Chapter 3

Propagation of Uncertainties

Most physical quantities usually cannot be measured in a single direct measurement but are instead found in two distinct steps. First, we measure one or more quantities that *can* be measured directly and from which the quantity of interest can be calculated. Second, we use the measured values of these quantities to calculate the quantity of interest itself. For example, to find the area of a rectangle, you actually measure its length l and height h and then calculate its area A as $A = lh$. Similarly, the most obvious way to find the velocity v of an object is to measure the distance traveled, d, and the time taken, t, and then to calculate v as $v = d/t$. Any reader with experience in an introductory laboratory can easily think of more examples. In fact, a little thought will show that almost all interesting measurements involve these two distinct steps of direct measurement followed by calculation.

When a measurement involves these two steps, the estimation of uncertainties also involves two steps. We must first estimate the uncertainties in the quantities measured directly and then determine how these uncertainties "propagate" through the calculations to produce an uncertainty in the final answer.[1] This propagation of errors is the main subject of this chapter.

In fact, examples of propagation of errors were presented in Chapter 2. In Section 2.5, I discussed what happens when two numbers x and y are measured and the results are used to calculate the difference $q = x - y$. We found that the uncertainty in q is just the *sum* $\delta q \approx \delta x + \delta y$ of the uncertainties in x and y. Section 2.9 discussed the product $q = xy$, and Problem 2.13 discussed the sum $q = x + y$. I review these cases in Section 3.3; the rest of this chapter is devoted to more general cases of propagation of uncertainties and includes several examples.

Before I address error propagation in Section 3.3, I will briefly discuss the estimation of uncertainties in quantities measured directly in Sections 3.1 and 3.2. The methods presented in Chapter 1 are reviewed, and further examples are given of error estimation in direct measurements.

Starting in Section 3.3, I will take up the propagation of errors. You will learn that almost all problems in error propagation can be solved using three simple rules.

[1] In Chapter 4, I discuss another way in which the final uncertainty can sometimes be estimated. If all measurements can be repeated several times, and if all uncertainties are known to be random in character, then the uncertainty in the quantity of interest can be estimated by examining the spread in answers. Even when this method is possible, it is usually best used as a check on the two-step procedure discussed in this chapter.

A single, more complicated, rule will also be presented that covers all cases and from which the three simpler rules can be derived.

This chapter is long, but its length simply reflects its great importance. Error propagation is a technique you will use repeatedly in the laboratory, and you need to become familiar with the methods described here. The only exception is that the material of Section 3.11 is not used again until Section 5.6; thus, if the ideas of this chapter are all new to you, consider skipping Section 3.11 on your first reading.

3.1 Uncertainties in Direct Measurements

Almost all direct measurements involve reading a scale (on a ruler, clock, or voltmeter, for example) or a digital display (on a digital clock or voltmeter, for example). Some problems in scale reading were discussed in Section 1.5. Sometimes the main sources of uncertainty are the reading of the scale and the need to interpolate between the scale markings. In such situations, a reasonable estimate of the uncertainty is easily made. For example, if you have to measure a clearly defined length l with a ruler graduated in millimeters, you might reasonably decide that the length could be read to the nearest millimeter but no better. Here, the uncertainty δl would be $\delta l = 0.5$ mm. If the scale markings are farther apart (as with tenths of an inch), you might reasonably decide you could read to one-fifth of a division, for example. In any case, the uncertainties associated with the reading of a scale can obviously be estimated quite easily and realistically.

Unfortunately, other sources of uncertainty are frequently much more important than difficulties in scale reading. In measuring the distance between two points, your main problem may be to decide where those two points really are. For example, in an optics experiment, you may wish to measure the distance q from the center of a lens to a focused image, as in Figure 3.1. In practice, the lens is usually several millimeters thick, so locating its center is hard; if the lens comes in a bulky mounting, as it often does, locating the center is even harder. Furthermore, the image may appear to be well-focused throughout a range of many millimeters. Even though the apparatus is mounted on an optical bench that is clearly graduated in millimeters, the uncertainty in the distance from lens to image could easily be a centimeter or so. Since this uncertainty arises because the two points concerned are not clearly defined, this kind of problem is called a *problem of definition.*

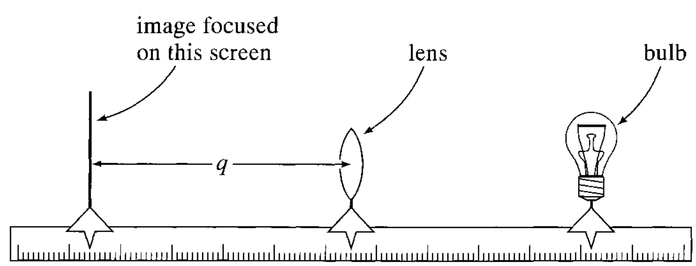

Figure 3.1. An image of the light bulb on the right is focused by the lens onto the screen at the left.

This example illustrates a serious danger in error estimation. If you look only at the scales and forget about other sources of uncertainty, you can badly underestimate the total uncertainty. In fact, the beginning student's most common mistake is to overlook some sources of uncertainty and hence *underestimate* uncertainties, often by a factor of 10 or more. Of course, you must also avoid *overestimating* errors. Experimenters who decide to play safe and to quote generous uncertainties on all measurements may avoid embarrassing inconsistencies, but their measurements may not be of much use. Clearly, the ideal is to find all possible causes of uncertainty and estimate their effects accurately, which is often not quite as hard as it sounds.

Superficially, at least, reading a digital meter is much easier than a conventional analog meter. Unless a digital meter is defective, it should display only significant figures. Thus, it is usually safe to say that the number of significant figures in a digital reading is precisely the number of figures displayed. Unfortunately, as discussed in Section 2.8, the exact meaning of significant figures is not always clear. Thus, a digital voltmeter that tells us that $V = 81$ microvolts could mean that the uncertainty is anything from $\delta V = 0.5$ to $\delta V = 1$ or more. Without a manual to tell you the uncertainty in a digital meter, a reasonable assumption is that the uncertainty in the final digit is ± 1 (so that the voltage just mentioned is $V = 81 \pm 1$).

The digital meter, even more than the analog scale, can give a misleading impression of accuracy. For example, a student might use a digital timer to time the fall of a weight in an Atwood machine or similar device. If the timer displays 8.01 seconds, the time of fall is apparently

$$t = 8.01 \pm 0.01 \text{ s.} \tag{3.1}$$

However, the careful student who repeats the experiment under nearly identical conditions might find a second measurement of 8.41 s; that is,

$$t = 8.41 \pm 0.01 \text{ s.}$$

One likely explanation of this large discrepancy is that uncertainties in the starting procedure vary the initial conditions and hence the time of fall; that is, the measured times really are different. In any case, the accuracy claimed in Equation (3.1) clearly is ridiculously too good. Based on the two measurements made, a more realistic answer would be

$$t = 8.2 \pm 0.2 \text{ s.}$$

In particular, the uncertainty is some 20 times larger than suggested in Equation (3.1) based on the original single reading.

This example brings us to another point mentioned in Chapter 1: Whenever a measurement can be repeated, it should usually be made several times. The resulting spread of values often provides a good indication of the uncertainties, and the average of the values is almost certainly more trustworthy than any one measurement. Chapters 4 and 5 discuss the statistical treatment of multiple measurements. Here, I emphasize only that if a measurement is repeatable, it should be repeated, both to obtain a more reliable answer (by averaging) and, more important, to get an estimate of the uncertainties. Unfortunately, as also mentioned in Chapter 1, repeating a measurement does not always reveal uncertainties. If the measurement is subject to a systematic error, which pushes all results in the same direction (such as a clock that runs slow), the spread in results will not reflect this systematic error. Eliminating such systematic errors requires careful checks of calibration and procedures.

3.2 The Square-Root Rule for a Counting Experiment

Another, different kind of direct measurement has an uncertainty that can be estimated easily. Some experiments require you to count events that occur at random but have a definite average rate. For example, the babies born in a hospital arrive in a fairly random way, but in the long run births in any one hospital probably occur at a definite average rate. Imagine that a demographer who wants to know this rate counts 14 births in a certain two-week period at a local hospital. Based on this result, he would naturally say that his best estimate for the expected number of births in two weeks is 14. Unless he has made a mistake, 14 is *exactly* the number of births in the two-week period he chose to observe. Because of the random way births occur, however, 14 obviously may *not* equal the actual average number of births in all two-week periods. Perhaps this number is 13, 15, or even a fractional number such as 13.5 or 14.7.

Evidently, the uncertainty in this kind of experiment is not in the observed number counted (14 in our example). Instead, the uncertainty is in how well this observed number approximates the true average number. The problem is to estimate how large this uncertainty is. Although I discuss the theory of these counting experiments in Chapter 11, the answer is remarkably simple and is easily stated here: The uncertainty in any counted number of random events, as an estimate of the true average number, is *the square root of the counted number.* In our example, the demographer counted 14 births in a certain two-week period. Therefore, his uncertainty is $\sqrt{14} \approx 4$, and his final conclusion would be

$$\text{(average births in a two-week period)} = 14 \pm 4.$$

To make this statement more general, suppose we count the occurrences of any event (such as the births of babies in a hospital) that occurs randomly but at a definite average rate. Suppose we count for a chosen time interval T (such as two weeks), and we denote the number of observed events by the Greek letter ν. (Pronounced "nu," this symbol is the Greek form of the letter n and stands for *number.*) Based on this experiment, our best estimate for the average number of events in time T is, of course, the observed number ν, and the uncertainty in this estimate is the square root of the number, that is, $\sqrt{\nu}$. Therefore, our answer for the average number of events in time T is

$$\boxed{\text{(average number of events in time } T) = \nu \pm \sqrt{\nu}.} \tag{3.2}$$

I refer to this important result as the *Square-Root Rule for Counting Experiments.*

Counting experiments of this type occur frequently in the physics laboratory. The most prominent example is in the study of radioactivity. In a radioactive material, each nucleus decays at a random time, but the decays in a large sample occur at a definite average rate. To find this rate, you can simply count the number ν of decays in some convenient time interval T; the expected number of decays in time T, with its uncertainty, is then given by the square-root rule, (3.2).

