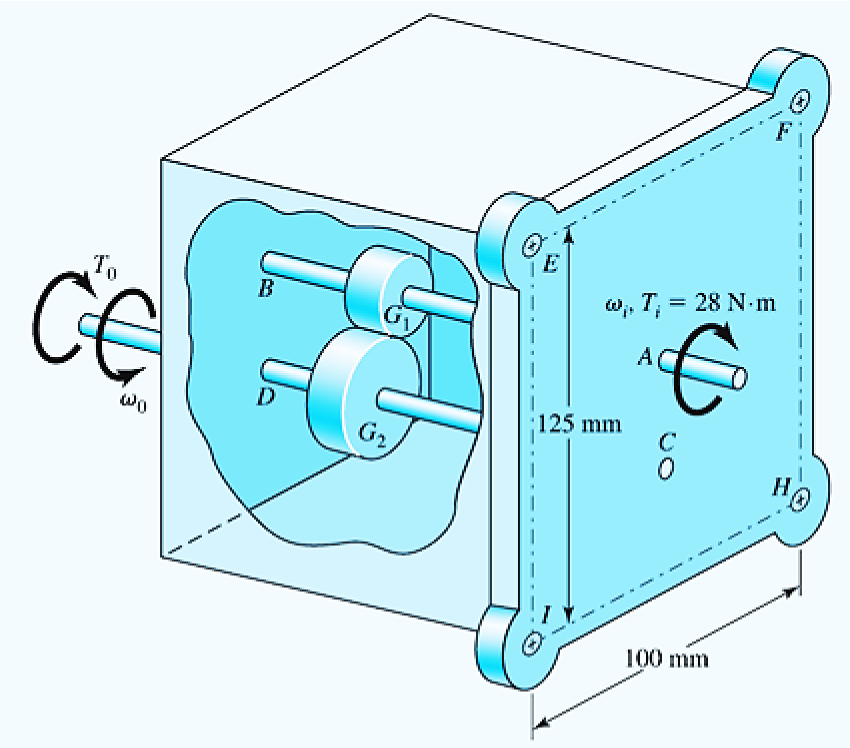
$$\sum \mathbf{F} = 0 \qquad \sum \mathbf{M} = 0$$

- **Free-Body Diagram**

  Free-body diagrams help simplifying the analysis of a very complex structure or machine by successively isolating each element and then studying and analyzing it.

Example 3-1

Figure 3–1*a* shows a simplified rendition of a gear reducer where the input and output shafts *AB* and *CD* are rotating at constant speeds ω_i and ω_o, respectively. The input and output torques (torsional moments) are $T_i = 28$ N · m and T_o, respectively. The shafts are supported in the housing by bearings at *A*, *B*, *C*, and *D*. The pitch radii of gears G_1 and G_2 are $r_1 = 20$ mm and $r_2 = 40$ mm, respectively. Draw the free-body diagrams of each member and determine the net reaction forces and moments at all points.



Example 3-1

Solution

First, we will list all simplifying assumptions.

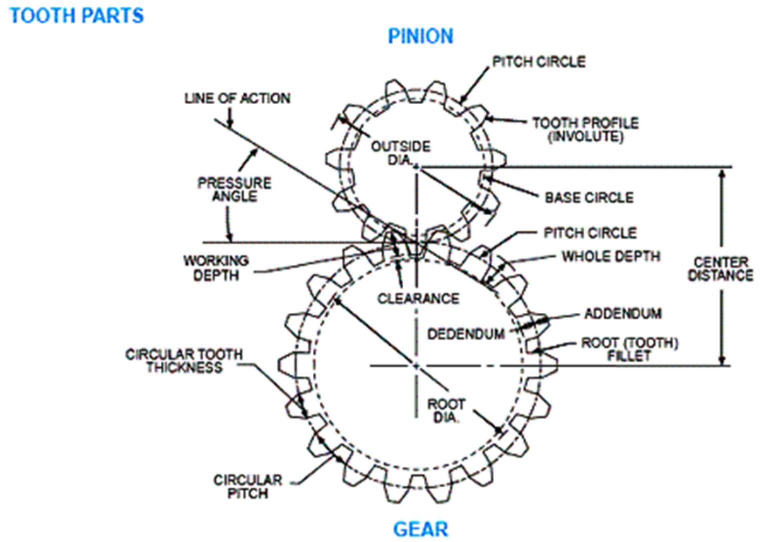
1 Gears G_1 and G_2 are simple spur gears with a standard pressure angle $\phi = 20°$ (see Sec. 13–5).

2 The bearings are self-aligning and the shafts can be considered to be simply supported.

3 The weight of each member is negligible.

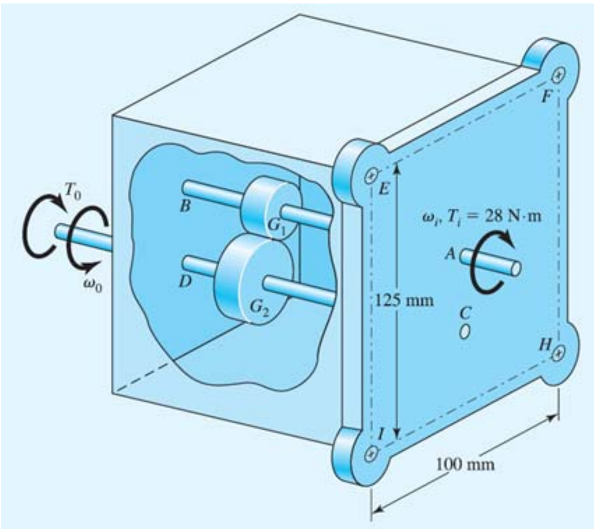
4 Friction is negligible.

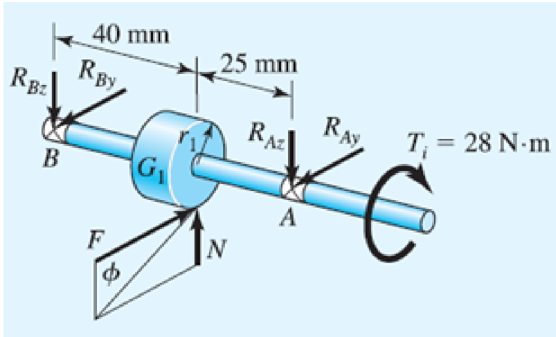
5 The mounting bolts at E, F, H, and I are the same size.



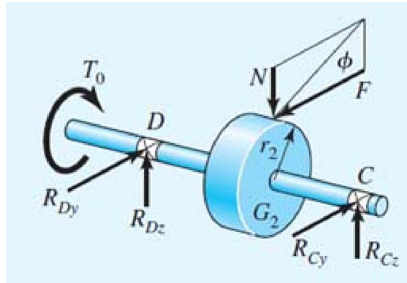
Example 3-1

The separate free-body diagrams of the members are shown in Figs. 3–1*b*–*d*. Note that Newton's third law, called *the law of action and reaction*, is used extensively where each member mates. The force transmitted between the spur gears is not tangential but at the pressure angle ϕ. Thus, $N = F \tan \phi$.

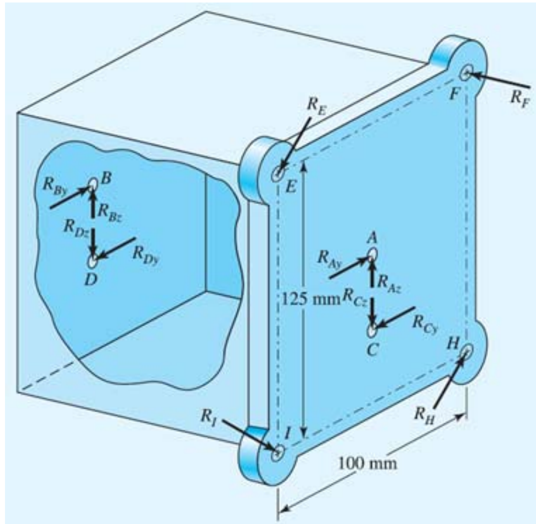


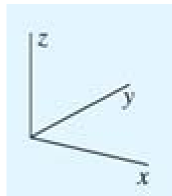


Input shaft



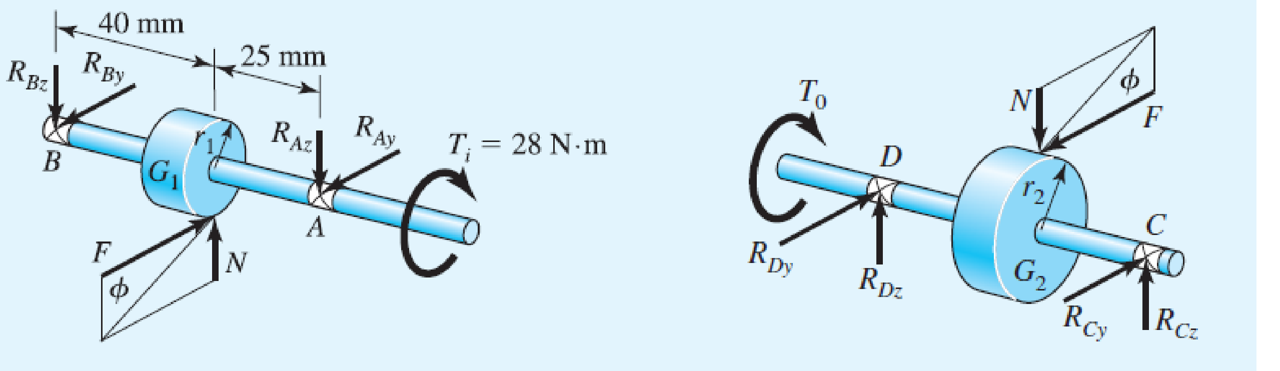
Output shaft





Gear box

Example 3-1



(*c*) Input shaft (*d*) Output shaft

Summing moments about the x axis of shaft AB in Fig. 3–1d gives

$$\sum M_x = F(0.02) - 28 = 0$$

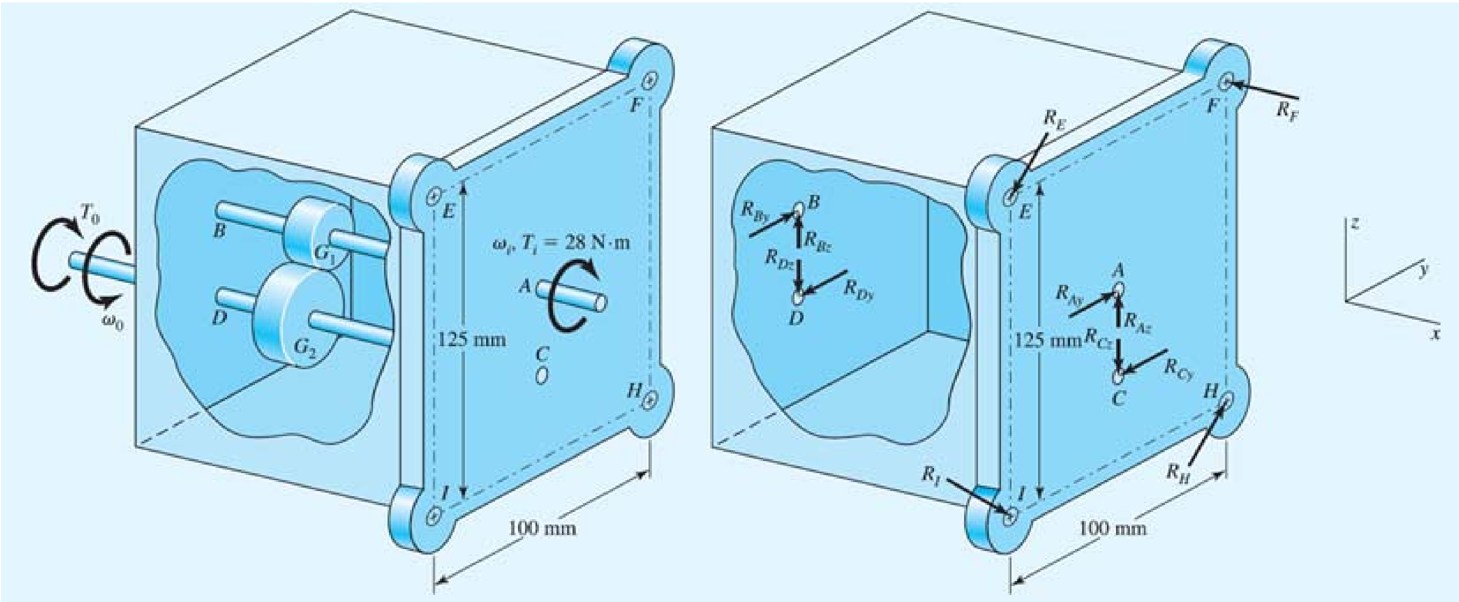
$$F = 1400 \text{ N}$$

The normal force is $N = 1400$ N tan 20° = 509.6 N.

Using the equilibrium equations for Figs. 3–1c and d, the reader should verify that: $R_{Ay} = 861.5$ N, $R_{Az} = 313.6$ N, $R_{By} = 538.5$ N, $R_{Bz} = 196$ N, $R_{Cy} = 861.5$ N, $R_{Cz} =$ 313.6 N, $R_{Dy} = 538.5$ N, $R_{Dz} = 196$ N, and $T_o = 56$ N · m. The direction of the output torque T_o is opposite ω_o because it is the resistive load on the system opposing the motion ω_o.

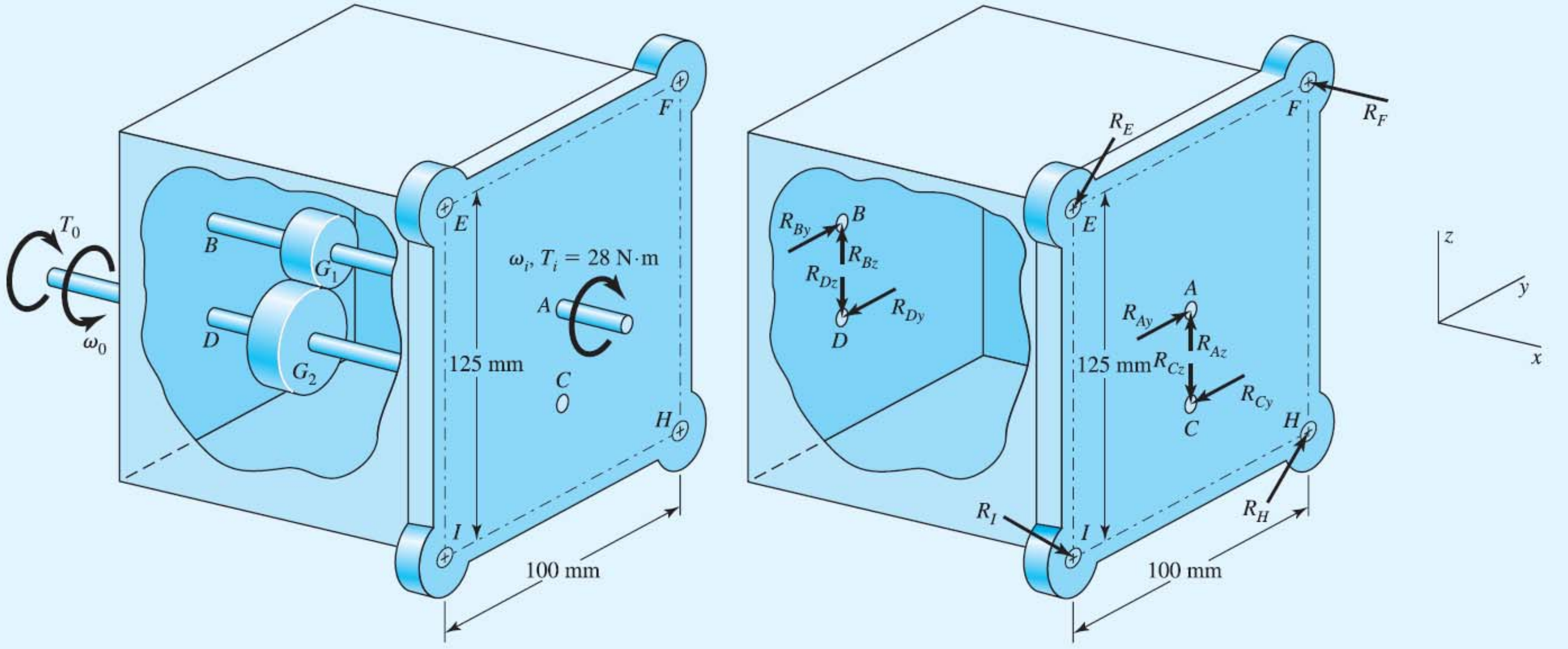
Example 3-1

Note in Fig. 3–1*b* the net force from the bearing reactions is zero whereas the net moment about the x axis is $0.06\ (861.5) + 0.06\ (538.5) = 84\ \text{N} \cdot \text{m}$. This value is the same as $T_i + T_o = 28 + 56 = 84\ \text{N} \cdot \text{m}$, as shown in Fig. 3–1*a*. The reaction forces R_E, R_F, R_H, and R_I, from the mounting bolts cannot be determined from the equilibrium equations as there are too many unknowns. Only three equations are available, $\sum F_y = \sum F_z = \sum M_x = 0$. In case you were wondering about assumption 5, here is where we will use it (see Sec. 8–12). The gear box tends to rotate about the x axis because of a pure torsional moment of $84\ \text{N} \cdot \text{m}$. The bolt forces must provide an



Example 3-1

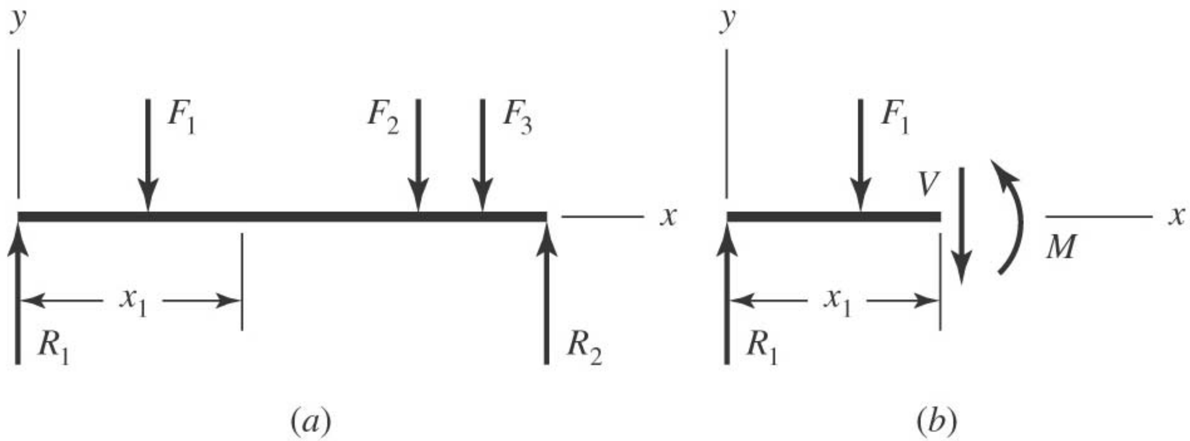
equal but opposite torsional moment. The center of rotation relative to the bolts lies at the centroid of the bolt cross-sectional areas. Thus if the bolt areas are equal: the center of rotation is at the center of the four bolts, a distance of $\sqrt{(100/2)^2 + (125/2)^2} = 80$ mm from each bolt; the bolt forces are equal ($R_E = R_F = R_H = R_I = R$), and each bolt force is perpendicular to the line from the bolt to the center of rotation. This gives a net torque from the four bolts of $4R\,(0.08) = 84$. Thus, $R_E = R_F = R_H = R_I = 262.5$ N.



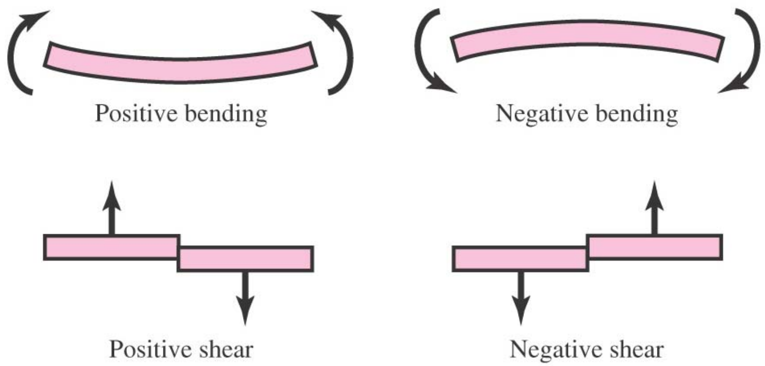
(a) Gear reducer (b) Gear box

Shear Force and Bending Moments in Beams

- Cut beam at any location x_1
- Internal shear force V and bending moment M must ensure equilibrium



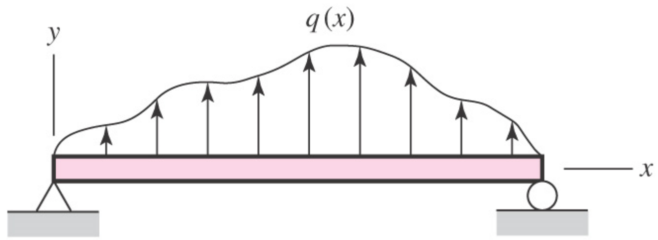
Sign Conventions



Shear Force and Bending Moments in Beams

Distributed Load on Beam

- Distributed load $q(x)$ called *load intensity*
- Units of force per unit length



Relationships between Load, Shear, and Bending

$$V = \frac{dM}{dx}$$

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q$$

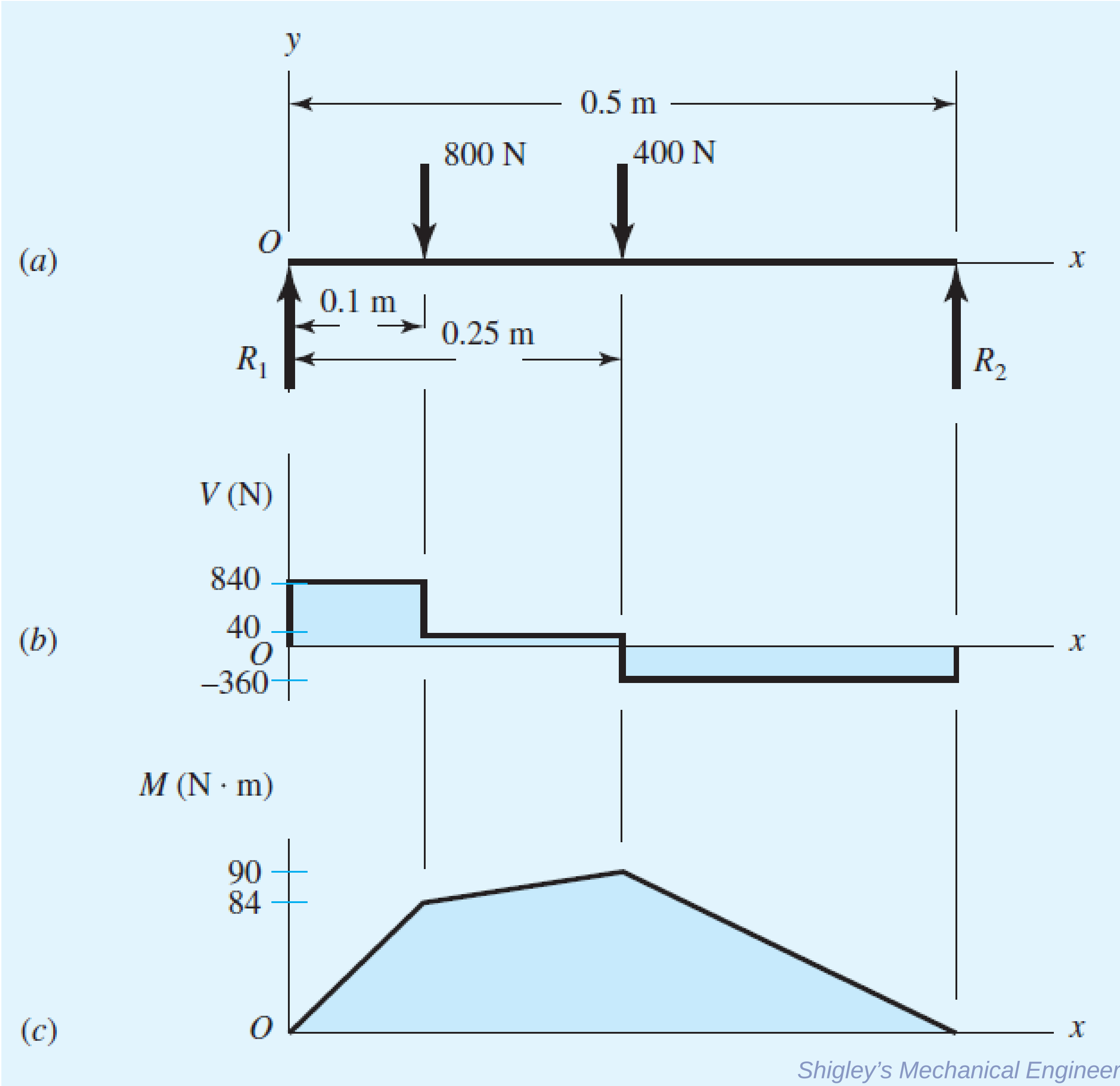
$$\int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q\,dx$$

$$\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V\,dx$$

- The change in ***shear force*** from A to B is equal to the *area of the loading diagram* between x_A and x_B.
- The change in ***moment*** from A to B is equal to the *area of the shear-force diagram* between x_A and x_B.

