Contents

3.1 Basic Rules of Differentiation

NOTATION

The following are all different ways of writing the derivative of $y = f(x)$:

$$f'(x), \qquad \frac{dy}{dx}, \qquad \frac{d[f(x)]}{dx}, \qquad y'$$

We have four basic rules for taking derivatives. These four rules are provided without proof.

Rule 1: Derivative of a constant

$$\frac{d}{dx}[c] = 0$$

Example 3.1.1. Find the derivatives of the following functions.

a) $y = 7$

b) $y = \pi$

Rule 2: The Power Rule

if n is any real number, then $\frac{d}{dx}[x^n] = nx^{n-1}$

Example 3.1.2. Find the derivatives of the following functions.

a) $y = x$

b) $y = x^3$

c) $y = \sqrt[3]{x^2}$

Rule 3: Derivative of a Constant Multiple of a Function

If c is a constant and $f(x)$ is a differentiable function then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

Example 3.1.3. Find the derivatives of the following functions.

a) $y = \frac{7x^3}{3}$

b) $y = \dfrac{4}{x^3}$

c) $y = \dfrac{1}{4x^3}$

Rule 4: The Sum and Difference Rule

If $f(x)$ and $g(x)$ are differentiable functions then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example 3.1.4. Find the derivatives of the following functions.

a) $y = 4x^5 + 3x^4 - 8x^2 + x + e^2$

b) $\dfrac{d}{dt}\left[\dfrac{4}{(5t)^7} + \dfrac{5}{t^3} - \sqrt[3]{t^5} + 25\pi^2\right]$

c) $f(t) = 2t^2 - \sqrt{t^3}$

Example 3.1.5. Find the tangent line to the curve at the given point. Put your answer in $y = mx + b$ form.

Point-slope form of the line with slope m passing through point (x_1, y_1):

$$y - y_1 = m(x - x_1)$$

1. $f(x) = -2, \qquad (2, -2)$

2. $f(x) = \sqrt[4]{x}, \qquad (16, 2)$

3. $f(x) = 8 - x^3, \qquad (2, 0)$

4. $f(x) = \dfrac{2}{9x^2}$, $\left(1, \dfrac{2}{9}\right)$

Example 3.1.6. Let $f(x) = x^3 - 4x^2$. Find the point(s) on the graph of f where the tangent line is horizontal.

Example 3.1.7. The supply function for a certain make of satellite radio is given by

$$p = f(x) = 0.0001x^{5/4} + 10$$

where x is the quantity supplied and p is the unit price in dollars.

a. Find $f'(x)$.

b. What is the rate of change of the unit price if the quantity supplied is 10,000 satelite radios?

3.2 The Product and Quotient Rules

The Product Rule

If $f(x)$ and $g(x)$ are differentiable functions then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

It is often easier to think of this rule in terms of words: "*The first times the derivative of the second plus the second times the derivative of the first.*"

Example 3.2.1. $s(t) = (t^5 - 3t^2 + t)(t^4 - 3t^3 + 2t^2 - t)$

Example 3.2.2. $f(x) = \left(x^3 + 2x^2 + \sqrt{x}\right)\left(x^2 + \frac{1}{x}\right)$

Example 3.2.3. $y = 9e^x\sqrt[3]{x^5}$

The Quotient Rule

If $f(x)$ and $g(x)$ are differentiable functions then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{[f(x)]^2}$$

NOTE: There is a negative sign in this equation and yes, it does matter which way you subtract.

This one is also easier to remember in words: ”*The bottom times the derivative of the top minus the top times the derivative of the bottom all divided by the bottom squared.*”

Example 3.2.4. $f(t) = \frac{t^2}{t^3+1} = \frac{top}{bottom}$

Example 3.2.5. $g(x) = \frac{\sqrt[3]{x}}{x+1} = \frac{top}{bottom}$

Rewrite this first. You need exponents that are numbers.

Example 3.2.6. $h(t) = \frac{1}{5t^2}$

Example 3.2.7. Find the equation of the tangent line to the graph of the curve of $y = \left(x^4 + 3x^3\right)(x + 5)$ when $x = 1$

Example 3.2.8. Find the equation of the tangent line to the graph of the curve of $y = \dfrac{2x}{x+4}$ when $x = -3$

Example 3.2.9. Suppose that $f(3) = 2$, $f'(3) = -7$, $g(3) = 10$ and $g'(3) = -3$. Find the values of

1. $(fg)'(3)$

2. $(f/g)'(3)$

3. $(g/f)'(3)$

Example 3.2.10. If $f(x) = \sqrt{x}\, g(x)$, where $g(9) = 6$ and $g'(9) = -8$, find $f'(9)$.