Quick Check 3.1. **(a)** To check the activity of a radioactive sample, an inspector places the sample in a liquid scintillation counter to count the number of decays in a two-minute interval and obtains 33 counts. What should he report as the number of decays produced by the sample in two minutes? **(b)** Suppose, instead, he had monitored the same sample for 50 minutes and obtained 907 counts. What would be his answer for the number of decays in 50 minutes? **(c)** Find the percent uncertainties in these two measurements, and comment on the usefulness of counting for a longer period as in part (b).

3.3 Sums and Differences; Products and Quotients

For the remainder of this chapter, I will suppose that we have measured one or more quantities $x, y, \ldots,$ with corresponding uncertainties $\delta x, \delta y, \ldots,$ and that we now wish to use the measured values of $x, y, \ldots,$ to calculate the quantity of real interest, q. The calculation of q is usually straightforward; the problem is how the uncertainties, $\delta x, \delta y, \ldots,$ propagate through the calculation and lead to an uncertainty δq in the final value of q.

SUMS AND DIFFERENCES

Chapter 2 discussed what happens when you measure two quantities x and y and calculate their sum, $x + y$, or their difference, $x - y$. To estimate the uncertainty in the sum or difference, we had only to decide on their highest and lowest probable values. The highest and lowest probable values of x are $x_{\text{best}} \pm \delta x$, and those of y are $y_{\text{best}} \pm \delta y$. Hence, the highest probable value of $x + y$ is

$$x_{\text{best}} + y_{\text{best}} + (\delta x + \delta y),$$

and the lowest probable value is

$$x_{\text{best}} + y_{\text{best}} - (\delta x + \delta y).$$

Thus, the best estimate for $q = x + y$ is

$$q_{\text{best}} = x_{\text{best}} + y_{\text{best}},$$

and its uncertainty is

$$\delta q \approx \delta x + \delta y. \tag{3.3}$$

A similar argument (be sure you can reconstruct it) shows that the uncertainty in the *difference* $x - y$ is given by the same formula (3.3). That is, the uncertainty in either the sum $x + y$ or the difference $x - y$ is the sum $\delta x + \delta y$ of the uncertainties in x and y.

If we have several numbers $x, \ldots, w$ to be added or subtracted, then repeated application of (3.3) gives the following provisional rule.

Uncertainty in Sums and Differences
(Provisional Rule)

If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w),$$

then the uncertainty in the computed value of q is the sum,

$$\delta q \approx \delta x + \cdots + \delta z + \delta u + \cdots + \delta w, \tag{3.4}$$

of all the original uncertainties.

In other words, when you add or subtract any number of quantities, the uncertainties in those quantities always *add*. As before, I use the sign $\approx$ to emphasize that this rule is only provisional.

Example: Adding and Subtracting Masses

As a simple example of rule (3.4), suppose an experimenter mixes together the liquids in two flasks, having first measured their separate masses when full and empty, as follows:

$$\begin{aligned}
M_1 &= \text{mass of first flask and contents} &&= 540 \pm 10 \text{ grams} \\
m_1 &= \text{mass of first flask empty} &&= 72 \pm 1 \text{ grams} \\
M_2 &= \text{mass of second flask and contents} &&= 940 \pm 20 \text{ grams} \\
m_2 &= \text{mass of second flask empty} &&= 97 \pm 1 \text{ grams}
\end{aligned}$$

He now calculates the total mass of liquid as

$$\begin{aligned}
M &= M_1 - m_1 + M_2 - m_2 \\
&= (540 - 72 + 940 - 97) \text{ grams} = 1{,}311 \text{ grams.}
\end{aligned}$$

According to rule (3.4), the uncertainty in this answer is the sum of all four uncertainties,

$$\begin{aligned}
\delta M \approx \delta M_1 + \delta m_1 + \delta M_2 + \delta m_2 &= (10 + 1 + 20 + 1) \text{ grams} \\
&= 32 \text{ grams.}
\end{aligned}$$

Thus, his final answer (properly rounded) is

$$\text{total mass of liquid} = 1{,}310 \pm 30 \text{ grams.}$$

Notice how the much smaller uncertainties in the masses of the empty flasks made a negligible contribution to the final uncertainty. This effect is important, and we will discuss it later on. With experience, you can learn to identify in advance those uncertainties that are negligible and can be ignored from the outset. Often, this can greatly simplify the calculation of uncertainties.

PRODUCTS AND QUOTIENTS

Section 2.9 discussed the uncertainty in the product $q = xy$ of two measured quantities. We saw that, provided the fractional uncertainties concerned are small, the fractional uncertainty in $q = xy$ is the sum of the fractional uncertainties in x and y. Rather than review the derivation of this result, I discuss here the similar case of the quotient $q = x/y$. As you will see, the uncertainty in a quotient is given by the same rule as for a product; that is, the fractional uncertainty in $q = x/y$ is equal to the sum of the fractional uncertainties in x and y.

Because uncertainties in products and quotients are best expressed in terms of fractional uncertainties, a shorthand notation for the latter will be helpful. Recall that if we measure some quantity x as

$$\text{(measured value of } x) \;=\; x_{\text{best}} \pm \delta x$$

in the usual way, then the fractional uncertainty in x is defined to be

$$\text{(fractional uncertainty in } x) \;=\; \frac{\delta x}{|x_{\text{best}}|}.$$

(The absolute value in the denominator ensures that the fractional uncertainty is always positive, even when x_{best} is negative.) Because the symbol $\delta x/|x_{\text{best}}|$ is clumsy to write and read, from now on I will abbreviate it by omitting the subscript "best" and writing

$$\text{(fractional uncertainty in } x) \;=\; \frac{\delta x}{|x|}.$$

The result of measuring any quantity x can be expressed in terms of its fractional error $\delta x/|x|$ as

$$\text{(value of } x) \;=\; x_{\text{best}}(1 \pm \delta x/|x|).$$

Therefore, the value of $q = x/y$ can be written as

$$\text{(value of } q) \;=\; \frac{x_{\text{best}}}{y_{\text{best}}}\,\frac{1 \pm \delta x/|x|}{1 \pm \delta y/|y|}.$$

Our problem now is to find the extreme probable values of the second factor on the right. This factor is largest, for example, if the numerator has its largest value, $1 + \delta x/|x|$, and the denominator has its *smallest* value, $1 - \delta y/|y|$. Thus, the largest probable value for $q = x/y$ is

$$(\text{largest value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}} \frac{1 + \delta x/|x|}{1 - \delta y/|y|}. \tag{3.5}$$

The last factor in expression (3.5) has the form $(1 + a)/(1 - b)$, where the numbers a and b are normally small (that is, much less than 1). It can be simplified by two approximations. First, because b is small, the binomial theorem[2] implies that

$$\frac{1}{(1 - b)} \approx 1 + b. \tag{3.6}$$

Therefore,

$$\begin{aligned} \frac{1 + a}{1 - b} \approx (1 + a)(1 + b) &= 1 + a + b + ab \\ &\approx 1 + a + b, \end{aligned}$$

where, in the second line, we have neglected the product ab of two small quantities. Returning to (3.5) and using these approximations, we find for the largest probable value of $q = x/y$

$$(\text{largest value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}}\left(1 + \frac{\delta x}{|x|} + \frac{\delta y}{|y|}\right).$$

A similar calculation shows that the smallest probable value is given by a similar expression with two minus signs. Combining these two, we find that

$$(\text{value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}}\left(1 \pm \left[\frac{\delta x}{|x|} + \frac{\delta y}{|y|}\right]\right).$$

Comparing this equation with the standard form,

$$(\text{value of } q) = q_{\text{best}}\left(1 \pm \frac{\delta q}{|q|}\right),$$

we see that the best value for q is $q_{\text{best}} = x_{\text{best}}/y_{\text{best}}$, as we would expect, and that the fractional uncertainty is

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}. \tag{3.7}$$

We conclude that when we divide or multiply two measured quantities x and y, the fractional uncertainty in the answer is the sum of the fractional uncertainties in x and y, as in (3.7). If we now multiply or divide a series of numbers, repeated application of this result leads to the following provisional rule.

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[2]The binomial theorem expresses $1/(1 - b)$ as the infinite series $1 + b + b^2 + \cdots$. If b is much less than 1, then $1/(1-b) \approx 1 + b$ as in (3.6). If you are unfamiliar with the binomial theorem, you can find more details in Problem 3.8.

Uncertainty in Products and Quotients (Provisional Rule)

If several quantities $x, \ldots, w$ are measured with small uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w},$$

then the fractional uncertainty in the computed value of q is the sum,

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|}, \qquad (3.8)$$

of the fractional uncertainties in $x, \ldots, w$.

Briefly, when quantities are multiplied or divided the *fractional uncertainties add.*

Example: A Problem in Surveying

In surveying, sometimes a value can be found for an inaccessible length l (such as the height of a tall tree) by measuring three other lengths l_1, l_2, l_3 in terms of which

$$l = \frac{l_1 l_2}{l_3}.$$

Suppose we perform such an experiment and obtain the following results (in feet):

$$l_1 = 200 \pm 2, \qquad l_2 = 5.5 \pm 0.1, \qquad l_3 = 10.0 \pm 0.4.$$

Our best estimate for l is

$$l_{\text{best}} = \frac{200 \times 5.5}{10.0} = 110 \text{ ft.}$$

According to (3.8), the fractional uncertainty in this answer is the sum of the fractional uncertainties in l_1, l_2, and l_3, which are 1%, 2%, and 4%, respectively. Thus

$$\frac{\delta l}{l} \approx \frac{\delta l_1}{l_1} + \frac{\delta l_2}{l_2} + \frac{\delta l_3}{l_3} = (1 + 2 + 4)\%$$
$$= 7\%,$$

and our final answer is

$$l = 110 \pm 8 \text{ ft.}$$

Quick Check 3.2. Suppose you measure the three quantities x, y, and z as follows:

$$x = 8.0 \pm 0.2, \qquad y = 5.0 \pm 0.1, \qquad z = 4.0 \pm 0.1.$$

Express the given uncertainties as percentages, and then calculate $q = xy/z$ with its uncertainty δq [as given by the provisional rule (3.8)].

3.4 Two Important Special Cases

Two important special cases of the rule (3.8) deserve mention. One concerns the product of two numbers, one of which has *no* uncertainty; the other involves a power (such as x^3) of a measured number.