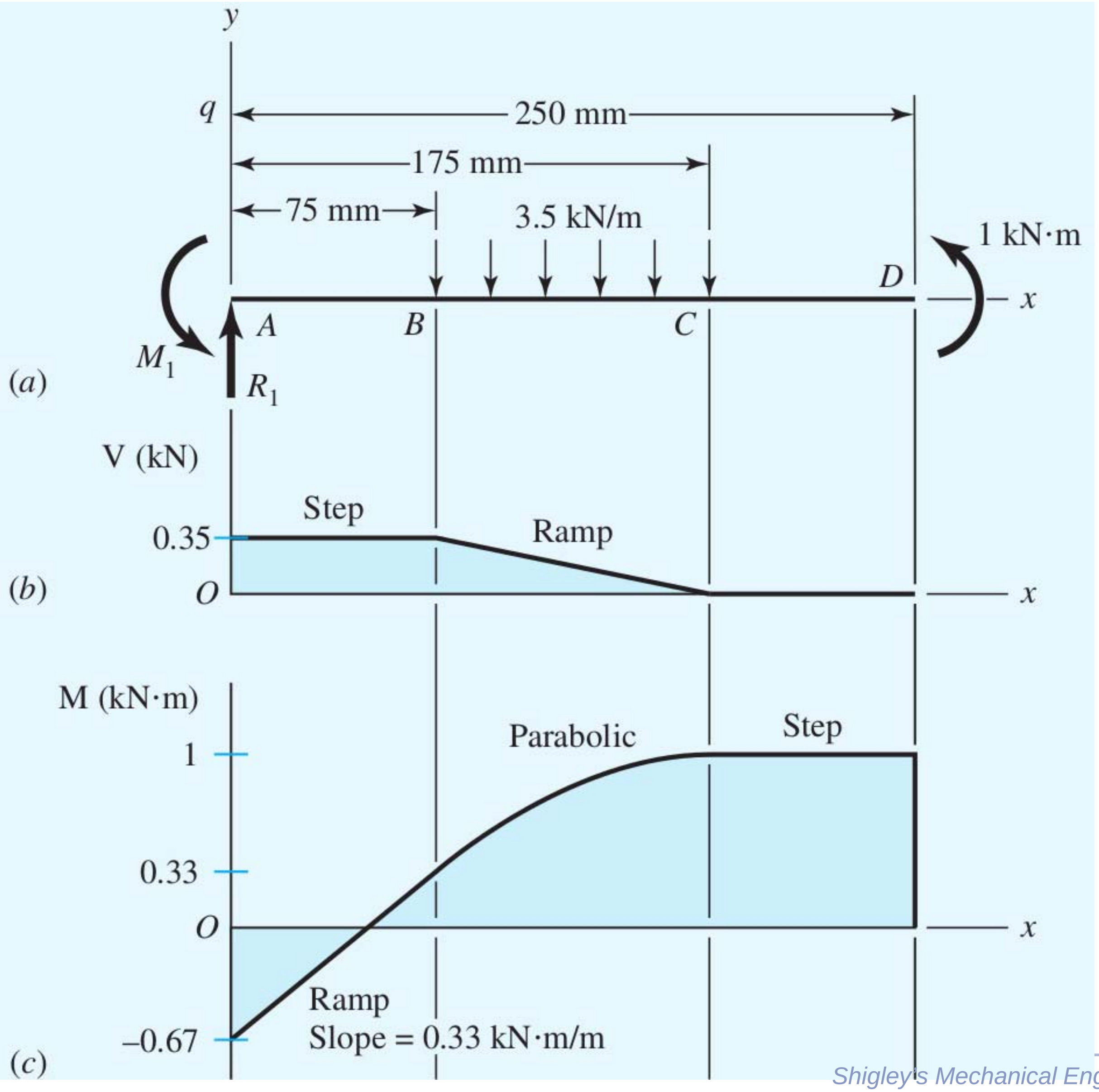
Shear Force and Bending Moments in Beams

Example 3-2



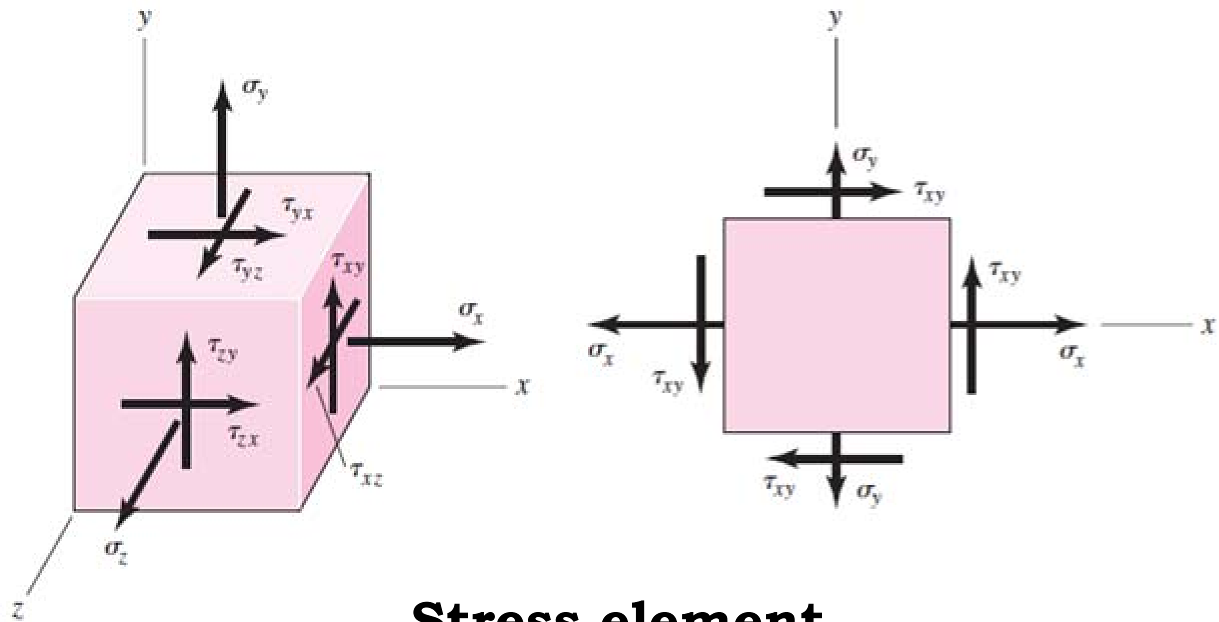
Shear Force and Bending Moments in Beams

Example 3-3



Cartesian Stress Component

- *Normal stress* is normal to a surface, designated by σ
- *Shear stress* is tangent to a surface, designated by τ



<u>Stress element</u>

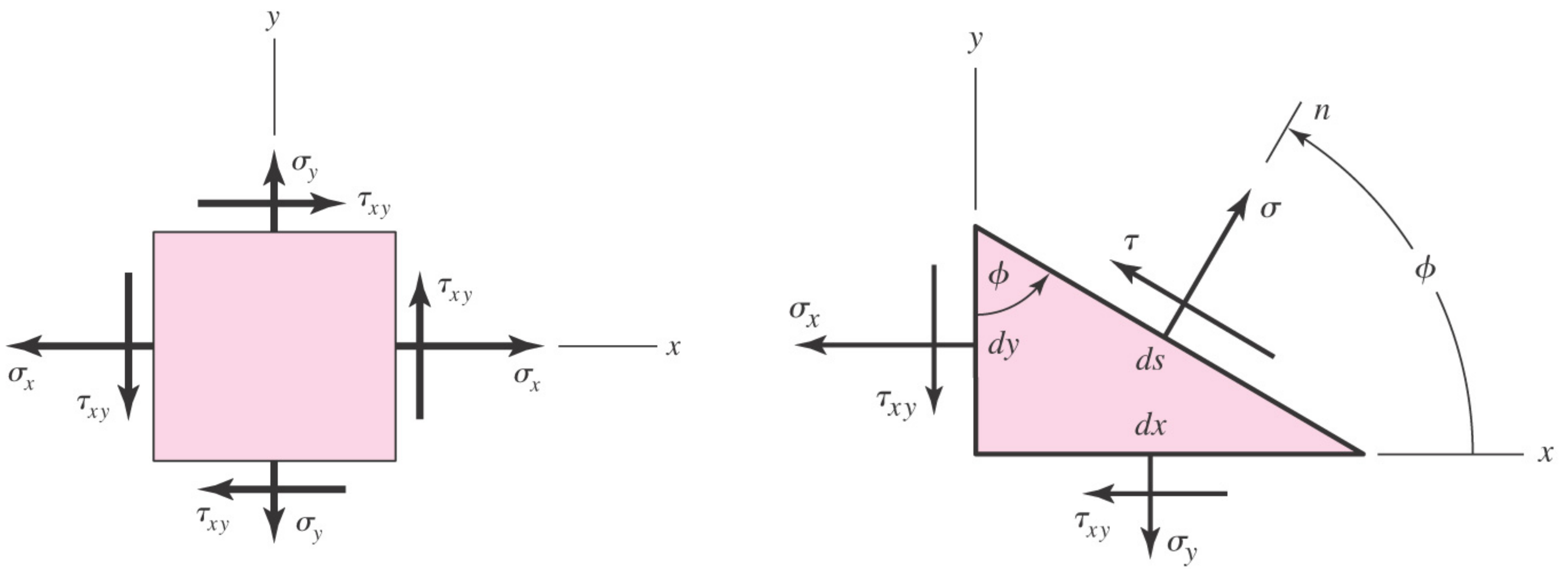
- Represents stress *at a point*
- Coordinate directions are arbitrary
- Choosing coordinates which result in zero shear stress will produce principal stresses

Plane-Stress Transformation Equations

- Cutting plane stress element at an arbitrary angle and balancing stresses gives *plane-stress transformation equations*

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\phi + \tau_{xy} \sin 2\phi \quad (3\text{–}8)$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2} \sin 2\phi + \tau_{xy} \cos 2\phi \quad (3\text{–}9)$$



Principal Stresses for Plane Stress

Principal stresses

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Principal directions (zero shear stresses)

$$\tan 2\phi_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Maximum shear stresses

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

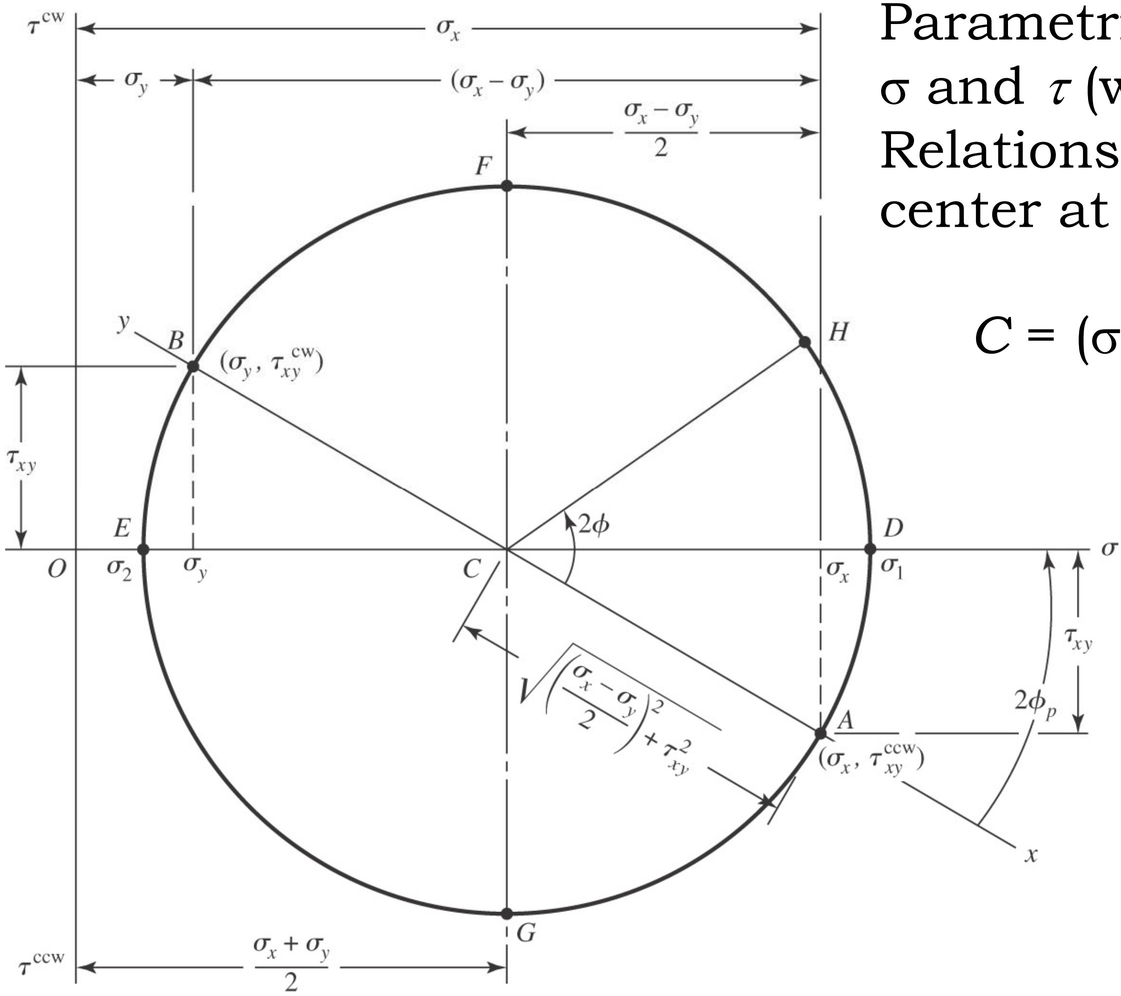
Maximum Shear Stress

- There are always three principal stresses. One is zero for plane stress.
- There are always three extreme-value shear stresses.

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2}$$

- The *maximum shear stress* is always the greatest of these three
- If principal stresses are ordered so that $\sigma_1 > \sigma_2 > \sigma_3$,
  then $\tau_{max} = \tau_{1/3}$

Mohr's Circle Diagram



Parametric relationship between σ and τ (with 2ϕ as parameter)
Relationship is a circle with center at

$$C = (\sigma, \tau) = [(\sigma_x + \sigma_y)/2, 0]$$

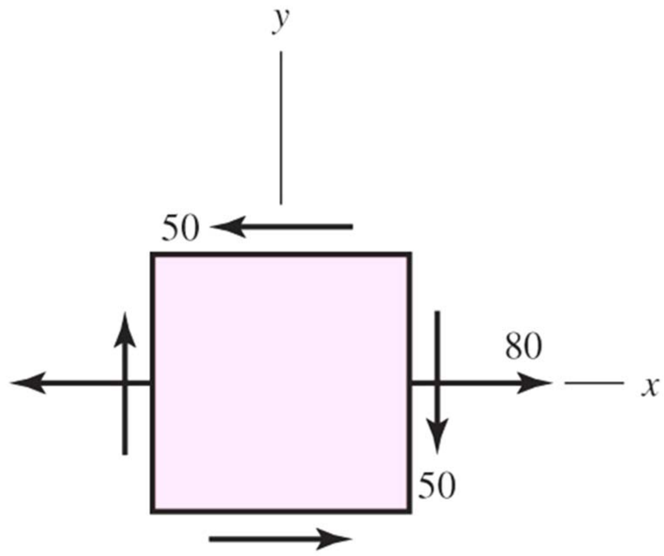
$$R = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

Example 3-4

A stress element has $\sigma_x = 80$ MPa and $\tau_{xy} = 50$ MPa cw, as shown in Fig. 3–11*a*.

(*a*) Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the *xy* coordinates. Draw another stress element to show τ_1 and τ_2, find the corresponding normal stresses, and label the drawing completely.

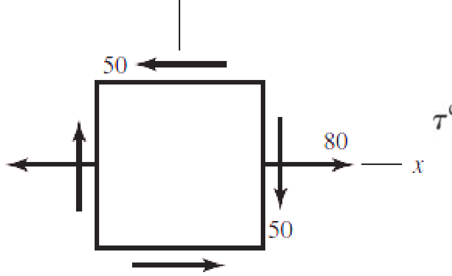
(*b*) Repeat part *a* using the transformation equations only.

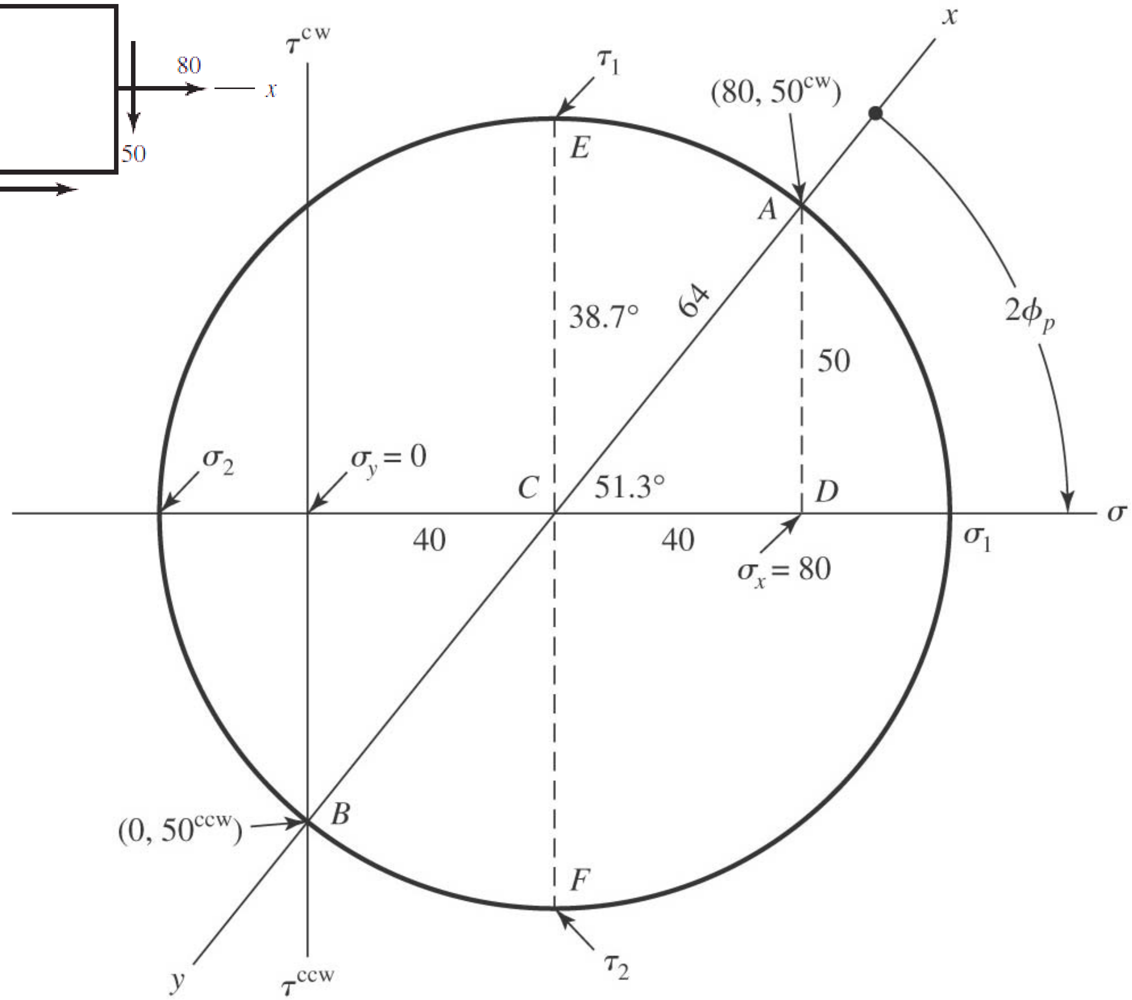


(*a*)

Example 3-4

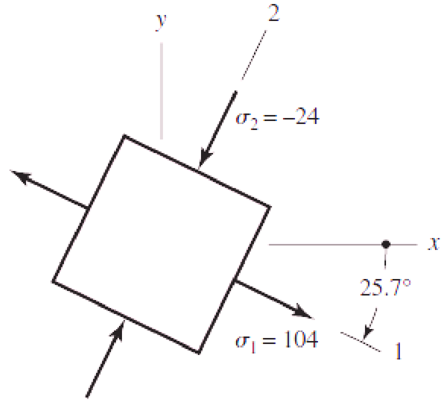
x-y **orientation**



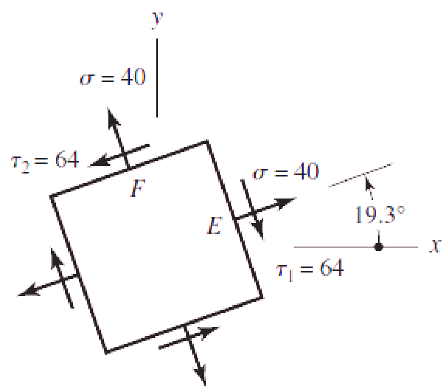


(*b*)

Principal stress orientation



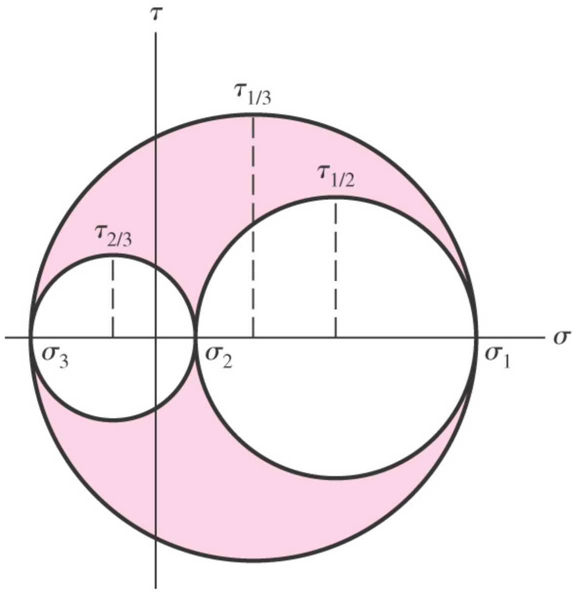
Max shear orientation



General Three-Dimensional Stress

- All stress elements are actually 3-D.
- Plane stress elements simply have one surface with zero stresses.
- For cases where there is no stress-free surface, the principal stresses are found from the roots of the cubic equation

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma - (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2) = 0 \tag{3–15}$$