Example 3.2.11. Determine the point(s) on the graph of $y = \dfrac{e^x}{1+x^2}$ has a horizontal tangent line(s).

Example 3.2.12. $f(x) = \left(4 + \dfrac{1}{x}\right)\left(2x - \dfrac{1}{x^2}\right)$

Example 3.2.13. $f(s) = \dfrac{\sqrt{s}+3}{\sqrt[3]{s^2}}$

Example 3.2.14. $g(t) = \dfrac{(t+1)\left(t^2+1\right)}{t-2}$

Example 3.2.15. $f(y) = \pi^4 y^{-4}$

Example 3.2.16. $g(t) = \dfrac{-5t^3 + 8t - 9 + \sqrt{t}}{t}$

3.3 Derivatives of Trigonometric Functions

3.3.1 Some Special Trigonometric Limits

(These will be on the test)

$$\lim_{t\to 0} \sin t = 0 \qquad \lim_{t\to 0} \cos t = 1$$

$$\lim_{t\to 0} \frac{\sin t}{t} = 1 \qquad \lim_{t\to 0} \frac{\cos t - 1}{t} = 0$$

Example 3.3.1. $\lim_{t\to 0+} \frac{\sin t - 1}{t} =$

Example 3.3.2. $\lim_{t\to 0} \frac{\cos^2 t - 1}{t} =$

Example 3.3.3. $\lim_{\theta\to 0} \frac{\sin(7\theta)}{\theta} =$

Example 3.3.4. $\lim_{x\to 0} \frac{\sin(3x)}{4x} =$

Example 3.3.5. $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right) =$

3.3.2 Derivatives of Trigonometric Functions

$D[\sin u] = \cos u$ $D[\cos u] = -\sin u$

$D[\tan u] = \sec^2 u$ $D[\cot u] = -\csc^2 u$

$D[\sec u] = \sec u \tan u$ $D[\csc u] = -\csc u \cot u$

Example 3.3.6. $y = \cot x + 3x - \sec x$

Example 3.3.7. $h(t) = t^2 \sin t$

Example 3.3.8. $g(x) = \dfrac{e^x}{\tan x}$

Example 3.3.9. $r = \dfrac{\sin\phi + \cos\phi}{\cos\phi}$

Example 3.3.10. Show that $\frac{d}{dx}(\tan x) = \sec^2 x$

Example 3.3.11. Find the equation of the tangent line to the graph of the curve of $y = 3 + \frac{x}{2} + \sin x$ at the point $(0, 3)$.

Example 3.3.12. Find the point(s) where $y = \cos x + \sqrt{3}\sin x$ has horizontal tangent line(s).

3.4 The Chain Rule

3.4.1 Chain Rule: 2 methods

The chain rule is used for composition of functions: $y = (f \circ g)(x) = f(g(x))$

There are two "methods" that can be used to solve for the derivative of a composition of functions. Both methods are called the Chain Rule and are essentially the same using different notation.

Method 1: If we rewrite the functions as $y = f(u)$ and $u = g(x)$ where both y and u are differentiable functions then we can take the derivatives separately: $\frac{dy}{du}$ and $\frac{du}{dx}$ If we multiply them together we will get an expression for $\frac{dy}{dx}$:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}}$$

Example 3.4.1. Differentiate $y = (x^5 - 8x)^2$ using Method 1. Indicate which function is y and which is u.

Here we have a composition of functions. We can think of this as

$$y = f(u) = u^2 \qquad \text{and} \qquad u = g(x) = x^5 - 8x$$

If we take the composition of these two functions we get:

Example 3.4.2. Differentiate $f(\theta) = \cos(5\theta)$ using Method 1. Indicate which function is $f(u)$ and which is u.

Q. Why use the chain rule? In the previous example we could have easily multiplied out the function and taken a derivative.

A. Because it makes $y = (x^5 - 8x)^{25}$ possible without multiplying it out. When we have small exponents there are few problems but when we have an exponent such as 25 it becomes more complicated.