MEASURED QUANTITY TIMES EXACT NUMBER

Suppose we measure a quantity x and then use the measured value to calculate the product $q = Bx$, where the number B has *no uncertainty*. For example, we might measure the diameter of a circle and then calculate its circumference, $c = \pi \times d$; or we might measure the thickness T of 200 identical sheets of paper and then calculate the thickness of a single sheet as $t = (1/200) \times T$. According to the rule (3.8), the fractional uncertainty in $q = Bx$ is the sum of the fractional uncertainties in B and x. Because $\delta B = 0$, this implies that

$$\frac{\delta q}{|q|} = \frac{\delta x}{|x|}.$$

That is, the fractional uncertainty in $q = Bx$ (with B known exactly) is the same as that in x. We can express this result differently if we multiply through by $|q| = |Bx|$ to give $\delta q = |B|\delta x$, and we have the following useful rule:[3]

> **Measured Quantity Times Exact Number**
>
> If the quantity x is measured with uncertainty δx and is used to compute the product
>
> $$q = Bx,$$
>
> where B has no uncertainty, then the uncertainty in q is just $|B|$ times that in x,
>
> $$\delta q = |B|\delta x. \qquad (3.9)$$

[3] This rule (3.9) was derived from the rule (3.8), which is provisional and will be replaced by the more complete rules (3.18) and (3.19). Fortunately, the same conclusion (3.9) follows from these improved rules. Thus (3.9) is already in its final form.

This rule is especially useful in measuring something inconveniently small but available many times over, such as the thickness of a sheet of paper or the time for a revolution of a rapidly spinning wheel. For example, if we measure the thickness T of 200 sheets of paper and get the answer

$$\text{(thickness of 200 sheets)} = T = 1.3 \pm 0.1 \text{ inches},$$

it immediately follows that the thickness t of a single sheet is

$$\begin{aligned}\text{(thickness of one sheet)} = t &= \frac{1}{200} \times T \\ &= 0.0065 \pm 0.0005 \text{ inches}.\end{aligned}$$

Notice how this technique (measuring the thickness of several identical sheets and dividing by their number) makes easily possible a measurement that would otherwise require quite sophisticated equipment and that this technique gives a remarkably small uncertainty. Of course, the sheets must be known to be equally thick.

Quick Check 3.3. Suppose you measure the diameter of a circle as

$$d = 5.0 \pm 0.1 \text{ cm}$$

and use this value to calculate the circumference $c = \pi d$. What is your answer, with its uncertainty?

POWERS

The second special case of the rule (3.8) concerns the evaluation of a power of some measured quantity. For example, we might measure the speed v of some object and then, to find its kinetic energy $\frac{1}{2}mv^2$, calculate the square v^2. Because v^2 is just $v \times v$, it follows from (3.8) that the fractional uncertainty in v^2 is *twice* the fractional uncertainty in v. More generally, from (3.8) the general rule for any power is clearly as follows.

Uncertainty in a Power

If the quantity x is measured with uncertainty δx and the measured value is used to compute the power

$$q = x^n,$$

then the fractional uncertainty in q is n times that in x,

$$\frac{\delta q}{|q|} = n\frac{\delta x}{|x|}. \tag{3.10}$$

The derivation of this rule required that n be a positive integer. In fact, however, the rule generalizes to include *any* exponent n, as we will see later in Equation (3.26).

Quick Check 3.4. To find the volume of a certain cube, you measure its side as 2.00 ± 0.02 cm. Convert this uncertainty to a percent and then find the volume with its uncertainty.

Example: Measurement of g

Suppose a student measures g, the acceleration of gravity, by measuring the time t for a stone to fall from a height h above the ground. After making several timings, she concludes that

$$t = 1.6 \pm 0.1 \text{ s},$$

and she measures the height h as

$$h = 46.2 \pm 0.3 \text{ ft}.$$

Because h is given by the well-known formula $h = \frac{1}{2}gt^2$, she now calculates g as

$$\begin{aligned} g &= \frac{2h}{t^2} \\ &= \frac{2 \times 46.2 \text{ ft}}{(1.6 \text{ s})^2} = 36.1 \text{ ft/s}^2. \end{aligned}$$

What is the uncertainty in her answer?

The uncertainty in her answer can be found by using the rules just developed. To this end, we need to know the fractional uncertainties in each of the factors in the expression $g = 2h/t^2$ used to calculate g. The factor 2 has no uncertainty. The fractional uncertainties in h and t are

$$\frac{\delta h}{h} = \frac{0.3}{46.2} = 0.7\%$$

and

$$\frac{\delta t}{t} = \frac{0.1}{1.6} = 6.3\%.$$

According to the rule (3.10), the fractional uncertainty of t^2 is twice that of t. Therefore, applying the rule (3.8) for products and quotients to the formula $g = 2h/t^2$, we find the fractional uncertainty

$$\begin{aligned} \frac{\delta g}{g} &= \frac{\delta h}{h} + 2\frac{\delta t}{t} \\ &= 0.7\% + 2 \times (6.3\%) = 13.3\%, \end{aligned} \qquad (3.11)$$

and hence the uncertainty

$$\delta g = (36.1 \text{ ft/s}^2) \times \frac{13.3}{100} = 4.80 \text{ ft/s}^2.$$

Thus, our student's final answer (properly rounded) is

$$g = 36 \pm 5 \text{ ft/s}^2.$$

This example illustrates how simple the estimation of uncertainties can often be. It also illustrates how error analysis tells you not only the size of uncertainties but also how to reduce them. In this example, (3.11) shows that the largest contribution comes from the measurement of the time. If we want a more precise value of g, then the measurement of t must be improved; any attempt to improve the measurement of h will be wasted effort.

Finally, the accepted value of g is 32 ft/s^2, which lies within our student's margins of error. Thus, she can conclude that her measurement, although not especially accurate, is perfectly consistent with the known value of g.

3.5 Independent Uncertainties in a Sum

The rules presented thus far can be summarized quickly: When measured quantities are added or subtracted, the *uncertainties add;* when measured quantities are multiplied or divided, the *fractional uncertainties add.* In this and the next section, I discuss how, under certain conditions, the uncertainties calculated by using these rules may be unnecessarily large. Specifically, you will see that if the original uncertainties are *independent* and *random,* a more realistic (and smaller) estimate of the final uncertainty is given by similar rules in which the uncertainties (or fractional uncertainties) are *added in quadrature* (a procedure defined shortly).

Let us first consider computing the sum, $q = x + y$, of two numbers x and y that have been measured in the standard form

$$(\text{measured value of } x) = x_{\text{best}} \pm \delta x,$$

with a similar expression for y. The argument used in the last section was as follows: First, the best estimate for $q = x + y$ is obviously $q_{\text{best}} = x_{\text{best}} + y_{\text{best}}$. Second, since the highest probable values for x and y are $x_{\text{best}} + \delta x$ and $y_{\text{best}} + \delta y$, the highest probable value for q is

$$x_{\text{best}} + y_{\text{best}} + \delta x + \delta y. \tag{3.12}$$

Similarly, the lowest probable value of q is

$$x_{\text{best}} + y_{\text{best}} - \delta x - \delta y.$$

Therefore, we concluded, the value of q probably lies between these two numbers, and the uncertainty in q is

$$\delta q \approx \delta x + \delta y.$$

To see why this formula is likely to overestimate δq, let us consider how the actual value of q could equal the highest extreme (3.12). Obviously, this occurs if we have underestimated x by the full amount δx *and* underestimated y by the full δy, obviously, a fairly unlikely event. If x and y are measured independently and our errors are random in nature, we have a 50% chance that an *underestimate* of x is accompanied by an *overestimate* of y, or *vice versa*. Clearly, then, the probability we will underestimate both x and y by the full amounts δx and δy is fairly small. Therefore, the value $\delta q \approx \delta x + \delta y$ overstates our probable error.

What constitutes a better estimate of δq? The answer depends on precisely what we mean by uncertainties (that is, what we mean by the statement that q is "probably" somewhere between $q_{\text{best}} - \delta q$ and $q_{\text{best}} + \delta q$). It also depends on the statistical laws governing our errors in measurement. Chapter 5 discusses the normal, or Gauss, distribution, which describes measurements subject to random uncertainties. It shows that if the measurements of x and y are made independently and are both governed by the normal distribution, then the uncertainty in $q = x + y$ is given by

$$\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}. \tag{3.13}$$

When we combine two numbers by squaring them, adding the squares, and taking the square root, as in (3.13), the numbers are said to be *added in quadrature.* Thus, the rule embodied in (3.13) can be stated as follows: If the measurements of x and y are independent and subject only to random uncertainties, then the uncertainty δq in the calculated value of $q = x + y$ is the *sum in quadrature* or *quadratic sum* of the uncertainties δx and δy.

Compare the new expression (3.13) for the uncertainty in $q = x + y$ with our old expression,

$$\delta q \approx \delta x + \delta y. \tag{3.14}$$

First, the new expression (3.13) is always smaller than the old (3.14), as we can see from a simple geometrical argument: For any two positive numbers a and b, the numbers a, b, and $\sqrt{a^2 + b^2}$ are the three sides of a right-angled triangle (Figure 3.2). Because the length of any side of a triangle is always less than the sum of the

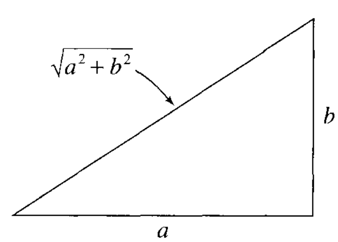

Figure 3.2. Because any side of a triangle is less than the sum of the other two sides, the inequality $\sqrt{a^2 + b^2} < a + b$ is always true.

other two sides, it follows that $\sqrt{a^2 + b^2} < a + b$ and hence that (3.13) is always less than (3.14).

Because expression (3.13) for the uncertainty in $q = x + y$ is always smaller than (3.14), you should always use (3.13) *when* it is applicable. It is, however, *not* always applicable. Expression (3.13) reflects the possibility that an overestimate of x can be offset by an underestimate of y or vice versa, but there are measurements for which this cancellation is not possible.

Suppose, for example, that $q = x + y$ is the sum of two lengths x and y measured with the same steel tape. Suppose further that the main source of uncertainty is our fear that the tape was designed for use at a temperature different from the present temperature. If we don't know this temperature (and don't have a reliable tape for comparison), we have to recognize that our tape may be longer or shorter than its calibrated length and hence may yield readings under or over the correct length. This uncertainty can be easily allowed for.[4] The point, however, is that if the tape is too long, then we *underestimate both* x and y; and if the tape is too short, we *overestimate both* x and y. Thus, there is no possibility for the cancellations that justified using the sum in quadrature to compute the uncertainty in $q = x + y$.