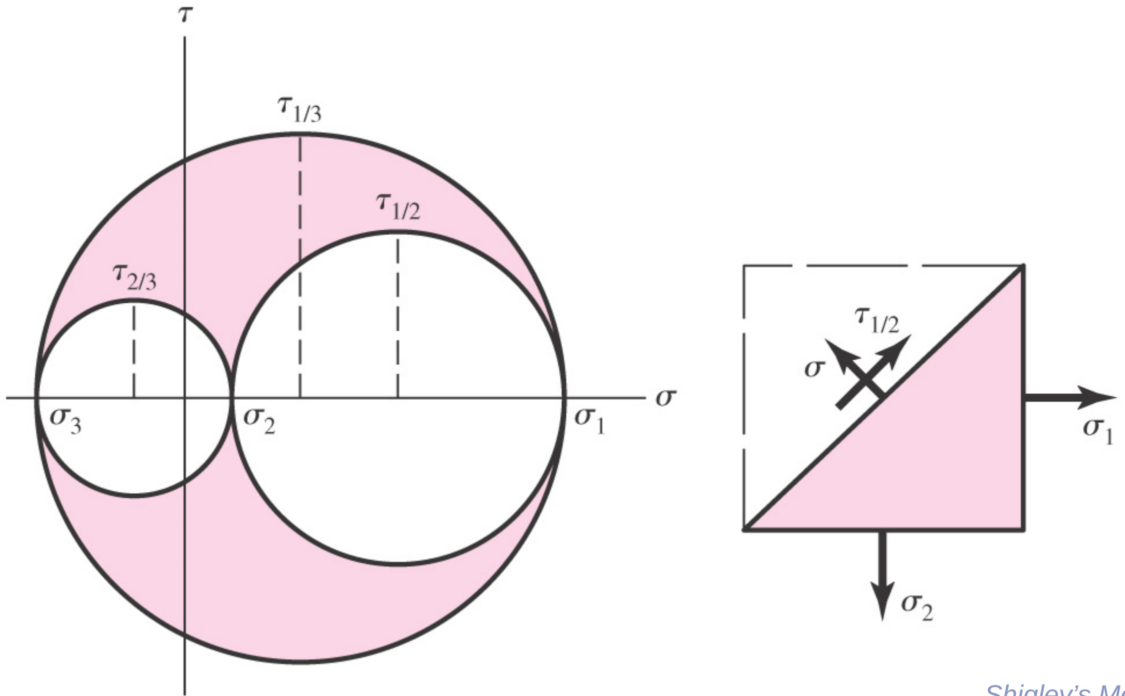


General Three-Dimensional Stress

- Always three extreme shear values

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \qquad (3\text{–}16)$$

- *Maximum Shear Stress* is the largest
- Principal stresses are usually ordered such that $\sigma_1 > \sigma_2 > \sigma_3$, in which case $\tau_{max} = \tau_{1/3}$



Homework

3-6

$$\Sigma F_y = 0$$

$$R_0 = 2 + 4(0.15) = 2.6 \text{ kN} \quad Ans.$$

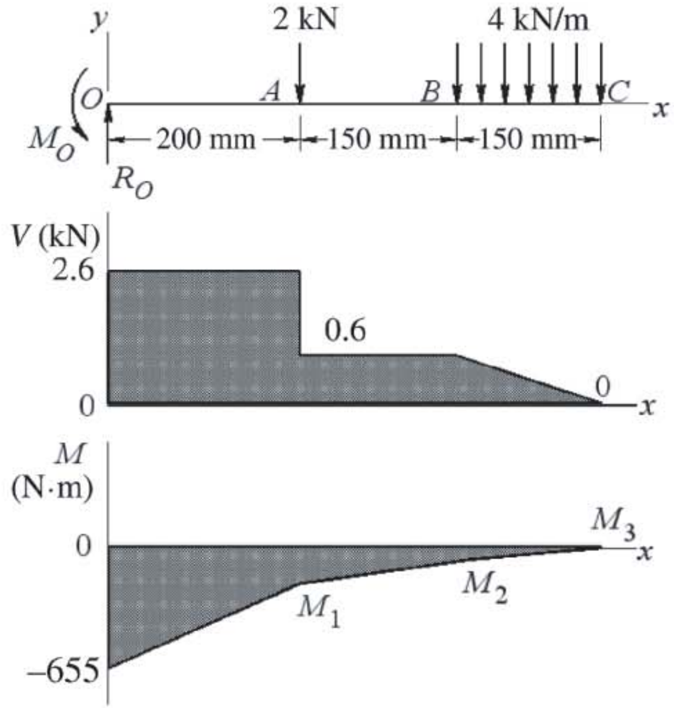
$$\Sigma M_0 = 0$$

$$M_0 = 2000(0.2) + 4000(0.15)(0.425) = 655 \text{ N}\cdot\text{m} \quad Ans.$$

$$M_1 = -655 + 2600(0.2) = -135 \text{ N}\cdot\text{m} \quad Ans.$$

$$M_2 = -135 + 600(0.15) = -45 \text{ N}\cdot\text{m} \quad Ans.$$

$$M_3 = -45 + \frac{1}{2}(600)(0.15) = 0 \quad \text{checks!}$$



Homework

(a)

$$C = \frac{-8+7}{2} = -0.5 \text{ MPa}$$

$$CD = \frac{8+7}{2} = 7.5 \text{ MPa}$$

$$R = \sqrt{7.5^2 + 6^2} = 9.60 \text{ MPa}$$

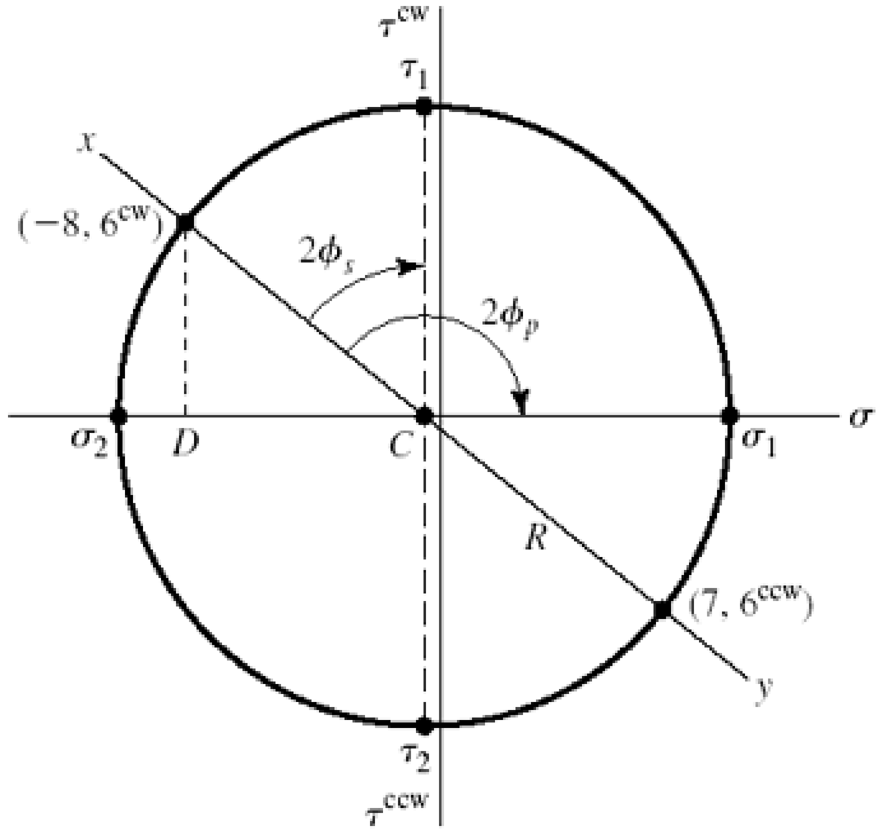
$$\sigma_1 = 9.60 - 0.5 = 9.10 \text{ MPa}$$

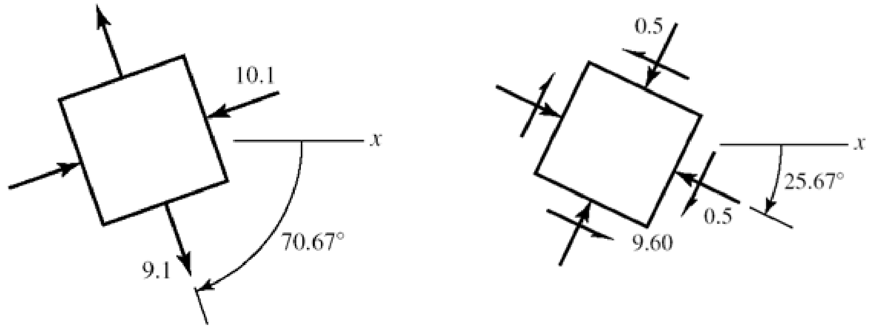
$$\sigma_2 = -0.5 - 9.6 = -10.1 \text{ Mpa}$$

$$\phi_p = \frac{1}{2}\left[90^\circ + \tan^{-1}\left(\frac{7.5}{6}\right)\right] = 70.67^\circ \text{ cw}$$

$$\tau_1 = R = 9.60 \text{ MPa}$$

$$\phi_s = 70.67^\circ - 45^\circ = 25.67^\circ \text{ cw}$$





Homework

(b)

$$C = \frac{9-6}{2} = 1.5 \text{ MPa}$$

$$CD = \frac{9+6}{2} = 7.5 \text{ MPa}$$

$$R = \sqrt{7.5^2 + 3^2} = 8.078 \text{ MPa}$$

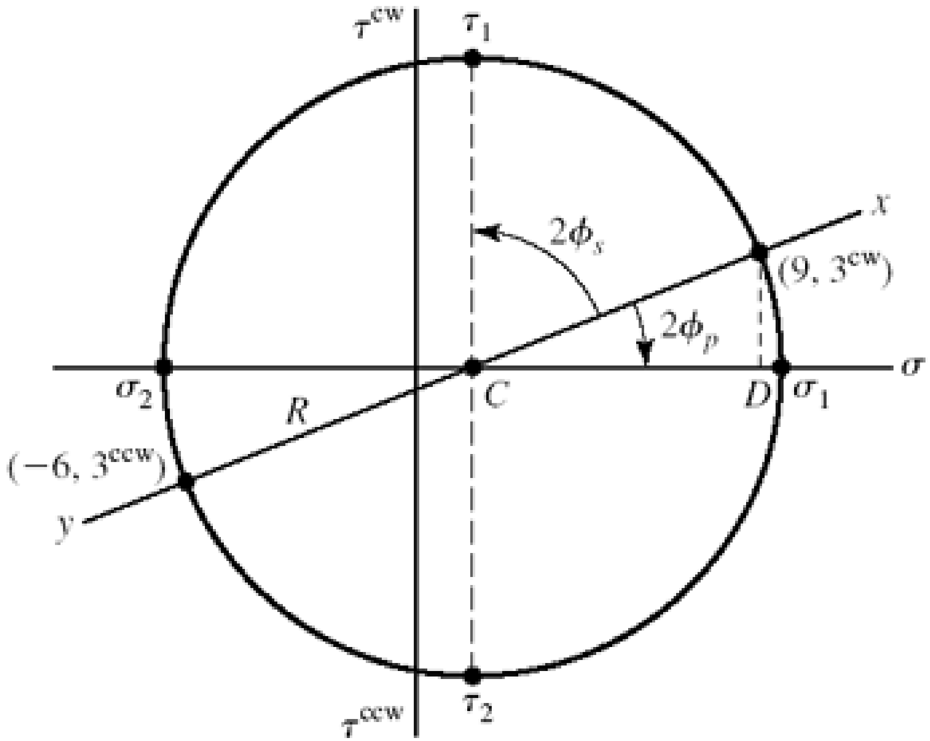
$$\sigma_1 = 1.5 + 8.078 = 9.58 \text{ MPa}$$

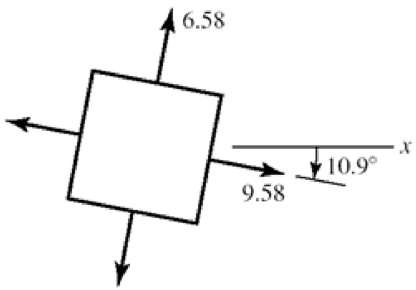
$$\sigma_2 = 1.5 - 8.078 = -6.58 \text{ MPa}$$

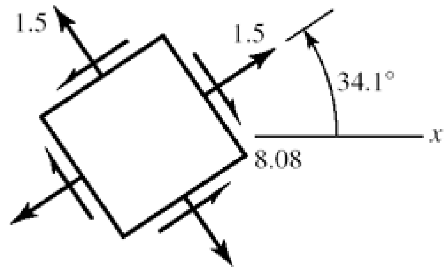
$$\phi_p = \frac{1}{2}\tan^{-1}\left(\frac{3}{7.5}\right) = 10.9^\circ \text{ cw}$$

$$\tau_1 = R = 8.078 \text{ MPa}$$

$$\phi_s = 45^\circ - 10.9^\circ = 34.1^\circ \text{ ccw}$$







Homework

(c)

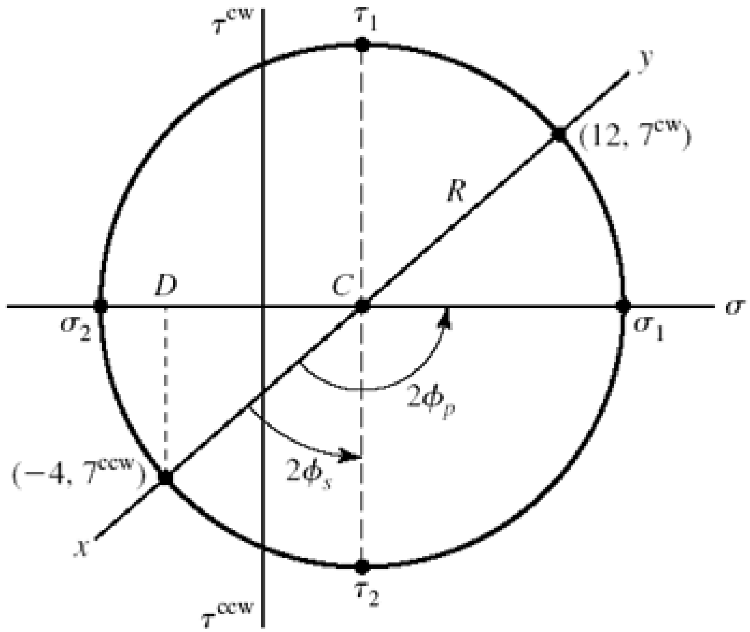
$$C = \frac{12-4}{2} = 4 \text{ MPa}$$

$$CD = \frac{12+4}{2} = 8 \text{ MPa}$$

$$R = \sqrt{8^2 + 7^2} = 10.63 \text{ MPa}$$

$$\sigma_1 = 4 + 10.63 = 14.63 \text{ MPa}$$

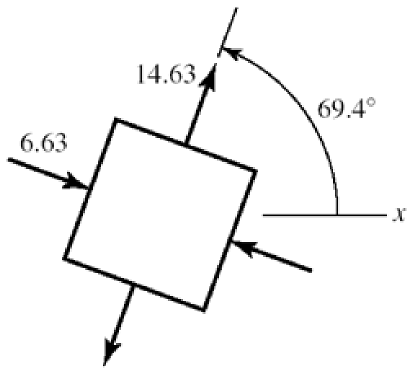
$$\sigma_2 = 4 - 10.63 = -6.63 \text{ MPa}$$

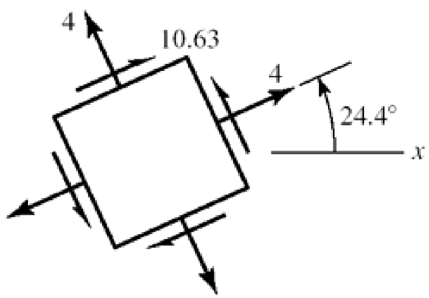


$$\phi_p = \frac{1}{2}\left[90^\circ + \tan^{-1}\left(\frac{8}{7}\right)\right] = 69.4^\circ \text{ ccw}$$

$$\tau_1 = R = 10.63 \text{ MPa}$$

$$\phi_s = 69.4^\circ - 45^\circ = 24.4^\circ \text{ ccw}$$





Homework

(d)

$$C = \frac{6-5}{2} = 0.5 \text{ MPa}$$

$$CD = \frac{6+5}{2} = 5.5 \text{ MPa}$$

$$R = \sqrt{5.5^2 + 8^2} = 9.71 \text{ MPa}$$

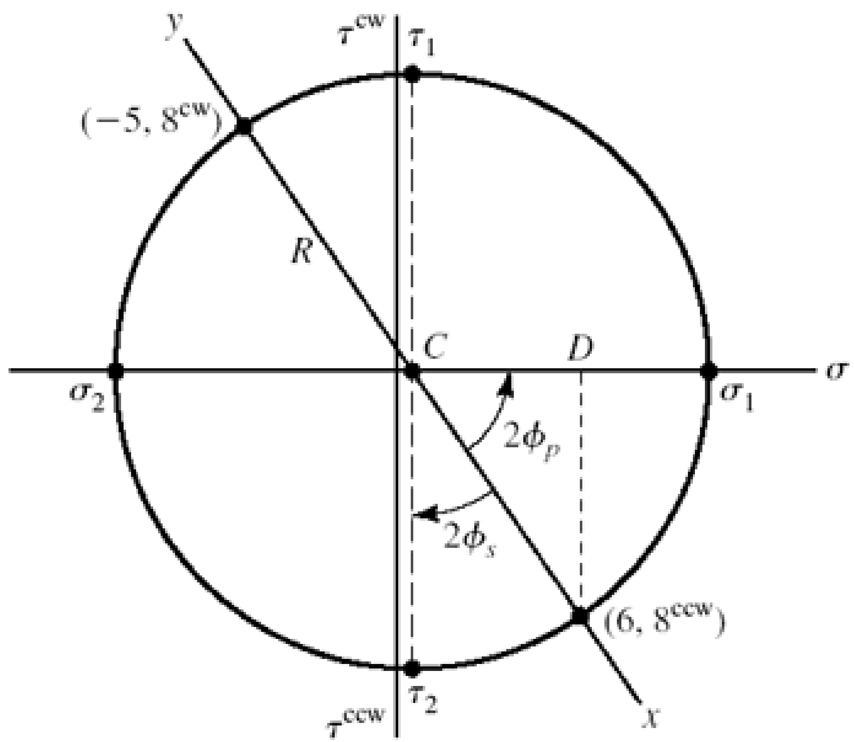
$$\sigma_1 = 0.5 + 9.71 = 10.21 \text{ MPa}$$

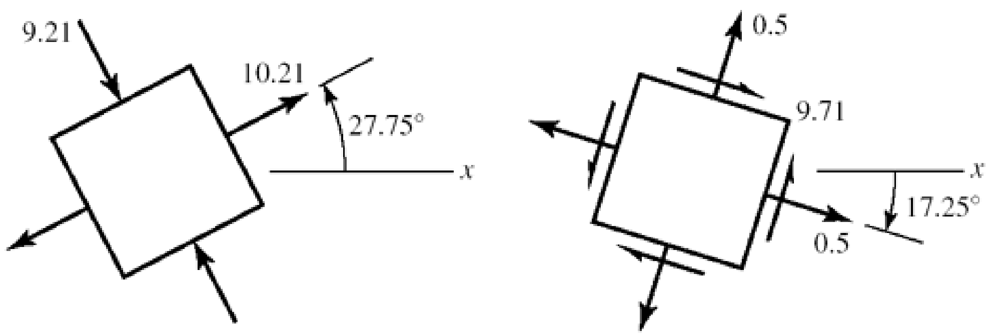
$$\sigma_2 = 0.5 - 9.71 = -9.21 \text{ MPa}$$

$$\phi_p = \frac{1}{2}\tan^{-1}\left(\frac{8}{5.5}\right) = 27.75^\circ \text{ ccw}$$

$$\tau_1 = R = 9.71 \text{ MPa}$$

$$\phi_s = 45^\circ - 27.75^\circ = 17.25^\circ \text{ cw}$$





Elastic Strain

- *Hooke's law*

$$\sigma = E\epsilon \tag{3-17}$$

- E is Young's modulus, or modulus of elasticity
- Tension in on direction produces negative strain (contraction) in a perpendicular direction.
- For axial stress in x direction,

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\nu \frac{\sigma_x}{E} \tag{3-18}$$

- The constant of proportionality ν is *Poisson's ratio*
- See Table A-5 for values for common materials.

Elastic Strain

- For a stress element undergoing σ_x, σ_y, and σ_z, simultaneously,

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$
$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$
$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

- Hooke's law for shear:

$$\tau = G\gamma$$

- *Shear strain* G is the change in a right angle of a stress element when subjected to pure shear stress.
- G is the *shear modulus of elasticity* or *modulus of rigidity*.
- For a linear, isotropic, homogeneous material, $E = 2G(1 + \nu)$

Uniformly Distributed Stresses

- Uniformly distributed stress distribution is often assumed for pure tension, pure compression, or pure shear.
- For tension and compression,

$$\sigma = \frac{F}{A}$$

- For direct shear (no bending present),

$$\tau = \frac{V}{A}$$

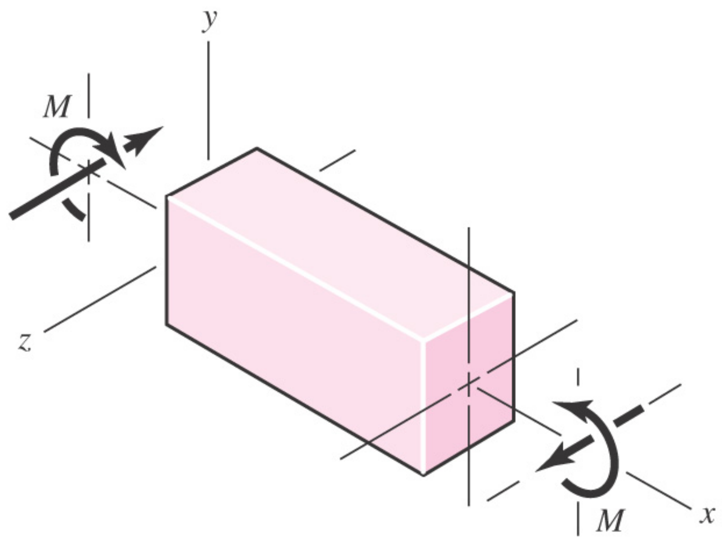
Normal Stresses for Beams in Bending

- Straight beam in positive bending
- *x* axis is *neutral axis*
- *xz* plane is *neutral plane*
- *Neutral axis* is coincident with the *centroidal axis* of the cross section
- Bending stress varies linearly with distance from neutral axis, *y*

$$\sigma_x = -\frac{My}{I}$$

- *I* is the *second-area moment* about the *z* axis

$$I = \int y^2 dA$$



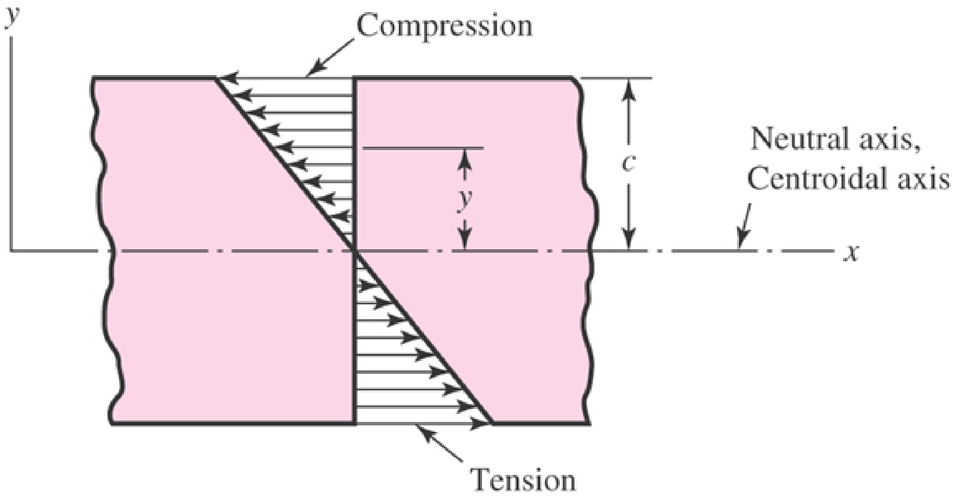
Normal Stresses for Beams in Bending

- Maximum bending stress is where y is greatest.

$$\sigma_{max} = \frac{Mc}{I}$$

$$\sigma_{max} = \frac{M}{Z}$$

- c is the magnitude of the greatest y
- $Z = I/c$ is the *section modulus*

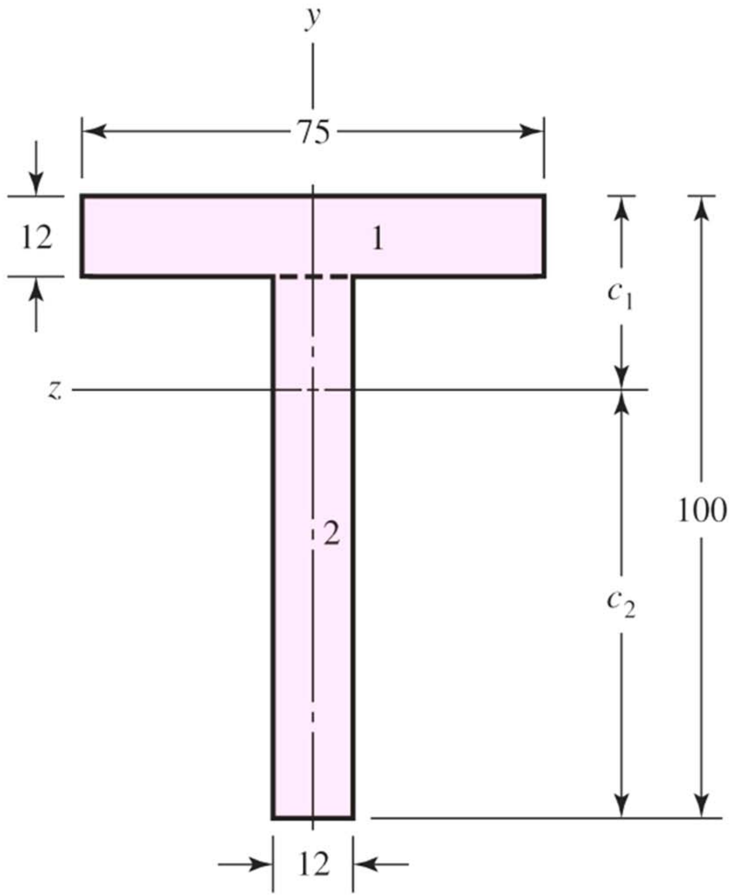


Normal Stresses for Beams in Bending

- Pure bending (though effects of axial, torsional, and shear loads are often assumed to have minimal effect on bending stress)
- Material is isotropic and homogeneous
- Material obeys Hooke's law
- Beam is initially straight with constant cross section
- Beam has axis of symmetry in the plane of bending
- Proportions are such that failure is by bending rather than crushing, wrinkling, or sidewise buckling
- Plane cross sections remain plane during bending

Example 3-5

A beam having a T section with the dimensions shown in Fig. 3–15 is subjected to a bending moment of 1600 N · m, about the negative z axis, that causes tension at the top surface. Locate the neutral axis and find the maximum tensile and compressive bending stresses.