Example 3.4.3. Differentiate $y = (x^5 - 8x)^{25}$ using Method 1. Indicate which function is y and which is u.

Method 2: If $y = (f \circ g)(x) = f(g(x))$ then

$$\boxed{y' = f'(g(x)) \cdot g'(x)}$$

Both method 1 and method 2 are known as the chain rule. The second method may be easier.

In the second method it is often convenient to think of these functions as the "Outside function" and the "Inside function". The outside function is $f(u)$ and the inside function is $u = g(x)$.

It is often easiest to remember the chain rule in words:

Take the derivative of the outside function (leave the inside function alone) then multiply by the derivative of the inside function.

The chain rule can be written as:

The Chain Rule

If $y = f(u)$ then
$$y' = f'(u) \cdot u'$$

Example 3.4.4. Differentiate $y = \sqrt{(2x^5 - 8x + 5)^5}$. Indicate which function is f and which is u from the definition.

Example 3.4.5. Differentiate $f(x) = e^{3x^5 - 4x^2 + 2}$. Indicate which function is f and which is u from the definition.

Using the chain rule we can write a more general form of the power rule. So far we can only take derivatives of a variable raised to an exponent: $y = x^n$, but with the chain rule we can take the derivative of a function raised to an exponent.

The General Power Rule

Given the function
$$y = [u(x)]^n$$
then
$$\frac{dy}{dx} = n\,[u(x)]^{n-1} \cdot \frac{du}{dx}$$
or
$$y' = n \cdot u^{n-1} \cdot u'$$

3.4.2 The function a^x

Recall: $s = e^{\ln a}$ for any positive real number a. So we can think of a^x as

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$$

In fact we will consider this to be a definition:

Definition 3.1. For any number a greater than zero and x any real number $a^x = e^{x \ln a}$

For example:

- $3^2 = e^{2 \ln 3}$
- $5^{(7x+1)} = e^{(7x+1) \ln 5}$

The derivative of a^x: Use the chain rule.

Let $y = a^x = e^{x \ln a}$. Then by the chain rule $D[e^u] = e^u \, du$ we get:

$$\begin{aligned} \frac{d}{dx}\left[a^x\right] &= \frac{d}{dx}\left[e^{x \ln a}\right] \\ &= e^{x \ln a}\left(\frac{d}{dx}\left[x \ln a\right]\right) \\ &= e^{x \ln a} \ln a \\ &= a^x \ln a \end{aligned}$$

If u is a differentiable function then by the chain rule:

$$D\left[a^u\right] = a^u \ln a \, du$$

Example 3.4.6. Differentiate the following

1. $y = 3^x$

2. $f(t) = 5^{t^3 - 7t + 8}$

Example 3.4.7. More Examples

1. $y = \dfrac{1}{\left(\sqrt[3]{x^3 - 6x + 5}\right)^{10}}$

2. $f(x) = 8^{\cos(4x)}$

3. $f(t) = t^3\sqrt{3t^2 - t - 5}$

4. $f(\theta) = \cos^4\theta + \cos\theta^4$

5. $y = \tan^3(5x^2 + x)$

6. $h(y) = 5e^{6y+1}\left(y^3 - 8\right)^4$

7. $g(x) = \left(\dfrac{4x^2 - 5}{9 - x^3}\right)^5$

3.5 Implicit Differentiation and Inverse Trigonometric Functions

3.5.1 Implicit Differentiation

Explicit Functions:

Definition 3.2. An **explicit function** is a function in which one variable is defined only in terms of the other variable.

(a) $y = x^2 + 7$ Parabola

(b) $y = \sqrt{9 - x^2}$ Top half of the circle centered at the origin with radius 3.

(c) $y = -\sqrt{9 - x^2}$ Bottom half of the circle centered at the origin with radius 3.

Implicit Functions:

Definition 3.3. An **implicit function** is a function in which one variable is not defined only in terms of another variable.

(a) $y - x^2 = 7$ Parabola

(b) $y^2 + x^2 = 9$ Circle centered at the origin with radius 3.

(c) $y^4 + 7y^2x - y^2x^4 - 9x^5 = \log x$

Some implicit functions can be written explicitly:

- both examples (a) represent the same parabola and
- (b) can be solved for either the top or the bottom of the circle.
- (c) can not be solved explicitly for y in terms of x.