I will prove later (in Chapter 9) that, whether or not our errors are independent and random, the uncertainty in $q = x + y$ is *certainly no larger* than the simple sum $\delta x + \delta y$:

$$\delta q \leqslant \delta x + \delta y. \tag{3.15}$$

That is, our old expression (3.14) for δq is actually an *upper bound* that holds in all cases. If we have any reason to suspect the errors in x and y are *not* independent and random (as in the example of the steel tape measure), we are not justified in using the quadratic sum (3.13) for δq. On the other hand, the bound (3.15) guarantees that δq is certainly no worse than $\delta x + \delta y$, and our safest course is to use the old rule

$$\delta q \approx \delta x + \delta y.$$

Often, whether uncertainties are added in quadrature or directly makes little difference. For example, suppose that x and y are lengths both measured with uncertainties $\delta x = \delta y = 2$ mm. If we are sure these uncertainties are independent and random, we would estimate the error in $x + y$ to be the sum in quadrature,

$$\sqrt{(\delta x)^2 + (\delta y)^2} = \sqrt{4 + 4} \text{ mm} = 2.8 \text{ mm} \approx 3 \text{ mm},$$

but if we suspect that the uncertainties may not be independent, we would have to use the ordinary sum,

$$\delta x + \delta y \approx (2 + 2) \text{ mm} = 4 \text{ mm}.$$

In many experiments, the estimation of uncertainties is so crude that the difference between these two answers (3 mm and 4 mm) is unimportant. On the other hand, sometimes the sum in quadrature is significantly smaller than the ordinary sum. Also, rather surprisingly, the sum in quadrature is sometimes easier to compute than the ordinary sum. Examples of these effects are given in the next section.

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[4] Suppose, for example, that the tape has a coefficient of expansion $\alpha = 10^{-5}$ per degree and that we decide that the difference between its calibration temperature and the present temperature is unlikely to be more than 10 degrees. The tape is then unlikely to be more than 10^{-4}, or 0.01%, away from its correct length, and our uncertainty is therefore 0.01%.

Quick Check 3.5. Suppose you measure the volumes of water in two beakers as

$$V_1 = 130 \pm 6 \text{ ml} \quad \text{and} \quad V_2 = 65 \pm 4 \text{ ml}$$

and then carefully pour the contents of the first into the second. What is your prediction for the total volume $V = V_1 + V_2$ with its uncertainty, δV, assuming the original uncertainties are independent and random? What would you give for δV if you suspected the original uncertainties were not independent?

3.6 More About Independent Uncertainties

In the previous section, I discussed how independent random uncertainties in two quantities x and y propagate to cause an uncertainty in the sum $x + y$. We saw that for this type of uncertainty the two errors should be added in quadrature. We can naturally consider the corresponding problem for differences, products, and quotients. As we will see in Section 5.6, in all cases our previous rules (3.4) and (3.8) are modified only in that the sums of errors (or fractional errors) are replaced by quadratic sums. Further, the old expressions (3.4) and (3.8) will be proven to be upper bounds that always hold whether or not the uncertainties are independent and random. Thus, the final versions of our two main rules are as follows:

Uncertainty in Sums and Differences

Suppose that $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$ and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w).$$

If the uncertainties in $x, \ldots, w$ are known to be *independent and random,* then the uncertainty in q is the quadratic sum

$$\delta q = \sqrt{(\delta x)^2 + \cdots + (\delta z)^2 + (\delta u)^2 + \cdots + (\delta w)^2} \tag{3.16}$$

of the original uncertainties. In any case, δq is never larger than their ordinary sum,

$$\delta q \leqslant \delta x + \cdots + \delta z + \delta u + \cdots + \delta w. \tag{3.17}$$

and

Uncertainties in Products and Quotients

Suppose that $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w}.$$

If the uncertainties in $x, \ldots, w$ are *independent and random,* then the fractional uncertainty in q is the sum in quadrature of the original fractional uncertainties,

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \cdots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \cdots + \left(\frac{\delta w}{w}\right)^2}. \tag{3.18}$$

In any case, it is never larger than their ordinary sum,

$$\frac{\delta q}{|q|} \leq \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|}. \tag{3.19}$$

Notice that I have not yet justified the use of addition in quadrature for independent random uncertainties. I have argued only that when the various uncertainties are independent and random, there is a good chance of partial cancellations of errors and that the resulting uncertainty (or fractional uncertainty) should be smaller than the simple sum of the original uncertainties (or fractional uncertainties); the sum in quadrature does have this property. I give a proper justification of its use in Chapter 5. The bounds (3.17) and (3.19) are proved in Chapter 9.

Example: Straight Addition vs Addition in Quadrature

As discussed, sometimes there is no significant difference between uncertainties computed by addition in quadrature and those computed by straight addition. Often, however, there is a significant difference, and—surprisingly enough—the sum in quadrature is often much simpler to compute. To see how this situation can arise, consider the following example.

Suppose we want to find the efficiency of a D.C. electric motor by using it to lift a mass m through a height h. The work accomplished is mgh, and the electric energy delivered to the motor is VIt, where V is the applied voltage, I the current, and t the time for which the motor runs. The efficiency is then

$$\text{efficiency, } e = \frac{\text{work done by motor}}{\text{energy delivered to motor}} = \frac{mgh}{VIt}.$$

Let us suppose that m, h, V, and I can all be measured with 1% accuracy,

$$(\text{fractional uncertainty for } m, h, V, \text{ and } I) = 1\%,$$

and that the time t has an uncertainty of 5%,

$$(\text{fractional uncertainty for } t) \; = \; 5\%.$$

(Of course, g is known with negligible uncertainty.) If we now compute the efficiency e, then according to our old rule ("fractional errors add"), we have an uncertainty

$$\begin{aligned} \frac{\delta e}{e} &\approx \frac{\delta m}{m} + \frac{\delta h}{h} + \frac{\delta V}{V} + \frac{\delta I}{I} + \frac{\delta t}{t} \\ &= (1 + 1 + 1 + 1 + 5)\% \; = \; 9\%. \end{aligned}$$

On the other hand, if we are confident that the various uncertainties are independent and random, then we can compute $\delta e/e$ by the quadratic sum to give

$$\begin{aligned} \frac{\delta e}{e} &= \sqrt{\left(\frac{\delta m}{m}\right)^2 + \left(\frac{\delta h}{h}\right)^2 + \left(\frac{\delta V}{V}\right)^2 + \left(\frac{\delta I}{I}\right)^2 + \left(\frac{\delta t}{t}\right)^2} \\ &= \sqrt{(1\%)^2 + (1\%)^2 + (1\%)^2 + (1\%)^2 + (5\%)^2} \\ &= \sqrt{29}\% \; \approx \; 5\%. \end{aligned}$$

Clearly, the quadratic sum leads to a significantly smaller estimate for δe. Furthermore, to one significant figure, the uncertainties in m, h, V, and I *make no contribution at all* to the uncertainty in e computed in this way; that is, to one significant figure, we have found (in this example)

$$\frac{\delta e}{e} = \frac{\delta t}{t}.$$

This striking simplification is easily understood. When numbers are added in quadrature, they are squared first and then summed. The process of squaring greatly exaggerates the importance of the larger numbers. Thus, if one number is 5 times any of the others (as in our example), its square is 25 times that of the others, and we can usually neglect the others entirely.

This example illustrates how combining errors in quadrature is usually better and often easier than computing them by straight addition. The example also illustrates the type of problem in which the errors *are* independent and for which addition in quadrature is justified. (For the moment I take for granted that the errors are random and will discuss this more difficult point in Chapter 4.) The five quantities measured (m, h, V, I, and t) are physically distinct quantities with different units and are measured by entirely different processes. For the sources of error in any quantity to be correlated with those in any other is almost inconceivable. Therefore, the errors can reasonably be treated as independent and combined in quadrature.

Quick Check 3.6. Suppose you measure three numbers as follows:

$$x = 200 \pm 2, \quad y = 50 \pm 2, \quad z = 20 \pm 1,$$

where the three uncertainties are independent and random. What would you give for the values of $q = x + y - z$ and $r = xy/z$ with their uncertainties?

3.7 Arbitrary Functions of One Variable

You have now seen how uncertainties, both independent and otherwise, propagate through sums, differences, products, and quotients. However, many calculations require more complicated operations, such as computation of a sine, cosine, or square root, and you will need to know how uncertainties propagate in these cases.

As an example, imagine finding the refractive index n of glass by measuring the critical angle θ. We know from elementary optics that $n = 1/\sin\theta$. Therefore, if we can measure the angle θ, we can easily calculate the refractive index n, but we must then decide what uncertainty δn in $n = 1/\sin\theta$ results from the uncertainty $\delta\theta$ in our measurement of θ.

More generally, suppose we have measured a quantity x in the standard form $x_{\text{best}} \pm \delta x$ and want to calculate some known function $q(x)$, such as $q(x) = 1/\sin x$ or $q(x) = \sqrt{x}$. A simple way to think about this calculation is to draw a graph of $q(x)$ as in Figure 3.3. The best estimate for $q(x)$ is, of course, $q_{\text{best}} = q(x_{\text{best}})$, and the values x_{best} and q_{best} are shown connected by the heavy lines in Figure 3.3.

To decide on the uncertainty δq, we employ the usual argument. The largest probable value of x is $x_{\text{best}} + \delta x$; using the graph, we can immediately find the largest probable value of q, which is shown as q_{max}. Similarly, we can draw in the smallest probable value, q_{min}, as shown. If the uncertainty δx is small (as we always suppose it is), then the section of graph involved in this construction is approximately straight, and q_{max} and q_{min} are easily seen to be equally spaced on either side of q_{best}. The uncertainty δq can then be taken from the graph as either of the lengths shown, and we have found the value of q in the standard form $q_{\text{best}} \pm \delta q$.