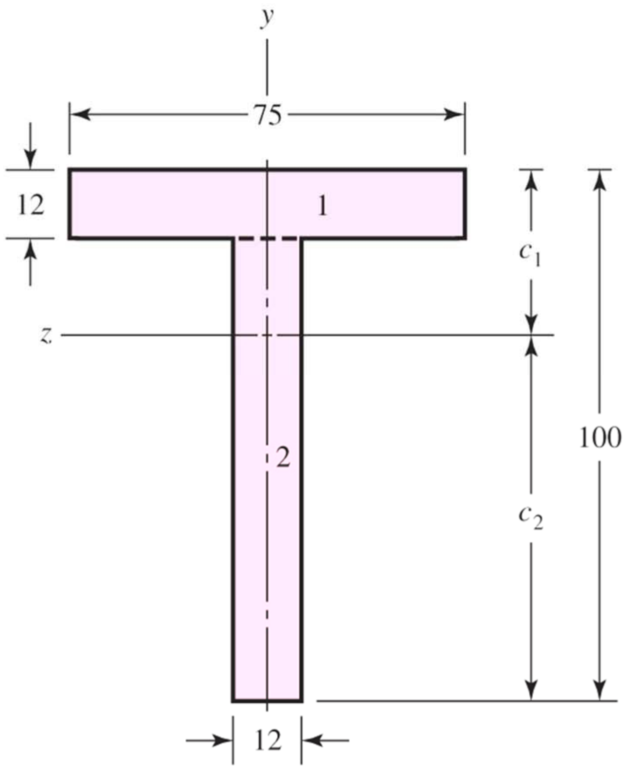
Dimensions in mm

Example 3-5

Dividing the T section into two rectangles, numbered 1 and 2, the total area is $A = 12(75) + 12(88) = 1956\ \text{mm}^2$. Summing the area moments of these rectangles about the top edge, where the moment arms of areas 1 and 2 are 6 mm and $(12 + 88/2) = 56$ mm respectively, we have

$$1956c_1 = 12(75)(6) + 12(88)(56)$$

and hence $c_1 = 32.99$ mm. Therefore $c_2 = 100 - 32.99 = 67.01$ mm.



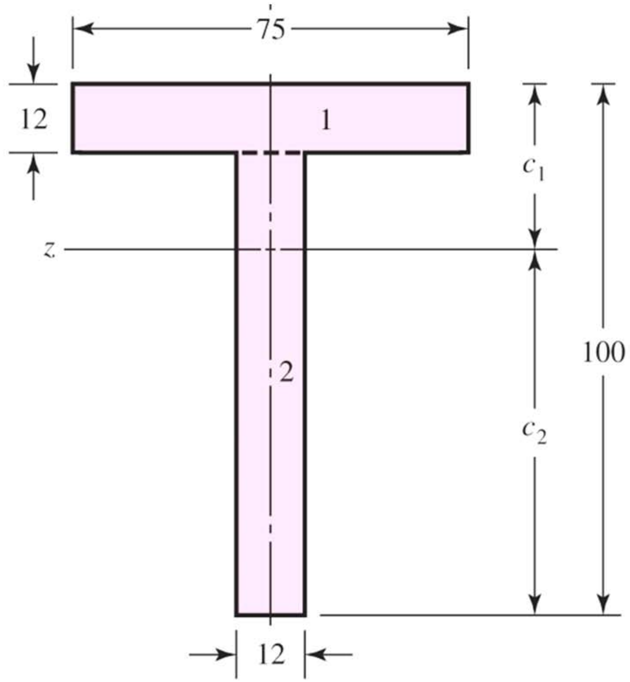
Example 3-5

Next we calculate the second moment of area of each rectangle about its own centroidal axis. Using Table A–18, we find for the top rectangle

$$I_1 = \frac{1}{12}bh^3 = \frac{1}{12}(75)12^3 = 1.080 \times 10^4 \text{ mm}^4$$

For the bottom rectangle, we have

$$I_2 = \frac{1}{12}(12)88^3 = 6.815 \times 10^5 \text{ mm}^4$$



Example 3-5

We now employ the *parallel-axis theorem* to obtain the second moment of area of the composite figure about its own centroidal axis. This theorem states

$$I_z = I_{ca} + Ad^2$$

where I_{ca} is the second moment of area about its own centroidal axis and I_z is the second moment of area about any parallel axis a distance d removed. For the top rectangle, the distance is

$$d_1 = 32.99 - 6 = 26.99 \text{ mm}$$

and for the bottom rectangle,

$$d_2 = 67.01 - \frac{88}{2} = 23.01 \text{ mm}$$

Using the parallel-axis theorem for both rectangles, we now find that

$$\begin{aligned} I &= [1.080 \times 10^4 + 12(75)26.99^2] + [6.815 \times 10^5 + 12(88)23.01^2] \\ &= 1.907 \times 10^6 \text{ mm}^4 \end{aligned}$$

Example 3-5

Finally, the maximum tensile stress, which occurs at the top surface, is found to be

$$\sigma = \frac{Mc_1}{I} = \frac{1600(32.99)10^{-3}}{1.907(10^{-6})} = 27.68(10^6)\ \text{Pa} = 27.68\ \text{MPa}$$

Similarly, the maximum compressive stress at the lower surface is found to be

$$\sigma = -\frac{Mc_2}{I} = -\frac{1600(67.01)10^{-3}}{1.907(10^{-6})} = -56.22(10^6)\ \text{Pa} = -56.22\ \text{MPa}$$

Two-Plane Bending

- Consider bending in both *xy-plane* and *xz*-plane
- Cross sections with one or two planes of symmetry only

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- For solid circular cross section, the maximum bending stress is

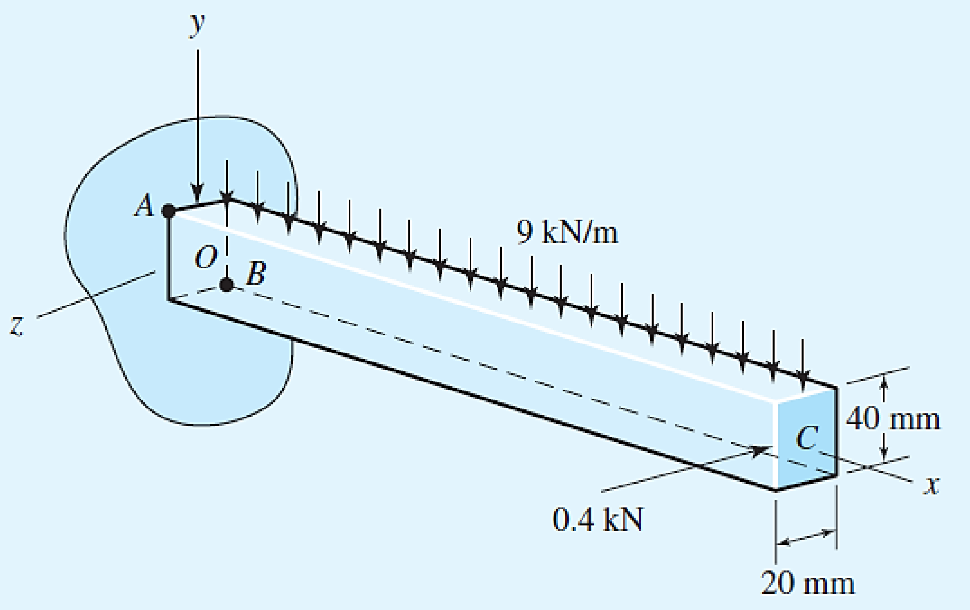
$$\sigma_m = \frac{Mc}{I} = \frac{(M_y^2 + M_z^2)^{1/2}(d/2)}{\pi d^4/64} = \frac{32}{\pi d^3}(M_y^2 + M_z^2)^{1/2}$$

Example 3-6

As shown in Fig. 3–16*a*, beam *OC* is loaded in the *xy* plane by a uniform load of 9 kN/m, and in the *xz* plane by a concentrated force of 0.4 kN at end *C*. The beam is 0.2 m long.

(*a*) For the cross section shown determine the maximum tensile and compressive bending stresses and where they act.

(*b*) If the cross section was a solid circular rod of diameter, $d = 30$ mm, determine the magnitude of the maximum bending stress.

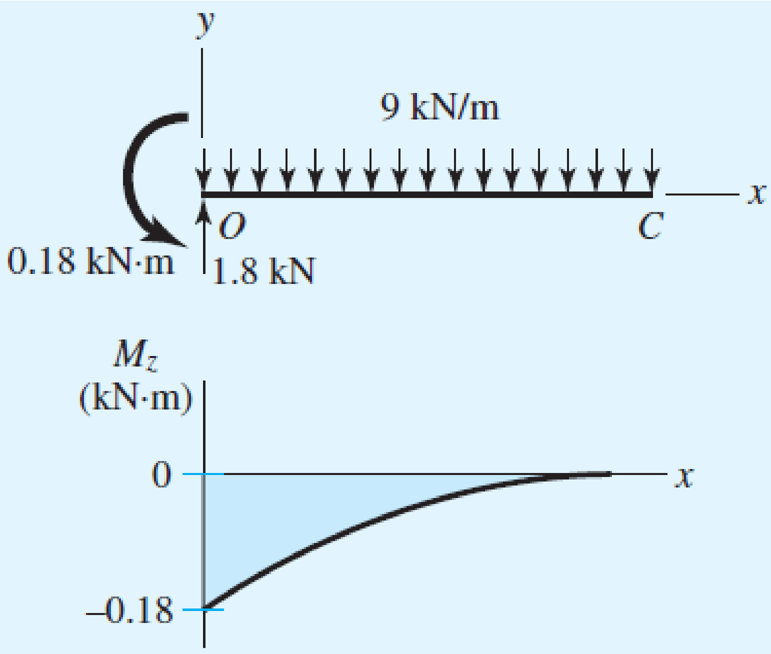


(*a*)

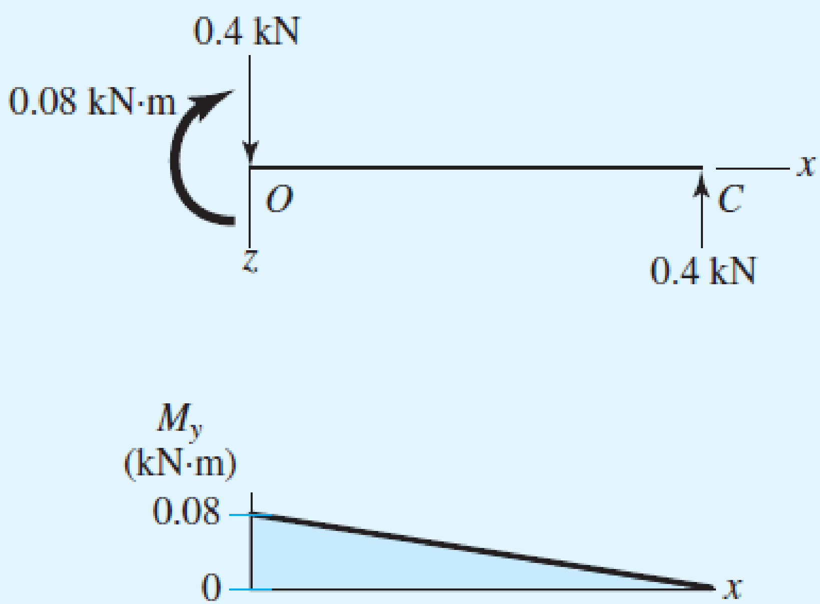
Example 3-6

(a) The reactions at O and the bending-moment diagrams in the xy and xz planes are shown in Figs. 3–16b and c, respectively. The maximum moments in both planes occur at O where

$$(M_z)_O = -\frac{1}{2}(9)0.2^2 = -0.18 \text{ kN}\cdot\text{m} \qquad (M_y)_O = 0.4(0.2) = 0.08 \text{ kN}\cdot\text{m}$$



(b)



(c)

Example 3-6

The second moments of area in both planes are

$$I_z = \frac{1}{12}(0.02)0.04^3 = 106.7 \times 10^{-9}\ \text{m}^4 \qquad I_y = \frac{1}{12}(0.04)0.02^3 = 26.7 \times 10^{-9}\ \text{m}^4$$

The maximum tensile stress occurs at point A, shown in Fig. 3–16a, where the maximum tensile stress is due to both moments. At A, $y_A = 0.02$ m and $z_A = 0.01$ m. Thus, from Eq. (3–27)

$$(\sigma_x)_A = -\frac{-0.18(0.02)}{106.7 \times 10^{-9}} + \frac{0.08(0.01)}{26.7 \times 10^{-9}} = 63\ 702\ \text{kPa} = 63.7\ \text{MPa}$$ **Answer**

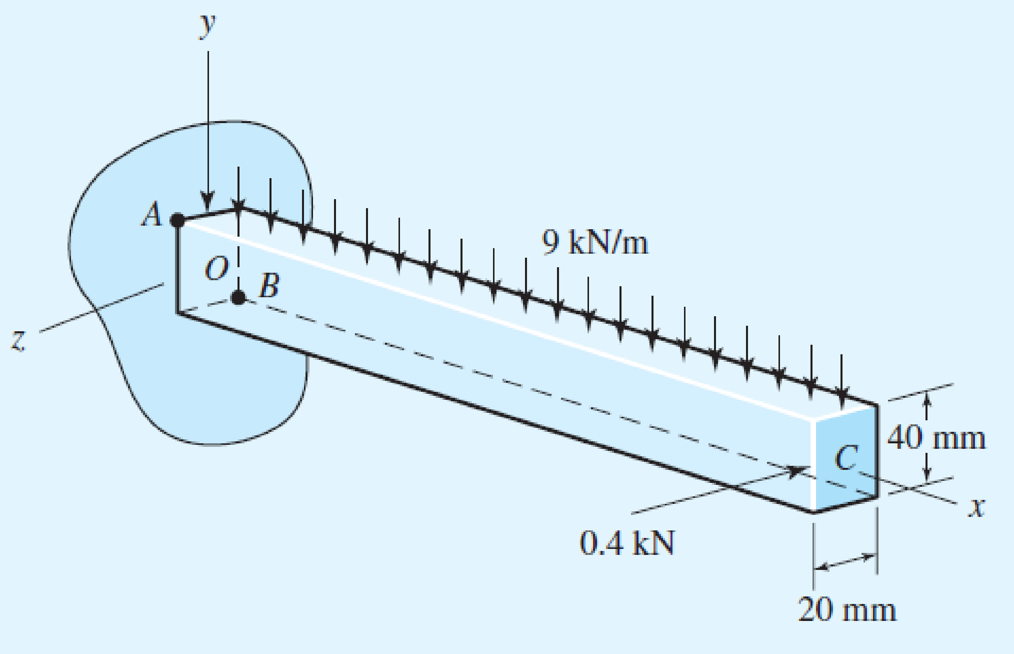
The maximum compressive bending stress occurs at point B where, $y_B = -0.02$ m and $z_B = -0.01$ m. Thus

$$(\sigma_x)_B = -\frac{-0.18(-0.02)}{106.7 \times 10^{-9}} + \frac{0.08(-0.01)}{26.7 \times 10^{-9}} = -\ 63\ 702\ \text{kPa} = -\ 63.7\ \text{MPa}$$ **Answer**

Example 3-6

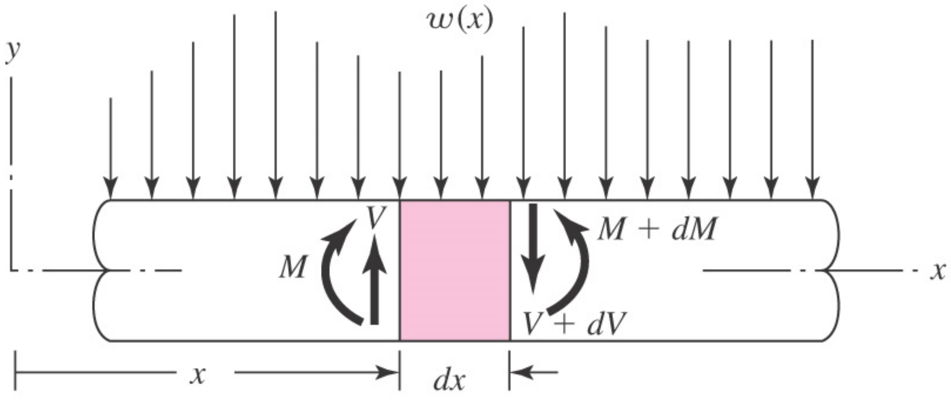
(*b*) For a solid circular cross section of diameter, $d = 30$ mm, the maximum bending stress at end O is given by Eq. (3–28) as

$$\sigma_m = \frac{32}{\pi(0.03)^3}\left[0.08^2 + (-0.18)^2\right]^{1/2} = 74310.8 \text{ kPa} = 74.31 \text{ MPa}$$

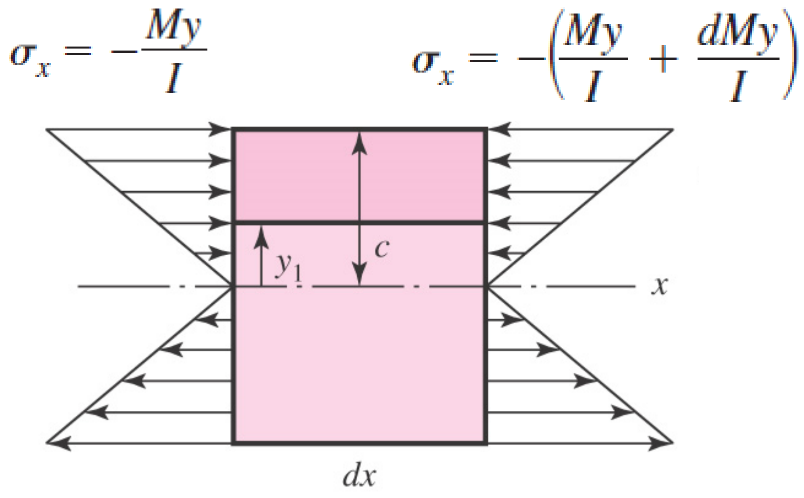


(*a*)

Shear Stress for Beams in Bending



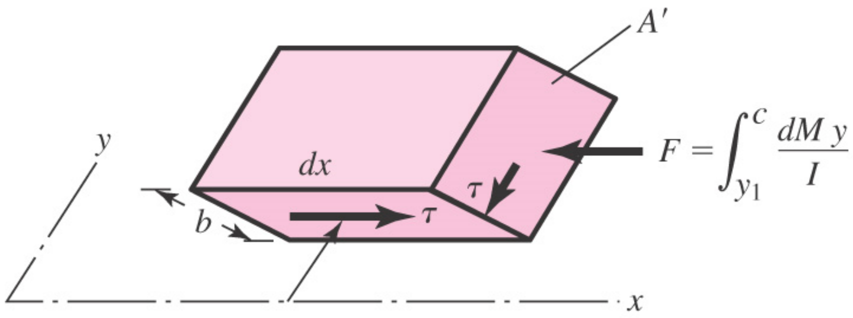
(a)



(b)

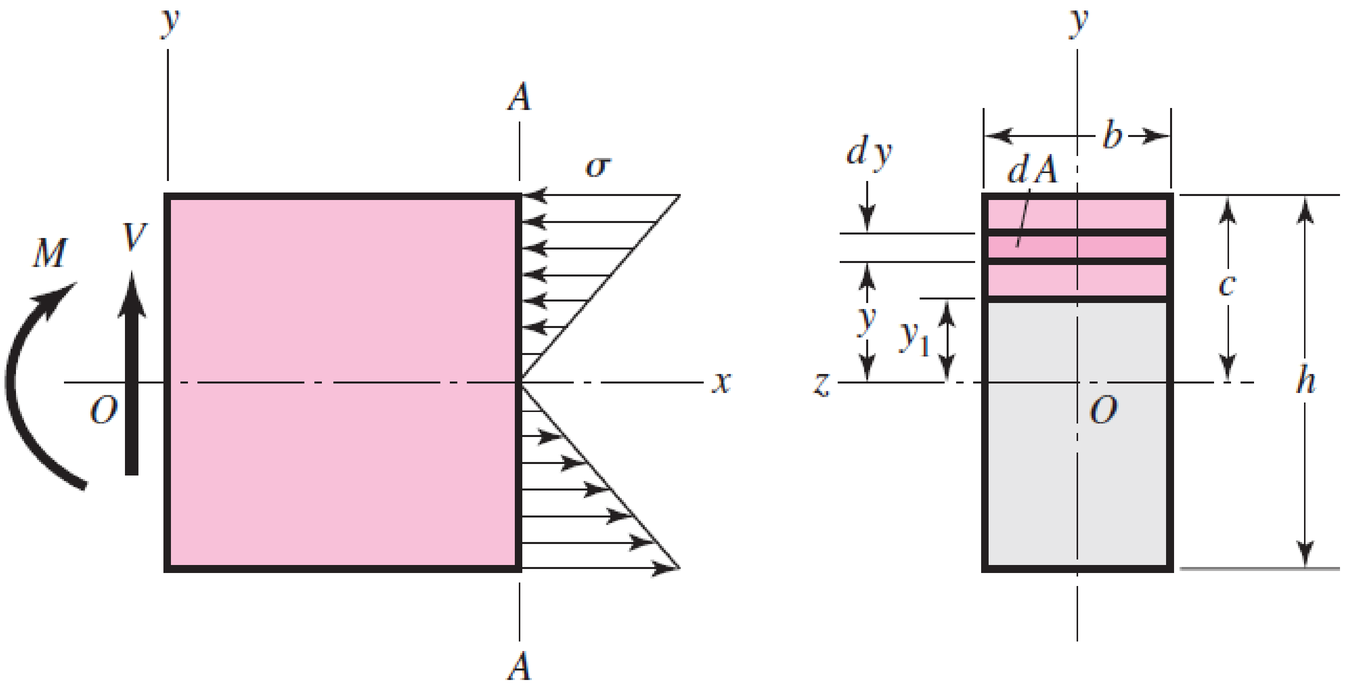
$$\tau b\, dx = \int_{y_1}^{c} \frac{(dM)y}{I} dA$$

$$\tau = \frac{V}{Ib} \int_{y_1}^{c} y dA$$



(c)

Shear Stress for Beams in Bending

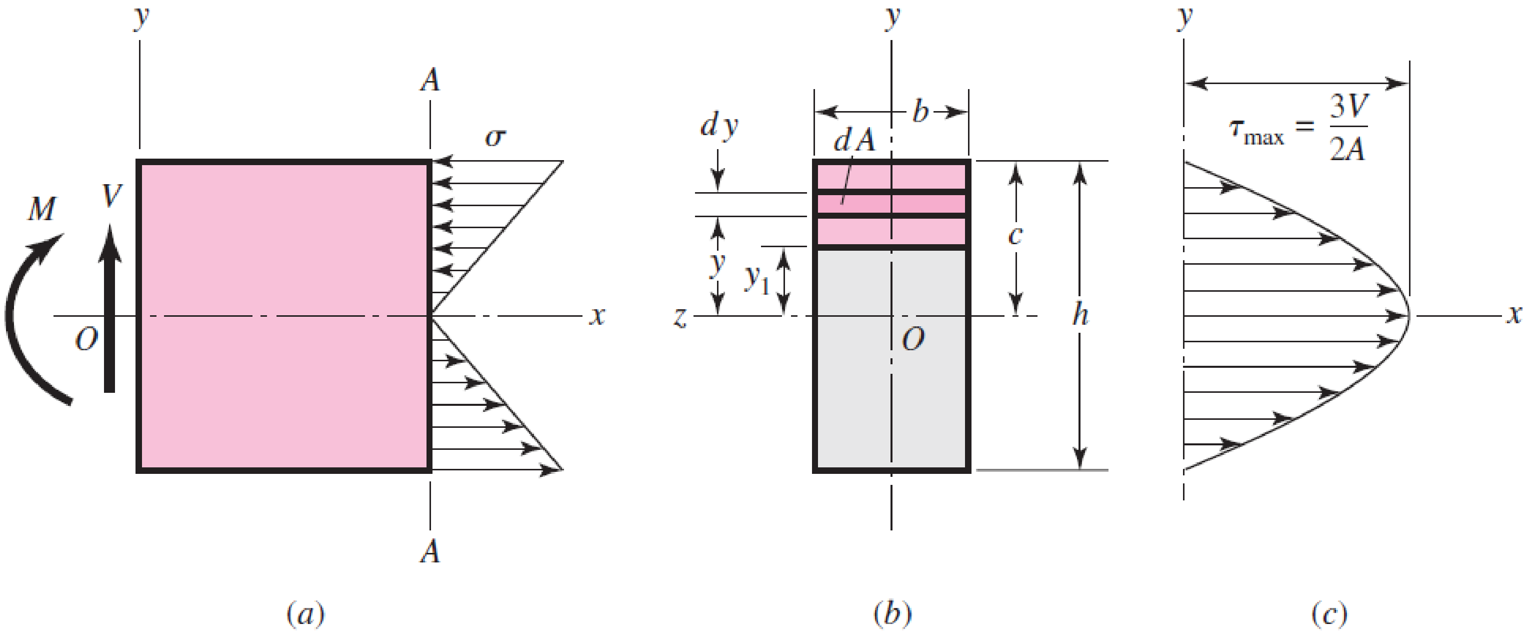


$$Q = \int_{y_1}^{c} y\,dA = \bar{y}' A'$$

$$\tau = \frac{VQ}{Ib}$$

- Transverse shear stress is always accompanied with bending stress.