Implicit Differentiation:

We have seen how to differentiate functions of the form $y = f(x)$. We also want to be able to differentiate functions that either can't be written explicitly in terms of x or the resulting function is too complicated to deal with. To do this we use implicit differentiation.

Key to implicit differentiation (chain rule):

$$\frac{d}{dx}\left[y^n\right] = ny^{n-1}\frac{dy}{dx} \qquad \text{and} \qquad \frac{d}{dx}\left[x^n\right] = nx^{n-1}\frac{dx}{dx} = nx^{n-1}$$

Steps to find $\frac{dy}{dx}$ of implicit functions involving the variables x and y.

1. Treat y as a differentiable function of x.
2. Differentiate both sides of the equation with respect to x using all the rules we have previously used. i.e. Apply the operator $\frac{d}{dx}$ to every term and use the product, quotient and chain rules.
3. Solve for $\frac{dy}{dx}$ as if it were a variable.
   (a) Simplify each side of the equal sign.
   (b) Bring all $\frac{dy}{dx}$ terms to one side of the equal sign and all non-$\frac{dy}{dx}$ terms to the other.
   (c) Factor out $\frac{dy}{dx}$.
   (d) Divide to isolate $\frac{dy}{dx}$.

Example 3.5.1. Find $\frac{dy}{dx}$ for $x^3 - x^2 - xy = 4$ (a) by solving the equation explicitly in terms of y and (b) by implicit differentiation.

Example 3.5.2. Find $\dfrac{dy}{dx}$ for $\quad 4y^3 + x^5 = x^3y^3$

Example 3.5.3. Find $\dfrac{dy}{dx}$ for $\quad 2xy^3 + y^2 = \cos(x + y)$

Example 3.5.4. Find the slope of the tangent line to the curve $x^2 - y^3 = 1$ at the point $(3, 2)$

3.5.2 Derivatives of Inverse Functions

The six trigonometric functions have inverses only when the domain is restricted to make the function 1 to 1. The following table gives the restriction.

Function	Domain	Range
$y = \sin x$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$y = \cos x$	$[0, \pi]$	$[-1, 1]$
$y = \tan x$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	$(-\infty, \infty)$
$y = \csc x$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$	$(-\infty, -1] \cup [1, \infty)$
$y = \sec x$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$	$(-\infty, -1] \cup [1, \infty)$
$y = \cot x$	$(0, \pi)$	$(-\infty, \infty)$

What about the derivatives of inverse trigonometric functions?

Find $\frac{d}{dx}\left[\sin^{-1} x\right]$

Start with $y = \sin^{-1} x \implies \sin(y) = x$. Now use implicit differentiation.

$$\frac{d}{dx}[\sin(y)] = \frac{d}{dx}[x]$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Of course we want our answer in terms of x so what we will do is convert $\cos y$ using the Pythagorean identity $\cos^2 y + \sin^2 y = 1$ to get:

$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

Substituting back into the derivative we get:

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ OR $\frac{d}{dx}\left[\sin^{-1} x\right] = \frac{1}{\sqrt{1-x^2}}$

Similarly we can generate the following derivatives which we will write using the chain rule:

$$D\left[\sin^{-1} u\right] = \frac{du}{\sqrt{1-u^2}} \qquad D\left[\cos^{-1} u\right] = -\frac{du}{\sqrt{1-u^2}}$$

$$D\left[\tan^{-1} u\right] = \frac{du}{1+u^2} \qquad D\left[\cot^{-1} u\right] = -\frac{du}{1+u^2}$$

$$D\left[\sec^{-1} u\right] = \frac{du}{|u|\sqrt{u^2-1}} \qquad D\left[\csc^{-1} u\right] = -\frac{du}{|u|\sqrt{u^2-1}}$$

Recall: $\sin^{-1} u = \arcsin u$, etc.

Example 3.5.5. Differentiate the following:

1. $y = \sin^{-1}\sqrt{u}$
2. $f(t) = \cos^{-1}(4t^3)$
3. $g(x) = \sec^{-1}(e^{2x^3})$
4. Find $\frac{dr}{d\theta}$ for $\theta^2 + 3r^4 + \sin(r\theta) = \dfrac{r^3}{\theta}$