Occasionally, uncertainties are calculated from a graph as just described. (See Problems 3.26 and 3.30 for examples.) Usually, however, the function $q(x)$ is known

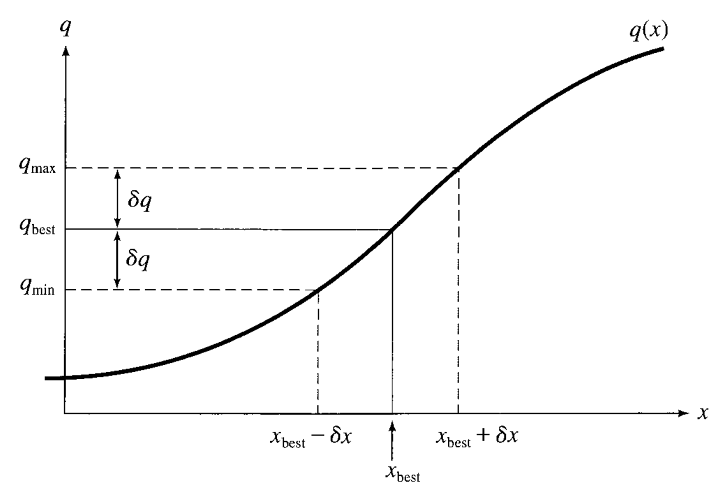

Figure 3.3. Graph of $q(x)$ vs x. If x is measured as $x_{\text{best}} \pm \delta x$, then the best estimate for $q(x)$ is $q_{\text{best}} = q(x_{\text{best}})$. The largest and smallest probable values of $q(x)$ correspond to the values $x_{\text{best}} \pm \delta x$ of x.

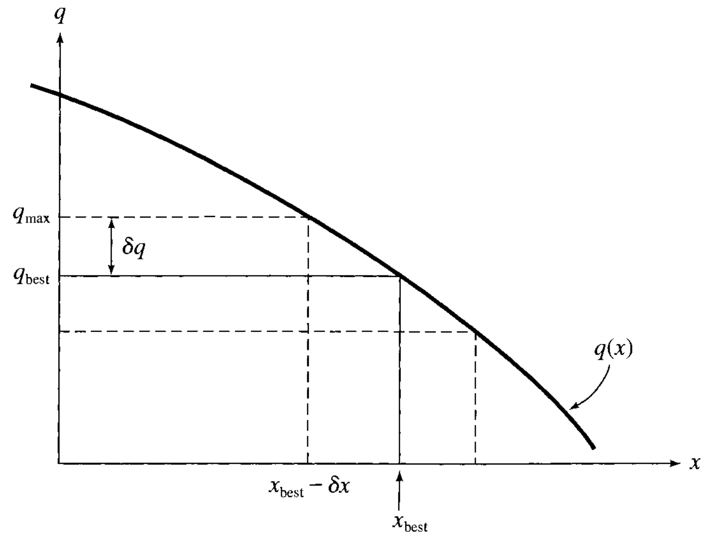

Figure 3.4. If the slope of $q(x)$ is negative, the maximum probable value of q corresponds to the minimum value of x, and vice versa.

explicitly—$q(x) = \sin x$ or $q(x) = \sqrt{x}$, for example—and the uncertainty δq can be calculated analytically. From Figure 3.3, we see that

$$\delta q \;=\; q(x_{best} + \delta x) - q(x_{best}). \tag{3.20}$$

Now, a fundamental approximation of calculus asserts that, for any function $q(x)$ and any sufficiently small increment u,

$$q(x + u) - q(x) \;=\; \frac{dq}{dx}\,u.$$

Thus, provided the uncertainty δx is small (as we always assume it is), we can rewrite the difference in (3.20) to give

$$\delta q \;=\; \frac{dq}{dx}\,\delta x. \tag{3.21}$$

Thus, to find the uncertainty δq, we just calculate the derivative dq/dx and multiply by the uncertainty δx.

The rule (3.21) is not quite in its final form. It was derived for a function, like that of Figure 3.3, whose slope is positive. Figure 3.4 shows a function with negative slope. Here, the maximum probable value q_{max} obviously corresponds to the minimum value of x, so that

$$\delta q \;=\; -\frac{dq}{dx}\,\delta x. \tag{3.22}$$

Because dq/dx is negative, we can write $-dq/dx$ as $|dq/dx|$, and we have the following general rule.

Uncertainty in Any Function of One Variable

If x is measured with uncertainty δx and is used to calculate the function $q(x)$, then the uncertainty δq is

$$\delta q = \left| \frac{dq}{dx} \right| \delta x. \tag{3.23}$$

This rule usually allows us to find δq quickly and easily. Occasionally, if $q(x)$ is very complicated, evaluating its derivative may be a nuisance, and going back to (3.20) is sometimes easier, as we discuss in Problem 3.32. Particularly if you have programmed your calculator or computer to find $q(x)$, then finding $q(x_{\text{best}} + \delta x)$ and $q(x_{\text{best}})$ and their difference may be easier than differentiating $q(x)$ explicitly.

Example: Uncertainty in a Cosine

As a simple application of the rule (3.23), suppose we have measured an angle θ as

$$\theta = 20 \pm 3°$$

and that we wish to find $\cos\theta$. Our best estimate of $\cos\theta$ is, of course, $\cos 20° = 0.94$, and according to (3.23), the uncertainty is

$$\begin{aligned} \delta(\cos\theta) &= \left| \frac{d\cos\theta}{d\theta} \right| \delta\theta \\ &= |\sin\theta|\, \delta\theta \text{ (in rad).} \end{aligned} \tag{3.24}$$

We have indicated that $\delta\theta$ must be expressed in radians, because the derivative of $\cos\theta$ is $-\sin\theta$ only if θ is expressed in radians. Therefore, we rewrite $\delta\theta = 3°$ as $\delta\theta = 0.05$ rad; then (3.24) gives

$$\begin{aligned} \delta(\cos\theta) &= (\sin 20°) \times 0.05 \\ &= 0.34 \times 0.05 = 0.02. \end{aligned}$$

Thus, our final answer is

$$\cos\theta = 0.94 \pm 0.02.$$

Quick Check 3.7. Suppose you measure x as 3.0 ± 0.1 and then calculate $q = e^x$. What is your answer, with its uncertainty? (Remember that the derivative of e^x is e^x.)

As another example of the rule (3.23), we can rederive and generalize a result found in Section 3.4. Suppose we measure the quantity x and then calculate the power $q(x) = x^n$, where n is any known, fixed number, positive or negative. According to (3.23), the resulting uncertainty in q is

$$\delta q = \left|\frac{dq}{dx}\right| \delta x = |nx^{n-1}|\delta x.$$

If we divide both sides of this equation by $|q| = |x^n|$, we find that

$$\frac{\delta q}{|q|} = |n| \frac{\delta x}{|x|}; \tag{3.25}$$

that is, the fractional uncertainty in $q = x^n$ is $|n|$ times that in x. This result (3.25) is just the rule (3.10) found earlier, except that the result here is more general, because n can now be any number. For example, if $n = 1/2$, then $q = \sqrt{x}$, and

$$\frac{\delta q}{|q|} = \frac{1}{2} \frac{\delta x}{|x|};$$

that is, the fractional uncertainty in $\sqrt{x}$ is *half* that in x itself. Similarly, the fractional uncertainty in $1/x = x^{-1}$ is the same as that in x itself.

The result (3.25) is just a special case of the rule (3.23). It is sufficiently important, however, to deserve separate statement as the following general rule.

Uncertainty in a Power

If x is measured with uncertainty δx and is used to calculate the power $q = x^n$ (where n is a fixed, known number), then the fractional uncertainty in q is $|n|$ times that in x,

$$\frac{\delta q}{|q|} = |n| \frac{\delta x}{|x|}. \tag{3.26}$$

Quick Check 3.8. If you measure x as 100 ± 6, what should you report for $\sqrt{x}$, with its uncertainty?

3.8 Propagation Step by Step

We now have enough tools to handle almost any problem in the propagation of errors. Any calculation can be broken down into a sequence of steps, each involving just one of the following types of operation: (1) sums and differences; (2) products and quotients; and (3) computation of a function of one variable, such as x^n, $\sin x$,

e^x, or $\ln x$. For example, we could calculate

$$q = x(y - z\sin u) \tag{3.27}$$

from the measured quantities x, y, z, and u in the following steps: Compute the *function* $\sin u$, then the *product* of z and $\sin u$, next the *difference* of y and $z\sin u$, and finally the *product* of x and $(y - z\sin u)$.

We know how uncertainties propagate through each of these separate operations. Thus, provided the various quantities involved are independent, we can calculate the uncertainty in the final answer by proceeding in steps from the uncertainties in the original measurement. For example, if the quantities x, y, z, and u in (3.27) have been measured with corresponding uncertainties $\delta x, \ldots, \delta u$, we could calculate the uncertainty in q as follows. First, find the uncertainty in the function $\sin u$; knowing this, find the uncertainty in the product $z\sin u$, and then that in the difference $y - z\sin u$; finally, find the uncertainty in the complete product (3.27).

Quick Check 3.9. Suppose you measure three numbers as follows:

$$x = 200 \pm 2, \quad y = 50 \pm 2, \quad z = 40 \pm 2,$$

where the three uncertainties are independent and random. Use step-by-step propagation to find the quantity $q = x/(y - z)$ with its uncertainty. [First find the uncertainty in the difference $y - z$ and then the quotient $x/(y - z)$.]

Before I discuss some examples of this step-by-step calculation of errors, let me emphasize three general points. First, because uncertainties in sums or differences involve absolute uncertainties (such as δx) whereas those in products or quotients involve fractional uncertainties (such as $\delta x/|x|$), the calculations will require some facility in passing from absolute to fractional uncertainties and vice versa, as demonstrated below.

Second, an important simplifying feature of all these calculations is that (as repeatedly emphasized) uncertainties are seldom needed to more than one significant figure. Hence, much of the calculation can be done rapidly in your head, and many smaller uncertainties can be completely neglected. In a typical experiment involving several trials, you may need to do a careful calculation on paper of all error propagations for the first trial. After that, you will often find that all trials are sufficiently similar that no further calculation is needed or, at worst, that for subsequent trials the calculations of the first trial can be modified in your head.

Finally, you need to be aware that you will sometimes encounter functions $q(x)$ whose uncertainty cannot be found reliably by the stepwise method advocated here. These functions always involve at least one variable that appears more than once. Suppose, for example, that in place of the function (3.27), we had to evaluate

$$q = y - x\sin y.$$

This function is the difference of two terms, y and $x \sin y$, but these two terms are definitely *not* independent because both depend on y. Thus, to estimate the uncertainty, we would have to treat the terms as dependent (that is, add their uncertainties directly, not in quadrature). Under some circumstances, this treatment may seriously overestimate the true uncertainty. Faced with a function like this, we must recognize that a stepwise calculation may give an uncertainty that is unnecessarily big, and the only satisfactory procedure is then to use the general formula to be developed in Section 3.11.