Transverse Shear Stress in a Rectangular Beam

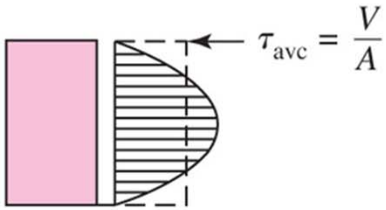
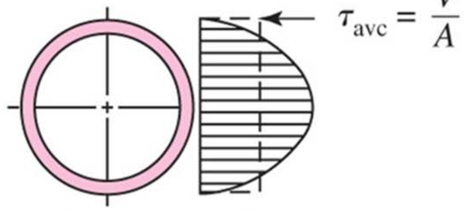
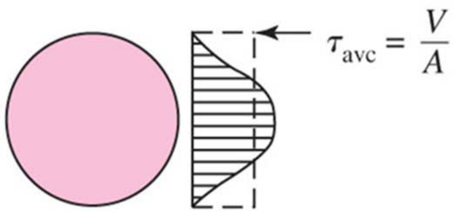
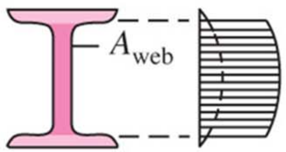


$$Q = \int_{y_1}^{c} y\,dA = b\int_{y_1}^{c} y\,dy = \frac{by^2}{2}\bigg|_{y_1}^{c} = \frac{b}{2}\left(c^2 - y_1^2\right)$$

$$\tau = \frac{VQ}{Ib} = \frac{V}{2I}\left(c^2 - y_1^2\right) \qquad I = \frac{Ac^2}{3}$$

$$\tau = \frac{3V}{2A}\left(1 - \frac{y_1^2}{c^2}\right)$$

Maximum Values of Transverse Shear Stress

Beam Shape	Formula	Beam Shape	Formula
 $\tau_{avc} = \frac{V}{A}$ Rectangular	$\tau_{max} = \frac{3V}{2A}$	 $\tau_{avc} = \frac{V}{A}$ Hollow, thin-walled round	$\tau_{max} = \frac{2V}{A}$
 $\tau_{avc} = \frac{V}{A}$ Circular	$\tau_{max} = \frac{4V}{3A}$	 A_{web} Structural I beam (thin-walled)	$\tau_{max} \doteq \frac{V}{A_{web}}$

Significance of Transverse Shear Compared to Bending

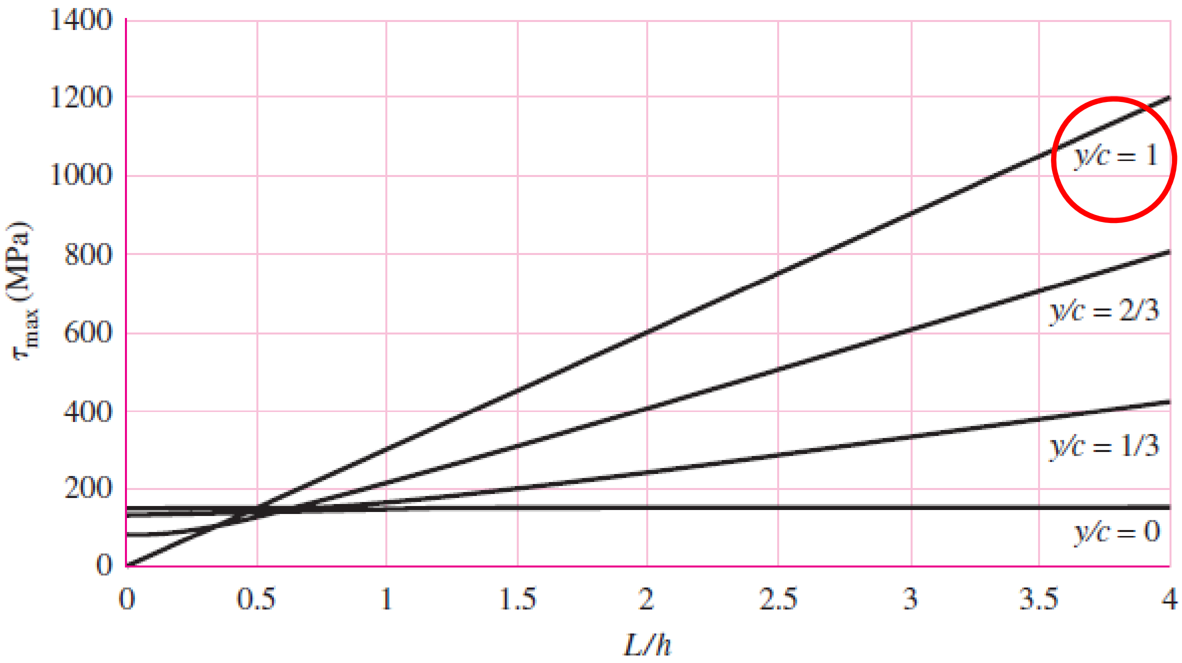
Example: Cantilever beam, rectangular cross section

- Maximum shear stress, including *bending stress* (*My/I*) and *transverse shear stress* (*VQ/Ib*),

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{3F}{2bh}\sqrt{4(L/h)^2(y/c)^2 + [1 - (y/c)^2]^2}$$

Figure 3–19

Plot of maximum shear stress for a cantilever beam, combining the effects of bending and transverse shear stresses.

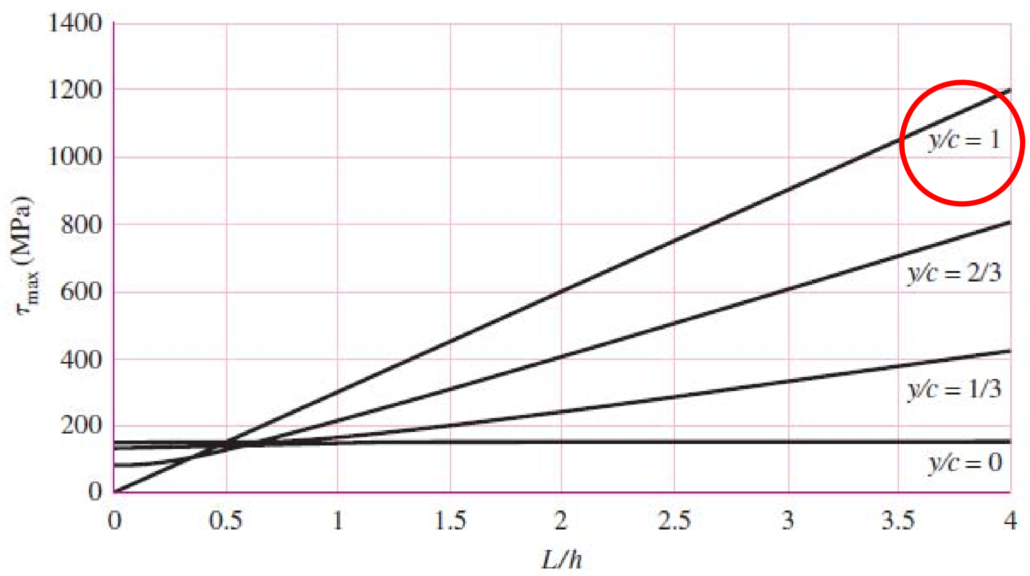


Significance of Transverse Shear Compared to Bending

- Critical stress element (largest τ_{max}) will always be either
  - Due to bending, on the outer surface ($y/c=1$), where the transverse shear is zero
  - Or due to transverse shear at the neutral axis ($y/c=0$), where the bending is zero
- Transition happens at some critical value of L/h
- Valid for any cross section that does not increase in width farther away from the neutral axis.
  - Includes round and rectangular solids, but not I beams and channels

Figure 3–19

Plot of maximum shear stress for a cantilever beam, combining the effects of bending and transverse shear stresses.



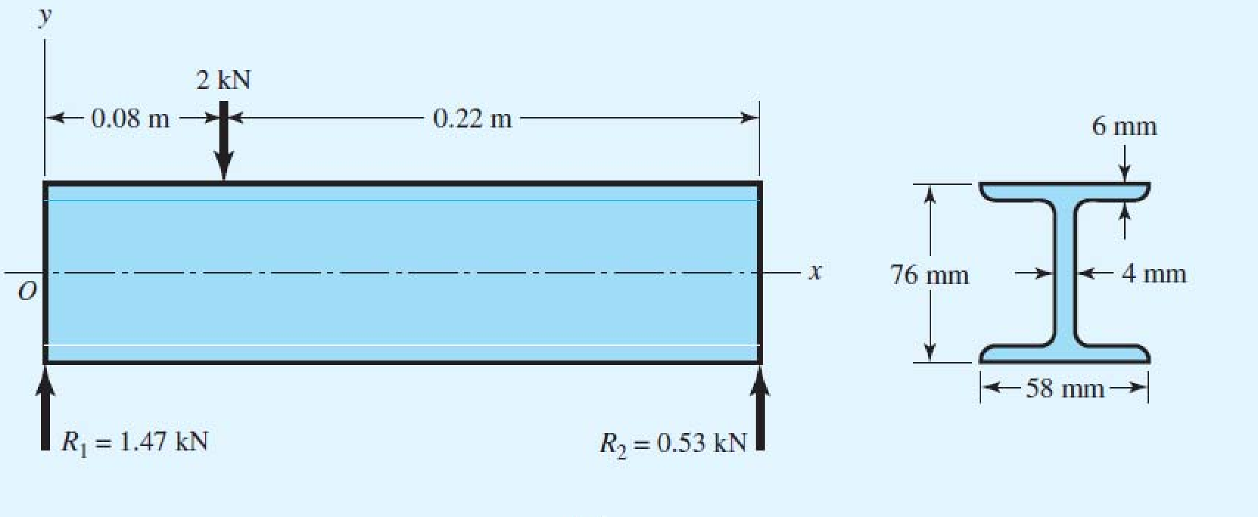
Example 3-7

A beam 0.3 m long is to support a load of 2 kN acting 80 mm from the left support, as shown in Fig. 3–20*a*. The beam is an I beam with the cross-sectional dimensions shown. To simplify the calculations, assume a cross section with square corners, as shown in Fig. 3–20*c*. Points of interest are labeled (*a*, *b*, *c*, and *d*) at distances *y* from the neutral axis of 0 mm, 32^- mm, 32^+ mm, and 38 mm (Fig. 3–20*c*). At the critical axial location along the beam, find the following information.

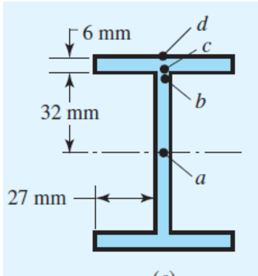
(*a*) Determine the profile of the distribution of the transverse shear stress, obtaining values at each of the points of interest.

(*b*) Determine the bending stresses at the points of interest.

(*c*) Determine the maximum shear stresses at the points of interest, and compare them.



(*a*)



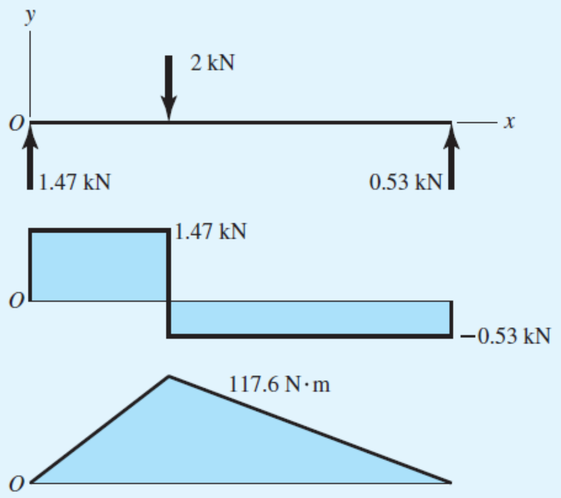
(*c*)

Example 3-7

First, we note that the transverse shear stress is not likely to be negligible in this case since the beam length to height ratio is much less than 10, and since the thin web and wide flange will allow the transverse shear to be large. The loading, shear-force, and bending-moment diagrams are shown in Fig. 3–20*b*. The critical axial location is at $x = 0.08$ m where the shear force and the bending moment are both maximum.

(*a*) We obtain the area moment of inertia I by evaluating I for a solid 76-mm × 58-mm rectangular area, and then subtracting the two rectangular areas that are not part of the cross section.

$$I = \frac{(58)(76)^3}{12} - 2\left[\frac{(27)(64)^3}{12}\right] = 942\,069 \text{ mm}^4$$



(*b*)

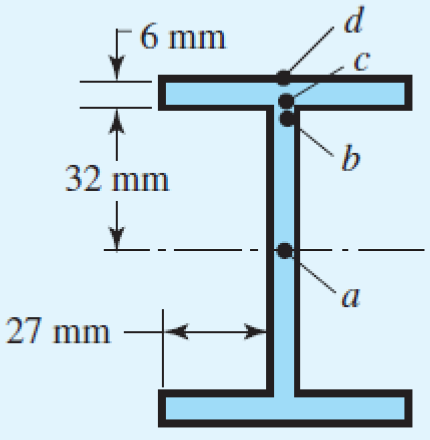
Example 3-7

Finding Q at each point of interest using Eq. (3–30) gives

$$Q_a = \left(32 + \frac{6}{2}\right)[(58)(6)] + \left(\frac{32}{2}\right)[(32)(4)] = 14\,228 \text{ mm}^3$$

$$Q_b = Q_c = \left(32 + \frac{6}{2}\right)[(58)(6)] = 12\,180 \text{ mm}^3$$

$$Q_d = (38)(0) = 0 \text{ mm}^3$$



(c)

Example 3-7

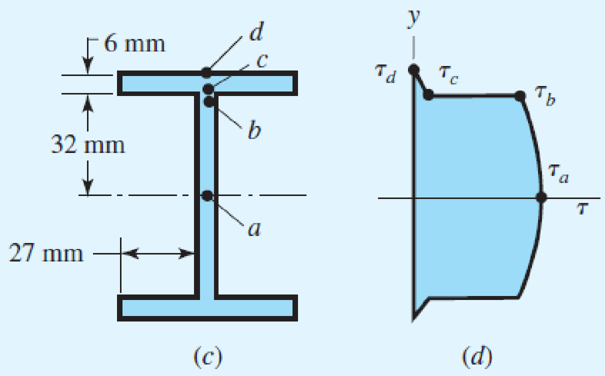
Applying Eq. (3–31) at each point of interest, with V and I constant for each point, and b equal to the width of the cross section at each point, shows that the magnitudes of the transverse shear stresses are

$$\tau_a = \frac{VQ_a}{Ib_a} = \frac{(1470)\,(14\,228 \times 10^{-9})}{(942\,069 \times 10^{-12})\,(0.004)} = 5.55 \text{ MPa}$$

$$\tau_b = \frac{VQ_b}{Ib_b} = \frac{(1470)\,(12\,180 \times 10^{-9})}{(942\,069 \times 10^{-12})\,(0.004)} = 4.75 \text{ MPa}$$

$$\tau_c = \frac{VQ_c}{Ib_c} = \frac{(1470)\,(12\,180 \times 10^{-9})}{(942\,069 \times 10^{-12})\,(0.058)} = 0.33 \text{ MPa}$$

$$\tau_d = \frac{VQ_d}{Ib_d} = \frac{(1470)\,(0)}{(942\,069 \times 10^{-12})\,(0.058)} = 0 \text{ MPa}$$



The magnitude of the idealized transverse shear stress profile through the beam depth will be as shown in Fig. 3–20*d*.

Example 3-7

(*b*) The bending stresses at each point of interest are

$$\sigma_a = \frac{My_a}{I} = \frac{(117.6)\,(0)}{942\,069 \times 10^{-12}} = 0 \text{ MPa}$$

$$\sigma_b = \sigma_c = -\frac{My_b}{I} = -\frac{(117.6)\,(0.032)}{942\,069 \times 10^{-12}} = -3.99 \text{ MPa}$$

$$\sigma_d = -\frac{My_d}{I} = -\frac{(117.6)\,(0.038)}{942\,069 \times 10^{-12}} = -4.74 \text{ MPa}$$

Example 3-7

(*c*) Now at each point of interest, consider a stress element that includes the bending stress and the transverse shear stress. The maximum shear stress for each stress element can be determined by Mohr's circle, or analytically by Eq. (3–14) with $\sigma_y = 0$,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Thus, at each point

$$\tau_{\max,a} = \sqrt{0 + (5.55)^2} = 5.55 \text{ MPa}$$

$$\tau_{\max,b} = \sqrt{\left(\frac{-3.99}{2}\right)^2 + (4.75)^2} = 5.15 \text{ MPa}$$

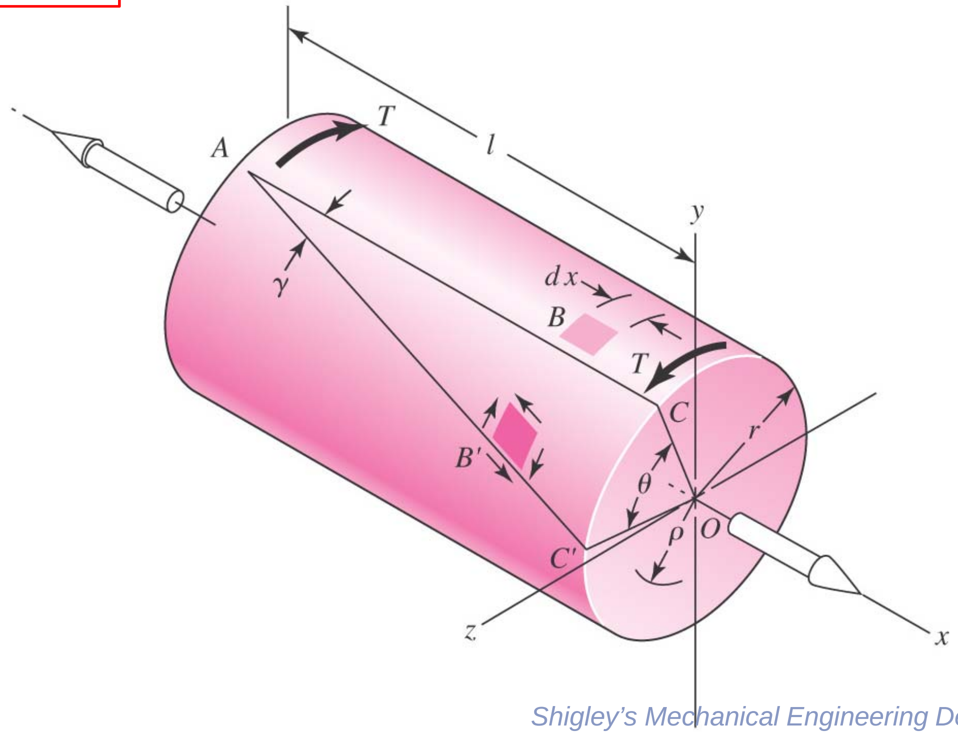
$$\tau_{\max,c} = \sqrt{\left(\frac{-3.99}{2}\right)^2 + (0.33)^2} = 2.02 \text{ MPa}$$

$$\tau_{\max,d} = \sqrt{\left(\frac{-4.74}{2}\right)^2 + 0} = 2.37 \text{ MPa}$$

Torsion

- *Torque vector* – a moment vector collinear with axis of a mechanical element
- A bar subjected to a torque vector is said to be in *torsion*
- *Angle of twist*, in radians, for a solid round bar

$$\theta = \frac{Tl}{GJ}$$



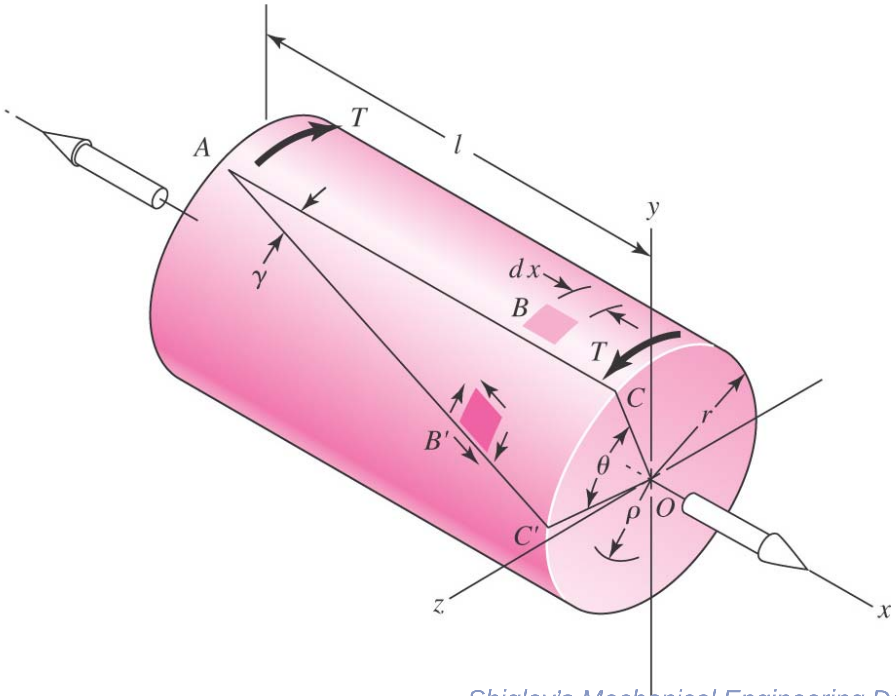
Torsion

- For round bar in torsion, torsional shear stress is proportional to the radius r

$$\tau = \frac{T\rho}{J}$$

- Maximum torsional shear stress is at the outer surface

$$\tau_{\max} = \frac{Tr}{J}$$



Assumption for Torsion Equations

❑ Torsional Equations are only applicable for the following conditions

- Pure torque
- Remote from any discontinuities or point of application of torque
- Material obeys Hooke's law
- Adjacent cross sections originally plane and parallel remain plane and parallel
- Radial lines remain straight
  - Depends on axisymmetry, so does not hold true for noncircular cross sections