3.9 Examples

In this and the next section, I give three examples of the type of calculation encountered in introductory laboratories. None of these examples is especially complicated; in fact, few real problems are much more complicated than the ones described here.

Example: Measurement of g with a Simple Pendulum

As a first example, suppose that we measure g, the acceleration of gravity, using a simple pendulum. The period of such a pendulum is well known to be $T = 2\pi\sqrt{l/g}$, where l is the length of the pendulum. Thus, if l and T are measured, we can find g as

$$g = 4\pi^2 l/T^2. \tag{3.28}$$

This result gives g as the product or quotient of three factors, $4\pi^2$, l, and T^2. If the various uncertainties are independent and random, the fractional uncertainty in our answer is just the quadratic sum of the fractional uncertainties in these factors. The factor $4\pi^2$ has no uncertainty, and the fractional uncertainty in T^2 is twice that in T:

$$\frac{\delta(T^2)}{T^2} = 2\frac{\delta T}{T}.$$

Thus, the fractional uncertainty in our answer for g will be

$$\frac{\delta g}{g} = \sqrt{\left(\frac{\delta l}{l}\right)^2 + \left(2\frac{\delta T}{T}\right)^2}. \tag{3.29}$$

Suppose we measure the period T for one value of the length l and get the results[5]

$$l = 92.95 \pm 0.1 \text{ cm},$$
$$T = 1.936 \pm 0.004 \text{ s}.$$

[5] Although at first sight an uncertainty $\delta T = 0.004$ s may seem unrealistically small, you can easily achieve it by timing several oscillations. If you can measure with an accuracy of 0.1 s, as is certainly possible with a stopwatch, then by timing 25 oscillations you will find T within 0.004 s.

Our best estimate for g is easily found from (3.28) as

$$g_{\text{best}} = \frac{4\pi^2 \times (92.95 \text{ cm})}{(1.936 \text{ s})^2} = 979 \text{ cm/s}^2.$$

To find our uncertainty in g using (3.29), we need the fractional uncertainties in l and T. These are easily calculated (in the head) as

$$\frac{\delta l}{l} = 0.1\% \quad \text{and} \quad \frac{\delta T}{T} = 0.2\%.$$

Substituting into (3.29), we find

$$\frac{\delta g}{g} = \sqrt{(0.1)^2 + (2 \times 0.2)^2}\,\% = 0.4\%;$$

from which

$$\delta g = 0.004 \times 979 \text{ cm/s}^2 = 4 \text{ cm/s}^2.$$

Thus, based on these measurements, our final answer is

$$g = 979 \pm 4 \text{ cm/s}^2.$$

Having found the measured value of g and its uncertainty, we would naturally compare these values with the accepted value of g. If the latter has its usual value of 981 cm/s^2, the present value is entirely satisfactory.

If this experiment is repeated (as most such experiments should be) with different values of the parameters, the uncertainty calculations usually do not need to be repeated in complete detail. We can often easily convince ourselves that all uncertainties (in the answers for g) are close enough that no further calculations are needed; sometimes the uncertainty in a few representative values of g can be calculated and the remainder estimated by inspection. In any case, the best procedure is almost always to record the various values of l, T, and g and the corresponding uncertainties in a single table. (See Problem 3.40.)

Example: Refractive Index Using Snell's Law

If a ray of light passes from air into glass, the angles of incidence i and refraction r are defined as in Figure 3.5 and are related by Snell's law, $\sin i = n \sin r$, where n is the refractive index of the glass. Thus, if you measure the angles i and r, you

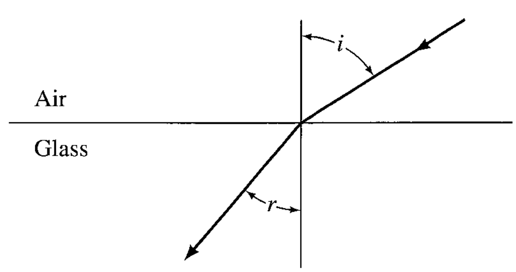

Figure 3.5. The angles of incidence i and refraction r when a ray of light passes from air into glass.

can calculate the refractive index n as

$$n = \frac{\sin i}{\sin r}. \tag{3.30}$$

The uncertainty in this answer is easily calculated. Because n is the quotient of $\sin i$ and $\sin r$, the fractional uncertainty in n is the quadratic sum of those in $\sin i$ and $\sin r$:

$$\frac{\delta n}{n} = \sqrt{\left(\frac{\delta \sin i}{\sin i}\right)^2 + \left(\frac{\delta \sin r}{\sin r}\right)^2}. \tag{3.31}$$

To find the fractional uncertainty in the sine of any angle θ, we note that

$$\begin{aligned}\delta \sin\theta &= \left|\frac{d\sin\theta}{d\theta}\right| \delta\theta \\ &= |\cos\theta|\, \delta\theta \text{ (in rad).}\end{aligned}$$

Thus, the fractional uncertainty is

$$\frac{\delta \sin\theta}{|\sin\theta|} = |\cot\theta|\ \delta\theta \text{ (in rad).} \tag{3.32}$$

Suppose we now measure the angle r for a couple of values of i and get the results shown in the first two columns of Table 3.1 (with all measurements judged to be uncertain by $\pm 1°$, or 0.02 rad). The calculation of $n = \sin i/\sin r$ is easily carried out as shown in the next three columns of Table 3.1. The uncertainty in n can then be found as in the last three columns; the fractional uncertainties in $\sin i$ and $\sin r$ are calculated using (3.32), and finally the fractional uncertainty in n is found using (3.31).

Table 3.1. Finding the refractive index.

i (deg) all ± 1	r (deg) all ± 1	$\sin i$	$\sin r$	n	$\frac{\delta \sin i}{\lvert\sin i\rvert}$	$\frac{\delta \sin r}{\lvert\sin r\rvert}$	$\frac{\delta n}{n}$
20	13	0.342	0.225	1.52	5%	8%	9%
40	23.5	0.643	0.399	1.61	2%	4%	5%

Before making a series of measurements like the two shown in Table 3.1, you should think carefully how best to record the data and calculations. A tidy display like that in Table 3.1 makes the recording of data easier and reduces the danger of mistakes in calculation. It is also easier for the reader to follow and check.

If you repeat an experiment like this one several times, the error calculations can become tedious if you do them for each repetition. If you have a programmable calculator, you may decide to write a program to do the repetitive calculations automatically. You should recognize, however, that you almost never need to do the error calculations for all the repetitions; if you find the uncertainties in n corresponding to the smallest and largest values of i (and possibly a few intermediate values), then these uncertainties suffice for most purposes.

3.10 A More Complicated Example

The two examples just given are typical of many experiments in the introductory physics laboratory. A few experiments require more complicated calculations, however. As an example of such an experiment, I discuss here the measurement of the acceleration of a cart rolling down a slope.[6]

Example: Acceleration of a Cart Down a Slope

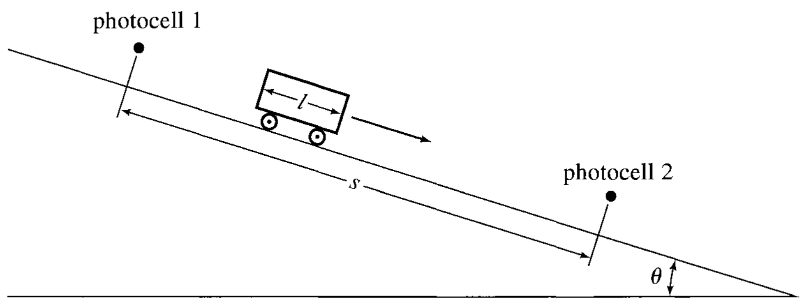

Figure 3.6. A cart rolls down an incline of slope θ. Each photocell is connected to a timer to measure the time for the cart to pass it.

Let us consider a cart rolling down an incline of slope θ as in Figure 3.6. The expected acceleration is $g\sin\theta$ and, if we measure θ, we can easily calculate the expected acceleration and its uncertainty (Problem 3.42). We can measure the actual acceleration a by timing the cart past two photocells as shown, each connected to a timer. If the cart has length l and takes time t_1 to pass the first photocell, its speed there is $v_1 = l/t_1$. In the same way, $v_2 = l/t_2$. (Strictly speaking, these speeds are the cart's *average* speeds while passing the two photocells. However, provided l is small, the difference between the average and instantaneous speeds is unimportant.) If the distance between the photocells is s, then the well-known formula $v_2^2 = v_1^2 + 2as$ implies that

$$\begin{aligned} a &= \frac{v_2^2 - v_1^2}{2s} \\ &= \left(\frac{l^2}{2s}\right)\left(\frac{1}{t_2^2} - \frac{1}{t_1^2}\right). \end{aligned} \tag{3.33}$$

Using this formula and the measured values of l, s, t_1, and t_2, we can easily find the observed acceleration and its uncertainty.

[6]If you wish, you could omit this section without loss of continuity or return to study it in connection with Problem 3.42.