❑ Consequently, only applicable for round cross sections

Torsional Shear in Rectangular Section

- Shear stress does not vary linearly with radial distance for rectangular cross section
- Shear stress is zero at the corners
- Maximum shear stress is at the middle of the longest side
- For rectangular b x c bar, where b is longest side

$$\tau_{\max} = \frac{T}{\alpha b c^2} \approx \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) \qquad (3\text{–}40)$$

$$\theta = \frac{Tl}{\beta b c^3 G} \qquad (3\text{–}41)$$

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Power, Speed, and Torque

- Power equals torque times speed

$$H = T\omega \tag{3–43}$$

where H = power, W

T = torque, N · m

ω = angular velocity, rad/s

- A convenient conversion with speed in rpm

$$T = 9.55\frac{H}{n} \tag{3–44}$$

where H = power, W

n = angular velocity, revolutions per minute

Power, Speed, and Torque

- In U.S. Customary units, with unit conversion built in

$$H = \frac{FV}{33\,000} = \frac{2\pi Tn}{33\,000(12)} = \frac{Tn}{63\,025} \qquad (3\text{–}42)$$

where
H = power, hp
T = torque, lbf · in
n = shaft speed, rev/min
F = force, lbf
V = velocity, ft/min

Example 3-8

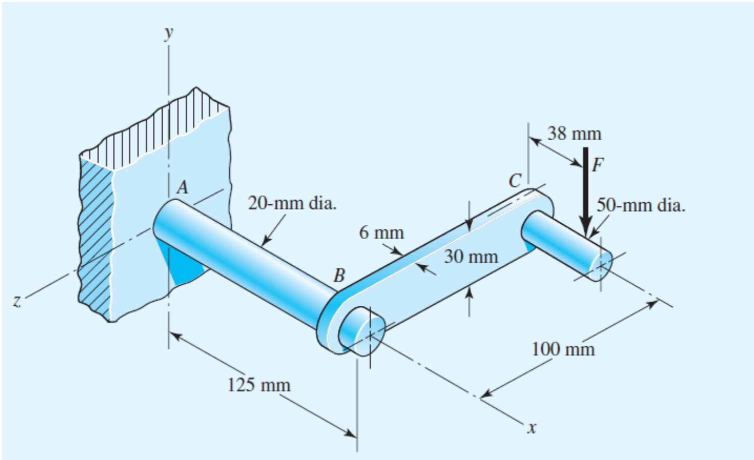
Figure 3–22 shows a crank loaded by a force $F = 1.3$ kN that causes twisting and bending of a 20-mm-diameter shaft fixed to a support at the origin of the reference system. In actuality, the support may be an inertia that we wish to rotate, but for the purposes of a stress analysis we can consider this a statics problem.

(*a*) Draw separate free-body diagrams of the shaft *AB* and the arm *BC*, and compute the values of all forces, moments, and torques that act. Label the directions of the coordinate axes on these diagrams.

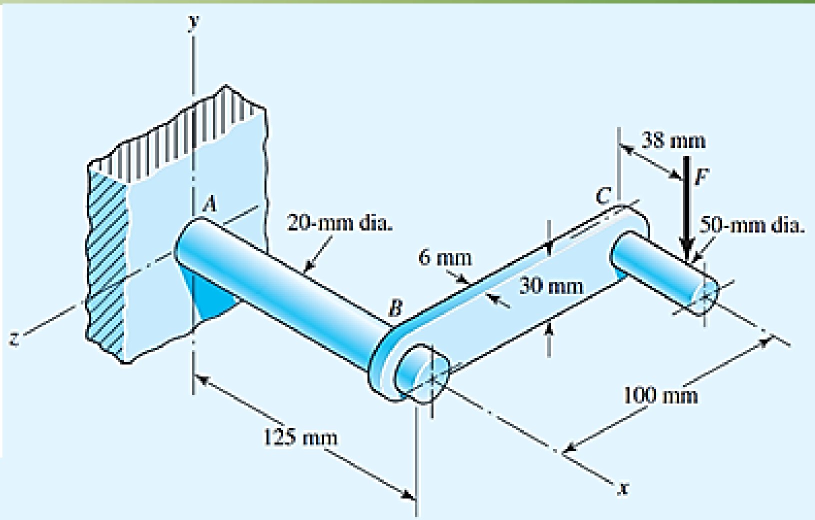
(*b*) Compute the maxima of the torsional stress and the bending stress in the arm *BC* and indicate where these act.

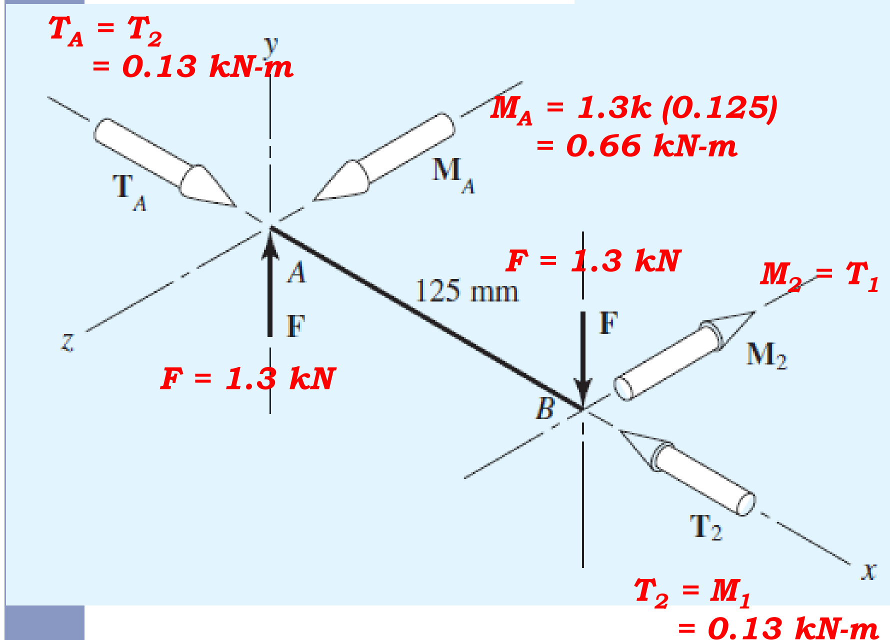
(*c*) Locate a stress element on the top surface of the shaft at *A*, and calculate all the stress components that act upon this element.

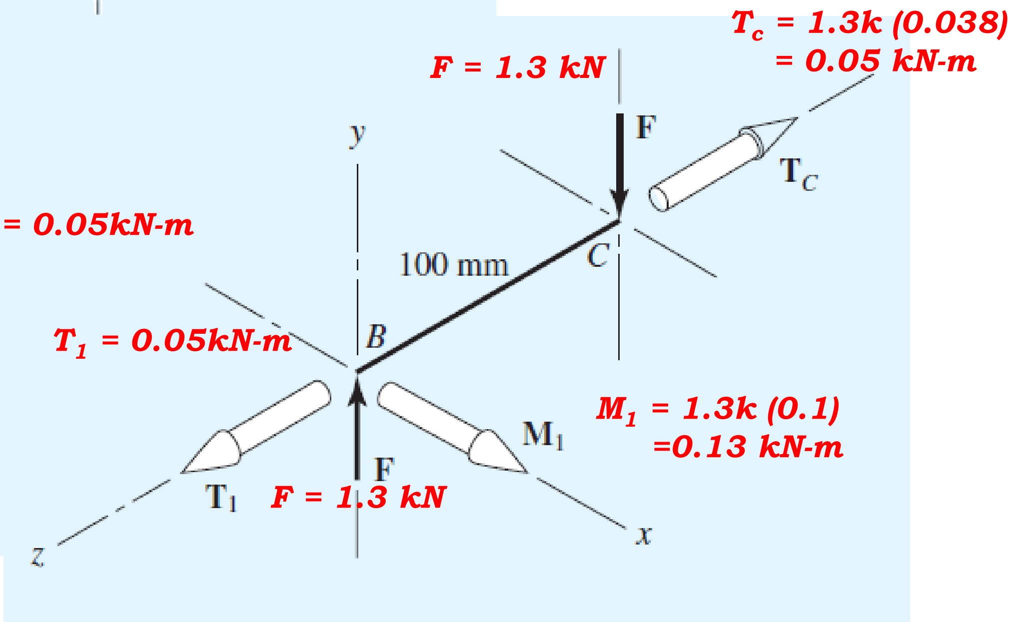
(*d*) Determine the maximum normal and shear stresses at *A*.



Example 3-8


y
A
20-mm dia.
6 mm
30 mm
B
C
38 mm
F
50-mm dia.
z
125 mm
100 mm
x

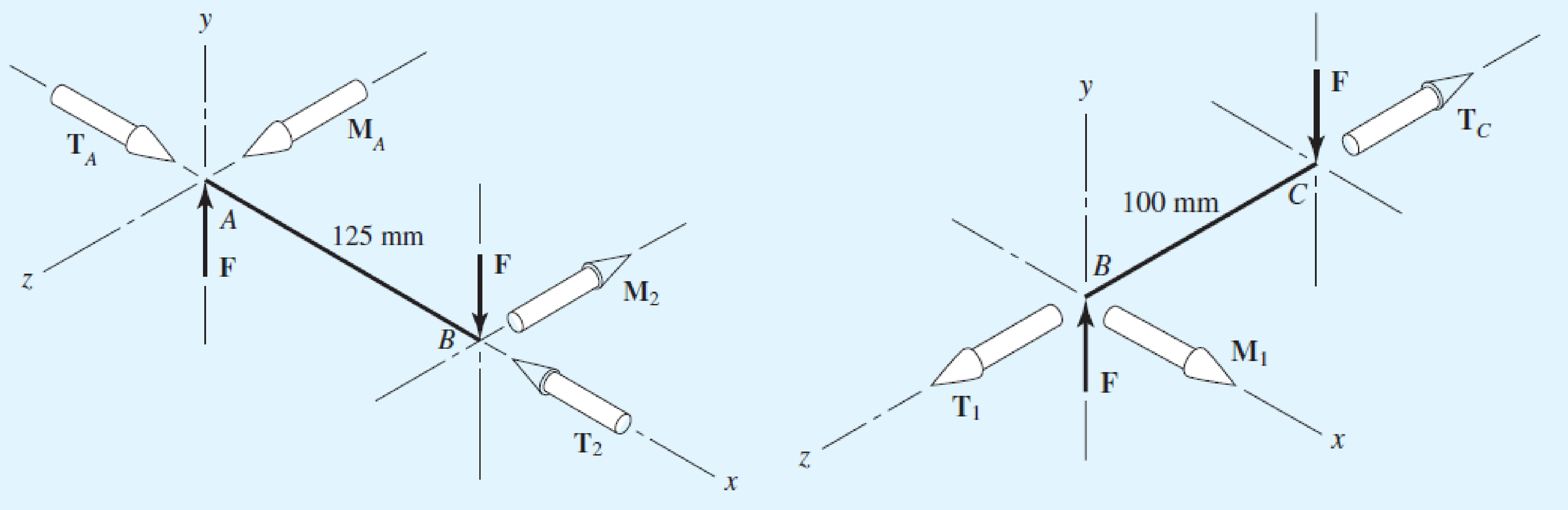

T_A = T_2
= 0.13 kN-m
M_A = 1.3k (0.125)
= 0.66 kN-m
y
T_A
M_A
A
125 mm
F
z
F = 1.3 kN
F = 1.3 kN
F
M_2 = T_1 = 0.05kN-m
M_2
B
T_2
x
T_2 = M_1
= 0.13 kN-m


T_c = 1.3k (0.038)
= 0.05 kN-m
F = 1.3 kN
F
T_C
y
100 mm
C
T_1 = 0.05kN-m
B
M_1 = 1.3k (0.1)
=0.13 kN-m
M_1
T_1
F
F = 1.3 kN
z
x

Example 3-8

(*a*) The two free-body diagrams are shown in Fig. 3–23. The results are

At end *C* of arm *BC*:	$\mathbf{F} = -1.3\mathbf{j}$ kN, $\mathbf{T}_C = -0.05\mathbf{k}$ kN · m
At end *B* of arm *BC*:	$\mathbf{F} = 1.3\mathbf{j}$ kN, $\mathbf{M}_1 = 0.13\mathbf{i}$ kN · m, $\mathbf{T}_1 = 0.05\mathbf{k}$ kN · m
At end *B* of shaft *AB*:	$\mathbf{F} = -1.3\mathbf{j}$ kN, $\mathbf{T}_2 = -0.13\mathbf{i}$ kN · m, $\mathbf{M}_2 = -0.05\mathbf{k}$ kN · m
At end *A* of shaft *AB*:	$\mathbf{F} = 1.3\mathbf{j}$ kN, $\mathbf{M}_A = 0.66\mathbf{k}$ kN · m, $\mathbf{T}_A = 0.13\mathbf{i}$ kN · m

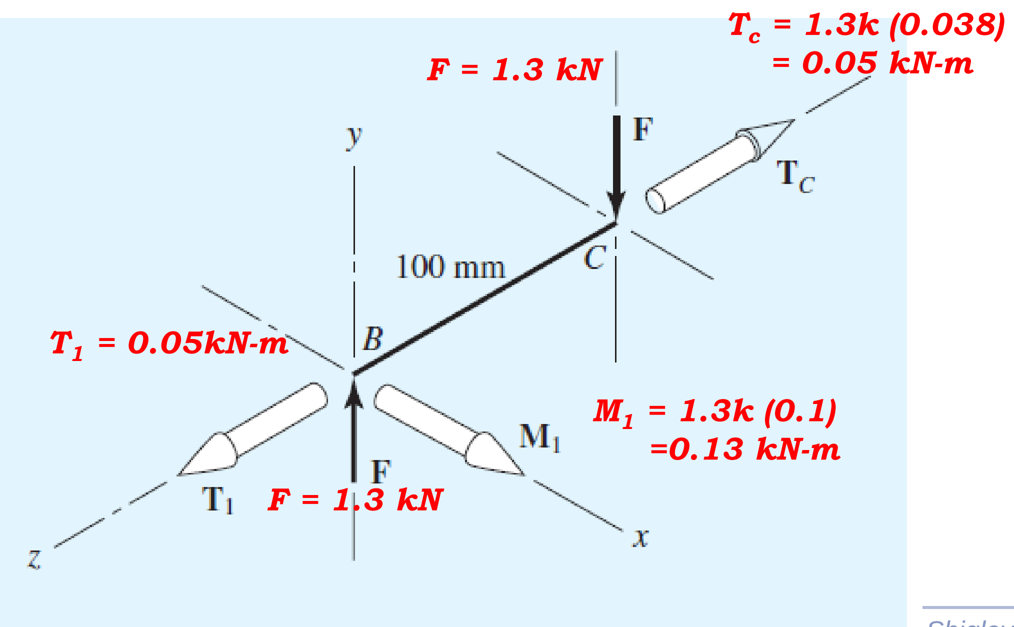


Example 3-8

(*b*) For arm *BC*, the bending moment will reach a maximum near the shaft at *B*. If we assume this is 0.13 kN · m, then the bending stress for a rectangular section will be

$$\sigma = \frac{M}{I/c} = \frac{6M}{bh^2} = \frac{6(130)}{0.006(0.03)^2} = 144.4 \text{ MPa}$$

Of course, this is not exactly correct, because at *B* the moment is actually being transferred into the shaft, probably through a weldment.

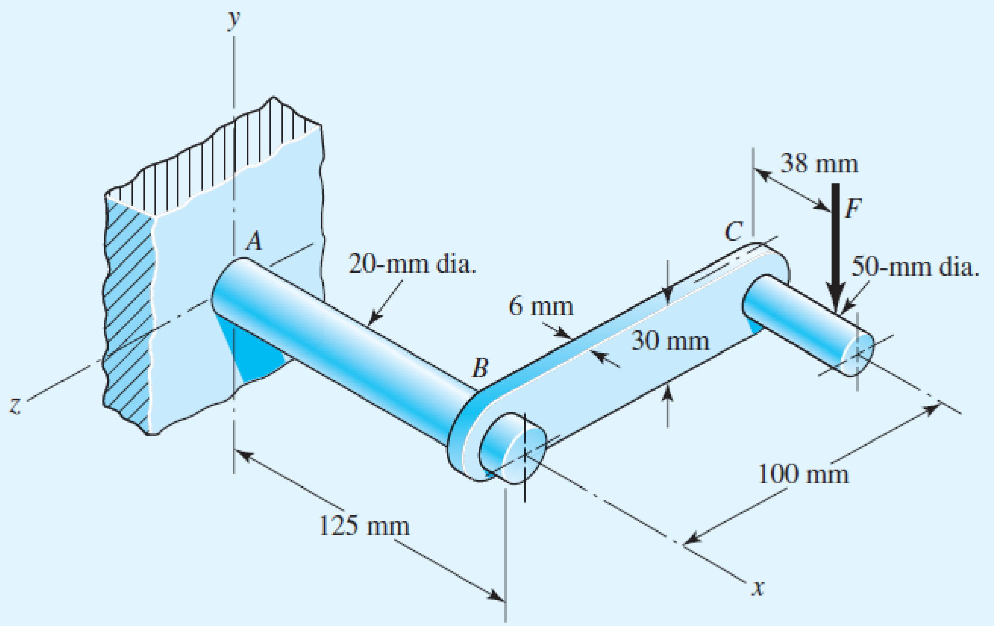


Example 3-8

For the torsional stress, use Eq. (3–43). Thus

$$\tau_{\max} = \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) = \frac{50}{0.03(0.006)^2}\left(3 + \frac{1.8}{0.03/0.006}\right) = 155.6 \text{ MPa}$$

This stress occurs at the middle of the 30-mm side.



Example 3-8

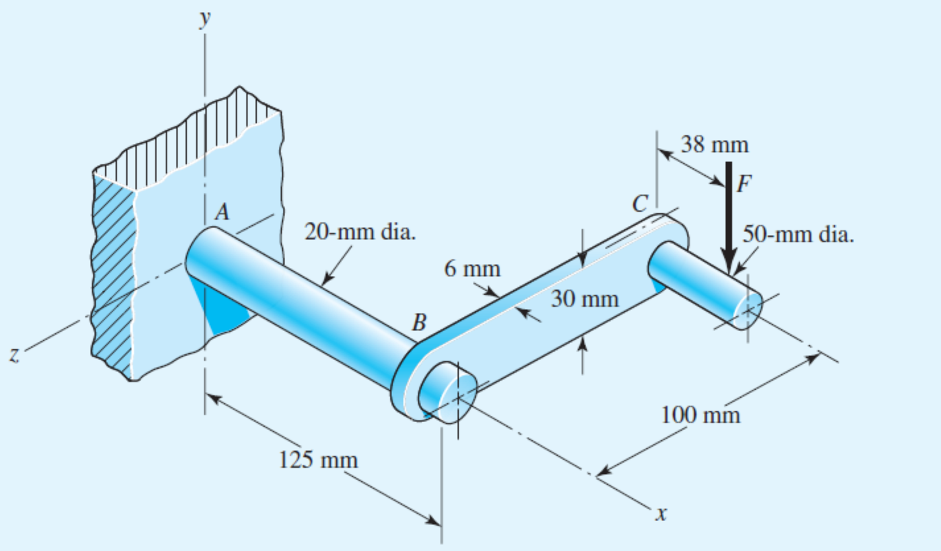
(*c*) For a stress element at *A*, the bending stress is tensile and is

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(660)}{\pi(0.02)^3} = 840.3 \text{ MPa}$$

The torsional stress is

$$\tau_{xz} = \frac{-T}{J/c} = \frac{-16T}{\pi d^3} = \frac{-16(130)}{\pi(0.02)^3} = -82.8 \text{ MPa}$$

where the reader should verify that the negative sign accounts for the direction of τ_{xz}.



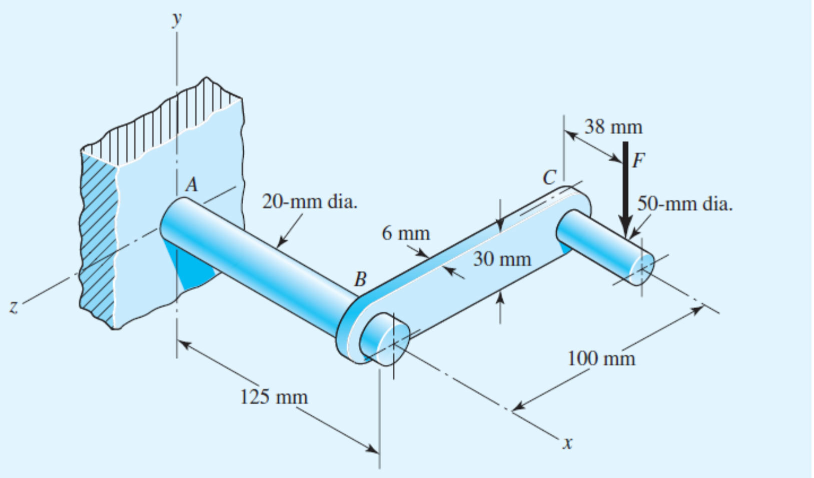
Example 3-8

(*d*) Point *A* is in a state of plane stress where the stresses are in the *xz* plane. Thus the principal stresses are given by Eq. (3–13) with subscripts corresponding to the *x*, *z* axes.

The maximum normal stress is then given by

$$\sigma_1 = \frac{\sigma_x + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$$

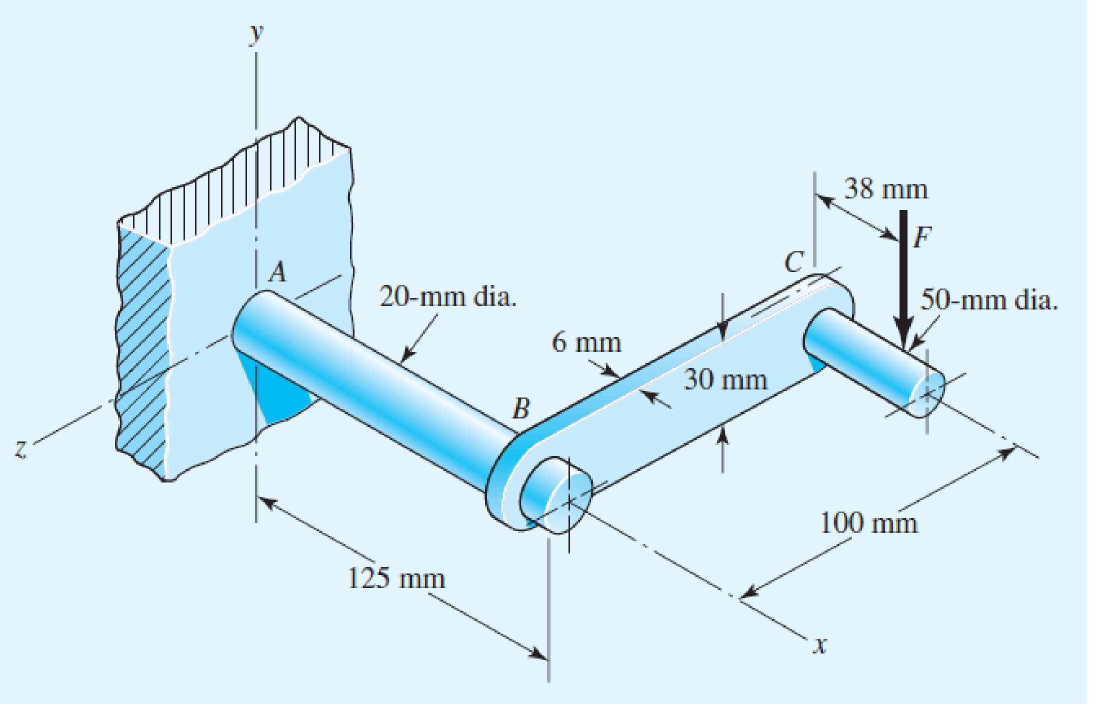
$$= \frac{840.3 + 0}{2} + \sqrt{\left(\frac{840.3 - 0}{2}\right)^2 + (-82.8)^2} = 848.4 \text{ MPa}$$



Example 3-8

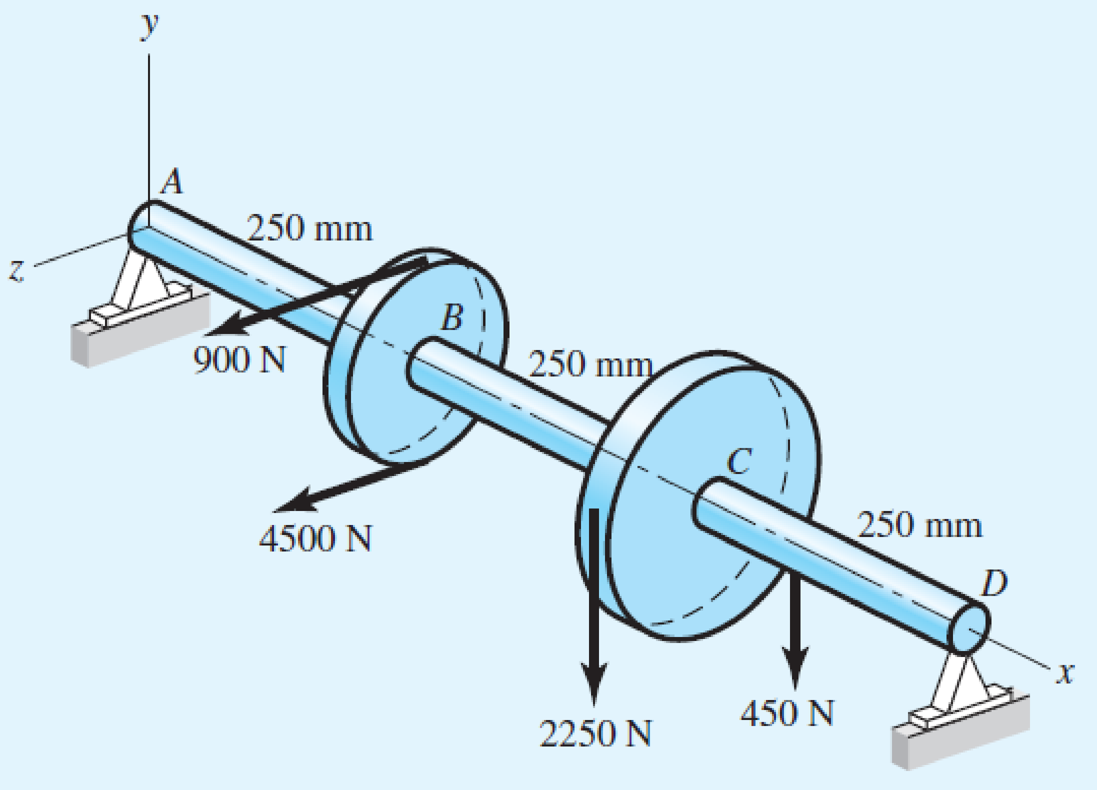
The maximum shear stress at A occurs on surfaces different than the surfaces containing the principal stresses or the surfaces containing the bending and torsional shear stresses. The maximum shear stress is given by Eq. (3–14), again with modified subscripts, and is given by

$$\tau_1 = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \sqrt{\left(\frac{840.3 - 0}{2}\right)^2 + (-82.8)^2} = 428.2 \text{ MPa}$$



Example 3-9

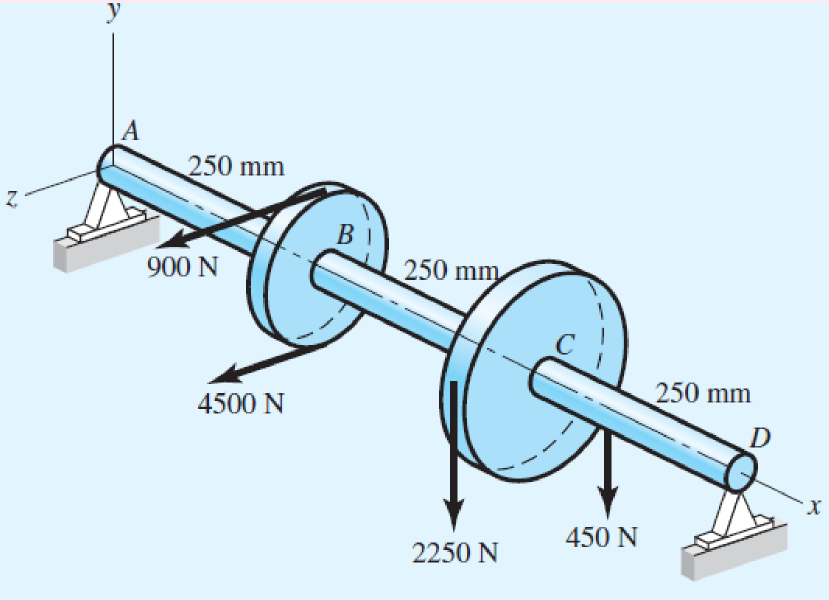
The 40-mm-diameter solid steel shaft shown in Fig. 3–24*a* is simply supported at the ends. Two pulleys are keyed to the shaft where pulley *B* is of diameter 100 mm and pulley *C* is of diameter 200 mm. Considering bending and torsional stresses only, determine the locations and magnitudes of the greatest tensile, compressive, and shear stresses in the shaft.



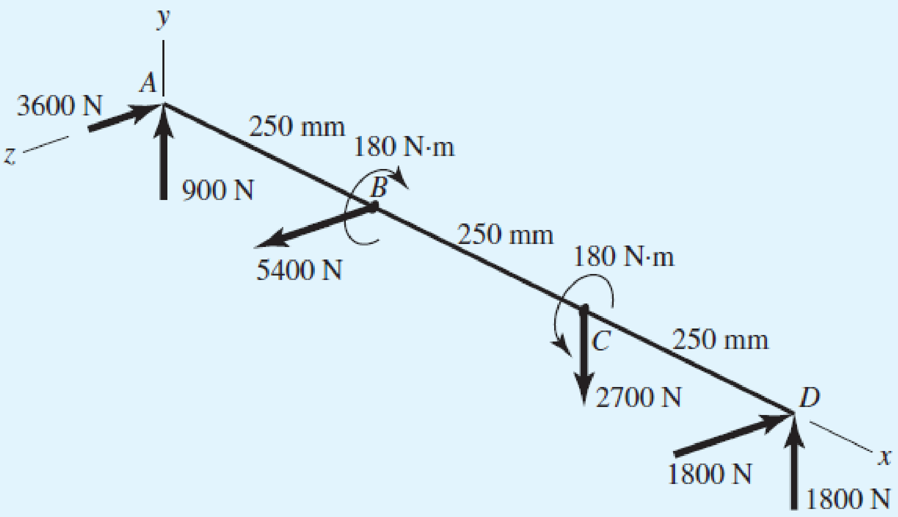
(*a*)

Example 3-9

Figure 3–24*b* shows the net forces, reactions, and torsional moments on the shaft. Although this is a three-dimensional problem and vectors might seem appropriate, we will look at the components of the moment vector by performing a two-plane analysis.

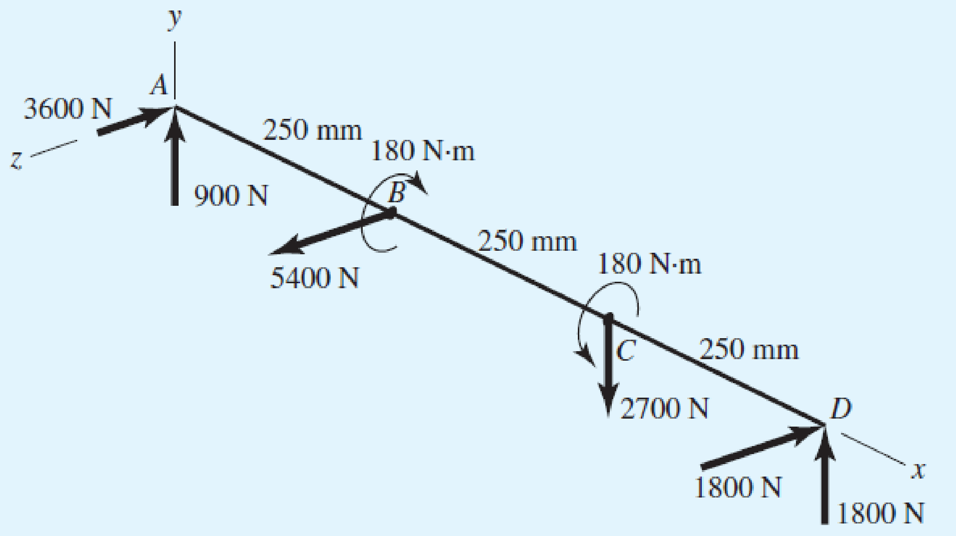


(*a*)

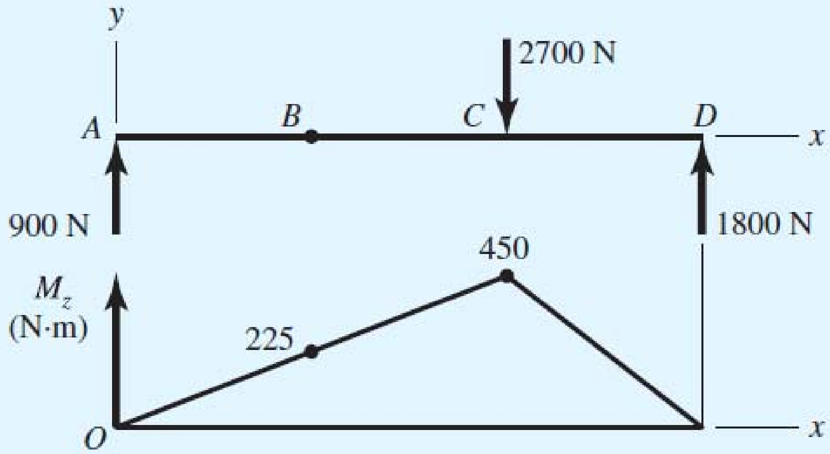


(*b*)

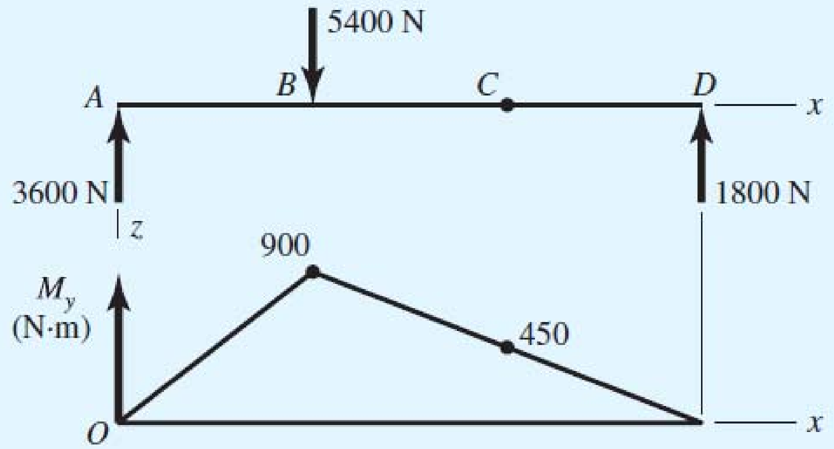
Example 3-9



(b)



(c)



(d)

Example 3-9

The net moment on a section is the vector sum of the components. That is,

$$M = \sqrt{M_y^2 + M_z^2} \quad (1)$$

At point B,

$$M_B = \sqrt{225^2 + 900^2} = 928\ \text{N}\cdot\text{m}$$

At point C,

$$M_C = \sqrt{450^2 + 450^2} = 636\ \text{N}\cdot\text{m}$$

Thus the maximum bending moment is 928 N·m and the maximum bending stress at pulley B is

$$\sigma = \frac{M\,d/2}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{32(928)}{\pi(0.04^3)} = 147.7\ \text{MPa}$$

The maximum torsional shear stress occurs between B and C and is

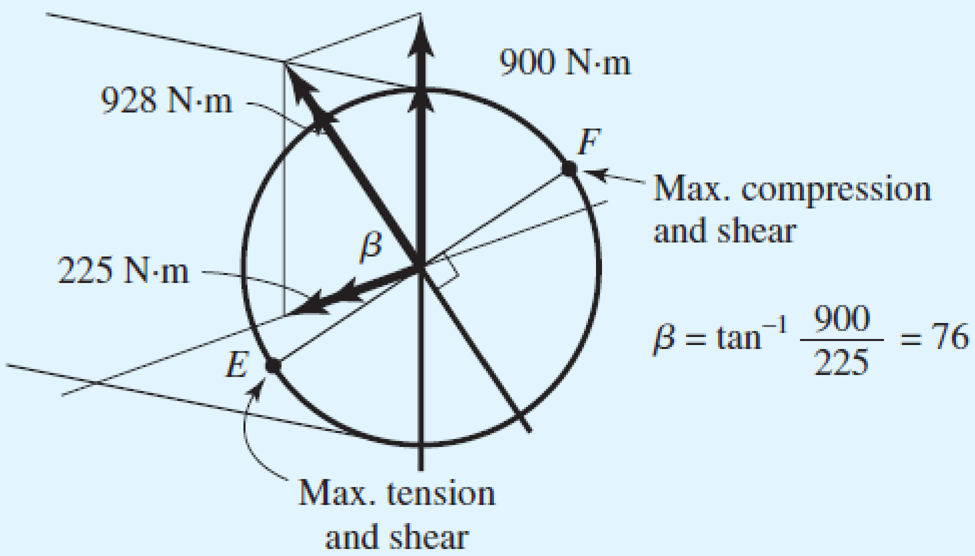
$$\tau = \frac{T\,d/2}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{16(180)}{\pi(0.04^3)} = 14.3\ \text{MPa}$$

Example 3-9

The maximum bending and torsional shear stresses occur just to the right of pulley *B* at points *E* and *F* as shown in Fig. 3–24*e*. At point *E*, the maximum tensile stress will be σ_1 given by

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{147.7}{2} + \sqrt{\left(\frac{147.7}{2}\right)^2 + 14.3^2} = 149.1 \text{ MPa}$$

Location: at B ($x = 250$ mm$^+$)



(*e*)

Example 3-9

At point F, the maximum compressive stress will be σ_2 given by

$$\sigma_2 = \frac{\sigma}{2} - \sqrt{\left(\frac{-\sigma}{2}\right)^2 + \tau^2} = \frac{147.7}{2} - \sqrt{\left(\frac{-147.7}{2}\right)^2 + 14.3^2} = -1.4 \text{ MPa}$$

The extreme shear stress also occurs at E and F and is

$$\tau_1 = \sqrt{\left(\frac{\pm\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{\pm 147.7}{2}\right)^2 + 14.3^2} = 75.2 \text{ MPa}$$

Closed Thin-Walled Tubes

- Wall thickness t << tube radius r ***(r/t > 10)***
- Product of shear stress times wall thickness is constant

Shear flow (q)

$$(q = \tau \cdot t = \text{constant})$$

- Shear stress is inversely proportional to wall thickness
- Total torque T is

$$T = \int \tau t r \, ds = (\tau t) \int r \, ds = \tau t (2A_m) = 2A_m t \tau$$

- A_m is the area enclosed by the section median line

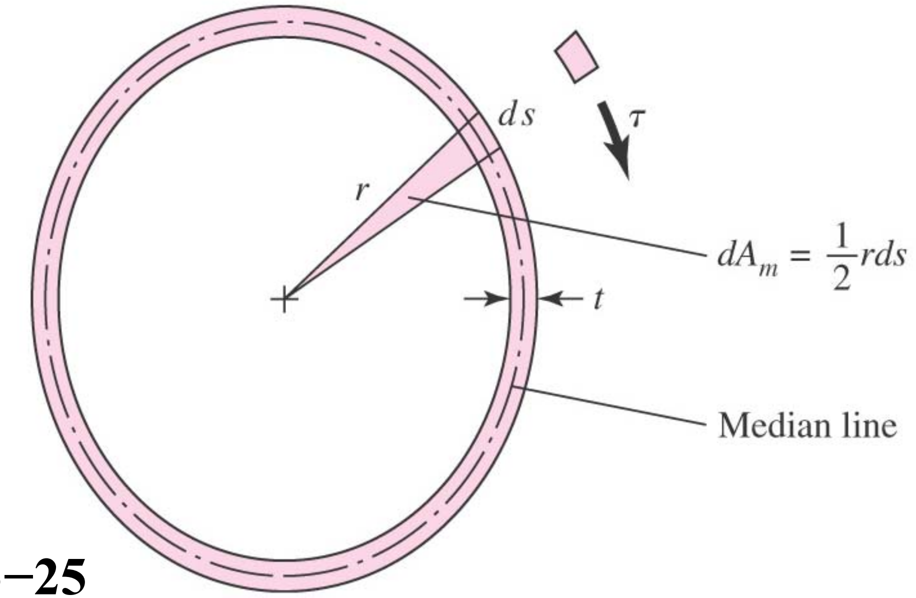


Fig. 3−25

Closed Thin-Walled Tubes

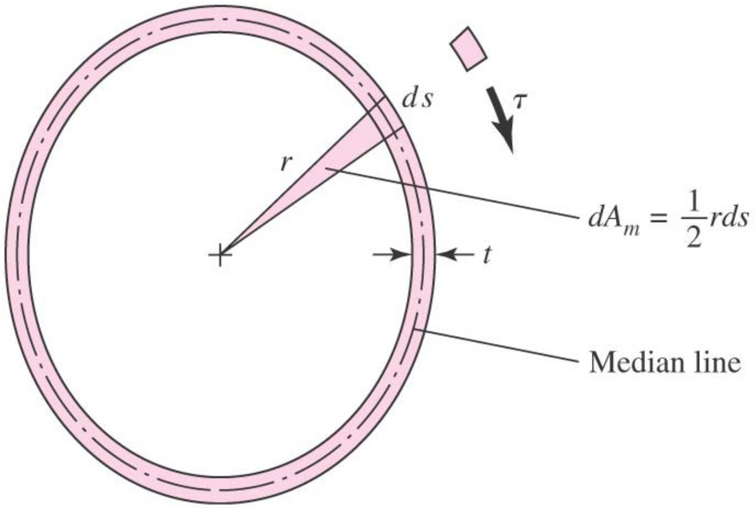
- Solving for **shear stress**

$$\tau = \frac{T}{2A_m t} \qquad (3\text{–}45)$$

- **Angular twist (radians) per unit length**

$$\theta_1 = \frac{T L_m}{4G A_m^2 t} \qquad (3\text{–}46)$$

- L_m is the length of the section median line



Example 3–10

A welded steel tube is 1 m long, has a 5-mm wall thickness, and a 45 mm by 70 mm rectangular cross section as shown in Fig. 3–26. Assume an allowable shear stress of 90 MPa and a shear modulus of 100 GPa.

(*a*) Estimate the allowable torque *T*.

(*b*) Estimate the angle of twist due to the torque.

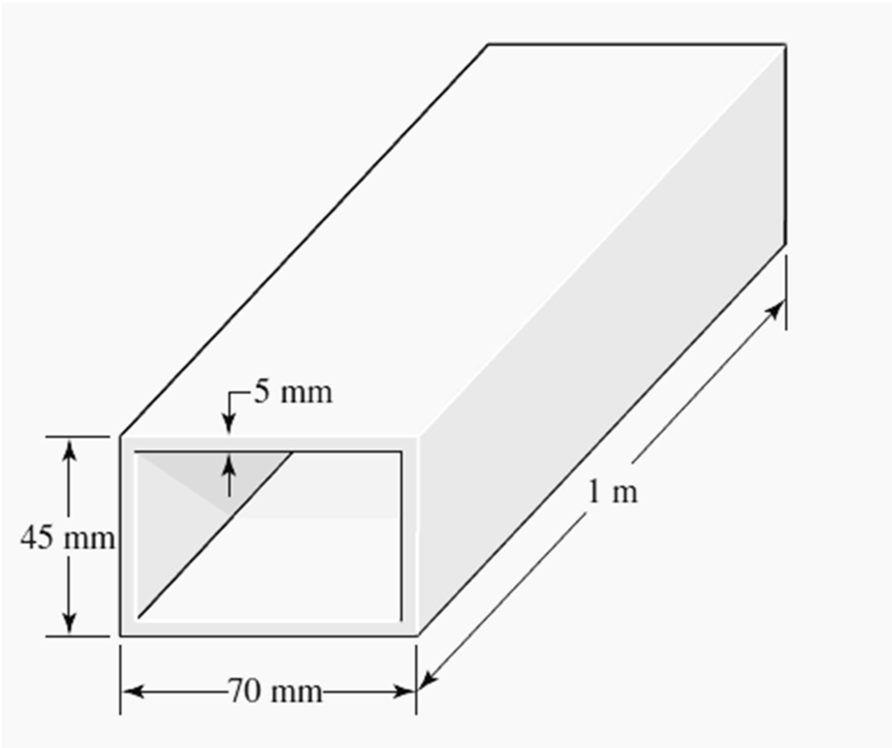


Fig. 3–26

Example 3–10

(a) Within the section median line, the area enclosed is

$$A_m = (45 - 5)(70 - 5) = 2600 \text{ mm}^2$$

and the length of the median perimeter is

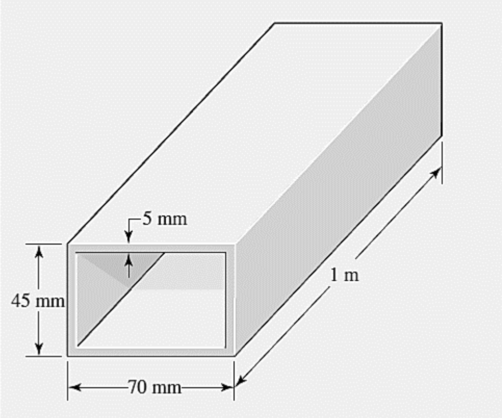
$$L_m = 2[(45 - 5) + (70 - 5)] = 210 \text{ mm}$$

From Eq. (3–45) the torque T is

$$T = 2A_m t\tau = 2(2600 \times 10^{-6})(0.005)(90 \times 10^6) = 2340 \text{ N}\cdot\text{m}$$

(b) The angle of twist θ from Eq. (3–46) is

$$\theta = \theta_1 l = \frac{TL_m}{4GA_m^2 t} l = \frac{2340(0.210)}{4(90 \times 10^9)(2600 \times 10^{-6})^2(0.005)}(1) = 0.036 \text{ rad} = 2.08^\circ$$



Example 3–11

Compare the shear stress on a circular cylindrical tube with an outside diameter of 37 mm and an inside diameter of 32 mm, predicted by Eq. (3–37), to that estimated by Eq. (3–45).

Solution

From Eq. (3–37),

$$\tau_{\max} = \frac{Tr}{J} = \frac{Tr}{(\pi/32)\left(d_o^4 - d_i^4\right)} = \frac{T\,(0.0185)}{(\pi/32)(0.037^4 - 0.032^4)} = 228249.6\ T$$

From Eq. (3–45),

$$\tau = \frac{T}{2A_m t} = \frac{T}{2(\pi 0.0345^2/4)0.0025} = 213944.89\ T$$

Taking Eq. (3–37) as correct, the error in the thin-wall estimate is −6.27 percent.