One set of data for this experiment, including uncertainties, was as follows (the numbers in parentheses are the corresponding percentage uncertainties, as you can easily check):

$$\begin{aligned} l &= 5.00 \pm 0.05 \text{ cm } (1\%) \\ s &= 100.0 \pm 0.2 \text{ cm } (0.2\%) \\ t_1 &= 0.054 \pm 0.001 \text{ s } (2\%) \\ t_2 &= 0.031 \pm 0.001 \text{ s } (3\%). \end{aligned} \tag{3.34}$$

From these values, we can immediately calculate the first factor in (3.33) as $l^2/2s = 0.125$ cm. Because the fractional uncertainties in l and s are 1% and 0.2%, that in $l^2/2s$ is

$$\begin{aligned} (\text{fractional uncertainty in } l^2/2s) &= \sqrt{\left(2\frac{\delta l}{l}\right)^2 + \left(\frac{\delta s}{s}\right)^2} \\ &= \sqrt{(2 \times 1\%)^2 + (0.2\%)^2} = 2\%. \end{aligned}$$

(Note how the uncertainty in s makes no appreciable contribution and could have been ignored.) Therefore,

$$l^2/2s = 0.125 \text{ cm} \pm 2\%. \tag{3.35}$$

To calculate the second factor in (3.33) and its uncertainty, we proceed in steps. Because the fractional uncertainty in t_1 is 2%, that in $1/t_1^2$ is 4%. Thus, since $t_1 = 0.054$ s,

$$1/t_1^2 = 343 \pm 14 \text{ s}^{-2}.$$

In the same way, the fractional uncertainty in $1/t_2^2$ is 6% and

$$1/t_2^2 = 1041 \pm 62 \text{ s}^{-2}.$$

Subtracting these (and combining the errors in quadrature), we find

$$\frac{1}{t_2^2} - \frac{1}{t_1^2} = 698 \pm 64 \text{ s}^{-2} \quad (\text{or } 9\%). \tag{3.36}$$

Finally, according to (3.33), the required acceleration is the product of (3.35) and (3.36). Multiplying these equations together (and combining the fractional uncertainties in quadrature), we obtain

$$\begin{aligned} a &= (0.125 \text{ cm} \pm 2\%) \times (698 \text{ s}^{-2} \pm 9\%) \\ &= 87.3 \text{ cm/s}^2 \pm 9\% \end{aligned}$$

or

$$a = 87 \pm 8 \text{ cm/s}^2. \tag{3.37}$$

This answer could now be compared with the expected acceleration $g \sin\theta$, if the latter had been calculated.

When the calculations leading to (3.37) are studied carefully, several interesting features emerge. First, the 2% uncertainty in the factor $l^2/2s$ is completely swamped by the 9% uncertainty in $(1/t_2^2) - (1/t_1^2)$. If further calculations are needed for subsequent trials, the uncertainties in l and s can therefore be ignored (so long as a quick check shows they are still just as unimportant).

Another important feature of our calculation is the way in which the 2% and 3% uncertainties in t_1 and t_2 grow when we evaluate $1/t_1^2$, $1/t_2^2$, and the difference $(1/t_2^2) - (1/t_1^2)$, so that the final uncertainty is 9%. This growth results partly from taking squares and partly from taking the difference of large numbers. We could imagine extending the experiment to check the constancy of a by giving the cart an initial push, so that the speeds v_1 and v_2 are both larger. If we did, the times t_1 and t_2 would get smaller, and the effects just described would get worse (see Problem 3.42).

3.11 General Formula for Error Propagation[7]

So far, we have established three main rules for the propagation of errors: that for sums and differences, that for products and quotients, and that for arbitrary functions of one variable. In the past three sections, we have seen how the computation of a complicated function can often be broken into steps and the uncertainty in the function computed stepwise using our three simple rules.

In this final section, I give a single general formula from which all three of these rules can be derived and with which any problem in error propagation can be solved. Although this formula is often rather cumbersome to use, it is useful theoretically. Furthermore, there are some problems in which, instead of calculating the uncertainty in steps as in the past three sections, you will do better to calculate it in one step by means of the general formula.

To illustrate the kind of problem for which the one-step calculation is preferable, suppose that we measure three quantities x, y, and z and have to compute a function such as

$$q = \frac{x + y}{x + z} \tag{3.38}$$

in which a variable appears more than once (x in this case). If we were to calculate the uncertainty δq in steps, then we would first compute the uncertainties in the two sums $x + y$ and $x + z$, and then that in their quotient. Proceeding in this way, we would completely miss the possibility that errors in the numerator due to errors in x may, to some extent, cancel errors in the denominator due to errors in x. To understand how this cancellation can happen, suppose that x, y, and z are all positive numbers, and consider what happens if our measurement of x is subject to error. If we *overestimate x*, we *overestimate both* $x + y$ and $x + z$, and (to a large extent) these overestimates cancel one another when we calculate $(x + y)/(x + z)$. Similarly, an *underestimate* of x leads to *underestimates* of *both* $x + y$ and $x + z$, which again cancel when we form the quotient. In either case, an error in x is substantially

[7] You can postpone reading this section without a serious loss of continuity. The material covered here is not used again until Section 5.6.

canceled out of the quotient $(x + y)/(x + z)$, and our stepwise calculation completely misses these cancellations.

Whenever a function involves the same quantity more than once, as in (3.38), some errors may cancel themselves (an effect sometimes called *compensating errors*). If this cancellation is possible, then a stepwise calculation of the uncertainty may overestimate the final uncertainty. The only way to avoid this overestimation is to calculate the uncertainty in one step by using the method I will now develop.[8]

Let us suppose at first that we measure two quantities x and y and then calculate some function $q = q(x, y)$. This function could be as simple as $q = x + y$ or something more complicated such as $q = (x^3 + y)\sin(xy)$. For a function $q(x)$ of a *single* variable, we argued that if the best estimate for x is the number x_{best}, then the best estimate for $q(x)$ is $q(x_{\text{best}})$. Next, we argued that the extreme (that is, largest and smallest) probable values of x are $x_{\text{best}} \pm \delta x$ and that the corresponding extreme values of q are therefore

$$q(x_{\text{best}} \pm \delta x). \tag{3.39}$$

Finally, we used the approximation

$$q(x + u) \approx q(x) + \frac{dq}{dx}u \tag{3.40}$$

(for any small increment u) to rewrite the extreme probable values (3.39) as

$$q(x_{\text{best}}) \pm \left|\frac{dq}{dx}\right| \delta x, \tag{3.41}$$

where the absolute value is to allow for the possibility that dq/dx may be negative. The result (3.41) means that $\delta q \approx |dq/dx|\delta x$.

When q is a function of two variables, $q(x, y)$, the argument is similar. If x_{best} and y_{best} are the best estimates for x and y, we expect the best estimate for q to be

$$q_{\text{best}} = q(x_{\text{best}}, y_{\text{best}})$$

in the usual way. To estimate the uncertainty in this result, we need to generalize the approximation (3.40) for a function of two variables. The required generalization is

$$q(x + u, y + v) \approx q(x, y) + \frac{\partial q}{\partial x}u + \frac{\partial q}{\partial y}v, \tag{3.42}$$

where u and v are any small increments in x and y, and $\partial q/\partial x$ and $\partial q/\partial y$ are the so-called *partial derivatives* of q with respect to x and y. That is, $\partial q/\partial x$ is the result of differentiating q with respect to x while treating y as fixed, and vice versa for $\partial q/\partial y$. [For further discussion of partial derivatives and the approximation (3.42), see Problems 3.43 and 3.44.]

The extreme probable values for x and y are $x_{\text{best}} \pm \delta x$ and $y_{\text{best}} \pm \delta y$. If we insert these values into (3.42) and recall that $\partial q/\partial x$ and $\partial q/\partial y$ may be positive or

[8] Sometimes a function that involves a variable more than once can be rewritten in a different form that does not. For example, $q = xy - xz$ can be rewritten as $q = x(y - z)$. In the second form, the uncertainty δq can be calculated in steps without any danger of overestimation.

negative, we find, for the extreme values of q,

$$q(x_{\text{best}}, y_{\text{best}}) \pm \left(\left| \frac{\partial q}{\partial x} \right| \delta x + \left| \frac{\partial q}{\partial y} \right| \delta y \right).$$

This means that the uncertainty in $q(x, y)$ is

$$\delta q \approx \left| \frac{\partial q}{\partial x} \right| \delta x + \left| \frac{\partial q}{\partial y} \right| \delta y. \tag{3.43}$$

Before I discuss various generalizations of this new rule, let us apply it to rederive some familiar cases. Suppose, for instance, that

$$q(x, y) = x + y; \tag{3.44}$$

that is, q is just the sum of x and y. The partial derivatives are both one,

$$\frac{\partial q}{\partial x} = \frac{\partial q}{\partial y} = 1, \tag{3.45}$$

and so, according to (3.43),

$$\delta q \approx \delta x + \delta y. \tag{3.46}$$

This is just our original provisional rule that the uncertainty in $x + y$ is the sum of the uncertainties in x and y.

In much the same way, if q is the product $q = xy$, you can check that (3.43) implies the familiar rule that the fractional uncertainty in q is the sum of the fractional uncertainties in x and y (see Problem 3.45).

The rule (3.43) can be generalized in various ways. You will not be surprised to learn that when the uncertainties δx and δy are independent and random, the sum (3.43) can be replaced by a sum in quadrature. If the function q depends on more than two variables, then we simply add an extra term for each extra variable. In this way, we arrive at the following general rule (whose full justification will appear in Chapters 5 and 9).

Uncertainty in a Function of Several Variables

Suppose that $x, \ldots, z$ are measured with uncertainties $\delta x, \ldots, \delta z$ and the measured values are used to compute the function $q(x, \ldots, z)$. If the uncertainties in $x, \ldots, z$ are independent and random, then the uncertainty in q is

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x} \delta x\right)^2 + \cdots + \left(\frac{\partial q}{\partial z} \delta z\right)^2}. \tag{3.47}$$

In any case, it is never larger than the ordinary sum

$$\delta q \leqslant \left| \frac{\partial q}{\partial x} \right| \delta x + \cdots + \left| \frac{\partial q}{\partial z} \right| \delta z. \tag{3.48}$$

Although the formulas (3.47) and (3.48) look fairly complicated, they are easy to understand if you think about them one term at a time. For example, suppose for a moment that among all the measured quantities, $x, y, \ldots, z$, only x is subject to any uncertainty. (That is, $\delta y = \ldots = \delta z = 0$.) Then (3.47) contains only one term and we would find

$$\delta q = \left| \frac{\partial q}{\partial x} \right| \delta x \quad (\text{if } \delta y = \cdots = \delta z = 0). \tag{3.49}$$

In other words, the term $|\partial q/\partial x|\delta x$ by itself is the uncertainty, or partial uncertainty, in q caused by the uncertainty in x alone. In the same way, $|\partial q/\partial y|\delta y$ is the partial uncertainty in q due to δy alone, and so on. Referring back to (3.47), we see that the total uncertainty in q is the quadratic sum of the partial uncertainties due to each of the separate uncertainties $\delta x, \delta y, \ldots, \delta z$ (provided the latter are independent). This is a good way to think about the result (3.47), and it suggests the simplest way to use (3.47) to calculate the total uncertainty in q: First, calculate the partial uncertainties in q due to $\delta x, \delta y, \ldots, \delta z$ separately, using (3.49) and its analogs for $y, \ldots, z$; then simply combine these separate uncertainties in quadrature to give the total uncertainty as in (3.47).