Open Thin-Walled Sections

- When the median wall line is not closed, the section is said to be an *open section*
- Some common open thin-walled sections

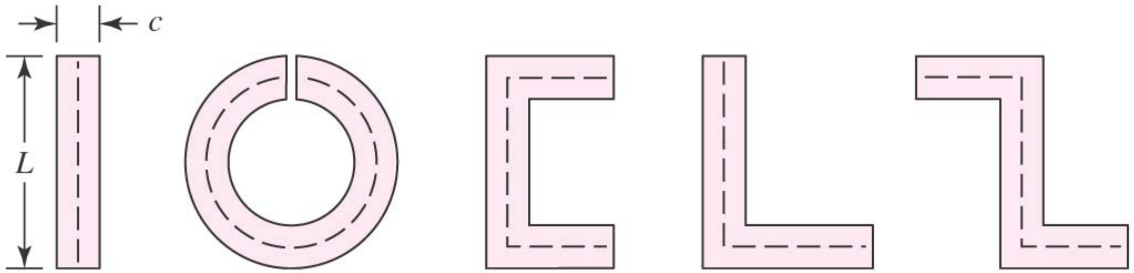


Fig. 3–27

- **Torsional shear stress**

$$\tau = G\theta_1 c = \frac{3T}{Lc^2} \qquad (3\text{–}47)$$

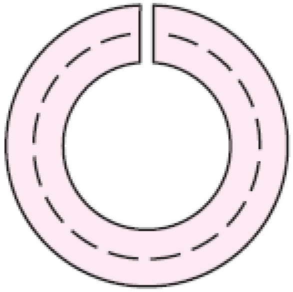
where

T = Torque, L = length of median line, c = wall thickness,

G = shear modulus, and θ_1 = angle of twist per unit length

Open Thin-Walled Sections

- *Shear stress* is inversely proportional to c^2
- *Angle of twist* is inversely proportional to c^3
- For small wall thickness, stress and twist can become quite large
- **Example**:
  - Compare thin round tube with and without slit
  - Ratio of wall thickness to outside diameter of 0.1
  - Stress with slit is 12.3 times greater
  - Twist with slit is 61.5 times greater



Example 3-12

A 0.3-m-long strip of steel is 3 mm thick and 25 mm wide, as shown in Fig. 3–28. If the allowable shear stress is 80 MPa and the shear modulus is 80 GPa, find the torque corresponding to the allowable shear stress and the angle of twist, in degrees, (*a*) using Eq. (3–47) and (*b*) using Eqs. (3–40) and (3–41).

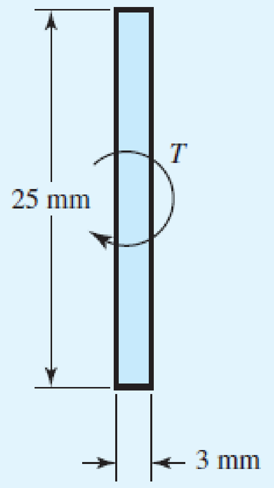


Figure 3–28

The cross-section of a thin strip of steel subjected to a torsional moment T.

Example 3-12

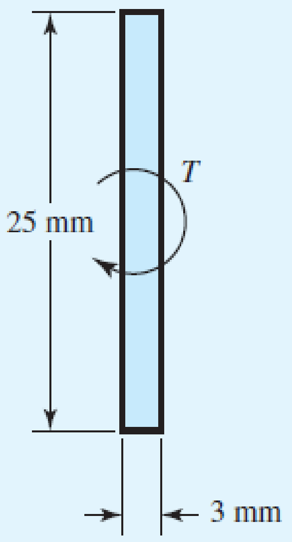
(*a*) The length of the median line is 25 mm. From Eq. (3–47),

$$T = \frac{Lc^2\tau}{3} = \frac{(0.025)\,(0.003)^2\,80 \times 10^6}{3} = 6\ \text{N}\cdot\text{m}$$

$$\theta = \theta_1 l = \frac{\tau l}{Gc} = \frac{80 \times 10^6\,(0.3)}{80 \times 10^9\,(0.003)} = 0.1\ \text{rad} = 5.7^\circ$$

A torsional spring rate k_t can be expressed as T/θ:

$$k_t = 6/0.1 = 60\ \text{N}\cdot\text{m/rad}$$



Example 3-12

(b) From Eq. (3–40),

$$T = \frac{\tau_{\max} bc^2}{3 + 1.8/(b/c)} = \frac{80 \times 10^6 (0.025)(0.003)^2}{3 + 1.8/(25/3)} = 5.6 \text{ N} \cdot \text{m}$$

From Eq. (3–41), with $b/c = 25/3 = 8.3$,

$$\theta = \frac{Tl}{\beta bc^3 G} = \frac{5.6 (0.3)}{0.307 (0.025)(0.003)^3\, 80 \times 10^9} = 0.1011 \text{ rad} = 5.8°$$

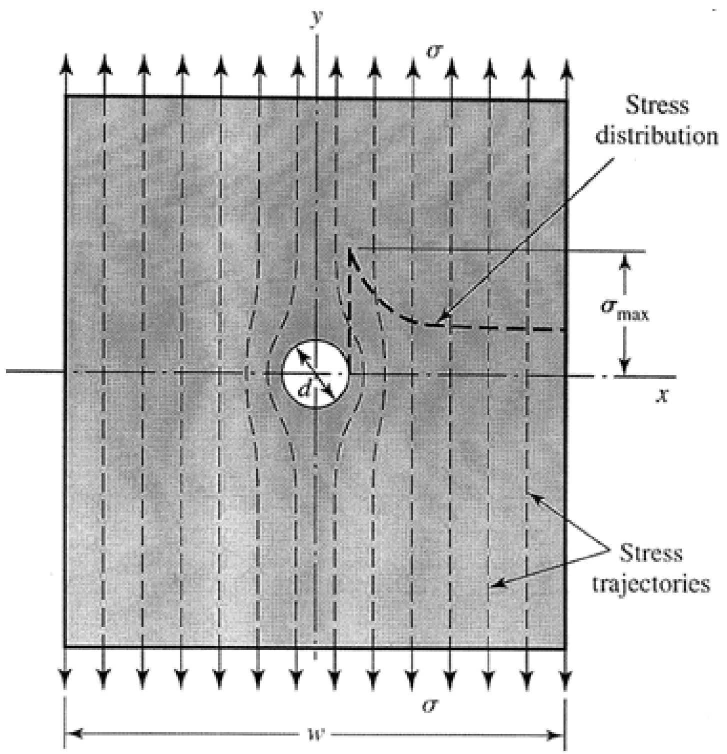
$$k_t = 5.6/0.1011 = 55.4 \text{ N} \cdot \text{m/rad}$$

The cross section is not thin, where b should be greater than c by at least a factor of 10. In estimating the torque, Eq. (3–47) provides a value of 7.1 percent higher than Eq. (3–40), and is 8.0 percent higher than when the table on page 102 is used.

Stress Concentration

- Localized increase of stress near discontinuities
- $\boldsymbol{K_t}$ is Theoretical (Geometric) Stress Concentration Factor

$$K_t = \frac{\sigma_{\max}}{\sigma_0} \qquad K_{ts} = \frac{\tau_{\max}}{\tau_0}$$



Theoretical Stress Concentration Factor

- Graphs available for standard configurations
- See Appendix A-15 and A-16 for common examples
- Many more in *Peterson's Stress-Concentration Factors*
- Note the trend for higher *Kt* at sharper discontinuity radius, and at greater disruption

Figure A–15–1

Bar in tension or simple compression with a transverse hole. $\sigma_0 = F/A$, where $A = (w - d)t$ and t is the thickness.

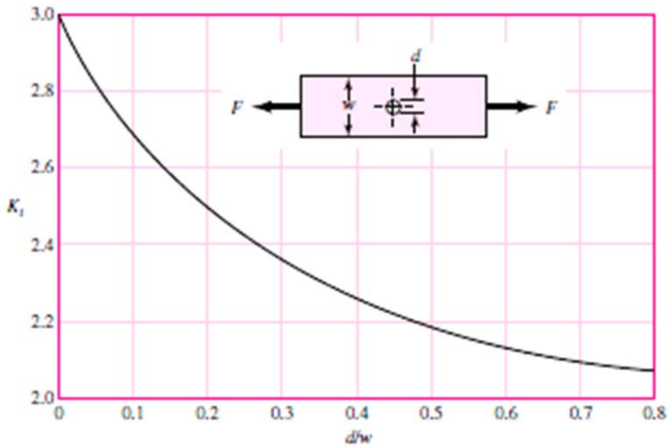
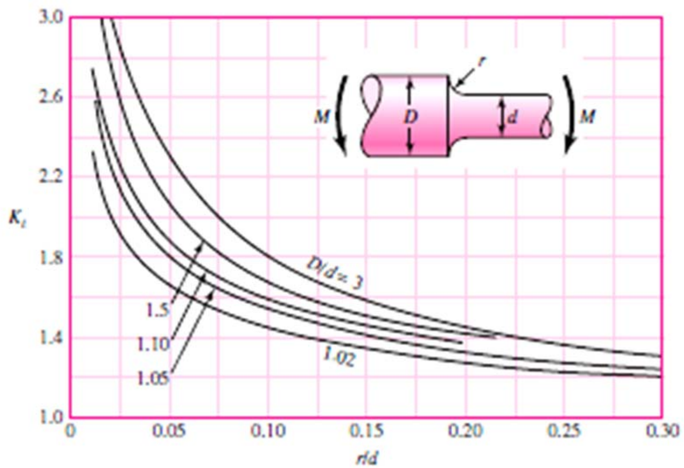


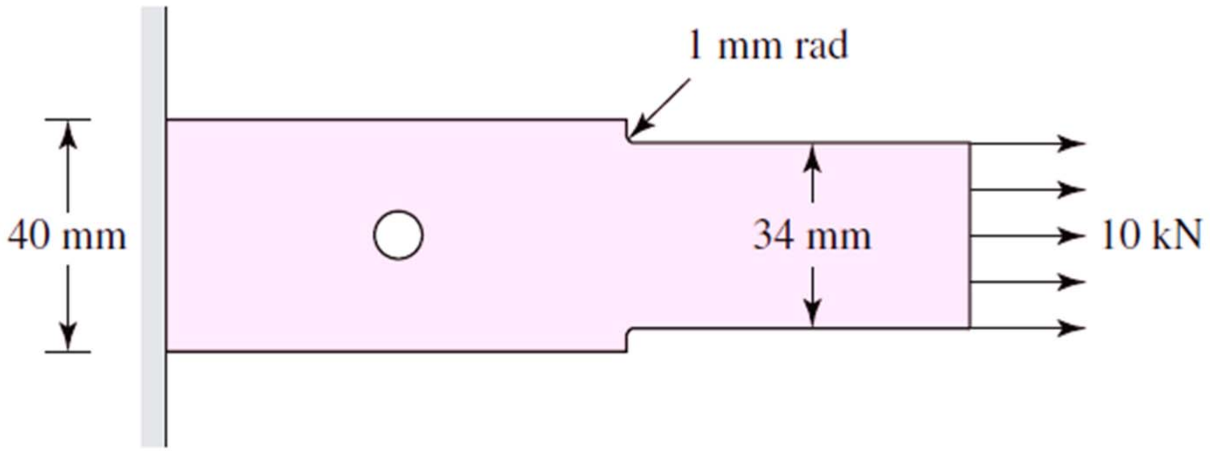
Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



Example 3-13

The 2-mm-thick bar shown in Fig. 3–30 is loaded axially with a constant force of 10 kN. The bar material has been heat treated and quenched to raise its strength, but as a consequence it has lost most of its ductility. It is desired to drill a hole through the center of the 40-mm face of the plate to allow a cable to pass through it. A 4-mm hole is sufficient for the cable to fit, but an 8-mm drill is readily available. Will a crack be more likely to initiate at the larger hole, the smaller hole, or at the fillet?



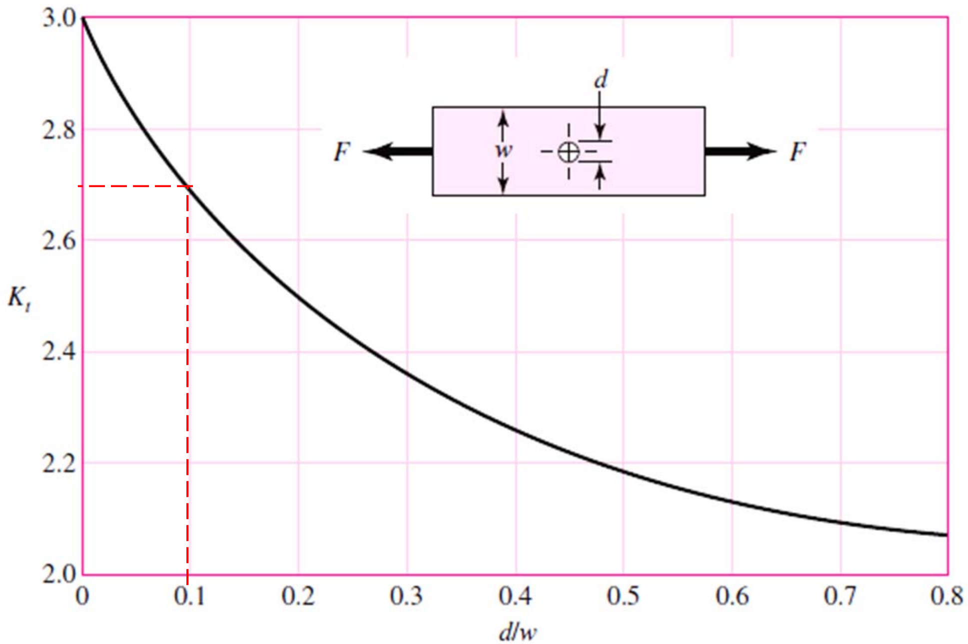
Example 3-13

Since the material is brittle, the effect of stress concentrations near the discontinuities must be considered. Dealing with the hole first, for a 4-mm hole, the nominal stress is

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w-d)t} = \frac{10\ 000}{(40-4)2} = 139 \text{ MPa}$$

The theoretical stress concentration factor, from Fig. A–15–1, with $d/w = 4/40 = 0.1$, is $K_t = 2.7$. The maximum stress is

$$\sigma_{max} = K_t\sigma_0 = 2.7(139) = 380 \text{ MPa}$$



Example 3-13

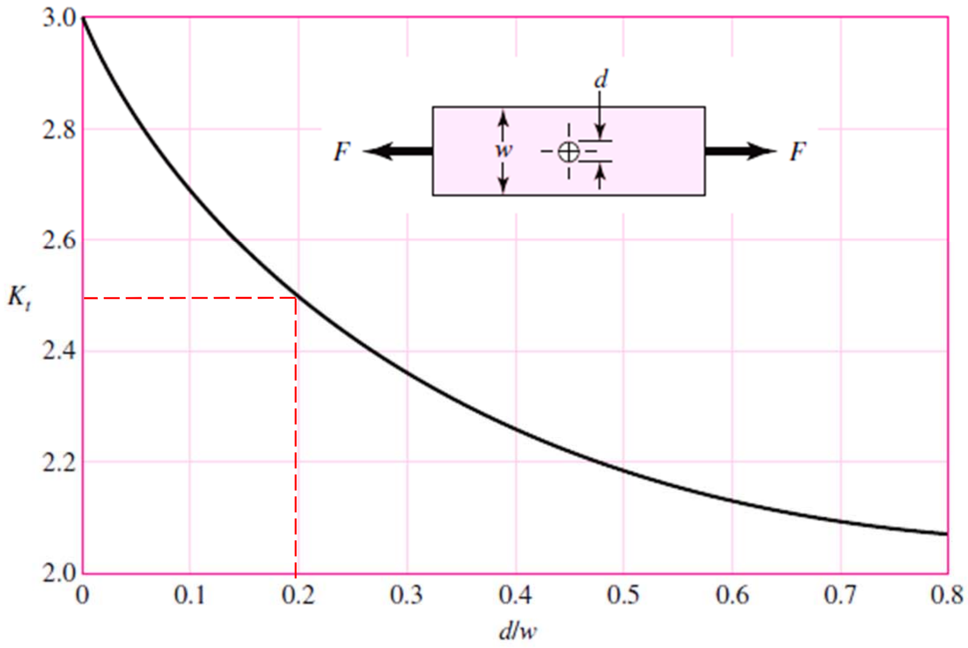
Similarly, for an 8-mm hole,

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w-d)t} = \frac{10\ 000}{(40-8)2} = 156 \text{ MPa}$$

With $d/w = 8/40 = 0.2$, then $K_t = 2.5$, and the maximum stress is

$$\sigma_{\max} = K_t \sigma_0 = 2.5(156) = 390 \text{ MPa}$$

Though the stress concentration is higher with the 4-mm hole, in this case the increased nominal stress with the 8-mm hole has more effect on the maximum stress.



Example 3-13

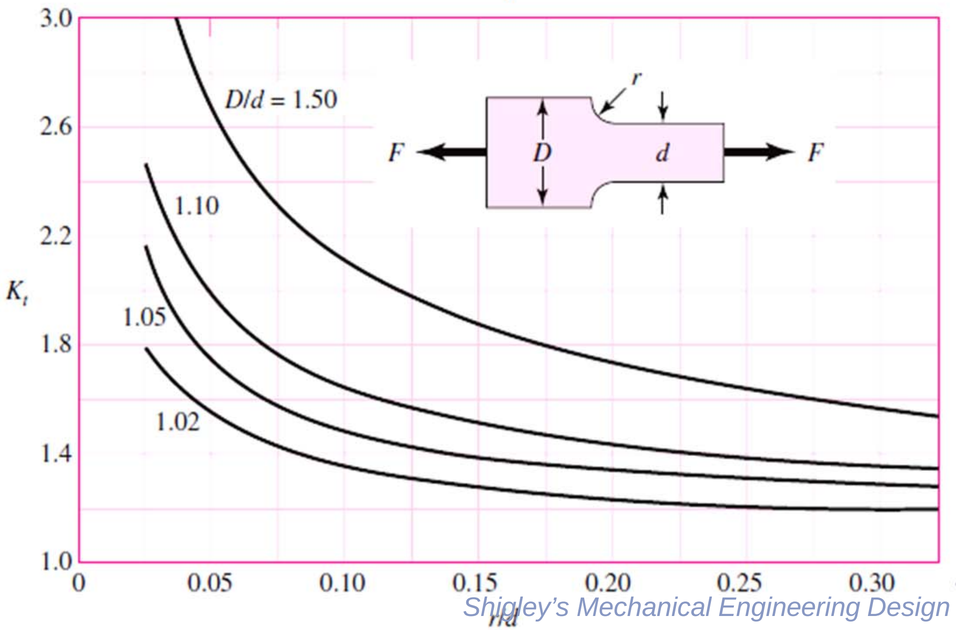
For the fillet,

$$\sigma_0 = \frac{F}{A} = \frac{10\ 000}{(34)2} = 147 \text{ MPa}$$

From Table A–15–5, $D/d = 40/34 = 1.18$, and $r/d = 1/34 = 0.026$. Then $K_t = 2.5$.

$$\sigma_{\max} = K_t \sigma_0 = 2.5(147) = 368 \text{ MPa}$$

The crack will most likely occur with the 8-mm hole, next likely would be the 4-mm hole, and least likely at the fillet.



Contact Stress

- Two bodies with curved surfaces pressed together
- Point or line contact changes to area contact
- Stresses developed are three-dimensional
- Called *contact stresses* or *Hertzian stresses*
- Common examples
  - Wheel rolling on rail
  - Mating gear teeth
  - Rolling bearings

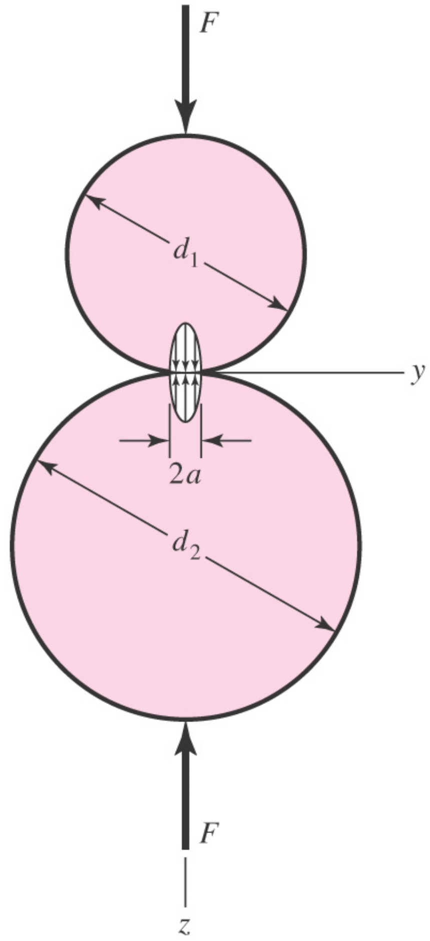
Spherical Contact Stress

- **Two solid spheres** of diameters d_1 and d_2 are pressed together with force F
- Circular area of contact of radius a

$$a = \sqrt[3]{\frac{3F}{8}\frac{\left(1-\nu_1^2\right)/E_1 + \left(1-\nu_2^2\right)/E_2}{1/d_1 + 1/d_2}}$$

- Pressure distribution is *hemispherical*
- Maximum pressure at the center of contact area

$$p_{\max} = \frac{3F}{2\pi a^2}$$



Spherical Contact Stress

- Maximum stresses on the *z* axis
- **Principal stresses**

$$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = -p_{\max}\left[\left(1 - \left|\frac{z}{a}\right| \tan^{-1}\frac{1}{|z/a|}\right)(1+\nu) - \frac{1}{2\left(1+\dfrac{z^2}{a^2}\right)}\right]$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{1+\dfrac{z^2}{a^2}}$$

- From Mohr's circle, **maximum shear stress** is

$$\tau_{\max} = \tau_{1/3} = \tau_{2/3} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_2 - \sigma_3}{2}$$

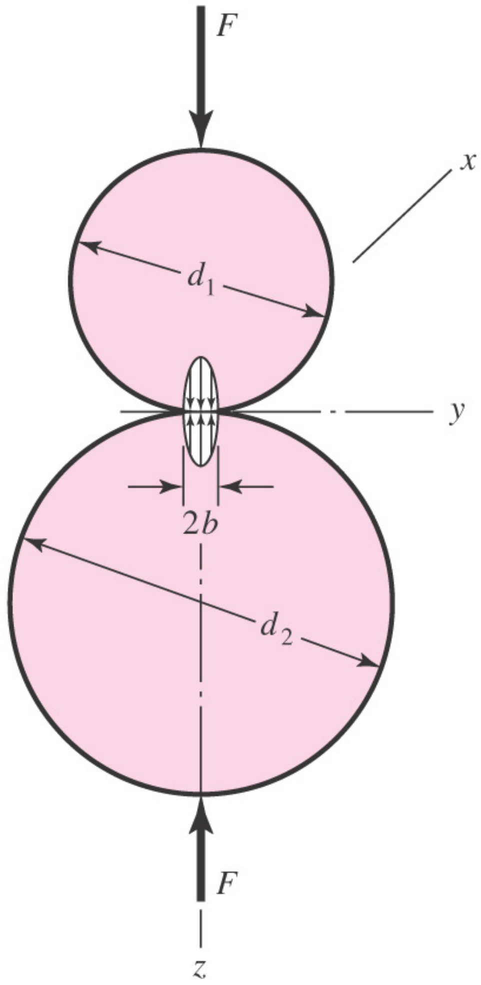
Cylindrical Contact Stress

- **Two right circular cylinders** with length l and diameters d_1 and d_2
- Area of contact is a narrow rectangle of width $2b$ and length l
- Pressure distribution is *elliptical*
- Half-width b

$$b = \sqrt{\frac{2F}{\pi l} \frac{\left(1 - \nu_1^2\right)/E_1 + \left(1 - \nu_2^2\right)/E_2}{1/d_1 + 1/d_2}}$$

- Maximum pressure

$$p_{\max} = \frac{2F}{\pi b l}$$



Cylindrical Contact Stress

- Maximum stresses on *z* axis

$$\sigma_x = -2\nu p_{\max}\left(\sqrt{1+\frac{z^2}{b^2}} - \left|\frac{z}{b}\right|\right)$$

$$\sigma_y = -p_{\max}\left(\frac{1+2\frac{z^2}{b^2}}{\sqrt{1+\frac{z^2}{b^2}}} - 2\left|\frac{z}{b}\right|\right)$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{\sqrt{1+z^2/b^2}}$$