In the same way, whether or not the uncertainties $\delta x, \delta y, \ldots, \delta z$ are independent, the rule (3.48) says that the total uncertainty in q never exceeds the simple sum of the partial uncertainties due to each of $\delta x, \delta y, \ldots, \delta z$ separately.

Example: Using the General Formula (3.47)

To determine the quantity

$$q = x^2 y - xy^2,$$

a scientist measures x and y as follows:

$$x = 3.0 \pm 0.1 \quad \text{and} \quad y = 2.0 \pm 0.1.$$

What is his answer for q and its uncertainty, as given by (3.47)?

His best estimate for q is easily seen to be $q_{\text{best}} = 6.0$. To find δq, we follow the steps just outlined. The uncertainty in q due to δx alone, which we denote by δq_x, is given by (3.49) as

$$\begin{aligned} \delta q_x &= (\text{error in } q \text{ due to } \delta x \text{ alone}) \\ &= \left| \frac{\partial q}{\partial x} \right| \delta x \\ &= |2xy - y^2|\delta x = |12 - 4| \times 0.1 = 0.8. \end{aligned} \tag{3.50}$$

Similarly, the uncertainty in q due to δy is

$$\begin{aligned} \delta q_y &= (\text{error in } q \text{ due to } \delta y \text{ alone}) \\ &= \left| \frac{\partial q}{\partial y} \right| \delta y \\ &= |x^2 - 2xy|\delta y = |9 - 12| \times 0.1 = 0.3. \end{aligned} \tag{3.51}$$

Finally, according to (3.47), the total uncertainty in q is the quadratic sum of these two partial uncertainties:

$$\begin{aligned} \delta q &= \sqrt{(\delta q_x)^2 + (\delta q_y)^2} \qquad (3.52) \\ &= \sqrt{(0.8)^2 + (0.3)^2} = 0.9. \end{aligned}$$

Thus, the final answer for q is

$$q = 6.0 \pm 0.9.$$

The use of (3.47) or (3.48) to calculate uncertainties is reasonably straightforward if you follow the procedure used in this example; that is, first calculate each separate contribution to δq and only then combine them to give the total uncertainty. This procedure breaks the problem into calculations small enough that you have a good chance of getting them right. It has the further advantage that it lets you see which of the measurements $x, y, \ldots, z$ are the main contributors to the final uncertainty. (For instance, in the example above, the contribution $\delta q_y = 0.3$ was so small compared with $\delta q_x = 0.8$ that the former could almost be ignored.)

Generally speaking, when the stepwise propagation described in Sections 3.8 to 3.10 is possible, it is usually simpler than the general rules (3.47) or (3.48) discussed here. Nevertheless, you must recognize that if the function $q(x, \ldots, z)$ involves any variable more than once, there may be compensating errors; if so, a stepwise calculation may overestimate the final uncertainty, and calculating δq in one step using (3.47) or (3.48) is better.

Principal Definitions and Equations of Chapter 3

THE SQUARE-ROOT RULE FOR A COUNTING EXPERIMENT

If we observe the occurrences of an event that happens at random but with a definite average rate and we count ν occurrences in a time T, our estimate for the true average number is

$$(\text{average number of events in time } T) = \nu \pm \sqrt{\nu}. \qquad \text{[See (3.2)]}$$

RULES FOR ERROR PROPAGATION

The rules of error propagation refer to a situation in which we have found various quantities, $x, \ldots, w$ with uncertainties $\delta x, \ldots, \delta w$ and then use these values to calculate a quantity q. The uncertainties in $x, \ldots, w$ "propagate" through the calculation to cause an uncertainty in q as follows:

Sums and Differences: If

$$q = x + \cdots + z - (u + \cdots + w),$$

then

$$\delta q = \sqrt{(\delta x)^2 + \cdots + (\delta z)^2 + (\delta u)^2 + \cdots + (\delta w)^2}$$

(provided all errors are independent and random)

and

$$\delta q \leqslant \delta x + \cdots + \delta z + \delta u + \cdots + \delta w$$

(always). [See (3.16) & (3.17)]

Products and Quotients: If

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w},$$

then

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \cdots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \cdots + \left(\frac{\delta w}{w}\right)^2}$$

(provided all errors are independent and random)

and

$$\frac{\delta q}{|q|} \leqslant \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|}$$

(always). [See (3.18) & (3.19)]

Measured Quantity Times Exact Number: If B is known exactly and

$$q = Bx,$$

then

$$\delta q = |B|\,\delta x \quad \text{or, equivalently,} \quad \frac{\delta q}{|q|} = \frac{\delta x}{|x|}. \qquad \text{[See (3.9)]}$$

Uncertainty in a Power: If n is an exact number and

$$q = x^n,$$

then

$$\frac{\delta q}{|q|} = |n|\,\frac{\delta x}{|x|}. \qquad \text{[See (3.26)]}$$

Uncertainty in a Function of One Variable: If $q = q(x)$ is any function of x, then

$$\delta q = \left|\frac{dq}{dx}\right| \delta x. \qquad \text{[See (3.23)]}$$

Sometimes, if $q(x)$ is complicated and if you have written a program to calculate $q(x)$ then, instead of differentiating $q(x)$, you may find it easier to use the equivalent formula,

$$\delta q = |q(x_{\text{best}} + \delta x) - q(x_{\text{best}})|. \qquad \text{[See Problem 3.32]}$$

General Formula for Error Propagation: If $q = q(x, \ldots, z)$ is any function of $x, \ldots, z$, then

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \cdots + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

(provided all errors are independent and random)

and

$$\delta q \leqslant \left|\frac{\partial q}{\partial x}\right| \delta x + \cdots + \left|\frac{\partial q}{\partial z}\right| \delta z$$

(always). [See (3.47) & (3.48)]

Problems for Chapter 3

For Section 3.2: The Square-Root Rule for a Counting Experiment

3.1. ★ To measure the activity of a radioactive sample, two students count the alpha particles it emits. Student A watches for 3 minutes and counts 28 particles; Student B watches for 30 minutes and counts 310 particles. **(a)** What should Student A report for the average number emitted in 3 minutes, with his uncertainty? **(b)** What should Student B report for the average number emitted in 30 minutes, with her uncertainty? **(c)** What are the fractional uncertainties in the two measurements? Comment.

3.2. ★ A nuclear physicist studies the particles ejected by a beam of radioactive nuclei. According to a proposed theory, the average rates at which particles are ejected in the forward and backward directions should be equal. To test this theory, he counts the total number ejected forward and backward in a certain 10-hour interval and finds 998 forward and 1,037 backward. **(a)** What are the uncertainties associated with these numbers? **(b)** Do these results cast any doubt on the theory that the average rates should be equal?

3.3. ★ Most of the ideas of error analysis have important applications in many different fields. This applicability is especially true for the square-root rule (3.2) for counting experiments, as the following example illustrates. The normal average incidence of a certain kind of cancer has been established as 2 cases per 10,000 people per year. The suspicion has been aired that a certain town (population 20,000) suffers a high incidence of this cancer because of a nearby chemical dump. To test this claim, a reporter investigates the town's records for the past 4 years and finds 20 cases of the cancer. He calculates that the expected number is 16 (check this) and concludes that the observed rate is 25% more than expected. Is he justified in claiming that this result proves that the town has a higher than normal rate for this cancer?

3.4. ★★ As a sample of radioactive atoms decays, the number of atoms steadily diminishes and the sample's radioactivity decreases in proportion. To study this effect, a nuclear physicist monitors the particles ejected by a radioactive sample for 2 hours. She counts the number of particles emitted in a 1-minute period and repeats the measurement at half-hour intervals, with the following results:

Time elapsed, t (hours):	0.0	0.5	1.0	1.5	2.0
Number counted, ν, in 1 min:	214	134	101	61	54

(a) Plot the number counted against elapsed time, including error bars to show the uncertainty in the numbers. (Neglect any uncertainty in the elapsed time.) **(b)** Theory predicts that the number of emitted particles should diminish exponentially as $\nu = \nu_o \exp(-rt)$, where (in this case) $\nu_o = 200$ and $r = 0.693\ \text{h}^{-1}$. On the same graph, plot this expected curve and comment on how well the data seem to fit the theoretical prediction.

For Section 3.3: Sums and Differences; Products and Quotients

3.5. ★ Using the provisional rules (3.4) and (3.8), compute the following:
- **(a)** $(5 \pm 1) + (8 \pm 2) - (10 \pm 4)$
- **(b)** $(5 \pm 1) \times (8 \pm 2)$
- **(c)** $(10 \pm 1)/(20 \pm 2)$
- **(d)** $(30 \pm 1) \times (50 \pm 1)/(5.0 \pm 0.1)$

3.6. ★ Using the provisional rules (3.4) and (3.8), compute the following:
- **(a)** $(3.5 \pm 0.1) + (8.0 \pm 0.2) - (5.0 \pm 0.4)$
- **(b)** $(3.5 \pm 0.1) \times (8.0 \pm 0.2)$
- **(c)** $(8.0 \pm 0.2)/(5.0 \pm 0.4)$
- **(d)** $(3.5 \pm 0.1) \times (8.0 \pm 0.2)/(5.0 \pm 0.4)$

3.7. ★ A student makes the following measurements:

$$a = 5 \pm 1 \text{ cm}, \quad b = 18 \pm 2 \text{ cm}, \quad c = 12 \pm 1 \text{ cm},$$
$$t = 3.0 \pm 0.5 \text{ s}, \quad m = 18 \pm 1 \text{ gram}$$

Using the provisional rules (3.4) and (3.8), compute the following quantities with their uncertainties and percentage uncertainties: **(a)** $a + b + c$, **(b)** $a + b - c$, **(c)** ct, and **(d)** mb/t.

3.8. ★★ The binomial theorem states that for any number n and any x with $|x| < 1$,

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots$$

(a) Show that if n is a positive integer, this infinite series terminates (that is, has only a finite number of nonzero terms). Write the series down explicitly for the cases $n = 2$ and $n = 3$. **(b)** Write down the binomial series for the case $n = -1$. This case gives an infinite series for $1/(1 + x)$, but when x is small, you get a good approximation if you keep just the first two terms:

$$\frac{1}{1 + x} \approx 1 - x,$$